Waves, optics and atomic physics

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Contents

Chapter 1

Systems with finite number of degrees of freedom

1.1 Harmonic oscillator

A system described by the harmonic oscillator equation,

$$\ddot{x} + \omega^2 x = 0, \tag{1.1}$$

where $x(t) : \mathbb{R} \to \mathbb{R}$, is the simplest system exhibiting oscillatory motion. This mathematical model is used to describe a number of physical systems more or less accurately. For example, to describe the oscillation of a weight of mass m on a spring of stiffness k,

$$m\ddot{x} + kx = 0, \qquad \omega = \sqrt{\frac{k}{m}}, \tag{1.2}$$

or to describe the motion of a mathematical pendulum near the equilibrium position,

$$ml\ddot{\varphi} + mg\varphi = 0, \qquad \omega = \sqrt{\frac{g}{l}}, \tag{1.3}$$

or to describe the current flow in an LC circuit,

$$L\ddot{I} + \frac{1}{C}I = 0, \qquad \omega = \frac{1}{\sqrt{LC}}. \tag{1.4}$$

Thus, different physical systems lead to the same mathematical description of their behavior. This is characteristic of a number of wave phenomena that we will deal with. Typically, we will consider a particular physical system, often mechanical, create a mathematical model of it, and investigate the wave phenomena that arise from it. The knowledge obtained will then have general validity for any system behaving according to the same mathematical model.

1.2 Mathematical supplement: Complex numbers and exponentials

Complex number $z \in \mathbb{C}$ is a number of the form $z = a + ib$, where $a, b \in \mathbb{R}$ and i is an imaginary unit with the property $i^2 = -1$. The addition and multiplication of these numbers is defined by "naturally".

A complex conjugate number $\bar{z}$ is a complex number $\bar{z} = a - ib$. The formula $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ holds. The magnitude of a complex number is defined as $|z| = \sqrt{a^2 + b^2}$, this expression can be written as $|z| = \sqrt{z\bar{z}}$.

Real and imaginary parts. We define the functions $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ and $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ called the real and imaginary parts using the rules

$$\mathrm{Re}\, z = a, \qquad \mathrm{Im}\, z = b$$

(note that the imaginary part does not contain an imaginary unit!). If the real part is zero, we call the number *pure imaginary*. The functions Re and Im are *real* linear, i.e.

$$\mathrm{Re}\,(z_1 + z_2) = \mathrm{Re}\, z_1 + \mathrm{Re}\, z_2, \qquad \mathrm{Re}\,(\alpha z) = \alpha\, \mathrm{Re}\, z, \qquad \alpha \in \mathbb{R},$$

identically for Im. Note $\mathrm{Re}\,(z_1 z_2) \neq (\mathrm{Re}\, z_1)(\mathrm{Re}\, z_2)$ (same for Im). These functions can be expressed simply by complex conjugation:

$$\mathrm{Re}\, z = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}\, z = \frac{z - \bar{z}}{2i}.$$

Complex exponential. Consider $z = a + ib \in \mathbb{C}$. We define the complex exponential by the following relation:

$$e^z := e^{a+ib} = e^a e^{ib} = e^a \left(\cos b + i \sin b\right).$$

Special case for $a = 0$,

$$e^{ib} = \cos b + i \sin b, \quad b \in \mathbb{R},$$

is called Euler's formula [1]. The following holds $|e^{ib}| = 1$ and we can therefore write $|e^z| = e^a$.

Polar form of a complex number. Any complex number $z \in \mathbb{C}$ can be written in the form $z = |z|e^{i\varphi}$, $\varphi \in \mathbb{R}$. We call the number φ the *argument* of a complex number (this number is not uniquely given, any integer multiple of 2π can be added). The argument φ is the solution of the equations

$$\cos\varphi = \frac{\mathrm{Re}\, z}{|z|}, \qquad \sin\varphi = \frac{\mathrm{Im}\, z}{|z|}.$$

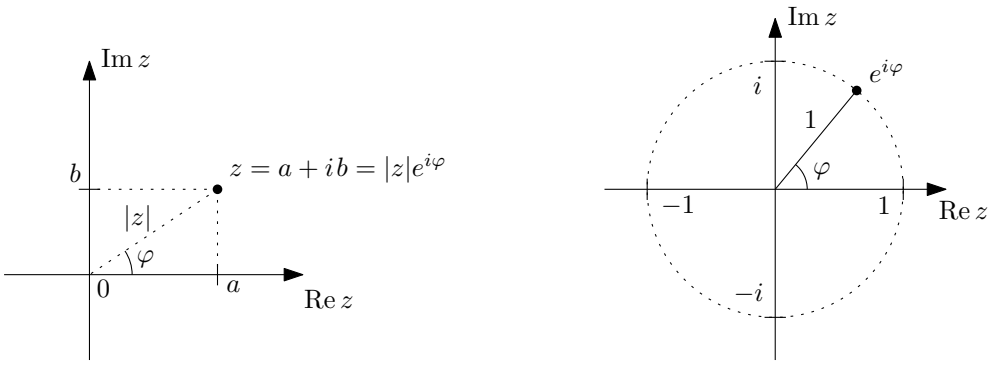
These equations are often formally[2] combined into the equation

$$\mathrm{tg}\,\varphi = \frac{\mathrm{Im}\, z}{\mathrm{Re}\, z}.$$

Gaussian (complex) plane. Complex numbers can be represented as points on a (two-dimensional) plane, where the Cartesian axes are the real and imaginary parts of the complex numbers, see Figure 1.1.

[1] The special case of which is "the most beautiful mathematical identity" $e^{i\pi} = -1$.

[2] In this notation, we lose information about whether $\varphi \in \langle 0, \pi)$ and/or $\varphi \in \langle \pi, 2\pi)$.



(a) Cartesian and goniometric representations of the complex number z shown in the Gaussian plane.

(b) The unit circle in the Gaussian plane formed by complex numbers of the form $e^{i\varphi}$, $\varphi \in \mathbb{R}$.

Figure 1.1: The Gaussian plane is used to graphically represent complex numbers, where we plot the real part on the horizontal axis and the imaginary part on the vertical axis.

The addition of complex numbers then has the geometric meaning of adding two-dimensional vectors in the Gaussian plane. The number $e^{i\varphi}$ represents a number on a unit circle. An intuitive notion of multiplication of complex numbers is obtained from goniometric notation:

$$z_1 z_2 = |z_1|e^{i\varphi_1}|z_2|e^{i\varphi_2} = |z_1||z_2|e^{i(\varphi_1+\varphi_2)}.$$

Thus, multiplication by the number $e^{i\varphi}$ represents a rotation by the angle φ in the complex plane. Multiplication by $|z|$ represents scaling in this plane.

Complex notation for goniometric functions. The following relations follow directly from Euler's formula:

$$\cos\varphi = \operatorname{Re} e^{i\varphi} = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \qquad \sin\varphi = \operatorname{Im} e^{i\varphi} = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

1.3 Mathematical supplement: Ordinary linear differential equations with constant coefficients

Consider the following differential equation for the function $x(t)$

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0, \tag{1.5}$$

where the coefficients $a_i \in \mathbb{R}$ are **real constants**, $a_n \neq 0$. Let us look for a solution of the form $x(t) = e^{\lambda t}$. Substituting into (1.5) (and factoring out $e^{\lambda t}$) we obtain the so-called *characteristic polynomial* of this equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{1.6}$$

Let λ be the root of this polynomial with *real coefficients*, then $\bar{\lambda}$ is also a root. That is, either $\lambda \in \mathbb{R}$ or $\lambda, \bar{\lambda} \in \mathbb{C}$ are a pair of complex conjugated roots.

If all the roots are different from each other (have multiplicity one) we get the so-called *system of fundamental solutions*

$$\left\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\right\}, \tag{1.7}$$

Since the polynomial (1.6) is of degree n, we have n fundamental solutions[3].

Superposition Principle. If $x_1(t)$, $x_2(t)$ are solutions of the equation (1.5), then so is $c_1x_1(t)+c_2x_2(t)$, where c_1 and c_2 are arbitrary constants (generally complex). The superposition principle applies only to linear differential equations.

The general solution of the equation (1.5) is a general complex linear combination of the fundamental solutions:

$$x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \ldots + c_ne^{\lambda_n t}. \tag{1.8}$$

Complex and real solutions. If $x(t)$ is a solution of equation (1.5), then so are $\bar{x}(t)$, $\operatorname{Re} x(t)$ and $\operatorname{Im} x(t)$[4] solutions as well. Thus, if we have complex fundamental solutions (corresponding to complex-conjugated roots of $\lambda = a + ib, \bar{\lambda} = a - ib$) of $e^{\lambda t}$ and $e^{\bar{\lambda} t}$, we can change to real fundamental solutions:

$$\operatorname{Re} e^{\lambda t} = \frac{e^{\lambda t} + e^{\bar{\lambda} t}}{2} = e^{at}\cos bt, \qquad \operatorname{Im} e^{\lambda t} = \frac{e^{\lambda t} - e^{\bar{\lambda} t}}{2i} = e^{at}\sin bt. \tag{1.9}$$

This is equivalent to the restriction of the complex integration constants by the condition $c_2 = \bar{c}_1$ in the linear combination $c_1e^{\lambda t} + c_2e^{\bar{\lambda} t}$.

The general real solution is then a real linear combination of real fundamental solutions.

Initial conditions. The general solution of an ordinary differential equation of n-th order (i.e., the highest derivative is of n-th order) depends on n integration constants. These are determined from the initial conditions. Typically by specifying the values of the zeroth through to $n-1$-th derivatives at a given time t_0:

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = v_0, \quad \ldots, \quad x^{(n-1)}(t_0) = x_0^{(n-1)}. \tag{1.10}$$

Homogeneous equation. The general solution we wrote was a solution of the so-called *homogeneous equation* – the equation with vanishing right-hand side. If we add a specified function $f(t)$ to the right-hand side,

$$a_nx^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_2\ddot{x} + a_1\dot{x} + a_0x = f(t), \tag{1.11}$$

we speak of the so-called *inhomogeneous equation.* The solution of the inhomogeneous equation can be split into two parts because of linearity:

$$x(t) = x_{\mathrm{hom}}(t) + x_{\mathrm{part}}(t). \tag{1.12}$$

Part $x_{\mathrm{hom}}(t)$ is the solution of the original equation with vanishing right-hand side. Part $x_{\mathrm{part}}(t)$ is an arbitrary (particular) solution satisfying the equation with right-hand side (1.11), the function $x_{\mathrm{part}}(t)$ is called *particular solution.*

Harmonic oscillator. The harmonic oscillator equation (1.1) has the characteristic polynomial of the form

$$\lambda^2 + \omega^2 = 0 \tag{1.13}$$

with roots $\lambda_{1,2} = \pm i\omega$ and a fundamental system of solutions

$$\{e^{i\omega t}, e^{-i\omega t}\}. \tag{1.14}$$

[3]If λ is a root with multiplicity k, then this root has k fundamental solutions of the form

$$\{e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}\}.$$

[4]If we write down the complex conjugate of the equation (1.5), then if $x(t)$ was a solution of the original equation (1.5), then by the realness of the coefficients of a_i, i.e., $a_i = \bar{a}_i$, the $\bar{x}(t)$ is a solution as well. If $x(t)$ and $\bar{x}(t)$ are solutions, then by the superposition principle, $\operatorname{Re} x(t) = \frac{1}{2}(x(t) + \bar{x}(t))$ and $\operatorname{Im} x(t) = \frac{1}{2i}(x(t) - \bar{x}(t))$ are solutions as well.

Thus, the general complex solution is of the form

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}. \tag{1.15}$$

We shift to the real solution by applying Re and Im to one of the solutions (e.g., $e^{i\omega t}$) or by restricting the constants c_1 and c_2 by the condition $c_1 = \bar{c_2} = \frac{a-ib}{2}$:

$$x(t) = a\,\mathrm{Re}\, e^{i\omega t} + b\,\mathrm{Im}\, e^{i\omega t} = a\cos\omega t + b\sin\omega t = \frac{1}{2}(a-ib)e^{i\omega t} + \frac{1}{2}(a+ib)e^{-i\omega t}. \tag{1.16}$$

The initial conditions leading to a particular solution are usually the initial position and initial velocity at time $t = 0$:

$$x(0) = x_0, \qquad \dot{x}(0) = v_0. \tag{1.17}$$

1.4 Mathematical supplement: Mean values

Consider the function $f(x) : \mathbb{R} \to \mathbb{R}$. Its mean value on the interval $\langle x_1, x_2\rangle$ is defined as

$$\langle f\rangle_{\langle x_1,x_2\rangle} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx.$$

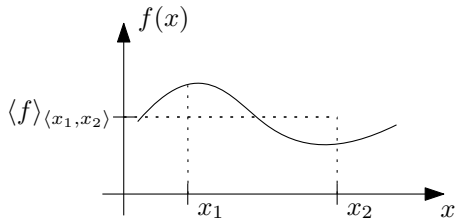


Figure 1.2: Illustration of the mean value of a function. The area under the graph of the function f between the points x_1 and x_2, $S_1 = \int_{x_1}^{x_2} f(x)\,dx$, is the same as the area of the rectangle $S_2 = \langle f\rangle_{\langle x_1,x_2\rangle}(x_2 - x_1)$.

You can define the mean over the whole $\mathbb{R}$ by a limit

$$\langle f\rangle \equiv \langle f\rangle_{\langle -\infty,\infty\rangle} = \lim_{x'\to\infty}\frac{1}{2x'}\int_{-x'}^{x'} f(x)\,dx.$$

If the function f is periodic with period L, its mean is given by the mean over an arbitrary interval of length L:

$$\langle f\rangle = \langle f\rangle_{\langle x,x+L\rangle} = \frac{1}{L}\int_{x}^{x+L} f(x')\,dx', \qquad \text{where } x \in \mathbb{R} \text{ is arbitrary.}$$

For trigonometric functions, the following relations hold:

$$\langle \sin\omega t\rangle = \langle\cos\omega t\rangle = 0, \qquad \langle\sin^2\omega t\rangle = \langle\cos^2\omega t\rangle = \frac{1}{2}. \tag{1.18}$$

For functions of several variables it is necessary to specify in which variable the averaging is performed. In this text, we will only perform time averaging, so we will often not mention it explicitly. For example, if we have a function $f(\vec{r}, t)$, then after time averaging we are left with a function of only the position vector $\vec{r}$: $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, $\langle f\rangle : \mathbb{R}^3 \to \mathbb{R}$.

1.5 Damped harmonic oscillator

The equation of a damped harmonic oscillator is

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0. \tag{1.19}$$

The notation $\Gamma = 2\delta$ is sometimes introduced, where δ is the so-called *decrement of attenuation*. Assuming a solution of the form $x(t) = e^{\lambda t}$, the characteristic polynomial is then of the form

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0. \tag{1.20}$$

If we consider a small (so-called *subcritical*) damping $\delta < \omega_0$, then the solution (1.20) is a pair of complex-conjugated roots

$$\lambda_{1,2} = -\delta \pm i\sqrt{\omega_0^2 - \delta^2}. \tag{1.21}$$

Thus, the resulting complex and real solution is

$$x(t) = e^{-\delta t}\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right), \qquad x(t) = e^{-\delta t}\left(a\cos\omega t + b\sin\omega t\right), \tag{1.22}$$

where we have defined $\omega = \sqrt{\omega_0^2 - \delta^2}$.

1.6 Driven harmonic oscillator

The equation of the driven harmonic oscillator is obtained by adding a driving force to the right-hand side of the damped oscillator equation:

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = B(t) \tag{1.23}$$

Consider the harmonic driving "force"[5] $B(t) = B\cos(\Omega t)$, or its complex form

$$\hat{B}(t) = Be^{i\Omega t}, \quad B \in \mathbb{R}. \tag{1.24}$$

Let us look only for a partial solution of this equation, which will represent the steady state oscillation of the driven harmonic oscillator. Due to the complexification of the driving force, this solution will also be complex. If we take its real part, we get the solution for the original real driving force. Assume a solution of the form

$$\hat{x}(t) = Ae^{i\Omega t}, \tag{1.25}$$

where $A \in \mathbb{C}$. We can write the number A in polar form, $A = |A|e^{-i\varphi}$, and thus

$$\hat{x}(t) = |A|e^{i(\Omega t - \varphi)}, \tag{1.26}$$

where $|A|$ is the amplitude of the forced oscillations and φ is the phase delay of the oscillations after the driving force.

Substituting our *ansatz* (1.25) into Eq. (1.23), we get

$$A(i\Omega)^2 + 2\delta A(i\Omega) + \omega_0^2 A = B, \tag{1.27}$$

[5]The word force is in quotes because the quantity $B(t)$ has an acceleration dimension, since we have the isolated term $\ddot{x}$ in the differential equation.

From this relation, we trivially express the complex amplitude A:

$$A = \frac{B}{\omega_0^2 - \Omega^2 + 2i\delta\Omega}. \tag{1.28}$$

The amplitude of the driven oscillations $|A|$ is

$$|A| = \sqrt{A\bar{A}} = \frac{B}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2}}. \tag{1.29}$$

Next, we denote C and $-D$ as the real and imaginary parts of A, $A = C - iD$. We expand the expression for A with a complex conjugated denominator and obtain

$$A = B\frac{\omega_0^2 - \Omega^2 - 2i\delta\Omega}{(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2} = C - iD. \tag{1.30}$$

We now easily read off the coefficients of C and D:

$$C = \frac{\omega_0^2 - \Omega^2}{(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2}B, \quad D = \frac{2\delta\Omega}{(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2}B. \tag{1.31}$$

The coefficient C is called the *elastic amplitude* and the coefficient D is called the *absorption amplitude*. The real solution is then of the form

$$x(t) = \mathrm{Re}\,[\hat{x}(t)] = \mathrm{Re}\,[(C - iD)e^{i\Omega t}] = |A|\cos(\Omega t - \varphi) = C\cos\Omega t + D\sin\Omega t. \tag{1.32}$$

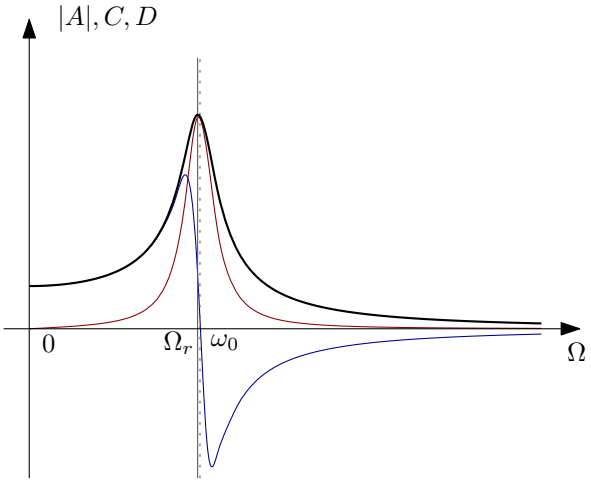


Figure 1.3: The figure shows the resonance curves of a harmonic oscillator driven by a harmonic driving force. The total amplitude $|A|$ is shown in black, the absorption amplitude D in red, and the elastic amplitude C in blue. The maximum amplitude $|A|$ is at $\Omega_r = \sqrt{\omega_0^2 - 2\delta^2} < \omega_0$.

For the phase delay of the driven oscillations, we have the relation

$$\mathrm{tg}\,(-\varphi) = \frac{-D}{C} \quad \rightarrow \quad \mathrm{tg}\,\varphi = \frac{D}{C} = \frac{2\delta\Omega}{\omega_0^2 - \Omega^2}. \tag{1.33}$$

The power delivered by the driving force. Why are the coefficients C and D named elastic and absorption amplitude? Let us study the power delivered to the system by the driving force. Let the physical system considered be an oscillating weight on a spring with damping.

The instantaneous value of the mechanical power is $P(t) = F(t)v(t)$, where $F(t) = mB(t)$. Substituting the expressions for $B(t)$ and $v(t)$, we have

$$P(t) = F(t)v(t) = mB\cos\Omega t\,(-C\Omega\sin\Omega t + D\Omega\cos\Omega t)\,. \tag{1.34}$$

If we now calculate the time-averaged value of this power (over one period) we get the result

$$\langle P\rangle = \frac{1}{2}mB\,\Omega D. \tag{1.35}$$

Thus, the time-averaged value of the delivered power is proportional only to the absorption amplitude D. The power corresponding to the elastic amplitude C only transfers from the driving source to the driven system and back, so that on average no power transfer occurs. If we want to deliver as much energy as possible to the system, we drive the system at a frequency that corresponds to the maximum of the absorption amplitude D.

Total oscillator energy. The instantaneous energy of an oscillator is given by the sum of its kinetic and potential energy:

$$E(t) = \frac{1}{2}mv(t)^2 + \frac{1}{2}m\omega_0^2x(t)^2 = \frac{1}{2}m|A|^2\left(\Omega^2\sin^2(\Omega t-\alpha) + \omega_0^2\cos^2(\Omega t-\alpha)\right). \tag{1.36}$$

If we again calculate the time-average value over one period, we obtain

$$\langle E\rangle = \frac{1}{4}m|A|^2\left(\Omega^2+\omega_0^2\right). \tag{1.37}$$

Thus we see that the total energy is proportional to the total amplitude of the oscillation. If we want to accumulate as much energy as possible in the oscillator (resonance phenomenon), we drive the system at a frequency that corresponds to the maximum of the total amplitude $|A|$.

The maximum of the amplitude $|A|$ is at point $\Omega_A = \sqrt{\omega_0^2 - 2\delta^2}$, the approximate value of the maximum of the amplitude D is at point[6] $\Omega_D \approx \sqrt{\omega_0^2-\delta^2}$. For small values of damping δ we can consider $\Omega_A \approx \Omega_D \approx \omega_0$.

Quality factor. A quantity called quality factor is often introduced,

$$Q = 2\pi\frac{\langle E\rangle}{\langle P\rangle T_0} = 2\pi\frac{\langle E\rangle}{\langle E_0\rangle}, \tag{1.38}$$

which (up to a multiple 2π) indicates how much energy delivered per oscillation period $\langle E_0\rangle$ is stored in the oscillating system.

Example. *Wireless charging.* A schematic of a wireless charging circuit is shown in Figure 1.4. The driving force is generated by the charging station with the transmitting coil, which due to mutual inductance induces the voltage in the coil L in the RLC charging circuit. The resistor R represents the appliance to be charged. Tuning the driving frequency of the voltage $U(t)$ to near resonance in the charging circuit causes not only the largest energy transfer (maximum absorption amplitude D) but also the largest voltage amplitude in the charging circuit (maximum total amplitude $|A|$).

[6]The exact value is given by the solution of equation $\frac{dD}{d\Omega} = 0$ with the result

$$\Omega_D = \frac{\sqrt{\omega_0^2 - 2\delta^2 + 2\sqrt{\omega_0^4 - \delta^2\omega_0^2 + \delta^4}}}{\sqrt{3}}.$$

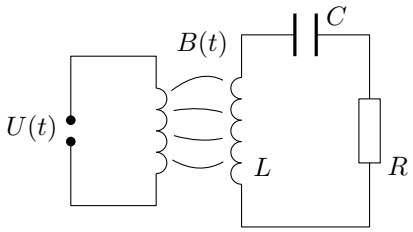


Figure 1.4: Wireless charging principle.

1.7 Systems with n degrees of freedom

Let us now turn our attention to systems with more degrees of freedom. The harmonic oscillator had one degree of freedom – its position was described by a single coordinate x. For a system with n degrees of freedom, we describe its position by Cartesian coordinates $\vec{x} = (x_1, \ldots, x_n)$.

Again, we will consider simple mechanical models such as a system of two weights on springs as in Figure 1.5 described by a pair of coordinates $\vec{x} = (x_1, x_2)$. Or a pair of pendulums connected by a spring as in Figure 1.6 requiring four coordinates $\vec{x} = (x_1, x_2, x_3, x_4)$.

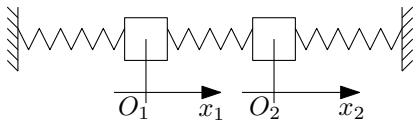


Figure 1.5: Two longitudinally oscillating weights on springs.

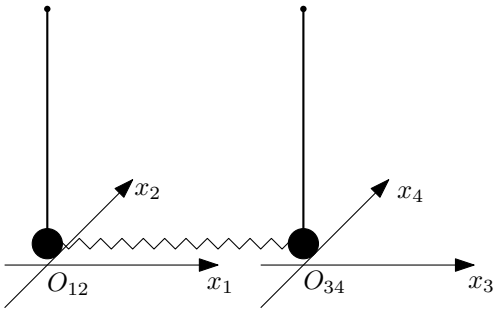


Figure 1.6: Two pendulums connected by a spring.

Consider Newton's equations of motion for a general system of the form

$$m_i \ddot{x}_i = F_i = -\frac{\partial U}{\partial x_i}, \quad i \in \{1, \ldots, n\}, \tag{1.39}$$

where m_i is mass of the body corresponding to the coordinate x_i and the forces in the system are described by a potential function $U(x_1, \ldots, x_n)$.

We will be concerned with the motion (i.e., the solution of the equations of motion) of these systems in the neighborhood of *stable equilibrium position*.

Definition. *Equilibrium position* is the position $\vec{x}_0$ of the system for which $F_i(\vec{x}_0) = 0$ holds for $\forall i \in \{1, \ldots, n\}$, i.e., no forces are acting at this position (if the system was at rest at $\vec{x}_0$, it will remain at this position). The forces are vanishing when the potential at $\vec{x}_0$ has *stationary point*, i.e.

$$F_i(\vec{x}_0) = \left.\frac{\partial U}{\partial x_i}\right|_{\vec{x}=\vec{x}_0} = 0, \qquad \forall i \in \{1, \ldots, n\}. \tag{1.40}$$

Definition. *Stable equilibrium position* is the equilibrium position where the matrix

$$\mathbb{U}_{ij} := \left.\frac{\partial^2 U}{\partial x_i \partial x_j}\right|_{\vec{x}=\vec{x}_0} \tag{1.41}$$

is *positive definite*. In other words, the potential at the point $\vec{x}_0$ has a *local minimum*. Due to the commutation of partial derivatives, this matrix is symmetric $\mathbb{U}_{ij} = \mathbb{U}_{ji}$.

The method for solving the equations of motion, which will be described below, has one technical requirement – the equilibrium position must be at the origin of the coordinates. This is achieved by introducing new coordinates $\vec{\xi}$ by substitution $\vec{\xi} = \vec{x} - \vec{x}_0$, where $\vec{x}_0$ are the constant coordinates of the chosen stable equilibrium position. Substituting into the equations (1.39), we have trivially on the left hand side

$$m_i \ddot{x}_i = m_i \frac{d^2}{dt^2}(\xi_i + x_{0i}) = m_i \ddot{\xi}_i. \tag{1.42}$$

On the right-hand side we have to make the substitution in the potential function $U(\vec{x})$:

$$\tilde{U}(\vec{\xi}) := U \circ \vec{x}(\vec{\xi}) = U(\vec{\xi} + \vec{x}_0), \tag{1.43}$$

where we have defined a new function $\tilde{U}$ of the variables $\vec{\xi}$. The inverse relation is $U(\vec{x}) = \tilde{U}(\vec{x} - \vec{x}_0)$, and after differentiating this relation (where on the right hand side we have the derivative of the composite function) we get[7] (using Einstein's summation rule)

$$\frac{\partial U}{\partial x_i} = \frac{\partial \tilde{U}}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \frac{\partial \tilde{U}}{\partial \xi_j}\delta_{ij} = \frac{\partial \tilde{U}}{\partial \xi_i}. \tag{1.44}$$

Thus, the equation of motion has the same form as in the original coordinates $\vec{x}$:

$$m_i \ddot{\xi}_i = -\frac{\partial \tilde{U}}{\partial \xi_i}. \tag{1.45}$$

1.7.1 Approximation of small oscillations

For a general potential, the equations (1.45) can be very difficult to solve. We will resort to the so-called *small oscillations* approximation – we will study the behavior of the system near the equilibrium position. This is done by expanding the potential function $\tilde{U}$ into a Taylor series[8] (around the equilibrium position, i.e., around the point $\vec{\xi} = 0$) and we keep only the first

[7] Consider a function $f(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ and k functions $g_i(y_1, \ldots, y_l) : \mathbb{R}^l \to \mathbb{R}$. We get the function $h(y_1, \ldots, y_l) : \mathbb{R}^l \to \mathbb{R}$ by composition

$$h(y_1, \ldots, y_l) = f\left(g_1(y_1, \ldots, y_l), \ldots, g_k(y_1, \ldots, y_l)\right).$$

Then the *chain rule* states

$$\frac{\partial h}{\partial y_i} = \sum_{m=1}^{k} \frac{\partial f}{\partial x_m}\frac{\partial g_m}{\partial y_i}.$$

This rule is an extension of the rule for differentiating a composite function of one variable, $[f(g(x))]' = f'(g(x))\, g'(x)$, for multiple variables. Here, we have $\tilde{U}(\vec{\xi}) \sim f(x_1, \ldots, x_n)$, $\xi_i(\vec{x}) \sim g_i(y_1, \ldots, y_n)$.

[8] Compare with the Taylor series expansion of a function of one variable! For the function $f(x)$ around the point 0, it has the following form:

$$f(x) = f(0) + \left.\frac{df}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_{x=0} x^2 + \ldots$$

nonzero term:

$$\tilde{U}(\vec{\xi}) = \tilde{U}(0) + \sum_{i=1}^{n} \left.\frac{\partial \tilde{U}}{\partial \xi_i}\right|_{\vec{\xi}=0} \xi_i + \frac{1}{2} \sum_{i,j=1}^{n} \left.\frac{\partial^2 \tilde{U}}{\partial \xi_i \partial \xi_j}\right|_{\vec{\xi}=0} \xi_i\, \xi_j + \dots \tag{1.46}$$

Let us now examine the different orders of the expansion. The zero order represents the value of the potential at the equilibrium position $U_0 = \tilde{U}(0)$. We can choose this to be zero, $U_0 := 0$, since a shift of the potential by a constant does not alter the equations of motion. The first order derivative represents the (minus) force evaluated at the equilibrium position,

$$F_i(0) = -\left.\frac{\partial \tilde{U}}{\partial \xi_i}\right|_{\vec{\xi}=0}, \tag{1.47}$$

but this is zero by the definition of the equilibrium position! The first non-zero order is therefore the second. If we denote the matrix $\mathbb{U}$ as

$$\mathbb{U}_{ij} = \left.\frac{\partial^2 \tilde{U}}{\partial \xi_i \partial \xi_j}\right|_{\vec{\xi}=0} \tag{1.48}$$

(which is a constant numerical matrix), which we will call *the potential energy matrix*, we can write the potential $\tilde{U}$ in the form

$$\tilde{U}(\vec{\xi}) = \frac{1}{2} \sum_{i,j=1}^{n} \mathbb{U}_{ij} \xi_i \xi_j + \dots, \tag{1.49}$$

where the three dots indicate higher orders of the Taylor expansion, which we neglect in the *small oscillations* approximation. The matrix $\mathbb{U}$ is symmetric due to the commutativity of partial derivatives, $\mathbb{U}_{ij} = \mathbb{U}_{ji}$. We have replaced the arbitrarily complicated potential function $\tilde{U}$ by a quadratic polynomial in the deviations from the equilibrium position $\vec{\xi}$.

Let us now substitute the approximated potential (1.49) to the right-hand side of the equations of motion (1.47) and manipulate it (using Einstein's summation rule):

$$-F_i = \frac{\partial \tilde{U}}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\frac{1}{2}\mathbb{U}_{jk}\xi_j\xi_k\right) = \frac{1}{2}\mathbb{U}_{jk}\left(\delta_{ij}\xi_k + \xi_j\delta_{ik}\right) = \frac{1}{2}\left(\mathbb{U}_{ik}\xi_k + \mathbb{U}_{ji}\xi_j\right) = \mathbb{U}_{ij}\xi_j, \tag{1.50}$$

where we have used the symmetry of the matrix $\mathbb{U}$ (and renaming the summation index) in the last equation. By approximating the potential, we have achieved that the equations of motion are *linear*!

$$m_i\ddot{\xi}_i + \sum_{j=1}^{n} \mathbb{U}_{ij}\xi_j = 0. \tag{1.51}$$

If we introduce another matrix $\mathbb{T} = \mathrm{diag}\,(m_1, \dots, m_n)$, which we will call *kinetic energy matrix*[9], we can write

$$\sum_{j=1}^{n} \left(\mathbb{T}_{ij}\ddot{\xi}_j + \mathbb{U}_{ij}\xi_j\right) = 0,$$

[9]The kinetic energy of the system can be written as

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{\xi}_i^2 = \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{T}_{ij}\dot{\xi}_i\dot{\xi}_j.$$

or written in a matrix form

$$\boxed{\mathbb{T}\ddot{\vec{\xi}} + \mathbb{U}\vec{\xi} = 0.} \tag{1.52}$$

This is the final form[10] of the equations of motion we will try to solve. The matrices $\mathbb{T}$ and $\mathbb{U}$ are symmetric positive definite constant matrices.

1.7.2 Method of modes

Intuitively, we expect that when the system is slightly perturbed from a stable equilibrium position, the system will oscillate about this position. Let us try to assume a solution containing harmonic oscillations $e^{i\omega t}$ (of an as yet undetermined angular frequency ω). Consider a vector function of the form

$$\vec{\xi}(t) = \vec{a}\, e^{i\omega t}, \tag{1.53}$$

where $\vec{a} \in \mathbb{R}^n$ is a constant vector. This form of the solution is called a *mode*. All parts of the system oscillate with the same angular frequency ω and the same phase. The real solutions are obtained, for example, by acting with Re and Im on the obtained complex solutions. By substituting the ansatz (1.53) into the equations (1.52) we obtain

$$(\mathbb{T}\vec{a}(i\omega)^2 + \mathbb{U}\vec{a})e^{i\omega t} = 0. \tag{1.54}$$

After simple manipulation we get:

$$\left(\mathbb{U} - \omega^2\mathbb{T}\right)\vec{a} = 0. \tag{1.55}$$

Of course, we require a non-trivial (non-zero) solution, so we look for frequencies ω such that the problem has a non-zero vector $\vec{a}$ as a solution. If we multiply the equation by $\mathbb{T}^{-1}$ and denote by $\mathbb{A} = \mathbb{T}^{-1}\mathbb{U}$ and $\lambda = \omega^2$, we get the form

$$(\mathbb{A} - \lambda)\vec{a} = 0. \tag{1.56}$$

This is thus the problem of finding the eigenvalues and their corresponding eigenvectors of the matrix $\mathbb{A}$. Thus, we proceed in the same way as in linear algebra. The requirement of non-triviality of the vector $\vec{a}$ is a requirement of non-triviality of the kernel of the operator $\mathbb{U} - \omega^2\mathbb{T}$, which is equivalent to its singularity, which is easily ensured by its zero determinant:

$$\det\left(\mathbb{U} - \omega^2\mathbb{T}\right) = 0. \tag{1.57}$$

This equation is called a *secular equation*. On the left-hand side is the n-th degree polynomial in the variable ω^2. Denote the roots of this polynomial by ω_k^2, $k \in \{1, \ldots, n\}$. We denote the corresponding kernel vectors to these eigenvalues by $\vec{a}_k$, i.e. we solve the equations

$$\left(\mathbb{U} - \omega_k^2\mathbb{T}\right)\vec{a_k} = 0. \tag{1.58}$$

For a given mode, the general solution is a linear superposition

$$\vec{\xi_k}(t) = \vec{a}_k\left(c_1 e^{i\omega_k t} + c_2 e^{-i\omega_k t}\right). \tag{1.59}$$

Moving on to the real solution (by choosing $c_2 = \bar{c}_1$):

$$\vec{\xi_k}(t) = A_k\vec{a}_k\cos(\omega_k t + \varphi_k). \tag{1.60}$$

[10] We could multiply the equations by $\mathbb{T}^{-1}$ to get the form $\ddot{\vec{\xi}} + (\mathbb{T}^{-1}\mathbb{U})\vec{\xi} = 0$.

The general solution found by the method of modes is then the linear superposition of all modes:

$$\vec{\xi}(t) = \sum_{k=1}^{n} A_k \vec{a}_k \cos(\omega_k t + \varphi_k). \tag{1.61}$$

The constants of the angular velocities ω_k and amplitude vectors $\vec{a}_k$ are given by the physical system, i.e., for example, the masses of the individual weights and the stiffnesses of the individual springs. The integration constants of mode amplitudes A_k and phase shifts φ_k are given by the initial conditions, i.e., for example, the initial positions and velocities of the individual weights.

Have we found a complete solution of the equations of motion (1.52) by the method of modes? There are n equations of motion and they are of second order. Thus, we expect the complete solution to depend on the $2n$ integration constants. We found n modes and each contains two integration constants of the mode amplitude and its phase shift, for a total of $2n$ integration constants. Thus, we can conclude that we have found the complete (general) solution of the equations (1.52).

Since we are looking for roots of the form ω^2, we need these to come out positive. This is ensured if the matrices $\mathbb{T}$ and $\mathbb{U}$ are positive definite. For physical systems that we perturb from a stable equilibrium position, this is always satisfied; see Section 1.9 for details.

It may be that some ω_k is a multiple root of a secular equation. Then it is a so-called degenerate problem. However, the only thing that changes is that the corresponding angular frequency ω_k has multiple linearly independent vectors of amplitude $\vec{a}$ (i.e., for the given ω_k the kernel matrix $\mathbb{U} - \omega_k^2 \mathbb{T}$ is multidimensional).

1.7.3 Cookbook

Let's quickly repeat the steps to get to the general solution using the method of modes.

1. Introduce the coordinates $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ that measure the deviation from the equilibrium position.

2. Write the equations of motion in the form $\mathbb{T}\ddot{\vec{\xi}} + \mathbb{U}\vec{\xi} = 0$, where $\mathbb{T}, \mathbb{U} \in \mathbb{R}^{n,n}$ are symmetric constant matrices. If necessary, use the small oscillations approximation.

3. Assume solutions of the form $\vec{\xi}(t) = \vec{a}e^{i\omega t}$, $\vec{a} \in \mathbb{R}^n$ is a constant vector of amplitude ratios.

4. Substitute it into the equations of motion and require non-triviality of the solution, i.e., $\vec{a} \neq 0$. We get $\left(\mathbb{U} - \omega^2 \mathbb{T}\right) \vec{a} = 0$. These conditions lead to the so-called secular equation $\left|\mathbb{U} - \omega^2 \mathbb{T}\right| = 0$.

5. The secular equation is a polynomial of n-th degree in ω^2. Find the corresponding roots ω_k^2. Find the corresponding eigenvectors $\vec{a}_k$ as solutions of equations $\left(\mathbb{U} - \omega_k^2 \mathbb{T}\right) \vec{a}_k = 0$.

6. The general solution of the motion is of the form

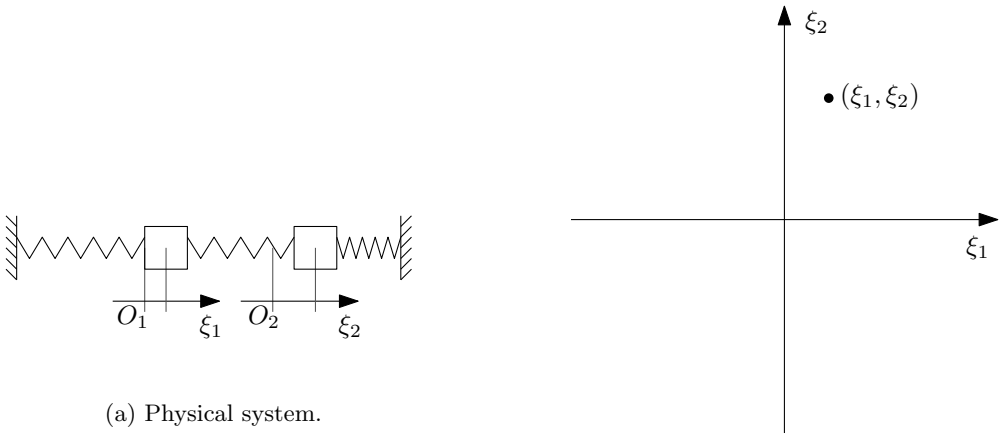
$$\vec{\xi}(t) = \sum_{k=1}^{n} A_k \vec{a}_k \cos\left(\omega_k t + \varphi_k\right).$$

1.8 Normal coordinates

We represent the position of our physical system by n coordinates $\vec{\xi}$. Instead of representing this position, for example, for a mechanical system as the concrete position of the individual bodies,

we introduce the abstract notion of a *configuration space* $\mathcal{C}$. This will represent the abstract set of all possible positions of a given physical system. Each point $p \in \mathcal{C}$ then represents a particular position of, for example, weights and springs.

In our case, the situation is quite simple. Our coordinates $\vec{\xi} \in \mathbb{R}^n$ measure the Cartesian displacements of the bodies from the equilibrium position and thus uniquely determine the position of these bodies. Thus, we can directly consider the space of coordinates[11], $\mathcal{C} = \mathbb{R}^n$ as the configuration space. As an example, Figure 1.7 shows the configuration space for a simple mechanical system.



(a) Physical system.

(b) Abstract configuration space $\mathcal{C} = \mathbb{R}^2$.

Figure 1.7: Configuration space for a longitudinally oscillating system of two masses and three springs.

Normal coordinates $\vec{\eta} \in \mathbb{R}^n$ are defined such that when the system oscillates in i-th mode, the following holds

$$\eta_i = A\cos(\omega_i t + \varphi), \qquad \eta_k = 0, \quad k \neq i. \tag{1.62}$$

This behavior is achieved if we introduce the new coordinate axes η_i in the directions of the vectors $\vec{a}_i$ (the original coordinate axes point in the directions of the standard basis vectors $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$). Thus, the representation of the i-th mode in the original and normal coordinates is

$$\vec{\xi}(t) = A\,\vec{a}_i\cos(\omega_i t + \varphi), \qquad \vec{\eta}(t) = A\,\vec{e}_i\cos(\omega_i t + \varphi). \tag{1.63}$$

The transformation relation between the original $\vec{\xi}$-coordinates and the normal coordinates looks as follows:

$$\vec{\xi} = \mathbb{A}\vec{\eta}, \tag{1.64}$$

where matrix $\mathbb{A}$,

$$\mathbb{A} = \left(\begin{pmatrix} \\ \vec{a}_1 \\ \\ \end{pmatrix} \quad \ldots \quad \begin{pmatrix} \\ \vec{a}_n \\ \\ \end{pmatrix} \right), \tag{1.65}$$

is a matrix consisting of column vectors $\vec{a}_i$. This transformation relation has exactly the property described above – it converts a solution of the form (1.63) for $\vec{\eta}(t)$ to $\vec{\xi}(t)$.

In normal coordinates, the equations of motion are of the form

$$\ddot{\eta}_k + \omega_k^2 \eta_k = 0, \tag{1.66}$$

[11]It depends, of course, whether it makes sense to choose a completely arbitrary position $\vec{\xi} \in \mathbb{R}^n$ as the configuration space. For example, in the small oscillation approximation we require that $\vec{\xi}$ be close to 0. However, we can also look at it in such a way that in the small oscillations approximation we have mathematical equations of motion that make (mathematical) sense for any $\vec{\xi} \in R^n$.

i.e., the system appears mathematically as a set of n independent harmonic oscillators.

The following normalization condition is often added to the definition of normal coordinates

$$\vec{a}_i^T \mathbb{T} \vec{a}_i = 1, \quad \forall i \in \{1, \ldots, n\}. \tag{1.67}$$

This fixes the magnitude of the vectors $\vec{a}_i$ – that is, we fix the scale on the new coordinate axes η_i. These coordinates are still not uniquely defined. We are free to choose the signs of the vectors $\vec{a}_i$, and also in the case of the degenerate problem we are free to choose the basis of the corresponding multidimensional eigenspace. However, the formal definition is as follows:

Definition: Let $\vec{a}_i, i \in \{1, \ldots, n\}$ be the eigenvectors of the problem $(\mathbb{U} - \omega^2 \mathbb{T})\vec{a} = 0$ normalized by the condition $\vec{a}_i^T \mathbb{T} \vec{a}_i = 1, i \in \{1, \ldots, n\}$. *Normal coordinates* $\vec{\eta}$ are defined by the relation

$$\vec{\xi} = \mathbb{A}\vec{\eta}, \tag{1.68}$$

where $\mathbb{A}$ is the matrix defined in (1.65).

In these coordinates

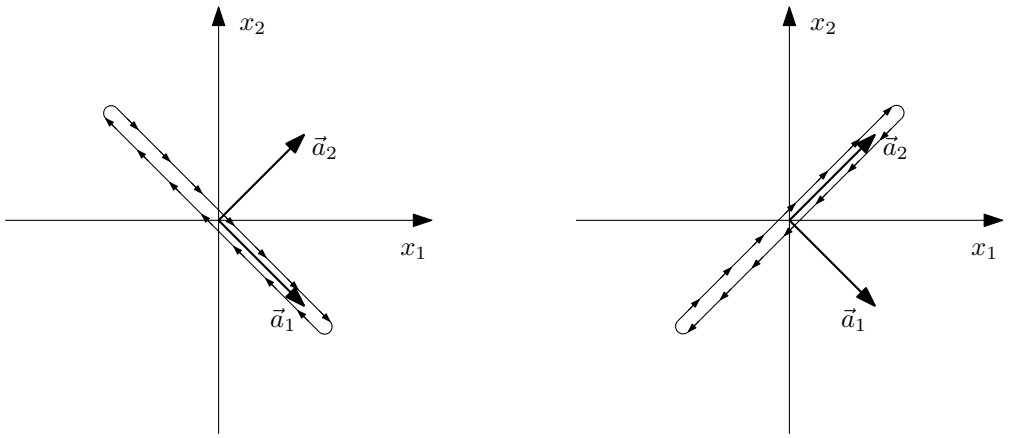
$$\mathbb{A}^T \mathbb{T} \mathbb{A} = \mathbb{I}, \qquad \mathbb{A}^T \mathbb{U} \mathbb{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2), \tag{1.69}$$

see the next section. Substituting the definition (1.68) into the equations of motion (1.52) (and multiplying by the matrix $\mathbb{A}^T$), we arrive at the form of the equations of motion in normal coordinates (1.66) already mentioned, using the relations (1.69).

Example. Let us illustrate the concept of normal coordinates with the example of longitudinal oscillations of two weights on three springs (see Figure 1.7). The general solution is of the form (for $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$):

$$\vec{x}(t) = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_1 t + \varphi_1) + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_2 t + \varphi_2). \tag{1.70}$$

If we excite the first mode, the system will inscribe a line segment in the direction of the vector $\vec{a}_1$; analogously if we build the second mode. See Figure 1.8.



(a) Oscillation of the system in the first mode. (b) Oscillation of the system in the second mode.

Figure 1.8: Schematically plotted trajectories in the configuration space of the system oscillating in each mode.

We would now like to introduce new coordinates (η_1, η_2), which will have the property that if the system oscillates in the first mode, then only the coordinate η_1 will describe the entire motion and the second coordinate η_2 will be zero. Similarly, for a system excited to the second

mode, we want $\eta_1 = 0$ and only η_2 describing the position of the system. If we look again at Figure 1.8, we see that it is sufficient to introduce the new coordinate axes in the directions of the vectors $\vec{a}_1$ and $\vec{a}_2$, as shown in Figure 1.9.

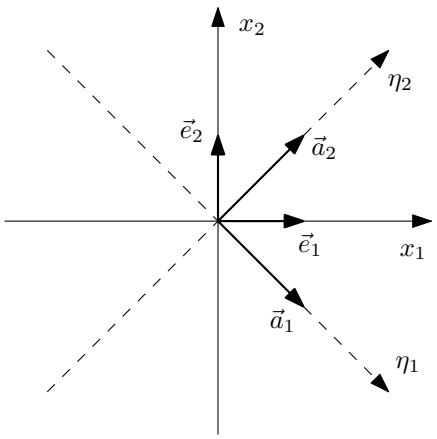


Figure 1.9: Normal coordinates (η_1, η_2) pointing in the directions of the amplitude ratio vectors $\vec{a}_1$ and $\vec{a}_2$. Basis vectors $\vec{e}_1$ and $\vec{e}_2$ pointing in the directions of axes x_1 and x_2.

So we want to go from the coordinates $\vec{x} = (x_1, x_2)$ to the new coordinates $\vec{\eta} = (\eta_1, \eta_2)$ using the transition matrix $\mathbb{A}$ defined as $\vec{x} = \mathbb{A}\,\vec{\eta}$ (and hence also $\vec{\eta} = \mathbb{A}^{-1}\vec{x}$).

We need a matrix $\mathbb{A}$ such that when we pass it the coordinates $\vec{\eta} = (1, 0)^T$ we get the vector $\vec{a}_1$ and when we pass it the vector $\vec{\eta} = (0, 1)^T$ we get the vector $\vec{a}_2$. This condition is obviously satisfied by the following matrix

$$\mathbb{P} = \left(\left(\vec{a}_1\right)\left(\vec{a}_2\right)\right) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{1.71}$$

i.e., we put the individual vectors $\vec{a}_i$ in the columns. Broken down by components (into individual coordinates) we have

$$x_1 = \eta_1 + \eta_2, \quad x_2 = -\eta_1 + \eta_2, \qquad \eta_1 = \frac{x_1 - x_2}{2}, \quad \eta_2 = \frac{x_1 + x_2}{2}. \tag{1.72}$$

In these normal coordinates, the motion of the system looks as follows:

$$\vec{\eta}(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(\omega_1 t + \varphi_1) + A_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(\omega_2 t + \varphi_2) = \begin{pmatrix} A_1 \cos(\omega_1 t + \varphi_1) \\ A_2 \cos(\omega_2 t + \varphi_2) \end{pmatrix}. \tag{1.73}$$

In this example, we have omitted the step of normalizing the vectors $\vec{a}$ for simplicity.

1.9 Small oscillations theoretically

In this section we answer a few nagging questions. Will the eigenvalues $\lambda_k = \omega_k^2$ always be real and positive? Will we always have enough eigenvectors to form a basis $\mathbb{R}^n$? How does the transition to normal coordinates simultaneously diagonalize the matrices $\mathbb{T}$ and $\mathbb{U}$?

In the small oscillations problem we work with the coordinates $\vec{\xi} \in \mathbb{R}^n$. Consider this space as a vector space $V = \mathbb{R}^n$ of dimension n. In it we have a standard basis $\mathcal{E} = (\vec{e}_i)_{i=1}^n$, where

$$\vec{e}_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th component}}, 0, \ldots 0)^T. \tag{1.74}$$

Let us now define the bilinear forms of T and U,

$$T : V \times V \to \mathbb{R}, \qquad U : V \times V \to \mathbb{R}, \tag{1.75}$$

so that $(T)_{\mathcal{E}} = \mathbb{T}$ and $(U)_{\mathcal{E}} = \mathbb{U}$, i.e., the matrices of these bilinear forms[12] in the standard basis $\mathcal{E}$ are just the matrices of kinetic and potential energy. Since the matrix $\mathbb{T}$ is positively definite, so is the form T. The symmetric positive definite bilinear form T defines a scalar product on V, $\langle \bullet, \bullet \rangle_T$. Let us denote by[13] orthonormal basis (according to the scalar product T) $\mathcal{F} = (\vec{f}_i)_{i=1}^n$. That is, $\langle \vec{f}_i, \vec{f}_j \rangle_T = T(\vec{f}_i, \vec{f}_j) = \delta_{ij}$.

The relation between bases $\mathcal{E}$ and $\mathcal{F}$ is given as follows by the regular transition matrix $\mathbb{S}$,

$$\vec{f}_i = \sum_{j=1}^{n} \mathbb{S}_{ji} \vec{e}_j \tag{1.76}$$

(this relationship defines the matrix $\mathbb{S}$). In the transition between bases, the matrices of bilinear forms are transformed

$$(B)_{\mathcal{F}} = \mathbb{S}^T (B)_{\mathcal{E}} \mathbb{S}. \tag{1.77}$$

Specifically, for bilinear forms T and U:

$$(T)_{\mathcal{F}} = \mathbb{S}^T \mathbb{T}\, \mathbb{S} = \mathbb{I}, \qquad (U)_{\mathcal{F}} = \mathbb{S}^T \mathbb{U}\, \mathbb{S}. \tag{1.78}$$

Since $\mathcal{F}$ is an ON basis with respect to T, the matrix $(T)_{\mathcal{F}}$ is of identity. From the transformation relation of the form T we can express $\mathbb{T}^{-1} = \mathbb{S}\,\mathbb{S}^T$. Denoting $\tilde{\mathbb{U}} = (U)_{\mathcal{F}}$, this matrix is symmetric:

$$\tilde{\mathbb{U}}^T = (\mathbb{S}^T \mathbb{U}\, \mathbb{S})^T = \mathbb{S}^T \mathbb{U}^T\, \mathbb{S} = \mathbb{S}^T \mathbb{U}\, \mathbb{S} = \tilde{\mathbb{U}} \tag{1.79}$$

(and real). The theory of linear algebra says that a symmetric matrix has real eigenvalues λ_k and one can choose vectors $\vec{a}_k$ from eigensubspaces to form an orthonormal basis $\mathcal{A} = (\vec{a}_i)_{i=1}^n$ of the vector space V with respect to the scalar product T. Since the form U is positive definite, all its eigenvalues must be positive, $\lambda_i = \omega_i^2 > 0$.

Here, we were looking for the eigenvalues of the matrix $\tilde{\mathbb{U}}$, i.e., we were solving the problem

$$\tilde{\mathbb{U}} (\vec{a})_{\mathcal{F}} = \lambda (\vec{a})_{\mathcal{F}} \quad \to \quad \mathbb{S}^T \mathbb{U}\, \mathbb{S}\, (\vec{a})_{\mathcal{F}} = \lambda (\vec{a})_{\mathcal{F}}. \tag{1.80}$$

(we use the symbol $(\vec{a})_{\mathcal{F}}$ to indicate that the solutions of these linear equations are the components of the vectors $\vec{a}$ in the basis $\mathcal{F}$). After multiplying this equation by the matrix $\mathbb{S}$, we have

$$\mathbb{S}\mathbb{S}^T \mathbb{U} \left(\mathbb{S}\, (\vec{a})_{\mathcal{F}} \right) = \lambda \left(\mathbb{S}\, (\vec{a})_{\mathcal{F}} \right) \quad \to \quad \mathbb{T}^{-1} \mathbb{U} \left(\mathbb{S}\, (\vec{a})_{\mathcal{F}} \right) = \lambda \left(\mathbb{S}\, (\vec{a})_{\mathcal{F}} \right). \tag{1.81}$$

For the transformation relation of the vectors between bases $\mathcal{E}$ and $\mathcal{F}$, $(\vec{v})_{\mathcal{F}} = \mathbb{S}^{-1} (\vec{v})_{\mathcal{E}}$ holds. We will denote the components of the vectors in the standard basis $\mathbb{R}^n$ by $\vec{v}$ only. Thus, $\mathbb{S} (\vec{a})_{\mathcal{F}} = \vec{a}$ and equation (1.80) implies

$$\mathbb{T}^{-1} \mathbb{U} \vec{a} = \lambda \vec{a} \quad \leftrightarrow \quad (\mathbb{U} - \lambda \mathbb{T}) \vec{a} = 0. \tag{1.82}$$

Thus we have shown that the vectors $\vec{a}$ found in the method of modes are nothing but the component representations of the abstract vectors $\vec{a}$ in the standard basis $\mathcal{E}$ in the vector space equipped with the forms T and U.

[12] The matrix of the bilinear form B in basis $\mathcal{E} = (\vec{e}_i)_{i=1}^n$ is defined as $(B)_{\mathcal{E}} = \mathbb{B}_{ij} = B(\vec{e}_i, \vec{e}_j)$

[13] An arbitrary but fixed. This basis will be only auxiliary in our endeavour.

In the basis $\mathcal{A} = (\vec{a}_i)_{i=1}^n$ it holds $(T)_\mathcal{A} = \mathbb{I}$, since it is an ON basis ($\langle \vec{a}_i, \vec{a}_j \rangle_T = \delta_{ij}$), and also $(U)_\mathcal{A} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$, since it is a basis formed by eigenvectors of the form U. Defining the transition matrix $\mathbb{A}$ as

$$\vec{a}_i = \sum_{j=1}^{n} \mathbb{A}_{ji} \vec{e}_j, \tag{1.83}$$

then the following will hold

$$\mathbb{A}^T \mathbb{T} \mathbb{A} = \mathbb{I}, \qquad \mathbb{A}^T \mathbb{U} \mathbb{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2). \tag{1.84}$$

The standard basis has components $(\vec{e}_i)_k = \delta_{ik}$, and after substituting into the definition of the transition matrix $\mathbb{A}$ (1.83) we have

$$(\vec{a}_i)_k = \mathbb{A}_{ki}, \tag{1.85}$$

i.e., that the matrix $\mathbb{A}$ is obtained by stacking the vectors $\vec{a}$ (or their components in the standard basis) side by side in columns:

$$\mathbb{A} = \begin{pmatrix} \begin{pmatrix} \\ \vec{a}_1 \\ \\ \end{pmatrix} & \ldots & \begin{pmatrix} \\ \vec{a}_n \\ \\ \end{pmatrix} \end{pmatrix}. \tag{1.86}$$

1.10 Dampened small oscillations

Equations of motion with damping are

$$\mathbb{T} \ddot{\vec{\xi}} + \mathbb{\Gamma} \dot{\vec{\xi}} + \mathbb{U} \vec{\xi} = 0, \tag{1.87}$$

where the **damping matrix** $\mathbb{\Gamma}$ is symmetric, $\mathbb{\Gamma}^T = \mathbb{\Gamma}$, positively definite (and real). Typically $\mathbb{\Gamma} = \mathrm{diag}\,(2\delta_1, \ldots, 2\delta_n)$.

Consider again *ansatz* in the form of a mode $\vec{\xi}(t) = \vec{a}\, e^{\lambda t}$, where $\vec{a} \in \mathbb{C}^n$ is a constant (generally complex) vector. The substitution yields the so-called quadratic eigenvalue problem

$$\left(\lambda^2 \mathbb{T} + \lambda \mathbb{\Gamma} + \mathbb{U}\right) \vec{a} = 0 \tag{1.88}$$

leading to the secular equation

$$\det\left(\lambda^2 \mathbb{T} + \lambda \mathbb{\Gamma} + \mathbb{U}\right) = 0, \tag{1.89}$$

where the left-hand side is a polynomial of degree $2n$. With sufficiently weak damping, the roots are complex. Since we have a polynomial with real coefficients, the roots are made up of pairs of conjugated roots (if λ is a root, then $\bar{\lambda}$ is a root). To a complex-conjugated root $\bar{\lambda}$ belongs a complex-conjugated vector $\overline{\vec{a}}$. Thus we always have pairs of solutions

$$\vec{\xi}_1(t) = \vec{a}\, e^{\lambda t}, \qquad \vec{\xi}_2(t) = \overline{\vec{a}}\, e^{\bar{\lambda} t}. \tag{1.90}$$

Consider a general linear combination of these solutions (which, by the linearity of the equations of motion, is also a solution) for a given root λ (and its complex-conjugated $\bar{\lambda}$):

$$\vec{\xi}(t) = c_1 \vec{a} e^{\lambda t} + c_2 \overline{\vec{a}} e^{\bar{\lambda} t}. \tag{1.91}$$

If we now write $\lambda = -\kappa + i\omega$, where $\kappa > 0$, and the components of the vector $a_j \in \mathbb{C}$ in polar form as $a_j = |a_j| e^{i\alpha_j}$ we get

$$\xi_j(t) = |a_j| e^{-\kappa t} \left(c_1 e^{i(\omega t + \alpha_j)} + c_2 e^{-i(\omega t + \alpha_j)} \right). \tag{1.92}$$

If we require a real solution, then again the condition $c_2 = \overline{c_1}$ holds. In the same way as for the solution of the harmonic oscillator equation, we can proceed to the solution of the form

$$\xi_j(t) = A|a_j|e^{-\kappa t}\cos(\omega t + \alpha_j + \varphi), \tag{1.93}$$

where the constants A and φ (arising from the constant $c_1 = a - ib$) are given by the initial conditions. We see that for a damped system, in general, not all parts of the system oscillate in phase! Each degree of freedom is phase shifted by an angle α_j!

The general solution is then given by the superposition of all modes (of which there are n):

$$\xi_j(t) = \sum_{k=1}^{n} A_k|a_j^{(k)}|e^{-\kappa_k t}\cos\left(\omega_k t + \alpha_j^{(k)} + \varphi_k\right), \tag{1.94}$$

where we have denoted the individual roots as $\lambda_k = -\kappa_k + i\omega_k$ and their corresponding eigenvectors as $\vec{a}^{(k)}$ and their components as $a_j^{(k)} = |a_j^{(k)}|e^{i\alpha_j^{(k)}}$.

1.11 Driven small oscillations

We now consider the equations of motion of damped small oscillations with a non-zero right-hand side in the form of a harmonic driving force

$$\mathbb{T}\ddot{\vec{\xi}} + \mathbb{\Gamma}\dot{\vec{\xi}} + \mathbb{U}\vec{\xi} = \vec{F}e^{i\Omega t}, \tag{1.95}$$

where $\vec{F} \in \mathbb{C}^n$. Writing $F_j = |F_j|e^{i\beta_j}$ we can interpret the numbers $|F_j|$ as the amplitudes of the driving force at each degree of freedom and β_j as the phase shift of the harmonic driving force at each degree of freedom.

Consider now *ansatz*

$$\vec{\xi}(t) = \vec{a}\,e^{i\Omega t}, \tag{1.96}$$

which is a combination of *ansatz* from the driven oscillations of the harmonic oscillator and from the method of modes, $\vec{a} \in \mathbb{C}^n$. When substituted into the equations of motion, we get

$$\left(-\Omega^2\mathbb{T} + i\Omega\mathbb{\Gamma} + \mathbb{U}\right)\vec{a} = \vec{F}. \tag{1.97}$$

The vector $\vec{a}$ is then obtained from the previous equation by simply inverting the matrix $\mathbb{A} = -\Omega^2\mathbb{T} + i\Omega\mathbb{\Gamma} + \mathbb{U}$:

$$\vec{a} = \left(-\Omega^2\mathbb{T} + i\Omega\mathbb{\Gamma} + \mathbb{U}\right)^{-1}\vec{F}. \tag{1.98}$$

This inversion exists since the determinant of matrix $\mathbb{A}$ is non-zero. Why is this so? The matrix $\mathbb{A}$ is actually the matrix in equation (1.89) when we replace $\lambda = i\Omega$. For weak damping, there are only complex roots λ with a non-zero real part. Thus, the real Ω (i.e., the purely imaginary λ) cannot be a root and hence $\det\mathbb{A}$ must be nonzero.

The actual solution for the real driving force is then

$$\xi_j(t) = \mathrm{Re}\left[|a_j|e^{i\alpha_j}e^{i\Omega t}\right] = |a_j|\cos(\Omega t + \alpha_j), \tag{1.99}$$

where we have again written the components of the vector $\vec{a}$ in polar form as $a_j = |a_j|e^{i\alpha_j}$. The numbers $|a_j|$ then represent the excited amplitudes at each degree of freedom, and the numbers α_j then represent the phase shift relative to the driving force (which itself could be variously shifted at each degree of freedom using the constants β_j).

Example. Consider again the longitudinal oscillations of two weights on springs with equal masses of the bodies and equal stiffnesses of the springs. Let the damping matrix be

$\mathbf{\Gamma} = \mathrm{diag}\,(\gamma, \gamma)$. We have two degrees of freedom and the driving force has two components $\vec{F} = (F_1, F_2)$. Consider $\beta_i = 0$ and hence $F_i \in \mathbb{R}$, $i \in \{1, 2\}$.

Figures 1.10, 1.11 and 1.12 show the resonance curves for three different forms of the driving force $\vec{F}$. The figures plot the absolute values of the individual components of the vector $\vec{a} = (a_1, a_2)$. The component $|a_1|$ is shown in red and the component $|a_2|$ in blue.

Note that the two modes of the unperturbed undamped system have the amplitudes $\vec{a} = (1, 1)$ and $\vec{a} = (1, -1)$. Depending on the form of "driving vector" $\vec{F}$, resonance peaks over the individual eigenfrequencies of the driven system may be present or absent.

This distinguishes the individual figures. In the first one, the driving force is set to excite primarily the mode with vector $\vec{a} = (1, 1)$. In the second, on the other hand, we observe a resonant peak over the second mode $\vec{a} = (1, -1)$. The third figure shows the situation with a driving force that "does not prefer" either mode.

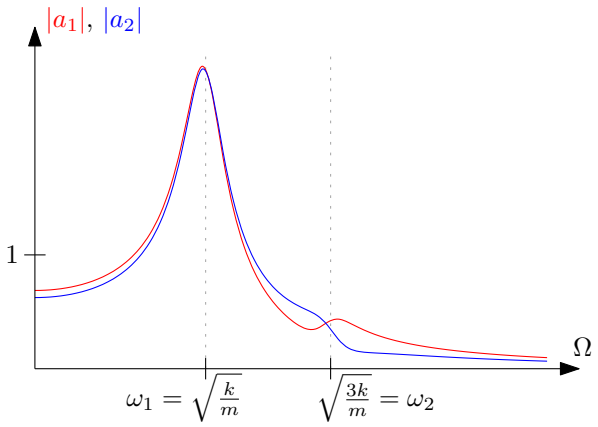


Figure 1.10: The driving force of the form $\vec{F} = (1; 0,75)$ has a resonance peak near the frequency of the first mode.

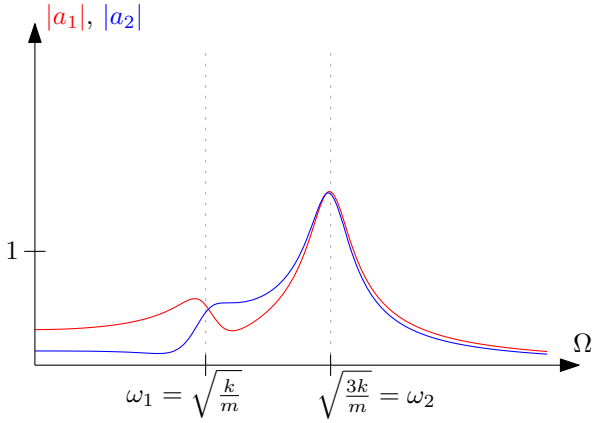


Figure 1.11: The driving force of the form $\vec{F} = (1; -0.75)$ has a resonance peak near the frequency of the second mode.

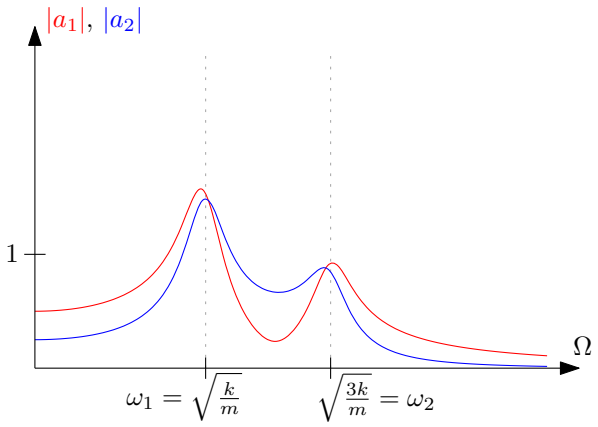


Figure 1.12: The driving force $\vec{F} = (1; 0)$ can excite both modes.

Chapter 2

String vibrations and travelling waves

2.1 The chain of atoms

Let us first examine the behavior of a series of weights of masses m connected by springs of stiffness k, see Figure 2.1. This model can be thought of as a 1D crystal – a so-called chain of atoms. Also, this physical system can be viewed as a discretely modeled string, rope, etc.

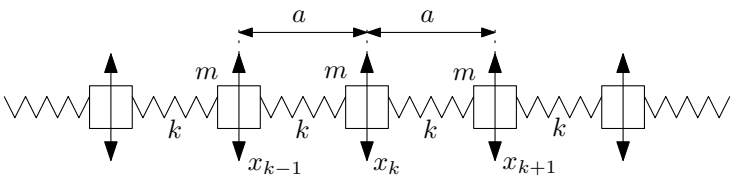


Figure 2.1: A chain of atoms aka transverse oscillations of a series of weights.

Let us consider the transverse oscillations of this system and find the equations of motion for the transverse motion of the k-th weight x_k. Figure 2.2 shows the forces $\vec{F}_1$ and $\vec{F}_2$ from adjacent weights, including their transverse projections $\vec{F}_{1x}$, $\vec{F}_{2x}$.

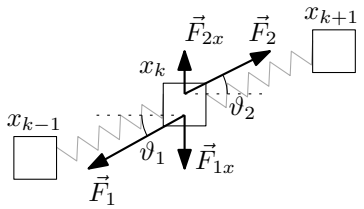


Figure 2.2: Forces acting on the k-th weight.

The equation of motion will be of the form

$$m\ddot{x}_k = F_{1x} + F_{2x} = -|F_1| \sin \vartheta_1 + |F_2| \sin \vartheta_2, \tag{2.1}$$

where ϑ_1 and ϑ_2 are angles between spring directions and horizontal direction. Let the length of the unstretched spring be a_0, and in the equilibrium state the springs are thus stretched to the tension $T = k(a - a_0)$. In the approximation of small oscillations we may consider[1],

[1]The same result would be obtained without much geometrical consideration by expanding the potential

$$U(\Delta x) = \frac{1}{2} k \left(\sqrt{a^2 + \Delta x^2} - a_0 \right)^2$$

that $|F_1| \approx |F_2| \approx T$ and that $\sin\vartheta \approx \operatorname{tg}\vartheta = \frac{\Delta x}{a}$, where Δx denotes the difference of adjacent positions of the weights connected by a given spring. Substituting these assumptions into equation (2.1), we obtain

$$m\ddot{x}_k = \frac{T}{a}\left(x_{k+1} - 2x_k + x_{k-1},\right). \tag{2.2}$$

where

$$\frac{T}{a} = k\left(1 - \frac{a_0}{a}\right) = k' \tag{2.3}$$

represents "effective" spring stiffness of transverse oscillations. If we were to write the corresponding kinetic and potential energy matrices, they would take the following form:

$$\mathbb{T} = \begin{pmatrix} \ddots & & & & \\ & m & & & \\ & & m & & \\ & & & \ddots & \end{pmatrix}, \qquad \mathbb{U} = \frac{T}{a}\begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.4}$$

where, in the case of a finite number of weights in the chain, we would have to add the appropriate boundary conditions for e.g. fixed ends, i.e. $x_0 = 0$ and $x_{N+1} = 0$ (where N is the number of weights).

2.1.1 Solution of the chain motion

Let us look for solutions of the equations of motion (2.2) for an infinite chain of atoms, i.e. we have an infinite set of equations, one for each index $k \in \mathbb{Z}$. Let us take inspiration from the method of modes, where solutions were the form

$$\vec{x}(t) = \vec{a}\, e^{i\omega t}. \tag{2.5}$$

In our case, the vector $\vec{a}$ has infinitely many components. The derivatives of the components of *ansatz* (2.5) look like this

$$x_l(t) = a_l e^{i\omega t}, \qquad \dot{x}_l(t) = i a_l \omega e^{i\omega t}, \qquad \ddot{x}_l(t) = -\omega^2 a_l e^{i\omega t}, \tag{2.6}$$

and after substituting into the equations of motion (and factoring out the exponentials):

$$-m\omega^2 a_l = \frac{T}{a}\left(a_{l+1} - 2a_l + a_{l-1}\right). \tag{2.7}$$

Spatial *ansatz*. Let's try to find the shapes of modes $\vec{a}$ by assuming that harmonic waves can be excited on the chain. In complex notation, we consider $\operatorname{Re} e^{i(kz+\varphi)}$, where we substitute the respective (horizontal) positions of the individual weights, $z = la$, for the coordinate z:

$$a_l = \operatorname{Re} e^{i(kla+\varphi)} = \cos(kla + \varphi), \tag{2.8}$$

the as yet undetermined constant $k = \frac{2\pi}{\lambda}$ is called *wavenumber*. After substituting the ansatz (2.8) into (2.7):

$$\operatorname{Re}\left[\left(2 - \frac{am}{T}\omega^2\right)e^{i(kla+\varphi)}\right] = \operatorname{Re}\left[e^{i(kla+\varphi)}\underbrace{\left(e^{ika} + e^{-ika}\right)}_{2\cos ka}\right]. \tag{2.9}$$

to the second order of the Taylor expansion.

Factoring out e^{ikla} and expressing the angular frequency ω, we have

$$\frac{1-\cos ka}{2} = \frac{am}{4T}\omega^2. \tag{2.10}$$

We rewrite the cosine term using the goniometric formula for double the angle and get the relationship between the angular frequency ω and the wave number k:

$$\omega^2 = \frac{4T}{am}\sin^2\frac{ka}{2}. \tag{2.11}$$

This relation is generally called *dispersion relation*. The pair of parameters ω and k must satisfy the relation (2.11) for the expression

$$x_l(t) = (\mathrm{Re}\, e^{i(kla+\varphi)})(\mathrm{Re}\, e^{i\omega t}) = \cos(kla+\varphi)\cos\omega t \tag{2.12}$$

to be a solution of the equations of motion (2.2). The solution (2.12) is of the form *standing wave*, $X(z)\cos(\omega t+\varphi)$, i.e. the amplitude of the standing waveform $X(z)$ varies harmonically.

The dispersion relation (2.11) has a solution only for a limited range of angular frequencies:

$$\omega \in \left\langle 0, \sqrt{\frac{4T}{am}} \right\rangle. \tag{2.13}$$

In the language of wave numbers k (and wavelengths $\lambda = \frac{2\pi}{k}$ of excited waves), this corresponds to an interval:

$$\frac{ka}{2} \in \left\langle 0, \frac{\pi}{2} \right\rangle \quad \leftrightarrow \quad k \in \left\langle 0, \frac{\pi}{a} \right\rangle \quad \leftrightarrow \quad \lambda \in \langle 2a, +\infty \rangle. \tag{2.14}$$

Thus, on an infinite chain of atoms we can excite standing waves with a continuous range of wavelengths (and to them there is always a corresponding given angular frequency also from the continuous range). But we cannot build a standing wave with a wavelength shorter than twice the distance between the weights $2a$ (and also with an angular frequency greater than $\sqrt{\frac{4T}{am}}$).

For the region of frequencies (and wavelengths) for which we have found solutions to the equations of motion, we say that the chain of atoms is a *transparent medium* – waves of given parameters can exist (propagate) in this medium. For frequencies (and wavelengths) outside this region, the environment is called *reactive*. We will learn more about these two types of environments in the chapter on dispersion relations.

2.1.2 Continuous limit

The weight chain model is a good microscopic model for a string. Now we would like to move to a continuous description by bringing the atoms closer together – we will consider the limit $a \to 0$.

If we consider a finite chain of length L, then the number of weights is approximately $N = \frac{L}{a}$. We would like to keep the tension on the chain constant, $T = \text{const.}$, so We must increase the stiffness of the springs accordingly[2]. $k' = \frac{T}{a}$. We also want to preserve the total string mass M, so we reduce the mass of the individual weights such that const. $= \rho L = M = mN$, i.e., $m = \frac{M}{N}$ (we introduce the notation ρ as the string linear density, $[\rho] = \mathrm{kg.m^{-1}}$).

[2]It's good to remember that this is not some kind of magic "to make it work". If you take a spring of length a and stiffness k and split it in half, you get two springs of length $\frac{a}{2}$ and stiffness $2k$! Increasing the stiffness of the springs by the prescription $\frac{T}{a}$ therefore just means that we keep the springs of the same type, but shorten them. This is a simple consequence of the definition of stiffness k as a force per unit change of length of the spring. It is easy to derive "spring composition laws" for the resulting stiffnesses of parallel and series spring connections.

Next, we introduce a continuous description of the position, where we move from a discrete set of functions describing the position of the individual weights $x_l(t)$, to a function of two variables $\psi(z,t)$ that describes the transverse displacement of the weights at the location z at time t. The weights are only at coordinates $z = la, l \in \mathbb{Z}$,

$$x_l(t) = \psi(la, t), \tag{2.15}$$

for the other points z, the function ψ can take arbitrary values[3]. See also Figure 2.3 for an illustration.

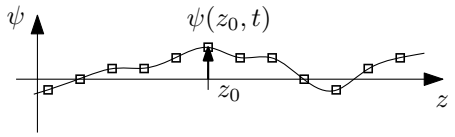


Figure 2.3: Function $\psi(z,t)$ describing the displacement of the weights.

After substituting (2.15) into the equations of motion (2.2), rewriting $m = \rho a$ and labeling $z = la$:

$$(\rho a)\ddot{\psi}(z,t) = T\left(\frac{\psi(z+a,t) - \psi(z,t)}{a} - \frac{\psi(z,t) - \psi(z-a,t)}{a}\right), \tag{2.16}$$

where by the symbol $\dot{\psi}$ we understand $\frac{\partial\psi}{\partial t}$ (and similarly by the symbol ψ' we will understand $\frac{\partial\psi}{\partial z}$). If we introduce the new function $\phi(z,t) = \frac{\psi(z+a,t)-\psi(z,t)}{a}$, we can further write the equations of motion as

$$\rho\,\ddot{\psi}(z,t) = T\frac{\phi(z,t) - \phi(z-a,t)}{a} \tag{2.17}$$

We now use Lagrange's mean value theorem[4]:

$$\rho\ddot{\psi}(z,t) = T\phi'(\xi,t) = T\frac{\psi'(\xi+a,t) - \psi'(\xi,t)}{a}, \tag{2.18}$$

where $\xi \in (z-a, z)$. Applying Lagrange's theorem once more to the new fraction on the right-hand side arising after the substitution from the definition of the function $\phi(z,t)$, we get

$$\rho\,\ddot{\psi}(z,t) = T\psi''(\eta,t), \tag{2.19}$$

where $\eta \in (\xi, \xi+a)$ and $\eta \in (z-a, z+a)$ overall. Schematically, the position of the points ξ and η is shown in Figure 2.4.

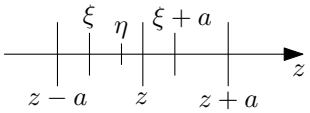


Figure 2.4: Positions of points ξ and η on the axis z. Lagrange's theorem says that $\xi \in (z-a, z)$ and $\eta \in (\xi, \xi+a)$. Overall, we can say that $\eta \in (z-a, z+a)$.

[3]We will need it to be at least twice differentiable.

[4]For a function f differentiable on the interval $\langle a, b\rangle$, there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

In the limit $a \to 0$ goes $\eta \to z$ and so the resulting equation is[5]

$$\boxed{\rho\,\ddot{\psi}(z,t) = T\psi''(z,t), \quad \text{neboli} \quad \rho\,\frac{\partial^2\psi}{\partial t^2}(z,t) = T\frac{\partial^2\psi}{\partial z^2}(z,t).} \tag{2.20}$$

This equation is called the *wave equation* and we will encounter it throughout the rest of the semester.

Note that the original discrete equation (2.16) determined the value of the function $\psi(z,t)$ only at points $z = la$, $l \in \mathbb{Z}$. In the limit $a \to 0$, these points have been condensed and the result is the equation for all $z \in \mathbb{R}$.

2.2 String oscillations and wave equation

Let us derive the wave equation for a string once more, but now straight from the continuous description. Let the string at rest be stretched along the axis z with tension T. The displacement of the point $(z,0,0)$ will be described by the vector $\vec{\psi}(z,t) = (\psi_x, \psi_y, \psi_z)$, see figure (2.5).

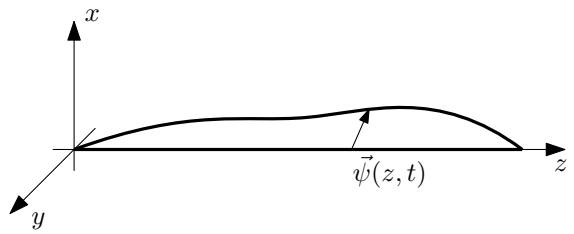


Figure 2.5: The displacement of the string from the equilibrium position is described by the vector $\vec{\psi}(z,t)$.

The components ψ_x and ψ_y represent two independent components of the transverse displacements – we speak of two polarizations of the transverse waves. Component ψ_z represents longitudinal displacements in the string. We will only consider transverse oscillations in one direction, i.e., we will restrict ourselves to the vector of the form $\vec{\psi} = (\psi_x, 0, 0)$ (and we stop writing the index x).

Consider the segment of the string between points z_1 and z_2. According to the first impulse theorem, the change in the total momentum of this piece of string is proportional to the resultant of the external forces,

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)} = \vec{F}_1 + \vec{F}_2 = \vec{F}_x, \tag{2.21}$$

where the forces $\vec{F}_1$ and $\vec{F}_2$, respectively, act on the left and right ends of the chosen section of string. See Figure 2.6.

[5] Alternatively, we can expand the function $\psi(z,t)$ into a Taylor series

$$\psi(z+\Delta z,t) = \psi(z,t) + \frac{\partial\psi}{\partial z}(z,t)\,\Delta z + \frac{1}{2}\frac{\partial^2\psi}{\partial z^2}(z,t)\Delta z^2 + O(\Delta z^3).$$

Substituting this expansion into (2.16) (for $\Delta z \in \{a, -a\}$) we get

$$\rho\ddot{\psi}(z,t) = T\left(\frac{\partial^2\psi}{\partial z^2}(z,t) + \frac{O(a^3)}{a^2}\right).$$

In the limit of $a \to 0$, the term $O(a^3)$ vanishes (the remainder in the Taylor expansion has the property that the limit of $\lim_{\Delta z\to 0}\frac{O(\Delta z^3)}{a^3}$ is finite).

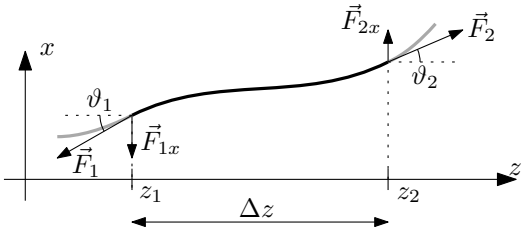


Figure 2.6: Forces $\vec{F}_1$ and $\vec{F}_2$ acting on the section of a string. At z_1 and z_2, the string is deflected by an angle ϑ_1 and ϑ_2, respectively, from the horizontal direction.

We will consider only small transverse oscillations – the following approximations will follow from this assumption. We are only interested in the transverse projections of the forces, $\vec{F}_{1x}$ and $\vec{F}_{2x}$:

$$F_{x1} = -|F_1| \sin\vartheta_1 \approx -T \operatorname{tg}\vartheta_1 = -T\frac{\partial\psi}{\partial z}(z_1,t), \qquad F_{x2} = |F_2| \sin\vartheta_2 \approx T \operatorname{tg}\vartheta_2 = T\frac{\partial\psi}{\partial z}(z_2,t), \tag{2.22}$$

where we have used the assumption that at small deflections the magnitude of the forces F_i differs little from the string tension at equilibrium T, and we can replace the sine function by the tangent function, which we will further replace by the derivative of the function ψ in the direction z.[6]

The longitudinal forces cancel exactly in the small deviations approximation:

$$F_{z1} = -|F_1| \cos\vartheta_1 \approx -T, \qquad F_{z2} = |F_2| \cos\vartheta_2 \approx T. \tag{2.23}$$

Thus, in the first impulse theorem (2.21), only the component x is nontrivial. Its right-hand side can be written as

$$F_x^{(e)} = T\left(\frac{\partial\psi}{\partial z}(z_2,t) - \frac{\partial\psi}{\partial z}(z_1,t)\right) = T\Delta z\frac{\partial^2\psi}{\partial z^2}(\xi,t), \tag{2.24}$$

where we have used Lagrange's mean value theorem, i.e., $\xi \in (z_1, z_2)$ (see the previous section on the continuous chain limit for details), and denoted by $\Delta z = z_2 - z_1$. Let us next look in more detail at the left-hand side of the impulse theorem. We can express the total momentum in terms of the center of gravity velocity $\vec{V}_{CM}$:

$$\vec{P} = M\vec{V}_{CM}, \qquad \vec{V}_{CM} = (V_x, 0, 0), \tag{2.25}$$

where M denotes the total mass of the selected string segment. The position of the center of gravity is given by

$$\vec{R} = \frac{1}{M}\int_l \rho\,\vec{r}\,dl = \left(\psi_{CM}, 0, z_{CM} = \frac{z_1+z_2}{2}\right), \tag{2.26}$$

where ψ_{CM} we denote the position of the center of gravity on the axis x, see Figure 2.7.

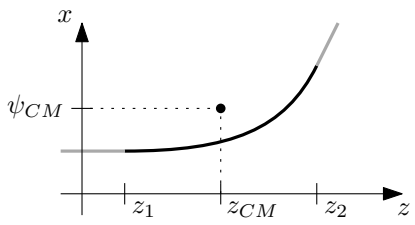


Figure 2.7: The center of gravity of the section of a string.

[6]The positive directions of the angles are chosen to correspond to the positive value of the derivative at a given point. Since the force F_{1x} points in the negative direction of the axis x, we have added an explicit sign at its expression.

Calculation of ψ_{CM} gives

$$\psi_{CM}(t) = \frac{\rho}{M}\int_{z_1}^{z_2} \psi(z,t)\,dz = \frac{1}{\Delta z}\int_{z_1}^{z_2} \psi(z,t)\,dz = \psi(\eta,t)\frac{\Delta z}{\Delta z}, \tag{2.27}$$

where we have used the integral mean value theorem[7], $\eta \in (z_1, z_2)$. Thus, the left-hand side of the impulse theorem has the form

$$\frac{dP_x}{dt} = M\frac{d^2}{dt^2}\psi_{CM} = M\frac{\partial^2\psi}{\partial t^2}(\eta,t). \tag{2.28}$$

The overall form of the first impulse theorem (its non-trivial component x) in the small deviations approximation is

$$\rho\Delta z\frac{\partial^2\psi}{\partial t^2}(\eta,t) = T\Delta z\frac{\partial^2\psi}{\partial z^2}(\xi,t). \tag{2.29}$$

After cancelling Δz, we can perform the limit $z_2 \to z_1$ and arrive at the *wave equation*

$$\boxed{\rho\frac{\partial^2\psi}{\partial t^2}(z,t) = T\frac{\partial^2\psi}{\partial z^2}(z,t).} \tag{2.30}$$

2.3 Longitudinal oscillations

Let us now compare the transverse oscillations with the longitudinal ones, which we have omitted in the continuous derivation of the wave equation. Let us work again with the discrete model of weights and springs and look at the right-hand side of the equations of motion (2.2). For transverse oscillations, the factor $k\left(1-\frac{a_0}{a}\right)$ comes out here, and for longitudinal oscillations, the simple factor k would be there. For springs whose rest length a_0 is close to a, the following would hold

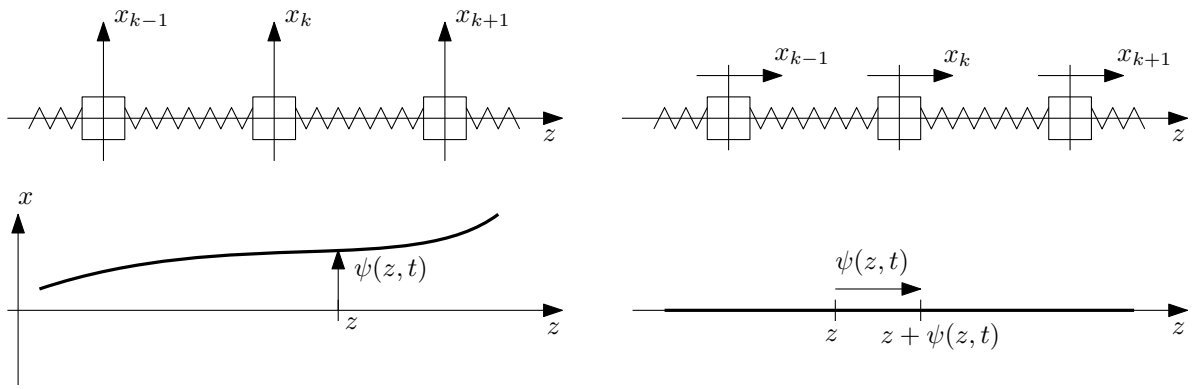
$$k\left(1-\frac{a_0}{a}\right) \ll k. \tag{2.31}$$

This assumption is usually satisfied for the string model. When the string is stretched, its elongation is much less than its total length, $\Delta l \ll l$. Thus, the effective stiffness of springs for transverse vibrations is much less than that of springs for longitudinal vibrations. It is therefore much more difficult to excite longitudinal oscillations of comparable amplitude than transverse oscillations. Therefore, we may have neglected them in the previous description.

Finally, let us still compare the descriptions of the deflections for longitudinal and transverse oscillations using the function $\psi(z,t)$ in Figure 2.8.

[7] For a function f continuous on the interval $\langle a,b\rangle$, there exists a point $c \in (a,b)$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a).$$



(a) Transverse oscillations of a chain and a string.

(b) Longitudinal oscillations of a chain and a string.

Figure 2.8: Comparison of longitudinal and transverse oscillations descriptions.

For transverse oscillations, the position of a piece of string is given by a vector $(z, \psi(z,t))$. For longitudinal oscillations, the deflected piece of string is at coordinate $z + \psi(z,t)$.

2.4 Sound

Sound is nothing but longitudinal waves in the material. Let us now derive the wave equation for the pressure changes caused by longitudinal waves in an ideal gas. Consider a tube of gas with a cross-section S oriented along the axis z. We describe the longitudinal movements of the gas from its equilibrium position by the function $\psi(z,t)$. Consider a small section of the tube $\langle z, z+dz\rangle$, then due to the displacements the gas moves to the position

$$\big\langle z+\psi(z,t), z+dz+\psi(z+dz,t)\big\rangle. \tag{2.32}$$

Pressure $p(z,t)$ is applied to the left side of the gas section, pressure $p(z+dz,t)$ is applied to the right side. See figure 2.9.

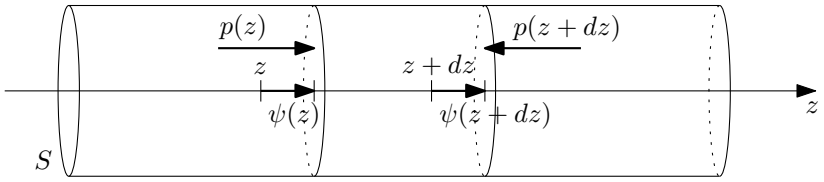


Figure 2.9: A tube of air and its small section between points $\langle z, z+dz\rangle$. Function $\psi(z,t)$ describes the longitudinal displacement of the gas particles from the equilibrium position. Function $p(z,t)$ describes the pressure at each point in the tube.

The equation of motion of the chosen section will again be given by the first impulse theorem:

$$dM\frac{\partial^2\psi}{\partial t^2} = p(z)S - p(z+dz)S, \tag{2.33}$$

where $dM = \rho\, dV = \rho_0 dV_0$. dV_0 denotes the original volume of the segment, $dV_0 = S\, dz$, dV is the volume after displacement by the deflections $\psi(z,t)$. The mass of the selected section remains constant, our imaginary boundary shifts as the gas molecules move. The volume dV can be expressed as a cross section times length as follows:

$$dV = S\Big(\big(z+dz+\psi(z+dz)\big) - \big(z+\psi(z)\big)\Big) = S\left(1+\frac{\psi(z+dz)-\psi(z)}{dz}\right)dz = S\left(1+\frac{\partial\psi}{\partial z}\right)dz. \tag{2.34}$$

Another important assumption is that the process in the gas is adiabatic – that is, that heat is not exchanged between the parts of the gas during pressure changes[8]. For adiabatic process in an ideal gas, the following holds

$$pV^\kappa = \text{const.}, \tag{2.35}$$

where κ is the Poisson constant appropriate to the gas. In our case, we consider the gas in our small section of volume dV, so we have

$$p_0(dV_0)^\kappa = p\,dV^\kappa, \tag{2.36}$$

where p_0 denotes the pressure of the gas at equilibrium. Expressing the pressure as p:

$$p = p_0\left(\frac{dV_0}{dV}\right)^\kappa = p_0\left(1+\frac{\partial\psi}{\partial z}\right)^{-\kappa} \approx p_0\left(1-\kappa\frac{\partial\psi}{\partial z}\right), \tag{2.37}$$

where we substituted dV from (2.34), for $dV_0 = S\,dz$, and used the Taylor expansion to the first order of the function $(1+x)^\alpha \approx 1+\alpha x$ in the last equation.

We can write the pressure difference at different locations on the right-hand side of equation (2.33) using the derivative:

$$\frac{p(z+dz)-p(z)}{dz} = \frac{\partial p}{\partial z} \approx -p_0\kappa\frac{\partial^2\psi}{\partial z^2}, \tag{2.38}$$

where we have used (2.37). If we now plug the previous result into the equation of motion (2.33) (after expanding $dM = \rho_0 S\,dz$):

$$\rho_0\frac{\partial^2\psi}{\partial t^2} = p_0\kappa\frac{\partial^2\psi}{\partial z^2}. \tag{2.39}$$

This is the wave equation describing the longitudinal oscillations in the gas tube. If we consider isothermal action (which satisfies $pV = \text{const.}$), we would not get the Poisson constant κ in the result.

We will see in Section 2.8 that if the wave equation is written in the form $\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial z^2}$, then the constant v represents the wave propagation velocity in a given medium. By comparison with (2.39) we can write that the speed of sound is $v = \sqrt{\frac{p_0\kappa}{\rho_0}}$. Substituting the atmospheric pressure $p_0 = 101,325\,\text{kPa}$, Poisson's constant $\kappa = 1,4$, and the density of dry air at temperature 20°C $\rho_0 = 1,2041\,\text{kg.m}^{-3}$, we get $v = 343,3\,\text{m.s}^{-1}$. For $\kappa = 1$, we would get $v = 290,1\,\text{m.s}^{-1}$.

2.5 Fixed-end string oscillations

Consider now a string of length L, which is stretched between points $z = 0$ and $z = L$. The motion of the inner points of the string is governed by the wave equation:

$$\frac{\partial^2\psi}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2\psi}{\partial z^2}, \qquad z\in(0,L),\ t\in\mathbb{R}, \tag{2.40}$$

where ρ_0 is the length density and T_0 is the tension on the string. We further consider the following *boundary conditions* of so called *fixed ends* – the string is fixed at the ends, i.e.

$$\psi(0,t) = 0, \qquad \psi(L,t) = 0, \qquad \forall t\in\mathbb{R}. \tag{2.41}$$

We try to find a solution by the method of *separation of variables*. We assume a solution of the form

$$\psi(z,t) = Z(z)\,T(t), \tag{2.42}$$

[8] Newton considered the process is isothermal. The speed of sound in air then came out wrong.

with unknown functions $Z(z)$ and $T(t)$ of one variable. If we substitute this *ansatz* into the wave equation (2.40), we obtain

$$Z(z)\ddot{T}(t) = \frac{T_0}{\rho_0} Z''(z)T(t). \tag{2.43}$$

If we now divide this equation by $Z(z)\,T(t)$ and multiply by $\frac{\rho_0}{T_0}$, we obtain the separated equation

$$\frac{Z''}{Z}(z) = \frac{\rho_0}{T_0}\frac{\ddot{T}}{T}(t), \qquad \forall z, t \in \mathbb{R}, \tag{2.44}$$

where the left-hand side depends only on the variable z and the right-hand side depends only on t. Since this equation must be satisfied for all $z, t \in \mathbb{R}$, the left and right sides must equal the common constant[9], let us denote it by $C \in \mathbb{R}$:

$$\frac{Z''}{Z}(z) = C = \frac{\rho_0}{T_0}\frac{\ddot{T}}{T}(t). \tag{2.45}$$

A simple manipulation leads to two ordinary differential equations for the functions $Z(z)$ and $T(t)$ with the not-yet-determined constant C:

$$Z'' - CZ = 0, \qquad \ddot{T} - C\frac{T_0}{\rho_0}T = 0. \tag{2.46}$$

Before we start solving the equations (2.46), let us look at the boundary conditions (2.41), into which we substitute *ansatz* (2.42):

$$\psi(0,t) = Z(0)T(t) = 0, \qquad \psi(L,t) = Z(L)T(t) = 0, \qquad \forall t \in \mathbb{R}. \tag{2.47}$$

If we require a non-trivial $T(t)$ (and hence a non-zero solution to $\psi(z,t)$), the boundary conditions (2.47) reduce to:

$$Z(0) = 0, \qquad Z(L) = 0. \tag{2.48}$$

We will therefore have to satisfy these when solving equation (2.46) for the function $Z(z)$ with the as yet undetermined constant C. The solution $Z(z)$ is

$$Z(z) = c_1 e^{\sqrt{C}z} + c_2 e^{-\sqrt{C}z}, \tag{2.49}$$

where c_1 and c_2 are the integration constants. The boundary conditions therefore look as follows

$$Z(0) = c_1 + c_2 = 0, \qquad Z(L) = c_1 e^{\sqrt{C}L} + c_2 e^{-\sqrt{C}L} = 0. \tag{2.50}$$

After substituting from the first equation into the second equation, we have the condition

$$c_1 e^{-\sqrt{C}L}\left(e^{2\sqrt{C}L} - 1\right) = 0. \tag{2.51}$$

[9] If we differentiate the equation (2.44) with respect to the variables z and t, respectively, we obtain:

$$\frac{d}{dz}\left(\frac{Z''}{Z}\right) = 0, \qquad 0 = \frac{d}{dt}\left(\frac{\rho_0}{T_0}\frac{\ddot{T}}{T}\right).$$

If a function has zero derivative, the function must be constant:

$$\frac{Z''}{Z} = C_1, \qquad \frac{\rho_0}{T_0}\frac{\ddot{T}}{T} = C_2.$$

Thus, we have shown that the left and right sides of the equation (2.44) are equal to the constants C_1 and C_2, respectively. But since the left-hand sides of the equations written above are equal for $\forall z, t$, it must be $C_1 = C_2$.

We require a non-trivial solution, i.e., $c_1(=-c_2)\neq 0$; the exponential of $e^{-\sqrt{C}L}$ is always non-zero; thus, it is necessary that the zero term be in parentheses. This is only possible for $C<0$, i.e. $\sqrt{C}=i\sqrt{|C|}$, and hence the function $Z(z)$ is of the form

$$Z(z)=c_1e^{i\sqrt{|C|}z}+c_2e^{-i\sqrt{|C|}z}=a\cos\left(\sqrt{|C|}z\right)+b\sin\left(\sqrt{|C|}z\right), \tag{2.52}$$

where in the second equation we have switched to the real solution by choosing $c_2=\bar{c}_1$. Now the boundary conditions (2.48) take the form:

$$Z(0)=a=0, \qquad Z(L)=b\sin\left(\sqrt{|C|}L\right)=0, \tag{2.53}$$

where in the second condition we have already used $a=0$. If we require a non-trivial solution, we need $b\neq 0$ and hence the sine must be zero, which leads to the condition:

$$\sqrt{|C|}L=m\pi, \quad m\in\mathbb{N}. \tag{2.54}$$

This means that the admissible constants C for which the solution satisfies the boundary conditions are numbered by natural numbers and specifically given as

$$C_m=-\left(\frac{m\pi}{L}\right)^2. \tag{2.55}$$

Denoting $k_m=\frac{m\pi}{L}$, so called *wavenumber*, then we can write $\sqrt{|C_m|}=k_m=\frac{m\pi}{L}$ and the form of the function $Z(z)$ is

$$Z_m(z)=b_m\sin(k_mz). \tag{2.56}$$

For each admissible C_m (i.e., those satisfying the boundary conditions), we still have to solve the corresponding time equation for $T(t)$, see (2.46):

$$\ddot{T}+\left(\frac{m\pi}{L}\right)^2\frac{T_0}{\rho_0}T=0. \tag{2.57}$$

This has a solution of the form:

$$T_m(t)=a_m\sin\left(\frac{m\pi}{L}\sqrt{\frac{T_0}{\rho_0}}\,t+\varphi_m\right), \tag{2.58}$$

where the angular frequency ω_m can be written using the wave number k_m as

$$\omega_m=\sqrt{\frac{T_0}{\rho_0}}k_m=\sqrt{\frac{T_0}{\rho_0}}\frac{m\pi}{L}, \qquad k_m=\frac{m\pi}{L}. \tag{2.59}$$

The resulting solution of the string motion $\psi_m(z,t)$ corresponding to the admissible value of the constant C_m is obtained by substituting into (2.42):

$$\psi_m(z,t)=Z_m(z)T_m(t)=A_m\sin\left(\frac{m\pi z}{L}\right)\sin\left(\frac{m\pi}{L}\sqrt{\frac{T_0}{\rho_0}}\,t+\varphi_m\right), \tag{2.60}$$

where we have grouped the dependent integration constants and named them $A_m=a_mb_m$. These solutions represent the vibrational modes of the string. The solutions came out in the form of so-called standing waves, i.e., in the form $\psi(z,t)=Z(z)\sin(\omega t+\varphi_0)$, where the wave retains its shape $Z(z)$ and only changes its amplitude by a harmonic function.

Since the wave equation (2.40) is linear, the solution is also a linear combination of all the solutions found above:

$$\psi(z,t) = \sum_{m=1}^{+\infty} \psi_m(z,t), \tag{2.61}$$

where the coefficients of the linear combination take the place of the amplitudes A_m already hidden in the functions $\psi_m(z,t)$ (2.60). The resulting general solution found by the method of separation of variables is

$$\boxed{\psi(z,t) = \sum_{m=1}^{\infty} A_m \sin(k_m z) \sin(\omega_m t + \varphi_m),} \tag{2.62}$$

where the angular frequency ω and the wave number k satisfy the following dispersion relation, and the admissible values of the wave numbers k_m are

$$\omega = \sqrt{\frac{T_0}{\rho_0}}\, k, \qquad k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N}. \tag{2.63}$$

The constants A_m and φ_m are given by the initial conditions, while the constants ω_m and k_m are given by the properties of the physical system we are investigating, here the length of the string L, its density ρ_0 and the tension in it T_0 (and the boundary conditions).

Figure 2.10 shows the first four modes of this solution (i.e., functions ψ_1 to ψ_4).

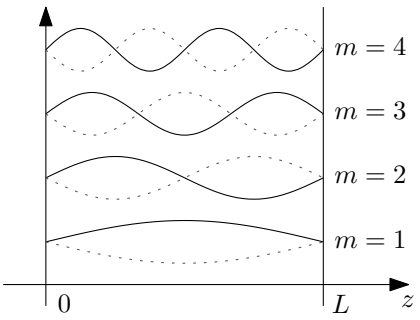


Figure 2.10: The first four modes of string motion with fixed ends. The dotted lines indicate the shapes of the modes shifted by half the time period.

2.5.1 Free end boundary condition

The second type of boundary condition on a finite length string is the free end boundary condition. Physically, this means that the end of the string is free to slide without friction on a rod perpendicular to the axis z. In what follows, without loss of generality, let us choose $z = 0$ to be the location of the boundary.

If we consider that the string attachment has a mass M, then Newton's equation of motion for the end of the string will take the form

$$M\frac{\partial^2 \psi}{\partial t^2}(0,t) = F_x = T\frac{\partial \psi}{\partial z}(0,t), \tag{2.64}$$

where on the right-hand side is the transverse force from the string acting on the string attachment.

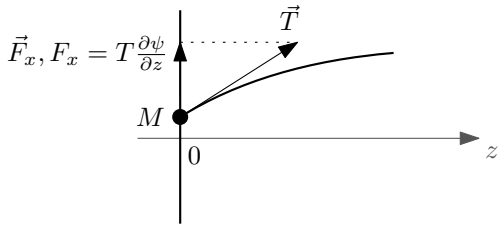


Figure 2.11: The transverse force acting on the suspension weight M.

For an massless attachment, $M = 0$, we obtain from the equation of motion a *free end* boundary condition of the form

$$\frac{\partial\psi}{\partial z}(0,t) = 0, \tag{2.65}$$

i.e., no transverse force acts on the attachment; geometrically – the string is attached horizontally.

2.6 Mathematical supplement: Fourier series

Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $2L$. Then we call the following function a Fourier series f_F of the function f:

$$\boxed{f_F(z) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} \left(a_m \cos\frac{m\pi z}{L} + b_m \sin\frac{m\pi z}{L} \right),} \tag{2.66}$$

where the coefficients a_m and b_m are given by the relations:

$$\boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(z)\cos\frac{m\pi z}{L}\,dz, \quad m \in \mathbb{N}_0; \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(z)\sin\frac{m\pi z}{L}\,dz, \quad m \in \mathbb{N}.} \tag{2.67}$$

For piecewise differentiable functions, at the points of continuity the Fourier series converges to the original function f, $f_F(z) = f(z)$. For the discontinuity point z_0, the following holds

$$f_F(z_0) = \frac{1}{2}\left(\lim_{z\to z_0+} f(z) + \lim_{z\to z_0-} f(z) \right), \tag{2.68}$$

thus, the Fourier series converges to the average of the one-sided limits of the function f at this point.

The Fourier series represents the expansion of the periodic function f into a discrete superposition of harmonic waves.

For even functions ($f(x) = f(-x)$) and odd functions ($f(x) = -f(-x)$), respectively, the Fourier series (2.66) and the formulas for the coefficients a_m and b_m (2.67) simplify. For **even functions** we get

$$a_m = \frac{2}{L}\int_0^L f(z)\cos\frac{m\pi z}{L}\,dz, \qquad b_m = 0, \qquad f_F(z) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} a_m \cos\frac{m\pi z}{L}. \tag{2.69}$$

For **odd functions**:

$$a_m = 0, \qquad b_m = \frac{2}{L}\int_0^L f(z)\sin\frac{m\pi z}{L}\,dz, \qquad f_F(z) = \sum_{m=1}^{+\infty} b_m \sin\frac{m\pi z}{L}. \tag{2.70}$$

*Note from a linear algebra perspective: the Fourier series is actually the decomposition of the vector f from the vector space of periodic functions $\mathcal{F}$ into an (infinite) basis

$$\mathcal{B} = \{(\cos k_m z)_{m=0}^{+\infty}, (\sin k_m z)_{m=1}^{+\infty}\}, \tag{2.71}$$

where $k_m = \frac{m\pi}{L}$. Moreover, if we introduce the scalar product of two periodic functions f and g ($f, g \in \mathcal{F}$) as

$$\langle f, g\rangle := \frac{1}{L}\int_{-L}^{L} f(z)g(z)\,dz, \tag{2.72}$$

then the coefficients a_m and b_m are nothing but the coefficients of the linear combination obtained as the projection of the vector f onto the basis vectors by the scalar product (2.72)[10]:

$$a_m = \langle f, \cos k_m z\rangle, \qquad b_m = \langle f, \sin k_m z\rangle, \tag{2.73}$$

In linear algebra, these coefficients are called Fourier coefficients.

2.6.1 Even and odd extensions

Let us have a function $f : \langle 0, L\rangle \to \mathbb{R}$. Let us define its so-called *even and odd extensions* $f_{\text{even}} : \mathbb{R} \to \mathbb{R}$ and $f_{\text{odd}} : \mathbb{R} \to \mathbb{R}$.

First, define the functions f_{even} and f_{odd} on the interval $\langle 0, L\rangle$ to agree with the original function f:

$$f_{\text{even}}|_{\langle 0,L\rangle} = f, \qquad f_{\text{odd}}|_{\langle 0,L\rangle} = f. \tag{2.74}$$

Then we redefine the functions f_{even} and f_{odd} on the interval $\langle -L, 0\rangle$ as follows:

$$f_{\text{even}}(z) = f(-z), \qquad f_{\text{odd}}(z) = -f(-z), \qquad z \in \langle -L, 0\rangle, \tag{2.75}$$

i.e., so that the function f_{even} is even on the interval $\langle -L, L\rangle$ and the function f_{odd} is odd[11] on $\langle -L, L\rangle$.

Finally, we uniquely redefine the functions f_{even} and f_{odd} to integer $\mathbb{R}$ so that they are periodic functions with period $2L$. The result of these extensions for the particular case of the function f in Figure 2.12 can be seen in Figure 2.13.

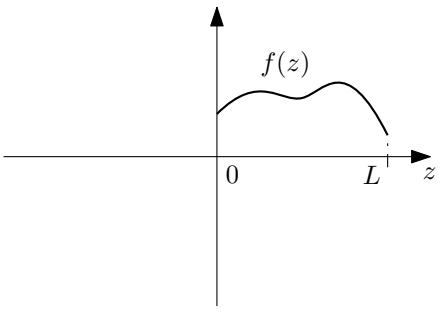
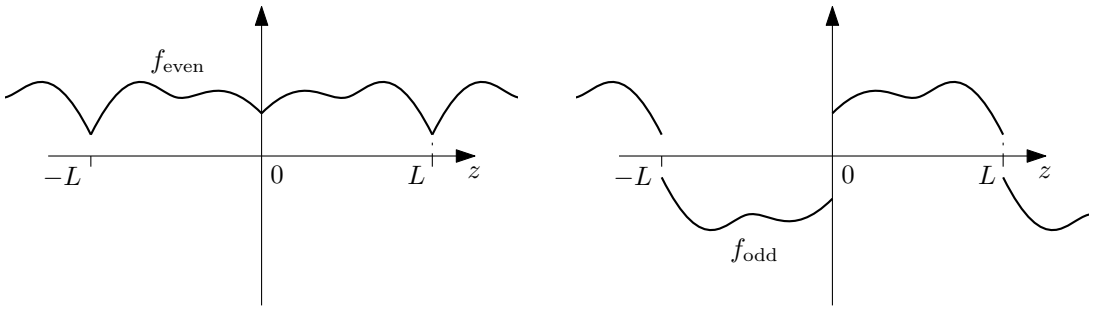


Figure 2.12: The original function $f : \langle 0, L\rangle \to \mathbb{R}$ defined only on the interval $\langle 0, L\rangle$.

[10] Except for the coefficient a_0, where we are bothered by the non-normality of the function $f(z) = 1$, $\langle f, f\rangle = 2$

[11] For the odd extension, there may be a problem if $f(0) \neq 0$ and $f(L) \neq 0$ (the function cannot then be odd or periodic, respectively). We can safely ignore this problem, since the coefficients of the Fourier series a_m and b_m are given by integral formulas and the integrals are not sensitive to a change in the functional value of the integrated function at a single point. Thus, we can imagine redefining the original function f at the odd extension by putting $f(0) = f(L) = 0$.



(a) Even extension f_{even}.

(b) Odd extension f_{odd}.

Figure 2.13: Even and odd extensions of function f.

From the given function $f : \langle 0, L\rangle \to \mathbb{R}$ we obtained the periodic even and odd functions f_{even} and f_{odd}, respectively. We can then compute their Fourier series, which, due to evenness and oddness, respectively, come out in the following forms:

$$f_{\text{even}}(z) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} a_m \cos\frac{m\pi z}{L}, \qquad f_{\text{odd}}(z) = \sum_{m=1}^{+\infty} b_m \sin\frac{m\pi z}{L}, \tag{2.76}$$

where the coefficients a_m and b_m are, according to (2.69) and (2.70), respectively, given by

$$\begin{aligned} a_m &= \frac{2}{L}\int_0^L f_{\text{even}}(z)\cos\frac{m\pi z}{L}\,dz = \frac{2}{L}\int_0^L f(z)\cos\frac{m\pi z}{L}\,dz, \\ b_m &= \frac{2}{L}\int_0^L f_{\text{odd}}(z)\sin\frac{m\pi z}{L}\,dz = \frac{2}{L}\int_0^L f(z)\sin\frac{m\pi z}{L}\,dz, \end{aligned} \tag{2.77}$$

where we have taken advantage of the fact that $f_{\text{even}}(z) = f_{\text{odd}}(z) = f(z)$ for $z \in \langle 0, L\rangle$. Then

$$f(z) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} a_m \cos\frac{m\pi z}{L} = \sum_{m=1}^{+\infty} b_m \sin\frac{m\pi z}{L} \quad \text{pro} \quad z \in \langle 0, L\rangle. \tag{2.78}$$

Thus, we were able to express the function f on the interval $\langle 0, L\rangle$ as a linear combination of either the sine functions only or the cosine functions only.

2.7 Initial value problem for fixed ends

Now we want to find a particular motion of the string, given the initial conditions. Let us write down the procedure of this problem for the boundary conditions of **fixed ends**.

The initial conditions consist of the initial position of the string and the initial velocity of the string (for simplicity, we choose that they are given in time $t = 0$). These are specified as a function of the initial position $f : \langle 0, L\rangle \to \mathbb{R}$ (we must specify the initial displacement of each point of the string) and as a function of the initial velocity $g : \langle 0, L\rangle \to \mathbb{R}$ (the same for the initial velocity of each point of the string). Thus, our particular solution must satisfy:

$$\psi(z, 0) = f(z), \quad \frac{\partial\psi}{\partial t}(z, 0) = g(z), \quad \forall z \in \langle 0, L\rangle. \tag{2.79}$$

In order to achieve this, we have the integration constants A_m and φ_m whose value we want to determine.

Initial conditions. Let us write the left-hand sides of the equations (2.79) explicitly, i.e., let us add the time $t = 0$ to the general solution (2.62) and its time derivative:

$$\begin{aligned}\psi(z,0) &= \sum_{m=1}^{+\infty} (A_m \sin\varphi_m) \sin\frac{m\pi z}{L} = f(z),\\ \frac{\partial\psi}{\partial t}(z,0) &= \sum_{m=1}^{+\infty} (A_m\omega_m \cos\varphi_m) \sin\frac{m\pi z}{L} = g(z).\end{aligned} \tag{2.80}$$

Odd extensions of the functions f and g. Now we need to write the functions f and g as Fourier series, which will contain only the functions $\sin\frac{m\pi z}{L}$. This is easily achieved if we compute the series of the functions f and g in **odd extension** (see Section 2.6.1):

$$f(z) = \sum_{m=1}^{+\infty} f_m \sin\frac{m\pi z}{L}, \qquad g(z) = \sum_{m=1}^{+\infty} g_m \sin\frac{m\pi z}{L}, \tag{2.81}$$

where the coefficients f_m and g_m are given by the following formulae:

$$f_m = \frac{2}{L}\int_0^L f(z)\sin\left(\frac{m\pi z}{L}\right)\,dz, \qquad g_m = \frac{2}{L}\int_0^L g(z)\sin\left(\frac{m\pi z}{L}\right)\,dz. \tag{2.82}$$

Equations for the coefficients A_m, φ_m. The equations for the coefficients A_m and φ_m are obtained by comparing the series (2.80) and (2.81) term by term:

$$A_m \sin\varphi_m = f_m, \qquad A_m\omega_m\cos\varphi_m = g_m. \tag{2.83}$$

These equations can be (formally[12]) solved:

$$A_m = \sqrt{f_m^2 + \frac{g_m^2}{\omega_m^2}}, \qquad \operatorname{tg}\varphi_m = \frac{f_m\omega_m}{g_m}. \tag{2.84}$$

2.8 d'Alembert's solution of the wave equation

Consider a wave equation of the form

$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial z^2}, \tag{2.85}$$

which we rewrite into the following form by starting to look at partial derivatives as differential operators:

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\psi = 0. \tag{2.86}$$

The differential operator on the left-hand side of the previous equation is called *d'Alembert operator*[13] and is denoted $\Box$:

$$\Box = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}. \tag{2.87}$$

[12] Formal is the second equation, which actually represents the two equations $\sin\varphi_m = \frac{f_m}{A_m}$ and $\cos\varphi_m = \frac{g_m}{A_m\omega_m}$ that uniquely define the angle φ_m for $A_m \neq 0$. If $A_m = 0$, the corresponding mode is missing and its phase does not matter.

[13] Another convention introduces a d'Alembert operator with the opposite sign

Due to the commutativity of partial derivatives, the identity $A^2 - B^2 = (A - B)(A + B)$ also applies to derivatives and we can write

$$\Box\psi = \left(\frac{1}{v}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\psi = 0. \tag{2.88}$$

Now, we would like to introduce new variables to simplify the form of the equation (2.88). We do this by introducing the variables ξ and η according to the following rules

$$\xi = z - vt, \quad \eta = z + vt, \qquad \left(z = \frac{\xi + \eta}{2}, \quad t = \frac{\eta - \xi}{2v}\right). \tag{2.89}$$

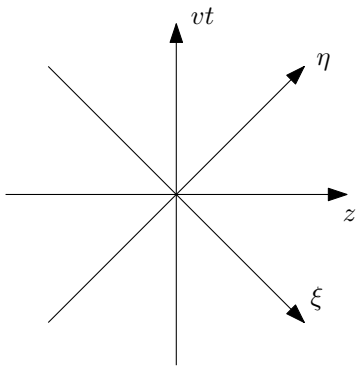


Figure 2.14: Original coordinates (z, vt) and new coordinates (ξ, η).

Let's rewrite the wave equation in the original variables z, t into the new variables ξ, η. We define a new function $\tilde{\psi}(\xi, \eta)$ by the substitution:

$$\tilde{\psi}(\xi, \eta) := \psi\left(z(\xi, \eta), t(\xi, \eta)\right) = \psi\left(\frac{\xi + \eta}{2}, \frac{\eta - \xi}{2v}\right). \tag{2.90}$$

The inverse substitution will then look like the following

$$\psi(z, t) = \tilde{\psi}(z - vt, z + vt) = \tilde{\psi}(\xi, \eta). \tag{2.91}$$

These substitution relations allow us to transform the derivatives. If we differentiate ψ according to the variables z and t, we obtain, according to the chain rule[14]

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \frac{\partial\tilde{\psi}}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\tilde{\psi}}{\partial\eta}\frac{\partial\eta}{\partial t} = -v\frac{\partial\tilde{\psi}}{\partial\xi} + v\frac{\partial\tilde{\psi}}{\partial\eta}, \\
\frac{\partial\psi}{\partial z} &= \frac{\partial\tilde{\psi}}{\partial\xi}\frac{\partial\xi}{\partial z} + \frac{\partial\tilde{\psi}}{\partial\eta}\frac{\partial\eta}{\partial z} = \frac{\partial\tilde{\psi}}{\partial\xi} + \frac{\partial\tilde{\psi}}{\partial\eta}.
\end{aligned} \tag{2.92}$$

[14]Let's have the function $f(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ and k-three functions $g_i(y_1, \ldots, y_l) : \mathbb{R}^l \to \mathbb{R}$. We obtain the function $h(y_1, \ldots, y_l) : \mathbb{R}^l \to \mathbb{R}$ by composing

$$h(y_1, \ldots, y_l) = f\left(g_1(y_1, \ldots, y_l), \ldots, g_k(y_1, \ldots, y_l)\right)$$

Then the *chain rule* states

$$\frac{\partial h}{\partial y_i} = \sum_{m=1}^{k} \frac{\partial f}{\partial x_m}\frac{\partial g_m}{\partial y_i}.$$

This rule is an extension of the rule for differentiating a composite function, $[f(g(x))]' = f'(g(x))\, g'(x)$, into multiple variables. Here we have $\tilde{\psi}(\xi, \eta) \sim f(x_1, x_2)$, $\xi(z, t) \sim g_1(y_1, y_2)$, $\eta(z, t) \sim g_2(y_1, y_2)$.

Combining these results into the form of the operators appearing in (2.88):

$$\left(\frac{1}{v}\frac{\partial}{\partial t}-\frac{\partial}{\partial z}\right)\psi=\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\tilde{\psi}=-2\frac{\partial}{\partial\xi}\tilde{\psi},$$
$$\left(\frac{1}{v}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)\psi=\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\tilde{\psi}=\quad 2\frac{\partial}{\partial\eta}\tilde{\psi}. \tag{2.93}$$

Thus, in the language of differential operators, we have

$$\frac{1}{v}\frac{\partial}{\partial t}-\frac{\partial}{\partial z}=-2\frac{\partial}{\partial\xi},\qquad \frac{1}{v}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}=2\frac{\partial}{\partial\eta}. \tag{2.94}$$

Substituting these expressions into the wave equation (2.88), we get

$$\frac{\partial^2\tilde{\psi}}{\partial\xi\partial\eta}=0. \tag{2.95}$$

This equation has a solution[15] in the form

$$\tilde{\psi}(\xi,\eta)=F(\xi)+G(\eta), \tag{2.96}$$

where $F,G:\mathbb{R}\longrightarrow\mathbb{R}$ are arbitrary real functions of one real variable (appropriately differentiable). Using the original variables, the function $\psi(z,t)$ is of the form

$$\boxed{\psi(z,t)=F(z-vt)+G(z+vt),} \tag{2.97}$$

We call this solution the *d'Alembert solution* of the wave equation. What is the physical meaning of this solution? Consider first the situation with $G\equiv 0$. In time $t=0$, the function $F(z)$ simply represents the shape of the excited wave. Next, let us consider the evolution of the location of constant phase. We call *phase* the argument of the function F, i.e., $z-vt$. Let us find the time dependence of the location $z_c(t)$ as a solution of equation $z-vt=c$, where c is an arbitrary constant. This is trivially

$$z_c(t)=c+vt. \tag{2.98}$$

(for the initial condition $z_c(0)=z_0$ we get $z_c(t)=z_0+vt$). Thus, a particular point on the wave propagates at speed v. Thus, the part of the solution $F(z-vt)$ represents a wave (of the shape of the form of the function $F(z)$) propagating as a whole in the positive direction of the axis z at speed v. This velocity is called *phase velocity*. By analogy, if we look at the function $G(z+vt)$, it represents a wave of the form $G(z)$ propagating with velocity v in the negative direction of the axis z.

[15] If we write the equation in the form

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\tilde{\psi}}{\partial\eta}\right)=0,$$

we see that the function $\frac{\partial\tilde{\psi}}{\partial\eta}$ is a function of the variable η only,

$$\frac{\partial\tilde{\psi}}{\partial\eta}=g(\eta).$$

We now integrate this with respect to η and get

$$\tilde{\psi}(\xi,\eta)=\underbrace{\int g(\eta)\,d\eta}_{G(\eta)}+F(\xi),$$

where $F(\xi)$ is the integration constant with respect to the integration variable η.

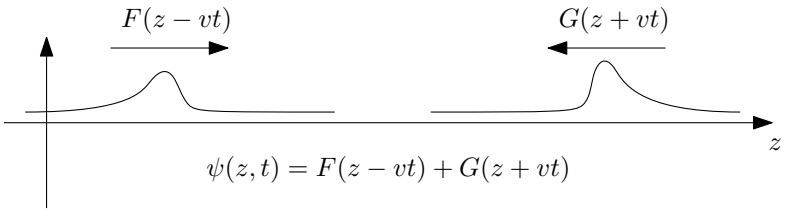


Figure 2.15: d'Alembert's solution representing the superposition of two oppositely propagating waves $F(z-vt)$ and $G(z+vt)$.

2.8.1 Emission of travelling waves

How can we build these travelling waves? Consider that our environment, e.g. a string, has an oscillating source at $z = 0$, which determines a displacement at this point by its motion:

$$\psi(0,t) = y(t), \tag{2.99}$$

$y(t)$ is an arbitrary but given function. A condition of this type, where we give the state of the system for a given position, is generally called a *boundary condition*. It determines the form of the solution at the edge of the system under study. This is specifically the condition in $z = 0$. To obtain an unambiguous solution to this emission problem, we need to prescribe a second boundary condition at the other "edge" of the system, here $z = +\infty$. We require that no waves come from infinity, i.e., we prescribe $G \equiv 0$. This boundary condition is called *radiation boundary condition*.

Now we can find the concrete form of d'Alembert's solution. After substituting $\psi(z,t) = F(z-vt)$ into (2.99):

$$\psi(0,t) = F(-vt) = y(t) \quad \longrightarrow \quad F(x) = y\left(-\frac{x}{v}\right). \tag{2.100}$$

Knowing the concrete form of the function $F(x)$ we can easily already write

$$\psi(z,t) = F(z-vt) = y\left(-\frac{z-vt}{v}\right) = y\left(t-\frac{z}{v}\right). \tag{2.101}$$

The resulting solution is therefore of the form

$$\boxed{\psi(z,t) = y\left(t-\frac{z}{v}\right).} \tag{2.102}$$

What is the expression in the argument of the driving function y? The fraction $\frac{z}{v}$ represents the time it takes for the signal emitted at point $z = 0$ to propagate to point z. This means that the wave we observe at point z at time t was radiated from the source at time $t - \frac{z}{v}$. We call this time *retarded time*:

$$t_r = t - \frac{z}{v}. \tag{2.103}$$

Symbolically, we can write

$$\psi(z,t) = y(t_r). \tag{2.104}$$

2.8.2 Harmonic travelling wave

If the source harmonically oscillates according to the prescription

$$y(t) = A\cos(\omega t + \varphi), \tag{2.105}$$

then the emitted wave is of the form

$$F(x) = y\left(-\frac{x}{v}\right) = A\cos\left[\omega\left(-\frac{x}{v}\right)+\varphi\right] = A\cos\left(\frac{\omega}{v}x-\varphi\right), \tag{2.106}$$

resp.

$$\psi(z,t) = y\left(t-\frac{z}{v}\right) = A\cos\left[\omega\left(t-\frac{z}{v}\right)+\varphi\right] = A\cos\left(\omega t-\frac{\omega}{v}z+\varphi\right). \tag{2.107}$$

If we denote the wave number by $k = \frac{\omega}{v}$ (dispersion relation), we get the *harmonic travelling waves*

$$\psi(z,t) = A\cos(\omega t - kz + \varphi). \tag{2.108}$$

For the function $G(x) = A\cos(kx+\varphi)$ we have a wave propagating in the opposite direction

$$\psi(z,t) = G(z+vt) = A\cos(\omega t + kz + \varphi). \tag{2.109}$$

2.8.3 Initial value problem for the d'Alembert solution

Consider a string extending along the whole axis z. The initial conditions for position and velocity in time $t = 0$ are of the form

$$\psi(z,0) = f(z), \qquad \frac{\partial\psi}{\partial t}(z,0) = g(z), \qquad \forall z\in\mathbb{R}, \tag{2.110}$$

where $f, g : \mathbb{R}\to\mathbb{R}$ are functions of the initial position and velocity. If we plug the d'Alembert solution (2.97) into the above initial conditions, we get

$$\psi(z,0) = F(z)+G(z) = f(z), \qquad \frac{\partial\psi}{\partial t}(z,0) = (-v)F'(z)+v\,G'(z) = g(z), \tag{2.111}$$

where by the symbols F' and G', respectively, we mean $\frac{dF}{dx}$ and $\frac{dG}{dx}$. We integrate the second of the initial conditions (with respect to the variable z),

$$G(z)-F(z) = \underbrace{\frac{1}{v}\int g(z)\,dz}_{\tilde{g}(z)} + c, \tag{2.112}$$

where we denote the primitive function (including the constant $\frac{1}{v}$) as $\tilde{g}(z)$ and the integration constant as c. Now we just need to add and subtract the left equation in (2.111) with the previous equation to get

$$F(x) = \frac{f(x)-\tilde{g}(x)-c}{2}, \qquad G(x) = \frac{f(x)+\tilde{g}(x)+c}{2}, \tag{2.113}$$

where we have additionally renamed the variable to x. The resulting concrete solution satisfying the given initial conditions is obtained by substituting it into the d'Alembert solution:

$$\psi(z,t) = F(z-vt)+G(z+vt) = \frac{f(z-vt)-\tilde{g}(z-vt)}{2}+\frac{f(z+vt)+\tilde{g}(z+vt)}{2}, \tag{2.114}$$

The constants $\frac{c}{2}$ canceled each other. Let us repeat that $\tilde{g}(x)$ is a primitive function to $\frac{1}{v}g(x)$, $\tilde{g}(x) = \frac{1}{v}\int g(x)\,dx$. Note that the resulting solution does not depend on the value of the integration constant c – it does not depend on the choice of the primitive function – and thus the solution is uniquely determined by the initial conditions.

2.9 Energy of waves

Let's now look at the waves on the string from an energy perspective – we find expressions for the mechanical energy in a given section of the string. We derived the equation of motion of the string from the first impulse theorem,

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)}. \tag{2.115}$$

Since the law of conservation of mechanical energy applies to systems where no external forces are acting, and since the chosen section of the string is subject to external forces from the rest of the string, the energy will not be conserved at that particular point on the string, but will be transferred from one point to another. We now quantify these considerations.

The total energy is the sum of kinetic and potential energy. The kinetic energy of a small section of string $\langle z, z + dz\rangle$ is

$$dT = \frac{1}{2} dm\, v^2 = \frac{1}{2}\rho \left(\frac{\partial \psi}{\partial t}\right)^2 dz, \tag{2.116}$$

where we have substituted $dm = \rho\, dz$.

We introduce the general notion of *energy density* ε as follows. If dE is the energy contained in the segment $\langle z, z + dz\rangle$, we define the energy density ε as

$$dE = \varepsilon\, dz. \tag{2.117}$$

The energy contained in the finite section $\langle z_1, z_2\rangle$ is then given by

$$E = \int_{z_1}^{z_2} dE(z) = \int_{z_1}^{z_2} \varepsilon\, dz. \tag{2.118}$$

While the unit of energy is $[E] = \mathrm{J}$, the unit of (linear) energy density is $[\varepsilon] = \mathrm{J.m^{-1}}$.

Thus, *kinetic energy density* according to the previous definition and expression (2.116) is

$$\boxed{\tau = \frac{1}{2}\rho \left(\frac{\partial \psi}{\partial t}\right)^2.} \tag{2.119}$$

Next, to derive an expression for the potential energy, we use the discrete model of a chain of atoms and its consequent continuous limit. In the string section $\langle z, z + dz\rangle$, the total potential is the sum of the potential energies of the individual springs,

$$dU = \sum_l U_l. \tag{2.120}$$

The linearized expression for the increase in potential energy at transverse displacements is[16]:

$$U_l = \frac{1}{2} k \left(1 - \frac{a_0}{a}\right) (\Delta\psi_l)^2, \tag{2.121}$$

where $\Delta\psi_l$ is the difference in displacements of adjacent weights, see Figure 2.16.

[16] This is an approximation of the small oscillations of the spring potential at transverse deflections $U(y) = \frac{1}{2}k\left(\sqrt{a^2 + y^2} - a_0\right)^2$

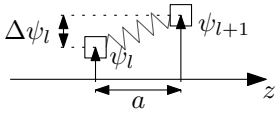


Figure 2.16: The displacements ψ_l of the individual weights of the chain and their difference $\Delta\psi_l$.

The number of springs between $\langle z, z+dz\rangle$ is $dN = \frac{dz}{a}$. For a small stretch of a string, we can consider that all $\Delta\psi_l$ are approximately equal[17] and hence

$$dU = dN\, U_{l_0} = \frac{1}{2}ka\left(1-\frac{a_0}{a}\right)\left(\frac{\Delta\psi_{l_0}}{a}\right)^2 dz. \tag{2.122}$$

Performing the continuous limit $a \to 0$ (and holding $T = ka\left(1-\frac{a_0}{a}\right) = \text{const.}$), the fraction $\frac{\Delta\psi}{a}$ becomes the derivative $\frac{\partial\psi}{\partial z}$ and we get

$$dU = \frac{1}{2}T\left(\frac{\partial\psi}{\partial z}\right)^2 dz. \tag{2.123}$$

Potential energy density is then

$$\boxed{u = \frac{1}{2}T\left(\frac{\partial\psi}{\partial z}\right)^2.} \tag{2.124}$$

We can now write *total energy density* on the string as

$$\boxed{\varepsilon = \tau + u = \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\psi}{\partial z}\right)^2.} \tag{2.125}$$

The total energy in the segment $\langle z_1, z_2\rangle$ is given by the integral

$$E_{\langle z_1,z_2\rangle}(t) = \int_{z_1}^{z_2} \varepsilon(z,t)\, dz. \tag{2.126}$$

Let us see how this energy changes with time:

$$\frac{dE_{\langle z_1,z_2\rangle}}{dt} = \int_{z_1}^{z_2} \frac{\partial\varepsilon}{\partial t} dz = \int_{z_1}^{z_2} \rho\frac{\partial\psi}{\partial t}\frac{\partial^2\psi}{\partial t^2} + T\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial z\partial t}\, dz. \tag{2.127}$$

Using the wave equation, we have

$$\frac{dE_{\langle z_1,z_2\rangle}}{dt} = T\int_{z_1}^{z_2} \frac{\partial\psi}{\partial t}\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial z\partial t}\, dz = T\int_{z_1}^{z_2} \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}\right) dz = T\left[\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}\right]_{z_1}^{z_2}. \tag{2.128}$$

[17] It can again be done more rigorously. The expression $\frac{\Delta\psi_k}{a}$ can be written as $\psi'(\xi_k, t)$ using the mean value theorem and then

$$\Delta U = \sum_k \frac{1}{2}T\psi'(\xi_k,t)^2\, a \quad \overset{a\to 0}{\longrightarrow} \quad \Delta U = \int_{z_1}^{z_2} \frac{1}{2}T\psi'(z,t)^2 dz = \frac{1}{2}T\psi'(\xi,t)^2\Delta z,$$

where we have used the definition of the Riemann integral as the limit of integral sums and the integral mean value theorem in the last equality. If we now $\Delta z \to 0$, we will go $\xi \to z$ and get

$$dU = \frac{1}{2}T\psi'(z,t)^2\, dz.$$

If we define the quantity *energy flux* S

$$S(z,t) = -T\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}, \tag{2.129}$$

then we can write the equation (2.128) as

$$\boxed{\frac{dE_{\langle z_1,z_2\rangle}(t)}{dt} = S(z_1,t) - S(z_2,t),} \tag{2.130}$$

i.e., that the temporal change of energy in a given string section is given by the balance of energy inflow and outflow at its ends. A positive flux at point z_1 increases the total energy and a positive flux at point z_2 decreases it. The equality (2.130) represents the integral law of conservation of energy on the string. Let us show that the energy flux S is nothing but the power of the tension forces on the string. This result will not be surprising, because it actually represents the third impulse theorem – the time change in the total energy of a system of particles is equal to the power of the external forces. Consider the situation at z_1 in Figure 2.17.

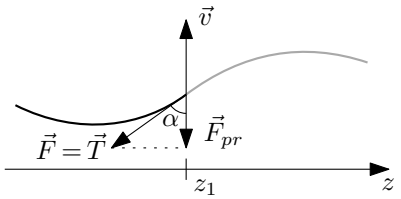


Figure 2.17: The tension force applied at point z_1 from the rest of the string to the section $\langle z_1, z_2\rangle$. The velocity of the string is $v = \frac{\partial\psi}{\partial t}$. The magnitude of the transverse projection is $F_{pr} = T\cos\alpha$.

When substituted into the formula for mechanical power:

$$P = \vec{F}\cdot\vec{v} = \vec{T}\cdot\vec{v} = -T\,v\cos\alpha = -F_{pr}\,v = -T\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial t} = S, \tag{2.131}$$

where we have explicitly included a minus sign in the scalar product since the vectors form an obtuse angle in the situation in Figure 2.17. From relation $P = S$ we can easily write the units of the energy flux $[S] = \mathrm{W} = \mathrm{J.s^{-1}}$.

We obtain the differential form of the conservation law by the limit $z_2 \to z_1$. We rewrite the left-hand side of the conservation law using the integral mean value theorem,

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\varepsilon(\xi,t)\Delta z, \tag{2.132}$$

where $\Delta z = z_2 - z_1$. After substituting into (2.130) (and dividing by Δz):

$$\frac{\partial\varepsilon}{\partial t}(\xi,t) = -\frac{S(z_2,t) - S(z_1,t)}{\Delta z} \quad \overset{\Delta z\to 0}{\longrightarrow} \quad \frac{\partial\varepsilon}{\partial t} + \frac{\partial S}{\partial z} = 0 \tag{2.133}$$

Performing the limits of $\Delta z \to 0$ we get the differential law of conservation of energy on the

string[18]:

$$\boxed{\frac{\partial \varepsilon}{\partial t} + \frac{\partial S}{\partial z} = 0.} \tag{2.134}$$

This is actually the equation of continuity[19]. The temporal change in energy density is given by the (spatial) change in energy flux at a given location.

The energy quantities (ε, τ, u, S) are quadratic in deviation ψ. Thus, the superposition principle does not hold! Consider the superposition of two waves $\psi = \psi_1 + \psi_2$, then

$$\varepsilon_{\psi_1+\psi_2} \neq \varepsilon_{\psi_1} + \varepsilon_{\psi_2}, \qquad \text{etc.} \tag{2.135}$$

For example, for the time derivative we have

$$\left(\frac{\partial \psi}{\partial t}\right)^2 = \left(\frac{\partial \psi_1 + \psi_2}{\partial t}\right)^2 = \left(\frac{\partial \psi_1}{\partial t}\right)^2 + \left(\frac{\partial \psi_2}{\partial t}\right)^2 + \underbrace{2\frac{\partial \psi_1}{\partial t}\frac{\partial \psi_2}{\partial t}}_{\text{interference term}}. \tag{2.136}$$

2.9.1 Energy in a travelling wave

Consider a wave propagating in one direction

$$\psi(z,t) = F(z - vt). \tag{2.137}$$

Denote by $F'(x) = \frac{dF}{dx}$, then the derivatives by z and t have the following form and the relation between them holds:

$$\frac{\partial \psi}{\partial z} = F'(z - vt), \qquad \frac{\partial \psi}{\partial t} = -vF'(z - vt) \quad \longrightarrow \quad -v\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial t}. \tag{2.138}$$

Consequently, the equality of kinetic and potential energy densities, $u = \tau$, holds, or

$$\tau = \frac{1}{2}\rho\left(\frac{\partial \psi}{\partial t}\right)^2 = \frac{1}{2}\rho v^2\left(\frac{\partial \psi}{\partial z}\right)^2 = \frac{1}{2}T\left(\frac{\partial \psi}{\partial z}\right)^2 = u \tag{2.139}$$

For the total energy, therefore, $\varepsilon = u + \tau = 2u = 2\tau$.

---

[18] We could also arrive at it in another way. Rewrite (2.130) as

$$\frac{dE}{dt} = \int_{z_1}^{z_2} \frac{\partial \varepsilon}{\partial t} dz = -\int_{z_1}^{z_2} \frac{\partial S}{\partial z} dz,$$

put all the terms on one side,

$$\int_{z_1}^{z_2} \left(\frac{\partial \varepsilon}{\partial t} + \frac{\partial S}{\partial z}\right) dz = 0,$$

and since this equality must hold for any z_1, z_2, the integrand must be zero.

[19] Remember the equation of continuity in electricity and magnetism:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0.$$

For a 1D problem (current flowing only in the direction of the axis z), we would get

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial z} = 0.$$

Let us next look at the energy flux in a travelling wave. We can choose two different manipulations:

$$S = -T\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial t} = Tv\left(\frac{\partial\psi}{\partial z}\right)^2 = 2\tau v = \varepsilon\, v \tag{2.140}$$

$$= \frac{T}{v}\left(\frac{\partial\psi}{\partial t}\right)^2 = Z\left(\frac{\partial\psi}{\partial t}\right)^2, \tag{2.141}$$

where we have introduced the quantity *impedance* $Z = \sqrt{T\rho}$ (after substitution from $v = \sqrt{\frac{T}{\rho}}$). Thus, one result is

$$S = \varepsilon\, v, \qquad \text{energy flux} = \text{energy density} \cdot \text{speed of propagation}. \tag{2.142}$$

We can say that the propagation of a wave on a string is actually the propagation of energy along the string. Or the second result:

$$S = Z\left(\frac{\partial\psi}{\partial t}\right)^2 \qquad \text{energy flux} = \text{impedance} \cdot \text{square of velocity}, \tag{2.143}$$

which tells us how to convert the velocity at a given point into a flux of energy at that point – through a constant of proportionality called *impedance*.

2.9.2 Example: Harmonic travelling wave

For a harmonic wave,

$$F(x) = A\cos kx, \qquad \psi(z,t) = F(z - vt) = A\cos(\omega t - kz), \tag{2.144}$$

we get the following forms of kinetic and potential energy densities and energy flux:

$$\begin{aligned}
\tau &= \frac{1}{2}\rho A^2\omega^2\sin^2(\omega t - kz),\\
u &= \frac{1}{2}TA^2k^2\sin^2(\omega t - kz) = \frac{1}{2}\frac{T}{v^2}A^2\omega^2\sin^2(\omega t - kz) = \tau,\\
S &= ZA^2\omega^2\sin^2(\omega t - kz).
\end{aligned} \tag{2.145}$$

For the mean values, we have

$$\boxed{\langle\tau\rangle = \langle u\rangle = \frac{1}{4}\rho A^2\omega^2, \qquad \langle S\rangle = \frac{1}{2}ZA^2\omega^2.} \tag{2.146}$$

Chapter 3

Dispersion relation, wave packets, group velocity

3.1 Phase velocity, dispersion relation

3.1.1 Travelling waves

Consider a harmonic travelling wave $\psi(z,t)$ (or its complexification) of the form:

$$\psi(z,t) = A\cos(\omega t - kz), \qquad \hat{\psi}(z,t) = Ae^{i(\omega t - kz)}, \tag{3.1}$$

Where ω is called the angular frequency, $[\omega] = \mathrm{s}^{-1}$, k the wave number, $[k] = \mathrm{m}^{-1}$. These give the period T, $[T] = \mathrm{s}$, and the wavelength λ, $[\lambda] = \mathrm{m}$, of this wave by the relations

$$\omega = \frac{2\pi}{T}, \qquad k = \frac{2\pi}{\lambda}. \tag{3.2}$$

This wave is shown in Figure 3.1.

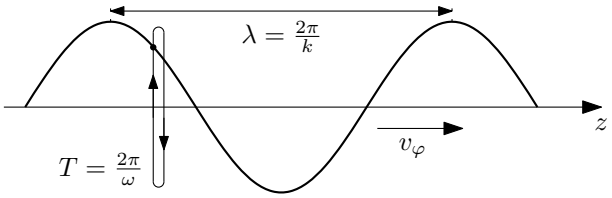


Figure 3.1: Harmonic travelling wave. At any given point $z = z_0$, we observe a harmonic oscillation with period T. In space, the harmonic wave has wavelength λ. The wave as a whole moves with phase velocity v_φ.

Let us now study the phase function $\varphi(t) = \omega t - kz$. Let us derive the relation for the velocity of motion of a place of constant phase, $\varphi(t) = \varphi_0 = \text{const}$. We express the function $z(t)$ implicitly given by this equation:

$$\varphi(t) = \omega t - kz = \varphi_0 \quad \longrightarrow \quad z(t) = \frac{\omega}{k}t - \frac{\varphi_0}{k} = v_\varphi t - \frac{\varphi_0}{k}. \tag{3.3}$$

From it we read the so-called *phase velocity* of magnitude

$$\boxed{v_\varphi = \frac{\omega}{k}.} \tag{3.4}$$

3.1.2 Dispersion relation

We have no restrictions on parameters ω and k yet (except that we consider only positive values). However, if we want to excite a harmonic travelling wave in some environment, we fail to do so for arbitrary combinations of angular velocity ω and wave number k.

Example. Consider the wave equation,

$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial z^2}, \tag{3.5}$$

and substitute a harmonic travelling wave (3.1) and require that the wave equation is satisfied for all points in space and time. The individual derivatives come out as follows

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi, \qquad \frac{\partial^2\psi}{\partial z^2} = -k^2\psi, \tag{3.6}$$

and thus

$$(\omega^2 - v^2k^2)\psi = 0. \tag{3.7}$$

If we require a non-trivial solution (i.e. a non-zero amplitude of the travelling wave), the following relation must be satisfied

$$\omega = vk. \tag{3.8}$$

This relation is called a *dispersion relation*. It specifies the admissible combinations of angular frequencies ω and wave numbers k for which the travelling wave is a solution of the wave equation. From the form of the dispersion relation, we see that the constant v in the wave equation has the meaning of phase velocity.

In general, a *dispersion relation* is of the form

$$\omega = \omega(k), \qquad k = k(\omega) \tag{3.9}$$

(one form is the inverse of the other). Thus, it is a function of $\omega(k)$, or $k(\omega)$ given by the physical environment. This relation characterizes the wave properties of a given medium in the sense that it gives us the admissible (i.e., solving the equations describing the medium) harmonic travelling waves propagating through that medium:

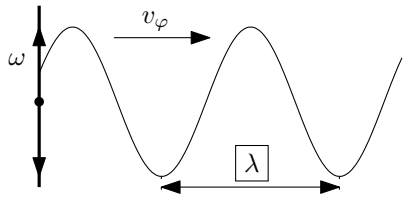
$$\psi(z,t) = Ae^{i(\omega(k)t-kz)} = Ae^{i(\omega t-k(\omega)z)}. \tag{3.10}$$

The phase velocity of these travelling waves is given by

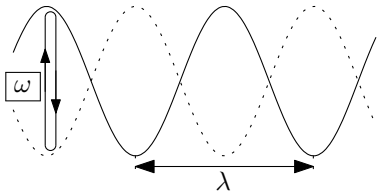
$$v_\varphi = \frac{\omega}{k}, \qquad v_\varphi(k) = \frac{\omega(k)}{k}, \qquad v_\varphi(\omega) = \frac{\omega}{k(\omega)}. \tag{3.11}$$

We see that, in general, the phase velocity can depend on the angular frequency ω, or wave number k, of the excited wave.

The interpretation of the two mutually inverse dispersion relations in (3.9) is shown in Figure 3.2. We can have a wave source oscillating at angular frequency ω, then the function $k(\omega)$ gives the wavenumber k (wavelength λ) of the resulting travelling waves. Or we can build up standing waves in the medium with a wavenumber k, then the expression $\omega(k)$ tells us what angular frequency ω they will oscillate at.



(a) Emission of a travelling wave by a source of angular frequency ω.



(b) Standing wave oscillations of wavelength λ.

Figure 3.2: The dispersion relation gives the relationship between angular frequency ω and wavenumber $k = \frac{2\pi}{\lambda}$ (and hence wavelength λ).

Examples of various dispersion relations can be found in Table 3.1.

Environment	**Dispersion relation**	**Comment**
string, sound, EM wave	$\omega = vk$	linear dispersion relation
chain of atoms	$\omega = \omega_{max} \sin \beta k$	
waveguide, ionosphere	$\omega^2 = \omega_{min}^2 + v^2 k^2$	
waves on shallow water	$\omega^2 = gk \tanh kh$	gravity g, depth h
waves on deep water	$\omega = \sqrt{gk}$, $\omega^2 = gk + \frac{\sigma}{\rho} k^3$	surface tension σ, density of water ρ
non-ideal string	$\omega^2 = \frac{T}{\rho} k^2 + \alpha k^4$	
light in matter	$\omega = \frac{c}{n} k$, $n(\omega) = \sqrt{1 + \alpha(\omega_0^2 - \omega^2)^{-1}}$	

Table 3.1: Examples of dispersion relations in various environments.

3.1.3 Reactive Environment

What happens if we have a source of angular frequency ω in a given environment (medium), but there is no wave number $k \in \mathbb{R}$ that satisfies the given dispersion relation? In this case, we speak of *reactive environment*. More precisely, for a given angular frequency, the environment appears to be reactive. In this environment, it is not possible to excite a travelling wave of the chosen angular frequency ω. An environment that allows the propagation of a travelling wave with angular frequency ω is called *transparent environment*.

Example. Consider an example of an environment that has the following form of the dispersion relation

$$\omega^2 = \omega_{min}^2 + v^2 k^2 \quad \leftrightarrow \quad \frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial z^2} - \omega_{min}^2 \psi, \tag{3.12}$$

where the given wave equation on the right leads to the dispersion relation given on the left.

For $\omega \in \langle \omega_{min}, +\infty)$ there exists $k \in \mathbb{R}$ satisfying the dispersion relation. Thus, for $\omega \geq \omega_{min}$ this is a transparent environment. For $\omega < \omega_{min}$ we cannot find a real wave number $k \in \mathbb{R}$, but there is a solution for $k \in \mathbb{C}$:

$$k = i\sqrt{\frac{\omega_{min}^2 - \omega^2}{v^2}} = i\kappa. \tag{3.13}$$

If I plug this solution for $k \in \mathbb{C}$ into the travelling wave (which gave rise to the dispersion relation) I get the wave that the source will induce in this reactive medium:

$$\psi(z,t) = e^{i(\omega t \pm kz)} = e^{i(\omega t \pm i\kappa z)} = e^{\mp \kappa z} e^{i\omega t}. \tag{3.14}$$

That is, an exponentially damped standing wave; this is shown in Figure 3.3. These non-propagating waves are called *evanescent waves*. Thus, in a reactive medium/environment, the wave does not propagate, but is exponentially damped – this is due to the complex solution $i\kappa$,

where the harmonic oscillations (in the form of a complex exponential) become a real exponential.

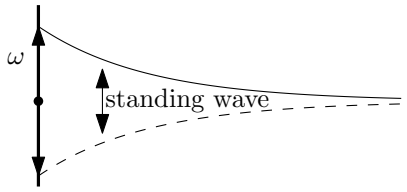


Figure 3.3: A wave source of angular frequency ω at the boundary of the reactive medium will produce an exponentially damped standing wave.

Figure 3.4 shows the propagation of a wave in a transparent medium that is obstructed by a reactive medium of finite width. Passing through the reactive medium will partially dampen the wave. We can define a distance called *depth of penetration*, $\delta = \frac{1}{\kappa}$, where the amplitude of the wave drops to $1/e$.

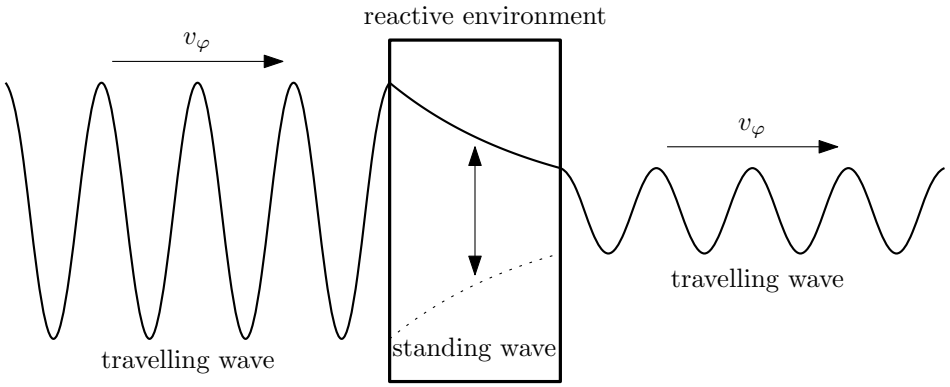


Figure 3.4: A travelling wave in a transparent medium that encounters a thin reactive medium will partially decay exponentially.

Example. Let's look at a more complicated example of a chain of atoms. The dispersion relation is of the form

$$\omega = \omega_{max} \sin \beta k, \qquad \beta = \frac{a}{2}, \tag{3.15}$$

which was derived by substituting a standing wave

$$\psi_l(t) = [\mathrm{Re}\, e^{ikla}]e^{i\omega t} \tag{3.16}$$

into the equations of motion[1], ψ_l gives the deflection of the l-th weight, a is the distance of the individual bodies, see figure 3.5. We see that the chain is a transparent medium for $\omega \in \langle 0, \omega_{max} \rangle$.

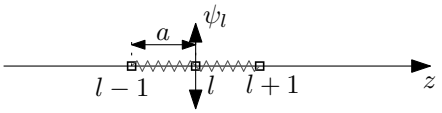


Figure 3.5: The chain of atoms. The individual atoms are numbered $l \in \mathbb{Z}$, their displacements are described by functions $\psi_l(t)$, and the distance between the atoms is a.

[1]The standing wave is produced by the superposition of travelling waves, so the dispersion relation encodes the same information for travelling and standing waves.

For $\omega \in \langle 0, \omega_{max} \rangle$ there exists $k \in \mathbb{R}$ satisfying the dispersion relation and hence it is a transparent medium. In the reactive regime for $\omega > \omega_{max}$, the simple trick $k \to i\kappa$ we used in the previous example will not work (try to work out that the dispersion relation will not be satisfied for $\omega, \kappa \in \mathbb{R}$). So let us first manipulate the dispersion relation (3.15) using complex exponentials:

$$\frac{\omega}{\omega_{max}} = \sin \beta k = \frac{e^{i\beta k} - e^{-i\beta k}}{2i} = \frac{(-i)e^{i\beta k} + ie^{-i\beta k}}{2} = \frac{e^{i(\beta k - \frac{\pi}{2})} + e^{-i(\beta k - \frac{\pi}{2})}}{2}, \tag{3.17}$$

where we have used the identity of $e^{\pm i\frac{\pi}{2}} = \pm i$. For the purely imaginary choice of $\beta k - \frac{\pi}{2}$, i.e., for example, of the form $\beta k - \frac{\pi}{2} = i\beta\kappa$, $\kappa \in \mathbb{R}$, we get

$$\frac{\omega}{\omega_{max}} = \frac{e^{\beta\kappa} + e^{-\beta\kappa}}{2} = \cosh \beta\kappa \geq 1. \tag{3.18}$$

Thus, by choosing $k = i\kappa - \frac{1}{\beta}\frac{\pi}{2}$ and for $\omega > \omega_{max}$ we satisfy the dispersion relation for $\omega, \kappa \in \mathbb{R}$, and it is of the form

$$\omega = \omega_{max} \cosh \beta\kappa. \tag{3.19}$$

The waveform of the evanescent wave (after substituting $k = i\kappa + \frac{1}{\beta}\frac{\pi}{2}$ into (3.16)) is

$$\psi_l(t) = e^{-\kappa l a} e^{i\frac{1}{\beta}\frac{\pi}{2} l a} e^{i\omega t} = (-1)^l e^{-\kappa l a} e^{i\omega t}, \tag{3.20}$$

this is shown in Figure 3.6.

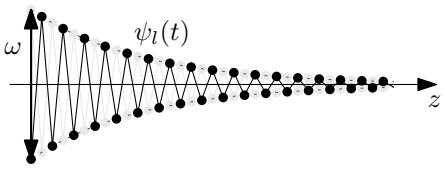


Figure 3.6: A wave on a chain of atoms in the reactive mode. The displacement of adjacent weights differs by a factor -1. The whole system oscillates as a standing wave with angular frequency ω (the gray positions of the weights are plotted half a period later than the black positions of the weights shown). The amplitude of the oscillation decreases exponentially with distance from the source of the wave.

3.2 Mathematical supplement: Fourier transform

We have already seen how to decompose a periodic signal into a (discrete) sum of harmonic waves. This was the decomposition of a periodic function into a Fourier series. Consider a function $f(t) : \mathbb{R} \to \mathbb{R}$ with period T, its Fourier series is then[2].

$$f(t) = \sum_{m=0}^{+\infty} a_m \cos\left(\frac{2m\pi t}{T}\right) + b_m \sin\left(\frac{2m\pi t}{T}\right), \tag{3.21}$$

Where the coefficients a_m, b_m are given by the formulae

$$a_m = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos\left(\frac{2m\pi t}{T}\right) dt, \qquad b_m = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin\left(\frac{2m\pi t}{T}\right) dt, \tag{3.22}$$

[2]To simplify the following explanation, we have included the coefficient a_0 directly in the sum, it should then be remembered that the formula for calculating a_0 must be divided by two.

where the index $m \in \mathbb{N}_0$ indexes the frequencies from which we superimpose the original signal. We denote the lowest frequency by $\omega_1 = \frac{2\pi}{T}$. Higher frequencies can then be written as $\omega_m = m\,\omega_1$. Decomposing a periodic signal into a Fourier series tells us that we need a discrete number of frequencies, $\{\omega_m \,|\, m \in \mathbb{N}\}$. See the schematic Figure 3.7.

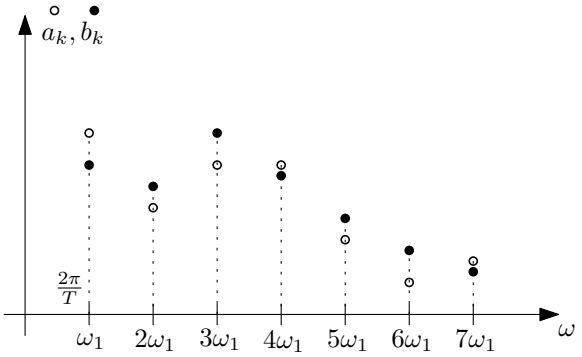


Figure 3.7: Schematic representation of discrete frequencies in a Fourier series.

Consider now a nonperiodic function $f(t) : \mathbb{R} \to \mathbb{R}$. For each $T \in \mathbb{R}^+$, we can calculate the coefficients of a_m, b_m according to the formulae (3.22) to obtain the Fourier series of the periodic extension of the function $f|_{\langle -\frac{T}{2}, \frac{T}{2}\rangle}$. In a non-rigorous procedure, we get *Fourier transform* as the limit of these Fourier series if we perform $T \to +\infty$. In this case $\omega_1 \to 0$ and the discrete frequencies $m\omega_1$, $m \in \mathbb{N}$, become a continuum of frequencies $\omega \in \langle 0, +\infty)$.

Thus, the nonperiodic function $f(t)$ can be written as the following limit of Fourier series

$$f(t) = \lim_{T\to+\infty} \sum_{m=0}^{+\infty} \left(\frac{a_m(T)}{\omega_1(T)} \cos(m\omega_1 t) + \frac{b_m(T)}{\omega_1(T)} \sin(m\omega_1 t) \right) \omega_1(T), \tag{3.23}$$

where we have expanded the terms in the series by the frequency ω_1. This expression is actually the Riemann integral[3]:

$$\boxed{f(t) = \int_0^{+\infty} A(\omega)\cos\omega t + B(\omega)\sin\omega t\, d\omega,} \tag{3.24}$$

where we denote $\omega = m\omega_1$ and the functions $A(\omega)$ and $B(\omega)$ are defined as

$$A(\omega) = \lim_{T\to+\infty} \frac{a_m(T)}{\omega_1(T)}, \qquad B(\omega) = \lim_{T\to+\infty} \frac{b_m(T)}{\omega_1(T)}, \tag{3.25}$$

Performing these limits, we obtain

$$\boxed{A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos\omega t\, dt, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\omega t\, dt.} \tag{3.26}$$

The relations (3.24) and (3.26) represent the Fourier transform and its inverse. We have shown that nonperiodic functions can be written as a continuous superposition of harmonic

[3]From the definition of the Riemann integral, the following holds

$$\lim_{N\to+\infty} \sum_{k=1}^{N} f\left(k\tfrac{a}{N}\right) \frac{a}{N} = \int_0^a f(x)\, dx.$$

Here the situation is complicated by the fact that we have infinite sums corresponding to the integration for $\omega \in \langle 0, +\infty)$.

waves. We will refer to the functions $A(\omega)$ and $B(\omega)$ as *spectral functions* or *spectra* of f. These functions give the amplitudes of the individual harmonic waves constituting the function f.

The non-rigorousness of our procedure lies in the double limit. First, we actually do a limit to obtain the functions $A(\omega)$ and $B(\omega)$, and then we do a limit to go from summation to integration. However, the rigorous procedure leads to the same expressions.

3.2.1 Fourier transform in complex notation

Let us write the Fourier transform (3.24) in the language of complex exponentials:

$$\begin{aligned} f(t) &= \int_0^{+\infty} A(\omega) \underbrace{\cos\omega t}_{\frac{e^{i\omega t}+e^{-i\omega t}}{2}} + B(\omega) \underbrace{\sin\omega t}_{\frac{e^{i\omega t}-e^{-i\omega t}}{2i}} \, d\omega \\ &= \int_0^{+\infty} \underbrace{\frac{A(\omega)-iB(\omega)}{2}}_{C(\omega)} e^{i\omega t} + \underbrace{\frac{A(\omega)+iB(\omega)}{2}}_{\bar{C}(\omega)} e^{-i\omega t} \, d\omega. \end{aligned} \tag{3.27}$$

Thus, we can define a complex spectral function,

$$C(\omega) = \frac{A(\omega) - iB(\omega)}{2}, \tag{3.28}$$

encoding the amplitudes $A(\omega)$ and $B(\omega)$ as the real and imaginary parts (except for the factor of two and minus). If we now look at the formulas for $A(\omega)$ and $B(\omega)$ (3.26), we see that they also make sense for $\omega < 0$ and the following hold:

$$A(-\omega) = A(\omega), \qquad B(-\omega) = -B(\omega). \tag{3.29}$$

For the function $C(\omega)$, the above implies $C(-\omega) = \bar{C}(\omega)$:

$$C(-\omega) = \frac{A(-\omega) - iB(-\omega)}{2} = \frac{A(\omega) + iB(\omega)}{2} = \bar{C}(\omega). \tag{3.30}$$

After substituting this fact into (3.27), the Fourier integral takes the form

$$f(t) = \int_0^{+\infty} C(\omega) e^{i\omega t} \, d\omega + \int_0^{+\infty} C(-\omega) e^{-i\omega t} \, d\omega \tag{3.31}$$

and after the substitution in the second integral $\tilde{\omega} = -\omega$, $d\tilde{\omega} = -d\omega$ we get this very simple form of *complex Fourier transform*

$$\boxed{f(t) = \int_{-\infty}^{+\infty} C(\omega) e^{i\omega t} \, d\omega.} \tag{3.32}$$

The condition $C(-\omega) = \bar{C}(\omega)$ is the necessary and sufficient condition for the reality of the function $f(t)$. The relation for the calculation of the function $C(\omega)$ is obtained by a simple combination of the relations (3.26):

$$\boxed{C(\omega) = \frac{A(\omega) - iB(\omega)}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \, dt.} \tag{3.33}$$

At this point it is worth noting that there are various conventions for introducing the complex Fourier transform. The factor $\frac{1}{2\pi}$ is often decomposed into $\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}}$ and one of the factors is

assigned to the formula for $f(t)$. This convention leads to a very symmetric form of the Fourier transform,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} C(\omega) e^{i\omega t}\, d\omega, \qquad C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt. \tag{3.34}$$

Another possible convention is to swap the functions $e^{i\omega t}$ and $e^{-i\omega t}$, or to use the frequency $f = \frac{\omega}{2\pi}$ as a variable in the Fourier transform. Thus, when studying the literature, it is always advisable to familiarize oneself with the convention used.

3.3 Wave packets and uncertainty relations

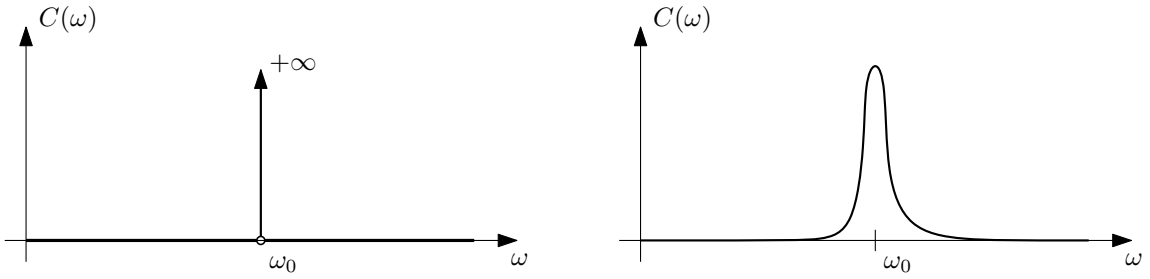
A source oscillating in a purely harmonic waveform $x(t)$ for an indefinite time – a source of harmonic travelling waves $\psi(z,t)$,

$$x(t) = A\cos(\omega_0 t + \varphi), \qquad \psi(z,t) = A\cos(\omega_0 t - k_0 z + \varphi), \qquad \forall t \in \mathbb{R}, \tag{3.35}$$

– is not a completely realistic model. Such waves are called *monochromatic* because their spectrum contains a single frequency ω_0, see Figure 3.8 (a). Therefore, we introduce the notion of *quasi-monochromatic* waves of the form

$$x(t) = A(t)\cos(\omega_0(t)\, t + \varphi(t)), \qquad \psi(z,t) = A(t_r)\cos(\omega_0(t_r)\, t - k_0(t_r) z + \varphi(t_r)), \tag{3.36}$$

where the amplitude A, frequency ω, and phase shift φ can in general change over time (the symbol t_r denotes retarded time). However, these changes must be slow enough so that over a period of time $\tau \gg T = \frac{2\pi}{\omega}$ these parameters can be considered nearly constant – and hence the chosen wave duration τ approximates the monochromatic wave accurately enough. The spectrum of the quasi-monochromatic wave is centered around the frequency ω_0, but contains the entire continuum of frequencies around it, see figure 3.8 (b).



(a) Monochromatic wave spectrum.

(b) Quasi-monochromatic wave spectrum.

Figure 3.8: Spectrum of monochromatic and quasi-monochromatic waves.

One possible example is the emission of a weakly damped harmonic oscillation

$$x(t) = e^{-\alpha t}\cos(\omega_0 t), \qquad \psi(z,t) = e^{-\alpha\left(t - \frac{z}{v}\right)}\cos(\omega_0 t - k_0 z), \tag{3.37}$$

see Figure 3.9. The weak damping condition is defined here by the fact that the decay time $\tau = \frac{1}{\alpha}$ of the wave to $1/e$ amplitude value is much larger than the period of the harmonic oscillations:

$$\tau = \frac{1}{\alpha} \gg T = \frac{2\pi}{\omega_0}, \qquad \alpha \ll \omega_0. \tag{3.38}$$

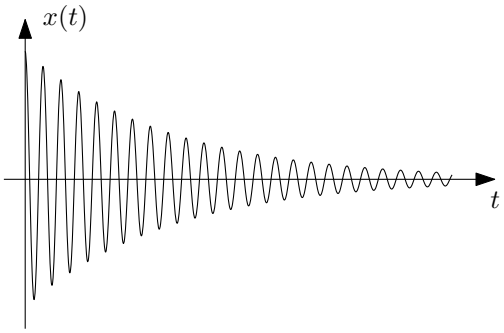


Figure 3.9: Exponentially damped oscillation.

The last concept we will need here is the notion of *wave packet*. We will use the notion to mean a temporally and spatially bounded oscillations. It is not directly meant that outside of a given temporal or spatial region the waves must completely disappear. Rather, it is meant that the "main" part of the wave (with the greatest amplitude, with the greatest energy) is concentrated in a limited temporal or spatial interval. Thus, for wave packets, we would like to define the quantities Δt–time width of the packet – and Δz–spatial width of the packet. These two quantities will have a simple relationship between them. If v is the packet's velocity through the environment, then $\Delta z = v\,\Delta t$ will hold.

We obtain quasi-monochromatic wave packets as a continuous superposition of harmonic waves,

$$x(t) = \int_{-\infty}^{+\infty} C(\omega)\, e^{i\omega t}\, d\omega, \qquad \psi(z,t) = x\left(t - \frac{z}{v}\right). \tag{3.39}$$

Now, we consider a simple model of a wave packet, where we choose a real spectral function $C(\omega)$ as in Figure 3.10. That is, a spectrum centered around the frequency ω_0 and the spectral width $\Delta\omega$.

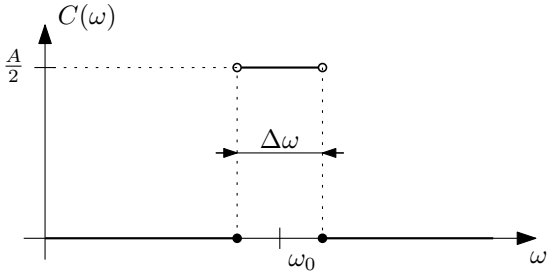


Figure 3.10: Rectangular spectrum.

The Fourier transform of this spectrum then follows

$$x(t) = \int_{0}^{+\infty} A(\omega)\cos(\omega t)\, d\omega = \int_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} A\cos(\omega t)\, d\omega \tag{3.40}$$

and after calculating and manipulating it using the sum formula, we get the resulting waveform of the packet,

$$x(t) = A\Delta\omega \frac{\sin\left(\frac{\Delta\omega}{2}t\right)}{\frac{\Delta\omega}{2}t}\cos(\omega_0 t), \tag{3.41}$$

which is shown in Figure 3.11. After it is emitted from the source, it will have the form[4].

$$\psi(z,t) = A\Delta\omega \frac{\sin\left(\frac{\Delta\omega}{2}t_r\right)}{\frac{\Delta\omega}{2}t_r}\cos(\omega_0 t - k_0 z), \qquad t_r = t - \frac{z}{v}, \tag{3.42}$$

where t_r is the retarded time.

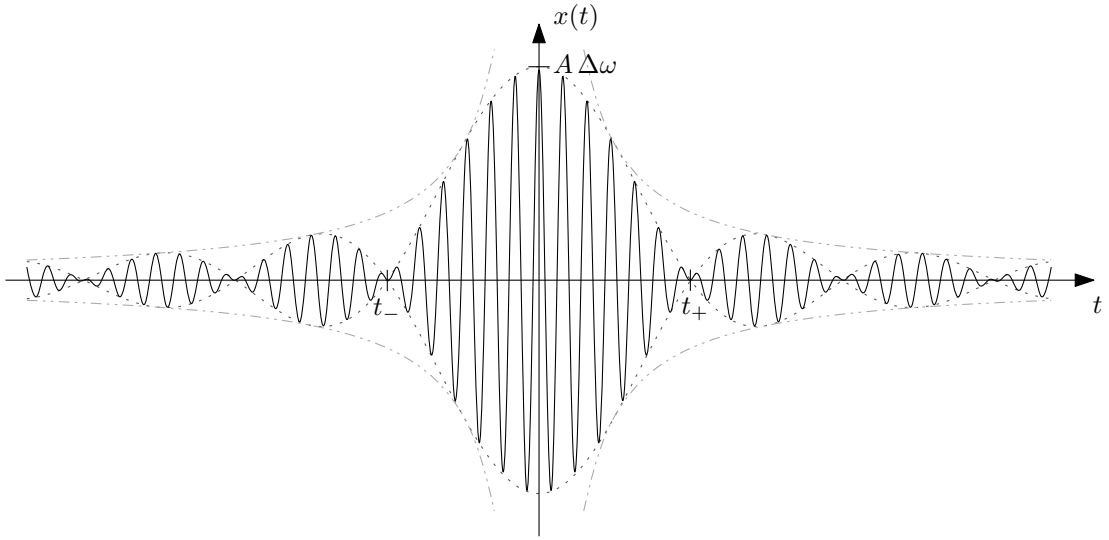


Figure 3.11: The wave packet generated from a rectangular spectrum. The actual waveform of the function $x(t)$ is shown by the solid line. The amplitude envelope is indicated in dashed lines, the envelope given by inverse proportion only is indicated in black. The times $t_\pm$ indicate the points where the amplitude envelope first crosses zero.

The functional form of the wave packet $x(t)$ (3.41) can be written in the form

$$x(t) = A(t)\cos(\omega_0 t), \quad \text{where} \quad A(t) = A\Delta\omega \frac{\sin\left(\frac{\Delta\omega}{2}t\right)}{\frac{\Delta\omega}{2}t}. \tag{3.43}$$

That is, the wave packet is in the form of a carrier harmonic wave of frequency ω_0 that is modulated by an amplitude envelope described by the function $A(t)$ (see the black dashed line in Figure 3.11 representing function $\pm A(t)$). Let us now find the width of this envelope by calculating the times t_+ and t_- ($t_+ > 0$, $t_- < 0$) when the amplitude envelope first crosses zero, and the difference of these times will represent the width of this envelope. The sine has the first zeros at $\pm\pi$ and thus

$$\frac{\Delta\omega}{2}t_\pm = \pm\pi. \tag{3.44}$$

The width of the packet is therefore

$$\Delta t = t_+ - t_- = \frac{4\pi}{\Delta\omega} \tag{3.45}$$

and after rearrangement we get the relation

$$\Delta t \cdot \Delta\omega = 4\pi = \text{const.} \tag{3.46}$$

This relation says that the time width of the packet and the width of its frequency spectrum are inversely proportional to each other! The shorter the wave packet, the more frequencies we

[4] At time $t = 0$, the function can be continuously defined according to the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

need to create it (and vice versa). The specific value of the constant on the right depends on the specific definition of the values Δt and $\Delta\omega$ and is not very important. The main conclusion is the qualitative relation of inverse proportionality.

In general, the following statement can be proved, which is called the *uncertainty relation*:

$$\Delta t \cdot \Delta\omega \geq \pi. \tag{3.47}$$

The product of the temporal width Δt and the spectral width $\Delta\omega$ will never be less than a certain constant (independent[5] of the shape of the signal or spectrum). Of course, the quantities Δt and $\Delta\omega$ are rigorously defined in the theorem. Unfortunately, there is no room for that here. But let us give a simple statement from the theory of Fourier analysis that intuitively illustrates the validity of the uncertainty relations, although it does not prove them:

Consider a function $f(t)$ and its corresponding spectral function $C(\omega)$. Consider the constant $a \in \mathbb{R}^+$. Then the spectral function $f(at)$ is of the form $\frac{1}{a}C\left(\frac{\omega}{a}\right)$. Let us prove this statement. Let us denote by $C_a(\omega)$ the spectral function to the function $f(at)$. From the definition of the Fourier integral, we have

$$C_a(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(at)\, e^{-i\omega t}\, dt. \tag{3.48}$$

Performing the substitution $t' = at$, we obtain the statement of the theorem:

$$C_a(\omega) = \frac{1}{a}\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t')e^{-i\frac{\omega}{a}t'}\, dt' = \frac{1}{a}C\left(\frac{\omega}{a}\right). \tag{3.49}$$

Now let us see what this theorem practically asserts. Let the time width of the wave packet $f(t)$ be of size Δt and the corresponding frequency spectrum width $\Delta\omega$,

$$\Delta t \quad \ldots \quad f(t) \quad \leftrightarrow \quad C(\omega) \quad \ldots \quad \Delta\omega. \tag{3.50}$$

Multiplying the argument of the function f by the constant $a > 0$ represents the scaling of the timeline. For $a > 1$ the function shrinks on the timeline, for $a < 1$ the function expands. Thus, the time width of the function $f(at)$ is $\frac{\Delta t}{a}$. From the assertion of the theorem, the spectrum of the scaled function is of the form $\frac{1}{a}C\left(\frac{\omega}{a}\right)$, so the scaling of the frequency spectrum is inverse to that of the time course of the function. If the original width of the frequency spectrum is $\Delta\omega$, for the scaled function it will be $a\Delta\omega$,

$$\frac{\Delta t}{a} \quad \ldots \quad f(at) \quad \leftrightarrow \quad \frac{1}{a}\,C\left(\frac{\omega}{a}\right) \quad \ldots \quad a\,\Delta\omega. \tag{3.51}$$

Thus, the product of the time width and the spectral width remains constant for arbitrary value of a:

$$\left(\frac{\Delta t}{a}\right)\,(a\,\Delta\omega) = \Delta t\,\Delta\omega. \tag{3.52}$$

Finally, let us consider a simple example of the application of uncertainty relations concerning the order of magnitude estimation of achievable data transmission rates. Consider the primitive way of encoding a binary signal into a broadcast signal, where we encode 1 by sending a wave packet, while we encode 0 by not sending it, see Figure 3.12.

[5]The constant is in turn dependent on the particular convention used in the Fourier transform.

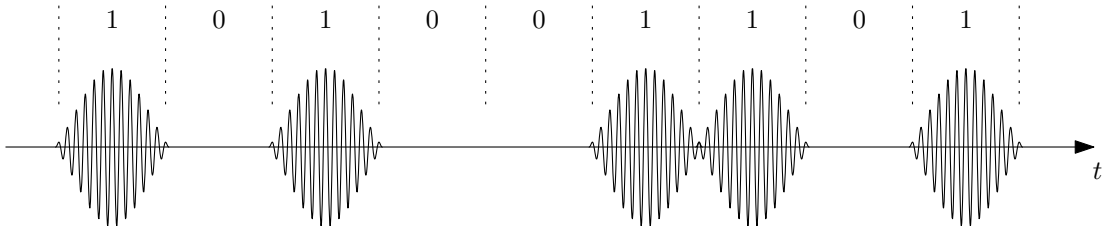


Figure 3.12: Binary signal encoded into transmitted/not-sent packets.

If the wave packets have a width of Δt we must wait approximately at least time Δt before possibly sending or not sending the next packet in the queue, so that the receiver can easily distinguish between the sent packets. This means that we can send $N = \frac{1}{\Delta t}$ packets, and thus bits of information, per unit time. If we have a bandwidth of $\Delta f = \frac{\Delta\omega}{2\pi}$ available for transmission, according to the uncertainty relations, the packet width is approximately at least $\Delta t = \frac{\pi}{\Delta\omega} = \frac{1}{2\Delta f}$. The transmission rate is therefore at most $N = 2\Delta f$ bits per second. For example, the bandwidth of a single Wi-fi channel is $\Delta f = 20\,\mathrm{MHz}$, which gives an estimate for the bit rate of $N = 40\,\mathrm{Mbps}$ when just one channel is used.

3.4 Group velocity

3.4.1 Superposition of two travelling harmonic waves

First, we will illustrate the phenomenon of *group velocity* with the simplest possible example – the superposition of two travelling harmonic waves. Consider an environment with dispersion relation $\omega(k)$ and take travelling waves with wave numbers k_1 and k_2, and obtain the corresponding angular frequencies from the dispersion relation:

$$k_1, k_2 \quad \longrightarrow \quad \omega_1 = \omega(k_1),\, \omega_2 = \omega(k_2). \tag{3.53}$$

Consider the following superposition of waves with the same amplitude and no phase shift[6] and manipulate it with the trigonometric formula for a sum of cosines:

$$\begin{aligned}\psi(z,t) =&\ A\cos\left(\omega_1 t - k_1 z\right) + A\cos\left(\omega_2 t - k_2 z\right)\\ =&\ 2A\cos\left(\frac{\omega_1+\omega_2}{2}\,t - \frac{k_1+k_2}{2}z\right)\cos\left(\frac{\omega_1-\omega_2}{2}\,t - \frac{k_1-k_2}{2}z\right).\end{aligned} \tag{3.54}$$

Now let us consider k_1 close to k_2, without any loss of generality we take $k_1 > k_2$. Let us denote by

$$k_0 = \frac{k_1+k_2}{2}, \qquad k_{mod} = \frac{k_1-k_2}{2}. \tag{3.55}$$

Then, $k_{mod} \ll k_0$ holds due to the closeness of k_1, k_2. Next, we denote

$$\omega_0 = \frac{\omega_1+\omega_2}{2}, \tag{3.56}$$

due to the closeness of k_1 and k_2, ω_1 and ω_2 will also be close and hence $\omega_1 \approx \omega_2 \approx \omega_0$ from the continuity of the dispersion relation. We further manipulate the expression

$$\frac{\omega_1-\omega_2}{2} = \frac{\omega(k_1)-\omega(k_2)}{k_1-k_2}\,\frac{k_1-k_2}{2} = \frac{d\omega}{dk}(\xi)\,k_{mod} \approx \frac{d\omega}{dk}(k_0)\,k_{mod}, \tag{3.57}$$

[6]The phase shift adds nothing except longer writing time.

where we have used Lagrange's theorem on the increment of a function, $\xi \in (k_1, k_2)$ and then approximated the derivative by substituting ξ for k_0. The superposition (3.54) will then be approximately of the form

$$\psi(z,t) \approx 2A \cos\left(\omega_0 t - k_0 z\right) \cos\left(\omega'(k_0) k_{mod}\, t - k_{mod}\, z\right). \tag{3.58}$$

This superposition takes the form of the product of two travelling harmonic waves. The first is a *carrier wave* with (high) angular velocity ω_0 and large wavenumber k_0 (small wavelength λ_0). The second is a *modulating wave*, which modulates the amplitude of the carrier wave – forming its amplitude envelope. This wave has much larger wavelength λ_{mod} (smaller wavenumber k_{mod}). We get the waveform of periodically repeating wave packets. See Figure 3.13.

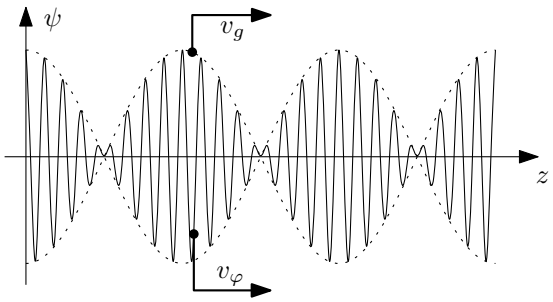


Figure 3.13: Superposition of two travelling harmonic waves in the form of periodically repeating wave packets.

What are the phase velocities of the carrier and modulation waves? According to relation $v = \frac{\omega}{k}$, we insert the respective angular frequencies and wave numbers of the individual waves and obtain:

$$v_{\text{carrier}} = v_\varphi = \frac{\omega_0}{k_0}, \qquad v_{modulating} = v_g = \frac{\omega'(k_0) k_{mod}}{k_{mod}} = \frac{d\omega}{dk}(k_0). \tag{3.59}$$

The phase velocity of the modulating wave is called *group velocity*. This gives the speed of motion of the wave packets, or their amplitude envelope, through the medium.

In the case where we have a linear dispersion relation of the form $\omega = vk$, the phase velocity of the carrier wave and the group velocity are equal. Otherwise, in general, they may differ.

The phase velocities of the original harmonic waves are

$$v_{\varphi 1} = \frac{\omega_1}{k_1} = \frac{\omega(k_1)}{k_1}, \qquad v_{\varphi 2} = \frac{\omega_2}{k_2} = \frac{\omega(k_2)}{k_2}. \tag{3.60}$$

Due to the proximity of k_1 and k_2, these velocities are close to the phase velocity of the carrier wave, $v_\varphi \approx v_{\varphi 1} \approx v_{\varphi 2}$.

3.4.2 General wave packet

Consider now a general wave packet, which is a continuous superposition of harmonic waves. Let there be a source in the environment that emits a signal of the form

$$f(t) = \int_{-\infty}^{+\infty} C(\omega) e^{i\omega t}\, d\omega, \tag{3.61}$$

where the spectrum function $C(\omega)$ is centered around the frequency ω_0, see Figure 3.14. The source emits travelling waves, so each harmonic component becomes a travelling wave:

$$f(t) \propto e^{i\omega t} \quad \longrightarrow \quad \psi(z,t) \propto e^{i(\omega t - kz)}, \tag{3.62}$$

where $k = k(\omega)$ is given by the dispersion relation. The full superposition of the travelling wave then takes the form

$$\psi(z,t) = \int_{-\infty}^{+\infty} C(\omega) e^{i(\omega t - k(\omega) z)}\, d\omega. \tag{3.63}$$

We find it more convenient to integrate over the wave numbers k, so by substitution,

$$\omega = \omega(k), \qquad d\omega = \frac{d\omega}{dk} dk, \qquad \tilde{C}(k) := C(\omega(k)) \frac{d\omega}{dk}, \tag{3.64}$$

we transform the expression to a form where we integrate the waves over their wavelengths (wavenumbers):

$$\psi(z,t) = \int_{-\infty}^{+\infty} \tilde{C}(k) e^{i(\omega(k) t - k z)}\, dk. \tag{3.65}$$

The spectral function $\tilde{C}(k)$ is also centered around a single wavenumber $k_0 = k(\omega_0)$, once again see Figure 3.14.

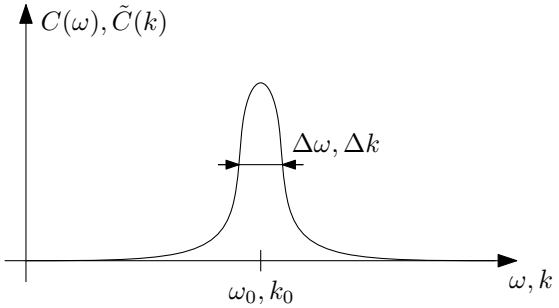


Figure 3.14: Wave packet spectrum. Schematic representation of the spectral functions $C(\omega)$ and $\tilde{C}(k)$.

We now expand the dispersion relation into a Taylor series with center k_0,

$$\omega(k) = \underbrace{\omega(k_0)}_{\omega_0} + \frac{d\omega}{dk}(k_0)(k - k_0) + O((k - k_0)^2), \tag{3.66}$$

and insert the expansion to the individual harmonic components forming the total wave:

$$\begin{aligned} e^{i(\omega(k)t - kz)} &= \underbrace{e^{ik_0 z} e^{-ik_0 z}}_{=1} e^{i\omega_0 t} e^{i(\omega'(k_0)(k-k_0)t - kz)} e^{iO((k-k_0)^2)t} \\ &= e^{i(\omega_0 t - k_0 z)} e^{i\left(\omega'(k_0)(k-k_0)t - (k-k_0)z\right)} e^{iO((k-k_0)^2)t}, \end{aligned} \tag{3.67}$$

where we have multiplied the expression by an appropriately chosen unity and rearranged the terms to factor out the complex carrier wave. We substitute the previous expression into the integral (3.65) and denote by $k' = k - k_0$ to shorten the notation:

$$\psi(z,t) = e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{+\infty} \tilde{C}(k' + k_0) e^{i(\omega'(k_0)k't - k'z)} e^{iO(k'^2)t} dk. \tag{3.68}$$

We now take advantage of the fact that the spectral function $\tilde{C}(k)$ is centered around the wave number k_0 (with spectral width Δk). Thus, we can approximate the function $\psi(z,t)$ by restricting the domain of integration to $k \in \langle k_0 - \frac{\Delta k}{2}, k_0 + \frac{\Delta k}{2} \rangle$ only. Since we have thus

restricted ourselves to values of k close to k_0, we can further neglect the higher orders of the Taylor expansion, $e^{iO(k'^2)}$. The result is:

$$\psi(z,t) \approx e^{i(\omega_0 t - k_0 z)} \int_{k_0-\frac{\Delta k}{2}}^{k_0+\frac{\Delta k}{2}} \tilde{C}(k'+k_0) \underbrace{e^{i(\omega'(k_0)k't-k'z)}}_{e^{-ik'(z-\omega'(k_0)t)}} dk. \tag{3.69}$$

The whole complicated expression under the integral depends on the variables z and t in only one place: the exponential contains the expression $z - \omega'(k_0)t$, which is constant with respect to the integration variable. We can therefore define the function $F(x)$,

$$F(x) = \int_{k_0-\frac{\Delta k}{2}}^{k_0+\frac{\Delta k}{2}} \tilde{C}(k) e^{-i(k-k_0)x} dk, \tag{3.70}$$

and write the resulting wave $\psi(z,t)$ as

$$\psi(z,t) \approx e^{i(\omega_0 t - k_0 z)} F\big(z - \omega'(k_0)t\big). \tag{3.71}$$

This is an analogous result to the superposition of two harmonic waves. Here we have a carrier wave $e^{i(\omega_0 t - k_0 z)}$ propagating at phase velocity $v_\varphi = \frac{\omega_0}{k_0}$ and it is amplitude modulated by a modulation function $F(x)$ which propagates at velocity $\omega'(k_0)$ (recall d'Alembert's solution $F(z-vt)$). Thus we have the same expressions for *phase and group velocity* (as for the superposition of two harmonic waves):

$$v_\varphi = \frac{\omega_0}{k_0}, \qquad v_g = \frac{d\omega}{dk}(k_0), \tag{3.72}$$

representing the velocities of propagation of the carrier and modulation waves.

3.4.3 Wave packet dispersion

In the previous section, after neglecting higher orders of Taylor expansion, we obtained a travelling wave packet whose amplitude envelope has a shape $F(x)$ that is constant in time. Under the integral, we have neglected a term of the form

$$\exp\left[i\, O\left((k-k_0)^2\right) t\right], \tag{3.73}$$

which makes the function $F(x)$ not constant in time, but has an additional time dependence,

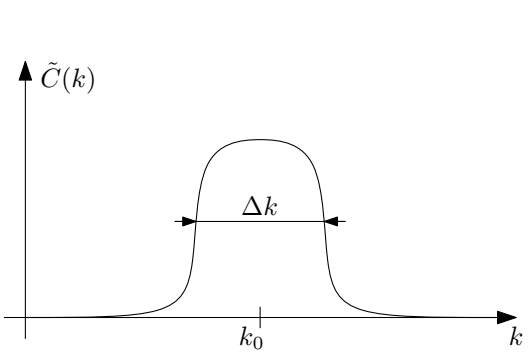
$$F(x) \quad \longrightarrow \quad F(x,t). \tag{3.74}$$

This additional time dependence causes the travelling amplitude envelope $F(z-v_g t, t)$ to change its shape over time – it deforms. This phenomenon is called *wave packet dispersion*. Let us try to quantify this phenomenon elementarily.

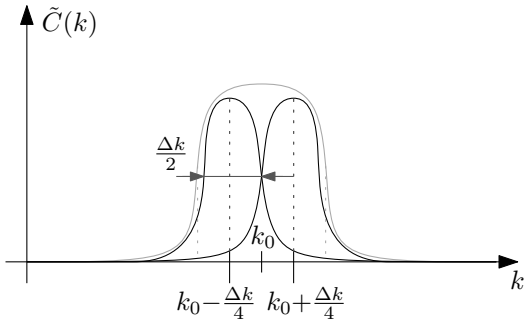
Consider again the spectrum of a wave packet $\tilde{C}(k)$ (shown again in Figure 3.15 (a)) centered around a wave number k_0 of frequency width Δk and its corresponding wave packet $\psi(z,t)$ (given by the Fourier transform of the spectrum $\tilde{C}(k)$). Let us decompose it into a superposition of the two packets,

$$\psi(z,t) = \psi_-(z,t) + \psi_+(z,t), \tag{3.75}$$

where the frequency spectrum of the ψ_- package is centered around $k_0 - \frac{\Delta k}{4}$ and ψ_+ around $k_0 + \frac{\Delta k}{4}$, both with spectral width $\frac{\Delta k}{2}$, see Figure 3.15 (b).



(a) Total spectrum of the wave packet.



(b) Decomposition of the wave packet spectrum into two sub-packets.

Figure 3.15: Decomposition of the spectrum for the study of wave packet dispersion.

Each of these sub-packets may move at a different group velocity:

$$v_{g-} = \frac{d\omega}{dk}(k_-), \qquad v_{g+} = \frac{d\omega}{dk}(k_+), \tag{3.76}$$

where v_{g-} and v_{g+} denote the group velocity ψ_- and ψ_+, respectively. What will be the difference of these velocities Δv_g?

$$\Delta v_g = v_{g+} - v_{g-} = \frac{\omega'(k_+) - \omega'(k_-)}{k_+ - k_-}(k_+ - k_-) = \omega''(\xi)\frac{\Delta k}{2} \approx \omega''(k_0)\frac{\Delta k}{2} = \frac{dv_g}{dk}(k_0)\frac{\Delta k}{2}. \tag{3.77}$$

In the manipulations, we have used Lagrange's theorem and the fact that $\xi \in (k_-, k_+)$, $\xi \approx k_0$, and the spectral centers of each subpackage are denoted by $k_- = k_0 - \frac{\Delta k}{4}$ and $k_+ = k_0 + \frac{\Delta k}{4}$, respectively. Thus, it holds

$$\Delta v_g = \frac{1}{2} v_g'(k_0)\, \Delta k. \tag{3.78}$$

Let us denote the (spatial) packet width at time t as $\Delta z(t)$ and investigate how this will change. Let the time width of the original signal $f(t)$ transmitted by the source be Δt. Its initial width is then

$$\Delta z(0) = v_g \Delta t, \tag{3.79}$$

as it travels at velocity v_g and takes approximately Δt to transmit. The sub-packet's centers are moving away from each other at a rate of Δv_g, so the packet width at time t is

$$\Delta z(t) = \Delta z(0) + \Delta v_g\, t \approx \Delta z(0) + \frac{1}{2}\omega''(k_0)\Delta k\, t. \tag{3.80}$$

Wave packet dispersion is a phenomenon that generally degrades our transmitted signal. If we transmit a number of packets in succession, the effect of their dispersion is to cause them to overlap and make it impossible to tell from the distorted signal whether or not a packet is present (we have primitively encoded a binary signal in this way).

Chapter 4

Wave reflections

In this chapter we will discuss the behavior of waves at the interface of environments. We will illustrate the concepts using a string model. We begin by studying a semi-infinite string that is terminated at a given point, see Figure 4.1. We then move on to the case where we have two different strings that are connected at a given location. Our goal will be to find expressions for the transmitted and reflected waves depending on the prescribed wave incident at the interface, see Figure 4.2.

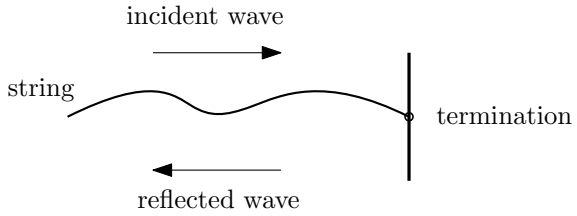


Figure 4.1: Model situation for studying reflections #1: Termination of a single string.

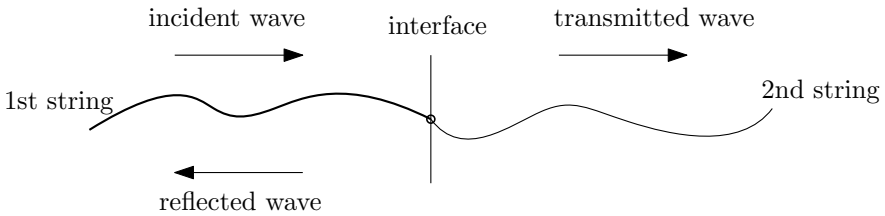


Figure 4.2: Model situation for studying reflections #2: Connection of two strings.

4.1 Termination of a string

Consider a string of density ρ and tension T, which for simplicity is terminated in $z = 0$. This means that the string itself spans at $z \in (-\infty, 0\rangle$. See Figure 4.3.

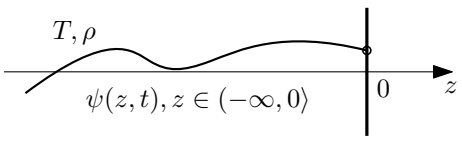


Figure 4.3: Termination of a string.

The motion of the string is described by the transverse displacement function $\psi(z,t)$, which is governed by the wave equation for $z \in (-\infty, 0)$,

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial z^2}, \qquad v = \sqrt{\frac{T}{\rho}}. \tag{4.1}$$

Consider the d'Alembert solution of the wave equation,

$$\psi(z,t) = \underbrace{F(z-vt)}_{\text{incident wave}} + \underbrace{G(z+vt)}_{\text{reflected wave}}, \tag{4.2}$$

where the wave $\psi_i(z,t) = F(z-vt)$ is the prescribed incident wave and the wave $\psi_r(z,t) = G(z+vt)$ is the reflected wave of interest. Next, we need to prescribe a boundary condition at the interface $z = 0$ to determine how the wave will be reflected at this interface. Meanwhile, for generality, consider that the string attachment (termination) has mass M and a velocity-dependent frictional force is applied at the attachment point,

$$F_{\text{friction}} = -\alpha \frac{\partial \psi}{\partial t}(0,t), \tag{4.3}$$

where α represents the coefficient of frictional force. The string itself also exerts a force on the attachment – the transverse projection of the tension force $F_x = -T\frac{\partial \psi}{\partial z}$, see Figure 4.4.

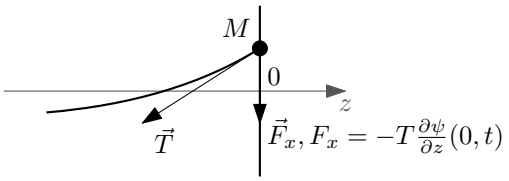


Figure 4.4: The transverse force from the string acting on the termination.

Let us now write down the general equation of motion for the termination. From Newton's second law we have:

$$M \frac{\partial^2 \psi}{\partial t^2}(0,t) = F_x + F_{\text{tření}} = -T \frac{\partial \psi}{\partial z}(0,t) - \alpha \frac{\partial \psi}{\partial t}(0,t). \tag{4.4}$$

Considering now for simplicity a massless termination, $M = 0$, we obtain at $z = 0$ a boundary condition of the form:

$$\boxed{T \frac{\partial \psi}{\partial z}(0,t) + \alpha \frac{\partial \psi}{\partial t}(0,t) = 0, \quad \forall t \in \mathbb{R}.} \tag{4.5}$$

We further substitute d'Alembert's solution (4.2) into this condition. By computing the corresponding derivatives,

$$\frac{\partial \psi}{\partial z}(0,t) = F'(-vt) + G'(vt), \qquad \frac{\partial \psi}{\partial t}(0,t) = -vF'(-vt) + vG'(vt), \tag{4.6}$$

where we've denoted by $F'(x) = \frac{dF}{dx}$, we obtain

$$T\left(F'(-vt) + G'(vt)\right) + \alpha\left(-vF'(-vt) + vG'(vt)\right) = 0. \tag{4.7}$$

This is the equation for the unknown shape of the reflected wave $G(x)$ given the shape of the incident wave $F(x)$. So we express the function G' from the previous equation:

$$G'(vt) = \frac{\alpha v - T}{\alpha v + T} F'(-vt). \tag{4.8}$$

We denote $x = vt$ and introduce the quantity *impedance* $Z = \sqrt{T\rho}$ (then $\frac{T}{v} = Z$). We can rewrite (4.8) as

$$G'(x) = \frac{\alpha - Z}{\alpha + Z} F'(-x). \tag{4.9}$$

After integrating with respect to the variable x, we find the shape of the reflected wave $G(x)$:

$$\boxed{G(x) = \frac{Z - \alpha}{Z + \alpha} F(-x),} \tag{4.10}$$

where we put the integration constant $c = 0$, because it only shifts the whole solution $\psi(z, t)$ along the vertical axis.

We have shown that the reflected wave is just a mirror flip of the incident wave, $G(x) \propto F(-x)$, and the amplitude is changed by the constant amplitude coefficient

$$R = \frac{Z - \alpha}{Z + \alpha}, \tag{4.11}$$

which is called *reflection coefficient*. The resulting solution is of the form

$$\psi(z, t) = F(z - vt) + RF(-(z + vt)). \tag{4.12}$$

If we require that there be no reflections, $R = 0$, then $\alpha = Z$ must hold. In this case, the frictional force takes such a form that it perfectly simulates the continuation of the string and no wave is reflected. This case is called *correct termination*. The case $\alpha \neq Z$ is called *non-correct termination*. If we consider zero friction, $\alpha = 0$, we get the free end condition and $R = 1$. For large friction, $\alpha \to +\infty$, we get the fixed end condition and $R = -1$.

4.2 Connection of two strings

Let's move on to the second model situation – connection of two different strings. The situation is shown in detail in Figure 4.5. The first string spans the coordinates $z \in (-\infty, 0\rangle$ and the second string spans the coordinates $z \in \langle 0, +\infty)$. The string parameters are denoted by ρ_i, T_i (and the complementary quantities v_i and Z_i), $i \in \{1, 2\}$.

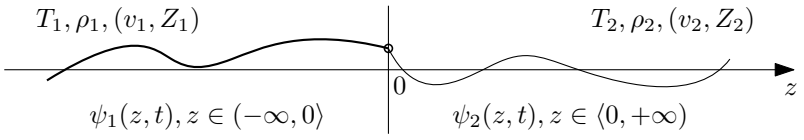


Figure 4.5: Connection of two strings.

Each of the strings is governed by an appropriate wave equation (with an appropriate value of phase velocity v_i):

$$\frac{\partial^2 \psi_i}{\partial t^2} = v_i^2 \frac{\partial^2 \psi_i}{\partial z^2}, \qquad i \in \{1, 2\}. \tag{4.13}$$

Let us write the corresponding d'Alembert solutions on the individual string sections and interpret each term in terms of studying the reflections of the waves coming from the first string (i.e., from $z = -\infty$):

$$\begin{aligned} \psi_1(z,t) &= \underbrace{F_1(z - v_1 t)}_{\text{incident wave}} + \underbrace{G_1(z + v_1 t)}_{\text{reflected wave}}, \\ \psi_2(z,t) &= \underbrace{F_2(z - v_2 t)}_{\text{transmitted wave}} + \underbrace{G_2(z + v_2 t)}_{=0}. \end{aligned} \tag{4.14}$$

The wave $G_2(z + v_2 t)$, i.e., the wave propagating from $z = +\infty$, has no interpretation in our model. So we impose the radiation boundary condition, i.e., we put $G_2 = 0$. We forbid the propagation of the wave from $+\infty$, which would interfere with our study of the reflection of waves coming from the first string. Our task, then, will be to find the shapes of the reflected wave $G_1(x)$ and the transmitted wave $F_2(x)$ given the prescribed incident waveform $F_1(x)$.

We now need to write the appropriate junction conditions at the location $z = 0$. The first is the continuity junction condition,

$$\boxed{\psi_1(0,t) = \psi_2(0,t), \quad \forall t \in \mathbb{R}.} \tag{4.15}$$

The second condition is the Newton's equation of motion at the point of connection. The situation is illustrated in Figure 4.6. Let us consider for the time being the general connection of the mass M. This connection is then acted upon by the transverse projections of the forces from the individual strings.

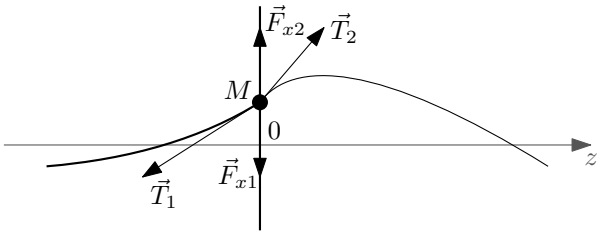


Figure 4.6: The transverse forces acting at the point of connection.

The equation of motion then takes the form

$$M \frac{\partial^2 \psi_{12}}{\partial t^2} = F_{x1} + F_{x2}, \tag{4.16}$$

where on the left side of the equation of motion we can choose whether to use the acceleration of the deflection ψ_1 or ψ_2 (indicated by the symbol ψ_{12}) due to the continuity condition. The expression for the transverse forces is as follows:

$$F_{x1} = -T_1 \frac{\partial \psi_1}{\partial z}(0,t), \qquad F_{x2} = T_2 \frac{\partial \psi_2}{\partial z}(0,t). \tag{4.17}$$

If we now take the mass of the connection to be zero, $M = 0$, (non-zero masses will be discussed later), we get the condition of equality of transverse forces at the connection point:

$$\boxed{T_1 \frac{\partial \psi_1}{\partial z}(0,t) = T_2 \frac{\partial \psi_2}{\partial z}(0,t), \quad \forall t \in \mathbb{R}.} \tag{4.18}$$

Based on these two junction conditions, the continuity condition (4.15) and the condition of equality of transverse forces (4.18), we express the shape of the reflected wave $G_1(x)$ and

the transmitted wave $F_2(x)$ using the shape of the incident wave $F_1(x)$. Let us substitute the d'Alembert solution (4.14) into the junction conditions:

$$F_1(-v_1t) + G_1(v_1t) = F_2(-v_2t), \qquad T_1F_1'(-v_1t) + T_1G_1'(v_1t) = T_2F_2'(-v_2t) \tag{4.19}$$

and perform the substitution $x = -v_1t$:

$$F_1(x) + G_1(-x) = F_2\left(\tfrac{v_2}{v_1}x\right), \qquad F_1'(x) + G_1'(-x) = \frac{T_2}{T_1}F_2'\left(\tfrac{v_2}{v_1}x\right) \tag{4.20}$$

Integrating the second of the equations with respect to x, we obtain

$$F_1(x) - G_1(-x) = \frac{T_2}{T_1}\frac{v_1}{v_2}F_2\left(\tfrac{v_2}{v_1}x\right), \tag{4.21}$$

where we have chosen the integration constant to be zero, because it ends up only shifting the entire resulting function $\psi(z,t)$ by a constant (try leaving it there and you'll see!). The constant term on the right-hand side can be rewritten with the impedances Z_1 and Z_2, $Z = \sqrt{T\rho}$:

$$\frac{T_2}{T_1}\frac{v_1}{v_2} = \frac{Z_2}{Z_1}. \tag{4.22}$$

The result is the following system of equations for the functions $G_1(x)$ and $F_2(x)$:

$$F_1(x) + G_1(-x) = F_2\left(\tfrac{v_2}{v_1}x\right), \qquad F_1(x) - G_1(-x) = \frac{Z_2}{Z_1}F_2\left(\tfrac{v_2}{v_1}x\right). \tag{4.23}$$

By summing the equations, we express the traveling wave

$$\boxed{F_2(x) = PF_1\left(\tfrac{v_1}{v_2}x\right), \qquad P = \frac{2Z_1}{Z_1+Z_2},} \tag{4.24}$$

where the constant amplitude coefficient P is called *transmission coefficient*. It is also denoted T, but it does not suit the notation here because of the tension in the strings T_i. We express the reflected wave from the first equation in (4.23) (arising from the continuity condition) after substituting from (4.24):

$$\boxed{G_1(x) = (P-1)F_1(-x) = RF_1(-x), \qquad R = \frac{Z_1-Z_2}{Z_1+Z_2},} \tag{4.25}$$

The coefficient R is again (as for the termination of a single string) called *reflection coefficient*. We see that the continuity condition implies a simple relationship between the coefficients of transmission and reflection:

$$\boxed{1 + R = P.} \tag{4.26}$$

For completeness, let us also write the waveforms $\psi_i(z,t)$ propagating along each string:

$$\psi_1(z,t) = F_1(z - v_1t) + RF_1(-(z + v_1t)), \qquad \psi_2(z,t) = PF_1\left(\tfrac{v_1}{v_2}(z - v_2t)\right). \tag{4.27}$$

It turns out that the reflection at the interface of two strings is in a sense a very simple phenomenon. The reflected wave is mirror-reversed and advances back along the first string. The transmitted wave is only deformed by the fraction of phase velocities $\frac{v_2}{v_1}$ – for $v_2 > v_1$ the wave is stretched on the second string, while for $v_2 < v_1$ it is shrunk. All the information about the passage and reflection of waves is encoded in the constant amplitude coefficients R and P

– these are determined only by the string parameters and not by the waveform of the incident wave.

The wave is not reflected if the impedances on the individual strings are matched:

$$R = 0 \quad \Leftrightarrow \quad Z_1 = Z_2 \quad \Leftrightarrow \quad \frac{T_1}{T_2} = \frac{\rho_2}{\rho_1}. \tag{4.28}$$

The reflection coefficient can be written in various forms

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{\frac{Z_1}{Z_2} - 1}{\frac{Z_1}{Z_2} + 1} = \frac{1 - \frac{Z_2}{Z_1}}{1 + \frac{Z_2}{Z_1}}; \tag{4.29}$$

from these forms, it follows that the coefficients depend on the ratio of impedances $\frac{Z_1}{Z_2}$ and the coefficients of passage and reflection have the following ranges

$$R \in \langle -1, 1 \rangle, \qquad P \in \langle 0, 2 \rangle. \tag{4.30}$$

With Z_1 fixed, we get a coefficient of $R = -1$ ($P = 0$) for $Z_2 = +\infty$ (a fixed end, an infinitely heavy or tensioned string), and $R = 1$ ($P = 2$) for $Z_2 = 0$ (a free end, a massless or untensioned string).

4.2.1 Harmonic incident wave

Consider an incident harmonic wave (here its complexification):

$$F_1(x) = Ae^{ik_1 x}, \qquad \psi_i(z,t) = F_1(z - v_1 t) = Ae^{-i(\omega t - k_1 z)}, \tag{4.31}$$

where we denote the incident part of the wave $\psi_1(z,t)$ as $\psi_i(z,t)$ and use the dispersion relation $\omega = v_1 k_1$. Let us now insert the incident wave $F_1(x)$ to the formulas (4.24) and (4.25) (and (4.27)). We denote the reflected and transmitted wave by $\psi_r(z,t)$ and $\psi_t(z,t)$:

$$\begin{aligned} G_1(x) &= RF_1(-x), \qquad \psi_r(z,t) = G_1(z + v_1 t) = RF_1(-z - v_1 t) = AR\, e^{-i(\omega t + k_1 z)}, \\ F_2(x) &= PF_1\left(\tfrac{v_1}{v_2} x\right), \quad \psi_t(z,t) = F_2(z - v_2 t) = PF_1\left(\tfrac{v_1}{v_2} z - v_1 t\right) = AP\, e^{-i(\omega t - k_2 z)}, \end{aligned} \tag{4.32}$$

where we reuse the dispersion relation $\omega = v_1 k_1$ and denote $k_2 = k_1 \frac{v_1}{v_2}$. Let us take a closer look at the wavenumber of the transmitted wave:

$$\frac{2\pi}{\lambda_2} = k_2 = \frac{v_1}{v_2} k_1 = \frac{v_1}{v_2} \frac{2\pi}{\lambda_1} \quad \longrightarrow \quad \lambda_2 = \frac{v_2}{v_1} \lambda_1. \tag{4.33}$$

The wavelength of the transmitted wave is changed by the ratio of the phase velocities on the first and second strings. This corresponds to the fact that the interface can be thought of as a source of waves excited by the incident wave from the first string. This imaginary source then radiates a transmitted wave propagating along the second string.

4.3 Energy relations

Let's look at the relations between the energy fluxes of incident, reflected and transmitted waves. We define the coefficients of *reflectivity* (*reflectance*) and *transmissivity* (*transmitance*) as the ratio of the respective energy fluxes:

$$\mathcal{R} = \frac{|\langle S_{\mathrm{ref}} \rangle|}{\langle S_{\mathrm{inc}} \rangle}, \qquad \mathcal{T} = \frac{\langle S_{\mathrm{tr}} \rangle}{\langle S_{\mathrm{inc}} \rangle}, \tag{4.34}$$

(for reflectivity, we put an absolute value around the reflected wave energy since the reflected wave energy flux is negative). For a harmonic progressive wave, we derived the following relation for the energy flux in the chapter on energy quantities on the string:

$$|\langle S\rangle| = \frac{1}{2}ZA^2\omega^2. \tag{4.35}$$

The energy fluxes for each wave are then

$$\langle S_{\mathrm{inc}}\rangle = \frac{1}{2}Z_1A^2\omega^2, \qquad |\langle S_{\mathrm{ref}}\rangle| = \frac{1}{2}Z_1A^2R^2\omega^2, \qquad \langle S_{\mathrm{tr}}\rangle = \frac{1}{2}Z_2A^2P^2\omega^2. \tag{4.36}$$

Substituting these relations into the definitions of reflectivity and transmissivity (4.34), where we substitute the appropriate values for the amplitudes and impedances:

$$\boxed{\mathcal{R} = R^2, \qquad \mathcal{T} = \frac{Z_2}{Z_1}P^2.} \tag{4.37}$$

We see that the amount of reflected energy (intensity) is simply given by the square of the amplitude coefficient R. On the other hand, the amount of energy passed cannot be inferred just from the change in amplitudes alone, but it must be taken into account that different environments are involved and that therefore the same amplitudes can carry different amounts of energy. There is therefore the additional factor of the ratio of impedances $\frac{Z_2}{Z_1}$.

Let us use the previous paragraph to explain "paradox" for the case of $R \to 1$ and $P \to 2$. This is the case where $Z_2 \to 0$. Almost the entire wave is reflected ($R \to 1$), but at the same time a wave with almost twice the amplitude ($P \to 2$) passes through! But at the same time we see that the transmissivity coefficient goes to zero due to the low impedance Z_2, $\mathcal{T} \to 0$, i.e. the energy carried by the transmitted wave is also limitingly close to zero. So there is no paradox. In different environments, it is not enough to compare amplitudes to infer the energies carried by individual waves. In the free-end limit we have $\rho = 0$ or $T = 0$. A massless or unstretched string carries no energy.

From the law of conservation of energy, the sum of the energy fluxes of the transmitted and reflected waves is equal to the energy flux of the incident wave (in absolute terms), thus:

$$\boxed{\mathcal{R} + \mathcal{T} = 1, \quad \text{i.e.} \quad R^2 + \frac{Z_2}{Z_1}P^2 = 1.} \tag{4.38}$$

4.4 Frequency-dependent coefficients of transmission and reflection

In the previous chapters we considered that the connection of two strings is massless. We now consider the opposite case. We will see that we will have to fundamentally change the procedure used so far to find reflected and transmitted waves. Let us consider the situation shown in Figure 4.7. For simplicity of calculation, this time we consider that the strings are stretched to the same tension T, have generally different densities ρ_1 and ρ_2, and the connection is realized by a point mass M.

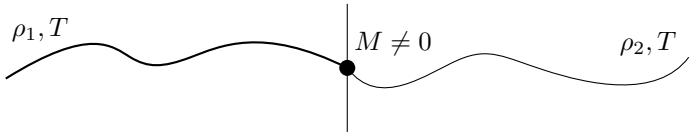


Figure 4.7: Connection with mass.

Let us repeat here the junction conditions – the continuity condition and the equation of motion of the point of connection:

$$\psi_1(0,t) = \psi_2(0,t), \qquad M\frac{\partial^2 \psi_{12}(0,t)}{\partial t^2} = T\left(\frac{\partial \psi_2}{\partial z}(0,t) - \frac{\partial \psi_1}{\partial z}(0,t)\right), \quad \forall t \in \mathbb{R}, \tag{4.39}$$

where the symbol ψ_{12} means that, due to the continuity condition, we can choose whether to consider the displacement ψ_1 or ψ_2. Let us focus on the second condition. We can see that the first and second derivatives are mixed here and thus the simple procedure used in the previous models, where we integrated the equation, will not be possible[1]. Let us try to take a step aside. Consider that each incident waveform $F_1(x)$ can be decomposed into a sum of harmonic waves using the Fourier transform:

$$F_1(x) = \int_{-\infty}^{+\infty} C(k)\, e^{ikx}\, dk. \tag{4.40}$$

So we will consider the incident wave in the form of a harmonic travelling wave with unit amplitude, see what is reflected and what passes through, and then write back the resulting solution as a (continuous) superposition of these elementary reflections. Moreover, let's do the following *ansatz* – assume that both the reflected and transmitted waves are of harmonic waveforms:

$$F_1(x) = e^{ik_1 x}, \qquad G_1(x) = R\, e^{-ik_1 x}, \qquad F_2(x) = P\, e^{ik_2 x}, \qquad k_2 = \frac{v_1}{v_2}k_1, \tag{4.41}$$

where the wave number k_2 is changed by the ratio of the phase velocities compared to the wave number of the incident wave k_1. If we can find a solution to the reflection problem in this form, we know that we have found the correct (and only) solution from the uniqueness of the solution. Let us write down the d'Alembert solutions in this ansatz,

$$\psi_1(z,t) = F_1(z - v_1 t) + G_1(z + v_1 t), \qquad \psi_2(z,t) = F_2(z - v_2 t), \tag{4.42}$$

where the individual waves have the form:

$$\begin{aligned} F_1(z - v_1 t) &= e^{-i(\omega t - k_1 z)}, \\ G_1(z + v_1 t) &= R\, e^{-i(\omega t + k_1 z)}, \\ F_2(z - v_2 t) &= P\, e^{-i(\omega t - k_2 z)}. \end{aligned} \tag{4.43}$$

We plug these into the junction conditions (4.39). From the continuity condition we have

$$e^{-i\omega t} + R\, e^{-i\omega t} = P\, e^{-i\omega t} \quad \longrightarrow \quad 1 + R = P, \tag{4.44}$$

and from the equation of motion of the connection (where we choose the function ψ_2 on the left-hand side, since it contains only one wave F_2 and thus we get an equation of simpler form):

$$-PM\omega^2 e^{-i\omega t} = T(ik_1)\left[R - 1 + \frac{v_1}{v_2}P\right] e^{-i\omega t}. \tag{4.45}$$

[1] After substituting the d'Alembert solutions (4.14) into the connection conditions (4.39), we arrive at the following inhomogeneous second order linear differential equation with constant coefficients:

$$F_2''(x) - \frac{T(1+\frac{v_2}{v_1})}{Mv_2^2}F_2'(x) = -\frac{2T}{Mv_2^2}F_1'\left(\tfrac{v_2}{v_1}x\right).$$

This equation can be solved. The homogeneous solution is found by the standard characteristic polynomial method, the inhomogeneous solution is found by the method of variations of constants. However, the resulting very complicated solution does not provide much insight into the reflection phenomenon in such a setting.

From this equation, we express the coefficient P (after substituting after $R = P - 1$) with the result:

$$\boxed{P(\omega) = \frac{2}{1 + \frac{v_1}{v_2} - i\frac{M\omega v_1}{T}} \in \mathbb{C}, \qquad R(\omega) = P(\omega) - 1,} \tag{4.46}$$

where we additionally got rid of the wave number k_1 by substituting from the dispersion relation $\omega = v_1 k_1$. We see two surprising facts. The coefficients came out dependent on the angular frequency of the incident wave ω and we also found the coefficients to be complex!

The complex nature of the coefficients simply means that the coefficients encode not only the change in amplitude of the transmitted and reflected waves, but also the phase shift. We can, for example, use the polar form of the complex number for the coefficient P,

$$P = |P|e^{i\varphi}, \tag{4.47}$$

which, when substituted into the transmitted wave $F_2(x)$, gives:

$$F_2(x) = Pe^{ik_2 x} = |P|e^{i(k_2 x + \varphi)}. \tag{4.48}$$

The magnitude of the coefficient $|P|$ thus has the original meaning of the amplitude change of the transmitted wave. The angle φ in the complex exponential $e^{i\varphi}$ gives the phase shift relative to the incident wave.

The dependence of the coefficients on the angular frequency of the incident wave ω means that each harmonic component generally behaves differently being reflected. This has an important implication if we return to the original problem, where we chose the general function $F_1(x)$ as the waveform of the incident wave. Consider as an example the shape of the reflected wave $G_1(x)$. In the Fourier integral (4.40), we replace each of the harmonic components e^{ikx} by $R(k)e^{-ikx}$ ($R(k)$ is the function formed by substituting $\omega = v_1 k$ for $R(\omega)$):

$$G_1(x) = \int_{-\infty}^{+\infty} C(k)R(k)e^{-ikx}dk \neq RF_1(-x). \tag{4.49}$$

Since the coefficient R is dependent on ω (on k), it cannot be extracted from the integral, and thus the resulting wave cannot be written as the total reflection coefficient R times the reflected incident wave $F_1(-x)$. For frequency-dependent coefficients, the incident wave deforms nontrivially upon reflection (by nontriviality we mean deformation beyond the mirroring for the reflected wave and expansion by the ratio $\frac{v_1}{v_2}$ for the transmitted wave).

4.5 Transmission matrix

In this chapter we want to encode the transmission and reflection coefficients into a suitably chosen matrix so that we can then start considering reflections at more than one interface in a very simple way.

So far, in the reflection problem, we have always considered the radiation condition, neglecting the wave arriving from the "right", since this had no interpretation in the study of reflections. Now we consider a more general situation where we have a given interface of two media and we let the waves on either side of the interface propagate in both directions, see Figure 4.8.

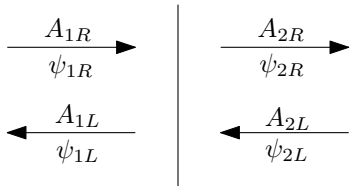


Figure 4.8: Interface between two environments. In both environments, travelling waves propagate in both directions, denoted ψ_{1R} (wave in the first environment advancing to the right), ψ_{1L} (wave in the first environment advancing to the left), ψ_{2R} and ψ_{2R} (analogous in the second environment). Their amplitudes are denoted A_{1R}, A_{1L}, A_{2R} and A_{2L}.

If, as in the previous section, we consider complexified harmonic travelling waves, these expressions have the following forms:

$$\begin{aligned} \psi_{1R} &= A_{1R}\, e^{i(\omega t - k_1 z)}, & \psi_{2R} &= A_{2R}\, e^{i(\omega t - k_2 z)}, \\ \psi_{1L} &= A_{1L}\, e^{i(\omega t + k_1 z)}, & \psi_{2L} &= A_{2L}\, e^{i(\omega t + k_2 z)}, \end{aligned} \tag{4.50}$$

where k_1 and k_2 are the wave number in the first and second environments, respectively.

Transmission matrix $\mathbb{D} \in \mathbb{C}^{2,2}$ converts the amplitudes of the waves (A_{2R}, A_{2L}) in the second medium to the amplitudes of the waves (A_{1R}, A_{1L}) in the first medium by the relation

$$\begin{pmatrix} A_{1R} \\ A_{1L} \end{pmatrix} = \mathbb{D} \begin{pmatrix} A_{2R} \\ A_{2L} \end{pmatrix}. \tag{4.51}$$

The transmission matrix $\mathbb{D}$ is practically found by solving the junction conditions between the two environments. At the end of this section we will show this by an example.

First, as a simple introductory example, let us return to the reflection problem for the connection of two strings of impedances Z_1 and Z_2 at location $z = 0$. Analogous schemes to the general Figure 4.8 are shown in Figure 4.9. Thus, we consider an incident wave of unit amplitude (either from the left or from the right), and the amplitude of the transmitted and reflected waves directly gives the coefficients of transmission and reflection.

(a) Incident wave from the left.

(b) Incident wave from the right.

Figure 4.9: Connecting two strings of impedances Z_1 and Z_2. We study reflections for a wave incident from the left and from the right. We denote the coefficients of transmission and reflection for a wave incident from the right by primes.

According to the definition of the transmission matrix (4.51), we can write the relations between the amplitudes of the individual waves as:

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D} \begin{pmatrix} P \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ P' \end{pmatrix} = \mathbb{D} \begin{pmatrix} R' \\ 1 \end{pmatrix}. \tag{4.52}$$

By solving the equations (4.52) for the coefficients of the matrix $\mathbb{D}$ after substituting the already known forms of the transmission and reflection coefficients [2] to get the resulting form of the transmission matrix

$$\mathbb{D} = \frac{1}{2Z_1}\begin{pmatrix} Z_1 + Z_2 & Z_1 - Z_2 \\ Z_1 - Z_2 & Z_1 + Z_2 \end{pmatrix}. \tag{4.53}$$

For the situation where we have only one interface between two environments, the transmission matrix is not very useful. But let's look at the situation where we take three adjacent environments – for example, three strings with impedances Z_1, Z_2, and Z_3. If we proceed by always splitting the "actual" incident wave into reflected and transmitted parts at each interface, we get the situation described in Figure 4.10.

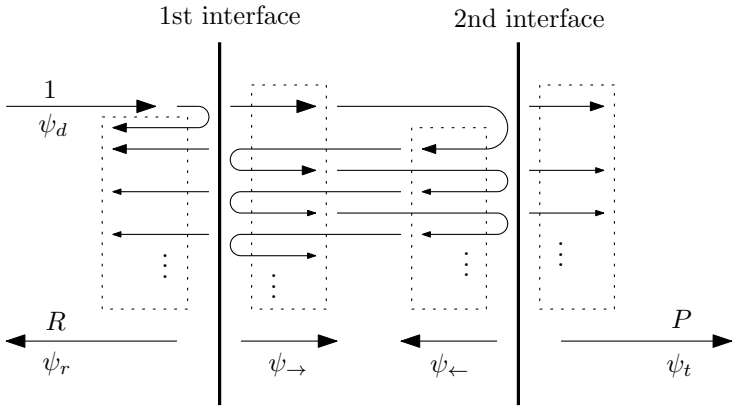


Figure 4.10: Two interfaces between three environments. A wave ψ_i of unit amplitude is incident from the left. There are infinitely many reflections (and transmissions) between the two interfaces, and the resulting reflected wave ψ_r, or transmitted wave ψ_t, is obtained as a superposition of all reflected, or transmitted, contributions. We denote the superposition of the waves all propagating in one direction and the other in the space between the interfaces by $\psi_\rightarrow$ and $\psi_\leftarrow$. After summing all the transmitted and reflected contributions (which again form a travelling wave), we can determine the "total" coefficient of transmission P and reflection R.

This procedure, where we sum infinitely many contributions, is tedious but possible. Let us look at the situation from the perspective of the transmission matrix. The transmission matrix converts the amplitudes of the resulting travelling waves from one medium to another. The situation is shown schematically in Figure 4.11.

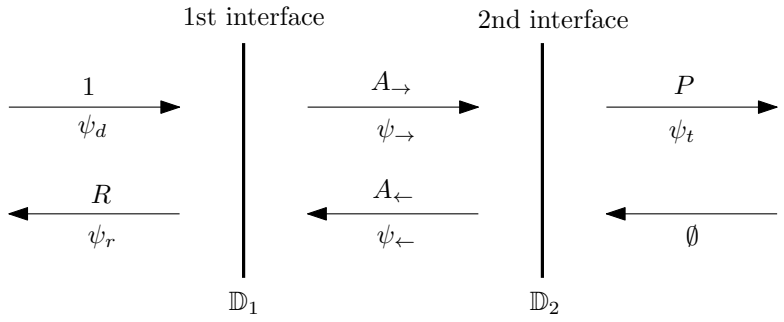


Figure 4.11: Two interfaces and the resulting propagating waves and their amplitudes. The amplitudes of the travelling waves between the two interfaces are denoted by $A_\rightarrow$ and $A_\leftarrow$. The transmission matrices at each interface are denoted by $\mathbb{D}_1$ and $\mathbb{D}_2$.

[2] Reflection coefficient $R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$ and transmission coefficient $P = 1 + R$. For the coefficients "from the other side", we just swap the indices for the impedances, $R' = \frac{Z_2 - Z_1}{Z_1 + Z_2}$, $P' = 1 + R'$.

The transmission matrices of the individual interfaces, $\mathbb{D}_1$ and $\mathbb{D}_2$, relate the respective amplitudes according to the definition (4.51) as follows:

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D}_1 \begin{pmatrix} A_{\rightarrow} \\ A_{\leftarrow} \end{pmatrix}, \qquad \begin{pmatrix} A_{\rightarrow} \\ A_{\leftarrow} \end{pmatrix} = \mathbb{D}_2 \begin{pmatrix} P \\ 0 \end{pmatrix}. \tag{4.54}$$

In the case that we combine these equations by excluding the amplitudes $A_{\rightarrow}$ and $A_{\leftarrow}$, we obtain

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D}_1 \mathbb{D}_2 \begin{pmatrix} P \\ 0 \end{pmatrix}. \tag{4.55}$$

In the language of transmission matrices, compounding of interfaces reduces to a mere multiplication of the individual matrices! From equations (4.55) we can already easily express the resulting coefficients R and P (the components of matrices $\mathbb{D}_1$ and $\mathbb{D}_2$ are known – they specify the interfaces in question).

Now let us look at the aforementioned example of finding the transfer matrix $\mathbb{D}$ by solving the junction conditions. We have already seen the derivation of the transition matrix from the already known transmission and reflection coefficients for the connection of two strings at the point $z = 0$. Let us show a more general derivation of the form of the matrix $\mathbb{D}$ for the interface of two strings at $z = L$. We will also construct the junction conditions by considering the general situation as in Figure 4.8. Thus, we take the waves on each string to be of the form (4.50). We have to adjust the junction conditions for the string for the interface position to $z = L$. The first is the continuity condition,

$$\psi_{1R}(L,t) + \psi_{1L}(L,t) = \psi_{2R}(L,t) + \psi_{2L}(L,t), \tag{4.56}$$

and the second is the condition of equality of transverse forces (we consider a massless connection),

$$T_1 \left(\frac{\partial \psi_{1R}}{\partial z}(L,t) + \frac{\partial \psi_{1L}}{\partial z}(L,t) \right) = T_2 \left(\frac{\partial \psi_{2R}}{\partial z}(L,t) + \frac{\partial \psi_{2L}}{\partial z}(L,t) \right). \tag{4.57}$$

For simplicity, consider that the stresses on both strings are equal, $T_1 = T_2 = T$. After substituting the harmonic waves (4.50) into the connection conditions (4.56) and (4.57) (and cancelling out the exponentials and $-iT$), we obtain the following set of equations relating the amplitude coefficients:

$$\begin{aligned} A_{1R} e^{-ik_1 L} + A_{1L} e^{ik_1 L} &= A_{2R} e^{-ik_2 L} + A_{2L} e^{ik_2 L}, \\ k_1 A_{1R} e^{-ik_1 L} - k_1 A_{1L} e^{ik_1 L} &= k_2 A_{2R} e^{-ik_2 L} - k_2 A_{2L} e^{ik_2 L}. \end{aligned} \tag{4.58}$$

Let us rewrite the left and right sides of the equations using matrix notation:

$$\underbrace{\begin{pmatrix} e^{-ik_1 L} & e^{ik_1 L} \\ k_1 e^{-ik_1 L} & -k_1 e^{ik_1 L} \end{pmatrix}}_{\mathbb{D}_L} \begin{pmatrix} A_{1R} \\ A_{1L} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-ik_2 L} & e^{ik_2 L} \\ k_2 e^{-ik_2 L} & -k_2 e^{ik_2 L} \end{pmatrix}}_{\mathbb{D}_R} \begin{pmatrix} A_{2R} \\ A_{2L} \end{pmatrix}, \tag{4.59}$$

where we have denoted the matrix on the left and right sides of the equation by $\mathbb{D}_L$ and $\mathbb{D}_R$, respectively. Comparing the form of the equations (4.51) defining the matrix $\mathbb{D}$ and the obtained equation (4.59) from the junction conditions, it is clear that the transmission matrix is of the form

$$\mathbb{D} = \mathbb{D}_L^{-1} \mathbb{D}_R. \tag{4.60}$$

Performing the inversion of the matrix[3] and multiplying them together we get the result

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} \left(1+\frac{k_2}{k_1}\right)e^{i(k_1-k_2)L} & \left(1-\frac{k_2}{k_1}\right)e^{i(k_1+k_2)L} \\ \left(1-\frac{k_2}{k_1}\right)e^{-i(k_1+k_2)L} & \left(1+\frac{k_2}{k_1}\right)e^{-i(k_1-k_2)L} \end{pmatrix}. \tag{4.61}$$

Substituting $L = 0$ gives the matrix

$$\mathbb{D}(L=0) = \frac{1}{2}\begin{pmatrix} 1+\frac{k_2}{k_1} & 1-\frac{k_2}{k_1} \\ 1-\frac{k_2}{k_1} & 1+\frac{k_2}{k_1} \end{pmatrix}, \tag{4.62}$$

which is of the same form as the matrix already obtained in (4.53), since the following relation holds

$$\frac{k_2}{k_1} = \frac{Z_2}{Z_1}. \tag{4.63}$$

[3]The inversion of the 2x2 matrix is

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \mathbb{A}^{-1} = \frac{1}{\det \mathbb{A}}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Chapter 5

Waves in space

5.1 Plane waves

Let us briefly review what we know about harmonic travelling waves in one dimension. They are a solution to the one-dimensional wave equation,

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial z^2}, \tag{5.1}$$

and can propagate in either the positive or negative direction of the axis z:

$$\psi(z,t) = e^{i(\omega t - kz)}, \qquad \psi(z,t) = e^{i(\omega t + kz)}. \tag{5.2}$$

If we introduce a (one-dimensional) propagation direction vector $\vec{n} = (\pm 1)$, a position vector $\vec{r} = (z)$, and a so-called *wave vector* $\vec{k} = k\,\vec{n}$, we can write the expression $\mp kz$ in a travelling wave as $-\vec{k} \cdot \vec{r}$. Thus, the travelling wave takes the form

$$\psi(z,t) = \psi(\vec{r},t) = e^{i(\omega t - \vec{k}\cdot\vec{r})}, \tag{5.3}$$

where by choosing the vector $\vec{n}$ (or $\vec{k} = k\vec{n}$) we choose the direction of the propagation. This expression is well defined for any dimension. We will now take advantage of this and examine it in 2D (and subsequently in 3D).

Let us move on to the two-dimensional situation. As a model for a two-dimensional wave, we can choose an elastic membrane that extends in the plane (y,z) and consider the transverse displacement of this membrane (in the direction of the axis x) described by the function $\psi(y,z,t)$. See Figure 5.1.

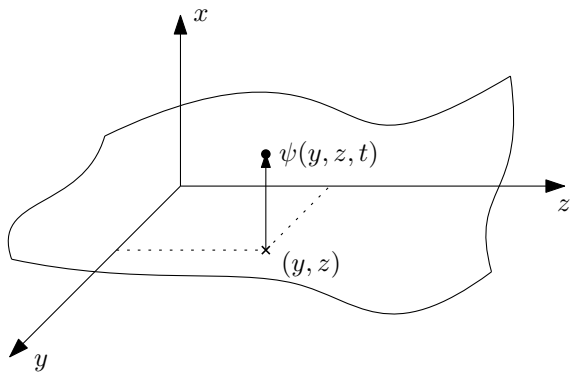


Figure 5.1: Model of the two-dimensional environment for transverse deflections of an elastic membrane.

Consider now a two-dimensional position vector $\vec{r} = (y, z)$, a unit vector $\vec{n} = (n_y, n_z)$, $|\vec{n}| = 1$, a wave vector $\vec{k} = k\,\vec{n}$ and take a wave again of the form (5.3), i.e.

$$\psi(\vec{r}, t) = e^{i(\omega t - \vec{k}\cdot\vec{r})}. \tag{5.4}$$

Let's look at the shape of the *wavefront*, i.e., a set of points with a constant value of phase. The phase function is $\varphi(\vec{r}, t) = \omega t - \vec{k}\cdot\vec{r}$. Let's put

$$\varphi(\vec{r}, t) = \omega t - \vec{k}\cdot\vec{r} = \varphi_0, \qquad \omega t - \varphi_0 = k_y y + k_z z = k(n_y y + n_z z). \tag{5.5}$$

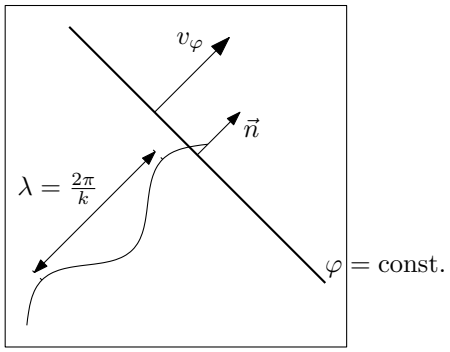
After rewriting, we get an algebraic equation of a line in the plane (y, z):

$$\frac{1}{k}(\omega t - \varphi_0) = n_y y + n_z z. \tag{5.6}$$

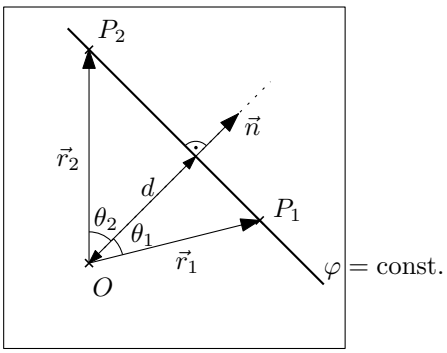
The constant phase curves are therefore lines with the vector $\vec{n}$ as their normal vector, see Figure 5.2 on the left. At a given time, the expression $\vec{n}\cdot\vec{r}$ is a constant. If we introduce notation as in Figure 5.2 on the right, we can write

$$\vec{n}\cdot\vec{r}_1 = \vec{n}\cdot\vec{r}_2 = d = r_i\cos\theta_i, \tag{5.7}$$

that is, for all points on the line, the scalar product $\vec{n}\cdot\vec{r}$ is constant and has the meaning of the perpendicular distance d of the line from the origin. However, the distance d changes with time; it is the left-hand side of the equation (5.6). The rate of advance of the constant phase location is $v_\varphi = \frac{\omega}{k}$. Thus, the straight line of the constant phase moves in the direction of the vector $\vec{n}$ – the vector of the wave's direction of propagation – at the phase velocity $v_\varphi = \frac{\omega}{k}$. The wave vector $\vec{k} = k\,\vec{n}$ thus encodes both the wavelength through its magnitude, $|\vec{k}| = k = \frac{2\pi}{\lambda}$, and the direction of propagation, $\vec{n}$.



(a) Schematic representation of the harmonic "plane" wave in 2D. A single line of constant phase is shown in bold. The harmonic wave propagates in the $\vec{n}$ direction (perpendicular to the line waveform) at v_φ.



(b) On a line of constant phase, $\vec{n}\cdot\vec{r} = r\cos\theta = d = v_\varphi t - \frac{\varphi_0}{k}$ holds

Figure 5.2: Harmonic travelling wave in 2D of the form given by (5.3).

This harmonic travelling "plane" wave[1] is a solution to the 2D wave equation

$$\frac{\partial^2\psi}{\partial t^2} = v^2\left(\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = v^2\Delta_{2\mathrm{D}}\psi, \tag{5.8}$$

[1]Perhaps we should call it a line wave in 2D?

where we have introduced the notation of the 2D Laplace operator Δ_{2D} (here in variables y and z), if the dispersion relation $\omega = v|\vec{k}|$ is satisfied. This wave equation is a direct generalization of the one-dimensional wave equation. This equation could be obtained, for example, by the continuous limit of a 2D lattice of weights on springs (a generalization of a chain of atoms in two dimensions).

Based on the analysis of the 2D case, we arrive at a straightforward generalization for the 3D case. We introduce three-dimensional vectors: the propagation direction $\vec{n}$, the wave vector $\vec{k} = k\,\vec{n}$ (and the position vector $\vec{r} = (x, y, z)$):

$$\vec{k} = (k_x, k_y, k_z) = k\,\vec{n} = k(n_x, n_y, n_z), \qquad |\vec{n}| = 1. \tag{5.9}$$

The wave $\psi(\vec{r}, t)$ has formally the same shape as in (5.4). The constant-phase surface (its *wavefront*) is this time the plane to which the vector $\vec{n}$ is a normal vector; this propagates through space at phase velocity $v_\varphi = \frac{\omega}{k}$, see schematic figure 5.3. Thus, waves of the form (5.4) are called *harmonic travelling plane waves*.

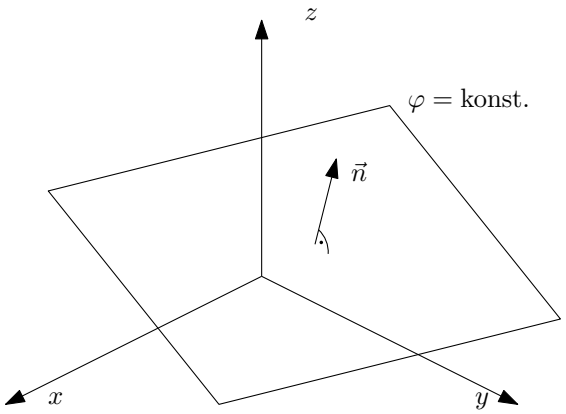


Figure 5.3: A plane wave in 3D space propagating through space in the direction of the vector $\vec{n}$ (or $\vec{k}$) at a velocity of $v_\varphi = \frac{\omega}{k}$.

This wave is a solution of the three-dimensional wave equation

$$\boxed{\frac{\partial^2\psi}{\partial t^2} = v^2\Delta\psi = v^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right),} \tag{5.10}$$

where Δ is the 3D Laplace operator, if the dispersion relation $\omega = v|\vec{k}|$ is satisfied.

Finally, let us write down a generalization of the d'Alembert solution of the one-dimensional wave equation. In one dimension, there were only two possible directions of wave propagation – in the positive and negative directions, i.e.

$$\psi(z, t) = F(z - vt) + G(z + vt). \tag{5.11}$$

In multiple dimensions, the propagation directions are given by the unit vector $\vec{n}$ and hence we assign a travelling plane wave to a given direction $\vec{n}$

$$\psi(\vec{r}, t) = F(\vec{n} \cdot \vec{r} - vt). \tag{5.12}$$

The general solution is then obtained as a superposition of travelling plane waves propagating in all possible directions:

$$\psi(\vec{r}, t) = \int_{(\vec{n})} F_{(\vec{n})}(\vec{n} \cdot \vec{r} - v\,t)\,d^3n. \tag{5.13}$$

5.2 Spherical waves

A harmonic travelling wave with spherical wavefronts is a wave of the following form

$$\psi(r,t) = e^{i(\omega t - kr)}. \tag{5.14}$$

Let us verify that the surfaces of the constant phase (wavefronts) are indeed spheres:

$$\varphi(r,t) = \omega t - k\sqrt{x^2+y^2+z^2} = \varphi_0, \tag{5.15}$$

After rewriting r^2, we obtain the algebraic equation of a sphere,

$$\underbrace{\left(\frac{1}{k}(\omega t - \varphi_0)\right)^2}_{R(t)^2=(vt-r_0)^2} = x^2+y^2+z^2, \tag{5.16}$$

where the radius $R(t)$ increases with the phase velocity $v = \frac{\omega}{k}$. The spherical wave is shown schematically in Figure 5.4. Unfortunately, a spherical wave with constant amplitude does not satisfy the wave equation,

$$\frac{\partial^2\psi}{\partial t^2} = v^2\Delta\psi. \tag{5.17}$$

Therefore, let us find a real solution to the wave equation for spherically symmetric waves, i.e., consider the function $\psi(r,t)$ as a function of only the distance from the origin r (and time t). To do this, we will need to express the Laplace operator $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$ using derivatives according to the radial coordinate r, $\frac{\partial}{\partial r}$. At TEF1, you have shown[2] that

$$\Delta\psi(r) = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}. \tag{5.18}$$

[2]First take the first derivative of the function $\psi(r,t)$

$$\frac{\partial\psi}{\partial x_i} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\partial\psi}{\partial r}\frac{x_i}{r},$$

where we used the identity $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$. The second derivative is then

$$\frac{\partial}{\partial x_i}\frac{\partial\psi}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial\psi}{\partial r}\frac{x_i}{r}\right) = \frac{\partial^2\psi}{\partial r^2}\frac{x_i}{r}\frac{x_i}{r} + \frac{\partial\psi}{\partial r}\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right).$$

We use Einstein's summation rule, i.e., we sum over the second derivative of $\frac{\partial^2}{\partial x_i^2}$ (i.e., over the index i). Then $x_i x_i = r^2$ holds. We still need to calculate the term

$$\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{\delta_{ii}r - x_i\frac{\partial r}{\partial x_i}}{r^2} = \frac{\delta_{ii}r - \frac{x_i x_i}{r}}{r^2} = \frac{1}{r}\left(\delta_{ii} - \frac{x_i x_i}{r^2}\right) = \frac{2}{r};$$

in the last equation we summed over i, i.e. $\delta_{ii} = 3$ and again $x_i x_i = r^2$. So in total we have

$$\Delta\psi(r,t) = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}.$$

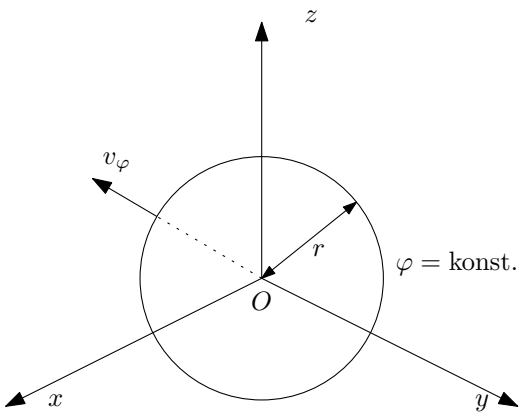


Figure 5.4: The spherical wavefront of a spherical wave in 3D space propagating at a velocity v_φ.

Thus, the wave equation for a wave with functional dependence $\psi(r,t)$ is of the form

$$\frac{\partial^2\psi}{\partial t^2} = v^2\left(\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}\right). \tag{5.19}$$

The right-hand side of the wave equation in coordinate r can be written as (verify!):

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2}. \tag{5.20}$$

Substituting back into the wave equation (5.17) and multiplying by r, we get

$$\boxed{\frac{\partial^2(r\psi)}{\partial t^2} = v^2\frac{\partial^2(r\psi)}{\partial r^2},} \tag{5.21}$$

which is the one-dimensional wave equation in the spatial coordinate r for the function $\Psi(r,t) = r\,\psi(r,t)$! We know the solution of the one-dimensional wave equation, it is the d'Alembert solution:

$$\Psi(r,t) = F(r-vt) + G(r+vt), \tag{5.22}$$

After expressing $\psi = \frac{1}{r}\Psi$, we get the solution of the original wave equation for spherically symmetric waves $\psi(r,t)$:

$$\boxed{\psi(r,t) = \frac{1}{r}F(r-vt) + \frac{1}{r}G(r+vt),} \tag{5.23}$$

where $F, G : \mathbb{R} \to \mathbb{R}$ are arbitrary functions (twice differentiable). The wave $\frac{1}{r}F(r-vt)$ represents a spherical wave propagating from the origin with phase velocity v with amplitude decaying as $\frac{1}{r}$. Wave $\frac{1}{r}G(r+vt)$ represents a spherical wave propagating from infinity towards the origin, we do not usually consider this wave – we prescribe the radiating condition, $G = 0$. If we choose the shape of the radiated wave to be a harmonic function, $F(x) = e^{-ikx}$, the corresponding wave $\psi(r,t)$ will have the form

$$F(r-vt) = \frac{1}{r}e^{i(\omega t - kr)}, \qquad \psi(r,t) = \frac{1}{r}e^{i(\omega t - kr)}, \tag{5.24}$$

where we have denoted $\omega = vk$ (dispersion relation).

Chapter 6

Electromagnetic Waves

6.1 Plane Electromagnetic Waves

We consider a homogeneous medium filling the entire space, composed of a linear dielectric and a linear magnetic material. This means that the medium can be described by two (constant) parameters: permittivity ε and permeability μ. We also introduce the parameters of relative permittivity ε_r and relative permeability μ_r using the relations

$$\varepsilon = \varepsilon_r \varepsilon_0, \qquad \mu = \mu_r \mu_0, \tag{6.1}$$

where ε_0 and μ_0 are the permittivity and permeability of vacuum. Maxwell's equations for the electromagnetic field $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ in this medium without free charges and currents take the form:

$$\begin{aligned} &\operatorname{div} \vec{E} = 0 \quad \text{(Gauss's law)}, && \operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's law of induction)}, \\ &\operatorname{div} \vec{B} = 0 \quad (\vec{B} \text{ is solenoidal}), && \operatorname{curl} \vec{B} = \varepsilon\mu \frac{\partial \vec{E}}{\partial t} \quad \text{(Ampère-Maxwell's law)}. \end{aligned} \tag{6.2}$$

Let's derive the wave equations resulting from these equations for the vectors $\vec{E}$ and $\vec{B}$. Apply the differential operator of curl to Faraday's law:

$$\operatorname{curl}\left(-\frac{\partial \vec{B}}{\partial t}\right) = \operatorname{curl}\operatorname{curl} \vec{E} = \operatorname{grad}\operatorname{div} \vec{E} - \Delta \vec{E} = -\Delta \vec{E}, \tag{6.3}$$

where we used the differential identity $\operatorname{curl}, \operatorname{curl} = \operatorname{grad}\operatorname{div} - \Delta$ and Gauss's law. By switching the derivatives on the left side (6.3) and using Ampère-Maxwell's law, we obtain

$$-\frac{\partial}{\partial t}\operatorname{curl} \vec{B} = -\varepsilon\mu \frac{\partial^2}{\partial t^2} \vec{E}. \tag{6.4}$$

Comparing the right-hand sides (6.3) and (6.4), we get the wave equation for the electric field vector $\vec{E}$:

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon\mu} \Delta \vec{E}. \tag{6.5}$$

By the same process, starting with Ampère-Maxwell's law, we arrive at the wave equation for the magnetic field vector $\vec{B}$:

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\varepsilon\mu} \Delta \vec{B}. \tag{6.6}$$

Thus, we have six independent wave equations for the individual components of vectors $\vec{E}$ and $\vec{B}$. The phase velocity of electromagnetic waves derived from the wave equations is

$$v_\varphi = \frac{1}{\sqrt{\varepsilon\mu}}. \tag{6.7}$$

In the case of vacuum, we have $v_\varphi = c$ and

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}. \tag{6.8}$$

We can define the *refractive index* n of the medium as the ratio of the speed of light to the phase velocity in the given medium:

$$n = \frac{c}{v_\varphi} = \sqrt{\varepsilon_r\mu_r} \approx \sqrt{\varepsilon_r}, \tag{6.9}$$

the last approximation comes from the fact that most materials have a relative magnetic permeability close to one.

Any solution to the wave equations (6.5) and (6.6) may not necessarily be a solution to the original Maxwell's equations (6.2). The validity of the wave equations was derived from the validity of Maxwell's equations. Any solution that satisfies our original Maxwell's equations will also be a solution to the wave equations, but the converse may not be true. Consider electrical and magnetic plane traveling waves as solutions to the wave equations (6.5) and (6.6) in the form:

$$\vec{E}(\vec{r},t) = \vec{E}_0\, F(\vec{n}\cdot\vec{r} - vt), \qquad \vec{B}(\vec{r},t) = \vec{B}_0\, F(\vec{n}\cdot\vec{r} - vt), \tag{6.10}$$

where vectors $\vec{E}_0$ and $\vec{B}0$ are constant vectors. The form of the traveling wave $F(x)$ is the same for all six components of vectors. Similarly, we consider the same direction of progression $\vec{n}$[1]. Now, let's verify whether these traveling waves satisfy Maxwell's equations. And if not, under what conditions they do so. Start with Gauss's law, into which we substitute $\vec{E}$ from (6.10):

$$\operatorname{div}\vec{E} = 0 = \partial_i E_i = E_{0i} n_i F'(\vec{n}\cdot\vec{r} - vt) = (\vec{E}_0\cdot\vec{n})\, F'(\vec{n}\cdot\vec{r} - vt), \quad \forall\vec{r},t, \tag{6.11}$$

where we utilized Einstein's summation convention and $\partial_i(\vec{n}\cdot\vec{r}) = n_i$ (symbol $\partial_i = \frac{\partial}{\partial x_i}$). If the form of our traveling wave $F(x)$ is non-constant, then certainly the derivative $F'(x)$ cannot be everywhere zero, and we must require the scalar product to vanish

$$\vec{E}_0\cdot\vec{n} = 0, \qquad \vec{E}_0 \perp \vec{n}. \tag{6.12}$$

From Gauss's law, it follows that the vector $\vec{E}_0$ (and therefore $\vec{E}(\vec{r},t)$) must be perpendicular to the direction of propagation $\vec{n}$. Identically, from the condition of the solenoidality of the magnetic field, we get the condition of perpendicularity of the vector $\vec{B}_0$ to the direction of propagation $\vec{n}$:

$$\operatorname{div}\vec{B} = 0 \quad \Rightarrow \quad \vec{B}_0\cdot\vec{n} = 0, \qquad \vec{B}_0 \perp \vec{n}. \tag{6.13}$$

The electromagnetic wave is therefore a transverse wave! Displacements are perpendicular to the direction of propagation.

Take Faraday's law next,

$$\operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \varepsilon_{ijk}\partial_j E_k = -\partial_t B_i, \tag{6.14}$$

[1]This ansatz, where we consider the same F and the same $\vec{n}$, is a consequence of Maxwell's equations. If we considered a different form of the traveling wave $F(x)$ and a different direction of progression $\vec{n}$ for each component of $\vec{E}$ and $\vec{B}$, Maxwell's equations would imply that they must be equal. For simplicity, we skip this step and directly assume the same F and $\vec{n}$ for all.

which we have written in index notation (using Einstein's summation convention). After substituting traveling waves and differentiating, we have:

$$\varepsilon_{ijk} E_{0k} n_j \, F'(\vec{n} \cdot \vec{r} - vt) = -(-v) B_{0i} \, F'(\vec{n} \cdot \vec{r} - vt). \tag{6.15}$$

Returning to vector notation,

$$\left(\vec{n} \times \vec{E}_0 - v\vec{B}_0 \right) F'(\vec{n} \cdot \vec{r} - vt) = 0, \tag{6.16}$$

after canceling $F'(x)$, we get a condition on vectors $\vec{E}_0$ and $\vec{B}_0$:

$$\vec{n} \times \vec{E}_0 = v\vec{B}_0. \tag{6.17}$$

The same condition would be reached using Ampère-Maxwell's law. This condition states that the vector $\vec{B}_0$ is perpendicular to the vector $\vec{E}_0$ and also provides the relationship between their magnitudes:

$$|\vec{E}_0| = v|\vec{B}_0|. \tag{6.18}$$

From conditions (6.12), (6.13), and (6.17) it also follows that the set $(\vec{E}, \vec{B}, \vec{n})$ forms a right-handed orthogonal set of vectors, see figure 6.1. Fulfilling these conditions ensures that the traveling wave (6.10) is a solution to Maxwell's equations.

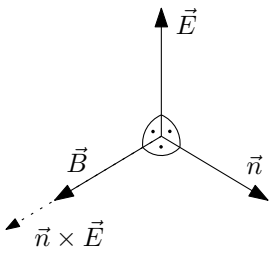


Figure 6.1: Vectors $\vec{E}$, $\vec{B}$, $\vec{n}$ forming a right-handed orthogonal system of vectors.

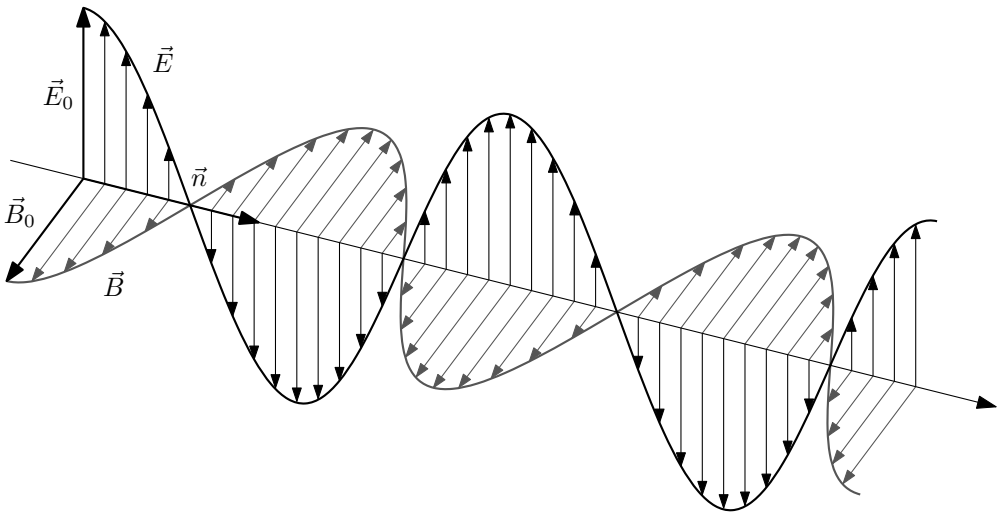


Figure 6.2: Harmonic traveling electromagnetic wave.

If we consider a harmonic traveling wave, i.e., choose (in complex notation)

$$F(x) = e^{ikx}, \tag{6.19}$$

then the resulting electromagnetic wave is of the form

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{-i(\omega t - \vec{k}\cdot\vec{r})}, \qquad \vec{B}(\vec{r},t) = \vec{B}_0\, e^{-i(\omega t - \vec{k}\cdot\vec{r})}, \tag{6.20}$$

where

$$\vec{E}_0, \vec{B}_0, \vec{n} \text{ form a right-handed OG set}, \qquad \omega = v|\vec{k}|, \qquad \vec{k} = k\,\vec{n}. \tag{6.21}$$

The relationship $\omega = v|\vec{k}|$ is the dispersion relation for EM waves. This wave is depicted in figure 6.2. For a special choice of direction of propagation in the direction of the z axis, we get

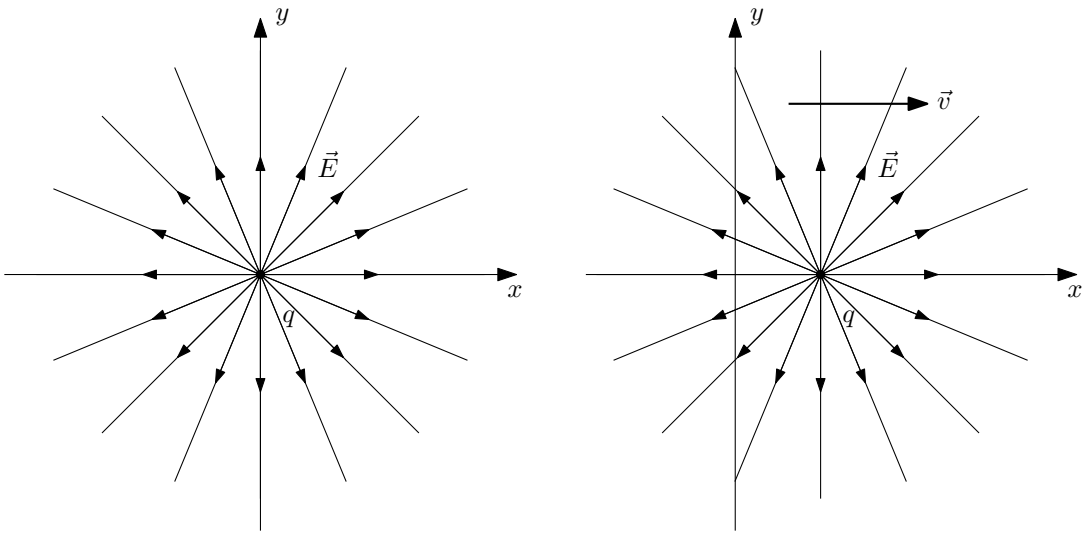
$$\vec{n} = (0,0,1), \qquad \vec{E} = \vec{E}_0\, e^{-i(\omega t - kz)}, \quad \vec{B} = \vec{B}_0\, e^{-i(\omega t - kz)}. \tag{6.22}$$

6.2 Radiation of Electromagnetic Waves

In this chapter, we will look at how to generate electromagnetic waves. First, a quick recap from the course on electricity and magnetism. For a charge q at rest at the origin, we obtain the Coulomb electric field

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}, \tag{6.23}$$

which is radial and decreases with distance as $\frac{1}{r^2}$, see Figure 6.3 on the left.



(a) Electric field of a static point charge.

(b) Electric field of a point charge moving with constant velocity.

Figure 6.3: Electric field of a point charge.

In the case of a charge moving with constant velocity $\vec{v}$ such that at time $t = 0$ it passes through the origin, the electric field at time $t = 0$ is of the form

$$\vec{E}(\vec{r},0) = \frac{q}{4\pi\varepsilon_0}\underbrace{\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}}_{\Gamma(\theta)}\frac{\vec{r}}{r^3}, \qquad \beta = \frac{v}{c}, \tag{6.24}$$

where $\Gamma(\theta)$ is the so-called Heaviside factor, θ is the angle between the velocity vector of the charge $\vec{v}$ and the (field) position vector $\vec{r}$, and the factor $\beta = \frac{v}{c}$. The field remains radial, only its magnitude changes in different directions. Here we will consider low speeds, $v \ll c$, so we

approximate the Heaviside factor with one, $\Gamma(\theta) \approx 1$. Furthermore, the electric field as a whole moves with the velocity $\vec{v}$ along with the charge. We can write

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}'}{r'^3}, \qquad \vec{r}' = \vec{r} - \vec{v}t, \tag{6.25}$$

where the meaning of the vector $\vec{r}'$ is illustrated in Figure 6.4. It is the vector connecting the current position of the charge q with the location determining the electric field.

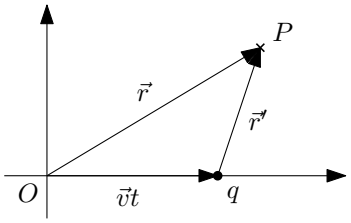


Figure 6.4: Relationship between vectors $\vec{r}$ and $\vec{r}'$. Vector $\vec{r}$ is the position vector of point P, vector $\vec{r}'$ connects charge q with point P.

The fact that the radial field moves through space with the charge only applies if the charge has been moving with constant velocity the entire time. Now, we will study the situation when:

- at time $t \leq 0$ the charge is at rest at the origin,
- at time $0 \leq t \leq \tau$ we let the charge accelerate to velocity v,
- for $t \geq \tau$ the charge will move with constant velocity.

We will consider constant acceleration a in the direction of the x axis, so the resultant velocity $\vec{v}$ will also be in the direction of the x axis. We will consider the acceleration period τ to be very short, so that the distance traveled by the charge during acceleration, $\frac{1}{2}a\tau^2$, is also considered small. Now, we cannot proceed further without introducing the following statement: The disturbance in the electromagnetic field propagates at speed c, and the field at point P and at time t is determined by what the charge was doing at the retarded time $t_r = t - \frac{R}{c}$, where R is the distance between point P and the position of the charge at time t_r, see Figure 6.5. Let's call this fact the *principle of retarded time*. Demonstrating that this principle applies in all generality is beyond our current scope. For a rigorous derivation, see the lecture TEF2.

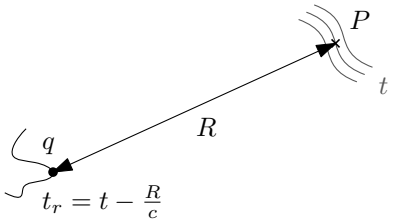


Figure 6.5: Principle of retarded time. The field created by the charge at time t_r propagates through space at speed c to arrive at location P at time $t = t_r + \frac{R}{c}$.

Based on this principle, we can divide the field around the charge at time $t > \tau$ into three regions illustrated in Figure 6.6. The first is a thin spherical shell of width $c\tau$ with a radius of approximately ct – this will represent the field radiated by the charge during its acceleration. The next is the field outside this shell – this is the field from when the charge was stationary

at the origin. And the last is the field inside this shell – the field of the charge moving with constant velocity $\vec{v}$.

Our task now is to determine the shapes of the fields in each region. But we already know the shapes of the fields outside and inside the shell, they are the fields (6.23) and (6.25). Thus, the main task will be to determine the acceleration field $\vec{E}_{acc}$ directly at the shell. We limit ourselves only to times $t \gg \tau$, when the shell of the acceleration field is far from the origin.

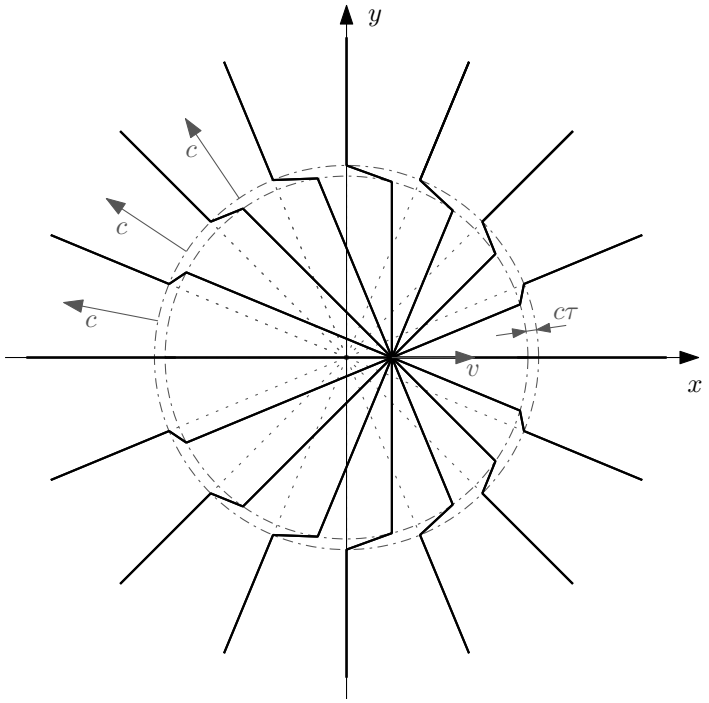


Figure 6.6: Radiation of an electromagnetic wave. The solid lines represent the electric field lines. The spherical shell of width $c\tau$ (dashed line) corresponding to the electromagnetic field of the accelerating charge expands at speed c and separates the field areas from the static and moving charge. The static electric field lines starting at the origin are marked with dots (inside the shell they are replaced by the field of the moving charge).

Let's look in detail at an electric field line at the location of the shell in Figure 6.7. The static field $\vec{E}_{\text{stat.}}$ and the "moving" field $\vec{E}_{\text{mov.}}$ having a field line under the same angle will be continuously connected by an acceleration field line. This fact is given by that if we gradually reduce the value of acceleration a to zero in our model, the field must transition to the field of a static charge everywhere in space. Furthermore, let's divide the acceleration field $\vec{E}_{\text{acc.}}$ present in the shell into a part parallel to the position vector $\vec{E}\parallel$ and a part perpendicular $\vec{E}\perp$.

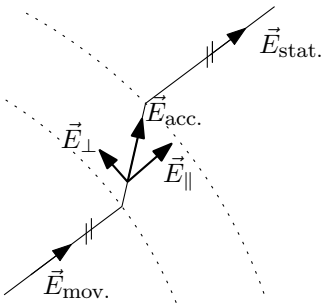


Figure 6.7: Detail of an electric field line near the shell. The acceleration field is divided into a perpendicular component $\vec{E}\perp$ and a parallel component $\vec{E}_{\parallel}$ to the position vector $\vec{r}$.

We determine the parallel field $\vec{E}_{\parallel}$ from Gauss's law,

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}, \tag{6.26}$$

where we choose the closed surface S as shown in Figure 6.8 – thus as a cylindrical surface passing through the outer edge of the shell. Inside this surface, no charge is enclosed, hence $Q = 0$.

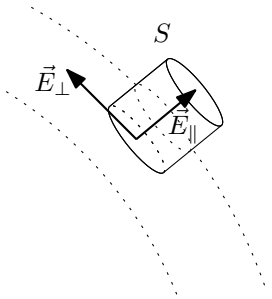


Figure 6.8: Surface S in Gauss's law. The cylinder is oriented perpendicular to the surface of the shell and passes through the outer side of the shell so that one base is directly in the shell and the other base is in the area with static field.

The flux through the mantle is zero – outside the shell, we only have the radial field $\vec{E}_{\text{stat.}}$, which lies in the mantle, $\vec{E}\parallel$ by definition also lies in the mantle, and finally $\vec{E}\perp$ on one side of the mantle flows in, but flows out equally on the other side (we consider S small so that $\vec{E}\perp$ does not change much). What remains are the fluxes through the bases, and by Gauss's law, these fluxes must equal (in absolute value):

$$E_{\parallel} S_{\text{base}} = E_{\text{stat.}} S_{\text{base}}. \tag{6.27}$$

Thus, the magnitude of $E_{\parallel}$ is exactly the same as that of the static electric field:

$$E_{\parallel} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}, \tag{6.28}$$

where r is the radius of the shell. If the shell is thin ($c\tau$ is small) and we consider large times ($t \gg \tau$), around the shell it will apply[2] $\vec{E}_{\text{mov.}} \approx \vec{E}\parallel \approx \vec{E}_{\text{stat.}}$. Next, let's look at the magnitude of $E_{\perp}$. Let's draw a selected electric field line even more detailedly – as in Figure 6.9.

[2]Gauss's law could also be used on the inner side of the shell with the field $\vec{E}_{\text{mov.}}$

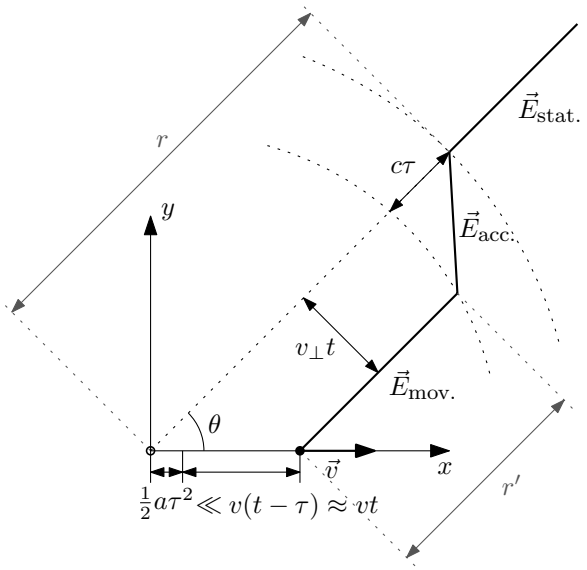


Figure 6.9: Detail of an electric field line for $t \gg \tau$. The angle θ denotes the deviation of the direction of the field lines of the static and "moving" field from the x axis (i.e., from the vector $\vec{v}$, respectively, $\vec{a}$). The width of the shell is $c\tau$. The radius of the outer side of the shell (separating the static electric field $\vec{E}_{\text{stat.}}$) is r, the radius of the inner side of the shell (separating the "moving" electric field $\vec{E}_{\text{mov.}}$) is r'. It holds that $r = ct$ and $r - r' \approx c\tau$, for large t ($\gg \tau$) we can consider $r \approx r'$. At the same time, we consider $\frac{1}{2}a\tau^2 \ll vt$, so the distance between the field lines of $\vec{E}_{\text{stat.}}$ and $\vec{E}_{\text{mov.}}$ is $v_\perp t$, where $v_\perp$ is the magnitude of the perpendicular projection of velocity $\vec{v}$, $v_\perp = v \sin\theta$, (i.e., we completely neglected the distance $\frac{1}{2}a\tau^2$).

We express the magnitude of $E_\perp$ using the similarity of two triangles. One is formed by vectors $\vec{E}\perp$ and $\vec{E}\parallel$ and the other formed by distances $v_\perp t$ and $c\tau$:

$$\frac{c\tau}{v_\perp t} = \frac{E_\parallel}{E_\perp}. \tag{6.29}$$

We express $E_\perp$

$$E_\perp = \frac{v_\perp t}{c\tau} E_\parallel = \frac{a_\perp \tau, r}{c^2 \tau} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{q}{c^2} \frac{a_\perp}{r}, \tag{6.30}$$

where we substituted for $E_\parallel$ from (6.28), $v_\perp = a_\perp \tau$ ($a_\perp = a \sin\theta$) and $t \approx \frac{r}{c}$. The perpendicular component of the electric field $\vec{E}_\perp$ is called the *radiation field* $\vec{E}_{\text{rad}}$:

$$\boxed{E_{rad} = \frac{1}{4\pi\varepsilon_0} \frac{q}{c^2} \frac{a_\perp}{r}.} \tag{6.31}$$

Now let's examine the properties of the radiation field by analyzing the relationship (6.31). We see that the field depends on the acceleration of the charge, not on its resultant velocity. Only an accelerated charge emits electromagnetic radiation.

Moreover, the radiation field varies in different directions – it is *anisotropic* – since it depends on the projection of acceleration $a_\perp$. Figure 6.10 illustrates the perpendicular component $\vec{a}\perp$ of the acceleration vector $\vec{a}$. If θ is the angle between vectors $\vec{r}$ and $\vec{a}$, then the magnitude of the perpendicular acceleration is $a\perp = a \sin\theta$. In the direction of acceleration, the charge does not radiate ($a_\perp(\theta = 0) = 0$), and it radiates the most perpendicular to the direction of acceleration ($a_\perp(\theta = \frac{\pi}{2}) = a$).

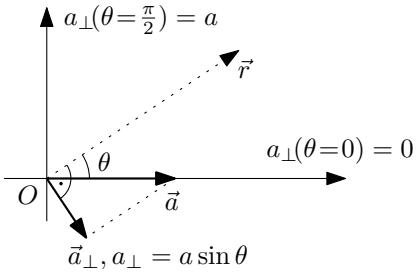


Figure 6.10: Projection of the acceleration vector into the perpendicular direction to the direction of the position vector $\vec{r}$.

The anisotropy of radiation is depicted using the so-called *radiation diagram*, which for our specific case is shown in Figure 6.11. The radiation diagram represents the intensity of radiation into different directions. In the chapter 6.3 on energy quantities, we will see that intensity is proportional to the square of the amplitude. Here the amplitude of the electric field is given by the perpendicular projection of acceleration $a_\perp$, so in the radiation diagram, we plot the quantity $a_\perp^2$.

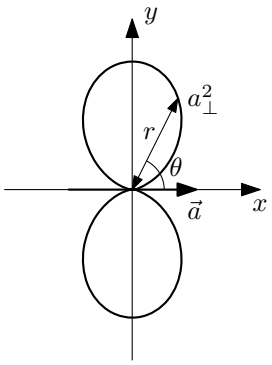


Figure 6.11: Radiation diagram of a charge accelerated in the direction of the x axis. It is a polar diagram $(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$, where the radius function $r(\theta)$ is chosen as the square of the perpendicular projection of acceleration, $r(\theta) = a_\perp^2(\theta) = a^2\sin^2\theta$. The distance of the radiation curve from the origin under angle θ expresses the intensity of radiation in that direction.

A very surprising property of the radiation field is that it decreases with distance as $\frac{1}{r}$! The parallel component $\vec{E}\|$ of the acceleration field $\vec{E}_{\text{acc}}$ decreases as $\frac{1}{r^2}$, as does the field outside the shell $\vec{E}_{\text{stat.}}$ and $\vec{E}_{\text{mov.}}$, see the schematic Figure 6.12. At a great distance from the source, therefore, the radiation field dominates over the static field!

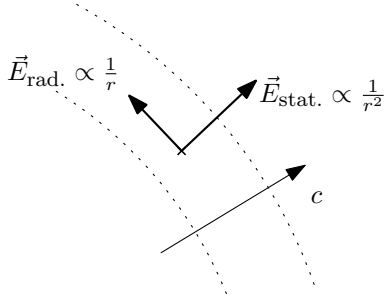


Figure 6.12: Decrease of the static and radiation field with distance.

It holds that the radiation field $\vec{E}_{\text{rad}}$ is, by definition, perpendicular to the direction of

propagation, $\vec{E}_{\mathrm{rad}} \perp \vec{s}$, where $\vec{s} = \vec{r}0 = \frac{\vec{r}}{r}$. Generally (without proof here), the same relationship applies to vectors $\vec{E}$, $\vec{B}$, and $\vec{s}$ as in a plane electromagnetic wave. The set of vectors $(\vec{E}_{\mathrm{rad}}, \vec{B}_{\mathrm{rad}}, \vec{s})$ forms a right-handed set of orthogonal vectors, and in terms of magnitudes, it holds that $E_{\mathrm{rad}} = cB_{\mathrm{rad}}$. See the schematic Figure 6.13.

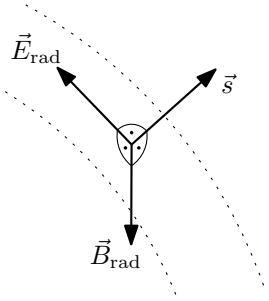


Figure 6.13: Directions of vectors $\vec{E}_{\mathrm{rad}}$, $\vec{B}_{\mathrm{rad}}$, and $\vec{s}$. Magnitudes fulfill $E_{\mathrm{rad}} = cB_{\mathrm{rad}}$.

The presented special case (when we have a charge at rest, then a period of short acceleration, and finally uniform motion) can be generalized in the following way: We consider the motion of a charge such that it always occurs near the origin and its velocity is small compared to the speed of light, $v \ll c$, then the radiation field at a large distance from the origin is of the form

$$\boxed{\vec{E}_{rad}(\vec{r}, t) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}\frac{\vec{a}\perp(t_r)}{r}, \qquad t_r = t - \frac{r}{c},} \tag{6.32}$$

where $\vec{a}\perp(t_r)$ is the perpendicular component of acceleration at the retarded time t_r and the minus sign is given by the fact that the radiation field points in the opposite direction to the vector $\vec{a}\perp$, see Figures 6.6, 6.7, and 6.9. This generalization corresponds to the fact that we imagine the radiation field as a series of shells connected to each other, where the radiation field in each shell corresponds to the acceleration of the charge at the respective retarded time. If the motion of the charge q is prescribed by the function $\vec{r}q(t)$, then the acceleration $\vec{a}(t) = \ddot{r}q(t)$ and the perpendicular component of acceleration is calculated as

$$\vec{a}\perp = \vec{a} - \vec{a}\| = \vec{a} - (\vec{a}\cdot\vec{r}_0)\vec{r}_0, \qquad \vec{r}0 = \frac{\vec{r}}{r}, \tag{6.33}$$

thus as the difference between the whole vector $\vec{a}$ and the parallel projection $\vec{a}\|$, which we easily calculate using the scalar product.

Consider as an example a charge oscillating along the z axis around the origin with acceleration

$$\vec{a}(t) = a_0\vec{z}\cos(\omega t), \qquad \vec{z} = (0, 0, 1). \tag{6.34}$$

The acceleration at the retarded time t_r then is

$$\vec{a}(t_r) = a_0\vec{z}\cos(\omega t_r) = a_0\vec{z}\cos(\omega t - kr), \tag{6.35}$$

where we denoted $k = \frac{\omega}{c}$ (dispersion relation). The magnitude of the perpendicular projection is

$$|\vec{a}\perp| = |\vec{a}|\sin\theta, \tag{6.36}$$

where θ is the angle between vectors $\vec{z}$ and $\vec{r}$ (and thus also the standard angle θ of spherical coordinates). After substituting (6.34) and (6.36) into the formula for the radiation field (6.32),

we obtain for the magnitude the result

$$E_{rad} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}a_0}_{\mathcal{E}_0}\frac{1}{r}\sin\theta\cos(\omega t_r) = \frac{\mathcal{E}_0}{r}\sin\theta\cos(\omega t - kr), \tag{6.37}$$

where we introduced the quantity $\mathcal{E}_0$ representing the amplitude of the electric field at a unit distance from the origin. We see that for a harmonically oscillating charge, we obtained a spherical harmonic traveling wave, which additionally has an angular dependence – anisotropy – given by the additional expression $\sin\theta$.

6.3 Energy Quantities in Electromagnetic Field

6.3.1 Energy Density

In the lecture on electricity and magnetism, you derived the relationship for the energy density of static electric and magnetic fields:

$$\boxed{w = \frac{1}{2}\left(\varepsilon\vec{E}^2 + \frac{1}{\mu}\vec{B}^2\right).} \tag{6.38}$$

Let's remind ourselves again of the definition of *energy density*. Now we have an electromagnetic field in 3D space, thus here (unlike the case on a string) energy density determines a small amount of energy dE in a volume dV:

$$dE = w\,dV, \qquad [w] = \mathrm{J.m^{-3}}. \tag{6.39}$$

The relationship (6.38) applies not only to static fields but to any fields. It is a general expression for the energy density $w(\vec{r},t)$ of any electromagnetic field (in a linear material medium with permittivity ε and permeability μ). We leave this statement without proof.

6.3.2 Energy Flow

Let's look at the change in energy density w over time, i.e., compute the derivative of the expression over time:

$$\frac{\partial w}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} + \frac{1}{\mu}\frac{\partial\vec{B}}{\partial t}\cdot\vec{B}, \tag{6.40}$$

where we used the rule for differentiating the scalar product:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}^2\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}E_iE_i\right) = \frac{1}{2}\left(\frac{\partial E_i}{\partial t}E_i + E_i\frac{\partial E_i}{\partial t}\right) = \frac{\partial E_i}{\partial t}E_i = \frac{\partial\vec{E}}{\partial t}\cdot\vec{E} \tag{6.41}$$

(and the same for the $\vec{B}$ field). Using Maxwell's equations (6.2), we replace the time derivatives with curls (from Faraday's and Ampère-Maxwell's laws):

$$\frac{\partial w}{\partial t} = \frac{1}{\mu}\left((\mathrm{curl}\,\vec{B})\cdot\vec{E} - (\mathrm{curl}\,\vec{E})\cdot\vec{B}\right). \tag{6.42}$$

Using Einstein's summation convention, it's easy to prove the identity[3] (see TEF1)

$$\mathrm{div}(\vec{B}\times\vec{E}) = (\mathrm{curl}\,\vec{B})\cdot\vec{E} - (\mathrm{curl}\,\vec{E})\cdot\vec{B}. \tag{6.43}$$

[3] Using index notation and Einstein's summation convention:

$$\mathrm{div}(\vec{B}\times\vec{E}) = \partial_i(\varepsilon_{ijk}B_jE_k) = \varepsilon_{ijk}((\partial_iB_j)E_k + B_j(\partial_iE_k)) = E_k\varepsilon_{kij}\partial_iB_j - B_j\varepsilon_{jik}\partial_iE_k = \vec{E}\cdot\mathrm{curl}\vec{B} - \vec{B}\cdot\mathrm{curl}\vec{E}.$$

Substituting into (6.42) we get the *continuity equation*:

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\frac{1}{\mu}\vec{E} \times \vec{B}\right) = 0, \tag{6.44}$$

where the term in brackets is interpreted as the *energy flow vector*. We denote it as $\vec{S}$ and call it the *Poynting vector*:

$$\boxed{\vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B}.} \tag{6.45}$$

The unit of energy flow $\vec{S}$ is $[\vec{S}] = \mathrm{W.m^{-2}}$. The amount of energy dE, which flows through the area dA with the normal vector $\vec{n}$ in time dt, is given by the relation:

$$dE = (\vec{S} \cdot \vec{n})\, dt\, dA. \tag{6.46}$$

The continuity equation (6.44) represents the differential form of the law of conservation of energy in the electromagnetic field. The integral form is obtained by integrating over a given volume V:

$$\frac{d}{dt}\int_V w\, dV = -\int_A \vec{S} \cdot d\vec{A}, \tag{6.47}$$

where the right side was obtained by applying Gauss's theorem to the volume integral, the surface A is the boundary of volume V, $A = \partial V$. The integral form of the law states that the change in the amount of field energy in volume V is given by the total energy flow through the boundary A.

6.3.3 Momentum Density

The electromagnetic field carries momentum in addition to energy. We introduce the concept of *momentum density* $\vec{g}$, which indicates the amount of momentum $d\vec{p}$ contained in a volume element dV:

$$d\vec{p} = \vec{g}\, dV. \tag{6.48}$$

The unit is

$$[\vec{g}] = [\vec{p}].\mathrm{m^{-3}} = (\mathrm{kg.m.s^{-1}}).\mathrm{m^{-3}} = \mathrm{kg.m^{-2}.s^{-1}}. \tag{6.49}$$

Consider only the case of an electromagnetic field in vacuum. We show that the electromagnetic field carries momentum that it can transfer to charged particles. General derivation see lecture TEF2, where it will be shown that generally

$$\boxed{\vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2}\frac{1}{\mu_0}\vec{E} \times \vec{B} = \varepsilon_0 \vec{E} \times \vec{B}.} \tag{6.50}$$

For simplicity and just for illustration, we will only show a very rough model of a plane harmonic electromagnetic wave and its interaction with a heavily damped point charge q. We will see how the momentum of the electromagnetic field is transferred to the charged particle due to this interaction. The charge is acted upon by the Lorentz force from the electromagnetic wave,

$$\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B}), \tag{6.51}$$

where $\vec{v}$ is the velocity of the charge. Additionally, a strong frictional force linearly dependent on the velocity of the charge $\vec{v}$ acts on the charge. The equation of motion is then of the form

$$m\vec{a} = -\alpha\vec{v} + \vec{F}_L. \tag{6.52}$$

For very strong damping, the inertial term $m\vec{a}$ is negligible compared to the frictional term $\alpha\vec{v}$. In such a case, we have the equation of motion

$$\alpha\vec{v} = \vec{F}_L, \tag{6.53}$$

where the velocity $\vec{v}$ is directly proportional to the applied force and not to the acceleration $\vec{a}$. Without loss of generality, consider that the charge is positive, $q > 0$. If there were only an electric field $\vec{E}$ present in the passing wave, the particle would only perform oscillatory motion in the direction of the electric field, see Figure 6.14

on the left. Now, let's superpose the effect of the magnetic part of the Lorentz force on this motion. In one half-period of the electromagnetic wave, the situation is depicted in Figure 6.14 in the middle, and in the second half-period, it is depicted in Figure 6.14 on the right. Since the direction of velocity flips simultaneously with the direction of the electric field $\vec{E}$, and therefore also with the direction of the magnetic field $\vec{B}$, the direction of the magnetic part of the Lorentz force remains the same – in the direction of the propagation of the electromagnetic wave!

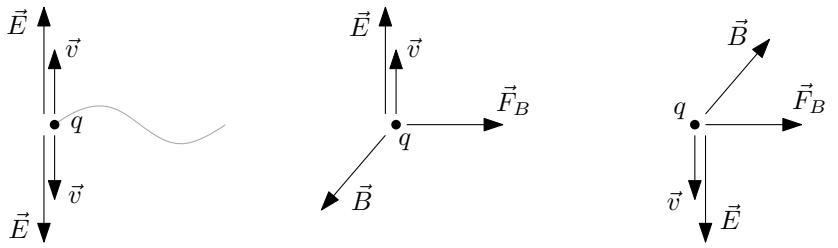


Figure 6.14: Interaction of a charge with a passing plane harmonic traveling wave. The EM wave propagates to the right. On the left, the velocity of the charge $\vec{v}$ due to the electric field $\vec{E}$ is depicted. The velocity vector is oriented in the direction of the field $\vec{E}$. The directions of the magnetic part of the Lorentz force are in the direction of propagation in both half-periods of the EM wave, as shown in the middle and right images.

Let's now quantify these considerations. Without loss of generality, consider an EM wave propagating in the direction $\vec{n} = \vec{z}$, with the electric field oscillating in the direction $\vec{x}$, $\vec{E} = E_x\vec{x}$, and thus the magnetic field oscillating in the direction $\vec{y}$, $\vec{B} = B_y\vec{y}$. The Lorentz force is then of the form

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = qE_x\vec{x} + qv_xB_y\vec{z} - qv_zB_y\vec{x}. \tag{6.54}$$

Let's calculate the time-average value of the Lorentz force $\vec{F}_L$ (which is also the time-average value of the momentum transferred due to this force):

$$\left\langle \frac{d\vec{p}}{dt} \right\rangle = \langle \vec{F}_L \rangle = 0 + q\langle v_xB_y \rangle \vec{z} - 0. \tag{6.55}$$

The first term is zero due to the harmonicity of the component E_x and the third term is zero because the component B_y is harmonic (i.e., changing sign), but v_z is positive due to the reason discussed above (with half the period compared to the EM wave).

Next, let's study the change in energy of the particle due to the acting Lorentz force:

$$\frac{dE}{dt} = P = \vec{F}_L \cdot \vec{v} = q\vec{v} \cdot \vec{E} + q\underbrace{\vec{v} \cdot (\vec{v} \times \vec{B})}_{\vec{B}\cdot(\vec{v}\times\vec{v})=0} = qv_xE_x, \tag{6.56}$$

where only the term from the electric field remains (the magnetic field does not do work; we

used the vector identity[4] $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b})$). For the time-average value, we get

$$\left\langle \frac{dE}{dt} \right\rangle = q\langle v_x E_x \rangle \tag{6.57}$$

and if we use the relationship between the magnitude of the electric and magnetic components in a plane electromagnetic wave, $E_x = cB_y$, we arrive at

$$\left\langle \frac{dE}{dt} \right\rangle = qc\langle v_x B_y \rangle. \tag{6.58}$$

Comparing (6.55) and (6.58) we get a relationship between the transferred momentum and energy:

$$\boxed{\left\langle \frac{d\vec{p}}{dt} \right\rangle = \frac{1}{c}\left\langle \frac{dE}{dt} \right\rangle \vec{z}.} \tag{6.59}$$

The momentum and energy transferred to the particle must occur at the expense of the momentum and energy of the electromagnetic field. If we multiply the previous relationship by dt and divide by dV, we get

$$\langle \vec{g} \rangle = \frac{1}{c}\langle w \rangle \, \vec{n}. \tag{6.60}$$

In the following chapter, we will see that for a plane wave, this result is equivalent (if we ignore the average values) to the general relationship (6.50).

6.3.4 Energy Quantities in a Plane Wave

Now, let's specify energy quantities for the case of a plane traveling wave. In this wave, the set of vectors $(\vec{E}, \vec{B}, \vec{n})$ forms a right-handed orthogonal system ($\vec{n}$ is the direction of propagation, $|\vec{n}| = 1$) and the relationship between the magnitudes of the field vectors is $E = vB$ (for vacuum $v = c$).

For the energy density w we get

$$w = \frac{1}{2}\Big(\varepsilon E^2 + \frac{1}{\mu}\underbrace{B^2}_{B^2=\frac{E^2}{v^2}=\varepsilon\mu E^2}\Big) = \varepsilon E^2. \tag{6.61}$$

For the Poynting vector $\vec{S}$, we have

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \sqrt{\frac{\varepsilon}{\mu}}\vec{E}\times(\vec{n}\times\vec{E}) = \sqrt{\frac{\varepsilon}{\mu}}E^2\,\vec{n}, \tag{6.62}$$

where we utilized the relationship $\vec{B} = \frac{1}{v}\vec{n}\times\vec{E}$ (stemming from the properties of vectors $\vec{E}$, $\vec{B}$, $\vec{n}$ for a plane EM wave) and also the relationship $\vec{E}\times(\vec{n}\times\vec{E}) = E^2\vec{n}$, which we easily deduce from figure 6.1. For a plane electromagnetic wave, the Poynting vector can be written using the energy density:

$$\vec{S} = v\,w\,\vec{n}. \tag{6.63}$$

[4] Which we again easily prove using the summation convention and index notation:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_i\varepsilon_{ijk}b_jc_k = \varepsilon_{ijk}a_ib_jc_k = c_k\varepsilon_{kij}a_ib_j = \vec{c}\cdot(\vec{a}\times\vec{b}).$$

For an electromagnetic plane traveling wave in vacuum, we can write the momentum density $\vec{g}$ after substituting expressions for the Poynting vector as

$$\vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c} w\,\vec{n} = \varepsilon_0 \vec{E} \times \vec{B}. \tag{6.64}$$

For a harmonic plane traveling wave propagating in the direction of the z axis (and the electric field vector oscillating in the direction of the x axis), the vectors $\vec{E}$ and $\vec{B}$ are of the form

$$\vec{E} = E_0 \vec{x} \cos \omega t, \qquad \vec{B} = B_0 \vec{y} \cos \omega t, \qquad E_0 = c B_0, \qquad \vec{n} = \vec{z}. \tag{6.65}$$

Then the energy density w and the Poynting vector $\vec{S}$ (and their time-average values) come out as

$$w = \varepsilon E^2 = \varepsilon E_0^2 \cos^2 \omega t, \qquad \vec{S} = \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \cos^2 \omega t\, \vec{z}, \tag{6.66}$$

$$\langle w \rangle = \frac{1}{2} \varepsilon E_0^2, \qquad \langle \vec{S} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} E_0^2\, \vec{z}. \tag{6.67}$$

The magnitude of the time-average value of the Poynting vector is called the *intensity* of the electromagnetic wave $I = \langle |\vec{S}| \rangle$.

Energy quantities of the electromagnetic field are quadratic in fields, just as in the case of a string. Again, we cannot use complexified waves for their calculation.

6.3.5 Radiation Pressure

Due to the carried momentum, an electromagnetic wave exerts pressure on the surface it strikes. First, consider the case where the wave strikes a surface perpendicularly and the entire wave is absorbed. Then all the momentum carried by this wave is transferred to the given surface. Let's quantify these considerations. Consider a small area dS and a small moment in time dt. Pressure is defined as force per unit area,

$$p = \frac{dF}{dS} = \frac{dp}{dS\,dt}, \tag{6.68}$$

and the acting force is the transferred momentum per unit time $dF = \frac{dp}{dt}$ (note, p denotes pressure and dp momentum). Now we need to look at how much momentum dp the surface absorbs in time dt. In time dt, radiation from the volume $dV = c\,dt\,dS$ hits the surface, see Figure 6.15. The momentum contained in this volume is $dp = g\,dV$.

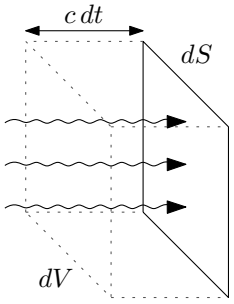


Figure 6.15: Radiation absorbed by an area of size dS over time dt comes from the volume $dV = c\,dt\,dS$.

Substituting into the expression (6.68) we get

$$p = \frac{g\,dV}{dS\,dt} = \frac{g\,c\,dt\,dS}{dS\,dt} = c\,g. \tag{6.69}$$

Using relationships for momentum density in a plane traveling wave (6.64), we can write the radiation pressure in several equivalent forms:

$$\boxed{p = c\,g = w = \frac{S}{c}.} \tag{6.70}$$

If we have a surface with reflectivity $\mathcal{R} \in \langle 0, 1\rangle$, the radiation pressure increases accordingly:

$$p = (1 + \mathcal{R})w. \tag{6.71}$$

If radiation strikes a surface at an angle α (angle of deviation from perpendicular), the radiation pressure decreases by a factor of $\cos^2\alpha$, $p = w\cos^2\alpha$. This is due to two effects: the amount of radiation hitting the area dS decreases by a factor of $\cos\alpha$ and the momentum transferred in the direction perpendicular to the area dS also decreases by a factor of $\cos\alpha$, see the schematic Figure 6.16.

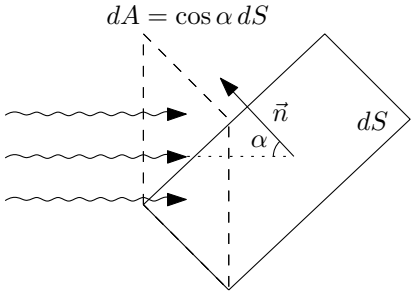


Figure 6.16: Radiation pressure on a surface inclined at an angle α. The size of the projection of area dS into the direction perpendicular to the radiation is $dA = \cos\alpha\, dS$.

Finally, for an idea of the magnitude of radiation pressure from the Sun at a distance of $1\,\mathrm{AU} \approx 150.10^6\,\mathrm{km}$ (i.e., at a distance where Earth orbits). The energy flow of solar radiation at this distance is $S = 1361\,\mathrm{W.m^{-2}}$ (also called the *solar constant*). The radiation pressure then is $p = \frac{S}{c} = 4,5.10^{-6}\,\mathrm{Pa}$. The closer we are to the center of the Sun, the greater the pressure. At a distance of $d = 10^6\,\mathrm{km}$ from the center of the Sun (the radius of the Sun $R_\odot \approx 0,7.10^6\,\mathrm{km}$) the pressure will be 150^2 times greater, $p = 0,1\,\mathrm{Pa}$. However, inside very massive stars, radiation pressure plays a significant role as one of the processes acting against the gravitational collapse of these stars.

6.3.6 Energy of the Radiated Wave

Let's now look at the energy flow $\vec{S}$ of the electromagnetic wave radiated by a point charge. The Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{6.72}$$

takes the form

$$\vec{S}(\vec{r}, t) = \frac{1}{\mu_0}EB\vec{r}_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}E^2\vec{r}_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}\right)^2\frac{1}{r^2}a_\perp^2(t_{\mathrm{ret}})\vec{r}_0 \tag{6.73}$$

after substituting from the relationship for the radiation field of a moving charge (6.32) and the relationship between the direction and magnitude of vectors $\vec{E}$ and $\vec{B}$ (see Figure 6.13), $t_{\mathrm{ret}} = t - \frac{r}{c}$ is the retarded time. We see that the energy flow is directed radially away from the source. Let's now calculate the energy flow passing through a sphere of radius r at time t. If

we now introduce spherical coordinates such that the z axis points in the direction of the vector $\vec{a}(t_{\text{ret}})$, then the expression for the magnitude of the perpendicular projection of acceleration $a_\perp$ is

$$a_\perp = a \sin\theta, \tag{6.74}$$

where θ is the standard angle of spherical coordinates measuring the deviation from the z axis. The energy flow dP through a small area dA, whose normal vector is $\vec{n}$, is given by the relationship

$$dP = \vec{S} \cdot d\vec{A}, \tag{6.75}$$

where $d\vec{A} = \vec{n}\, dA$. The area through which we calculate the flow is a sphere, the normal vector points also in the radial direction like the vector $\vec{S}$ (see Figure 6.17), so $dP = S\, dA$ applies.

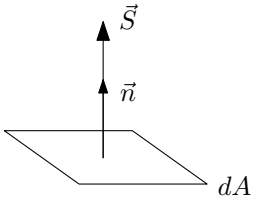


Figure 6.17: Area dA.

The total power $P(t, r)$ passing through the sphere of radius r at time t is obtained by integration:

$$\begin{aligned} P &= \int_A dP = \int_0^{2\pi} \int_0^{\pi} S \underbrace{r^2 \sin\theta \, d\varphi \, d\theta}_{dA} \\ &= \int_0^{2\pi} \int_0^{\pi} \sqrt{\frac{\varepsilon_0}{\mu_0}} \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{c^2} \right)^2 \frac{1}{r^2} a^2(t_r) \sin^2\theta \, r^2 \sin\theta \, d\varphi \, d\theta, \end{aligned} \tag{6.76}$$

where we substituted for S from (6.73) and (6.74). After simplifying, we get

$$P = 2\pi \sqrt{\frac{\varepsilon_0}{\mu_0}} \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{c^2} \right)^2 a^2(t_{\text{ret}}) \int_0^{\pi} \sin^3\theta \, d\theta, \tag{6.77}$$

where we integrated over the angle φ and pulled out all constants, including acceleration $a(t_r)$, which from the perspective of integration at a given t and on a sphere of constant radius r is also constant. The only thing left is the integral over the angle θ, which we easily calculate:

$$\int_0^{\pi} \sin^3\theta \, d\theta = \int_0^{\pi} \left(1 - \cos^2\theta\right) \sin\theta \, d\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^{\pi} = \frac{4}{3}. \tag{6.78}$$

After substituting and simplifying constants, the result is the so-called *Larmor formula* for the power radiated by a non-relativistic point charge:

$$\boxed{P(t, r) = \frac{\mu_0 q^2}{6\pi c} a^2(t_{\text{ret}}),} \tag{6.79}$$

the formula may appear in various forms depending on how one chooses to write the constant $\frac{\mu_0}{c}$ (using the relationship $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$). The power depends on the radius of the sphere r only through the retarded time t_{ret}. This means that the energy contained in the sphere expanding

at speed c remains constant[5]. This is due to the fact that the electric field decreases as $\frac{1}{r}$, therefore the Poynting vector decreases as $\frac{1}{r^2}$, but the surface area of the sphere is proportional to r^2, these two effects thus cancel each other out.

Consider a harmonically oscillating charge, then its position, the square of acceleration including the time-average value are:

$$x(t) = A\cos\omega t, \qquad a^2(t) = A^2\omega^4\cos^2\omega t, \qquad \langle a^2(t)\rangle = \frac{1}{2}A^2\omega^4. \tag{6.80}$$

The time-average value of the carried power then is

$$\langle P\rangle = \frac{\mu_0 q^2}{6\pi c}\langle a^2(t_r)\rangle = \frac{\mu_0 q^2}{12\pi c}A^2\omega^4. \tag{6.81}$$

This no longer depends on r and thus "on average" the same energy is contained in every shell of radius r.

6.4 Refractive Index in Materials and Plasma

Consider a simple model of light passing through a material as an interaction of the traveling electromagnetic wave with bound electrons in the atoms of the material. Initially, assume that there is only one type of electron present in the material, which has its own natural angular frequency of oscillation ω_0 due to binding in the electric field of the atomic nucleus.

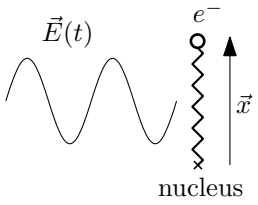


Figure 6.18: Model of an electron bound in a material oscillating under the influence of the electric field from the traveling electromagnetic wave.

The equation of motion for this electron will be

$$m\ddot{\vec{x}} + m\omega_0^2\vec{x} = \vec{F}_L, \tag{6.82}$$

where m is the mass of the electron and $\vec{F}_L$ is the Lorentz force acting on the electron in the traveling electromagnetic wave. Damping of the electron in this simple model is neglected. If we now in the Lorentz force,

$$\vec{F}_L = q\big(\vec{E} + \vec{v}\times\vec{B}\big), \tag{6.83}$$

estimate the magnitude of the magnetic part in the traveling electromagnetic wave,

$$|\vec{v}\times\vec{B}| \le v|\vec{B}| = \frac{v}{c}|\vec{E}| \ll |\vec{E}|, \tag{6.84}$$

we see that for non-relativistic speeds, $v \ll c$, due to the relationship $|\vec{E}| = c|\vec{B}|$, the magnetic force can be neglected. For a non-relativistic electron, the equation of motion (6.82) will approximately be

$$m\ddot{\vec{x}} + m\omega_0^2\vec{x} = -e\vec{E}(t), \tag{6.85}$$

[5]If I consider a fixed retarded time $t_{\text{ret}} = t - \frac{r}{c}$, then I can express $r(t) = -ct_{\text{ret}} + ct$ – thus shells of constant retarded time expand at speed c.

where e is the elementary electric charge. For a harmonic electromagnetic wave, we have at a given point

$$\vec{E}(t) = \vec{E}_0 \cos(\omega t), \tag{6.86}$$

meaning we have a harmonic oscillator driven by a harmonic driving force. Therefore, using the results from chapter 1.6, specifically equations (1.24), (1.25), and (1.28) (for zero damping $\delta = 0$, driving amplitude $B \to -\frac{e}{m}\vec{E}_0$, and driving frequency $\Omega = \omega$), we can write

$$\vec{x}(t) = -\frac{e}{m}\frac{\vec{E}_0}{\omega_0^2 - \omega^2}\cos\omega t. \tag{6.87}$$

The oscillating electron creates a variable dipole moment $\vec{p}$ of magnitude

$$\vec{p} = -e\vec{x} = \frac{e^2}{m}\frac{\vec{E}(t)}{\omega_0^2 - \omega^2}. \tag{6.88}$$

Heavy positively charged nuclei are considered stationary and therefore do not create any dipole moment. This way, the electromagnetic wave affects all electrons (with angular frequency ω_0) present in the material. If the number density of electrons in the material is N, then the polarization vector $\vec{P}$ (representing the volumetric density of dipole moment, $d\vec{p} = \vec{P}\,dV$) is given by

$$\vec{P} = N\,\vec{p}. \tag{6.89}$$

The permittivity ε of the medium can be expressed from the relationship for the electric displacement $\vec{D}$:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon\vec{E}. \tag{6.90}$$

(this relationship defines ε for the resulting $\vec{E}$
and $\vec{P}$). If we substitute the value of polarization $\vec{P}$ into (6.90), we get

$$\varepsilon_0\left(1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right)\vec{E} = \varepsilon\vec{E} \tag{6.91}$$

and thus

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right). \tag{6.92}$$

For the refractive index, we have an approximate relationship $n \approx \sqrt{\varepsilon_r} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$, meaning

$$\boxed{n(\omega) = \sqrt{1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2}}.} \tag{6.93}$$

Having the refractive index $n(\omega)$, we know the dispersion relation of the given medium

$$\omega = \frac{c}{n(\omega)}k. \tag{6.94}$$

This model quantifies the classical notion of the interaction of an electromagnetic wave with a material. The change in phase velocity of EM waves when passing through a material is interpreted as the effect of superposition of induced fields (from oscillating electrons) and the passing wave.

The singularity for $\omega = \omega_0$ would be removed by accounting for the non-zero damping of individual electrons. A schematic graph of the refractive index function $n(\omega)$ (6.93) is shown

in Figure 6.19. In the region $\omega < \omega_0$, the fraction under the square root is positive, and thus the refractive index $n > 1$ (and $v_\varphi < c$). In the region $\omega > \omega_0$, the fraction is negative. For frequencies in the range $\omega \in (\omega_0, \sqrt{\omega_0^2 + \frac{Ne^2}{m\varepsilon_0}})$, the expression under the square root is negative, and the entire medium behaves as reactive. For frequencies $\omega \geq \sqrt{\omega_0^2 + \frac{Ne^2}{m\varepsilon_0}}$, the medium is again transparent, this time with $n < 1$ ($v_\varphi > c$).

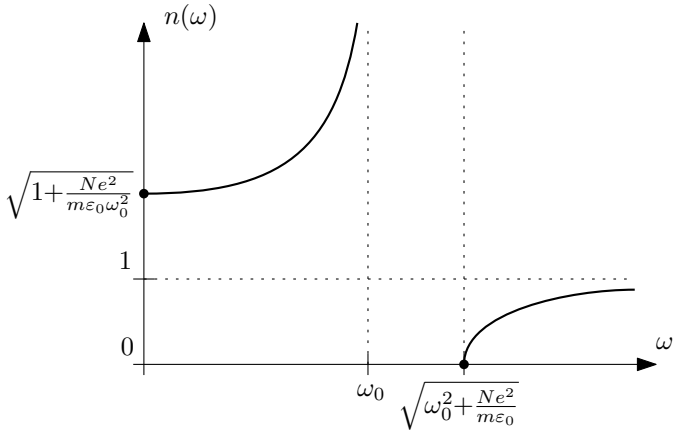


Figure 6.19: The function of the refractive index $n(\omega)$.

Typically, there are multiple types of electrons (differently "strongly" bound in the atom) in a material, thus generally we would consider the densities of each type of electron N_k with their own frequencies ω_k. This generalization would lead to a relationship for the refractive index

$$n(\omega) = \sqrt{1 + \sum_k \frac{N_k e^2}{m\varepsilon_0} \frac{1}{\omega_k^2 - \omega^2}}. \tag{6.95}$$

Let's proceed to the case of plasma. In plasma, we have electrons and positively charged ions that can move freely. Similarly to the case of the refractive index for materials, we neglect the motion of positively charged ions (in materials, these were atomic nuclei) due to their much higher mass compared to electrons[6]. For plasma, we use the previous results simply by setting $\omega_0 = 0$ – electrons are not bound. From the result (6.93), we therefore get

$$n(\omega) = \sqrt{1 - \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega^2}}. \tag{6.96}$$

From the general dispersion relation for the refractive index $\omega = \frac{c}{n}k$, we can express $n = \frac{1}{\omega}ck$ and after substituting into (6.96),

$$c^2k^2 \frac{1}{\omega^2} = 1 - \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega^2}, \tag{6.97}$$

and with a minor adjustment, we obtain the dispersion relation for waves in plasma

$$\boxed{\omega^2 = \omega_p^2 + c^2k^2,} \tag{6.98}$$

where we designated the so-called *plasma frequency* $\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$. For $\omega > \omega_p$, plasma behaves as a transparent medium, for $\omega < \omega_p$ as a reactive medium. In the Earth's ionosphere, depending

[6] A proton is approximately two thousand times heavier than an electron ($\frac{m_p}{m_e} \approx 1836$).

on the time of day, season, and solar activity, the plasma frequency is in the range of $10-30\,\mathrm{MHz}$. Waves of lower frequencies thus do not pass through the ionosphere and are reflected.

A waveguide is a device capable of guiding electromagnetic waves with minimal losses. Unlike in free space, where electromagnetic waves from a source propagate as spherical waves with amplitude decreasing with distance, a waveguide confines the wave propagation to a limited space along one direction, thus preventing amplitude reduction during propagation. Practically, it consists of a metallic tube or a plated conduit—a space bounded by a metallic shell. As a model, we will consider a long straight waveguide of rectangular cross-section, whose walls are made of perfectly conductive material (material with zero or negligible resistance).

Let's consider the propagation of the electric part of the electromagnetic wave in this environment. The electric field vector $\vec{E}$ must satisfy the wave equation and Gauss's law,

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \Delta \vec{E}, \qquad \operatorname{div} \vec{E} = 0, \tag{6.99}$$

thus entirely the same equations as, for example, for a planar electromagnetic wave. The significant difference is the presence of *boundary conditions*. We are looking for an electric field inside the waveguide with the requirement of fulfilling the boundary conditions of *perfectly conductive walls*. Perfectly conductive walls enforce the *vanishing of the tangential components of the electric field at the waveguide walls*. This fact follows from the differential form of Ohm's law,

$$\vec{j} = \sigma \vec{E}, \tag{6.100}$$

which relates the current density $\vec{j}$ in the conductor with the acting electric field $\vec{E}$ using the proportionality constant σ – the conductivity of the material. If we express the magnitude of the electric field $E = \frac{1}{\sigma} j$ and consider infinite conductivity (zero resistance), we get $E = 0$. Physically speaking, charges can follow the changing electric field and always precisely compensate for it with their movement[7]. However, since charges must remain within the waveguide wall, the perpendicular component of the electric field at the boundary cannot induce charge movement and thus remains undisturbed.

Let's introduce coordinates as in Figure 6.20, where the respective tangential components in the various walls of the waveguide are also illustrated.

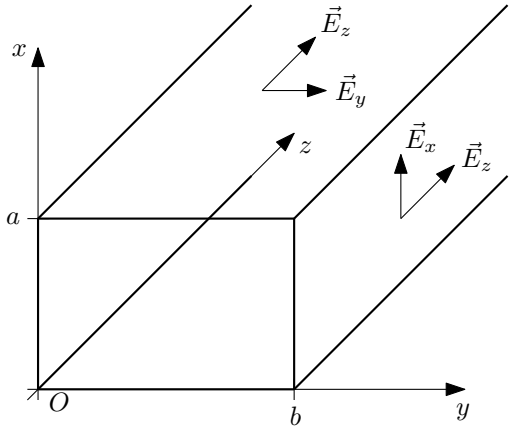


Figure 6.20: Rectangular waveguide of dimensions (a, b) oriented along the z axis. The corresponding tangential components of the electric field $\vec{E}$ are shown in the top (and bottom), and right (and left) walls.

[7]For static fields, this fact holds for any conductors. The static electric field in a conductor is always zero. For non-static fields, we additionally require the condition of perfect conductivity.

In these introduced coordinates, the boundary conditions are as follows. For the left ($y = 0$) and right ($y = b$) wall, we have:

$$\begin{aligned} E_x(x, 0, z, t) = E_x(x, b, z, t) = 0 \qquad & \forall x \in \langle 0, a\rangle, \quad \forall z, t \in \mathbb{R}, \\ E_z(x, 0, z, t) = E_z(x, b, z, t) = 0 \qquad & \forall x \in \langle 0, a\rangle, \quad \forall z, t \in \mathbb{R}. \end{aligned} \tag{6.101}$$

For the top ($x = a$) and bottom ($x = 0$) wall:

$$\begin{aligned} E_y(0, y, z, t) = E_y(a, y, z, t) = 0 \qquad & \forall y \in \langle 0, b\rangle, \quad \forall z, t \in \mathbb{R}, \\ E_z(0, y, z, t) = E_z(a, y, z, t) = 0 \qquad & \forall y \in \langle 0, b\rangle, \quad \forall z, t \in \mathbb{R}. \end{aligned} \tag{6.102}$$

We will now seek the simplest solution to equations (6.99) that satisfy boundary conditions (6.101) and (6.102) and that propagates along

the waveguide (i.e., along the z axis). We do not aim to find a completely general solution. We can simplify the search for a solution if we consider the electric field constant in some directions. Definitely, we want to preserve dependence on coordinates z and t, as we want to describe a wave traveling in the direction of the z axis. Unfortunately, if we were to leave the electric field constant in the xy planes, i.e., consider functions $\vec{E}(z, t)$, boundary conditions would force the solution to vanish (see below). Let's, therefore, add dependence on one more variable and study what happens. Let's take, for example, the electric field constant along the x axis, i.e., consider functions

$$\vec{E} = \vec{E}(y, z, t). \tag{6.103}$$

The ansatz $\vec{E}(y, z, t)$ leads to the nullification of the E_y and E_z components throughout the waveguide. Why? Boundary conditions for the top and bottom wall (6.102) talk about the field at points $x = 0$ and $x = a$, but our field is constant in the x direction. If it's zero at the edge, it must also be zero[8] for all x:

$$\begin{aligned} 0 = E_y(0, y, z, t) = E_y(y, z, t) = E_y(x, y, z, t) \quad & \forall x, y, z, t, \\ 0 = E_z(0, y, z, t) = E_z(y, z, t) = E_z(x, y, z, t) \quad & \forall x, y, z, t. \end{aligned} \tag{6.104}$$

Boundary conditions simplified the electric field $\vec{E}$ to the form

$$\vec{E} = (E_x, 0, 0), \qquad E_x = E_x(y, z, t). \tag{6.105}$$

Gauss's law is then automatically satisfied,

$$\operatorname{div} \vec{E} = \frac{\partial E_x(y, z, t)}{\partial x} = 0, \tag{6.106}$$

and from the wave equation, one component remains and the Laplace operator acts only in coordinates y and z:

$$\frac{\partial^2 E_x}{\partial t^2} = c^2 \Delta E_x = c^2 \left(\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right). \tag{6.107}$$

We are still left with the boundary condition for the E_x component from (6.101):

$$E_x(0, z, t) = E_x(b, z, t) = 0 \qquad \forall z, t \in \mathbb{R}. \tag{6.108}$$

[8] For the same reason, the E_x component also disappeared for a field that had dependence only $\vec{E}(z, t)$. From the boundary condition of the left and right wall (6.101), we would have gotten:

$$E_x(x, 0, z, t) = E_x(z, t) = E_x(x, y, z, t) \quad \forall x, y, z, t.$$

As we have already mentioned, we want to find an electric field propagating along the z axis. Let's consider the following ansatz:

$$E_x(y,z,t) = Y(y)e^{i(\omega t - k_z z)}. \tag{6.109}$$

We wrote the electric field as a traveling wave in the direction of the z axis with an unknown function Y of the coordinate y. We have denoted the wave number corresponding to the z direction with k_z. Individual derivatives in the wave equation (6.107) lead to expressions

$$\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 Y(y)e^{i(\omega t - k_z z)}, \qquad \frac{\partial^2 E_x}{\partial z^2} = -k_z^2 Y(y)e^{i(\omega t - k_z z)}, \qquad \frac{\partial^2 E_x}{\partial y^2} = Y''(y)e^{i(\omega t - k_z z)}, \tag{6.110}$$

and after substituting them into (6.107) and canceling the exponential, we get an ordinary differential equation for the function $Y(y)$ in the form

$$Y'' + \underbrace{\left(\tfrac{\omega^2}{c^2} - k_z^2\right)}_{K} Y = 0. \tag{6.111}$$

Boundary conditions (6.108) reduce to the following conditions on the function $Y(y)$:

$$Y(0) = 0, \qquad Y(b) = 0. \tag{6.112}$$

For $K \leq 0$, solutions are hyperbolic functions ($K < 0$) or linear functions ($K = 0$), and these cannot satisfy the boundary conditions non-trivially. For $K > 0$, we obtain the equation

$$Y'' + k_y^2 Y = 0, \tag{6.113}$$

where we have denoted as the wave number in the direction y,

$$k_y = \sqrt{\frac{\omega^2}{c^2} - k_z^2}. \tag{6.114}$$

The solution (6.113) can be written in the form

$$Y(y) = \alpha \cos(k_y y) + \beta \sin(k_y y). \tag{6.115}$$

Let's find the form of the solution that satisfies the boundary conditions (6.108):

$$0 = Y(0) = \alpha \quad \Rightarrow \quad 0 = Y(b) = \beta \sin(k_y b). \tag{6.116}$$

We must set $\alpha = 0$, and if we require a non-trivial solution, $\beta \neq 0$ and simultaneously

$$k_y b = m\pi, \qquad m \in \mathbb{N} \tag{6.117}$$

(since $k_y b > 0$, $m \in \mathbb{N}$). Thus, in the waveguide, through the form of the function $Y(y)$, there exist modes numbered by natural numbers $m \in \mathbb{N}$ in the form of sin functions. This is very similar to a string with fixed ends—fixed ends forced the displacement to vanish at the points of fixation. Here, the electric field must vanish at the walls due to the "fixed ends" of perfect conductivity. The permissible wave numbers for the function $Y(y)$ and the corresponding functions $Y_m(y)$ are thus in the form

$$k_{y(m)} = \frac{m\pi}{b}, \qquad Y_m(y) = \beta \sin\left(k_{y(m)} y\right) = \beta \sin\left(\frac{m\pi y}{b}\right), \qquad m \in \mathbb{N} \tag{6.118}$$

Now, let's put all the information together. The electric field in the waveguide can be excited in various modes numbered by a natural number $m \in \mathbb{N}$ in the form

$$\boxed{\vec{E} = (E_x, 0, 0), \qquad E_x(y, z, t) = E_0 \sin\left(\frac{m\pi y}{b}\right) e^{i(\omega t - k_z z)},} \tag{6.119}$$

(where we started writing E_0 instead of β). Meanwhile, constants $k_y = \frac{m\pi}{b}$, k_z, and ω must satisfy the dispersion relation (resulting from (6.114)):

$$\boxed{\omega^2 = c^2\left(k_y^2 + k_z^2\right) = \left(\frac{m\pi c}{b}\right)^2 + c^2 k_z^2.} \tag{6.120}$$

Denoting $\omega_{\min(m)} = \frac{m\pi c}{b}$, we can write the dispersion relation for the waveguide in the form

$$\boxed{\omega^2 = \omega_{\min(m)}^2 + c^2 k^2,} \tag{6.121}$$

where we also stopped writing the index on k_z, as, although we have an electric field in space, it effectively acts as a traveling wave only in the direction z, so there is no dispute that the wave number k (formerly k_z) describes the wavelength of the electric field in the direction z. We can therefore calculate the phase and group velocity of the electromagnetic wave traveling through the waveguide along the z axis as

$$v_\varphi = \frac{\omega}{k}, \qquad v_g = \frac{d\omega}{dk}, \tag{6.122}$$

thus entirely the same as for one-dimensional wave propagation.

From the dispersion relation (6.121), it follows that for a given mode (i.e., for a given natural number m), there exists a minimum angular frequency $\omega_{\min(m)}$, which can still propagate through the waveguide. For $\omega > \omega_{\min(m)}$, it is a transparent environment; for $\omega < \omega_{\min(m)}$, it is a reactive environment. The lowest mode for $m = 1$ also has the lowest minimum angular frequency

$$\omega_{\min} = \omega_{\min(1)} = \frac{\pi c}{b}. \tag{6.123}$$

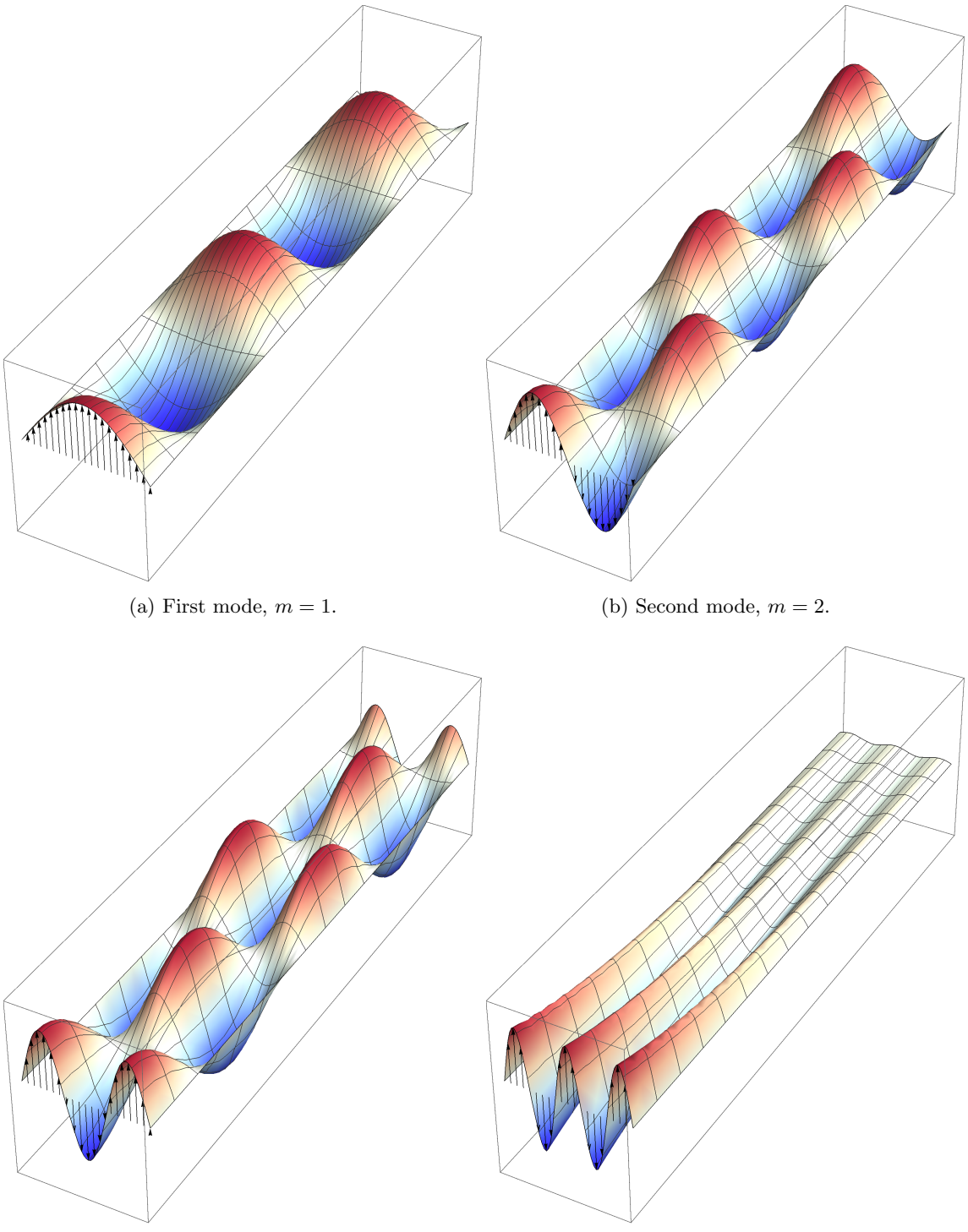
For a waveguide of size $b = 10\text{cm}$, that is $f_{\min} = \frac{\omega_{\min}}{2\pi} = 1,5\text{GHz}$. For frequencies ω in the reactive regime, we can easily find a solution to the dispersion relation in the form of $k = i\kappa$, i.e.,

$$\boxed{\omega^2 = \omega_{\min(m)}^2 - c^2\kappa^2,} \tag{6.124}$$

and the resulting wave in the waveguide will be an exponentially damped standing wave in the form

$$\boxed{E_x(y, z, t) = E_0 \sin\left(\frac{m\pi y}{b}\right) e^{-\kappa z} e^{i\omega t}.} \tag{6.125}$$

The waveguide thus leads the electromagnetic wave without reducing the amplitude, but its dimensions must be sufficient for it to lead the wave of a given frequency at all. If the dimensions of the waveguide are not sufficient, the wave does not propagate through the waveguide but forms an exponentially damped standing wave. Illustrations of individual modes are shown in Figure 6.21.



(a) First mode, $m = 1$.

(b) Second mode, $m = 2$.

(c) Third mode, $m = 3$.

(d) Fifth damped mode, $m = 5$.

Figure 6.21: Illustration of the electric field in the waveguide for several modes. Waves are shown for a fixed angular frequency ω. As the mode number m increases, the minimum angular frequency $\omega_{\min(m)}$ increases until inevitably, from a certain m_0, the wave stops propagating; here it happened for $m_0 = 5$ in the last image. The electric field is constant along the x axis, so the fields in a selected plane $x = x_0$ are illustrated in the images. At the beginning of the waveguide, vectors of the electric field intensity $\vec{E}$ are directly illustrated (in black), while elsewhere, the intensity of the electric field is continuously plotted (in color) without showing individual vectors. For propagating modes ($m < m_0$), the wave as a whole moves through the waveguide with the phase velocity $v_{\varphi(m)}$ (different for each mode), and for non-propagating modes ($m \geq m_0$), a standing wave is formed.

6.5 Boundary Conditions for EM Fields at the Interface of Non-conductive Media

Consider two homogeneous media composed of linear dielectrics and magnetics described by permittivities ε_1 and ε_2 and permeabilities μ_1 and μ_2. We will now derive the so-called *boundary conditions* for the electric and magnetic fields at the interface of these two media, which will specify the relationship between the values of the electric and magnetic fields on one side of the interface and the other.

Write Maxwell's equations in material media for generally variable permittivity $\varepsilon(\vec{r})$ and permeability $\mu(\vec{r})$:

$$\begin{aligned} &\operatorname{div}(\varepsilon\vec{E}) = \rho_{\text{free}} \quad \text{(Gauss's law)}, && \operatorname{div}\vec{B} = 0 \quad (\vec{B} \text{ is solenoidal}), \\ &\operatorname{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{(Faraday's law)}, && \operatorname{rot}\left(\frac{1}{\mu}\vec{B}\right) = \vec{j}_{\text{free}} + \frac{\partial(\varepsilon\vec{E})}{\partial t} \quad \text{(Ampère-Maxwell's law)}. \end{aligned} \tag{6.126}$$

Non-conductive media are characterized by the absence of free charges and currents:

$$\rho_{\text{free}} = 0, \qquad \vec{j}_{\text{free}} = 0. \tag{6.127}$$

The general strategy will be to rewrite each of Maxwell's equations into integral form, apply Gauss's or Stokes's theorem, and appropriately choose the volume/surface/curve of integration to provide relationships between fields on one and the other side of the interface.

Start with Gauss's law. Integrating over a given volume V and using Gauss's theorem, we get

$$\int_V \operatorname{div}(\varepsilon\vec{E})\, dV = \oint_S \varepsilon\vec{E}\cdot d\vec{S} = 0, \tag{6.128}$$

where the closed surface S is the boundary of volume V, $\partial V = S$. The right side is zero due to the absence of free charges. The element of surface in the integral is $d\vec{S} = \vec{n}\, dS$, where $\vec{n}$ is the unit normal vector to surface S. Now choose the volume/surface V/S as a cylinder whose axis is perpendicular to the interface of the two media as shown in Figure 6.22.

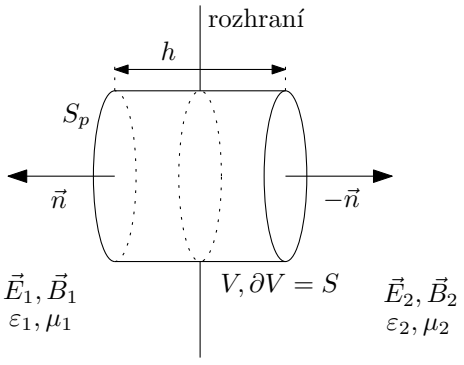


Figure 6.22: Cylindrical surface S placed with its axis perpendicular to the interface of two homogeneous media. The surface area of the base is S_p and the height of the cylinder is h. The unit normal vector to the base in the first medium is denoted $\vec{n}$, for the second base it is then $-\vec{n}$.

Divide the integration area into both bases and the mantle. Thus, the flux Φ through the entire cylinder can be split into $\Phi = \Phi_{\text{mantle}} + \Phi_{\text{p1}} + \Phi_{\text{p2}}$:

$$\Phi = \Phi_{\text{mantle}} + \left(\varepsilon_1\vec{E}_1\cdot\vec{n} - \varepsilon_2\vec{E}_2\cdot\vec{n}\right) S_p = 0, \tag{6.129}$$

where the integral over the small base area was written[9] as the product of the area S_p and the value of the scalar product $\vec{E}\cdot\vec{n}$ at that location. The flux through the mantle can be estimated as $|\Phi_{\text{mantle}}| \leq \alpha\, S_{\text{mantle}} = \alpha\,(2\pi r)h$, where α is the maximum of the function $|\vec{E}\cdot\vec{n}|$ on the surface of the mantle and r is the radius of the base. In the limit $h \to 0$, the flux through the mantle disappears and after cancelling S_p from (6.129), we obtain

$$\vec{n}\cdot\left(\varepsilon_1\vec{E}_1 - \varepsilon_2\vec{E}_2\right) = 0, \tag{6.130}$$

where $\vec{E}_1$ and $\vec{E}_2$ represent the values of the electric field from one and the other side of the interface. The scalar product $\vec{E}\cdot\vec{n}$ geometrically represents the magnitude of the projection of vector $\vec{E}$ in the direction of the normal vector, thus representing the magnitude of the normal component of the electric field $\vec{E}_\perp$. From Gauss's law, it follows that *the normal components of the electric field have a discontinuity at the interface* given by the ratio of the respective permittivities of the media:

$$\boxed{\varepsilon_1\vec{E}_{1\perp} = \varepsilon_2\vec{E}_{2\perp}, \qquad \frac{E_{1\perp}}{E_{2\perp}} = \frac{\varepsilon_2}{\varepsilon_1}.} \tag{6.131}$$

The same procedure is applied to the second divergent Maxwell's equation, i.e., the solenoidality of the magnetic field. The integral form

$$\int_V \operatorname{div}\vec{B}\,dV = \oint_S \vec{B}\cdot d\vec{S} = 0 \tag{6.132}$$

after choosing a cylindrical surface as before leads to

$$\vec{n}\cdot\left(\vec{B}_1 - \vec{B}_2\right) = 0, \tag{6.133}$$

or *the normal components of the magnetic field are continuous at the interface*:

$$\boxed{\vec{B}_{1\perp} = \vec{B}_{2\perp}.} \tag{6.134}$$

Next, consider Ampère-Maxwell's law in (6.126), whose integral form is obtained by integrating over the surface S,

$$\int_S \operatorname{rot}\left(\frac{1}{\mu}\vec{B}\right)\cdot d\vec{S} = \int_S \frac{\partial(\varepsilon\vec{E})}{\partial t}\cdot d\vec{S} \tag{6.135}$$

and applying Stokes's theorem to the left side of the equation, we get

$$\oint_l \frac{1}{\mu}\vec{B}\cdot d\vec{l} = \int_S \frac{\partial(\varepsilon\vec{E})}{\partial t}\cdot d\vec{S}, \tag{6.136}$$

where the closed curve l is the boundary of the surface S, $l = \partial S$. The element of length in the integral is $d\vec{l} = \vec{t}\,dl$, where $\vec{t}$ is the unit tangent vector to curve l, the element of surface

[9] We are essentially using the generalized mean value theorem of integration. The integral of a continuous scalar function $\vec{E}\cdot n$ over a compact (bounded and closed) surface S can be written as the size of this surface times the function value at some point of the surface:

$$\int_S \vec{E}\cdot\vec{n}\,dS = (\vec{E}\cdot\vec{n})(\vec{\xi})\,S = \vec{E}(\vec{\xi})\cdot\vec{n}(\vec{\xi})\,S,$$

where the point $\vec{\xi}$ is an unspecified point on the surface S. Performing the limit $S \to 0$ we get the function value $(\vec{E}\cdot\vec{n})$ at the chosen point around which we constructed the surface S.

is again $d\vec{S} = \vec{n}\, dS$. Choose the surface S (or its boundary l) as a rectangle whose surface is perpendicular to the interface of the media, as shown in Figure 6.23.

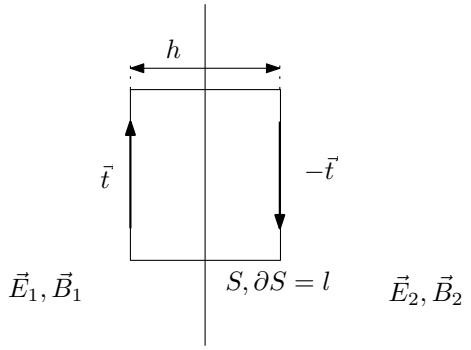


Figure 6.23: Rectangular surface S, or curve l, placed perpendicular to the interface of the media. Let's denote the dimension of the rectangle across the interfaces as h and along the interface as l. The unit tangent vector to the left side of the rectangle is $\vec{t}$, then the tangent vector to the right side maintaining the direction of traversal is $-\vec{t}$.

First, let's look at the integral on the right side of (6.136). This can be absolutely estimated from above by the expression $\alpha\, S = \alpha\, l\, h$, where α is the maximum of the function $\left|\frac{\partial(\varepsilon\vec{E})}{\partial t} \cdot \vec{n}\right|$ and S is the area of the chosen rectangle. In the limit $h \to 0$, this integral therefore disappears. The integration on the left side of (6.136) over the curve l is divided into individual sides. Analogous argumentation as before shows that the integrals over the top and bottom side of the rectangle disappear in the limit $h \to 0$. Therefore, in the integral form of Ampère-Maxwell's law after performing the limit $h \to 0$, only remains:

$$\frac{1}{\mu_1}\vec{B}_1 \cdot \vec{t}\, l - \frac{1}{\mu_2}\vec{B}_2 \cdot \vec{t}\, l = 0, \tag{6.137}$$

where (for reminder) l is the length of the left and right side of the rectangle and $\vec{B}_1$ and $\vec{B}_2$ are the values of magnetic fields from one and the other side of the interface. After cancelling[10] l:

$$\vec{t} \cdot \left(\frac{\vec{B}_1}{\mu_1} - \frac{\vec{B}_2}{\mu_2}\right) = 0. \tag{6.138}$$

The scalar product with vector $\vec{t}$ gives the magnitude of the projection in the direction tangent to the interface. Since we could have initially chosen the orientation of the rectangular surface S arbitrarily, the relation (6.138) must hold for any tangent vector $\vec{t}$. This gives us the following result

$$\boxed{\frac{B_{1\parallel}}{\mu_1} = \frac{B_{2\parallel}}{\mu_2}, \qquad \frac{B_{1\parallel}}{B_{2\parallel}} = \frac{\mu_1}{\mu_2},} \tag{6.139}$$

meaning that *the tangential components of the magnetic field have a discontinuity at the interface* given by the ratio of the respective permeabilities of the media.

Exactly the same procedure is also applied in the case of Faraday's law. The integral form

$$\oint_l \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \tag{6.140}$$

[10] And also the limit $l \to 0$, so that we again use the integral mean value theorem to get the values of fields $\vec{B}_1$ and $\vec{B}_2$ at well-defined places.

leads after choosing a rectangular surface as in the previous case to the condition

$$\vec{t}\cdot\left(\vec{E}_1-\vec{E}_2\right)=0 \tag{6.141}$$

and thus *the tangential components of the electric field are continuous at the interface*:

$$\boxed{\vec{E}_{1\parallel}=\vec{E}_{2\parallel}.} \tag{6.142}$$

Finally, let's summarize the found boundary conditions for the electric and magnetic fields at the interface of two non-conductive media into a table (with the law from which they derive):

$$\begin{aligned}
&\varepsilon_1\vec{E}_{1\perp}=\varepsilon_2\vec{E}_{2\perp}\quad\text{(Gauss's law)}, && \vec{B}_{1\perp}=\vec{B}_{2\perp}\quad(\vec{B}\text{ is solenoidal}),\\
&\vec{E}_{1\parallel}=\vec{E}_{2\parallel}\quad\text{(Faraday's law)}, && \frac{1}{\mu_1}\vec{B}_{1\parallel}=\frac{1}{\mu_2}\vec{B}_{2\parallel}\quad\text{(Ampère-Maxwell's law)}.
\end{aligned} \tag{6.143}$$

Chapter 7

Polarization

We have shown that an electromagnetic traveling wave is a transverse wave. The electric and magnetic fields are perpendicular to the direction of propagation. This means that there are two independent directions for the electric and magnetic field vectors – we say that we have two independent *polarizations*. We will focus on describing only the electric part of the wave $\vec{E}$, the magnetic part is then fully determined by the relation $\vec{B} = \frac{1}{v}\vec{s} \times \vec{E}$, where $\vec{s}$ is the vector of the direction of propagation. Let's choose coordinates so that the direction of propagation is $\vec{s} = \vec{z}$, then we can choose vectors $\vec{x}$ and $\vec{y}$ as the basis of the transverse plane in which the vectors $\vec{E}$ lie. Therefore, the general vector of the electric field can be decomposed into components in the direction of the x and y axes:

$$\vec{E} = E_x\vec{x} + E_y\vec{y}. \tag{7.1}$$

If we consider harmonic traveling waves, we can choose for each of these components a wave with different amplitude and with different phase shift on the principle of superposition:

$$\vec{E}(\vec{r}, t) = E_{x0}e^{i(\omega t - kz + \varphi_1)}\,\vec{x} + E_{y0}e^{i(\omega t - kz + \varphi_2)}\,\vec{y}. \tag{7.2}$$

Graphically, these two traveling components in mutually perpendicular directions $\vec{x}$ and $\vec{y}$ are illustrated in figure (7.1).

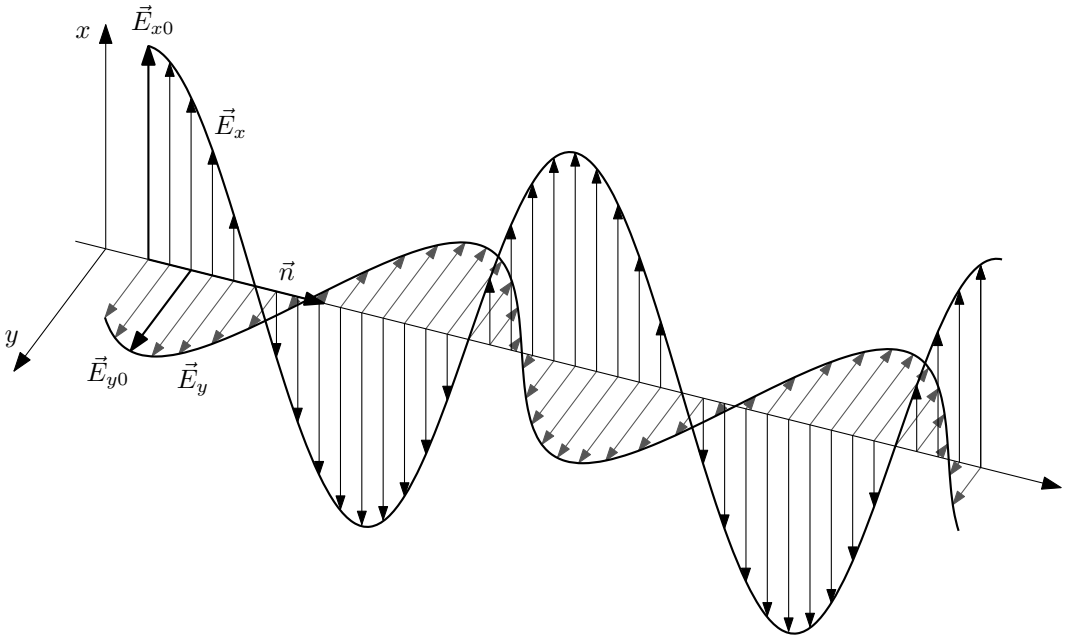


Figure 7.1: Two linearly independent directions of the electric field – each wave can generally have different amplitude, here E_{x0} and E_{y0}, and different phase shift.

The phase difference of these two components $\delta\varphi$,

$$\delta\varphi = (\omega t - kz + \varphi_1) - (\omega t - kz + \varphi_2) = \varphi_1 - \varphi_2, \tag{7.3}$$

does not depend on time or place in space. If we now choose any place $z = z_0$, we can observe the time evolution of the electric field $\vec{E}(t) = \vec{E}(z_0, t)$:

$$\vec{E}(t) = E_{x0}\,\vec{x}\,e^{i(\omega t+\varphi_1')} + E_{y0}\,\vec{y}\,e^{i(\omega t+\varphi_2')}, \tag{7.4}$$

where $\varphi_i' = \varphi_i - kz_0$, again it holds

$$\delta\varphi = \varphi_1 - \varphi_2 = \varphi_1' - \varphi_2'. \tag{7.5}$$

For the phase difference $\delta\varphi$, it does not matter in which specific place we observe the progression of the electric field. Therefore, let's stop distinguishing specific values of phases φ_1 and φ_2 (i.e., for example, stop writing the primes in expression (7.4)), but distinguish only their difference $\delta\varphi$. At the same time, a shift in space by Δz will achieve a phase shift in both waves of $-k\Delta z$, similarly, a shift in time by Δt will achieve a phase shift in both waves of $\omega\Delta t$. From this fact, it follows that in the expression (7.4) we can add a suitable phase to both exponentials (the same in both!), as it will be convenient in the given case – for example, to simplify the expression.

The real part from the expression (7.4),

$$\vec{E}(t) = E_{x0}\,\vec{x}\cos(\omega t + \varphi_1) + E_{y0}\,\vec{y}\cos(\omega t + \varphi_2), \tag{7.6}$$

is the parametric equation of an ellipse[1]. At a given point $z = z_0$, the electric field vector $\vec{E}(t)$ generally describes an elliptical curve, see figure 7.2. We say that the electromagnetic wave is *elliptically polarized*.

[1]This fact can be shown by converting the parametric form (where the parameter is time t) into the algebraic form for components E_x and E_y. We decompose the components of the electric field using sum formulas

$$E_x = E_{x0}\big(\cos\omega t\cos\varphi_1 - \sin\omega t\sin\varphi_1\big), \qquad E_y = E_{y0}\big(\cos\omega t\cos\varphi_2 - \sin\omega t\sin\varphi_2\big)$$

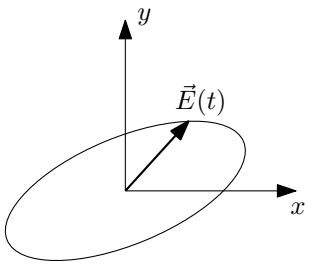


Figure 7.2: For general polarization, the electric field vector describes an ellipse at a selected point.

Let's look at the expression for the intensity of the electromagnetic wave in general polarization (7.6). The intensity is given as the time-average value of the absolute value of the energy flux:

$$I = \langle |\vec{S}| \rangle, \qquad \vec{S} = \sqrt{\frac{\varepsilon}{\mu}} |\vec{E}|^2 \, \vec{z}. \tag{7.7}$$

Therefore, let's compute the expression $|\vec{E}|^2 = \vec{E} \cdot \vec{E}$ for the wave (7.6),

$$\begin{aligned} |\vec{E}|^2 &= E_{x0}^2 \, \vec{x}^2 \cos^2(\omega t + \varphi_1) + E_{y0}^2 \, \vec{y}^2 \cos^2(\omega t + \varphi_2) + 2E_{x0}E_{y0}(\vec{x} \cdot \vec{y}) \cos(\omega t + \varphi_1) \cos(\omega t + \varphi_2) \\ &= E_{x0}^2 \cos^2(\omega t + \varphi_1) + E_{y0}^2 \cos^2(\omega t + \varphi_2) + 0, \end{aligned} \tag{7.8}$$

due to the perpendicularity of vectors $\vec{x}$ and $\vec{y}$ the interference term disappears and the total intensity is then simply the sum of intensities in the individual perpendicular directions (averaging over time gives a factor of $\frac{1}{2}$ from the $\cos^2$ functions):

$$I = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \big(E_{x0}^2 + E_{y0}^2 \big) = I_x + I_y. \tag{7.9}$$

The resulting intensity thus depends only on the amplitudes of the individual waves, not on their mutual phase. Specifically, the resulting intensity is proportional to the sum of the squares of the amplitudes

of the waves in mutually perpendicular directions.

Every wave of the form (7.4) can be represented by a complex two-component vector $\hat{\vec{E}} \in \mathbb{C}^2$, if we rewrite the form of the electric field $\vec{E}(t)$ (7.4) in the following way:

$$\vec{E}(z,t) = \big(E_{x0} e^{i\varphi_1} \vec{x} + E_{y0} e^{i\varphi_2} \vec{y} \big) \, e^{i(\omega t - kz)} = \hat{\vec{E}} e^{i(\omega t - kz)}, \qquad \hat{\vec{E}} = \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix} = \begin{pmatrix} E_{x0} \, e^{i\varphi_1} \\ E_{y0} \, e^{i\varphi_2} \end{pmatrix} \in \mathbb{C}^2. \tag{7.10}$$

The intensity of the wave using this vector is written as

$$I = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \|\hat{\vec{E}}\|^2 = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \big(|\hat{E}_1|^2 + |\hat{E}_2|^2 \big) = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} \big(E_{x0}^2 + E_{y0}^2 \big). \tag{7.11}$$

These relations can be seen as linear equations for functions $\sin \omega t$ and $\cos \omega t$. Solving these equations, we get

$$\cos \omega t = aE_x + bE_y, \qquad \sin \omega t = cE_x + dE_y,$$

where numbers a, b, c, d are given by specific values of E_{x0}, E_{y0}, φ_1, φ_2. Squaring and adding up, we get

$$1 = (aE_x + bE_y)^2 + (cE_x + dE_y)^2,$$

which is the equation of a conic section in variables E_x and E_y (it is a quadratic polynomial in E_x and E_y). Since the values E_x and E_y are bounded (by amplitudes E_{x0} and E_{y0}), it must be an ellipse or its degenerate cases (circle, line segment).

Now, let's look at two special cases of polarization. The case of *linear polarization* occurs if the electric field vector $\vec{E}(t)$ oscillates in one given direction, see figure 7.3, where the vector $\vec{E}(t)$ in the perpendicular plane xy is illustrated, and also see figure 7.4, where the electric field in space $\vec{E}(z, t_0)$ at a given time t_0 is shown.

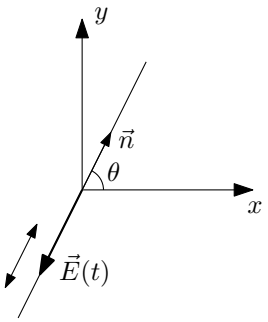


Figure 7.3: In linear polarization, the electric field vector $\vec{E}(t)$ harmonically oscillates in the xy plane (perpendicular to the direction of propagation along the z axis) along a unit direction vector $\vec{n} = (n_x, n_y)$.

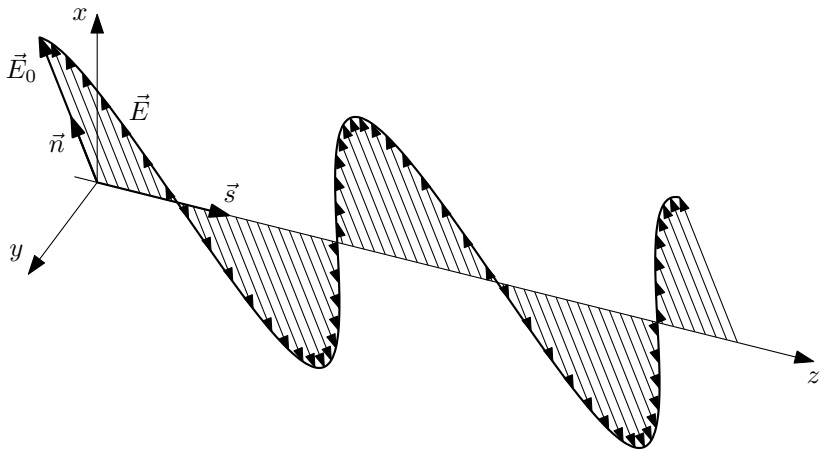


Figure 7.4: Spatial progression of the electric field in linear polarization. Illustrated is the electric field $\vec{E}(z, t_0)$ along the z axis at a given time t_0.

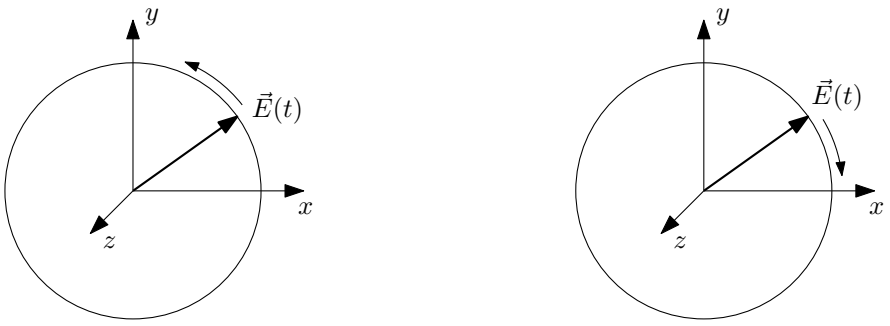
This case occurs for a phase shift $\delta\varphi \in \{0, \pi\}$, then the electric field (7.4) can be written in the form

$$\boxed{\vec{E}(t) = E_0\,\vec{n}\,e^{i(\omega t+\varphi)}, \qquad \hat{\vec{E}} = E_0\,\vec{n}\,e^{i\varphi} = E_0 \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} e^{i\varphi},} \tag{7.12}$$

where the unit vector $\vec{n} = (n_x, n_y) = (\cos\theta, \sin\theta)$ represents the direction of oscillation of the electric field $\vec{E}(t)$ in the xy plane and the angle θ is the deviation of this vector from the x axis.

The second special case is *circular polarization*, when the electric field vector $\vec{E}(t)$ describes a circle in the xy plane. In this case, we also distinguish two subcases according to the direction of rotation of the vector $\vec{E}(t)$. If, when viewed against the direction of propagation, the vector $\vec{E}(t)$ rotates counter-clockwise, we speak of *left-handed circular polarization*, and if it moves in the direction of clock hands, then it is *right-handed circular polarization*[2]. These polarizations illustrated in the xy plane are in figure 7.5. For circular polarization, the electric field vector $\vec{E}(z, t_0)$ at a given time forms a helix along the z axis, which is shown in figure 7.6.

[2]Unfortunately, there are two conventions. The second has definitions exactly opposite to what we have here.



(a) Left-handed circular polarization. (b) Right-handed circular polarization.

Figure 7.5: Circular polarization of the electromagnetic wave. The direction of rotation of the vector $\vec{E}(t)$ is determined when viewed against the direction of wave propagation (indicated by the z axis pointing towards us/out of the paper).

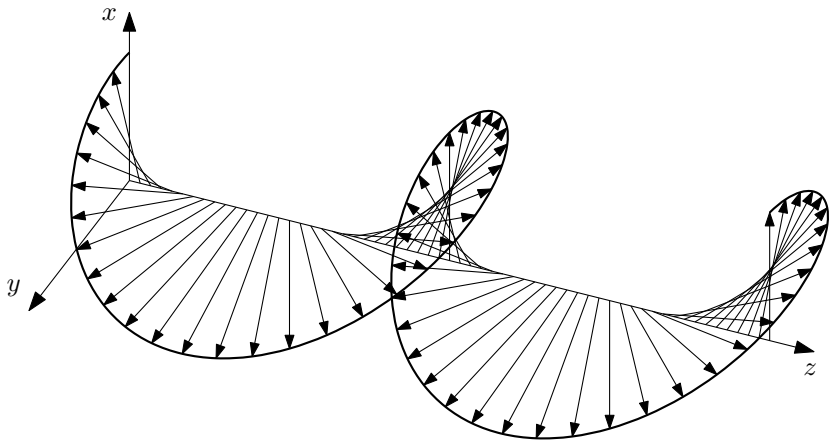


Figure 7.6: Spatial progression of the electric field for circular polarization. Illustrated is the electric field $\vec{E}(z, t_0)$ along the z axis at a given time t_0. In this case, it is right-handed polarization (the wave moves in the direction of the z axis and then in the given xy plane, the vector $\vec{E}(t)$ will rotate clockwise, whereas the helix itself along the z axis rotates counter-clockwise...).

Circular polarization is obtained for $\delta\varphi = \pm\frac{\pi}{2}$ and $E_{x0} = E_{y0} = E_0$. After substituting these conditions into (7.4), the result is the expression

$$\boxed{\vec{E}(t) = E_0\big(\vec{x}\cos(\omega t+\varphi) \underbrace{\pm\vec{y}\sin(\omega t+\varphi)}_{\vec{y}\cos(\omega t+\varphi\mp\frac{\pi}{2})}\big),} \tag{7.13}$$

which is the parametric equation of a circle. The plus sign with the sine (i.e., the phase $-\frac{\pi}{2}$) corresponds to left-handed polarization and the minus sign (i.e., the phase $+\frac{\pi}{2}$) to right-handed[3]. In the complex notation, we have

$$\vec{E}(t) = E_0\big(\vec{x}\,e^{i(\omega t+\varphi)} + \vec{y}\,e^{i(\omega t+\varphi\mp\frac{\pi}{2})}\big), \qquad \hat{\vec{E}} = E_0\begin{pmatrix}1\\ e^{\mp i\frac{\pi}{2}}\end{pmatrix}e^{i\varphi}. \tag{7.14}$$

[3]This fact is easily seen for small positive values of the phase $\omega t + \varphi$. Thanks to the cosine, the electric field vector then points in the direction of the x axis, and the sine of a small positive phase is also positive. Depending on the sign in front of the sine, we get a vector that has turned a little bit either into the positive or negative direction of the y axis.

If we do not have polarizations of the special types mentioned above, we speak, as already mentioned, of general elliptical polarization. This can again be divided into two categories: left-handed and right-handed,

see figure 7.7. In space, the electric field forms an elliptical helix.

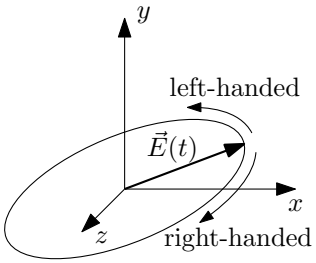


Figure 7.7: Left-handed and right-handed elliptical polarization.

7.1 Polarizer and Wave Plate

In the previous chapter, we introduced the formalism for describing polarization states, and now we will focus on how to change polarization states. For this purpose, we describe two basic optical elements, the polarizer and the wave plate, which we will place in the path of a traveling electromagnetic wave.

7.1.1 Polarizer

A *polarizer* is an optical element that allows only the component of the electric field that oscillates in the direction of the polarizer's axis – called the *transmission axis* – described by the direction vector $\vec{n} = (n_x, n_y)$, see Figure 7.8.

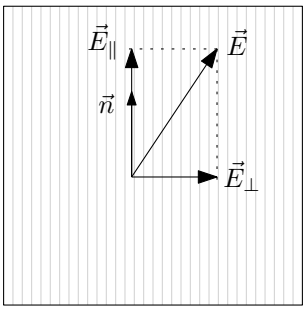
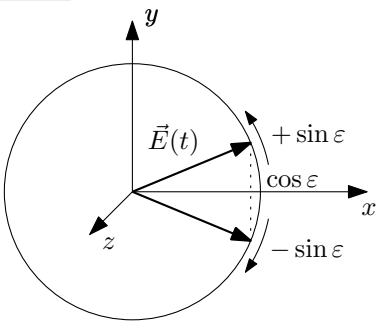


Figure 7.8: From the electric field $\vec{E}$ incident on the polarizer, only the component $\vec{E}_\parallel$ parallel to the transmission axis $\vec{n}$ is transmitted.



Thus, if the electric field entering the polarizer is decomposed into $\vec{E}_{\text{in}} = \vec{E}_{\parallel} + \vec{E}_{\perp}$, then the output field will be $\vec{E}_{\text{out}} = \vec{E}_{\parallel}$. The action of the polarizer can be described by projecting the vector $\vec{E}_{\text{in}}$ onto the direction $\vec{n}$ using the scalar product in the following way:

$$\boxed{\vec{E}_{\text{out}} = \left(\vec{E}_{\text{in}} \cdot \vec{n}\right) \vec{n},} \tag{7.15}$$

where $\vec{n}$ is the *unit* vector in the direction of the transmission axis. Furthermore, we want to encode the action of the polarizer into an appropriate transformation of vectors $\hat{\vec{E}}_{\text{in}}, \hat{\vec{E}}_{\text{out}} \in \mathbb{C}^2$ in complex notation. The transformation (7.15) is linear, and thus it can be encoded into a matrix $\mathbb{P}_{\vec{n}} \in \mathbb{C}^{2,2}$ – projector onto the axis $\vec{n}$, relating vectors $\hat{\vec{E}}_{\text{in}}, \hat{\vec{E}}_{\text{out}}$ as follows:

$$\hat{\vec{E}}_{\text{out}} = \mathbb{P}_{\vec{n}} \hat{\vec{E}}_{\text{in}}. \tag{7.16}$$

Let's find the expression of this matrix by expanding the relationship (7.15):

$$\vec{E}_{\text{out}} = \left(E_x n_x + E_y n_y\right) \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} n_x^2 E_x + n_x n_y E_y \\ n_x n_y E_x + n_y^2 E_y \end{pmatrix} = \underbrace{\begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix}}_{\mathbb{P}_{\vec{n}}} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathbb{P}_{\vec{n}} \vec{E}_{\text{in}}. \tag{7.17}$$

Introducing the angle θ as the angle of deviation of the vector $\vec{n}$ from the x-axis, we get the expression $\vec{n} = (\cos\theta, \sin\theta)$ and can write

$$\mathbb{P}_{\vec{n}} = \mathbb{P}_{\theta} = \begin{pmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{pmatrix}. \tag{7.18}$$

Let's consider examples of projectors on the basic transmission axes: $\vec{x}, \vec{y}, \frac{\vec{x}+\vec{y}}{\sqrt{2}}$ (
the transmission axis must be unitary):

$$\mathbb{P}_{\vec{x}} = \mathbb{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{P}_{\vec{y}} = P_{\frac{\pi}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{P}_{\frac{\vec{x}+\vec{y}}{\sqrt{2}}} = \mathbb{P}_{\frac{\pi}{4}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{7.19}$$

Consider a linearly polarized wave incident on a polarizer. How does the intensity of the wave change after passing through the polarizer? Without loss of generality, let's direct the x-axis in the direction of the vector of the incoming polarized light, i.e., $\vec{E}_{\text{in}} = E_0 \vec{x} \cos\omega t$. Let the transmission axis be generally $\vec{n} = (\cos\theta, \sin\theta)$, where the angle θ now describes the angle between the direction of oscillation of the linearly polarized wave and the transmission axis of the polarizer. Substituting into (7.15), the output amplitude vector comes out as

$$\vec{E}_{0\text{out}} = E_0(\vec{x} \cdot \vec{n})\, \vec{n} = (E_0 \cos\theta)\, \vec{n} \tag{7.20}$$

and thus the output wave will be in the form $\vec{E}_{\text{out}} = (E_0 \cos\theta)\, \vec{n} \cos\omega t$. Substituting the forms of the input and output waves into the relationship for wave intensity (7.7) we get

$$I_{\text{out}} = \sqrt{\frac{\varepsilon}{\mu}} \left\langle |\vec{E}_{\text{out}}|^2 \right\rangle = \sqrt{\frac{\varepsilon}{\mu}} \cos^2\theta \left\langle E_0^2 \cos^2\omega t \right\rangle = I_{\text{in}} \cos^2\theta. \tag{7.21}$$

This relationship between intensities is called *Malus's Law*:

$$\boxed{I_{\text{out}} = I_{\text{in}} \cos^2\theta.} \tag{7.22}$$

7.1.2 Wave Plate

Let's now move on to the second optical element, which is the *wave plate*. A wave plate allows changing the phase difference $\varphi_1 - \varphi_2$ between individual perpendicular components of the electric field. How does it achieve this? First, let's look at how much the actual phase of the wave changes after passing through a material of thickness d with a refractive index n. The situation is schematically shown in Figure 7.9. In the material, the wave takes the form $e^{i(\omega t - kz)}$, where k is the wave number given by the dispersion relation $\omega = \frac{c}{n}k$. Beyond the plate, the phase is shifted by $-kd$ compared to before the plate. However, the phase changes the same way for both perpendicular components of the electric field, so there is no change in the phase difference...

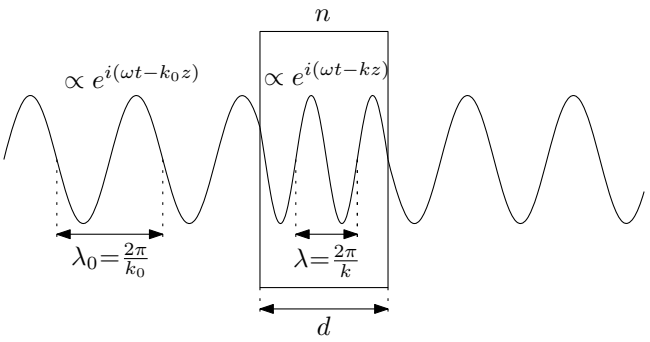


Figure 7.9: Wave passage through a material with refractive index n. In the material, the wave takes the form $e^{i(\omega t - kz)}$, where k is the wave number given by the dispersion relation $\omega = \frac{c}{n}k$.

To achieve this, we need a material, a so-called *birefringent crystal*, which behaves as a material with different refractive indices for waves polarized in different directions, see the schematic Figure 7.10. Such behavior arises due to the anisotropy of the given substance. The substance's response to the passing wave of the electric field varies depending on the orientation of the electric field vector (depending on the polarization of light).

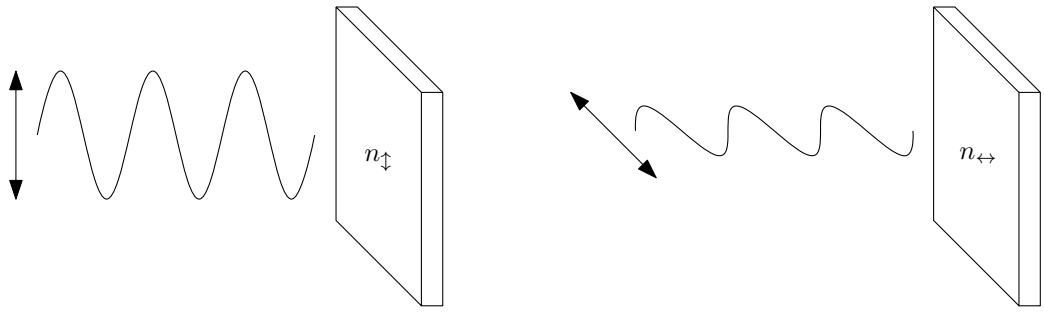


Figure 7.10: Birefringent crystal. For the electric field oriented in the vertical direction, it behaves as a material with refractive index $n_{\updownarrow}$, for the electric field oriented perpendicular to the vertical field, i.e., oscillating in the horizontal direction here, it behaves as a material with refractive index $n_{\leftrightarrow}$.

A wave plate made of birefringent crystal is parameterized by two perpendicular axes, $\vec{n}_1$ and $\vec{n}_2$, to which correspond different refractive indices n_1 and n_2 (note, these are not the magnitudes of vectors $\vec{n}_1$ and $\vec{n}_2$, which are unitary, $|\vec{n}_1| = |\vec{n}_2| = 1$) and its thickness d, or change in phase difference $\varphi_1 - \varphi_2$ by $\Delta\varphi$.

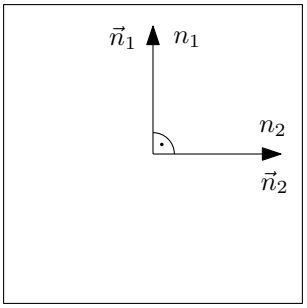


Figure 7.11: Labeling of the axes in the wave plate made of birefringent crystal. Perpendicular axes of the plate are in the direction of $\vec{n}_1$ and $\vec{n}_2$ and the corresponding refractive indices are n_1 and n_2.

How does the magnitude of the phase shift $\Delta\varphi$ depend on the values n_1, n_2, and d? As we have already mentioned, the phase before and after the plate generally differs by $-kd$, here for individual components of the electric field $-k_1 d$ and $-k_2 d$. Therefore, if the electric field at the beginning of the plate takes the form

$$\vec{E}_{\text{in}}(t) = E_1\,\vec{n}_1\cos(\omega t + \varphi_1) + E_2\,\vec{n}_2\cos(\omega t + \varphi_2), \tag{7.23}$$

the field at the end of the plate will be

$$\vec{E}_{\text{out}}(t) = E_1\,\vec{n}_1\cos(\omega t + \varphi_1 - k_1 d) + E_2\,\vec{n}_2\cos(\omega t + \varphi_2 - k_2 d). \tag{7.24}$$

The phase difference changes from $\varphi_1 - \varphi_2$ to $\varphi_1 - \varphi_2 + \Delta\varphi$, where $\Delta\varphi$ is

$$\Delta\varphi = (k_2 - k_1)d = \frac{\omega}{c}(n_2 - n_1)d. \tag{7.25}$$

Using the dispersion relation in vacuum $\omega = ck_0 = \frac{2\pi c}{\lambda_0}$, we can express the phase shift $\Delta\varphi$ in terms of the vacuum wavelength λ_0 as

$$\boxed{\Delta\varphi = \frac{2\pi}{\lambda_0}(n_2 - n_1)d.} \tag{7.26}$$

This phase shift $\Delta\varphi$ is added to the part of the wave corresponding to the refractive index n_1. This can be easily seen if we add the value $k_2 d$ to the phases of both waves in expression (7.24) (which is an operation not changing the polarization state), then we get

$$\vec{E}_{\text{out}}(t) = E_1\,\vec{n}_1\cos\big(\omega t + \varphi_1 + \underbrace{(k_2 - k_1)d}_{\Delta\varphi}\big) + E_2\,\vec{n}_2\cos\big(\omega t + \varphi_2\big). \tag{7.27}$$

Considering $\Delta\varphi$ to be positive only, which requires us to label the refractive indices such that $n_1 < n_2$, we obtain a practical rule: a positive phase shift $\Delta\varphi$ is added to the component of the electric wave corresponding to the smaller refractive index.

A plate causing a phase difference change of $\Delta\varphi = \frac{\pi}{2}$ is called a *quarter-wave plate*, for a phase shift $\Delta\varphi = \pi$, it is called a *half-wave plate*.

How do we write the action of a wave plate with a phase shift $\Delta\varphi$ in complex notation? Or how do we define the matrix $\mathbb{D}_{\Delta\varphi}$ transforming the complex vector $\hat{\vec{E}}_{\text{in}} \in \mathbb{C}^2$ into the vector $\hat{\vec{E}}_{\text{out}} \in \mathbb{C}^2$ according to the prescription

$$\hat{\vec{E}}_{\text{out}} = \mathbb{D}_{\Delta\varphi}\hat{\vec{E}}_{\text{in}}? \tag{7.28}$$

First, we need to decompose the electric field wave into the directions of the wave plate axes $\vec{n}_1$ and $\vec{n}_2$. This can be done using projectors onto these axes, $\mathbb{P}_{\vec{n}_1}$ and $\mathbb{P}_{\vec{n}_2}$. For perpendicular unit vectors $\vec{n}_1$ and $\vec{n}_2$, it holds that $\mathbb{P}_{\vec{n}_1} + \mathbb{P}_{\vec{n}_2} = \mathbb{I}$, so we can write

$$\hat{\vec{E}}_{\text{in}} = \mathbb{P}_{\vec{n}_1}\hat{\vec{E}}_{\text{in}} + \mathbb{P}_{\vec{n}_2}\hat{\vec{E}}_{\text{in}}. \tag{7.29}$$

We obtain the output field by adding the phase shift $\Delta\varphi$ to the part corresponding to the smaller refractive index (considering $n_1 < n_2$):

$$\hat{\vec{E}}_{\text{out}} = e^{i\Delta\varphi}\mathbb{P}_{\vec{n}_1}\hat{\vec{E}}_{\text{in}} + \mathbb{P}_{\vec{n}_2}\hat{\vec{E}}_{\text{in}} = \big(e^{i\Delta\varphi}\mathbb{P}_{\vec{n}_1} + \mathbb{P}_{\vec{n}_2}\big)\hat{\vec{E}}_{\text{in}}. \tag{7.30}$$

Comparing the right side (7.30) and the definition of the matrix $\mathbb{D}_{\Delta\varphi}$ (7.28), we see that

$$\mathbb{D}_{\Delta\varphi} = e^{i\Delta\varphi}\,\mathbb{P}_{\vec{n}_1} + \mathbb{P}_{\vec{n}_2}. \tag{7.31}$$

Considering, for simplicity, $\vec{n}_1 = \vec{x}$ and $\vec{n}_2 = \vec{y}$ (and using the definition of projectors $\mathbb{P}_{\vec{x}}$ and $\mathbb{P}_{\vec{y}}$ from (7.19)) we get

$$\mathbb{D}_{\Delta\varphi} = e^{i\Delta\varphi}\,\mathbb{P}_{\vec{x}} + \mathbb{P}_{\vec{y}} = \begin{pmatrix} e^{i\Delta\varphi} & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.32}$$

7.2 Polarization Measurement

The angular frequency ω of the electromagnetic wave is usually too large for us

to directly measure the actual course of the electric field (for the visible light spectrum, frequencies are in the order of hundreds of THz). We are inevitably confined to measuring only time-averaged values of certain quantities. To determine the polarization state (and total intensity) described by the relationship (7.6), we need to determine the amplitude values in individual directions, E_{x0} and E_{y0}, and also the phase difference value $\varphi_1 - \varphi_2$. Let's show that we can determine these quantities by measuring the following intensities:

$$I_x = \langle E_x^2\rangle, \qquad I_y = \langle E_y^2\rangle, \qquad I_{xy} = \langle E_x E_y\rangle, \qquad I_{\overline{xy}} = \langle E_x(\omega t - \tfrac{\pi}{2})E_y\rangle, \tag{7.33}$$

where under the expression $E_x(\omega t - \frac{\pi}{2})$ we understand that in the component E_x we shift the phase by $-\frac{\pi}{2}$. If we calculate these intensities by substituting from the generally elliptically polarized light (7.6) we get

$$\begin{aligned}
I_x = \langle E_x^2\rangle &= E_{x0}^2\langle\cos^2(\omega t + \varphi_1)\rangle = \frac{1}{2}E_{x0}^2,\\
I_y = \langle E_y^2\rangle &= E_{y0}^2\langle\cos^2(\omega t + \varphi_2)\rangle = \frac{1}{2}E_{y0}^2,\\
I_{xy} = \langle E_x E_y\rangle &= E_{x0}E_{y0}\big\langle \underbrace{\cos(\omega t + \varphi_1)\cos(\omega t + \varphi_2)}_{\frac{1}{2}\left(\cos(2\omega t + \varphi_1 + \varphi_2) + \cos(\varphi_1 - \varphi_2)\right)} \big\rangle = \frac{1}{2}E_{x0}E_{y0}\cos(\varphi_1 - \varphi_2),\\
I_{\overline{xy}} = \langle E_x(\omega t - \tfrac{\pi}{2})E_y\rangle &= E_{x0}E_{y0}\big\langle \underbrace{\overbrace{\cos(\omega t + \varphi_1 - \tfrac{\pi}{2})}^{\sin(\omega t + \varphi_1)}\cos(\omega t + \varphi_2)}_{\frac{1}{2}\left(\sin(2\omega t + \varphi_1 + \varphi_2) + \sin(\varphi_1 - \varphi_2)\right)} \big\rangle = \frac{1}{2}E_{x0}E_{y0}\sin(\varphi_1 - \varphi_2).
\end{aligned} \tag{7.34}$$

From intensities I_x and I_y, we can calculate amplitudes E_{x0} and E_{y0}. Then, intensities I_{xy} and $I_{\overline{xy}}$ give the sin and cos of the phase difference, which uniquely determines it (in the interval $\langle 0, 2\pi)$).

How can we practically measure these four intensities? Intensities I_x, respectively I_y, are measured by placing a polarizer with the transmission axis $\vec{x}$, respectively $\vec{y}$, in the path of the light, these only transmit the components E_x, respectively E_y. The intensity I_{xy} is measured by placing a linear polarizer with the transmission axis $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$ in the path of the light. The output intensity after passing through this polarizer then comes out[4]

$$I_{\text{out}} = \frac{1}{2}(I_x + I_y) + I_{xy}. \tag{7.35}$$

The last intensity $I_{\overline{xy}}$ is determined by first placing a quarter-wave plate with the axis $\vec{n}_1 = \vec{y}$ (the axis corresponding to the smaller refractive index) in the path of the light – this adds a phase shift of $\frac{\pi}{2}$ to the component E_y, which is equivalent to subtracting a phase of $\frac{\pi}{2}$ in the component E_x – and then again a polarizer with the transmission axis $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. In this configuration, the output intensity is given by the relationship[5]

$$I_{\overline{\text{out}}} = \frac{1}{2}(I_x + I_y) + I_{\overline{xy}}. \tag{7.36}$$

Thus, intensities I_x and I_y are measured directly by inserting an appropriately oriented polarizer, intensities I_{xy} and $I_{\overline{xy}}$ are calculated from the measured intensities I_{out} and $I_{\overline{\text{out}}}$.

7.3 Unpolarized Light

So far, our description only includes the concept of (completely, perfectly) polarized light. For any values of the parameters E_{x0}, E_{y0}, φ_1, φ_2 in a wave of form (7.4), respectively (7.6), we have (completely, perfectly) polarized light. How then do we describe unpolarized light? What exactly is it? Roughly speaking, unpolarized light is such light whose polarization randomly changes over time. Imagine an atom (electron in an atom) that emits linearly polarized light by its oscillation. After a while, another atom collides with it and causes it to start oscillating in a different direction, thus changing the plane of polarization of the emitted light. This process is continuously repeated in the material, and thus the plane of polarization of the emitted light is constantly randomly changed. See the schematic Figure 7.12. Now, let's try to define and quantify this rough idea more carefully.

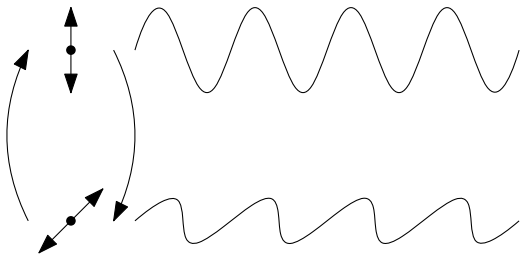


Figure 7.12: Oscillating atom emitting linearly polarized light. Due to external influences (collisions with other atoms), it randomly changes the direction of its oscillation, leading to a change in the plane of polarization of the emitted light.

First, let's introduce three important time scales that we will need when studying unpolarized light. The first scale is the period T of the electromagnetic oscillation itself. For visible light, it is approximately $T \approx 10^{-14}$ s.

[4] See example 9.4 in the exercises.

[5] Completely analogously as the previous relationship.

The second scale is the so-called *coherence time* t_{koh}. This time represents the duration for which a given polarization is preserved, i.e., in the initial illustrative example, the time during which the atom oscillates undisturbed. Typically, $t_{\mathrm{koh}} \approx 10^{-9} - 10^{-8}$ s. This time can also be defined using the concept of *temporal coherence*. We say that the field at a given location $\vec{E}(t)$ at times t_1 and t_2 are temporally coherent if knowing the field around time t_1 allows us to determine the field around time t_2 (and vice versa). The coherence time t_{koh} is thus the maximum distance of times $|t_1 - t_2|$ when the fields are still mutually coherent. If the field around t_2 cannot be predicted from knowing the field around t_1, we say that the fields are incoherent.

And the last scale is the resolution time t_{roz} of the instrument we use to measure the polarization state. The instrument measures the intensity of the incoming light, which is given by averaging over the resolution time, $I = \langle \vec{E}^2 \rangle_{t_{\mathrm{roz}}}$. Now let's distinguish two cases. First, if $t_{\mathrm{koh}} \gg t_{\mathrm{roz}}$ – in this case, we will talk about

a fast instrument. In this case, the instrument is capable of tracking changes in polarization, and we will simply measure completely polarized light, whose polarization changes with a period t_{koh}.

In the case that $t_{\mathrm{koh}} \ll t_{\mathrm{roz}}$ – we talk about a slow instrument, – the instrument is unable to track rapid random changes in polarization, and in this case, we will talk about the light hitting the instrument as unpolarized light (more precisely, as partially polarized or unpolarized light). Let's now look more closely at this intensity measurement by a slow instrument. We will show that by measuring a set of intensities I_x, I_y, I_{xy}, and $I_{\overline{xy}}$, we can distinguish polarized light from unpolarized (or partially polarized).

We generalize the notation of completely polarized light (7.6) so that the parameters E_{x0}, E_{y0}, φ_1, and φ_2 can be time-variable:

$$\vec{E}(t) = E_{x0}(t)\,\vec{x}\cos(\omega t + \varphi_1(t)) + E_{y0}(t)\,\vec{y}\cos(\omega t + \varphi_2(t)). \tag{7.37}$$

The functions $E_{x0}(t)$, $E_{y0}(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ change on the scale of coherence time t_{koh}. We can imagine that they change very slowly so that for the duration of t_{koh} they remain almost constant. Also, due to the inequality $T \ll t_{\mathrm{koh}}$, the change of these functions is much slower than the change in phase given by the term ωt. If we now perform averaging over one period T for the electric field (7.37), we can consider the functions $E_{x0}(t)$, $E_{y0}(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ as almost constant due to the relationship $T \ll t_{\mathrm{koh}}$, and the result of this averaging will be time-varying intensities,

$$\begin{aligned}
I_x(t) &= \frac{1}{2}E_{x0}^2(t),\\
I_y(t) &= \frac{1}{2}E_{y0}^2(t),\\
I_{xy}(t) &= \frac{1}{2}E_{x0}(t)E_{y0}(t)\cos(\varphi_1(t) - \varphi_2(t)),\\
I_{\overline{xy}}(t) &= \frac{1}{2}E_{x0}(t)E_{y0}(t)\sin(\varphi_1(t) - \varphi_2(t)),
\end{aligned} \tag{7.38}$$

which change on the scale t_{koh}. A slow instrument with a resolution time t_{roz} will then measure such intensities, which result from additional averaging over the resolution time t_{roz}:

$$I_x = \langle I_x(t)\rangle_{t_{\mathrm{roz}}}, \qquad I_y = \langle I_y(t)\rangle_{t_{\mathrm{roz}}}, \qquad I_{xy} = \langle I_{xy}(t)\rangle_{t_{\mathrm{roz}}}, \qquad I_{\overline{xy}} = \langle I_{\overline{xy}}(t)\rangle_{t_{\mathrm{roz}}}. \tag{7.39}$$

Let's see how much these intensities will be for the model of unpolarized light we outlined at the beginning – that is, for linearly polarized light, whose plane of polarization randomly changes. The electric field will take the form

$$\vec{E}(t) = E_0\,\vec{n}(t)\cos(\omega t + \varphi), \tag{7.40}$$

where we write the directional vector $\vec{n}(t)$ using the time-varying angle $\theta(t)$ as

$$\vec{n}(t) = (\cos\theta(t), \sin\theta(t)). \tag{7.41}$$

The function of angle $\theta(t)$ changes on the scale t_{koh}, and we consider that all angle values $\langle 0, 2\pi\rangle$ are evenly represented. The intensities (7.38) for light of form (7.40) are in the form

$$\begin{aligned}
I_x(t) &= \frac{1}{2}E_0^2\cos^2\theta(t),\\
I_y(t) &= \frac{1}{2}E_0^2\sin^2\theta(t),\\
I_{xy}(t) &= \frac{1}{2}E_0^2\cos\theta(t)\sin\theta(t)\cos(\varphi-\varphi) = \frac{1}{4}E_0^2\sin 2\theta(t),\\
I_{\overline{xy}}(t) &= \frac{1}{2}E_0^2\cos\theta(t)\sin\theta(t)\sin(\varphi-\varphi) = 0.
\end{aligned} \tag{7.42}$$

Averaging over the instrument's resolution time $t_{\mathrm{roz}} \gg t_{\mathrm{koh}}$ leads to

$$I_x = \frac{1}{4}E_0^2, \qquad I_y = \frac{1}{4}E_0^2, \qquad I_{xy} = 0, \qquad I_{\overline{xy}} = 0, \tag{7.43}$$

the average values $\langle\cos^2\theta(t)\rangle_{t_{\mathrm{roz}}} = \langle\sin^2\theta(t)\rangle_{t_{\mathrm{roz}}} = \frac{1}{2}$ and $\langle\sin 2theta(t)\rangle_{t_{\mathrm{roz}}} = 0$ come out exactly the same as standard time-averaged values of trigonometric functions over one period, since during the measurement duration t_{roz}, the angle θ changes many times and evenly covers the interval $\langle 0, 2\pi)$. Intensities I_{xy} and $I_{\overline{xy}}$ both came out zero, which is not possible for completely polarized light (sin and cos in (7.34) cannot be zero at the same time)! We see, therefore, that we can distinguish unpolarized light from polarized light by measurement.

In general, in unpolarized light, we have a condition on the functions $E_{x0}(t)$, $E_{y0}(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ arising precisely from the constancy of the total intensity:

$$I_0 = \langle\vec{E}^2\rangle = \langle E_x^2 + E_y^2\rangle = I_x + I_y = \mathrm{konst}. \tag{7.44}$$

In unpolarized light, the values of functions $E_{x0}(t)$, $E_{y0}(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ change randomly and evenly assume all permissible values. From this condition, it follows, among other things,

$$I_x = I_y \tag{7.45}$$

(it must not matter if we relabel the x and y axes). And further, for random angles $\varphi_1(t)$ and $\varphi_2(t)$, it holds that

$$\langle\cos(\varphi_1(t) - \varphi_2(t))\rangle_{t_{\mathrm{roz}}} = \langle\sin(\varphi_1(t) - \varphi_2(t))\rangle_{t_{\mathrm{roz}}} = 0. \tag{7.46}$$

From relationships (7.44), (7.45), and (7.46) for unpolarized light, it follows that

$$I_x = \frac{1}{2}I_0, \qquad I_y = \frac{1}{2}I_0, \qquad I_{xy} = 0, \qquad I_{\overline{xy}} = 0. \tag{7.47}$$

We can further consider a superposition of completely polarized light and unpolarized light. Then we get *partially polarized* light.

The polarization state and degree of polarization are described by so-called *Stokes parameters*, which are given by the relationships

$$P_1 = \frac{I_x - I_y}{I_x + I_y}, \qquad P_2 = \frac{2I_{xy}}{I_x + I_y}, \qquad P_3 = \frac{2I_{\overline{xy}}}{I_x + I_y}. \tag{7.48}$$

Generally, it holds (here without proof) that

$$P_1^2 + P_2^2 + P_3^2 \leq 1. \tag{7.49}$$

For completely polarized light, then $P_1^2 + P_2^2 + P_3^2 = 1$ (this part can be easily shown by substituting into (7.48) from (7.34)). For unpolarized light, we get $P_1 = P_2 = P_3 = 0$ (by substituting into (7.48) from (7.47)). Furthermore, we define the *degree of polarization* as the magnitude of the Stokes vector $\vec{P} = (P_1, P_2, P_3)$, i.e., $|\vec{P}| = \sqrt{P_1^2 + P_2^2 + P_3^2}$. In the case that $0 < |\vec{P}| < 1$, we call the light *partially polarized*. (For completely polarized light, we have $|\vec{P}| = 1$, and for unpolarized light, $|\vec{P}| = 0$.)

7.4 The problem of the reflection of an EM wave at a planar interface

In this chapter, we will deal with the problem of the transmission (refraction) and reflection of electromagnetic waves at the planar interface of two non-conductive media. From Maxwell's equations, we will derive the laws of reflection and refraction, find the amplitude coefficients of transmission and reflection depending on the angle of incidence and polarization of the incident electromagnetic wave. We predict the existence of a notable angle at which a certain polarization is not reflected and thus allows us to polarize light by reflection. In the solution, we will use the conditions of connection at the interface of non-conductive media, which we derived in chapter 6.5.

Let a plane traveling harmonic electromagnetic wave fall on a planar interface. As an *ansatz*, take that the reflected and transmitted wave will also be plane waves, whose electric components can therefore be written in the form

$$\vec{E}_\mathrm{d} = \vec{E}_{\mathrm{d}0} e^{i(\omega t - \vec{k}_\mathrm{d}\cdot\vec{r})}, \qquad \vec{E}_\mathrm{r} = \vec{E}_{\mathrm{r}0} e^{i(\omega_\mathrm{r} t - \vec{k}_\mathrm{r}\cdot\vec{r})}, \qquad \vec{E}_\mathrm{t} = \vec{E}_{\mathrm{t}0} e^{i(\omega_\mathrm{t} t - \vec{k}_\mathrm{t}\cdot\vec{r})}, \tag{7.50}$$

where $\vec{E}_\mathrm{d}$, $\vec{E}_\mathrm{r}$, and $\vec{E}_\mathrm{t}$ denote the incident, reflected, and transmitted wave, respectively. The constant amplitude vectors $\vec{E}_{\mathrm{d}0}$, $\vec{E}_{\mathrm{r}0}$, and $\vec{E}_{\mathrm{t}0}$ are left unspecified for now. The magnetic part of electromagnetic waves is uniquely given by the relationship

$$\vec{B} = \frac{1}{v}\,\vec{n} \times \vec{E} = \frac{1}{v}\frac{\vec{k}}{k} \times \vec{E} = \frac{1}{\omega}\vec{k} \times \vec{E} \tag{7.51}$$

(for respective values of ω, v, and $\vec{k} = k\,\vec{n}$). Wave vectors $\vec{k}_\mathrm{d}$, $\vec{k}_\mathrm{r}$, and $\vec{k}_\mathrm{t}$ indicate the direction of wave propagation, $\vec{k} = k\,\vec{n}$, and wavelength $\lambda = \frac{2\pi}{k}$. The relations between angular velocities and wave numbers are given by the respective dispersion relations

$$\omega = \frac{c}{n_1}|\vec{k}_\mathrm{d}| = \frac{k_\mathrm{d}}{\sqrt{\varepsilon_1\mu_1}}, \qquad \omega_\mathrm{r} = \frac{c}{n_1}|\vec{k}_\mathrm{r}| = \frac{k_\mathrm{r}}{\sqrt{\varepsilon_1\mu_1}}, \qquad \omega_\mathrm{t} = \frac{c}{n_2}|\vec{k}_\mathrm{t}| = \frac{k_\mathrm{t}}{\sqrt{\varepsilon_2\mu_2}}, \tag{7.52}$$

the refractive indexes are given by the properties of the medium as $n = \sqrt{\varepsilon_r\mu_r}$, the speed of light is $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$.

The *plane of incidence* is defined as the plane perpendicular to the planar interface and containing the wave vector of the incident wave $\vec{k}_\mathrm{d}$, see figure 7.13. We introduce Cartesian coordinates so that the plane of the interface is given as $z = 0$ and the plane of incidence is parallel to the yz coordinate plane. The whole problem is then translationally symmetric along the x

axis.

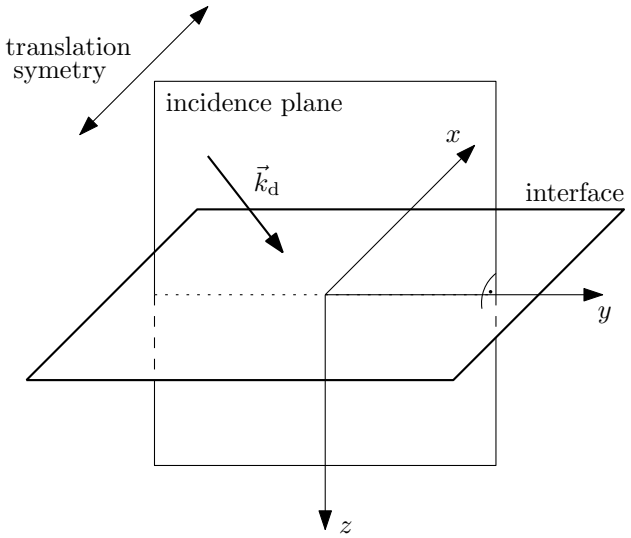


Figure 7.13: The interface plane and the plane of incidence perpendicular to it containing the wave vector of the incident wave $\vec{k}_\mathrm{d}$. Cartesian coordinates are introduced so that the plane of the interface is at $z = 0$ and the plane of incidence corresponds to planes parallel to the yz plane.

The translational symmetry along the x axis means that the waves (7.50) must not depend on the variable x. This means that the wave vectors must have a zero k_x component,

$$\vec{k} = (0, k_y, k_z) \tag{7.53}$$

(for the incident wave, this statement is trivial, since we introduced coordinates so that $k_{\mathrm{d}x} = 0$, this fact is non-trivial for the reflected and transmitted wave). The entire problem of transmission and reflection of an electromagnetic wave is thus planar. Next, we introduce the angles of incidence ϑ_d, reflection ϑ_r, and transmission (refraction) ϑ_t as deviations of the directions of propagation of the respective waves from the normal to the interface, see figure 7.14.

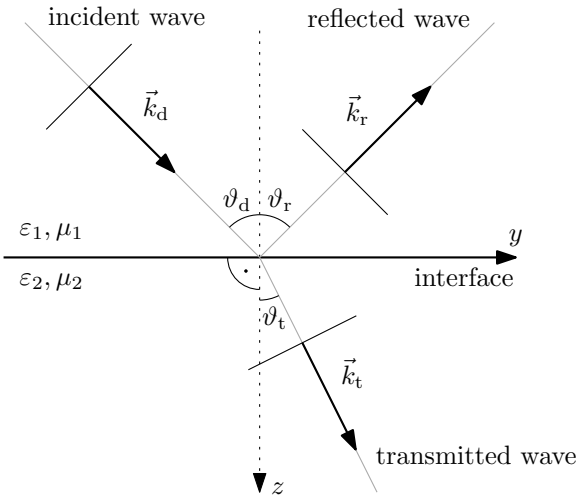


Figure 7.14: The electromagnetic wave incident on the interface in the direction of the wave vector $\vec{k}_\mathrm{d}$ with a deviation from the normal ϑ_d. For the reflected and transmitted wave, the angles of deviation from the normal are denoted ϑ_r and ϑ_t.

Using these angles, we can express the individual components of the wave vectors $\vec{k}_\mathrm{d}$, $\vec{k}_\mathrm{r}$,

and $\vec{k}_\mathrm{t}$ as follows:

$$\vec{k}_\mathrm{d} = (0, k_\mathrm{d} \sin\vartheta_\mathrm{d}, k_\mathrm{d} \cos\vartheta_\mathrm{d}), \quad \vec{k}_\mathrm{r} = (0, k_\mathrm{r} \sin\vartheta_\mathrm{r}, -k_\mathrm{r} \cos\vartheta_\mathrm{r}), \quad \vec{k}_\mathrm{t} = (0, k_\mathrm{t} \sin\vartheta_\mathrm{t}, k_\mathrm{t} \cos\vartheta_\mathrm{t}). \tag{7.54}$$

7.4.1 Law of reflection and refraction, critical angle, and total reflection

Let's start with the first condition of connection, the condition of continuity of the tangential components of the electric field at the interface,

$$\vec{E}_{1\parallel} = \vec{E}_{2\parallel}, \qquad \vec{E}_{\mathrm{d}\parallel} + \vec{E}_{\mathrm{r}\parallel} = \vec{E}_{\mathrm{t}\parallel}, \tag{7.55}$$

the electric field in the first medium $\vec{E}_1$ is the sum of the incident and reflected wave, $\vec{E}_\mathrm{d} + \vec{E}_\mathrm{r}$, and the field in the second medium $\vec{E}_2$ is given by the transmitted wave $\vec{E}_\mathrm{t}$. After substituting from the ansatz (7.50) we get

$$\vec{E}_{\mathrm{d}0\parallel} e^{i(\omega t - k_{\mathrm{d}y} y)} + \vec{E}_{\mathrm{r}0\parallel} e^{i(\omega_\mathrm{r} t - k_{\mathrm{r}y} y)} = \vec{E}_{\mathrm{t}0\parallel} e^{i(\omega_\mathrm{t} t - k_{\mathrm{t}y} y)}, \tag{7.56}$$

where we used the form of wave vectors (7.53) and the fact that at the interface $z = 0$. The condition (7.56) is a linear combination of exponentials (or a set of several linear combinations). For generally non-zero electric fields, we get that the exponentials must be pairwise linearly dependent, which is only possible if they are equal:

$$e^{i(\omega t - k_{\mathrm{d}y} y)} = e^{i(\omega_\mathrm{r} t - k_{\mathrm{r}y} y)} = e^{i(\omega_\mathrm{t} t - k_{\mathrm{t}y} y)}, \qquad \forall y, t \in \mathbb{R}. \tag{7.57}$$

If these functions are to be equal, the parameters in them must be equal:

$$\omega = \omega_\mathrm{r} = \omega_\mathrm{t}, \qquad k_{\mathrm{d}y} = k_{\mathrm{r}y} = k_{\mathrm{t}y}. \tag{7.58}$$

Thus, the angular frequencies of the individual waves must be the same (from now on we will write only ω) and the tangential components of the wave vectors must be equal (here, the tangential component is the y component):

$$\vec{k}_{\mathrm{d}\parallel} = \vec{k}_{\mathrm{r}\parallel} = \vec{k}_{\mathrm{t}\parallel}. \tag{7.59}$$

Now, substituting the expression of the tangential component of the wave vectors k_y from (7.54) we get

$$k_\mathrm{d} \sin\vartheta_\mathrm{d} = k_\mathrm{r} \sin\vartheta_\mathrm{r} = k_\mathrm{t} \sin\vartheta_\mathrm{t} \tag{7.60}$$

and further replacing the magnitudes of the wave vectors from the dispersion relations (7.52) (and canceling the common factor $\frac{\omega}{c}$):

$$n_1 \sin\vartheta_\mathrm{d} = n_1 \sin\vartheta_\mathrm{r} = n_2 \sin\vartheta_\mathrm{t}. \tag{7.61}$$

From the left equality follows the *law of reflection* – the angle of incidence equals the angle of reflection, the right equality represents the *Snell's law of refraction*:

$$\boxed{\vartheta_\mathrm{d} = \vartheta_\mathrm{r},} \qquad \boxed{n_1 \sin\vartheta_\mathrm{d} = n_2 \sin\vartheta_\mathrm{t}.} \tag{7.62}$$

In the following, we will denote the angle of incidence and reflection as ϑ_1, the angle of refraction as ϑ_2, thus Snell's law will be in the form

$$n_1 \sin\vartheta_1 = n_2 \sin\vartheta_2. \tag{7.63}$$

For $n_1 > n_2$, i.e., when passing from optically denser to optically rarer medium, Snell's law implies $\vartheta_2 > \vartheta_1$. However, it must hold that $\vartheta_2 \leq \frac{\pi}{2}$. We define the so-called *critical angle* ϑ_C as such an angle ϑ_1, at which $\vartheta_2 = \frac{\pi}{2}$. From Snell's law for the value of the critical angle follows

$$\boxed{\sin \vartheta_C = \frac{n_2}{n_1}.} \tag{7.64}$$

For the angle of incidence $\vartheta_1 > \vartheta_C$, the transmitted wave "has nowhere to refract" and leads to so-called *total reflection*, where the wave does not propagate into the second medium and is completely reflected. In this case, the second medium acts as a reactive medium. Let's look at this fact in more detail. Consider the dispersion relation of the second medium for the transmitted wave

$$\omega^2 = \frac{c^2}{n_2^2}|\vec{k}_\mathrm{t}|^2 = \frac{c^2}{n_2^2}(k_{ty}^2 + k_{tz}^2). \tag{7.65}$$

Let's express the component of the wave number $\vec{k}_\mathrm{t}$ in the direction of the z axis, indicating how much the wave progresses in the direction of the z axis,

$$k_{tz}^2 = \frac{\omega^2}{c^2} n_2^2 - k_{ty}^2. \tag{7.66}$$

Furthermore, we can set $k_{ty} = k_{\mathrm{d}y}$, from the expression (7.54) we have $k_{\mathrm{d}y} = k_\mathrm{d} \sin \vartheta_1$ and finally k_d can be expressed from the dispersion relation in the first medium (7.52), $k_\mathrm{d} = \frac{\omega}{c} n_1$. After making these adjustments, we get the relationship (7.66) for k_{tz}^2 in the form

$$k_{tz}^2 = \frac{\omega^2}{c^2} \left(n_2^2 - n_1^2 \sin^2 \vartheta_1 \right). \tag{7.67}$$

For $\vartheta_1 > \vartheta_C$, the dispersion relation (7.67) does not have a solution for real k_{tz}. Thus, indeed, for $\vartheta_1 > \vartheta_C$, the second medium acts as reactive, i.e., does not support the propagation of the electromagnetic wave in the direction of the z axis. The solution (7.67) is found for the ansatz $k_{tz} = -i\kappa$, after substituting we have

$$\boxed{\kappa^2 = \frac{\omega^2}{c^2} \left(n_1^2 \sin^2 \vartheta_1 - n_2^2 \right).} \tag{7.68}$$

The form of the electromagnetic wave in the second medium is obtained after substituting our ansatz $\vec{k}_\mathrm{t} = (0, k_{ty}, -i\kappa)$ into the form for the transmitted electric wave $\vec{E}_\mathrm{t}$ (7.50):

$$\vec{E}_\mathrm{t} = \vec{E}_{\mathrm{t}0}\, e^{i(\omega t - \vec{k}_\mathrm{t} \cdot \vec{r})} = \vec{E}_{\mathrm{t}0}\, e^{i(\omega t - k_{ty} y)} e^{-\kappa z}. \tag{7.69}$$

This form represents an electric wave traveling in the direction of the y axis, but exponentially attenuated in the direction of the z axis. We can again define the penetration depth δ as the distance at which the wave amplitude decreases to e^{-1} of its original value. Clearly, $\delta = \kappa^{-1}$.

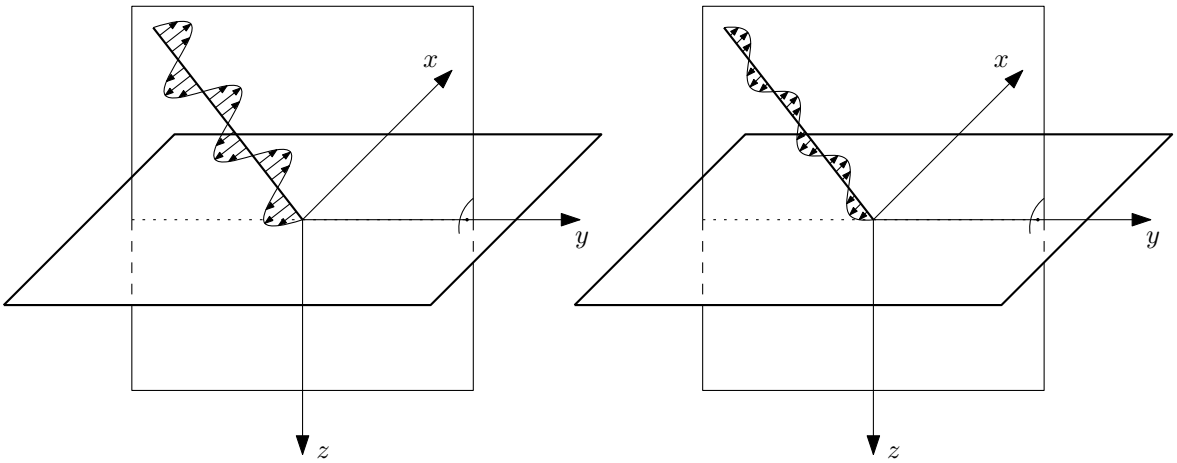
7.4.2 Connection conditions for individual polarizations

Now consider all the connection conditions of the electric and magnetic field at the interface,

$$\begin{aligned}
\vec{E}_{1\parallel} = \vec{E}_{2\parallel} \quad &\Rightarrow \quad \left(\vec{E}_{\mathrm{d}0} + \vec{E}_{\mathrm{r}0}\right)_{x,y} = \left(\vec{E}_{\mathrm{t}0}\right)_{x,y}, \\
\varepsilon_1 \vec{E}_{1\perp} = \varepsilon_2 \vec{E}_{2\perp} \quad &\Rightarrow \quad \varepsilon_1 \left(\vec{E}_{\mathrm{d}0} + \vec{E}_{\mathrm{r}0}\right)_z = \varepsilon_2 \left(\vec{E}_{\mathrm{t}0}\right)_z, \\
\vec{B}_{1\perp} = \vec{B}_{2\perp} \quad &\Rightarrow \quad \left(\vec{B}_{\mathrm{d}0} + \vec{B}_{\mathrm{r}0}\right)_z = \left(\vec{B}_{\mathrm{t}0}\right)_z, \\
\frac{1}{\mu_1} \vec{B}_{1\parallel} = \frac{1}{\mu_2} \vec{B}_{2\parallel} \quad &\Rightarrow \quad \frac{1}{\mu_1} \left(\vec{B}_{\mathrm{d}0} + \vec{B}_{\mathrm{r}0}\right)_{x,y} = \frac{1}{\mu_2} \left(\vec{B}_{\mathrm{t}0}\right)_{x,y},
\end{aligned} \tag{7.70}$$

where on the left side are the conditions without coordinates, on the right side we introduced the relationships using the individual components of the given vectors. In all waves, the exponentials at the interface are the same, so we can cancel them out, and only the amplitude vectors will remain in the connection conditions. The components x and y are the components tangential to the interface, the component z is the component perpendicular to the interface (see the introduction of coordinates in figure 7.13). For clarity of notation, we introduced the notation $(\vec{A}+\vec{B})_x = A_x + B_x$, etc. for other components.

Now we must split the approach into two separate parts according to the polarization of the incident wave. We consider a linearly polarized incident wave and once choose the plane of polarization so that the electric field oscillates in the plane of incidence, the second time we choose the plane of polarization with the vector of the electric field oscillating perpendicular to the plane of incidence, see figures 7.15.

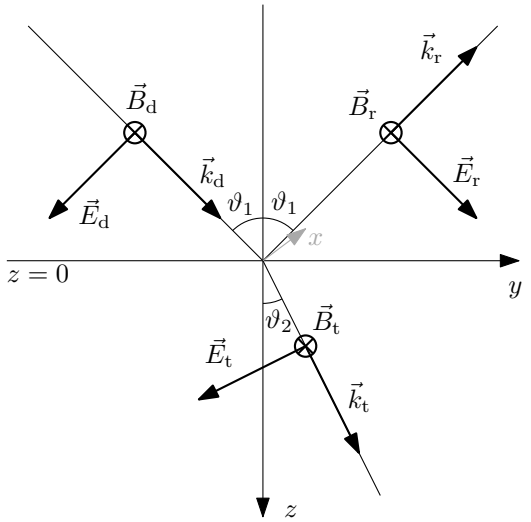


(a) Polarization in the plane of incidence.

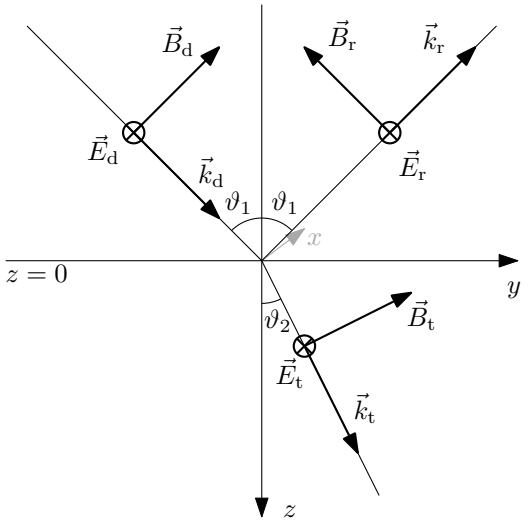
(b) Polarization perpendicular to the plane of incidence.

Figure 7.15: We distinguish two cases according to the direction of linear polarization of the incident wave.

Consider the ansatz that the polarization is preserved in the reflected and transmitted wave – this corresponds to the idea that the incident wave oscillates charges in the direction of its polarization, and these charges then emit waves with the same polarization. Now we can introduce positive directions of the electric and magnetic fields in individual electromagnetic waves. We want the triplet of vectors $\vec{E}$, $\vec{B}$, and $\vec{k}$ to form a right-handed system of vectors. The wave vectors $\vec{k}$ are given, the directions of the electric field vectors are limited by the choice of polarization in the plane of incidence or perpendicular to it. Let's then choose positive directions, for example, as in figures 7.16.



(a) The case of polarization in the plane of incidence. The directions of the magnetic field $\vec{B}$ are chosen to point in the positive direction of the x axis. The directions of the electric field $\vec{E}$ are then determined by the right-hand rule.



(b) The case of polarization perpendicular to the plane of incidence. The directions of the electric field $\vec{E}$ are chosen to point in the positive direction of the x axis. The directions of the magnetic field $\vec{B}$ are then determined by the right-hand rule.

Figure 7.16: Positive directions of the vectors of the electric field $\vec{E}$ and magnetic field $\vec{B}$ for individual polarizations. The plane of incidence with axes y and z is drawn.

Now we can specialize the general connection conditions in (7.70) for individual cases of polarization. On the left polarization in the plane of incidence, on the right polarization perpendicular to the plane of incidence:

$$\begin{aligned} E_{\mathrm{d}0y} + E_{\mathrm{r}0y} &= E_{\mathrm{t}0y}, & E_{\mathrm{d}0x} + E_{\mathrm{r}0x} &= E_{\mathrm{t}0x}, \\ \varepsilon_1\big(E_{\mathrm{d}0z} + E_{\mathrm{r}0z}\big) &= \varepsilon_2 E_{\mathrm{t}0z}, & 0 &= 0, \\ 0 &= 0, & B_{\mathrm{d}0z} + B_{\mathrm{r}0z} &= B_{\mathrm{t}0z}, \\ \frac{1}{\mu_1}\big(B_{\mathrm{d}0x} + B_{\mathrm{r}0x}\big) &= \frac{1}{\mu_2} B_{\mathrm{t}0x}, & \frac{1}{\mu_1}\big(B_{\mathrm{d}0y} + B_{\mathrm{r}0y}\big) &= \frac{1}{\mu_2} B_{\mathrm{t}0y}. \end{aligned} \tag{7.71}$$

Some connection conditions turned out to be trivial due to the fact that in a specific case the electric or magnetic field does not point in a certain direction at all, and the condition is therefore automatically met.

Further, let's express the individual components of the electric and magnetic field vectors using the angles of incidence ϑ_1 and refraction ϑ_2 from figures 7.16. For the case of polarization in the plane of incidence, we have:

$$\begin{aligned} &\vec{E}_{\mathrm{d}0} = E_{\mathrm{d}0}(0, -\cos\vartheta_1, \sin\vartheta_1), && \vec{E}_{\mathrm{r}0} = E_{\mathrm{r}0}(0, \cos\vartheta_1, \sin\vartheta_1), && \vec{E}_{\mathrm{t}0} = E_{\mathrm{t}0}(0, -\cos\vartheta_2, \sin\vartheta_2), \\ &\vec{B}_{\mathrm{d}0} = B_{\mathrm{d}0}(1, 0, 0), && \vec{B}_{\mathrm{r}0} = B_{\mathrm{r}0}(1, 0, 0), && \vec{B}_{\mathrm{t}0} = B_{\mathrm{t}0}(1, 0, 0), \end{aligned} \tag{7.72}$$

and for polarization perpendicular to the plane of incidence:

$$\begin{aligned} &\vec{E}_{\mathrm{d}0} = E_{\mathrm{d}0}(1, 0, 0), && \vec{E}_{\mathrm{r}0} = E_{\mathrm{r}0}(1, 0, 0), && \vec{E}_{\mathrm{t}0} = E_{\mathrm{t}0}(1, 0, 0), \\ &\vec{B}_{\mathrm{d}0} = B_{\mathrm{d}0}(0, \cos\vartheta_1, -\sin\vartheta_1), && \vec{B}_{\mathrm{r}0} = B_{\mathrm{r}0}(0, -\cos\vartheta_1, -\sin\vartheta_1), && \vec{B}_{\mathrm{t}0} = B_{\mathrm{t}0}(0, \cos\vartheta_2, -\sin\vartheta_2). \end{aligned} \tag{7.73}$$

Amplitudes of magnetic fields can be expressed using the amplitudes of electric fields

$$B_{\mathrm{d}0} = \frac{1}{v_1} E_{\mathrm{d}0}, \qquad B_{\mathrm{r}0} = \frac{1}{v_1} E_{\mathrm{r}0}, \qquad B_{\mathrm{t}0} = \frac{1}{v_2} E_{\mathrm{t}0}. \tag{7.74}$$

We introduce the coefficients of transmission P, and reflection R, as the ratio of the amplitude of the transmitted, respectively, reflected wave to the amplitude of the incident wave.

$$R = \frac{E_{\mathrm{r}0}}{E_{\mathrm{d}0}}, \qquad P = \frac{E_{\mathrm{t}0}}{E_{\mathrm{d}0}}. \tag{7.75}$$

Now, substitute the forms of the components of the electric and magnetic fields (7.72) and (7.73) into the connection conditions (7.71). Replace the amplitudes of the magnetic field according to (7.74) and finally, in these equations, introduce the coefficients R and P using the definitions (7.75). Denote the coefficients $R_\parallel$ and $P_\parallel$ for the coefficients of transmission and reflection for polarization in the plane of incidence and $R_\perp$ and $P_\perp$ for the coefficients for polarization perpendicular to the plane of incidence. After a series of these operations, we obtain these equations:

$$\begin{aligned}
(-1 + R_\parallel)\cos\vartheta_1 &= -P_\parallel \cos\vartheta_2, & 1 + R_\perp &= P_\perp,\\
\varepsilon_1(1 + R_\parallel)\sin\vartheta_1 &= \varepsilon_2 P_\parallel \sin\vartheta_2, & 0 &= 0,\\
0 &= 0, & \frac{1}{v_1}(1 + R_\perp)\sin\vartheta_1 &= \frac{1}{v_2}P_\perp \sin\vartheta_2,\\
\frac{1}{\mu_1}\frac{1}{v_1}(1 + R_\parallel) &= \frac{1}{\mu_2}\frac{1}{v_2}P_\parallel, & \frac{1}{\mu_1}\frac{1}{v_1}(1 - R_\perp)\cos\vartheta_1 &= \frac{1}{\mu_2}\frac{1}{v_2}P_\perp \cos\vartheta_2.
\end{aligned} \tag{7.76}$$

We always get three equations for the respective coefficients R and P. We will show that always two of the equations are the same and we are left with two independent equations for the coefficients of transmission and reflection. Sines in the equations (7.76) can be eliminated using Snell's law,

$$\frac{\sin\vartheta_1}{\sin\vartheta_2} = \frac{n_2}{n_1}, \tag{7.77}$$

and we arrive at

$$\frac{\varepsilon_1}{n_1}(1 + R_\parallel) = \frac{\varepsilon_2}{n_2}P_\parallel, \qquad \frac{1}{n_1 v_1}(1 + R_\perp) = \frac{1}{n_2 v_2}P_\perp. \tag{7.78}$$

Write the relations between the constants v, n, ε, and μ:

$$n = \frac{c}{v}, \qquad v = \frac{1}{\sqrt{\varepsilon\mu}}, \qquad \frac{\varepsilon}{n} = \frac{1}{c}\sqrt{\frac{\varepsilon}{\mu}}, \qquad \frac{1}{nv} = \frac{1}{c}, \qquad \frac{1}{\mu}\frac{1}{v} = \sqrt{\frac{\varepsilon}{\mu}}. \tag{7.79}$$

After substituting these relationships into equations (7.76) and (7.78), it will be shown that the equations (7.78) are not independent (they are the same as the fourth equations in (7.76)).

7.4.3 Transmission and reflection coefficients – Fresnel's formulas

The final form of the equations (after eliminating dependent equations (7.78) and substituting from (7.79)) for the coefficients of transmission and reflection for the respective polarizations is as follows:

$$\begin{aligned}
\sqrt{\frac{\varepsilon_1}{\mu_1}}(1 + R_\parallel) &= \sqrt{\frac{\varepsilon_2}{\mu_2}}P_\parallel, & 1 + R_\perp &= P_\perp,\\
(1 - R_\parallel)\cos\vartheta_1 &= P_\parallel \cos\vartheta_2, & \sqrt{\frac{\varepsilon_1}{\mu_1}}(1 - R_\perp)\cos\vartheta_1 &= \sqrt{\frac{\varepsilon_2}{\mu_2}}P_\perp \cos\vartheta_2.
\end{aligned} \tag{7.80}$$

We see that between the coefficients of transmission and reflection for polarization perpendicular to the plane of incidence, we obtain the "classical" relationship $P_\perp = 1 + R_\perp$ resulting from the continuity condition at the interface – here, it is the continuity of the components of the electric

field lying in the plane of the interface (the first of the connection conditions in (7.70)). In contrast, for polarization in the plane of incidence, the continuity condition does not apply, and the relationship between $R_\parallel$ and $P_\parallel$ is more complex. Solving the equations (7.80) we obtain

$$R_\parallel = \frac{\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\vartheta_1 - \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\vartheta_2}{\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\vartheta_1 + \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\vartheta_2}, \qquad R_\perp = \frac{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\vartheta_1 - \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\vartheta_2}{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\vartheta_1 + \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\vartheta_2},$$

$$P_\parallel = \frac{\sqrt{\frac{\varepsilon_1}{\mu_1}}}{\sqrt{\frac{\varepsilon_2}{\mu_2}}}(1 + R_\parallel), \qquad P_\perp = 1 + R_\perp. \tag{7.81}$$

This result represents the exact solution of the coefficients of transmission and reflection at the interface of two non-conductive media resulting from Maxwell's equations.

The quantity $\sqrt{\frac{\mu}{\varepsilon}}$ is called the impedance Z,

$$Z = \sqrt{\frac{\mu}{\varepsilon}}. \tag{7.82}$$

For most materials, it holds that $\mu_1 \approx \mu_2 \approx \mu_0$. Therefore, we can write

$$n = \sqrt{\varepsilon_r \mu_r} \approx \sqrt{\varepsilon_r} \approx \sqrt{\frac{\varepsilon_r}{\mu_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\varepsilon}{\mu}}. \tag{7.83}$$

By substituting the previous expression into (7.81), we arrive at the so-called *Fresnel formulas*:

$$R_\parallel = \frac{n_2\cos\vartheta_1 - n_1\cos\vartheta_2}{n_2\cos\vartheta_1 + n_1\cos\vartheta_2}, \qquad R_\perp = \frac{n_1\cos\vartheta_1 - n_2\cos\vartheta_2}{n_1\cos\vartheta_1 + n_2\cos\vartheta_2}, \tag{7.84}$$

where, of course, only one of the angles is the parameter, the other is determined from Snell's law.

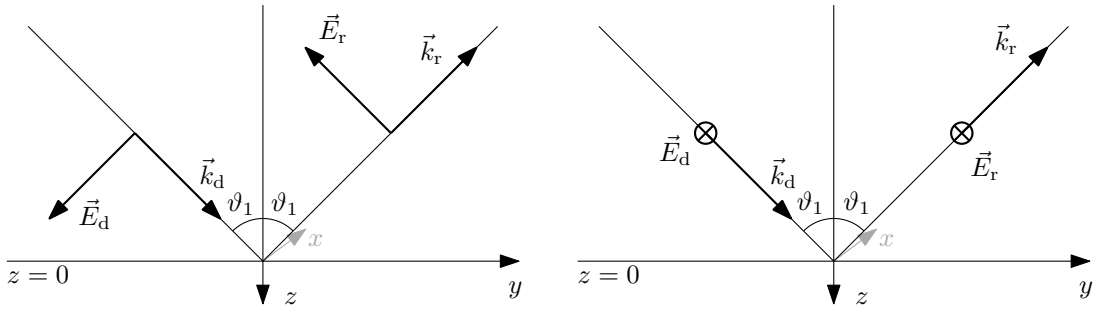
For normal incidence, $\vartheta_1 = 0$, from Snell's law we have $\vartheta_2 = 0$ and the Fresnel formulas reduce to a simple form

$$R = \pm\frac{n_1 - n_2}{n_1 + n_2}. \tag{7.85}$$

However, this result is quite peculiar. At normal incidence, the plane of incidence is not uniquely defined, and both polarizations are completely equivalent. They should therefore give the same results. The stumbling block is in the introduction of positive directions, see figure 7.16. For the case of polarization in the plane of incidence for $\vartheta_1 \to 0$, vectors $\vec{E}_\mathrm{d}$ and $\vec{E}_\mathrm{r}$ point in opposite directions, while for polarization perpendicular to the plane of incidence, they still point in the same direction (along the x axis). This situation points out the fact that for the interpretation of Fresnel formulas, it is necessary to have information on how positive directions were introduced. Their introduction then can change the signs of the R and P coefficients for different polarizations. And as usual, the convention in the literature is not uniform. From here on in the text, **let's change the positive direction of the vector $\vec{E}_\mathbf{r}$ for the case of polarization in the plane of incidence to the opposite.** This causes $R_\parallel \to -R_\parallel$, so from now on we work with the expression for $R_\parallel$, which has the opposite sign than is stated in (7.84), (for normal incidence thus consistently $R = \frac{n_1 - n_2}{n_1 + n_2}$). The resulting forms of the Fresnel formulas are then:

$$\boxed{R_\parallel = \frac{n_1\cos\vartheta_2 - n_2\cos\vartheta_1}{n_1\cos\vartheta_2 + n_2\cos\vartheta_1}, \qquad R_\perp = \frac{n_1\cos\vartheta_1 - n_2\cos\vartheta_2}{n_1\cos\vartheta_1 + n_2\cos\vartheta_2},} \tag{7.86}$$

in the figure 7.17, the final positive directions of the electric field of the incident and reflected wave necessary for interpreting the formulas (7.86) are drawn.



(a) The case of polarization in the plane of incidence.

(b) The case of polarization perpendicular to the plane of incidence.

Figure 7.17: Adjusted final positive directions of the electric field vectors $\vec{E}$ for each polarization of the incident and reflected wave. The plane of incidence with the y and z axes is drawn.

The forms of the Fresnel formulas can be further simplified if we substitute for the refractive index n_1 using Snell's law,

$$n_1 = n_2 \frac{\sin \vartheta_2}{\sin \vartheta_1}. \tag{7.87}$$

Thus, we arrive at simple forms[6]

$$R_\parallel = \frac{\mathrm{tg}\,(\vartheta_2 - \vartheta_1)}{\mathrm{tg}\,(\vartheta_2 + \vartheta_1)}, \qquad R_\perp = \frac{\sin(\vartheta_2 - \vartheta_1)}{\sin(\vartheta_2 + \vartheta_1)}. \tag{7.88}$$

The plots of these functions are shown in figures 7.18, where cases for $n_1 < n_2$ and $n_1 > n_2$ are separated. Furthermore, we define reflectivity as $\mathcal{R} = R^2$, indicating what portion of the intensity of the incident wave is reflected, $I_\mathrm{r} = \mathcal{R}\, I_\mathrm{d}$. The graphs of reflectivity are also shown in figure 7.18. For angles $\vartheta_1 \to \frac{\pi}{2}$ for the case $n_1 < n_2$, reflectivity approaches one, and the interface becomes a perfect mirror (similarly for $\vartheta_1 \to \vartheta_C$ for the case $n_1 > n_2$). The value of reflectivity differs depending on the polarization (except for $\vartheta_1 = 0$ and $\vartheta_1 = \frac{\pi}{2}$, resp. $\vartheta_1 = \vartheta_C$). This causes that if unpolarized light falls on the interface, it becomes partially linearly polarized after reflection.

7.4.4 Brewster's Angle, Polarization by Reflection

Looking at the graphs of reflection coefficients, respectively reflectivity, we see that there is a special value of the angle at which the polarization in the plane of incidence does not reflect at all. This angle is called *Brewster's angle*.

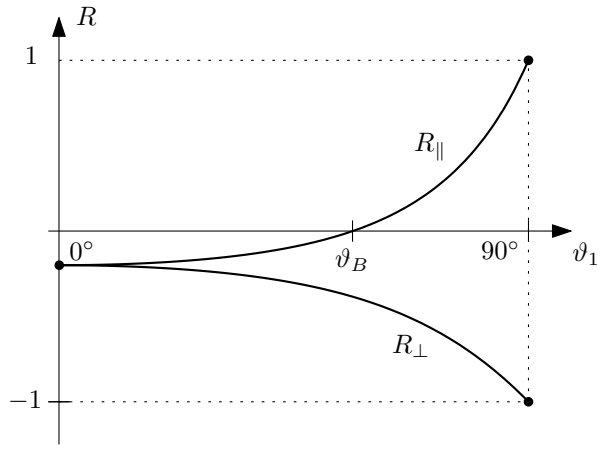
Let's find the angle value for which the reflection coefficient is zero. If we considered nullifying the numerator in the expressions for the reflection coefficient (whether for $R_\parallel$ or $R_\perp$), we get the condition $\vartheta_1 = \vartheta_2$. However, this only occurs for $\vartheta_1 = \vartheta_2 = 0$, and then $R = \frac{n_1 - n_2}{n_1 + n_2}$ is non-zero.

Another possibility is that the denominator becomes infinity. This situation can only occur with the function tangent, $\mathrm{tg}\,(\vartheta_2 + \vartheta_1) = +\infty$, hence for a combination of angles

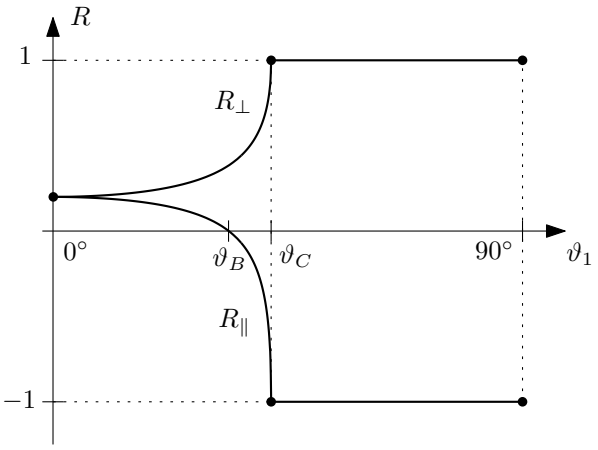
$$\vartheta_1 + \vartheta_2 = \frac{\pi}{2}, \tag{7.89}$$

[6]The form $R_\perp$ is obtained directly by using the sum formula, for the expression $R_\parallel$ it is necessary to first use the formula for the sine of a double angle and then the respective sum formula.

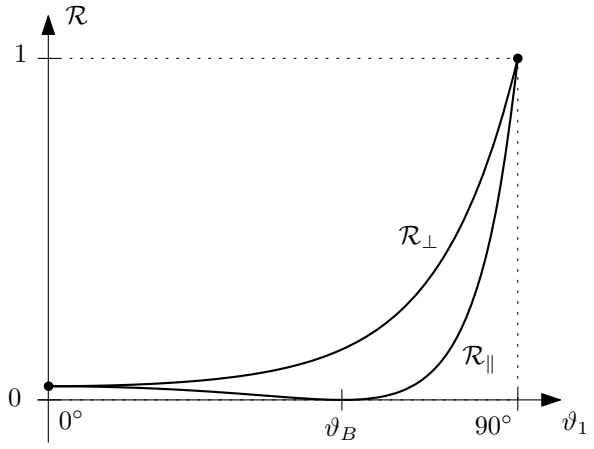
this condition thus defines Brewster's angle $\vartheta_1 = \vartheta_B$ and says that at incidence under Brewster's angle, the directions of the reflected and refracted waves form a right angle, see figure 7.19.



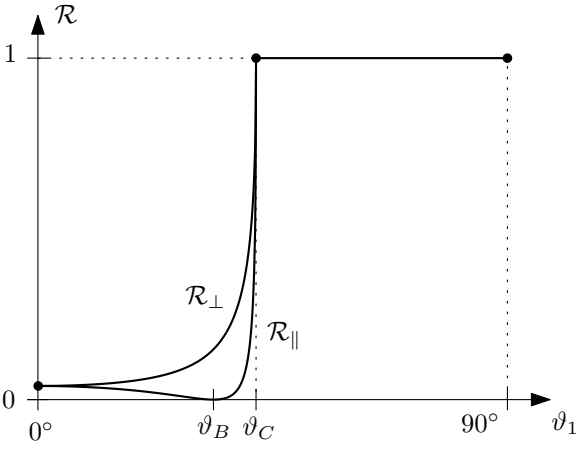
(a) Reflection coefficients for reflection on a optically denser medium, i.e., the situation for $n_1 < n_2$.



(b) Reflection coefficients for reflection on a optically rarer medium, i.e., the situation for $n_2 < n_1$. For angles $\vartheta > \vartheta_C$ total reflection occurs.



(c) Reflectivities for reflection on a optically denser medium, i.e., the situation for $n_1 < n_2$.



(d) Reflectivities for reflection on a optically rarer medium, i.e., the situation for $n_1 > n_2$. For angles $\vartheta > \vartheta_C$ total reflection occurs.

Figure 7.18: Fresnel's formulas. Graph of reflection coefficients R and reflectivity $\mathcal{R}$ depending on the angle of incidence ϑ_1 and polarization – in the plane of incidence $R_\parallel$ and $\mathcal{R}_\parallel$, perpendicular to the plane of incidence $R_\perp$ and $\mathcal{R}_\perp$. In the images on the left, the situation is for $n_1 < n_2$, on the right for $n_1 > n_2$. Brewster's angle ϑ_B denotes the angle at which $R_\parallel = \mathcal{R}_\parallel = 0$.

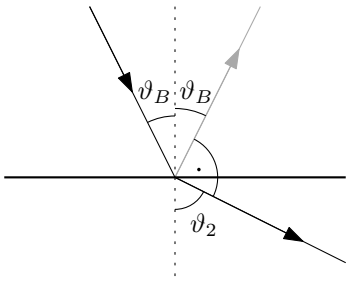


Figure 7.19: A beam incident at Brewster's angle ϑ_B refracts such that the direction of the transmitted and reflected waves forms a right angle. The reflected beam is linearly polarized perpendicular to the plane of incidence, as the reflection coefficient $R_{\parallel}$ is zero.

By substituting into Snell's law,

$$n_1 \sin \vartheta_B = n_2 \underbrace{\sin\left(\tfrac{\pi}{2} - \vartheta_B\right)}_{\cos \vartheta_B}, \tag{7.90}$$

we express

$$\boxed{\operatorname{tg} \vartheta_B = \frac{n_2}{n_1}.} \tag{7.91}$$

This equation has a solution for any combination of refractive indexes n_1 and n_2 – thus, for any interface, there exists a Brewster's angle.

The existence of Brewster's angle can be utilized in several ways. If we let unpolarized light fall under Brewster's angle, the electric field oscillating in the plane of incidence does not reflect, and we obtain light polarized perpendicular to the plane of incidence. This phenomenon is called *polarization by reflection*. As already mentioned, at other angles, the light becomes partially polarized. Furthermore, if we have linearly polarized light, we can let it fall under Brewster's angle in such a way that the polarization lies in the plane of incidence, in which case no light is reflected and the interface is perfectly transparent – this phenomenon is called *Brewster's window*.

Chapter 8

Interference and Diffraction

The linearity of Maxwell's equations results in the principle of superposition – if we take any two solutions of Maxwell's equations, their linear combination is also a solution. However, energy quantities are quadratic in fields – it does not hold that if we take the sum of solutions, we also get the corresponding sum of energy quantities. An additional interference term appears, which is responsible for the phenomenon of interference. In the following chapter, we will describe this phenomenon using the example of Michelson's interferometer and explore the conditions that may prevent the observation of interference, i.e., cause the disappearance of the interference term.

8.1 Michelson's Interferometer

Michelson's interferometer is illustrated and described in Figure 8.1.

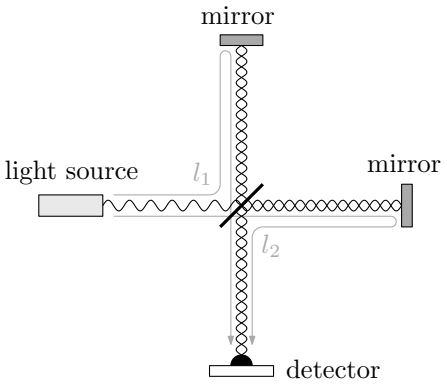


Figure 8.1: Michelson's interferometer. A beam from a light source is split by a beam splitter into two, reflects off mirrors, and then recombines to hit the detector. The path lengths of the individual beams are denoted l_1, l_2.

We will focus on a slightly more general and at the same time simpler arrangement illustrated in Figure 8.2. The main change is replacing a single light source with a beam splitter directly with two sources.

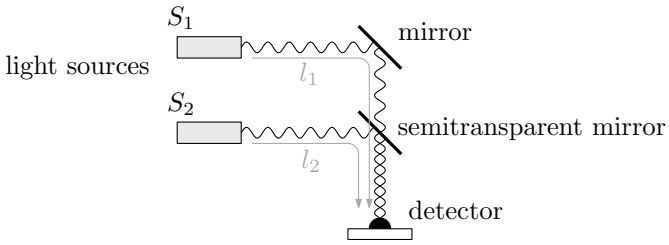


Figure 8.2: "Pedagogical" interferometer. Two light sources are directed into the detector with mirrors with path lengths l_1 and l_2.

Let's write down the electric fields generated by the individual light sources, assuming they have the same angular frequency ω:

$$E_1(t) = E_1 \cos(\omega t + \varphi_1), \qquad E_2(t) = E_2 \cos(\omega t + \varphi_2). \tag{8.1}$$

For now, let's assume that the phase shifts φ_1 and φ_2 are constants. We have also written the electric field as a scalar quantity – the phenomenon of polarization and the fact that the electric field quantity is vectorial are not important for the subsequent discussion. The electric field at a distance l along the path of the respective beam can be written as a traveling wave

$$E(l,t) = E \cos(\omega t - kl + \varphi). \tag{8.2}$$

The wave hitting the detector is then the superposition of waves from the individual sources:

$$E_D(t) = E_1(l_1,t) + E_2(l_2,t) = E_1 \cos(\omega t - kl_1 + \varphi_1) + E_2 \cos(\omega t - kl_2 + \varphi_2). \tag{8.3}$$

We are interested in the intensity measured by the detector with a resolution time $t_{\text{roz}} \gg T$. Due to the periodicity of the wave $E_D(t)$, it suffices to average over one period T:

$$\begin{aligned} I_D &= \langle E_D^2(t)\rangle_{t_{\text{roz}}} = \langle E_D^2(t)\rangle_T \\ &= \langle E_1^2 \cos^2(\ldots)\rangle + \langle E_2^2 \cos^2(\ldots)\rangle + 2E_1E_2\langle \cos(\omega t - kl_1 + \varphi_1)\cos(\omega t - kl_2 + \varphi_2)\rangle. \end{aligned} \tag{8.4}$$

Now denote

$$I_i = \frac{1}{2}E_i^2, \qquad E_i = \sqrt{2I_i}, \qquad i \in \{1,2\}, \tag{8.5}$$

thus expressing the intensities of the individual waves I_i using amplitudes E_i and vice versa (and ignoring the constant $\sqrt{\frac{\varepsilon}{\mu}}$, which only corresponds to a different choice of units). Also, use the sum formula for the product of cosines in (8.4):

$$I_D = I_1 + I_2 + 4\sqrt{I_1 I_2}\left[\frac{1}{2}\langle\cos(2\omega t - k(l_1+l_2) + \varphi_1 + \varphi_2\rangle + \frac{1}{2}\langle\cos(k(l_1-l_2) + \varphi_1 - \varphi_2)\rangle\right] \tag{8.6}$$

After averaging, the resulting intensity hitting the detector is:

$$I_D = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\big(k(l_1-l_2) + \varphi_1 - \varphi_2\big). \tag{8.7}$$

The resulting intensity depends on the phase difference function $\Delta\varphi$ of the form

$$\Delta\varphi = k(l_1 - l_2) + \varphi_1 - \varphi_2, \tag{8.8}$$

which depends on the difference in paths of the individual beams. Depending on this difference, we observe either constructive or destructive interference, the intensity can generally vary in the interval

$$I_1 + I_2 - 2\sqrt{I_1 I_2} = (\sqrt{I_1} - \sqrt{I_2})^2 \le I_D(\Delta\varphi) \le (\sqrt{I_1} + \sqrt{I_2})^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}. \tag{8.9}$$

For equal intensities of both sources, $I_1 = I_2 = I_0$, the expression simplifies to

$$I_D(\Delta\varphi) = 2I_0(1 + \cos\Delta\varphi) \in \langle 0, 4\rangle I_0. \tag{8.10}$$

8.1.1 Effect of Temporal and Spatial Coherence on the Visibility of Interference

In the previous text, we considered the phase shifts φ_1 and φ_2 to be constant. This is an idealized state that never occurs in reality. Let's consider that the phase shifts are functions of time, $\varphi_1(t)$ and $\varphi_2(t)$, such that they remain almost constant over a time $t_{\text{koh}} \gg T$ – the coherence time. After the coherence time has elapsed, the phase values randomly change. Recall that for thermal sources of visible light, we have $T \approx 10^{-14}\,\text{s}$ and $t_{\text{koh}} \approx 10^{-9}\,\text{s}$.

First, let's see how the expressions calculated in the previous chapter change. The electric fields emitted from the individual sources now take the form

$$E_{1,2}(t) = E_{1,2}\cos(\omega t + \varphi_{1,2}(t)). \tag{8.11}$$

Along the beams, we then have traveling waves of the form

$$E_{1,2}(l_{1,2}, t) = E_{1,2}\cos(\omega t - k l_{1,2} + \varphi_{1,2}(t_{\text{ret}1,2})), \tag{8.12}$$

where we have expressed the phase functions at the respective retarded times

$$t_{\text{ret}1} = t - \frac{l_1}{c}, \qquad t_{\text{ret}2} = t - \frac{l_2}{c}. \tag{8.13}$$

The interference part I_{int} of the intensity I_D hitting the detector (8.7), which resulted from averaging over one period T, does not change, as during one period the phases $\varphi_{1,2}(t)$ are constant. However, the interference intensity will now be time-dependent with a time scale of change t_{koh}:

$$I_{\text{int}}(t) = 2\sqrt{I_1 I_2}\cos\big(k(l_1 - l_2) + \varphi_1(t_{\text{ret}1}) - \varphi_2(t_{\text{ret}2})\big). \tag{8.14}$$

The resulting intensity observed in the detector is given by

additional averaging of relation (8.14) over the instrument's resolution time t_{roz}:

$$I_D(t) = I_1 + I_2 + \langle I_{\text{int}}(t)\rangle_{t_{\text{roz}}}. \tag{8.15}$$

Now we distinguish different situations and deal with how the averaged intensity $I_D(t)$ from relation (8.15) comes out.

1. Sources S_1 and S_2 are *spatially coherent.*

Spatial coherence describes the relationships between light (electric) fields at different points in space. We say that the fields at points P_1 and P_2 are *spatially coherent* if knowing the field at point P_1 allows us to predict the field at point P_2 (and vice versa). If this is not the case, we say that the sources are spatially incoherent.

If the light sources S_1 and S_2 are spatially coherent, it will hold for the phase functions[1] $\varphi_1(t) = \varphi_2(t) = \varphi(t)$. It should be noted that the spatial coherence of light sources is typically ensured by splitting the beam from one source, as in Michelson's interferometer. The phase difference function $\Delta\varphi$ is thus now of the form

$$\Delta\varphi = k(l_1 - l_2) + \varphi\big(t - \tfrac{l_1}{c}\big) - \varphi\big(t - \tfrac{l_2}{c}\big). \tag{8.16}$$

We distinguish two cases:

(a) The difference in retarded times is less than the coherence time, $|t_{\text{ret}1} - t_{\text{ret}2}| \ll t_{\text{koh}}$.

In such a case, the phase values at different (but close) times are the same, they cancel out in the phase function, and the interference intensity will not be time-dependent – the interference phenomenon will be constant and thus visible.

[1] More generally, $\varphi_1(t) - \varphi_2(t) = \text{const.}$, with a known value of the constant on the right side. Here, for simplicity, we choose the right side to be zero.

This situation occurs for $|l_1 - l_2| \ll ct_{\text{koh}}$, because

$$|t_{\text{ret1}} - t_{\text{ret2}}| = \left| t - \tfrac{l_1}{c} - t + \tfrac{l_2}{c} \right| = \frac{1}{c}|l_1 - l_2|. \tag{8.17}$$

The difference in paths of the individual beams must therefore be less than the distance light travels during the coherence time. For $t_{\text{koh}} \approx 10^{-9}\,\text{s}$, we have $|l_1 - l_2| \ll 30\,\text{cm}$.

(b) The difference in retarded times is greater than the coherence time, $|t_{\text{ret1}} - t_{\text{ret2}}| \gg t_{\text{koh}}$.

In this case, we are comparing phase shift values in different "windows" of temporal coherence, i.e., the phase managed to randomly change between times t_{ret1} and t_{ret2}. We then have a situation where the quantity

$$\delta\varphi(t) = \varphi(t_{\text{ret1}}) - \varphi(t_{\text{ret2}}) \tag{8.18}$$

randomly changes on the time scale t_{koh}. We must again distinguish two subcases:

(i) We have a "fast" instrument, i.e., $t_{\text{roz}} \ll t_{\text{koh}}$. In such a case, we average a constant in the expression for intensity (8.15), as the value $\delta\varphi$ remains unchanged during the averaging time. A fast instrument, therefore, observes rapid changes in interference on the scale of t_{koh}.

(ii) We have

a "slow" instrument, i.e., $t_{\text{roz}} \gg t_{\text{koh}}$. Now, during the instrument's resolution time, the phase value $\delta\varphi$ changes many times randomly, evenly filling the interval of angles $\langle 0, 2\pi)$. Averaging in intensity (8.15) causes the interference cosine to be nullified, and the interference phenomenon will not be observable. The total intensity measured in the detector will be a simple sum of the intensities of the individual beams, $I_D = I_1 + I_2$.

2. Sources S_1 and S_2 are not spatially coherent.

In such a case, knowing the field at source S_1 (and thus knowing the function $\varphi_1(t)$) does not allow determining the function $\varphi_2(t)$. The difference $\delta\varphi(t) = \varphi_1(t_{\text{ret1}}) - \varphi_2(t_{\text{ret2}})$ now also randomly changes on the time scale t_{koh} (regardless of the size of the difference $|l_1 - l_2|$, or $|t_{\text{ret1}} - t_{\text{ret2}}|$). Thus, the same case as in 1. (b) occurs, i.e., depending on the "speed" of the instrument, we either observe rapid changes in interference intensity or the observation of interference is prevented.

In practical cases for optical phenomena, we only have "slow" instruments, and we can therefore summarize the detailed discussion above into a simple rule: Incoherent waves (whether temporally or spatially) do not interfere with each other, the resulting intensity is given by a simple sum of the intensities of the individual waves.

This is also why the interference phenomenon is observed only under certain conditions in everyday life. Light emitted by various light sources (or also by non-point light sources) is spatially incoherent and thus we do not observe interference. On the other hand, we observe, for example, interference on thin films (oil slick on water), where the path difference of the individual beams is very small and thus the waves are temporally coherent.

8.2 Diffraction

Under the problem of diffraction, we will understand the arrangement in figure 8.3. A source of light emits electromagnetic waves, which we allow to pass through an opaque barrier with an aperture[2] and let the resulting wave hit a screen, where we observe so-called *diffraction pattern*, by which we mean the distribution of wave intensity falling on the screen depending on the position on the screen. Our task will be to predict the form of the diffraction pattern based on the arrangement of the experiment.

[2]The aperture can have very various and very complex forms.

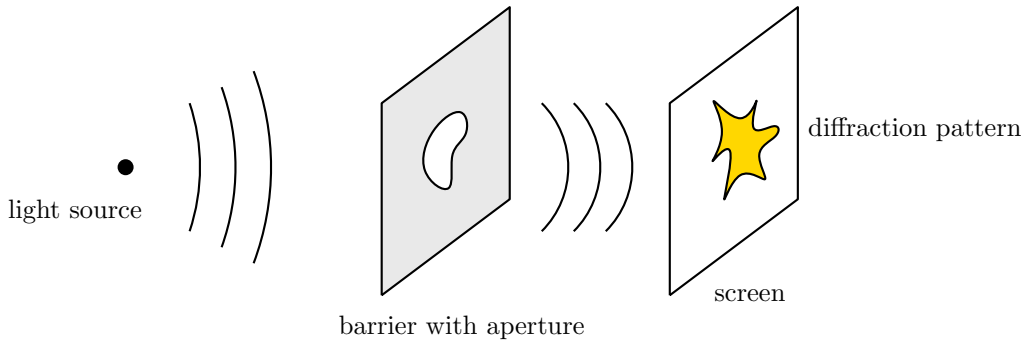


Figure 8.3: Diffraction problem. Light falls on a barrier with an aperture and creates a diffraction pattern on the screen.

8.2.1 Babinet's Principle

To be able to grasp the problem of diffraction in any way, let's look at how the opaque barrier actually works. First, consider a full barrier without any aperture. Denote $\vec{E}_{\text{dop}}$ as the field falling from the left on the opaque barrier. This field interacts with the atoms forming the barrier, and these atoms must emit a field $\vec{E}_{\text{ind}}$ such that **behind the barrier** both fields superimpose to zero:

$$\vec{E}_{\text{dop}} + \vec{E}_{\text{ind}} = 0. \tag{8.19}$$

I.e., by definition of an opaque barrier, the induced field must be such that it exactly cancels the incident field in the area behind the barrier.

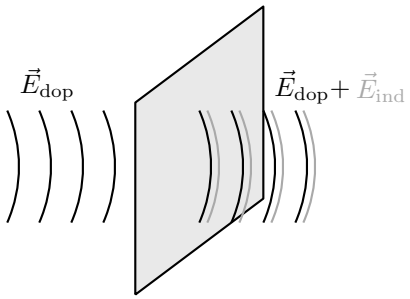


Figure 8.4: A full barrier must induce such a field that the total field behind the barrier is exactly zero.

Now conceptually divide the barrier into areas A and B, see figure 8.5. The induced field $\vec{E}_{\text{ind}}$ from the full barrier can be decomposed into induced fields from parts A and B (fields from atoms forming part A and B):

$$\vec{E}_{\text{ind}} = \vec{E}_{\text{ind}A} + \vec{E}_{\text{ind}B}. \tag{8.20}$$

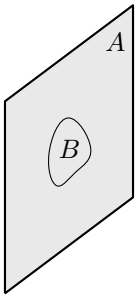


Figure 8.5: Barrier conceptually divided into two areas A and B.

Now we decide to remove part B of the barrier. What will now be the field behind barrier A, denote it $\vec{E}_A$? By removing part B, the induced field $\vec{E}_{\mathrm{ind}B}$ must disappear, and behind the barrier, there will be a superposition of the following fields:

$$\vec{E}_A = \vec{E}_{\mathrm{dop}} + \vec{E}_{\mathrm{ind}A}. \tag{8.21}$$

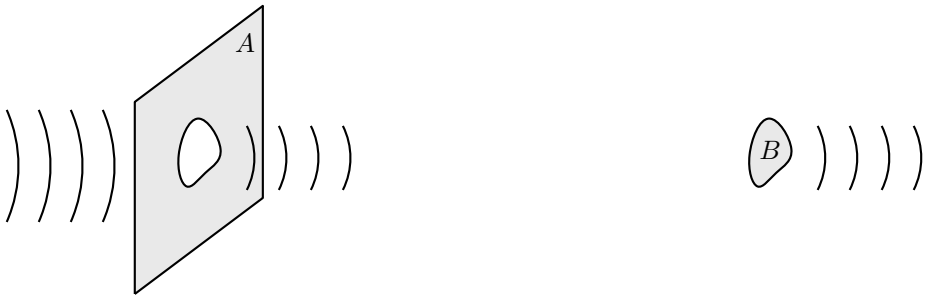
Substituting the decomposition (8.20) into (8.19), we get the relation

$$\vec{E}_{\mathrm{dop}} + \vec{E}_{\mathrm{ind}A} + \vec{E}_{\mathrm{ind}B} = 0. \tag{8.22}$$

Simply moving the term $\vec{E}_{\mathrm{ind}B}$ to the other side of the equation gives us the answer to our question:

$$\boxed{\vec{E}_A = -\vec{E}_{\mathrm{ind}B},} \tag{8.23}$$

meaning the field behind the barrier with the removed part B is exactly the same as if we only allowed part B of the barrier to emit (except for the sign, but the resulting intensity does not depend on the sign). See schematic figure 8.6. The equality (8.23) will greatly help us in solving the problem of diffraction.



(a) Field $\vec{E}_A = \vec{E}_{\mathrm{dop}} + \vec{E}_{\mathrm{ind}A}$.

(b) Field $\vec{E}_{\mathrm{ind}B}$.

Figure 8.6: Field behind the barrier shaped A (with aperture B), $\vec{E}_A$, is the same as the field emitted by plug B, $\vec{E}_{\mathrm{ind}B}$.

It is important to note that Babinet's principle only applies approximately. Why is that? Induced fields from parts of the barrier A and B influence each other. This means that after removing part B of the barrier, the field $\vec{E}_{\mathrm{ind}A}$ will necessarily change. See schematic figure 8.7. This effect is most pronounced at the boundary between A and B, the disappearance of the field $\vec{E}_{\mathrm{ind}B}$ will cause the most significant changes in the radiation of atoms around the boundary B. One of our assumptions will therefore be that the aperture B is sufficiently large compared to the wavelength of the passing light, $D \gg \lambda$, where we denoted the size of the aperture as D.

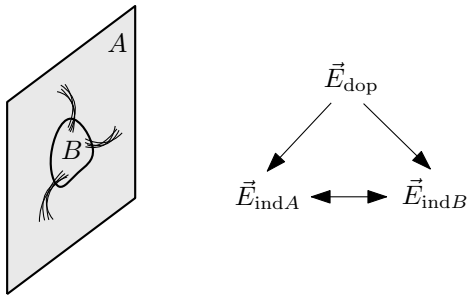


Figure 8.7: The incident field $\vec{E}_{\mathrm{dop}}$ induces fields radiated from barriers A and B, $\vec{E}_{\mathrm{ind}A}$ and $\vec{E}_{\mathrm{ind}B}$, and these influence each other.

8.2.2 Complementary Barriers

Barriers A and B, which complement each other to form a full barrier, are called *complementary*, see figure 8.8. Let's now look at how the diffraction patterns of these complementary barriers will differ.

(a) A without B. (b) B without A.

Figure 8.8: Complementary barriers.

Starting from the already proven relation (8.23) and by adding a suitable zero, we modify it:

$$\vec{E}_A = -\vec{E}_{\mathrm{ind}B} = -(\vec{E}_{\mathrm{ind}B} + \vec{E}_{\mathrm{dop}}) + \vec{E}_{\mathrm{dop}} = \vec{E}_{\mathrm{dop}} - \vec{E}_B, \tag{8.24}$$

where we denoted the field behind barrier B (without A), $\vec{E}_{\mathrm{dop}} + \vec{E}_{\mathrm{ind}B}$, as $\vec{E}_B$. The relation between fields for complementary barriers is thus

$$\boxed{\vec{E}_B = \vec{E}_{\mathrm{dop}} - \vec{E}_A.} \tag{8.25}$$

Babinet's principle is then identified as the identity $\vec{E}_A + \vec{E}_B = \vec{E}_{\mathrm{dop}}$, i.e., that by superimposing fields caused by the presence of barrier A, respectively B, we get the original field $\vec{E}_{\mathrm{dop}}$ in the absence of any barrier.

The significance of the identity (8.25) is as follows. In places where the incident field is zero, $\vec{E}_{\mathrm{dop}} = 0$, it applies behind the barrier

$$\vec{E}_B = -\vec{E}_A, \tag{8.26}$$

thus, the diffraction patterns from complementary screens are the same! (For clarity, we repeat that intensity, as a quantity proportional to the square of the electric field, does not depend on the signs.) Where the incident field is non-zero, there is interference between the incident field and the diffraction field from barrier A. The situation is illustrated by the following example with a laser beam falling on a screen. In the absence of any barrier, we observe on the screen the field $\vec{E}_{\mathrm{dop}}$ itself as a shining dot, see figure 8.9.

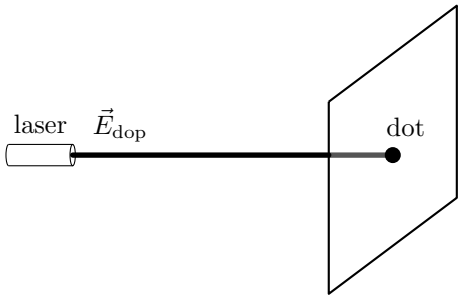
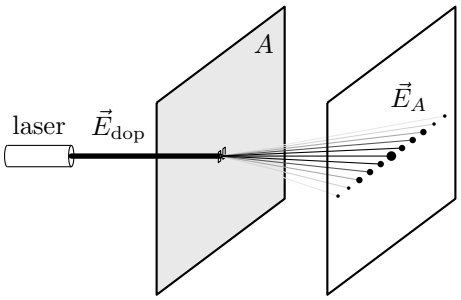
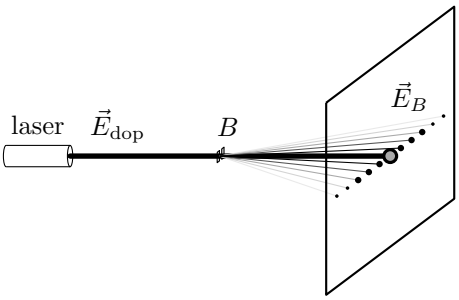


Figure 8.9: The laser beam representing the incident field $\vec{E}_{\mathrm{dop}}$ creates a shining dot on the screen.

If we now place barrier A in the path of the laser beam, we get the diffraction pattern $\vec{E}_A$, see figure 8.10 (a). Now, if we place the complementary screen in the path, besides the original shining dot, we observe exactly the same diffraction pattern, see figure 8.10 (b).



(a) Diffraction pattern $\vec{E}_A$ from barrier A.



(b) Diffraction pattern $\vec{E}_B$ from barrier B.

Figure 8.10: Diffraction patterns from complementary barriers – for example, two narrow slits and the corresponding complement. In the place on the screen outside the original shining dot, the patterns are identical. In the place of the original dot, the field $\vec{E}_B$ is given by the interference of the pattern $\vec{E}_A$ with the incident field $\vec{E}_{\mathrm{dop}}$.

8.2.3 Huygens-Fresnel Principle

Now we can proceed to the actual solution of the diffraction problem. For simplicity, we will only consider the case of a very distant light source in the axis of the aperture in the barrier. Because of this, a plane traveling electromagnetic wave falls perpendicularly on the barrier.

In the chapter about Babinet's principle, we showed (see relation (8.23)), that the diffraction pattern in the presence of barrier A (and thus the presence of an aperture at location B), $\vec{E}_A$, is given only by the induced field emitted by fictitious charges at the location of the aperture B, $\vec{E}_{\mathrm{ind}B}$. Charges are oscillated by the incident wave and then emit spherical waves themselves. Given that in the case of a full barrier the task has translational symmetry along the plane of the barrier, the amplitude of the emitted waves must be the same at every point of the barrier. The resulting field behind the barrier is then given by the superposition of spherical waves emitted by fictitious charges at the location of the aperture B. These considerations lead us to the following *diffraction integral* describing the field at point P behind the barrier:

$$\boxed{\vec{E}_P = \vec{E}_0 \int_B \frac{1}{r}\, e^{i(\omega t - kr)}\, dS,} \tag{8.27}$$

where B is the set of points forming the aperture in the barrier, dS is the area element in the plane of the barrier, and r is the distance between the current area element and point P. This distance both causes the amplitude of the emitted wave to decrease and the term $-kr$ determines the phase shift of the emitted wave at the location of point P. The amplitude $\vec{E}_0$ cannot be determined from our considerations. Therefore, our final prediction will be only the relative distribution of intensities on the screen.

In the diffraction integral, we completely neglected that the radiation of charges is not isotropic. Charges are oscillated in the plane of the barrier and will thus emit most strongly in the direction perpendicular to this plane. We limit ourselves to stating that our diffraction integral is valid only for such an area of the screen that is not too far from the perpendicular passing through the aperture in the barrier.

Now, we introduce Cartesian coordinates (X, Y) in the plane of the barrier and (x, y) in the plane of the screen as in figure 8.11. Denote the distance between the planes of the barrier and the screen as L. The area element has the expression $dS = dX\, dY$.

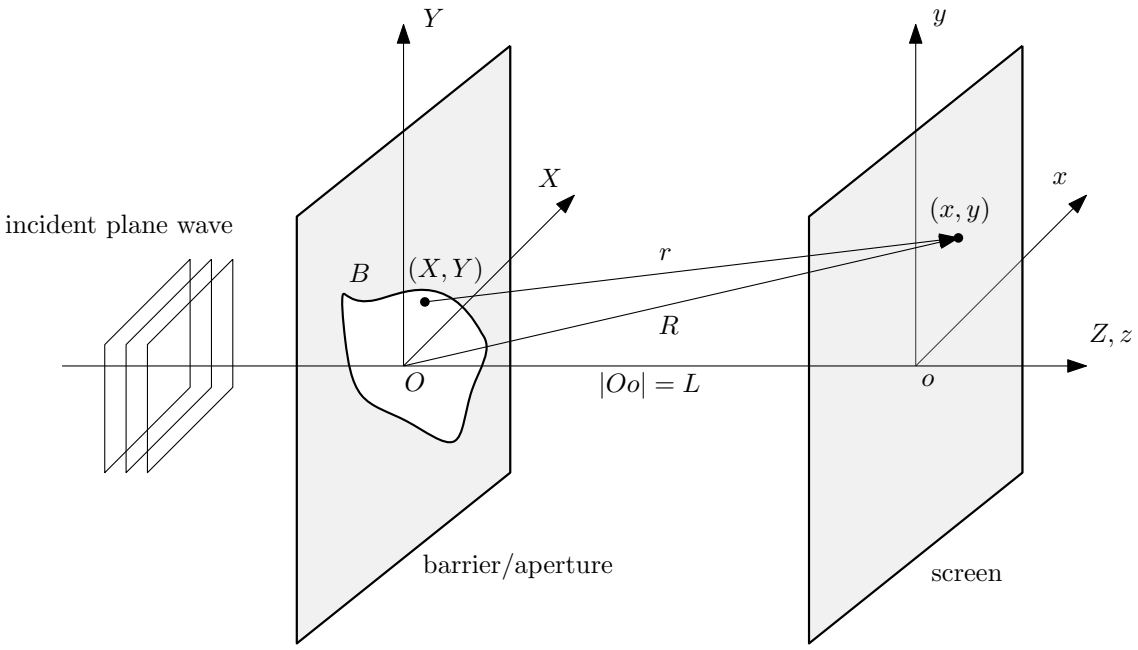


Figure 8.11: Introduction of Cartesian coordinates (X, Y) for the plane of the barrier and (x, y) for the plane of the screen. The distance r is then the distance between points (X, Y) and (x, y), the distance R is between the origin of the plane of the barrier and point (x, y) in the plane of the screen. The distance between the parallel planes of the barrier and the screen is L.

The distance r in the diffraction integral (8.27) has the expression

$$r = r(x, y, X, Y), \qquad r^2 = L^2 + (X - x)^2 + (Y - y)^2. \tag{8.28}$$

Thus, the diffraction task is generally solved (of course, with all the limitations that accompanied our derivation). For a given aperture B, we can use the diffraction integral (8.27) to determine the electric field $\vec{E}(x, y)$ on the screen and subsequently calculate the intensity distribution $I(x, y) = \langle \vec{E}^2(x, y) \rangle$. Due to not knowing the amplitude $\vec{E}_0$, we do not determine the absolute distribution of intensity but only the relative one.

Finally, let's explain the term *Huygens-Fresnel principle*. This principle says that points of the aperture in the barrier are sources of spherical waves (Huygens' principle) and the resulting field behind the barrier is obtained as their superposition (Huygens-Fresnel principle). Historically, this principle was postulated by Fresnel and directly leads to the diffraction integral (8.27). Here, we derived this integral (and thus the Huygens-Fresnel principle) based on Babinet's principle, the principle of superposition, and the study of radiation from oscillating charges.

8.2.4 Fraunhofer Diffraction

The diffraction integral (8.27) is generally very complex to compute. Here, through a series of approximations, we arrive at the simplest possible diffraction, which is the so-called *Fraunhofer diffraction*.

Let's now introduce the distance R, which indicates the distance of the location (x, y) on the screen from the origin O of the plane of the barrier, once again see figure 8.11. The coordinate expression of this distance is

$$R^2 = L^2 + x^2 + y^2. \tag{8.29}$$

We see that this does not depend on the coordinates (X, Y) and therefore from the perspective of integration in the plane of the barrier, the distance R is constant. We will now want to

express the distance r using the distance R and neglect some terms to simplify the expression under the integral. Substituting for L^2 in the expression for r (8.28) from the expression for R (8.29) we get

$$r^2 = R^2 + (X - x)^2 - x^2 + (Y - y)^2 - y^2. \tag{8.30}$$

By extracting the square root and extracting R, we obtain the sought expression for the relationship between distances r and R:

$$r = R\sqrt{1 + \frac{(X - x)^2 - x^2 + (Y - y)^2 - y^2}{R^2}}. \tag{8.31}$$

Now, let's approximate the square root using Taylor's expansion to the first order, $\sqrt{1+x} \approx 1 + \frac{x}{2}$; here, we must assume that the dimensions of the aperture in the barrier (ranges of coordinates X and Y) and the size of the area on the screen where we observe the diffraction pattern (ranges of coordinates x and y) are much smaller than the distance of the screen from the barrier L (R in the denominator we estimated by the distance L):

$$r \approx R\left(1 + \frac{(X^2 - 2Xx) + (Y^2 - 2Yy)}{2R^2}\right). \tag{8.32}$$

Additionally, if the aperture in the barrier is sufficiently small, we can further neglect the quadratic terms in the coordinates of the barrier X^2 and Y^2 compared to the linear terms $2Xx$ and $2Yy$. What exactly is meant by "sufficiently small" will be learned later when we derive the so-called criterion for Fraunhofer diffraction. After neglecting the quadratic terms, we arrive at the final approximation for the distance r in the form

$$r \approx R\left(1 - \frac{Xx + Yy}{R^2}\right) = R - \frac{Xx + Yy}{R}. \tag{8.33}$$

If we approximated even a bit more, we would simply get $r \approx R$. Now, let's substitute these approximations into the diffraction integral (8.27). Into the phase of the exponential, we substitute the approximation (8.33) and for the amplitude decrease, we use the even rougher approximation $r = R$. This leads us to the *Fraunhofer diffraction integral*:

$$\boxed{\vec{E}(x, y) = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \int_B e^{i\frac{k}{R}(Xx+Yy)}\, dX\, dY.} \tag{8.34}$$

Why did we use a rougher approximation for the amplitude than for the phase? The phase is more important for the interference phenomenon than the amplitude – the phase decides whether the interference will be constructive or destructive, whereas the amplitudes only decide how much contrast there will be in the interference[3]. Additionally, the phase function involves the wave number $k = \frac{2\pi}{\lambda}$, which is of the order of $10^7\ \mathrm{m}^{-1}$ for optical wavelengths, meaning even small phase differences are greatly amplified by the large wave number. The last addition is that if we used the approximation $r \approx R$ in the phase function, we would not get any non-trivial interference at all.

Mathematical digression. Denote new variables u and v as

$$u = \frac{k}{R}\, x, \qquad v = \frac{k}{R}\, y \tag{8.35}$$

and introduce the so-called *characteristic function* $f(X, Y)$ of set B, $f : \mathbb{R}^2 \to \mathbb{R}$, such that

$$f(X, Y) = \begin{cases} 1 & \text{for } (X, Y) \in B \\ 0 & \text{for } (X, Y) \notin B. \end{cases} \tag{8.36}$$

[3] I.e., how large will be the luminance difference between constructive and destructive interference.

Then the integral in Fraun

hofer's diffraction integral takes the form

$$E(u,v) \propto \int_{\mathbb{R}^2} f(X,Y)\, e^{i(uX+vY)}\, dX\, dY. \tag{8.37}$$

This is a 2D Fourier transform (compare with the 1D Fourier transform in section 3.2). From a mathematical point of view, the electric field on the screen predicted by Fraunhofer diffraction is given by the two-dimensional Fourier transform of the characteristic function f of the aperture in the barrier.

Now, let's derive the so-called *criterion for Fraunhofer diffraction*, i.e., under what circumstances we can confidently neglect the quadratic terms in the expression (8.32). If we did not neglect them, there would appear an additional phase shift in the phase function of the form

$$\Delta\varphi = k\frac{X^2+Y^2}{2R}. \tag{8.38}$$

If this additional term is to minimally affect the resulting interference pattern, it must hold[4] $\Delta\varphi \ll 1$. Express the wave number using the wavelength, $k = \frac{2\pi}{\lambda}$, and introduce the diameter D as the diameter of a thought circle into which the entire aperture B already fits, see figure 8.12.

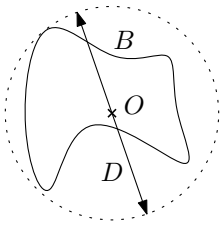


Figure 8.12: Diameter D of a thought circle into which the entire aperture in the barrier B fits.

Now, we can use the estimates $X^2+Y^2 \leq \frac{1}{4}D^2$ and $L \leq R$ in the expression for the phase shift (8.38):

$$\frac{2\pi}{\lambda}\frac{X^2+Y^2}{2R} \leq \frac{\pi}{4}\frac{D^2}{\lambda L} \ll 1 \tag{8.39}$$

We discard the numerical factor $\frac{\pi}{4}$ (increasing it to one) and thus arrive at the final criterion for Fraunhofer diffraction:

$$\boxed{L \gg \frac{D^2}{\lambda}, \quad D^2 \ll \lambda L.} \tag{8.40}$$

The resulting criterion thus quantifies how far the screen must be, or how small the aperture in the barrier must be, for us to confidently use Fraunhofer diffraction.

In the case that the criterion for Fraunhofer diffraction is not met, it is then referred to as *Fresnel diffraction*. Likewise, we talk about Fresnel diffraction in the case that the light source is not sufficiently far from the barrier with the aperture, so we cannot consider that plane waves are falling on the barrier. We will not deal with this more complex diffraction here.

Finally, let's look at the geometric meaning of the term $\frac{Xx+Yy}{R}$ in the Fraunhofer integral. Without loss of generality and for simplicity, let's set $Y, y = 0$ and study the expression only in

[4]Thus, we made a certain argumentative shift. In the end, it is not as important how large the quadratic term is compared to the linear one, but how much it contributes to the resulting phase shift, where a change in phase by π represents the difference between constructive and destructive interference.

the plane (x, z), respectively (X, Z). The situation is illustrated in figure 8.13. The main thing is the introduction of angle θ, which represents the angle under which we see a given point on the screen from the origin of the barrier, and the distance of the orthogonal projection l, again see figure 8.13. Then the relation $l = X \sin\theta = X\frac{x}{R}$ holds. The large distance of the screen causes that the rays emanating from the origin of the barrier and from a point at distance X are almost parallel and the difference in their lengths $R - r$ is approximately equal to l. In the Fraunhofer approximation, we thus imagine that all rays falling at a given point on the screen emanate parallelly from

the corresponding points in the aperture B under angle θ and the path differences are then simply given by the orthogonal projection between the individual rays.

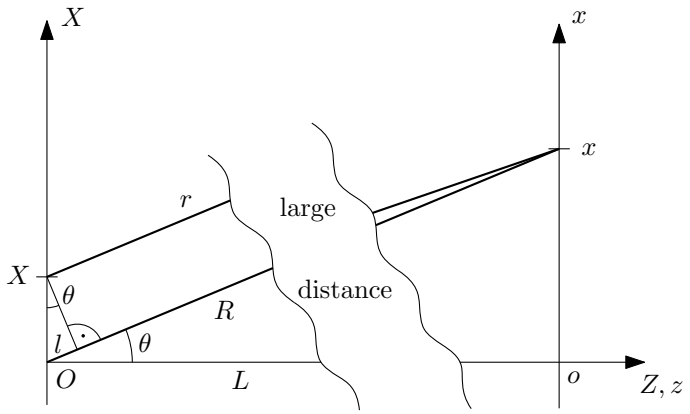


Figure 8.13: Geometric meaning of the phase function in Fraunhofer diffraction. The angle θ can be expressed as $\sin\theta = \frac{x}{R}$. The distance l is given as the size of the leg of the triangle formed by the orthogonal projection from point X to the ray emanating from the origin of the barrier. The size of l can be expressed as $l = X \sin\theta = \frac{Xx}{R}$.

In the following sections, we will study applications of the Fraunhofer integral to several basic shapes of the aperture in the barrier.

8.2.5 Young's Experiment

Young's experiment involves the study of diffraction at two rectangular slits. The main features of this experiment are preserved even if we simplify the situation by replacing rectangular openings with two point holes, see the schematic figure 8.14. The situation is further simplified by studying the resulting diffraction pattern only along the x axis, i.e., for $y = 0$. It is, of course, possible to calculate the entire situation with full preservation of the geometry of two rectangular slits, see exercise example 11.7.

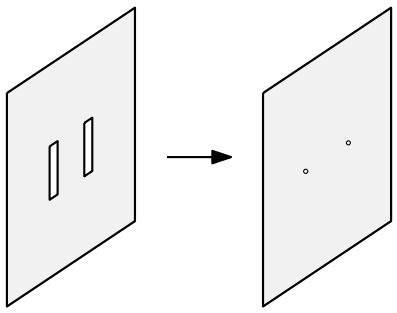


Figure 8.14: Young's experiment on a double slit. For simplicity, we replace rectangular slits with point holes.

Fraunhofer's integral (8.34) then changes to just a sum of two terms,

$$\int_B \quad \longrightarrow \quad \sum_{2 \text{ sources}}, \tag{8.41}$$

where the source positions are $Y = 0$ and $X = \pm\frac{d}{2}$, where d denotes the distance between point holes in the barrier, see figure 8.15.

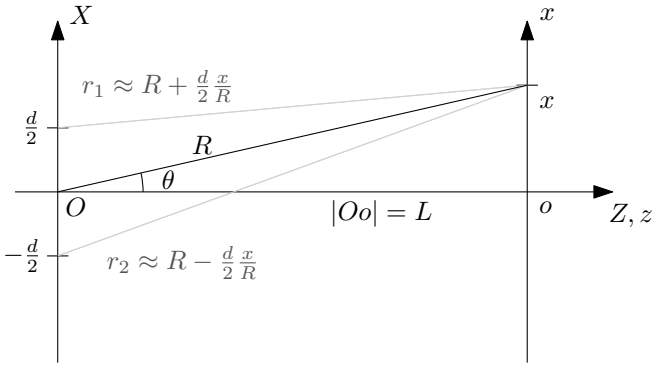


Figure 8.15: Young's experiment on a double slit.

Specifically, we get the following expression for the electric field along the x axis, $\vec{E}(x)$:

$$\begin{aligned}\vec{E}(x) &= \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \left(e^{i\frac{k}{R}\frac{d}{2}x} + e^{-i\frac{k}{R}\frac{d}{2}x} \right) = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} 2\cos\left(\frac{k}{R}\frac{d}{2}x\right) \\ &= \frac{2\vec{E}_0}{R} e^{i(\omega t - kR)} \cos\left(\frac{1}{2}kd\sin\theta\right), \end{aligned} \tag{8.42}$$

where we introduced the angle θ as in figure 8.13, i.e., as $\sin\theta = \frac{x}{R}$, see also figure 8.15. The intensity observed on the screen will be

$$I(\sin\theta) = \langle (\operatorname{Re}\vec{E})^2 \rangle = \frac{4E_0^2}{R^2} \langle \cos^2(\omega t - kR) \rangle \cos^2\left(\frac{1}{2}kd\sin\theta\right) = \frac{2E_0^2}{R^2}\cos^2\left(\frac{1}{2}kd\sin\theta\right). \tag{8.43}$$

The intensity obviously weakens with increasing distance from the screen as $\frac{1}{R^2}$, but the main feature is the interference term given by the square of the cosine. Points where this cosine reaches maxima are given by the condition

$$\frac{1}{2}kd\sin\theta = m\pi, \quad m \in \mathbb{Z}, \tag{8.44}$$

and thus diffraction maxima are seen on the screen approximately under angles

$$\boxed{\sin\theta_m = m\frac{\lambda}{d}, \quad m \in \mathbb{Z}.} \tag{8.45}$$

We see that the size of the "bending" (i.e., how much the maximum is deviated from the direct direction) is directly proportional to the wavelength λ – the longer the wavelength, the more light bends – and inversely proportional to the distance of the slits d – the closer the slits are, the more maxima are distanced from the direct direction. The number m is called

the *order of the maximum*, and the maximum observable order is evidently given by the following condition:

$$\sin\theta \le 1 \qquad m\frac{\lambda}{d} \le 1 \qquad m \le \frac{d}{\lambda}. \tag{8.46}$$

The course of intensity on the x axis on the screen is shown in figure 8.16 along with the appropriately marked maxima.

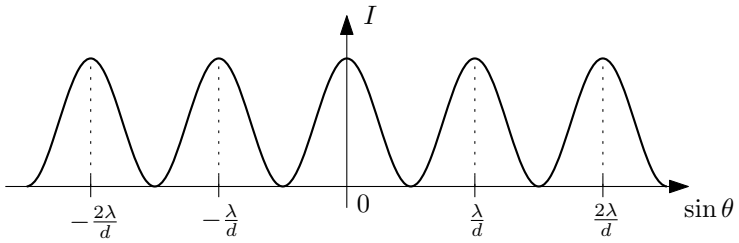


Figure 8.16: The course of intensity on the x axis on the screen for two point sources depicted as a function of the variable $\sin\theta$. Intensity maxima are at points $\sin\theta_m = m\frac{\lambda}{d}$, $m \in \mathbb{Z}$.

If we are directly interested in the Cartesian coordinates of the diffraction maxima on the screen, it is sufficient to express x from the definition $\sin\theta = \frac{x}{R}$:

$$x_m = R_m \sin\theta_m \approx L\, m\frac{\lambda}{d}, \tag{8.47}$$

where, for simplicity, we considered maxima near the origin, where we can approximate $R = \sqrt{L^2 + x^2} \approx L$. The distance between adjacent diffraction maxima on the screen is then

$$\boxed{\Delta x = x_{m+1} - x_m = L\,\frac{\lambda}{d}.} \tag{8.48}$$

8.2.6 Diffraction Grating

The generalization of Young's experiment is the case of a *diffraction grating*, where the number of rectangular slits is generalized to any natural number $N \in \mathbb{N}$, $N \geq 2$. Again, we simplify the situation by replacing finitely large rectangular slits with point holes, see figure 8.17. We will study the intensity pattern only along the x axis. The distance between adjacent points is again d.

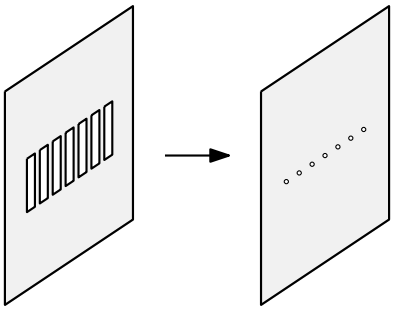


Figure 8.17: Diffraction grating. Again, for simplicity, we consider point holes and intensity on the screen along the x axis.

We introduce coordinates in the plane of the barrier so that the first hole lies at the origin and each subsequent one lies in the positive part of the X axis, i.e., at coordinates $X_j = j\,d$, $j \in \{0, 1, \ldots, N-1\}$, see figure 8.18.

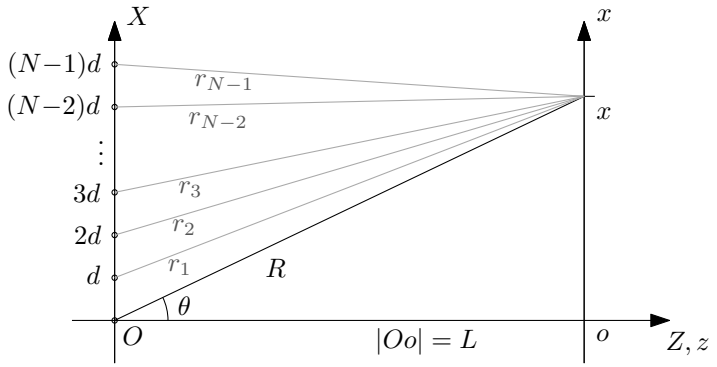


Figure 8.18: Introduction of coordinates for the case of a diffraction grating. Point holes lie at coordinates $X_j = j\,d,\ j \in \{0, 1, 2, \ldots, N-1\}$. Distances from individual holes to a point on the screen are in Fraunhofer's approximation $r_j \approx R - jd\frac{x}{R}$.

Fraunhofer's integral (8.34) is now, similarly to the case of Young's experiment (see e.g., (8.42)), a sum of spherical waves over individual point holes:

$$\vec{E}(x) = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \sum_{j=0}^{N-1} e^{i\frac{k}{R}(jd)x} = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \sum_{j=0}^{N-1} \left(e^{i\frac{k}{R}xd}\right)^j . \tag{8.49}$$

We apply the formula for the sum of a finite geometric series,

$$\sum_{k=0}^{N-1} x^k = \frac{x^N - 1}{x - 1}, \tag{8.50}$$

and suitably expand the resulting expression to eventually eliminate as many complex exponentials as possible by conversion to trigonometric functions:

$$\vec{E}(x) = \frac{\vec{E}_0}{R}\, e^{i(\omega t - kR)}\, \frac{e^{i\frac{k}{R}Nxd} - 1}{e^{i\frac{k}{R}xd} - 1} \cdot \frac{e^{-i\frac{k}{R}x\frac{d}{2}}}{e^{-i\frac{k}{R}x\frac{d}{2}}} \cdot \frac{e^{-i\frac{k}{R}Nx\frac{d}{2}}}{e^{-i\frac{k}{R}Nx\frac{d}{2}}}. \tag{8.51}$$

Then we just modify:

$$\vec{E} = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \frac{e^{i\frac{1}{2}kdN\frac{x}{R}} - e^{-i\frac{1}{2}kdN\frac{x}{R}}}{e^{i\frac{1}{2}kd\frac{x}{R}} - e^{-i\frac{1}{2}kd\frac{x}{R}}}\, e^{-i\frac{1}{2}kd(N-1)\frac{x}{R}} = \frac{\vec{E}_0}{R} e^{i(\omega t - kR - \frac{1}{2}kd(N-1)\frac{x}{R})} \frac{\sin\left(\frac{k}{R}Nx\frac{d}{2}\right)}{\sin\left(\frac{k}{R}x\frac{d}{2}\right)}. \tag{8.52}$$

The resulting intensity on the screen written using angle θ, $\sin\theta = \frac{x}{R}$, is

$$I = \langle(\mathrm{Re}\,\vec{E})^2\rangle = \frac{E_0^2}{2R^2} \left(\frac{\sin\left(\frac{1}{2}kNd\sin\theta\right)}{\sin\left(\frac{1}{2}kd\sin\theta\right)}\right)^2 . \tag{8.53}$$

Let's now examine, just as with Young's experiment, the location of diffraction maxima. These are now determined by a zero denominator (but the intensity limit obviously comes out finite). The denominator is zero under the condition

$$\frac{1}{2}kd\sin\theta = m\pi, \quad m \in \mathbb{Z}. \tag{8.54}$$

This condition is exactly the same as in the case of the double slit! See condition (8.44). Thus, we can adopt the results about the angular position of diffraction maxima (8.45), the number of diffraction maxima (8.46), and the distance of maxima on the screen (8.48):

$$\boxed{\sin\theta_m = m\frac{\lambda}{d}, \qquad m \leq \frac{d}{\lambda}, \qquad \Delta x = L\frac{\lambda}{d}.} \tag{8.55}$$

So, how does the case of a diffraction grating differ from Young's experiment with two slits? Let's look at the graph of intensity on the x axis on the screen in figure 8.19 for $N = 10$. Diffraction maxima are narrower depending on the number of slits, this narrowing is illustrated in figure 8.20.

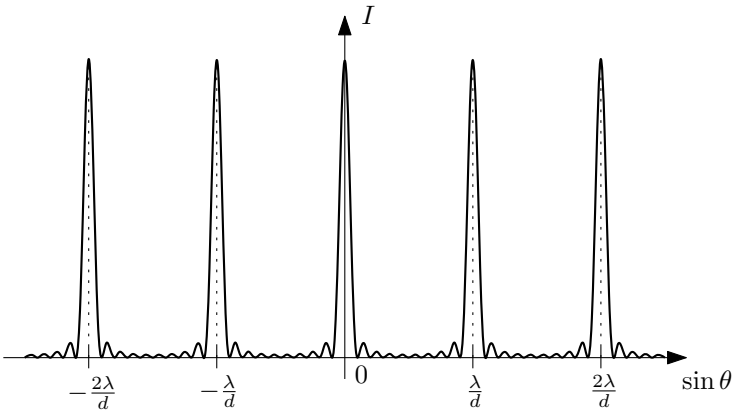


Figure 8.19: The course of intensity on the x axis on the screen for a diffraction grating depicted as a function of the variable $\sin\theta$. Intensity maxima are at points $\sin\theta_m = m\frac{\lambda}{d}$, $m \in \mathbb{Z}$. Here, specifically shown for $N = 10$ slits.

The width of the diffraction maximum is defined for simplicity as the distance between points where the intensity first reaches zero value around this maximum. These points are given by the first zeros of the numerator in the intensity (8.53), where the denominator is also non-zero. For the central maximum, we get:

$$\frac{1}{2}kNd\sin\theta_\pm = \pm\pi, \qquad \sin\theta_\pm = \pm\frac{1}{N}\frac{\lambda}{d}. \tag{8.56}$$

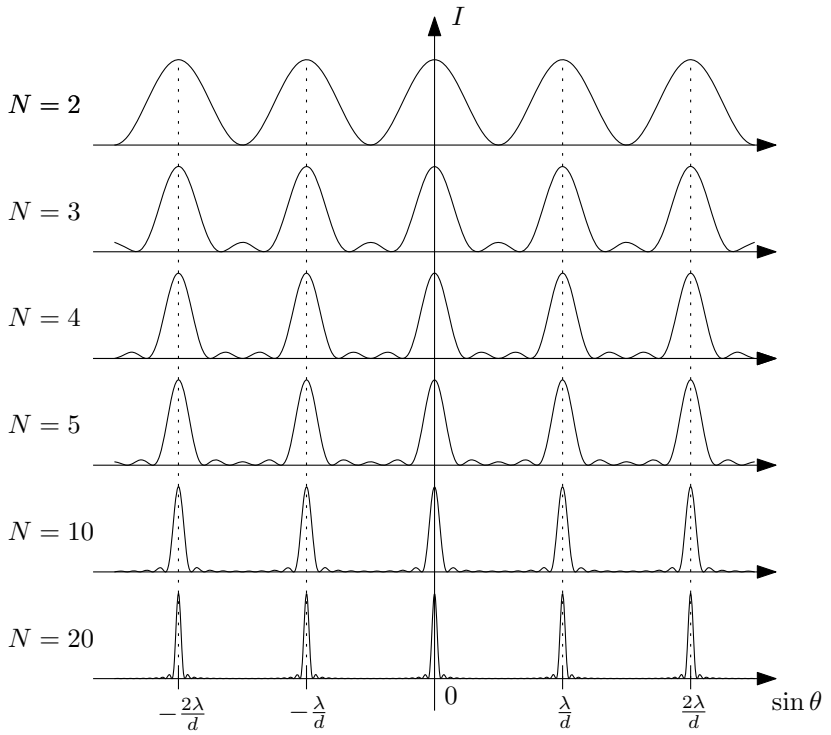


Figure 8.20: Width of diffraction maxima for different numbers of slits in the diffraction grating. The position of the maxima remains the same, only their width changes inversely proportional to the number of slits.

For the width of the maximum $\Delta(\sin\theta)$, we have:

$$\Delta(\sin\theta) = \sin\theta_+ - \sin\theta_- = \frac{2}{N}\frac{\lambda}{d}, \qquad \Delta\theta \approx \frac{2}{N}\frac{\lambda}{d}, \tag{8.57}$$

where we approximated $\sin\theta \approx \theta$.

Diffraction gratings are often used as spectrometers. The angles at which we observe diffraction maxima (except for the central maximum) depend on the wavelength of light. Therefore, if light composed of multiple wavelengths hits the diffraction grating, the maxima for different wavelengths will be displayed at different points on the screen. On the screen, we can determine from which wavelengths the light hitting the diffraction grating is composed – we can thus determine its spectrum.

However, we must not forget that the diffraction maxima have a finite width. Therefore, if the spectrum contains wavelengths that are too close to each other, we will not be able to distinguish them on the screen. A simple criterion for the resolving power of the grating is such that the distance between the diffraction maxima corresponding to different wavelengths must be greater than the width of these diffraction maxima. The position of the diffraction maxima for two wavelengths λ_1 and λ_2 is

$$\sin\theta_1 = m\frac{\lambda_1}{d}, \qquad \sin\theta_2 = m\frac{\lambda_2}{d}. \tag{8.58}$$

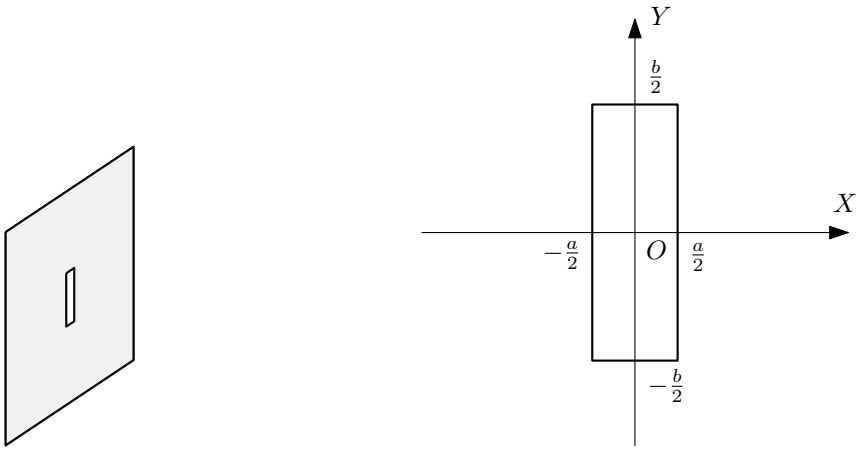
We want the distance between the maxima to be greater than the width of these maxima:

$$m\left|\frac{\lambda_1}{d} - \frac{\lambda_2}{d}\right| > \frac{2\lambda}{Nd}, \qquad |\lambda_1 - \lambda_2| > \frac{\lambda_1 + \lambda_2}{mN}. \tag{8.59}$$

where for the width of the maxima, we chose for simplicity the average wavelength $\lambda = \frac{\lambda_1+\lambda_2}{2}$.

8.2.7 Slit of finite width

The last application of the Fraunhofer integral, which we will look at in detail, is diffraction on a single rectangular slit, see the schematic figure 8.21 on the left.



(a) Rectangular slit.

(b) Coordinates of the rectangle in the barrier plane.

Figure 8.21: Diffraction on a rectangular slit with dimensions $a \times b$.

By direct application of the Fraunhofer integral (8.34), where the coordinate expression of the rectangular slit B is shown in the figure 8.21 on the right, we show:

$$\begin{aligned} E(x,y) &\propto \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} e^{i\frac{k}{R}(xX+yY)}\,dX\,dY = \left(\int_{-a/2}^{a/2} e^{i\frac{k}{R}xX}dX\right)\left(\int_{-b/2}^{b/2} e^{i\frac{k}{R}yY}dY\right) \\ &= \left[\frac{1}{i\frac{k}{R}x}e^{i\frac{k}{R}xX}\right]_{-a/2}^{a/2}\left[\frac{1}{i\frac{k}{R}y}e^{i\frac{k}{R}yY}\right]_{-b/2}^{b/2} = 4\frac{\sin\left(\frac{k}{R}\frac{a}{2}x\right)}{\frac{k}{R}x}\frac{\sin\left(\frac{k}{R}\frac{b}{2}y\right)}{\frac{k}{R}y} \\ &= ab\,\frac{\sin\left(\frac{1}{2}ka\frac{x}{R}\right)}{\frac{1}{2}ka\frac{x}{R}}\cdot\frac{\sin\left(\frac{1}{2}kb\frac{y}{R}\right)}{\frac{1}{2}kb\frac{y}{R}}. \end{aligned} \tag{8.60}$$

Since we cannot determine the amplitude $\vec{E}_0$ in the diffraction integral, we limited ourselves only to the calculation of the actual (phase) integral. If we are further only interested in the behavior of the electric field and intensity on the axis x (i.e., we consider $y = 0$), we get expressions

$$E(x,0) \propto \frac{\sin\left(\frac{1}{2}kd\sin\theta\right)}{\frac{1}{2}kd\sin\theta}, \qquad I(x,0) \propto \left(\frac{\sin\left(\frac{1}{2}kd\sin\theta\right)}{\frac{1}{2}kd\sin\theta}\right)^2, \tag{8.61}$$

where we denoted by $d = a$ the width of the slit in the direction of the axis x and again introduced the angle θ as $\sin\theta = \frac{x}{R}$. The intensity function (in the variable $\sin\theta$ is no longer periodic but has the form of the function $\left(\frac{\sin u}{u}\right)^2$, where $u = \frac{1}{2}kd\sin\theta$). The graph of intensity on the axis x is shown in the figure 8.22.

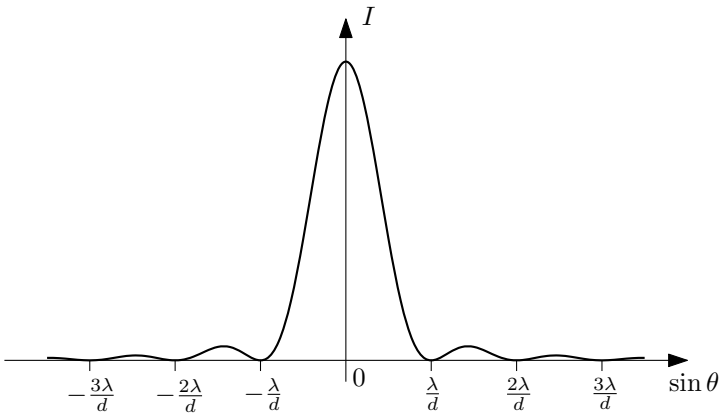


Figure 8.22: Intensity profile on the axis x on the screen for a slit of width depicted as a function of the variable $\sin\theta$. The position of the minima is at points $\sin\theta_m = m\frac{\lambda}{d}$, $m \in \mathbb{Z} \setminus \{0\}$.

The width of the central maximum is given by the distance of the first zeros of intensity, i.e.,

$$\frac{1}{2}kd\sin\theta_\pm = \pm\pi, \qquad \Delta(\sin\theta) = \sin\theta_+ - \sin\theta_- = \frac{2\lambda}{d}, \qquad \Delta\theta \approx \frac{2\lambda}{d}. \tag{8.62}$$

The angular size of the main maximum $\Delta\theta$ is also referred to as the size of the *angular divergence of the beam*. Once the propagation of the electromagnetic wave is limited by an aperture of some size, diffraction occurs at this aperture and the beam behind the aperture does not remain the same size but expands at a rate given by the corresponding angular divergence. Examples of limitations include the finite size of the output aperture of a laser system, the finite size of a telescope lens, etc. In other words, precise "parallelism" of a spatially limited beam cannot be achieved, but it will always diver

ge due to diffraction.

Let's look at the position of the minima next. These positions are given by the zeros of the numerator, i.e.,

$$\frac{1}{2}kd\sin\theta = m\pi, \quad m \in \mathbb{Z} \setminus \{0\}, \tag{8.63}$$

which is again the same condition as in the case of Young's experiment or a diffraction grating (only now it's about minima instead of maxima), i.e., the same relationships apply,

$$\boxed{\sin\theta_m = m\frac{\lambda}{d}, \qquad m \leq \frac{d}{\lambda}, \qquad \Delta x = L\frac{\lambda}{d},} \tag{8.64}$$

for the position of minima, the number of visible minima, and the distance of minima on the screen.

8.2.8 Diffraction on a circular aperture

Naturally, we may ask, what will diffraction on a circular aperture look like? Since the aperture has rotational symmetry, we expect the diffraction pattern to have the same symmetry.

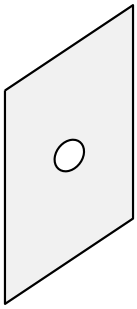


Figure 8.23: Diffraction on a circular aperture.

When calculating the diffraction pattern, it seems advantageous not to introduce Cartesian coordinates X, Y and x, y into the barrier and screen plane, but polar coordinates ρ, ϕ and r, φ. From the symmetry of the problem, then the resulting intensity will only depend on the coordinate r, $I(r)$. Unfortunately, despite the apparent simplicity of this problem, the resulting diffraction integral leads to special functions called Bessel functions $J_n(x)$, $n \in \mathbb{N}_0$. The resulting diffraction intensity has the form

$$I(\sin\theta) \propto \left(\frac{J_1(\frac{1}{2}kd\sin\theta)}{\frac{1}{2}kd\sin\theta} \right)^2, \tag{8.65}$$

where d is the diameter of the aperture and $\sin\theta = \frac{r}{R} = \frac{r}{\sqrt{L^2+r^2}}$. Bessel functions can be defined by integrals

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nu - x\sin u)\,du. \tag{8.66}$$

The shape of the function $I(\sin\theta)$ is illustrated in the figure 8.24. The central maximum, which is defined by the positions of the first zero points of intensity I, is determined by the first zero of the function $J_1(x)$ for $x > 0$. The value of x cannot be analytically calculated, so we express it in the form $x = \alpha\pi$, where $\alpha \approx 1.22$. Then the positions of the first minima on the screen will be at positions

$$\sin\theta = \pm\alpha\frac{\lambda}{d}. \tag{8.67}$$

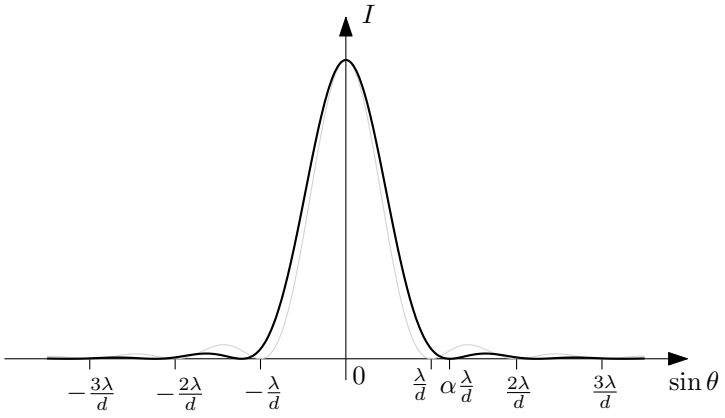


Figure 8.24: The shape of the function $I(\sin\vartheta)$ for diffraction on a circular aperture – depicted in black. For comparison, the course of the function $I(\sin\vartheta)$ for a (rectangular) slit of width d is shown in gray. The central maximum for the circular aperture is slightly wider than for the slit.

8.2.9 Influence of coherence on the visibility of the diffraction pattern

Now let's address the question of what can prevent the observation of the diffraction phenomenon. Since diffraction is an interference phenomenon, the coherence of the light source that causes diffraction will affect its visibility.

Diffraction is caused by path differences (and thus induced phase differences) of waves originating at the aperture in the barrier and hitting a specific spot on the screen. If we illuminate the barrier with a point source of light with a coherence time t_{koh}, then the difference in paths of the individual beams must be much less than $c\,t_{\mathrm{koh}}$. At the end of

the chapter 8.2.4 on Fraunhofer diffraction, we showed that the path difference is of the size $d \sin\theta$, where d is the size of the aperture in the barrier. For thermal sources, we have $t_{\mathrm{koh}} \approx 10^{-9}\,\mathrm{s}$, and thus $c\,t_{\mathrm{koh}} \approx 30\,\mathrm{cm}$. Therefore, the temporal coherence of thermal sources will not be a problem for apertures in the barrier much smaller than 30 centimeters. Otherwise, the contrast of the interference pattern and thus the visibility of diffraction will decrease.

What about in the case of a non-point light source? In a thermal non-point source, its individual points are sources of spatially incoherent waves – based on the knowledge of the wave emitted from one point of the source, we cannot predict the wave coming out of another place of the source. As a simple example, consider a non-point light source consisting of three discrete point sources as in figure 8.25.

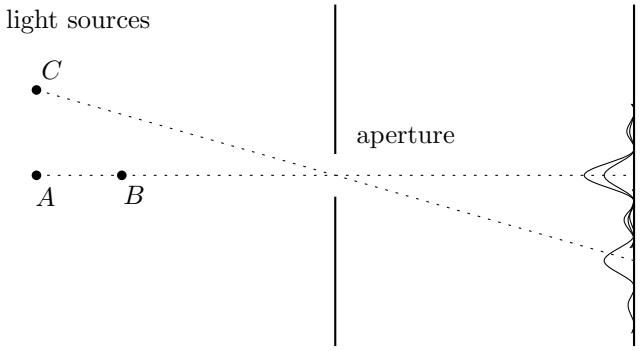


Figure 8.25: Non-point light source formed by points A, B, and C. Points A and B are placed on the axis of the aperture perpendicular to the barrier (and screen). Point C, on the other hand, is displaced off-axis. On the screen, the diffraction patterns from individual sources A, B, and C are then schematically depicted.

In the section on the influence of coherence on the visibility of interference, we showed that incoherent waves do not interfere with each other – thus, the intensities from individual waves simply add up. So, the patterns from sources A and B will simply overlap, and the interference pattern will remain unchanged (see schematic graphs on the screen in figure 8.25). It remains now to determine what the diffraction pattern from source C – that is, the source transversely displaced off the axis of the aperture in the barrier – will look like. We have not yet considered this situation – we always took plane waves falling perpendicularly on the barrier.

Fortunately, the answer is relatively simple. In such a case, the main diffraction maximum will be displaced off the axis of the aperture so that the line passing through source C and the maximum on the screen goes through the center of the aperture in the barrier, see figure 8.26.

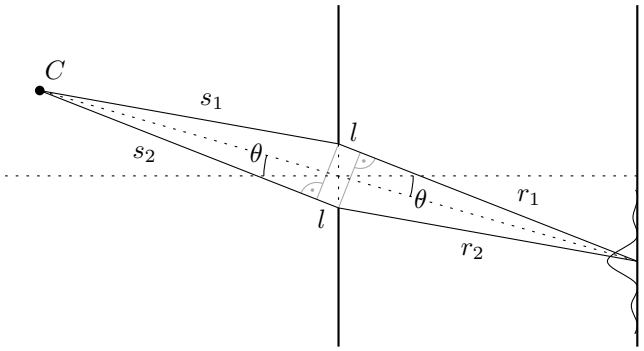


Figure 8.26: Paths of the extreme rays coming from source C hitting one spot on the screen. The lengths of these rays are $s_1 + r_1$ and $s_2 + r_2$. Constructive interference occurs where these paths equal. In this case, it also holds that $s_2 - s_1 = r_2 - r_1 = l = d \sin\theta$, where d is the size of the aperture.

Displacement of the source off the axis of the aperture in the barrier thus causes a displacement of the diffraction pattern on the screen. The superposition of diffraction patterns from very close sources A and C will appear as a blurring of the diffraction pattern caused only by source A. A continuously distributed (non-point) light source will then create an infinite superposition of non-interfering diffraction patterns on the screen – this superposition will appear as a blurring of the original diffraction pattern. If the size of the light source is so large that the position of the main diffraction maxima from the extreme points of the light source reaches the diffraction maxima of the first order from the central points of the light source, the diffraction pattern will completely disappear. The previous sentence will serve as a criterion for determining the limit of the size of the light source so that the blurring of the diffraction pattern is not too great. Let's look at figure 8.27, where we introduce the necessary geometric quantities.

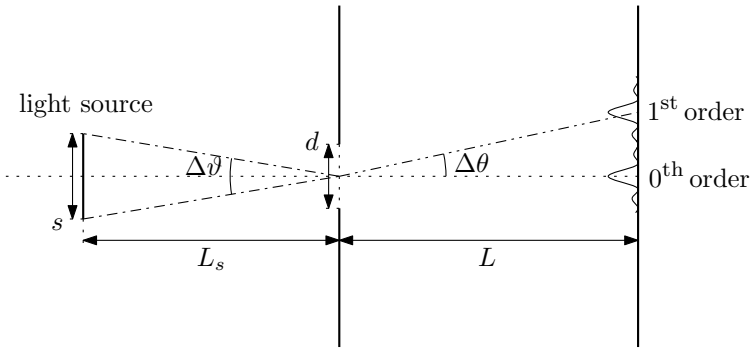


Figure 8.27: We consider a light source just wide enough s at a distance L_s from the barrier (with angular size $\Delta\vartheta$) that causes the overlap of the zeroth-order maximum from the edge points of the source with the first-order maximum from the central points of the source.

Quantitatively, the requirement for sufficient unblurredness is written as

$$\boxed{\Delta\vartheta \ll \Delta\theta.} \tag{8.68}$$

The distance of interference maxima for a point light source is $\Delta\theta = \frac{\lambda}{d}$ and for small $\Delta\vartheta$ we can write $\Delta\vartheta \approx \frac{s}{L_s}$. We can then get different forms of inequality (8.68) depending on the specific application:

$$\frac{s}{L_s} \ll \frac{\lambda}{d}, \qquad \text{resp.} \qquad s \ll L_s \frac{\lambda}{d}, \qquad \text{resp.} \qquad d \ll \frac{\lambda}{\Delta\vartheta}. \tag{8.69}$$

Finally, let's give a small example. The angular size of the Sun in the sky is about $\Delta\vartheta = 30'$. Considering that the dominant wavelength in visible light is $\lambda = 600\,\mathrm{nm}$, then the above criterion gives $d \ll 70\,\mu\mathrm{m}$. Thus, to be able to observe the diffraction of sunlight, it would have to pass through apertures smaller than $70\,\mu\mathrm{m}$ ¡ 0.1 mm!

Chapter 9

*Geometric Optics

9.1 *Transition from wave optics to geometric optics

The concepts we have worked with so far – in wave optics – were, for example: wave, amplitude, wavefront, interference. In contrast, in geometric optics, the main concept is the ray. Our main task will therefore be to introduce this concept – for this, we will have to make some approximations, which will mean that geometric optics will only be an approximate theory of light propagation. We will see that our approximation will require $\lambda \to 0$, thus in geometric optics, among other things, we completely neglect diffraction phenomena, which are directly proportional to the wavelength λ.

In the chapter on plane electromagnetic waves, we showed that the individual components of the electric and magnetic field are governed by the wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \Delta \psi, \qquad \psi \in \{E_i, B_j\}. \tag{9.1}$$

Let's generally decompose the wave $\psi(\vec{r}, t)$ solving the wave equation (9.1) into an "amplitude" function $A(\vec{r}, t)$ and a phase function $\varphi(\vec{r}, t)$ as

$$\psi(\vec{r}, t) = A(\vec{r}, t)\, e^{i\varphi(\vec{r}, t)}. \tag{9.2}$$

Without additional assumptions, this decomposition is completely ambiguous. We will therefore require that the amplitude function A changes only very slowly over distances of the order of λ. In contrast, the phase function φ will change very rapidly – by a value of 2π over a distance of λ.

Now we want to find, among other things, the so-called *local propagation directions* indicating in which direction the wave propagates at each point in space. In the case of a plane traveling harmonic wave, the situation is simple – the direction of propagation is the same everywhere in space and given by the wave vector $\vec{k}$, which is also perpendicular to the plane wavefronts (surfaces of constant phase) everywhere in space – in this case, we can talk about a global wave vector $\vec{k}$. Local directions for a general wavefront are obtained by approximating individual small parts of the wavefront with a plane and by comparing it with the corresponding equivalent plane traveling wave, we determine the corresponding local quantities.

Therefore, let's expand the phase function $\varphi(\vec{r}, t)$ to the first order of Taylor's expansion in variables $\vec{r}$ and t around the point $\vec{r}_0$ at time t_0:

$$\begin{aligned} \varphi(\vec{r}, t) &= \varphi(\vec{r}_0, t_0) + \sum_{j=1}^{3} \frac{\partial \varphi(\vec{r}_0, t_0)}{\partial x_j}(x_j - x_{j0}) + \frac{\partial \varphi(\vec{r}_0, t_0)}{\partial t}(t - t_0) + \dots \\ &= \varphi(\vec{r}_0, t_0) + \operatorname{grad} \varphi(\vec{r}_0, t_0) \cdot (\vec{r} - \vec{r}_0) + \frac{\partial \varphi(\vec{r}_0, t_0)}{\partial t}(t - t_0) + \dots \end{aligned} \tag{9.3}$$

We see that in the first order, we can write the phase function as

$$\varphi(\vec{r},t) = \varphi_0 + \vec{k}\cdot(\vec{r}-\vec{r}_0) - \omega(t-t_0), \tag{9.4}$$

where

$$\boxed{\vec{k} = \operatorname{grad}\varphi, \qquad \omega = -\frac{\partial\varphi}{\partial t}.} \tag{9.5}$$

By expanding the phase function $\varphi(\vec{r},t)$ to the first order of Taylor's polynomial around the point $(\vec{r}_0,t_0)$, we replaced the general wavefront defined by the equation $\varphi(\vec{r},t) = \text{const.}$ around the point $\vec{r}_0$ at time t_0 with a plane traveling wave, see schematic figure 9.1. The vector field $\vec{k}(\vec{r},t)$, or scalar field $\omega(\vec{r},t)$, we call *local wave vector*, or *local angular frequency*, respectively. Naturally, the local wave vector $\vec{k}$ defines the *local wave number* $k(\vec{r},t) = |\vec{k}(\vec{r},t)|$, *local wavelength* $\lambda(\vec{r},t) = \frac{2\pi}{k(\vec{r},t)}$ and the local direction of propagation $\vec{n}(\vec{r},t) = \frac{\vec{k}(\vec{r},t)}{k(\vec{r},t)}$. Similarly, we can define the so-called *local propagation speed* $v(\vec{r},t) = \frac{\omega(\vec{r},t)}{k(\vec{r},t)}$, or its vector form by adding the direction of propagation $\vec{v}(\vec{r},t) = v(\vec{r},t)\,\vec{n}(\vec{r},t)$. Since the vector $\vec{k}$ is given as the gradient of the phase function, $\vec{k} = \operatorname{grad}\varphi$, it is perpendicular to the wavefronts due to the property of the gradient; the same applies to the local speed $\vec{v}$, as it is just a multiple of the wave vector, $\vec{v} \propto \vec{k}$.

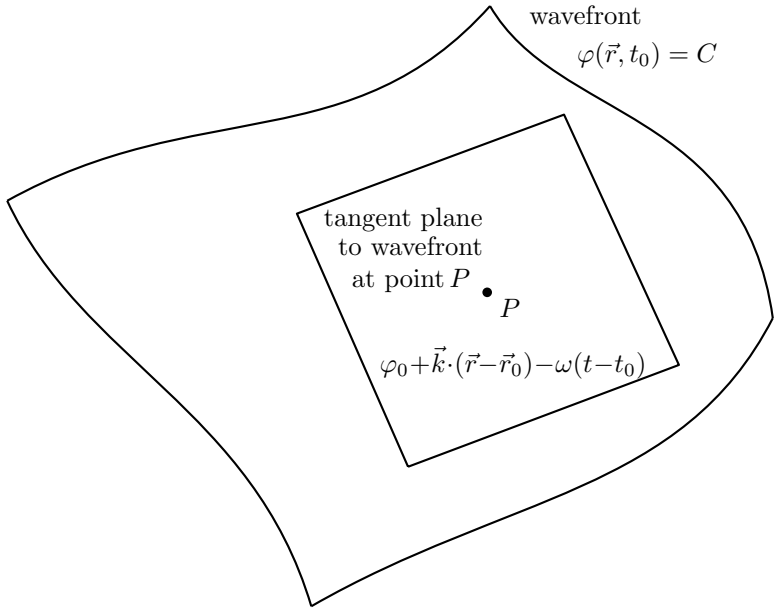


Figure 9.1: Replacement of the precise wavefront at time t_0 with a plane at point P (whose position is described by vector $\vec{r}_0$).

Rays, the central concept of geometric optics, are defined as curves $\vec{r}(t) : \mathbb{R} \to \mathbb{R}^3$ satisfying the differential equation

$$\frac{d\vec{r}(t)}{dt} = \vec{v}(\vec{r}(t),t), \tag{9.6}$$

where on the right appears the local propagation speed at the current location of the "end" of the ray. This equation is simply an expression of the requirement that the speed of the ray $\frac{d\vec{r}}{dt}$ is equal to the local propagation speed $\vec{v}$ at each point. Curves whose derivative equals a given vector field are generally called *integral curves* (of the given vector field) in mathematics. Since the speed $\vec{v}$ is perpendicular to the wavefronts everywhere, the rays are also perpendicular to the wavefronts they pass through at every point. The course of the rays relative to the wavefronts is schematically shown in figure 9.2.

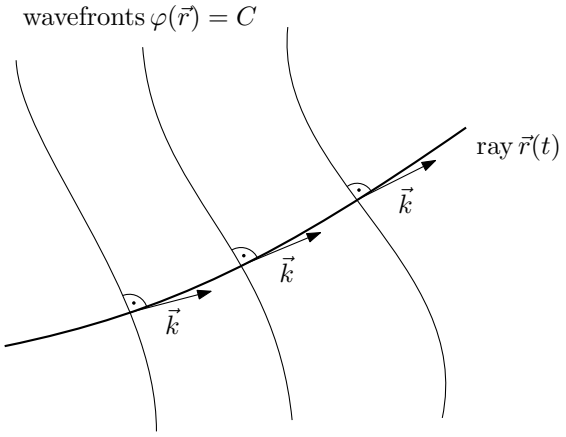


Figure 9.2: A ray as an integral curve of the vector field $\vec{v}$, or $\vec{k}$. Since the vector $\vec{k} = \operatorname{grad}\varphi$ is perpendicular to the wavefronts everywhere, the rays are also perpendicular to the wavefronts.

Purely based on the knowledge of the phase function $\varphi(\vec{r}, t)$, we were able to define local wave quantities and from them derive the paths of rays in the given wave. Now we need to find a differential equation that will determine the phase function. We defined the phase function as part of the "total" wave $\psi(\vec{r}, t)$ satisfying the wave equation (9.1) according to the ansatz (9.2). Let's first calculate the individual derivatives of this ansatz occurring in the wave equation:

$$\frac{\partial \psi}{\partial t} = \left[\frac{\partial A}{\partial t} + iA\frac{\partial \varphi}{\partial t}\right] e^{i\varphi}, \tag{9.7}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \left[\frac{\partial^2 A}{\partial t^2} + 2i\frac{\partial A}{\partial t}\frac{\partial \varphi}{\partial t} + iA\frac{\partial^2 \varphi}{\partial t^2} - A\left(\frac{\partial \varphi}{\partial t}\right)^2\right] e^{i\varphi}, \tag{9.8}$$

and analogously for spatial derivatives we get

$$\Delta\psi = \sum_{j=1}^{3}\left[\frac{\partial^2 A}{\partial x_j^2} + 2i\frac{\partial A}{\partial x_j}\frac{\partial \varphi}{\partial x_j} + iA\frac{\partial^2 \varphi}{\partial x_j^2} - A\left(\frac{\partial \varphi}{\partial x_j}\right)^2\right] e^{i\varphi}. \tag{9.9}$$

Now we want to estimate the magnitudes of individual terms in the derivatives (9.8) and (9.9) based on approximations resulting from both the slow change of the amplitude function A and from the approximation of geometric optics, when we consider $\lambda \to 0$ (and thus $k = \frac{2\pi}{\lambda} \to +\infty$, $\omega = vk \to +\infty$). According to (9.5), the first spatial derivatives of φ are proportional to the first power of the wave number k size, and the first time derivative of φ as well (since ω is directly proportional to k):

$$\frac{\partial \varphi}{\partial x_i} \propto k^1, \qquad \frac{\partial \varphi}{\partial t} \propto k^1, \qquad \left(\frac{\partial \varphi}{\partial x_j}\right)^2 \propto k^2, \qquad \left(\frac{\partial \varphi}{\partial t}\right)^2 \propto k^2. \tag{9.10}$$

The amplitude and its derivatives do not significantly change with the change in wavelength, so we can write

$$A \propto k^0, \qquad \frac{\partial A}{\partial x_j} \propto k^0, \qquad \frac{\partial A}{\partial t} \propto k^0, \qquad \frac{\partial^2 A}{\partial x_j^2} \propto k^0, \qquad \frac{\partial^2 A}{\partial t^2} \propto k^0. \tag{9.11}$$

What about the second derivatives of the phase function? These represent the change in the wave vector in space and the change in angular frequency over time. We also assume that these

change slowly and the magnitude of their change does not scale with the size k, so we can consider

$$\frac{\partial^2\varphi}{\partial x_j^2} \propto k^0, \qquad \frac{\partial^2\varphi}{\partial t^2} \propto k^0. \tag{9.12}$$

In the limit $k \to +\infty$, then only terms with the highest power of k will remain in the derivatives (9.8) and (9.9) – these are precisely the quadratic terms of the first derivatives, which are proportional to k^2. From the wave equation then in the approximation of geometric optics only remains

$$\boxed{\left(\frac{\partial\varphi}{\partial t}\right)^2 = v^2 \sum_{j=1}^{3} \left(\frac{\partial\varphi}{\partial x_j}\right)^2.} \tag{9.13}$$

This equation is called the *eikonal equation*. In this text, let's further consider that everywhere in space we have a monochromatic wave with angular frequency ω_0, thus

$$-\frac{\partial\varphi(\vec{r},t)}{\partial t} = \omega(\vec{r},t) = \omega_0 = \text{const.} \tag{9.14}$$

If we solve this simple differential equation, we get the following form of the phase function:

$$\varphi(\vec{r},t) = -\omega_0 t + \varphi_0(\vec{r}), \tag{9.15}$$

where the function $\varphi_0(\vec{r})$ emerged as an integration constant when integrating over time. In the phase function, we have separated the dependence on time and space, and the temporal dependence is very trivial. Substituting into the eikonal equation (9.13) we get

$$\boxed{\sum_{j=1}^{3} \left(\frac{\partial\varphi_0}{\partial x_j}\right)^2 = \frac{\omega_0^2}{v^2}.} \tag{9.16}$$

This equation we may call the *time-independent eikonal equation*. If we look into the definition of local quantities (9.5) (and below in the text) and substitute the form of the phase function (9.15) we find out that for monochromatic waves all local quantities are constant in time and only depend on spatial coordinates:

$$\vec{k}(\vec{r}) = \operatorname{grad}\varphi_0(\vec{r}), \qquad \omega = \omega_0, \qquad \lambda(\vec{r}) = \frac{2\pi}{|\vec{k}(\vec{r})|}, \qquad \textit{etc.} \tag{9.17}$$

We will look at the significance and solution of the (time-independent) eikonal equation after an important generalization in the following section.

9.2 *Inhomogeneous Media

So far, our results are not very interesting, since we have so far only considered homogeneous media, where the phase velocity v and therefore the refractive index $n = \frac{c}{v}$ are constant. Let's therefore consider a medium where the refractive index varies in space – $n(\vec{r})$. Since $n \approx \sqrt{\varepsilon_r} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$, we thus have a variable permittivity in space as well – $\varepsilon(\vec{r})$. How will these new assumptions change the derivations we have made so far?

Let's return to the derivation of the wave equation for the electric field $\vec{E}$ from the chapter 6.1 on electromagnetic waves. Gauss's law for non-constant permittivity ε in a medium without free charges has the form:

$$0 = \operatorname{div}(\varepsilon\vec{E}) = \varepsilon \operatorname{div}\vec{E} + (\operatorname{grad}\varepsilon)\cdot\vec{E} \quad \Rightarrow \quad \operatorname{div}\vec{E} = -\frac{1}{\varepsilon}(\operatorname{grad}\varepsilon)\cdot\vec{E}. \tag{9.18}$$

When calculating the double curl of the electric field $\vec{E}$ (6.3), the term $\operatorname{grad}\operatorname{div}\vec{E}$ does not disappear:

$$\operatorname{rotrot}\vec{E} = \underbrace{\operatorname{grad}\operatorname{div}\vec{E}}_{\neq 0} - \Delta\vec{E} = -\operatorname{grad}\left(\frac{1}{\varepsilon}(\operatorname{grad}\varepsilon)\cdot\vec{E}\right) - \Delta\vec{E}. \tag{9.19}$$

The resulting wave equation for the electric field then takes a modified form compared to (6.5):

$$\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} - \Delta\vec{E} = \operatorname{grad}\left(\frac{1}{\varepsilon}\operatorname{grad}\varepsilon\cdot\vec{E}\right). \tag{9.20}$$

How will the additional term manifest when deriving the eikonal equation? Fortunately, not at all. The term on the right side of the modified wave equation (9.20) contains zeroth, or first, spatial derivatives of the components of the electric field – E_i and $\frac{\partial E_i}{\partial x_j}$. These are proportional to at most the first power of the local wave number k: k^1. In the limit of geometric optics, when $\lambda \to 0$ and thus $k \to +\infty$, only terms that we have already retained in the original eikonal equation (9.16), i.e., terms proportional to k^2, will prevail. The eikonal equation thus remains the same, the only thing that appears additionally is the dependence of the refractive index on position:

$$\boxed{\sum_{j=1}^{3}\left(\frac{\partial\varphi_0}{\partial x_j}\right)^2 = \frac{\omega_0^2}{c^2}n^2(\vec{r}).} \tag{9.21}$$

The local propagation speed (local phase velocity) v is now variable – it can be different at every point in space – $v(\vec{r})$:

$$v(\vec{r}) = \frac{c}{n(\vec{r})}. \tag{9.22}$$

Equation (9.21) can be written in a more geometric manner. On its left side appears the square of the magnitude of the gradient of the phase function φ_0, so we can write:

$$\boxed{|\operatorname{grad}\varphi_0| = \frac{\omega_0}{v(\vec{r})}.} \tag{9.23}$$

The rate of growth of the phase function $\varphi_0(\vec{r})$ is therefore determined at each point in space by the reciprocal value of the instantaneous phase velocity $v(\vec{r})$. How to imagine the solution of the eikonal equation (9.23)? And how do we find the course of specific rays from the resulting phase function $\varphi_0(\vec{r})$? We will perform the solution in several steps (which can also be schematically followed in figure 9.3):

1. First, we need to choose an initial wavefront $\varphi_0(\vec{r}) = C = \text{const.}$ – i.e., we select points in space (forming a surface) and assign them an arbitrary initial phase value C. The simplest choices are, for example, a plane or a spherical surface. At each point of this surface, we construct unit normal vectors $\vec{n}$ – these will represent the directions of the greatest growth of the phase function φ_0. The magnitude of this growth is prescribed by the eikonal equation (9.23).

2. Now I want to get the "next" wavefront defined by the condition $\varphi_0(\vec{r}) = C + dC$ – i.e., look at what the super-surface with the phase infinitesimally increased by the value dC will look like. Since the magnitude of the gradient indicates the magnitude of growth of a given function along the normal vector from the given isosurface, it is necessary to move by a distance given by a simple equation $|\operatorname{grad}\varphi_0|\,dl = dC$, thus by a distance

$$dl = \frac{dC}{|\operatorname{grad}\varphi_0|} = \frac{v}{\omega_0}dC, \tag{9.24}$$

where in the second equality we substituted from the eikonal equation. In this way (by infinitesimally increasing the constant C and gradually constructing wavefronts), I "develop" the phase function $\varphi_0(\vec{r})$. This process is schematically shown in two steps in figure 9.3. This construction actually represents Huygens's construction of wavefronts, where every point of the "current" wavefront becomes a source of secondary spherical waves, whose common boundary forms a new wavefront.

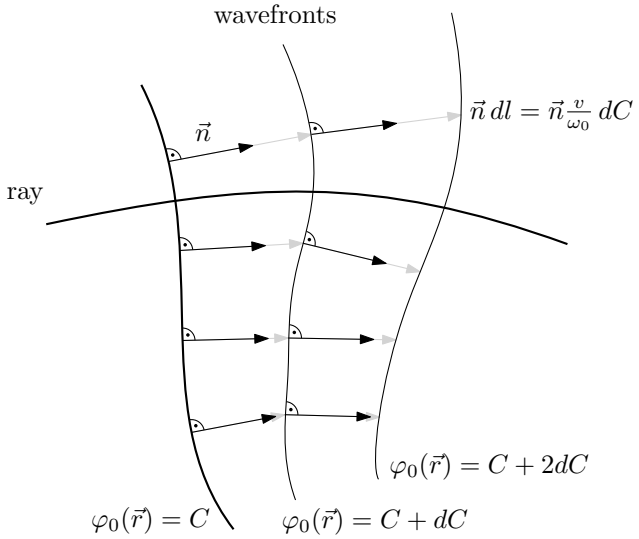


Figure 9.3: Solving the eikonal equation. Bold on the left is the initial super-surface $\varphi_0(\vec{r}) = C$. Perpendicular to it are shown the black unit normal vectors. Further wavefronts with the phase value infinitesimally increased by multiples of dC are obtained by moving in the direction of normal vectors by a distance $dl = \frac{dC}{|\mathrm{grad}\,\varphi_0|} = \frac{v}{\omega_0}dC$. The rays are then given as curves everywhere perpendicular to the found wavefronts.

3. Rays $\vec{r}(t)$ we defined as curves satisfying the differential equation (9.6), $\frac{d\vec{r}}{dt} = \vec{v}$. Now just substitute for the speed:

$$\vec{v} = v\,\vec{n} = v\frac{\vec{k}}{k} = \frac{v^2}{\omega_0}\vec{k} = \frac{v^2}{\omega_0}\mathrm{grad}\,\varphi_0, \tag{9.25}$$

thus using the phase function $\varphi_0(\vec{r})$ we find the rays as solutions to the differential equation

$$\frac{d\vec{r}}{dt} = \frac{v^2}{\omega_0}\,\mathrm{grad}\,\varphi_0. \tag{9.26}$$

Let's finally look at the above-described method of solving the eikonal equation on a specific case, which is described in figure 9.4.

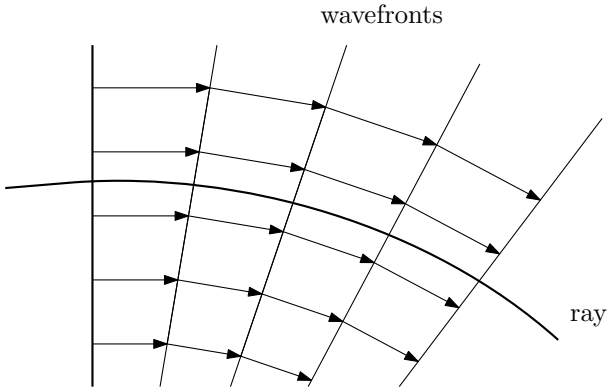


Figure 9.4: An example of a qualitative solution of the eikonal equation for an initial plane supersurface and an environment in which the refractive index n increases from top to bottom. Such an inhomogeneous environment causes the wavefronts (and thus the rays) to turn downwards. A simple realization of such an environment is a container with water, where salt is dissolved at the bottom. The highest concentration of salt is at the bottom and gradually decreases towards the surface. The gradient of salt concentration causes a variable refractive index – the highest salt concentration corresponds to the highest refractive index and vice versa.

9.3 *Fermat's Principle

The eikonal equation allowed us to find the phase function φ_0, from which we could then determine the paths of rays as curves perpendicular to the wavefronts. Fermat's principle, on the other hand, postulates that the actual paths of rays are such that they extremize the so-called *optical path* between given end points. Let's now formulate this principle more precisely. Consider two points in space P_1 and P_2 and we are interested in what the trajectory of the ray passing between these points will look like.

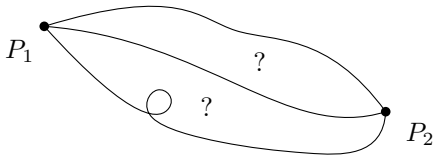


Figure 9.5: What trajectory will the ray follow from point P_1 to point P_2?

Generally, a ray is a curve $\vec{r}(\sigma) : \mathbb{R} \to \mathbb{R}^3$, where the parameter σ lies in the interval $\langle \sigma_1, \sigma_2 \rangle$ and for the end points of the curve it is true

$$\vec{r}(\sigma_1) = P_1, \qquad \vec{r}(\sigma_2) = P_2. \tag{9.27}$$

The optical path of the ray is the length of the ray weighted by the refractive index through which the ray passes. If we had a homogeneous medium with a constant refractive index and the length of the ray l, then the optical path s is simply $s = n \cdot l$. In the case of an inhomogeneous medium, where the refractive index is a function of spatial variables, $n(\vec{r})$, it is necessary to integrate

$$s = \int_{P_1}^{P_2} n \, dl. \tag{9.28}$$

For each curve $\vec{r}(\sigma)$, we can calculate its optical path s and get the so-called functional $S[\vec{r}(\sigma)] =$

s, which is a mapping from the space of all curves connecting points P_1 and P_2 into real numbers:

$$S[\vec{r}(\sigma)] = \int_{P_1}^{P_2} n\, dl = \int_{\sigma_1}^{\sigma_2} n(\vec{r}(\sigma)) \left| \frac{d\vec{r}}{d\sigma} \right| d\sigma, \tag{9.29}$$

where in the second equality we used the definition of the line integral[1]. Fermat's principle then states that the actual path of the ray is such that among all other rays it has a minimum value of the optical path. In the language of variational calculus, the actual trajectory of the ray is such that the variation of the optical path is zero:

$$\boxed{\delta S = \delta \int_1^2 n\, dl = 0.} \tag{9.30}$$

If we were to substitute the definition of the refractive index $n = \frac{c}{v}$ into the functional of the optical path, we would arrive at the fact that, equivalently, actual trajectories minimize the time that the ray needs to travel between points P_1 and P_2:

$$S = \int_1^2 n\, dl = c \int_1^2 \frac{dl}{v} = c \int_1^2 dt. \tag{9.31}$$

Fermat's principle (9.30) is nothing other than Hamilton's principle[2], known from Lagrangian mechanics. From the expanded form of the optical path functional (9.29), we can deduce the form of the so-called *optical Lagrangian*:

$$L(\vec{r}, \dot{\vec{r}}) = n(\vec{r})\, |\dot{\vec{r}}|, \tag{9.32}$$

where under the symbol $\dot{\vec{r}}$ we understand $\frac{d\vec{r}}{d\sigma}$. The actual path of the ray is then given by Euler-Lagrange equations, $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\vec{r}}}) - \frac{\partial L}{\partial \vec{r}} = 0$, which for the optical Lagrangian take the form

$$\frac{d}{dt}\left(n \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \right) = |\dot{\vec{r}}|\, \mathrm{grad}\, n. \tag{9.33}$$

The functional S (9.29) is so-called invariant with respect to reparametrization – the resulting optical path is determined only by the trajectory of the ray and is not influenced by the specific parametrization of the curve $\vec{r}(\sigma)$[3]. The resulting differential equation (9.33) for the path of

[1] The integral of a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ along a curve l given as $\vec{r}(\sigma) : \mathbb{R} \to \mathbb{R}^3$ with end points $\vec{r}(\sigma_1) = P_1$ and $\vec{r}(\sigma_2) = P_2$ is defined as:

$$\int_1^2 f\, dl := \int_{\sigma_1}^{\sigma_2} f(\vec{r}(\sigma)) \left| \frac{d\vec{r}(\sigma)}{d\sigma} \right| d\sigma,$$

where the integral on the right is now a standard one-dimensional Riemann integral.

[2] Hamilton's principle states that a physical system evolves along such paths that extremize the action functional

$$S = \int_{t_1}^{t_2} L(q_j(t), \dot{q}_j(t), t)\, dt, \qquad \delta S = 0.$$

The function L is called the Lagrangian of the given system. In geometric optics, time t does not appear as an independent variable but rather the curve parameter σ.

[3] This is an implicit property of the definition of the line integral, but it can also be shown explicitly. Consider the original curve $\vec{r}(\sigma)$ and its reparametrization $\vec{R}(\kappa) = \vec{r}(\sigma(\kappa))$ given by the function $\sigma(\kappa) : \mathbb{R} \to \mathbb{R}$, where $\vec{R}(\kappa_{1,2}) = P_{1,2}$ and $\sigma_{1,2} = \sigma(\kappa_{1,2})$. Then it holds

$$\int_{\sigma_1}^{\sigma_2} n(\vec{r}(\sigma)) \left| \frac{d\vec{r}}{d\sigma} \right| d\sigma = \left| \sigma = \sigma(\kappa), d\sigma = \frac{d\sigma(\kappa)}{d\kappa}\, d\kappa \right| = \int_{\kappa_1}^{\kappa_2} n(\vec{R}(\kappa)) \left| \frac{d\vec{r}}{d\sigma} \right| \frac{d\sigma(\kappa)}{d\kappa}\, d\kappa = \int_{\kappa_1}^{\kappa_2} n(\vec{R}(\kappa)) \left| \frac{d\vec{R}}{d\kappa} \right| d\kappa.$$

the ray can then be simplified by choosing a suitable parametrization, for example, by choosing so-called parametrization by path $|\dot{\vec{r}}| = 1$:

$$\frac{d}{dt}\left(n \cdot \dot{\vec{r}}\right) = \operatorname{grad} n. \tag{9.34}$$

What is the relationship between the formulation of geometric optics through the eikonal equation and through Fermat's principle? These formulations provide identical predictions regarding the trajectories of rays[4]. The eikonal equation also predicts the temporal course of light along the trajectory of the ray. This information is technically not provided by Fermat's principle due to reparametrization invariance, however, the temporal dependence can be easily calculated once the trajectory is known. This statement regarding the equivalence of two formulations of geometric optics is not at all obvious at first glance and we will not delve into it in detail[5].

From Fermat's principle, 5 rules of ray behavior in geometric optics easily follow:

1. In a homogeneous medium, light propagates in a straight line. This fact immediately follows from equation (9.34) for $n = \text{const.}$ the relation $\dot{\vec{r}} = \vec{\text{const.}}$ or from Fermat's principle the constant refractive index can be factored out, $\int_1^2 n\,dl = n\int_1^2 dl$ and the shortest connection between two points is a straight line.

2. The principle of the independence of ray paths applies – rays do not interact with each other. Fermat's principle obviously does not include the interaction of different rays.

3. The principle of reversibility of ray paths applies. The value of the functional S does not depend on the direction of integration, $\int_1^2 n\,dl = \int_2^1 n\,dl$ and thus rays are an extremum of the functional in both direct and reverse directions.

4. The law of reflection applies. Consider the class of rays shown in figure 9.6. The total length of the ray is $l(x) = \sqrt{h^2 + x^2} + \sqrt{h^2 + (l-x)^2}$ (the optical path is simply $s(x) = n \cdot l(x)$). The requirement of extremization, $\frac{dl(x)}{dx} = 0$, leads to the condition $x = \frac{l}{2}$ and thus also the equality of angles of incidence and reflection.

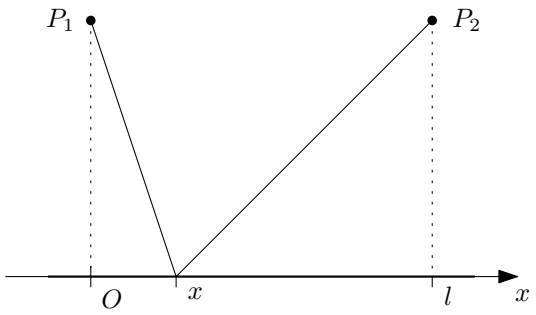


Figure 9.6: Class of rays for deriving the law of reflection from Fermat's principle. From the first rule of ray behavior, the resulting ray must consist of two segments. The parameter $x \in \mathbb{R}$ determines the position of reflection on the plane interface. The perpendicular distances of points P_1 and P_2 from the interface are h.

5. The law of refraction applies. Now consider the class of rays shown in figure 9.7. Here we must consider the optical path of the ray directly: $s(x) = n_1\sqrt{h_1^2 + x^2} + n_2\sqrt{h_2^2 + (l-x)^2}$.

[4] Of course, assuming the setting of corresponding initial, or boundary, conditions.

[5] Very roughly speaking, solving the eikonal equation is equivalent to Huygens's construction of wavefronts (and rays) and Huygens's construction implicitly creates rays so that their paths extremize the propagation time.

The extremization condition $\frac{ds(x)}{dx} = 0$ leads to the equation

$$n_1 \frac{x}{\sqrt{h_1^2 + x^2}} = n_2 \frac{l - x}{\sqrt{h_2^2 + (l - x)^2}}. \tag{9.35}$$

The fractions are direct expressions of the sines of the angle of incidence and refraction, $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

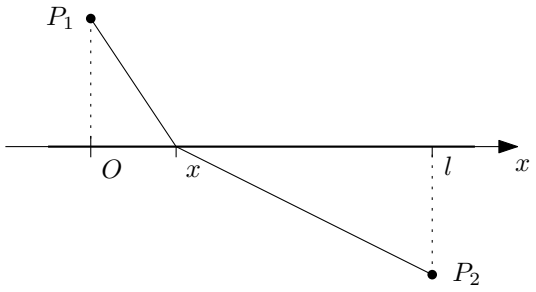


Figure 9.7: Class of rays for deriving the law of refraction from Fermat's principle. From the first rule of ray behavior, the resulting ray must consist of two segments. The parameter $x \in \mathbb{R}$ determines the position of the ray's transition from one medium to another on the plane interface. The perpendicular distances of points P_1 and P_2 from the interface are h_1 and h_2.

Chapter 10

Limits of Classical Physics

In previous chapters, we have thoroughly addressed wave phenomena and the theory of electromagnetic waves, and we have thus managed to explain a number of physical phenomena. Now, we will look at several cases where the wave character of light cannot explain the results of experimental observations. To resolve these discrepancies, it will be necessary to postulate the so-called quantum hypothesis – that the energy of certain systems under certain conditions cannot assume a continuum of values, but only discretely separated levels. In a figurative sense, this leads us back to the particle character of light, where light is composed of a stream of particles, called photons. However, the purely particle character of light cannot explain the interference phenomena we dealt with in previous chapters. This contradiction is resolved by adopting the so-called wave-particle duality, where a physical field or particle can simultaneously have the characteristics of both waves and particles. The resulting theory born from these contradictions is called quantum mechanics.

10.1 Photoelectric Effect

The photoelectric effect is a process in which light falling on a certain material releases electrons from its surface. Let's first describe how this effect is explained by classical (wave) physics.

The classical idea of this effect, schematically shown in Figure 10.1, considers the incoming electromagnetic wave oscillating a bound electron on the principle of a driven harmonic oscillator. If the kinetic energy of the oscillating electron exceeds the binding energy, the electron is released. Thus, the character of the effect should be determined by the properties of the resonance curve: at low energy of the incoming wave (low amplitude), the electrons do not oscillate sufficiently and will not be released, and further, the value of the threshold amplitude of the incoming wave at which electrons begin to be released should depend on the frequency of the incoming radiation – the closer to resonance, the smaller the amplitude of the incoming wave needed. This prediction is completely wrong.

Experiments show that electrons are released only if the frequency of the incoming radiation is higher than a certain threshold frequency $\nu_{\min}$ (determined by the properties of the material on which light falls), and the release of electrons occurs at any intensity of the incoming radiation and the number of released electrons depends linearly on this intensity. These experimental facts can be fully explained by the particle character of light – considering that light consists of a stream of particles, called photons, whose energy linearly depends on the frequency of light[1] ν as $E_{\mathrm{f}} = h\nu$. If the binding energy of an electron in an atom is W, then electrons are released only if $E_{\mathrm{f}} \geq W$ – that is, the photon of the incoming radiation has sufficient energy to release

[1] The linear dependence is explained in the results of the experiment at the end of this section.

the bound electron by its absorption. The threshold frequency is thus given by the relationship $E_\mathrm{f} = W$, $h\nu_\mathrm{min} = W$. The "excess" energy of the photon is converted into kinetic energy E_k of the released electron. By these considerations, we have arrived at the basic equation of the photoelectric effect (for $\nu \geq \nu_\mathrm{min}$):

$$\boxed{h\nu = W + E_k,} \tag{10.1}$$

where $E_k = \frac{1}{2} m_e v^2$, v is the speed of the released electron.

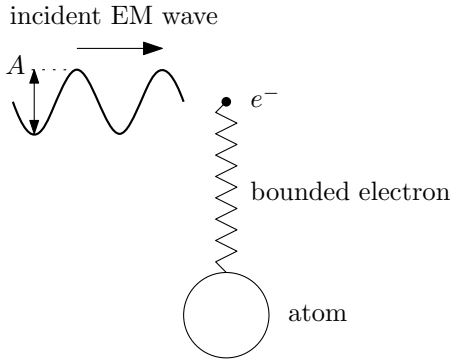


Figure 10.1: Classical idea of the photoelectric effect. The incoming electromagnetic wave oscillates an electron bound in an atom on the principle of a driven harmonic oscillator.

Finally, consider the experimental setup, which is described in Figure 10.2.

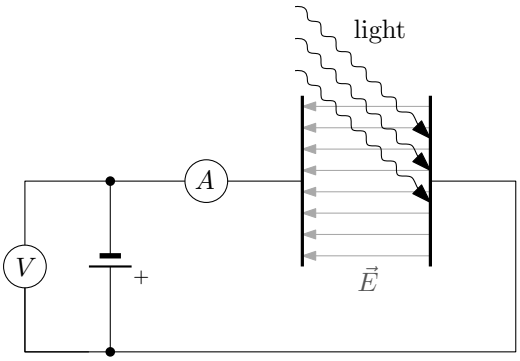


Figure 10.2: Experimental setup for measuring the photoelectric effect. Light falling on the electrode of a phototube releases electrons, which must overcome the potential difference U between the electrodes caused by the connected voltage. We measure the current I, which flows through the circuit due to the release of electrons and their transport between the electrodes.

In this experiment, for different frequencies ν of the incoming radiation, we measure the voltage U, at which the current flowing through the circuit just becomes null. The released electrons have energy $E_k = h\nu - W$ and these are able to overcome the potential difference U between the electrodes only if $E_k > Ue$. The voltage U_0, at which the current just becomes null, is clearly given by the condition $U_0 e = E_k$, i.e., combining with the equation of the photoelectric effect (10.1) we get a prediction for the stopping voltage:

$$U_0(\nu) = \frac{h}{e}\nu - \frac{W}{e}. \tag{10.2}$$

For frequencies $\nu < \nu_\mathrm{min}$, no frequencies are released, so we can set $U(\nu < \nu_\mathrm{min}) = 0$. The illustration of this simple relationship for voltage $U_0(\nu)$ is in Figure 10.3.

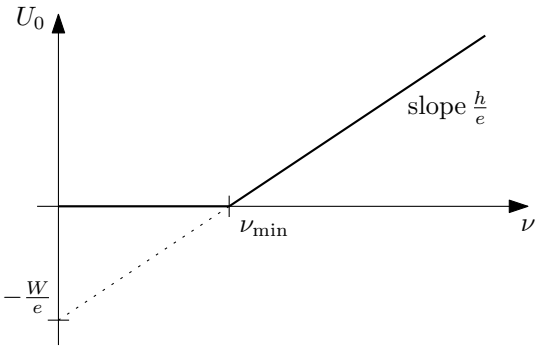


Figure 10.3: Graph of the stopping voltage U_0, at which the current flowing through the circuit in Figure 10.2 just disappears.

Experiments indeed confirm this shape of the dependency $U_0(\nu)$ and from it, we can draw the following three conclusions. For $\nu > \nu_{\min}$, the graph is linear, thus confirming the assumption of a linear dependence of the photon's energy on its frequency ν. From the slope of the line, we can determine the value of Planck's constant h from the knowledge of the electron's charge! And finally, from the value of $\nu_{\min}$, we can easily calculate the material constant of the work function, $W = h\nu_{\min}$.

10.2 Energy, Momentum, and Mass of a Photon

Let's ask what the momentum of a photon is if we accept the quantum hypothesis that the energy of a photon is given as

$$\boxed{E_{\mathrm{f}} = h\nu.} \tag{10.3}$$

If a traveling electromagnetic wave is to be composed of photons, the energy and momentum of this electromagnetic wave must be composed of the energy and momentum of individual photons. In chapter 6.3, we derived the relationship between momentum density $\vec{g}$ and energy density w in a plane traveling EM wave (6.64):

$$\vec{g} = \frac{w}{c}\vec{n}, \qquad g = \frac{w}{c}, \tag{10.4}$$

where $\vec{n}$ is the direction of travel of the EM wave. This relationship leads[2] to the conclusion that the energy and momentum of a photon, E_{f} and $\vec{p}_{\mathrm{f}}$, must be in the same relationship:

$$\vec{p}_{\mathrm{f}} = \frac{E_{\mathrm{f}}}{c}\vec{n} = \frac{h\nu}{c}\vec{n}, \qquad \boxed{p_{\mathrm{f}} = \frac{E_{\mathrm{f}}}{c} = \frac{h\nu}{c}.} \tag{10.5}$$

What will be the mass of the photon m_{f}? The relativistic relationship between energy and momentum[3] is

$$\left(mc^2\right)^2 = E^2 - p^2c^2. \tag{10.6}$$

[2] It can also be done slightly more rigorously (but at the same time slightly superfluously). Consider that in the EM wave, the number density of photons is n, i.e., in a small volume dV there are in total $dN = n\,dV$ photons. The total energy dE and momentum dp of these photons is $dE = E_{\mathrm{f}}\,dN = h\nu n\,dV$, $dp = p_{\mathrm{f}}\,dN = p_{\mathrm{f}} n\,dV$. At the same time, from the definitions of energy and momentum density of the EM field, it follows $dE = w\,dV$, $dp = g\,dV$. By comparing these relationships, we obtain $w = h\nu n$ and $g = pn$. Since $g = \frac{w}{c}$, we get (after canceling the number densities n) $p = \frac{h\nu}{c}$.

[3] The four-momentum of a particle has the form $p^\mu = (\frac{E}{c}, \vec{p})$. The so-called relativistic invariant, i.e., a quantity that remains the same in all reference frames, is the quantity

$$p^\mu p_\mu = -p_0^2 + p_1^2 + p_2^2 + p_3^2.$$

If we substitute the values of energy E_f (10.3) and momentum p_f (10.5) of a photon,

$$(m_\mathrm{f}c^2)^2 = E^2 - p^2c^2 = (h\nu)^2 - \left(\tfrac{h\nu}{c}\right)^2 c^2 = 0, \tag{10.7}$$

we see[4], that the photon is a massless particle:

$$\boxed{m_\mathrm{f} = 0.} \tag{10.8}$$

For completeness, let's finally explicitly state the form of the four-momentum of a photon p^μ:

$$p^\mu = \left(\tfrac{E}{c}, \vec{p}\right) = \left(\tfrac{h\nu}{c}, \tfrac{h\nu}{c}\vec{n}\right) = \frac{h\nu}{c}(1, \vec{n}). \tag{10.9}$$

10.3 Compton Scattering

Consider an experiment in which we allow electromagnetic radiation of frequency ν to fall on a given material and we examine the frequency ν' with which the wave scatters depending on the angle of deviation θ from the original direction – the so-called *scattering angle* θ, see Figure 10.4.

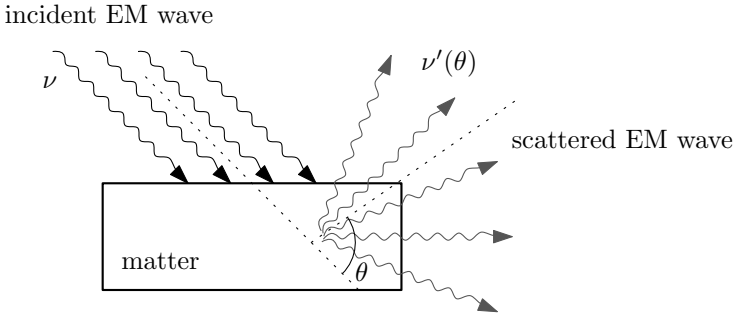


Figure 10.4: Scattering of the incident electromagnetic wave of frequency ν on the substance. We are interested in the frequency of the scattered wave ν' depending on the scattering angle θ.

The classical explanation, considering the wave nature of electromagnetic radiation, goes as follows: The incident electromagnetic wave oscillates the electrons present in the material, and these, as accelerated charges, emit electromagnetic radiation into their surroundings, forming a scattered electromagnetic wave, schematically see Figure 10.5. This classical process is called

For the four-momentum of a particle expressed in any two reference frames p^μ and p'^μ, the equality $p^\mu p_\mu = p'^\mu p'_\mu$ holds. If we choose the rest frame of the chosen particle as the system (S'), then $E' = m_0c^2$ and $\vec{p}' = 0$, thus $p'^\mu = (m_0c, \vec{0})$. By substituting into the equality of invariants:

$$-\frac{E^2}{c^2} + |\vec{p}|^2 = -(mc)^2 + \vec{0}^2.$$

After a simple rearrangement, we obtain the final form of the relativistic relationship between energy and momentum:

$$\boxed{(mc^2)^2 = E^2 - |\vec{p}|^2c^2.}$$

[4]Strictly speaking, our deduction is erroneous, as the relativistic relationship between energy and momentum was derived by transitioning to the rest frame of the particle, which is not possible with a photon – it is not possible to transition to a frame moving relative to another at the speed of light c. Also, we took the expression for rest energy as $E_0 = mc^2$, which is a quantity that does not make sense for a photon – in every reference frame, a photon moves at speed c, so it cannot be observed at rest. However, the only consistent value for the mass of the photon in the relationship (10.6) with the expressions for energy and momentum of the photon is zero.

Thomson scattering. Therefore, the prediction of the properties of the scattered EM wave is based on a combination of the excited harmonic oscillator formed by the bound electrons in the material oscillated by the incident EM wave and the EM wave emitted by this accelerated electron. This prediction says that the frequency of the scattered radiation does not depend on the scattering angle θ at all – the accelerated charge emits an EM wave of the same frequency in all directions, and this accelerated charge oscillates with the same frequency as the incident wave. That means, Thomson scattering simply predicts $\nu' = \nu$ without dependence on the angle θ.

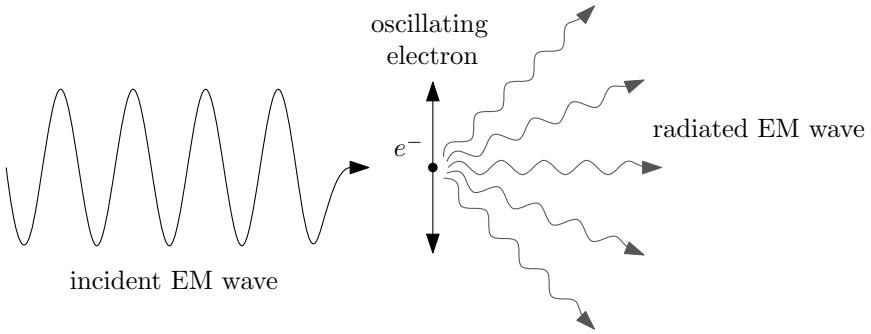


Figure 10.5: Classical explanation of scattering – Thomson scattering.

A. H. Compton conducted the aforementioned experiment in 1923, where he allowed X-rays to scatter. The result of the experiment contradicted the classical explanation of Thomson scattering! The frequency of the scattered radiation depended on the scattering angle θ! For high-frequency radiation, the particle nature of light begins to manifest in this experiment. Let us now look at the course of the experiment, assuming that the incident EM wave is composed of a stream of photons. For X-ray radiation, the energy of photons is many orders of magnitude greater than the binding energy of electrons (photon energy $E_\mathrm{f} = h\nu \approx 10 - 100\,\mathrm{keV}$, electron work function $W \approx 1\,\mathrm{eV}$), so we can effectively consider the electrons in the material as free. Figure 10.6 shows the geometry of the particle explanation of Compton scattering.

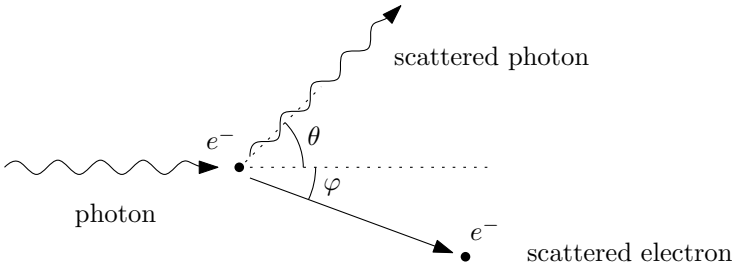


Figure 10.6: Geometry of Compton scattering. The scattered photon deviates from its original direction by an angle θ, the scattered electron begins to move in the direction making an angle φ with the original direction of the photon.

The main task, as already mentioned in the introduction, will be to predict the function of the frequency of the scattered radiation (photons) depending on the scattering angle θ, $\nu'(\theta)$. We will use the conservation laws of energy and momentum:

$$E_\mathrm{f} + E_e = E'_\mathrm{f} + E'_e, \qquad \vec{p}_\mathrm{f} = \vec{p}'_\mathrm{f} + \vec{p}'_e, \tag{10.10}$$

where unprimed, resp. primed, quantities denote values before, resp. after, the collision; indices f, resp. e, denote quantities for the photon, resp. electron. Consider the electron initially at

rest, thus $\vec{p}_e = 0$. From the law of conservation of momentum, it immediately follows that the problem is planar[5], i.e., the scattering angles θ and φ fully describe the directions in which the scattered photon and electron will move.

Let's now modify the law of conservation of momentum and energy into forms where momentum and energy of the scattered electron appear on the right sides:

$$\begin{aligned} \vec{p}_\mathrm{f} - \vec{p}'_\mathrm{f} &= \vec{p}'_e, \\ p_\mathrm{f}^2 + p_\mathrm{f}'^2 - 2\underbrace{\vec{p}_\mathrm{f} \cdot \vec{p}'_\mathrm{f}}_{p_\mathrm{f} p'_\mathrm{f} \cos\theta} &= p_e'^2, \end{aligned} \tag{10.11}$$

$$E_\mathrm{f} - E'_\mathrm{f} + E_e = E'_e. \tag{10.12}$$

The momentum and energy of the scattered electron in the material into which the radiation falls cannot be simply measured, so it is appropriate to eliminate them from the relationships by substitution into the relativistic relationship between energy and momentum:

$$E'_e - p_e'^2 c^2 = E_e^2, \tag{10.13}$$

where the rest energy of the electron $E_e = m_e c^2$ (m_e denotes its rest mass) appears on the right side. Thus, by substituting (10.11) and (10.12) into (10.13), we get:

$$\left(E_\mathrm{f} - E'_\mathrm{f} + m_e c^2\right)^2 - \left(p_\mathrm{f}^2 + p_\mathrm{f}'^2 - 2 p_\mathrm{f} p'_\mathrm{f} \cos\theta\right) c^2 = m_e^2 c^4. \tag{10.14}$$

After substituting the relationship between the energy and momentum of the photon $p_\mathrm{f} = \frac{E_\mathrm{f}}{c}$, $p'_\mathrm{f} = \frac{E'_\mathrm{f}}{c}$, and rearranging:

$$2E_\mathrm{f}E'_\mathrm{f}(\cos\theta - 1) + 2(E_\mathrm{f} - E'_\mathrm{f})m_e c^2 = 0. \tag{10.15}$$

Finally, we express the photon energies using the frequencies of the incident and scattered waves ν and ν', $E_\mathrm{f} = h\nu$ and $E'_\mathrm{f} = h\nu'$ and after a minor rearrangement we arrive at an equation of the form:

$$\nu\nu'(\cos\theta - 1) + (\nu - \nu')\frac{m_e c^2}{h} = 0. \tag{10.16}$$

From this, we can easily express the dependency of the frequency of the scattered wave ν' on the scattering angle θ describing Compton scattering:

$$\boxed{\nu'(\theta) = \frac{\nu}{1 + \frac{h\nu}{m_e c^2}(1 - \cos\theta)}.} \tag{10.17}$$

We see that for incident radiation, whose photons have much less energy than the rest energy of the electron, $h\nu \ll m_e c^2$, Compton's formula transitions to the prediction of classical Thomson scattering $\nu' = \nu$. Only for high frequencies of the incident radiation does the particle nature of light begin to manifest, and the classical explanation ceases to work.

From the conservation laws of energy and momentum, it is not possible to uniquely determine the angle at which the photon scatters. Initially, we had four equations (1x energy conservation and 3x momentum conservation). The planarity of the problem limited the number of momentum equations to two. Thus, we have three equations for four unknowns φ, θ, ν', and E'_e. From the equations, we can find functions such as $\nu'(\theta)$, $E'_e(\theta)$, and $\varphi(\theta)$, i.e., the dependencies of the remaining variables on the photon scattering angle θ (we focused only on the first of these in the text).

[5]Vector $\vec{p}_f$ is a linear combination of $\vec{p}'_f$ and $\vec{p}'_e$, thus it lies in the plane formed by these vectors.

In reality, there is no further equation that would uniquely determine the scattering angle θ – scattering has a probabilistic nature – a photon scatters with different probabilities at different angles θ – these probabilities are predicted by quantum electrodynamics (QED, a quantum field theory describing electromagnetism).

In conclusion, we can express the relationship (10.17) in terms of wavelength variables, $\lambda = \frac{c}{\nu}$, $\lambda' = \frac{c}{\nu'}$, and obtain the relation $\lambda'(\theta)$:

$$\lambda'(\theta) = \lambda + \frac{h}{m_e c}(1 - \cos\theta) = \lambda + \lambda_C(1 - \cos\theta). \tag{10.18}$$

The quantity $\lambda_C = \frac{h}{m_e c}$ is called the *Compton wavelength*. Photons with the Compton wavelength have energy equal to the rest energy of the electron, $E_C = h\nu_C = \frac{hc}{\lambda_C} = m_e c^2$. The strength of the Compton effect depending on the frequency of the incident radiation can be determined, for example, by the relative change in wavelength upon scattering:

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \frac{\lambda_C}{\lambda}(1 - \cos\theta). \tag{10.19}$$

10.4 Stability of atoms

In Rutherford's planetary model of the atom, electrons orbit around a very small and massive nucleus in circular paths like planets around the Sun. Let us show that this idea, without additional assumptions, is completely incompatible with the fact that accelerated charges constantly emit electromagnetic radiation and thus lose energy. Since the electrons move in curved (circular) paths, they are constantly subjected to centripetal acceleration, which then leads to the emission of electromagnetic radiation, the energy of which must be taken at the expense of the mechanical energy of the electron's orbital motion. Therefore, the electron inevitably spirals into the nucleus of the atom. By quantifying the above considerations, we calculate the time it takes for an electron to fall into the nucleus – the so-called lifetime of a classical atom. If this time were sufficiently long, i.e., the fall of the electron is sufficiently slow, the idea of the Rutherford atom would be sustainable in this respect. Unfortunately, we will see that this is not the case.

As a simple example, take an atom with atomic number Z (i.e., the number of protons Z in the nucleus), but only one electron in the atomic shell[6]. Consider Newton's equation of motion in the radial direction for this electron; it includes the Coulomb attractive force between the nucleus and the electron:

$$\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2} = m_e a_r, \tag{10.20}$$

where a_r denotes the radial component of acceleration (with the positive direction pointing towards the nucleus). Writing the radial acceleration in polar coordinates (r, φ) and considering that the electron spirals into the nucleus very slowly[7], we get

$$a_r = -\ddot{r} + r\dot{\varphi}^2 \approx \frac{v_\varphi^2}{r} \approx \frac{v^2}{r}, \tag{10.21}$$

where we considered $\ddot{r} \approx 0$, the tangential component of velocity $v_\varphi = r\dot{\varphi}$, and also $v \approx v_\varphi$. Moreover, again considering slow falling, we can take the total acceleration $a \approx a_r$. By

[6] That is, the atom is always ionized so that only one electron remains in it.

[7] That is, it falls into the nucleus much more slowly than it orbits the nucleus.

substituting these considerations into the motion equation (10.20), we can express the velocity and acceleration of motion as a function of the distance from the nucleus r:

$$a^2 = \left(\frac{Ze^2}{4\pi\varepsilon_0 r^2 m_e}\right)^2 = \frac{r_0^2 c^4}{r^4} Z^2, \qquad v^2 = ra = \frac{Ze^2}{4\pi\varepsilon_0 r m_e} = \frac{r_0}{r} c^2 Z, \tag{10.22}$$

where, for simplification, we introduced a helper constant $r_0 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e c^2}$ called the *classical electron radius*.

Now let's focus on the electron's energy balance equation, $\frac{dE}{dt} = P$, where E is the total mechanical energy of the electron and P is the power of the emitted electromagnetic radiation due to the accelerated motion of the electron. From this balance equation, we will eventually derive the time dependence of the electron's orbit radius $r(t)$. The emitted power is given by Larmor's formula

$$P = -\frac{\mu_0 q^2}{6\pi c} a^2 = -\frac{q^2}{6\pi\varepsilon_0 c^3} a^2 = -\frac{2}{3}\frac{r_0^3}{r^4} m_e c^3 \tag{10.23}$$

(the minus sign here is given conventionally because the emitted power occurs at the expense of mechanical energy). The total mechanical energy of the electron is given by the sum of its kinetic and potential energy:

$$E = T + U = \frac{1}{2} m_e v^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r} = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{2r} = -\frac{1}{2} m_e c^2 Z \frac{r_0}{r}, \tag{10.24}$$

where we used the expression for the electrostatic energy of two charges $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$ and substituted the expression for velocity from (10.22). The energy balance equation, after combining (10.23) and (10.24), becomes:

$$\frac{dE}{dt} = \frac{1}{2} m_e c^2 Z \frac{r_0}{r^2}\dot{r} = P = -\frac{2}{3}\frac{r_0^3}{r^4} m_e c^3. \tag{10.25}$$

After canceling common factors on both sides, we get a simple differential equation for the orbit radius of the electron $r(t)$:

$$r^2 \dot{r} = -\frac{4}{3Z} r_0^2 c. \tag{10.26}$$

This can be easily solved by the transformation $r^2\dot{r} = \frac{1}{3}\frac{dr^3}{dt}$ and integrating:

$$\frac{dr^3}{dt} = -\frac{4}{Z} r_0^2 c \quad \longrightarrow \quad \boxed{r^3(t) = a_0^3 - \frac{4}{Z} r_0^2 ct,} \tag{10.27}$$

where we used the initial condition $r^3(0) = a_0^3$, a_0 thus denotes the initial orbit radius of the electron (initial "size" of the atom). The time it takes for the electron to fall into the nucleus of the atom t_{fall} is given by the condition $r(t_{\text{fall}}) = 0$, by solving it we obtain the final classical prediction:

$$\boxed{t_{\text{fall}} = \frac{Z a_0^3}{4 r_0^2 c}.} \tag{10.28}$$

The classical electron radius is numerically $r_0 = 2.8 \times 10^{-15}\,\text{m}$, for the so-called Bohr radius of the hydrogen atom $a_0 = 5.3 \times 10^{-11}\,\text{m}$ (and thus also $Z = 1$) the result is $t_{\text{fall}} = 1.6 \times 10^{-11}\,\text{s}$. In classical physics, therefore, atoms would disappear in the order of tens of picoseconds! If we were to discuss the approximation $a \approx a_r$ more closely or

consider relativistic corrections[8], the problem becomes even worse (t_{fall} comes out even smaller)! This situation is resolved by Bohr's model of the atom, where it is postulated that electrons can orbit only along paths with certain discretely distributed radii. Electrons for jumps between these paths must absorb or emit a finite quantum of energy, and this effectively prevents the emission of electromagnetic radiation and a continuous decrease in the orbit radius.

[8] In our Newtonian model, the electron in the hydrogen atom reaches the speed of light for $r = r_0$, see (10.22).

10.4.1 Bohr's model of the atom

In Bohr's model, the principle of quantization of the angular momentum of individual electron orbits is postulated – electrons can orbit only along such paths whose angular momentum has the following size

$$L = n\hbar, \qquad n \in \mathbb{N}, \tag{10.29}$$

where $\hbar = \frac{h}{2\pi}$ is called the *reduced Planck constant*. Based on this postulate, let's derive the permissible radii of electron orbits (and also their energies). For a circular orbit of radius r and orbital velocity v, the angular momentum is $L = m|\vec{r} \times \vec{v}| = mvr$. For an atom with atomic number Z, we have already derived the orbital velocity v in (10.22) on the right, by substituting this velocity into the quantization condition (10.29), $m_e v r = n\hbar$, we get

$$m_e \sqrt{\frac{Ze^2}{4\pi\varepsilon_0 r m_e}}\, r = n\hbar, \tag{10.30}$$

from this equation we can easily express the permissible radii of electron orbits r_n:

$$r_n = \frac{4\pi\varepsilon_0 n^2 \hbar^2}{Ze^2 m_e}, \qquad n \in \mathbb{N}. \tag{10.31}$$

The energies E_n of these individual orbits, after substituting for r into (10.24):

$$E_n = -\frac{1}{(4\pi\varepsilon_0)^2}\frac{Z^2 e^4 m_e}{2\hbar^2 n^2} = -R_E \frac{Z^2}{n^2}, \qquad n \in \mathbb{N}, \tag{10.32}$$

where we introduced the quantity $R_E = \frac{1}{(4\pi\varepsilon_0)^2}\frac{m_e e^4}{2\hbar^2} = 13.6\,\mathrm{eV}$ called the *Rydberg energy*. Electrons can then jump between these energy levels upon absorption/emission of photons with energy

$$h\nu_{mn} = \Delta E_{mn} = |E_m - E_n| = R_E Z^2 \left|\frac{1}{m^2} - \frac{1}{n^2}\right|, \qquad m, n \in \mathbb{N}, \tag{10.33}$$

where m, n are the indices of the initial and final levels. For $Z = 1$, this formula is called the *Rydberg formula* and accurately predicts the emission/absorption spectrum shape of the hydrogen atom. For higher atomic numbers $Z > 1$, this formula remains valid for ionized atoms, in whose atomic shell only one electron remains, i.e., for atoms $X^{(Z-1)+}$ – for example, He^+, Li^{2+}, etc. For atoms, in whose shell more than one electron remains, the interaction between individual electrons in the atomic shell causes various shifts (and splitting) of individual spectral lines, so then their emission/absorption spectra are much more complex than given by the simple Rydberg formula.

10.4.2 Schrödinger's quantum model of the atom

In quantum mechanics, it is possible to predict the energy values of individual levels E_n (10.32) by solving the problem of the "motion" of a charged quantum particle in a Coulomb central field, where quantized levels directly emerge as a consequence of the calculation and not as an ad hoc added assumption. This calculation is the basis of the so-called Schrödinger quantum model of the atom, where individual "orbits" (better said states) of electrons are described by quantum numbers (n, l, m, s):

- *Principal quantum number* $n \in \mathbb{N}$ describes the energy of the given electron state E_n in accordance with formula (10.32).

- *Orbital quantum number* $l \in \{0, 1, \ldots, n-1\}$ indicates the square of the total angular momentum of the electron $L^2 = \hbar^2 l(l+1)$.

- *Magnetic quantum number* $m \in \{-l, -l+1, \ldots, l-1, l\}$ determines one component from the angular momentum of the electron $\vec{L}$, $L_z = m\hbar$ (typically the third component is taken by convention).

- *Spin quantum number* $s \in \{-\frac{1}{2}, +\frac{1}{2}\}$ represents an "additional" intrinsic angular momentum (called *spin*) of the electron. The square of its magnitude is $S = \hbar^2 s(s+1)$ and one of its components $S_z = \hbar s$.

10.5 Black Body Radiation

A *black body* (sometimes also referred to as a *perfectly black body*) is a physical idealization where all radiation incident on the surface of this body is absorbed. At the same time, this body emits thermal radiation – the so-called *black body radiation*. The figure 10.7 on the right shows an approximate physical realization of a perfectly black body – a dark cavity specially formed so that the radiation that gets into the cavity is absorbed as much as possible by multiple reflections. The entrance aperture of this cavity then behaves as the surface of a perfectly black body. The surface of the black body then emits thermal radiation, the form of which depends only on the temperature T of this body.

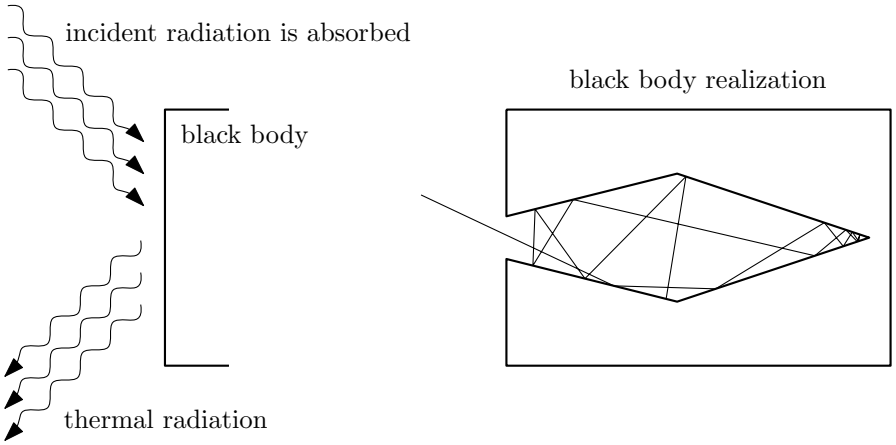


Figure 10.7: A black body absorbs all incident radiation, in contrast, it emits black body thermal radiation. The cavity in the right picture represents the physical realization of a black body, multiple reflections cause almost perfect absorption of incident radiation and the cavity opening then behaves as the surface of a perfectly black body.

Furthermore, every real body is described by a parameter called *emissivity* ε, which indicates the ratio between the actual emitted radiation power and the radiation emitted by a perfectly black body. Another parameter that can be introduced is called *absorptivity* α, which, on the contrary, indicates the ratio between the actual absorbed power and the total power of radiation incident on the body. Kirchhoff's law of thermal radiation states that the absorptivity α is always equal to the emissivity ε, $\alpha = \varepsilon$. From this, it follows that the emissivity is always less than or equal to one, $\varepsilon \leq 1$, since it is not possible for a body to absorb more energy than is incident on it, $\varepsilon = \alpha \leq 1$.

In reality, the emissivity can depend on the frequency of the emitted radiation, $\varepsilon(\nu)$, thus each frequency can be emitted in a different ratio to the radiation of a black body. The function

of emissivity $\varepsilon(\nu)$ is a characteristic of the body and does not depend on the temperature T. If we know the intensity of black body radiation $I(T)$, then a body with emissivity ε will emit power $\varepsilon I(T)$; alternatively, for frequency-dependent emissivity $\varepsilon(\nu)$ we need to know the emission intensities of the black body at individual frequencies, the so-called *spectral density* $i(\nu, T)$, then the real body will emit intensity of density $\varepsilon(\nu)\, i(\nu, T)$.

Let's now look more closely at the meaning of the term spectral density. The total intensity of radiation indicates the power of radiation emitted from a unit area of the body, $[I] = \mathrm{W.m^{-2}}$. The spectral density $i(\nu)$ of this intensity then indicates the power dI emitted in the frequency range $\langle \nu, \nu + d\nu \rangle$ as:

$$dI = i(\nu)\, d\nu. \tag{10.34}$$

Clearly, the unit of spectral density is the unit of radiation intensity per Hertz: $[i(\nu)] = \mathrm{W.m^{-2}.Hz^{-1}}$. If we want to return from the spectral density back to the total emitted power, we must integrate over all frequencies:

$$I = \int_0^{+\infty} dI(\nu) = \int_0^{+\infty} i(\nu)\, d\nu. \tag{10.35}$$

The figure 10.8 schematically shows the relationship between the spectral density $i(\nu)$ and the emitted power dI.

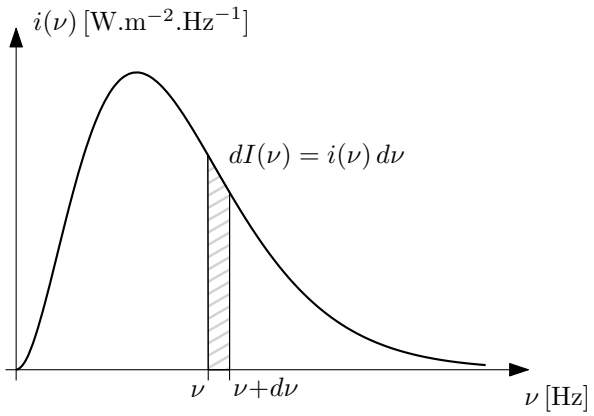


Figure 10.8: The spectral density $i(\nu)$ and the emitted power dI in the range of frequencies $\langle \nu, \nu + d\nu \rangle$ – this is given by the area of the graph between the values of frequency ν and $\nu + d\nu$. Similarly, the emitted power in the range of frequencies $\langle \nu_1, \nu_2 \rangle$ is given by the area between these frequencies, i.e., $I_{\langle \nu_1, \nu_2 \rangle} = \int_{\nu_1}^{\nu_2} i(\nu)\, d\nu$.

10.5.1 Planck's Law of Radiation

Planck's law of radiation specifies the spectral density[9] of black body radiation. Its form is as follows:

$$\boxed{i(\nu, T) = \frac{2\pi h \nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1},} \qquad [i] = \mathrm{W.m^{-2}, Hz^{-1}}. \tag{10.36}$$

[9] Planck's law of radiation is often stated using a quantity called *spectral radiance* i_r. While spectral density i indicates the total emitted power in all directions (thus into the entire hemisphere above the surface of the body), spectral radiance normalizes this power to a unit of solid angle, i.e., the unit of spectral radiance is $[i_r] = [i].\mathrm{sr^{-1}}$. For isotropically emitting bodies (which a perfectly black body is), a simple relationship $i = \pi\, i_r$ applies. Therefore, some formulas presented here may differ by a factor of π from formulas in other literature. The emitted power dP from an area of size dS, into the frequency interval width $d\nu$, into a small solid angle of size $d\Omega$ is given as $dP = i\, d\nu\, dS\, d\Omega \cos\theta$, where the angle θ is the angle of deviation of the direction of emission from the normal vector to the area dS. If the radiation is emitted at an angle θ, then the effective area from which the radiation comes is reduced to size $dA = dS \cos\theta$.

where h is Planck's constant, c is the speed of light, and k is Boltzmann's constant. The graphical representation of this function as a function of variable ν is in the figure 10.9.

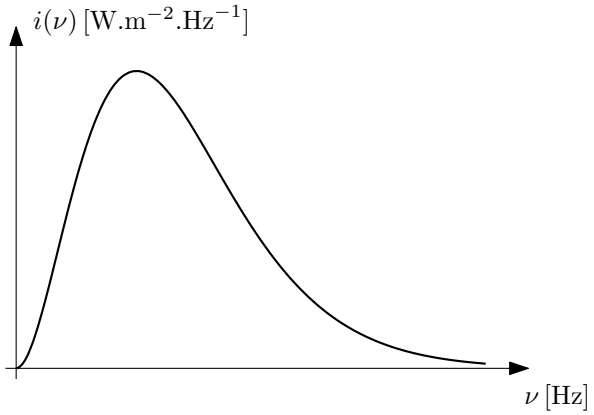


Figure 10.9: Planck's law of radiation. The graph shows the spectral density $i(\nu, T)$ for a constant temperature T.

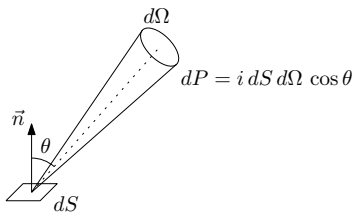
The derivation of this law requires knowledge of statistical physics[10] and will not be presented here. Perhaps just mention that the formula (10.36) is composed of two fundamental parts: $i = n\langle E\rangle$. The first part $n = \frac{2\pi\nu^2}{c^2}$ is the density of the number of modes of the electromagnetic field in the frequency range $\langle\nu, \nu+d\nu\rangle$, which follows from classical electromagnetic theory, when considering EM radiation confined in a cavity. The second part $\langle E\rangle = \frac{h\nu}{\exp(\frac{h\nu}{kT})-1}$ is the average energy of one electromagnetic mode in the cavity, as predicted by statistical physics. Planck's key contribution to the result $\langle E\rangle$ was the consideration that the energy levels of a given mode of the electromagnetic field must be quantized in the form $E_n = nh\nu$, $n \in \mathbb{N}_0$. The classical notion, where the energy of the electromagnetic field is continuous, led to an incorrect prediction of the spectral density of thermal radiation in the form of the so-called Rayleigh-Jeans law (see also the section 10.5.3 on the limits of Planck's law).

10.5.2 Spectral Density in Other Variables

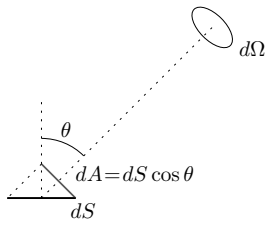
In Planck's law of radiation, we have frequency ν as the variable in which we express spectral density (and thus how we divide the total intensity), and we could more precisely call the function $i(\nu)$ *frequency spectral density*. However, we could also divide the intensity so that we are interested in how much energy is emitted in the range of wavelengths $\langle\lambda, \lambda + d\lambda\rangle$:

$$dI = i(\lambda)\, d\lambda \tag{10.37}$$

to obtain *"wavelength" spectral density* $i(\lambda)$. As a result of this definition, the unit is clearly $[i(\lambda)] = \mathrm{W.m^{-2}.m^{-1}} = \mathrm{W.m^{-3}}$. Naturally, the question arises, what is the relationship between



(a) Emission into a small solid angle $d\Omega$.



(b) Graphical explanation of the $\cos\theta$ factor.

[10] Which you will gain in 02TSFA.

these densities ($i(\nu)$ and $i(\lambda)$)? We do not switch between these functions by a mere substitution! It is due to the very definition of spectral density – we require that the power emitted into the corresponding intervals of frequencies $\langle \nu, \nu + d\nu \rangle$ and wavelengths $\langle \lambda, \lambda + d\lambda \rangle$ be the same[11]:

$$dI = i(\nu)\, d\nu = i(\lambda)\, d\lambda. \tag{10.38}$$

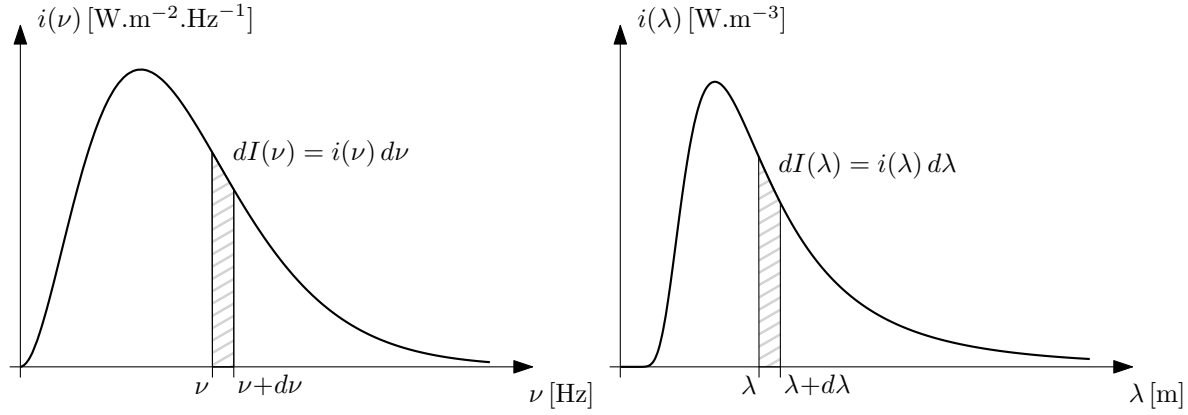
What is meant by corresponding? Naturally, we require $\nu = \frac{c}{\lambda}$, but the infinitesimal increments of these variables must also correspond, thus if we differentiate:

$$d\nu = -\frac{c}{\lambda^2}\, d\lambda. \tag{10.39}$$

The minus sign is naturally given by the fact that as the frequency of radiation increases, its wavelength decreases. However, we are more interested in the relationship between the magnitudes of these changes, hence in the following we will take their absolute values,

$$|d\nu| = \frac{c}{\lambda^2}|d\lambda|, \tag{10.40}$$

and further in the text, we will not explicitly write them out.



(a) Frequency spectral density $i(\nu)$.

(b) "Wavelength" spectral density $i(\lambda)$.

Figure 10.10: Schematic representation of the relationship between frequency spectral density and "wavelength" spectral density. It is not about the identity of function values, but about the equivalence of the area dI in corresponding intervals $\langle \nu, \nu + d\nu \rangle$ and $\langle \lambda, \lambda + d\lambda \rangle$, i.e., it must hold $dI = i(\nu)\, d\nu = i(\lambda)\, d\lambda$.

Now we can take Planck's law of radiation (10.36) as an example of a specific frequency spectral density $i(\nu)$ and convert it into a wavelength function $i(\lambda)$:

$$dI = i(\nu, T)\, d\nu = \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}\, d\nu = \frac{2\pi h}{c^2} \frac{c^3}{\lambda^3} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \frac{c}{\lambda^2}\, d\lambda = i(\lambda, T)\, d\lambda. \tag{10.41}$$

From the last equality, we deduce the spectral density of Planck's law of radiation for wavelengths $i(\lambda, T)$:

$$\boxed{i(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}.} \tag{10.42}$$

Similarly, functions of spectral densities could be converted into other variables such as angular frequency ω, wave number k, etc.

[11] And similarly for finite ranges of frequencies and wavelengths:

$$\int_{\nu_1}^{\nu_2} i(\nu)\, d\nu = \int_{\lambda_1}^{\lambda_2} i(\lambda)\, d\lambda.$$

10.5.3 Limits of Planck's Law

Historically, inaccurate predictions of the spectral density of black body radiation were known, which turned out to be mere approximations of Planck's law in the limit of low and high frequencies. Let's now look at these approximations.

Low frequencies: Rayleigh-Jeans Law

For low frequencies (high wavelengths), we can expand the exponential in the denominator of Planck's law (10.36) to the first order of Taylor's expansion,

$$e^{\frac{h\nu}{kT}} \approx 1 + \frac{h\nu}{kT}, \tag{10.43}$$

and after substitution, we obtain the *Rayleigh-Jeans Law*:

$$i(\nu, T) \approx \frac{2\pi kT\nu^2}{c^2}, \qquad \left(i(\lambda, T) \approx \frac{2\pi kTc}{\lambda^4}\right)^{12} \tag{10.44}$$

As already indicated in the chapter on the exact Planck law, the Rayleigh-Jeans law historically emerged as a classical prediction of the spectral density of black body radiation, assuming a continuum of energy[13] of electromagnetic modes in a black body cavity. This law predicts that the power dI emitted in the frequency range $\langle \nu, \nu + d\nu \rangle$ increases quadratically with the frequency ν. If we look at how much energy per unit of time a black body emits altogether at all frequencies, we get

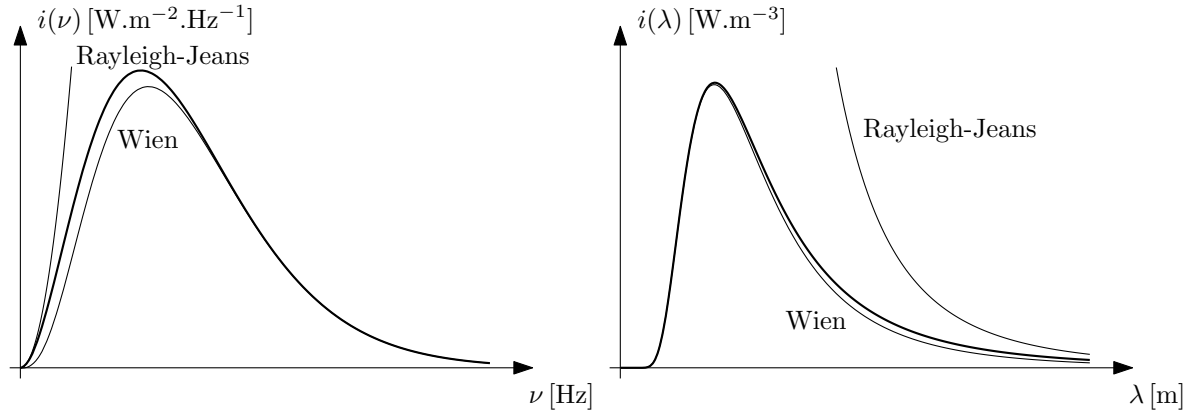
$$I(T) = \int_0^{+\infty} i(\nu, T)\, d\nu = +\infty, \quad \forall T \neq 0. \tag{10.45}$$

This result is called the *ultraviolet catastrophe*. The prediction of classical theory leads to an absurd result: a body, which is not at absolute zero temperature, necessarily emits an infinite amount of energy! Let us repeat that the derivation was corrected by Planck, who ad hoc added the so-called *quantum hypothesis*, assuming that the energies of electromagnetic wave modes in a black body cavity are quantized.

A comparison of the exact Planck's law and the approximate Rayleigh-Jeans formula can be seen in Figure 10.11.

[12] If we did the same for the spectral density for wavelengths, then in the limit of large wavelengths we obtain this expression in brackets.

[13] For a continuous distribution of EM mode energies, statistical physics predicts the average energy of one mode in the form $\langle E \rangle = kT$ (k is again the Boltzmann constant). Which after substitution into $i = n\langle E \rangle$ leads to the Rayleigh-Jeans law.



(a) Spectral density $i(\nu)$.

(b) Spectral density $i(\lambda)$.

Figure 10.11: Approximations of Planck's law of spectral density of black body radiation $i(\nu, T)$, resp. $i(\lambda, T)$: Rayleigh-Jeans Law for low frequencies (high wavelengths) and Wien's Law for high frequencies (low wavelengths).

High frequencies: Wien's Law

For high frequencies (low wavelengths) $e^{\frac{h\nu}{kT}} \gg 1$ and therefore we can neglect the one in the denominator in Planck's law (10.36):

$$e^{\frac{h\nu}{kT}} \gg 1 \qquad e^{\frac{h\nu}{kT}} - 1 \approx e^{\frac{h\nu}{kT}} \tag{10.46}$$

The resulting approximation is called *Wien's Law*:

$$i(\nu, T) \approx \frac{2\pi h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}, \qquad \left(i(\lambda, T) \approx \frac{2\pi hc^2}{\lambda^5} e^{-\frac{hc}{\lambda kT}} \right). \tag{10.47}$$

Wien's Law was derived by Wilhelm Wien in 1896 by combining the Maxwell-Boltzmann distribution of molecular velocities and the empirically known Stefan-Boltzmann law.

A comparison of the exact Planck's law and the approximate Wien formula can be seen in Figure 10.11.

10.5.4 Wien's Displacement Law

How will the value of the wavelength $\lambda_{\max}$ with the highest spectral density evolve with the temperature T of the black body? I.e., at what wavelength does the black body emit the most? Let's find the maximum of the spectral density $i(\lambda, T)$ (10.42):

$$\frac{\partial\, i(\lambda, T)}{\partial \lambda} = 2\pi hc^2 \left[-5\lambda^{-6} \frac{1}{e^x - 1} + \lambda^{-5} \frac{\frac{x}{\lambda} e^x}{(e^x - 1)^2} \right] = \frac{2\pi hc^2 \lambda^{-6}}{(e^x - 1)^2} \left[e^x x - 5(e^x - 1) \right] \stackrel{!}{=} 0, \tag{10.48}$$

where we denoted $x = \frac{hc}{\lambda kT} \in \mathbb{R}^+$. The condition of zero derivative obviously leads to the equation

$$\frac{xe^x}{e^x - 1} = 5. \tag{10.49}$$

This equation cannot be solved using elementary functions[14], but the numerical solution is in the form

$$x_{\max} \doteq 4,965114. \tag{10.50}$$

[14]However, it can be solved using special functions! The Lambert W function (otherwise also known as the product logarithm) is defined as the inverse function to the function $f(x) = xe^x$, $W(x) := f^{-1}(x)$, i.e.,

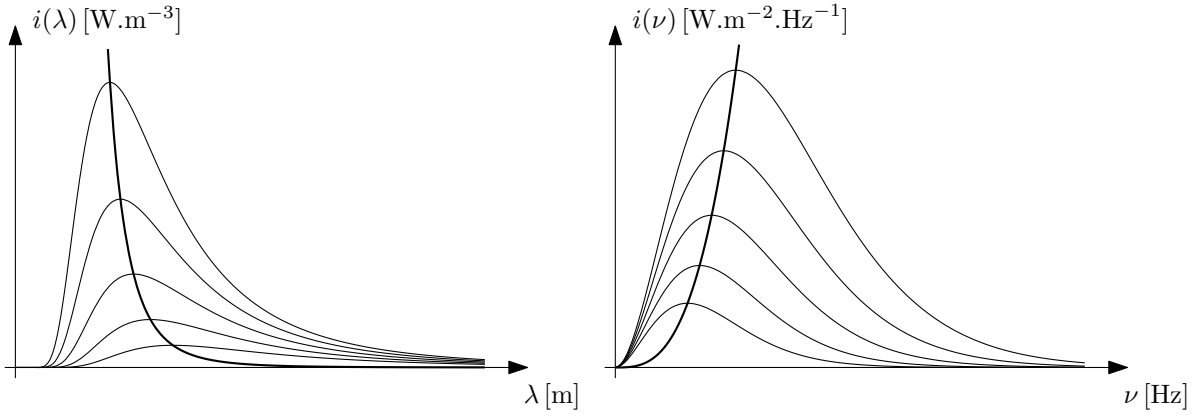
By substituting this value back into the notation $x = \frac{hc}{\lambda kT}$, we obtain the sought relation for the wavelength $\lambda_{\max}$ with the highest spectral density:

$$x_{\max} = \frac{hc}{\lambda_{\max} kT} \quad \Rightarrow \quad \lambda_{\max} T = \frac{hc}{x_{\max} k} = b, \tag{10.51}$$

where we denoted a new constant as $b = \frac{hc}{x_{\max} k}$. This relation is called bfWien's Displacement Law:

$$\boxed{\lambda_{\max} T = b = \text{const.} \doteq 2,89777.10^{-3}\,\text{K.m.}} \tag{10.52}$$

In Figure 10.12 on the left, the progression of maxima of spectral densities $i(\lambda, T)$ for various temperatures T is illustrated.



(a) Spectral densities $i(\lambda, T)$ and their maxima. (b) Spectral densities $i(\nu, T)$ and their maxima.

Figure 10.12: Spectral intensity of black body radiation for various temperatures T. The curve connecting the maxima of individual curves as a function of temperature is highlighted.

Note: The same question as at the beginning of this chapter can also be asked for the frequency $\nu_{\max}$ with the highest spectral density. We proceed in exactly the same way only with the density $i(\nu, T)$ (10.36). If I denote $y = \frac{h\nu}{kT}$, then the requirement of zero derivative,

$$\frac{\partial\, i(\nu, T)}{\partial \nu} \stackrel{!}{=} 0, \tag{10.53}$$

leads to the following equation and its solution $y_{\max}$:

$$e^y y - 3(e^y - 1) = 0, \qquad 3 = \frac{y e^y}{e^y - 1}, \qquad y_{\max} = 2,82144. \tag{10.54}$$

Substitution into the substitution and expressing $\nu_{\max}$ leads to a law analogous to Wien's:

$$\boxed{\frac{\nu_{\max}}{T} = \frac{k y_{\max}}{h} = a = \text{const.} = 0,058789\,\text{THz.K}^{-1}.} \tag{10.55}$$

$W(y) = x$, where x and y satisfy $y = xe^x$. The function f is not injective over the whole $\mathbb{R}$! It is necessary to take $D_f = H_W = \langle -1, +\infty)$ and $H_f = D_W = \langle -\frac{1}{e}, +\infty)$.

The equation (10.49), where on the right side we can write a general c (for us $c \in \{3, 5\}$), can then be solved as

$$\frac{xe^x}{e^x - 1} = c, \quad xe^x = ce^x - c, \quad \text{after subst. } x = x' + c: \quad x' e^{x'} = -\frac{c}{e^c}, \quad x' = W\left(-\frac{c}{e^c}\right), \quad \boxed{x = c + W\left(-\frac{c}{e^c}\right).}$$

The argument of the W function must be within its domain, i.e., $-\frac{c}{e^c} = (-c)e^{-c} = f(-c) \geq -\frac{1}{e}$, which is always satisfied.

At this point, it is appropriate to note that since the relationship between spectral densities $i(\lambda, T)$ and $i(\nu, T)$ is not given by a mere substitution, then the resulting values of $\lambda_{\max}$ and $\nu_{\max}$ are not given by a simple conversion $\lambda_{\max} \neq \frac{c}{\nu_{\max}}$. Let's demonstrate this with the example of the Sun with a surface temperature of $T = 6000\,\mathrm{K}$: $\lambda_{\max} \doteq 483\,\mathrm{nm}$ (and the corresponding frequency of approximately 621 THz) and $\nu_{\max} \doteq 353$ THz (and the corresponding wavelength of about 850 nm).

10.5.5 Stefan-Boltzmann Law

Let's now look at the total power $I(T)$ emitted by a black body at temperature T regardless of the frequency or wavelength of the radiation. It is sufficient to only integrate the spectral density $i(\nu, T)$ over all frequencies ν (or the density $i(\lambda, T)$ over all wavelengths λ):

$$I(T) = \int_0^{+\infty} i(\nu, T)\, d\nu, \qquad \left(I(T) = \int_0^{+\infty} i(\lambda, T)\, d\lambda \right). \tag{10.56}$$

After substituting from Planck's radiation law (10.36) we have the expression

$$I(T) = \frac{2\pi h}{c^2} \int_0^{+\infty} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} d\nu. \tag{10.57}$$

By performing the substitution $x = \frac{h\nu}{kT}$, $\nu = \frac{kTx}{h}$, $d\nu = \frac{kT}{h}\, dx$, we convert the integral into the form

$$I(T) = \left[\frac{2\pi k^4}{c^2 h^3} \int_0^{+\infty} \frac{x^3}{e^x - 1}\, dx \right] T^4 = \sigma\, T^4. \tag{10.58}$$

The expression in brackets is already a mere numerical constant, which we denoted by σ and is called the Stefan-Boltzmann constant. The total power emitted by a body at temperature T thus depends on the fourth power of this temperature. This fact is called the **Stefan-Boltzmann Law**:

$$\boxed{I(T) = \sigma\, T^4.} \tag{10.59}$$

Utilizing the mathematical statement

$$\int_0^{+\infty} \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}, \tag{10.60}$$

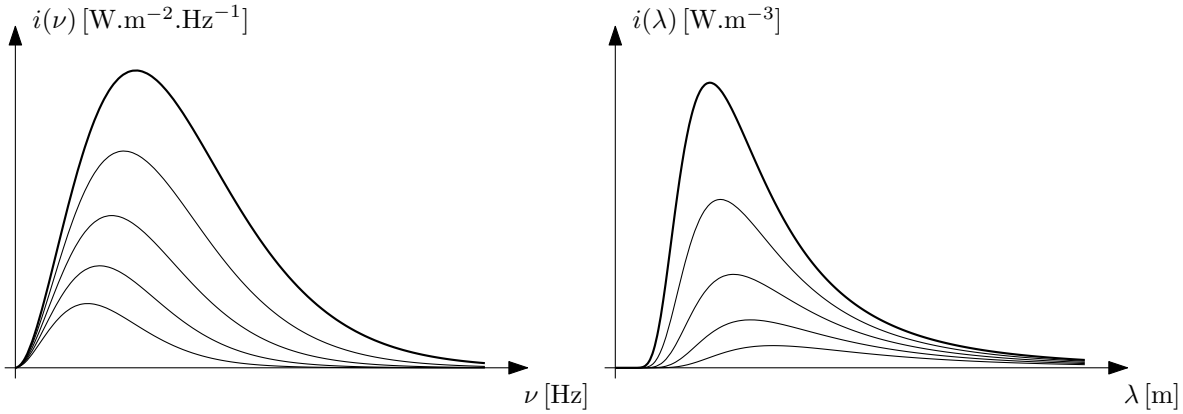
which we will not derive here, we obtain the Stefan-Boltzmann constant in the form

$$\sigma = \frac{2k^4\pi^5}{15c^2h^3} = 5,6704.10^{-8}\,\mathrm{W.m^{-2}.K^{-4}}. \tag{10.61}$$

The expression for this constant thus fully follows from Planck's radiation law and its value is fully determined by fundamental physical constants[15].

In Figure 10.13, the curves of spectral intensities for various temperatures T are illustrated.

[15] And fundamental mathematical constants 2 and 15. And also π.



(a) Spectral intensity $i(\nu)$ for various temperatures T.

(b) Spectral intensity $i(\lambda)$ for various temperatures T.

Figure 10.13: Graphs of spectral intensity $i(\nu, T)$ and $i(\lambda, T)$ for various temperatures T. The total power emitted (and thus the area under these graphs) according to the Stefan-Boltzmann law grows with the fourth power of temperature T^4, $I(T) = \sigma T^4$.

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