Exercises from Waves

September 27, 2023

1 Mean values

Consider function $f(x) : \mathbb{R} \to \mathbb{R}$. Its mean value in the interval $\langle x_1, x_2 \rangle$ is defined as

$$\langle f \rangle_{\langle x_1, x_2 \rangle} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\, dx.$$

We can define the mean over the whole $\mathbb{R}$ by a limit

$$\langle f \rangle \equiv \langle f \rangle_{\langle -\infty, \infty \rangle} = \lim_{x' \to \infty} \frac{1}{2x'} \int_{-x'}^{x'} f(x)\, dx.$$

If the function f is periodic with period L, its mean value is given by the mean value over an arbitrary interval of length L:

$$\langle f \rangle = \langle f \rangle_{\langle x', x'+L \rangle} = \frac{1}{L} \int_{x'}^{x'+L} f(x)\, dx, \qquad x' \in \mathbb{R}.$$

By definition, the following rules obviously hold

$$\langle cf \rangle = c \langle f \rangle, \qquad \langle f + g \rangle = \langle f \rangle + \langle g \rangle.$$

Exercise 1.1 Calculate $\langle 1 \rangle$, $\langle \cos \omega t \rangle$, $\langle \sin \omega t \rangle$, $\langle \sin(\omega t + \varphi_0) \rangle$, $\langle \cos^2 \omega t \rangle$, $\langle \sin^2 \omega t \rangle$, $\langle \cos^2 \omega t + \sin^2 \omega t \rangle$.

2 Complex numbers

Complex number $z \in \mathbb{C}$ is a number of the form $z = a + ib$, where $a, b \in \mathbb{R}$ and i is an imaginary unit with the property $i^2 = -1$. The addition and multiplication of these numbers is defined by "natural" extension.

A complex number $\bar{z}$ is a complex number $\bar{z} = a - ib$. The formula $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ applies. The magnitude of a complex number is defined as $|z| = \sqrt{a^2 + b^2}$, this expression can be written as $|z| = \sqrt{z\bar{z}}$.

Real and imaginary parts. We define the functions Re : $\mathbb{C} \to \mathbb{R}$ and Im : $\mathbb{C} \to \mathbb{R}$ called the real and imaginary parts using the rules

$$\operatorname{Re} z = a, \qquad \operatorname{Im} z = b$$

(note that the imaginary part does not contain an imaginary unit!). If the real part is zero, we call the number *pure imaginary.* The functions Re and Im are *real* linear, i.e.

$$\operatorname{Re}(z_1 + z_2) = \operatorname{Re} z_1 + \operatorname{Re} z_2, \qquad \operatorname{Re}(\alpha z) = \alpha \operatorname{Re} z, \qquad \alpha \in \mathbb{R},$$

equally for Im. Note $\operatorname{Re}(z_1 z_2) \neq (\operatorname{Re} z_1)(\operatorname{Re} z_2)$ (equally for Im). These functions can be expressed simply by complex conjugation:

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

Complex exponential. Consider $z \in \mathbb{C}$. Let us now consider e^z and manipulate it:

$$e^z = e^{a+ib} = e^a e^{ib}.$$

The first term is an ordinary real exponential function. The question is what is the exponential of a purely imaginary number. The answer is given by Euler's formula [1]:

$$e^{i\varphi} = \cos\varphi + i \sin\varphi,$$

Moreover, $|e^{i\varphi}| = 1$ holds. We can thus write $e^z = e^a(\cos b + i \sin b)$. For the magnitude of this number, $|e^z| = e^a$ holds.

Polar form of a complex number. Any complex number $z \in \mathbb{C}$ can be written in the form $z = |z|e^{i\varphi}$, $\varphi \in \mathbb{R}$. We call the number φ the *argument* of a complex number (this number is not given uniquely, any integer multiple of 2π can be added). The argument φ is the solution of the equations

$$\cos\varphi = \frac{\operatorname{Re} z}{|z|}, \qquad \sin\varphi = \frac{\operatorname{Im} z}{|z|}.$$

These equations are often formally[2] combined into one equation

$$\operatorname{tg}\varphi = \frac{\operatorname{Im} z}{\operatorname{Re} z}.$$

Gaussian (complex) plane. Complex numbers can be represented as points on a (two-dimensional) plane, where the Cartesian axes are the real and imaginary parts of the complex numbers.

[1] Of which "the most beautiful mathematical identity" $e^{i\pi} = -1$ is a special case.

[2] In this notation, we lose information about whether $\varphi \in \langle 0, \pi)$ and/or $\varphi \in \langle \pi, 2\pi)$.

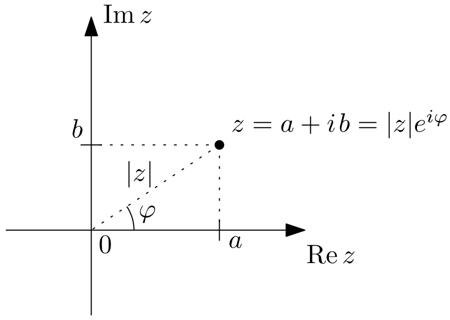


Figure 2.1: Gaussian plane.

The addition of complex numbers then has the geometric meaning of adding two-dimensional vectors in the Gaussian plane. The number $e^{i\varphi}$ represents a number on the unit circle. An intuitive idea of multiplication of complex numbers is obtained from goniometric notation:

$$z_1 z_2 = |z_1| e^{i\varphi_1} |z_2| e^{i\varphi_2} = |z_1||z_2| e^{i(\varphi_1+\varphi_2)}.$$

Thus, multiplication by the number $e^{i\varphi}$ represents a rotation by the angle φ in the complex plane. Multiplication by $|z|$ represents scaling in this plane.

Complex notation for goniometric functions. The following relations follow directly from Euler's formula:

$$\cos\varphi = \mathrm{Re}\, e^{i\varphi} = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \qquad \sin\varphi = \mathrm{Im}\, e^{i\varphi} = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

Replacing $\varphi \in \mathbb{R}$ by the general $z \in \mathbb{C}$, we can use the previous formulas to define the sine and cosine functions on the entire complex plane.

Exercise 2.1 Find the real and imaginary parts of a number

$$w = \frac{a + ib}{c + id}.$$

Exercise 2.2 Calculate $\mathrm{Re}\left[(C - iD)e^{i\Omega t}\right]$, where $C, D, \Omega t \in \mathbb{R}$.

Exercise 2.3 Show that $\overline{e^z} = e^{\bar{z}}$ holds, specially $\overline{e^{ib}} = e^{-ib}$.

Exercise 2.4* Prove the validity of the relations

$$\mathrm{Re}\,(iz) = -\mathrm{Im}\, z, \qquad \mathrm{Im}\,(iz) = \mathrm{Re}\, z.$$

Use these relations to show the validity of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right).$$

Exercise 2.5 Derive the summation formulas for the sines and cosines of the sum (and difference) of angles using the trivial identity

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}.$$

Exercise 2.6 Derive the summation formulas for sums of sines and cosines by manipulating the expression as follows

$$e^{i\alpha} + e^{i\beta} = e^{i\frac{\alpha}{2}} e^{i\frac{\beta}{2}} \left(e^{i\frac{\alpha-\beta}{2}} + e^{i\frac{\beta-\alpha}{2}}\right).$$

Exercise 2.7* Prove the validity of the relations

$$\sin ix = i \sinh x, \qquad \cos ix = \cosh x, \qquad \sinh ix = i \sin x, \qquad \cosh ix = \cos x.$$

Exercise 2.8 Consider the expression $c_1 e^{i\omega t} + c_2 e^{-i\omega t}$, where $c_1, c_2 \in \mathbb{C}$ and $\omega t \in \mathbb{R}$. What conditions must the constants c_1 and c_2 satisfy for the above expression to be real for all $t \in \mathbb{R}$?

Exercise 2.9 The solution of the harmonic oscillator equation can be written in several equivalent forms:

$$x(t) = A\cos(\omega t + \varphi) = A\sin(\omega t + \phi) = a\cos\omega t + b\sin\omega t = ce^{i\omega t} + \bar{c}e^{-i\omega t},$$

$A, a, b, \varphi, \phi, \omega t \in \mathbb{R}$, $c \in \mathbb{C}$. Find the relationships between the constants A, φ, ϕ, a, b, and c.

Exercise 2.10* "Prove" Euler's formula using differential identity

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}.$$

Instructions: show that the function $f(x) = \cos x + i \sin x$ satisfies the differential equation for the exponential with the appropriate initial condition.

Exercise 2.11* Write expressions $\cos^2 x$, $\cos^3 x$, in general** $\cos^n x$, $n \in \mathbb{N}$, using only functions $\cos kx$, $k \in \mathbb{N}_0$.

Exercise 2.12 Sum the series

$$\sum_{m=0}^{N} \cos mx.$$

Exercise 2.13 Calculate the following definite integrals:

$$\int_0^{+\infty} e^{-ax} \cos bx \, dx, \qquad \int_0^{\infty} e^{-ax} \sin bx \, dx, \qquad a, b \in \mathbb{R}, a > 0.$$

*Calculate the corresponding indefinite integrals (primitive functions).

3 Small oscillations and method of modes

Method of modes Cookbook

1. Introduce coordinates $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that measure the deviation from the equilibrium position.

2. Write the equations of motion in the form $\mathbb{T}\ddot{\vec{x}} + \mathbb{U}\vec{x} = 0$, where $\mathbb{T}, \mathbb{U} \in \mathbb{R}^{n,n}$ are constant matrices.

3. Assume a solution of the form $\vec{x}(t) = A\,\vec{a}\cos(\omega t + \varphi)$, $\vec{a} \in \mathbb{R}^n$ is a constant vector of amplitudes.

4. Substitute the ansatz into the equations of motion and require nontriviality of the solutions, i.e., $A \neq 0$ and $\vec{a} \neq 0$. We get $\left(\mathbb{U} - \omega^2\mathbb{T}\right)\vec{a} = 0$. These conditions lead to the so-called secular equation $\left|\mathbb{U} - \omega^2\mathbb{T}\right| = 0$.

5. The secular equation is a polynomial of n-th degree in ω^2. Find the corresponding roots ω_k^2. Find the corresponding eigenvectors $\vec{a}_k$ as solutions of equations $\left(\mathbb{U} - \omega_k^2\mathbb{T}\right)\vec{a}_k = 0$.

6. The general solution of the motion is of the form

$$\vec{x}(t) = \sum_{k=1}^{n} A_k \vec{a}_k \cos(\omega_k t + \varphi_k) .$$

Small oscillations

In the Taylor series of the potential function $U(\vec{x})$, the first nonzero term is just the second order of the expansion. If we denote $\mathbb{U}_{ij}$ as

$$\mathbb{U}_{ij} = \left.\frac{\partial^2 U}{\partial x_i \partial x_j}\right|_{\vec{x}=0} , \tag{1}$$

we have the expansion of the function $U(\vec{x})$ of the form

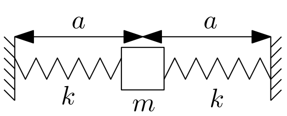
$$U(\vec{x}) = \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{U}_{ij} x_i x_j + \ldots \tag{2}$$

Neglecting all higher orders, we get an approximate expression

$$U_{\text{small}}(\vec{x}) = \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{U}_{ij} x_i x_j = \frac{1}{2}\vec{x}^T\,\mathbb{U}\,\vec{x}, \tag{3}$$

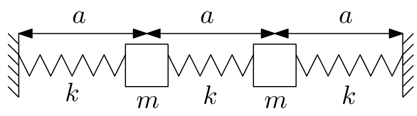
which is exactly the form we need for the method of modes.

Exercise 3.1 Construct the potential for the longitudinal and transverse vibrations of the weights on the springs as in the figure. The length of the unstretched springs is a_0.



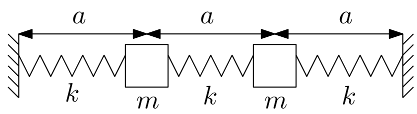
Find the forms of these potentials in the small oscillation approximation.

Exercise 3.2 Construct the equations of motion for the longitudinal oscillations of the system in the figure. The length of the unstretched springs is a_0.



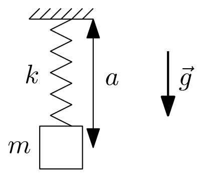
Find their solutions by the method of modes.

Exercise 3.3 Write the potential for the transverse oscillations of the system in the figure. The length of the unstretched springs is a_0.

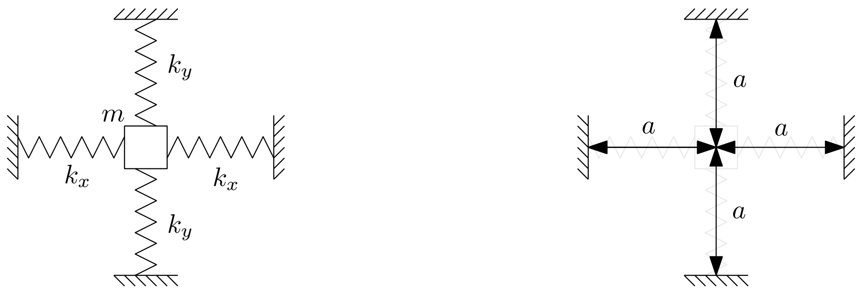


Find its form in the small oscillations approximation. How does it differ from the potential for longitudinal oscillations?

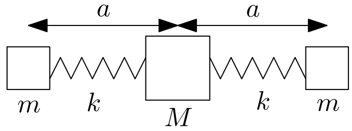
Exercise 3.4 Find the potential of a spring pendulum (see figure) in the small oscillations approximation. The pendulum can perform 2D motion in the vertical plane.



Exercise 3.5* Find the potential of the spring pendulum (see figure) in the small oscillation approximation. The pendulum can perform 2D motion in the vertical plane.



Exercise 3.6* Find the solution to the equations of motion of the following mechanical system using the method of modes. Only longitudinal motion is allowed.



Is the solution found complete? "Where did the problem occur?"

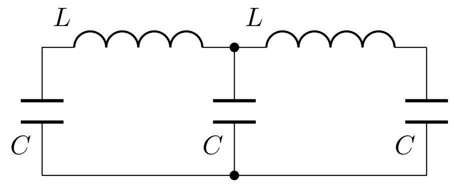
Exercise 3.7 Consider a general solution to the motion of a system of the form

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \varphi_1) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \varphi_2).$$

Find a specific solution for the initial conditions

$$x_1(0) = A \neq 0, \quad x_2(0) = 0, \quad \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 0.$$

Exercise 3.8* Find a general solution for the currents in each branch in the following LC circuit.



Further examples for home practice:

Exercise 3.9 Equations of motion. Find the equations of motion and their corresponding matrices $\mathbb{T}$ and $\mathbb{U}$, defined using $\mathbb{T}\ddot{\vec{x}} + \mathbb{U}\vec{x} = 0$.

a) Construct the equations of motion for three longitudinally oscillating weights on four springs.

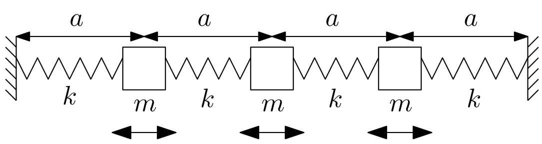


Figure 3.2: Longitudinal vibrations of three weights on four springs.

b) Find the equations for the currents in the following triple LC circuit.

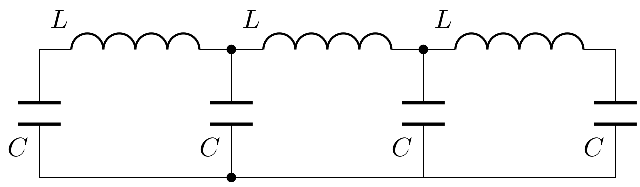


Figure 3.3: Triple LC circuit.

Exercise 3.10 System modes. Find the modes and the general solution of the form

$$\vec{x}(t) = \sum_{k=1}^{N} A_k \vec{a}_k \cos(\omega_k t + \varphi_k)$$

for the systems described by the following matrices $\mathbb{T}$ and $\mathbb{U}$:

a)

$$\mathbb{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 3k & -2k \\ -2k & 6k \end{pmatrix};$$

b)

$$\mathbb{T} = \begin{pmatrix} 2m & 0 \\ 0 & 3m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 5k & -2k \\ -2k & 5k \end{pmatrix};$$

c)

$$\mathbb{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 2k & -k \\ -k & 4k \end{pmatrix};$$

d)

$$\mathbb{T} = \begin{pmatrix} m & 0 \\ 0 & 2m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 3k & -2k \\ -2k & 6k \end{pmatrix};$$

e)

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 3k & -2k & 0 \\ -2k & 4k & -2k \\ 0 & -2k & 5k \end{pmatrix};$$

Hint: One of the angular frequencies is $\omega = \sqrt{\frac{k}{m}}$.

f)

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}.$$

Help: One of the angular frequencies is $\omega = \sqrt{\frac{2k}{m}}$.

g)

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 4k \end{pmatrix}.$$

Hint: One of the angular frequencies is $\omega = \sqrt{\frac{2k}{m}}$.

h)

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 3m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 2k & -k & 0 \\ -k & 4k & -3k \\ 0 & -3k & 6k \end{pmatrix}.$$

Help: One of the angular frequencies is $\omega = \sqrt{\frac{2k}{m}}$.

Exercise 3.11 Small oscillations. Find the matrices $\mathbb{U}$ for the following potential energy functions U:

a)

$$U(x_1, x_2) = \frac{1}{2}k\left[x_1^2 + 2\left(x_2 - x_1\right)^2 + 4x_2^2\right],$$

b)

$$U(x_1, x_2, x_3) = \frac{1}{2}k\left[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2\right],$$

c)

$$U(y) = k\left[\sqrt{a^2 + y^2} - a_0\right]^2$$

d)

$$U(x_1, x_2) = \frac{1}{2}k\left[\sqrt{\left(a + l\sin\frac{x_2}{l} - l\sin\frac{x_1}{l}\right)^2 + l^2\left(\cos\frac{x_2}{l} - \cos\frac{x_1}{l}\right)^2} - a\right]^2 - mgl\left(\cos\frac{x_1}{l} + \cos\frac{x_2}{l}\right).$$

4 Strings and Fourier series

String with fixed ends. A string of length L with fixed ends at $z = 0$ and $z = L$ has a general solution of its equations of motion in the form of the following superposition of modes

$$\psi(z,t) = \sum_{m=1}^{\infty} A_m \sin(k_m z) \sin(\omega_m t + \varphi_m), \tag{4}$$

where

$$\omega = \sqrt{\frac{T_0}{\rho_0}}\, k, \qquad k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N}. \tag{5}$$

Thus, the angular frequency ω and the wavenumber k satisfy the given dispersion relation and the wavelengths of the modes are given by the discrete wavenumbers k_m.

Boundary conditions. Consider $z_0 \in \{0, L\}$. Fixed-end boundary condition: $\psi(z_0, t) = 0, \forall t \in \mathbb{R}$, free-end boundary condition: $\frac{\partial \psi(z_0,t)}{\partial z} = 0, \forall t \in \mathbb{R}$.

Fourier series. Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $2L$. Then we call the following function f_F a Fourier series of f:

$$f_F(z) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} \left(a_m \cos\frac{m\pi z}{L} + b_m \sin\frac{m\pi z}{L} \right), \tag{6}$$

where the coefficients a_m and b_m are the given relations:

$$a_m = \frac{1}{L}\int_{-L}^{L} f(z)\cos\frac{m\pi z}{L}\, dz, \quad m \in \mathbb{N}_0; \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(z)\sin\frac{m\pi z}{L}\, dz, \quad m \in \mathbb{N}. \tag{7}$$

For even functions ($f(x) = f(-x)$) and odd functions ($f(x) = -f(-x)$), respectively, the Fourier series (6) and the formulas for the coefficients a_m and b_m (7) are simplified. For **even functions** we get

$$a_m = \frac{2}{L}\int_{0}^{L} f(z)\cos\frac{m\pi z}{L}\, dz, \qquad b_m = 0, \qquad f_F(z) = \frac{a_0}{2} + \sum_{m=1}^{+\infty} a_m \cos\frac{m\pi z}{L}. \tag{8}$$

For **odd functions**:

$$a_m = 0, \qquad b_m = \frac{2}{L}\int_{0}^{L} f(z)\sin\frac{m\pi z}{L}\, dz, \qquad f_F(z) = \sum_{m=1}^{+\infty} b_m \sin\frac{m\pi z}{L}. \tag{9}$$

Initial value problem. The initial conditions consist of the initial position of the string and the initial velocity of the string (for simplicity, we choose that they are specified in time $t = 0$). These are specified as a function of the initial position $f : \langle 0, L\rangle \to \mathbb{R}$ (we must specify the initial deflection of each point on the string) and as a function of the initial velocity $g : \langle 0, L\rangle \to \mathbb{R}$ (the same for the initial velocity of each point on the string). Thus, our search for a particular solution must satisfy:

$$\psi(z, 0) = f(z), \quad \frac{\partial \psi}{\partial t}(z, 0) = g(z), \quad \forall z \in \langle 0, L\rangle. \tag{10}$$

In order to achieve this, we have the integration constants A_m and φ_m whose value we want to determine.

Let us write explicitly the left-hand sides of the equations (10), i.e., let us substitute the time $t = 0$ to the general solution (4) and its time derivative:

$$\psi(z,0) = \sum_{m=1}^{+\infty} (A_m \sin\varphi_m) \sin\frac{m\pi z}{L} = f(z),$$
$$\frac{\partial\psi}{\partial t}(z,0) = \sum_{m=1}^{+\infty} (A_m\omega_m \cos\varphi_m) \sin\frac{m\pi z}{L} = g(z). \tag{11}$$

Extensions of the functions f and g. Now we need to write the functions f and g as Fourier series, which contain only the functions $\sin\frac{m\pi z}{L}$. This is easily achieved if we compute series of functions f and g in **odd extension**:

$$f(z) = \sum_{m=1}^{+\infty} f_m \sin\frac{m\pi z}{L}, \qquad g(z) = \sum_{m=1}^{+\infty} g_m \sin\frac{m\pi z}{L}, \tag{12}$$

where the coefficients of f_m and g_m are given by the following formulas:

$$f_m = \frac{2}{L}\int_0^L f(z)\sin\left(\frac{m\pi z}{L}\right) dz, \qquad g_m = \frac{2}{L}\int_0^L g(z)\sin\left(\frac{m\pi z}{L}\right) dz. \tag{13}$$

Equations for the coefficients A_m, φ_m. The equations for the coefficients A_m and φ_m are obtained by comparing the series (11) and (12) term by term:

$$A_m \sin\varphi_m = f_m, \qquad A_m\omega_m \cos\varphi_m = g_m. \tag{14}$$

Exercise 4.1 If we shorten the string by $\Delta l = 10\,\mathrm{cm}$, its vibration frequency increases to $\alpha = 150\%$. Calculate the original length of the string L. Assume that the string tension remains the same.

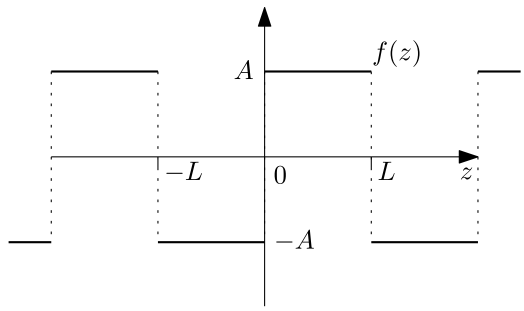
Exercise 4.2 A piano string of length $L = 1\,\mathrm{m}$ has a diameter of $d = 0,5\,\mathrm{mm}$ and makes a fundamental tone of C with a frequency of $f = 256\,\mathrm{Hz}$. The volume density of this string is $\rho_{vol} = 9\,\mathrm{g/cm^3}$. What is the tension T of the string?

Exercise 4.3 Find the mode shapes for a string of length L (stretched between $z \in \langle 0, L\rangle$) for the free ends. Assume a mode-shaped (standing wave) solution $\psi(z,t) = X(z)\cos(\omega t + \varphi)$ (or a complexification $\hat{\psi}(z,t) = X(z)e^{i\omega t}$). Write the general solution as a superposition of these modes. Is there anything missing in the solution?

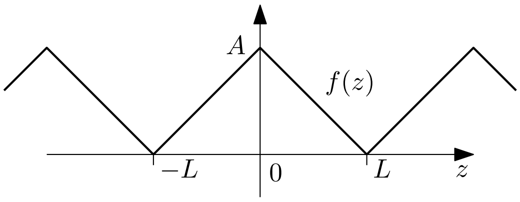
Exercise 4.4* Same problem as the previous exercise, except that now you consider one end fixed and the other free.

Exercise 4.5 Calculate the Fourier series of the following functions f with period $2L$.

a) Square wave .



b) *Triangular wave .



Exercise 4.6 Consider a string with fixed ends. Find a particular solution of its motion if you let it oscillate so that at time $t = 0$ it is at rest and has the form $\psi(z, 0) = A$, where A is a constant.

Exercise 4.7 Consider a string with fixed ends. Find a particular solution of its motion if it is in equilibrium at time $t = 0$ and at the same time you strike it with a hammer to give a segment of string length Δz centred about a point $L/2$ a velocity of v_0.

Exercise 4.8* *Initial value problem for a string with free ends.* Modify the procedure to find a particular solution from the given initial conditions for a string of length L with free ends. For example, the general solution from the separation of variables method comes out as

$$\psi(z,t) = z_0 + v_0 t + \sum_{m=1}^{+\infty} A_m \cos k_m z \, \sin(\omega_m t + \varphi_m), \quad \text{where} \quad k_m = \frac{m\pi}{L} \quad \text{and} \quad \omega_m = \sqrt{\frac{T_0}{\rho_0}} k_m.$$

5 Travelling and standing waves

The d'Alembert solution of the wave equation is a solution of the form

$$\psi(z,t) = F(z - vt) + G(z + vt),$$

where the functions $F, G : \mathbb{R} \to \mathbb{R}$ represent a waveform of a travelling wave propagating in the positive and negative directions of the axis z, respectively.

A wave source located at $z = 0$ oscillating according to the prescription $x(t)$ emits in the positive direction of the axis z a travelling wave of the form

$$\psi(z,t) = x(t_r),$$

where $t_r = t - \frac{z}{v}$ is the so-called retarded time.

A harmonic travelling wave is a wave of the form

$$\psi(z,t) = A\cos(\omega t - kz + \varphi), \qquad \hat{\psi}(z,t) = Ae^{i(\omega t - kz + \varphi)},$$

where the angular velocity ω gives the period T by the relation $T = \frac{2\pi}{\omega}$, the wave number k gives the wavelength λ by the relation $\lambda = \frac{2\pi}{k}$.

The energy flux on a string is given by

$$S = -T \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial z}.$$

Exercise 5.1 Two tuning forks produce 20 beats in 10 seconds. One tuning fork has a frequency of 256 Hz. What is the frequency of the second tuning fork?

Exercise 5.2 What is the amplitude, period, phase velocity, and wavelength of a wave, expressed in the SI system by the equation

$$\psi(z,t) = 4.10^{-2} \sin[2\pi(8t + 5z)]\,?$$

Exercise 5.3 *The superposition of travelling waves in the same direction is a travelling wave.* Show that the sum of

$$A_1 \cos(\omega t - kz + \varphi_1) + A_2 \cos(\omega t - kz + \varphi_2)$$

can be written as $A\cos(\omega t - kz + \varphi)$. Determine the values of the constants A and φ.

Exercise 5.4 *The superposition of opposing travelling waves is a standing wave.* Show that the sum of

$$A\cos(\omega t - kz + \varphi_1) + A\cos(\omega t + kz + \varphi_2)$$

is of the form $X(z)\cos(\omega t + \varphi)$. Determine the form of the function $X(z)$ and the value of the constant φ.

Exercise 5.5 Two sources on the z axis at $z = -d$ and $z = d$ oscillate according to the prescription $x_1(t) = x_2(t) = A\cos(\omega t)$ and emit waves in both directions. Determine the resulting travelling waves from each source and discuss the nature of their superposition.

Exercise 5.6 Consider a homogeneous string stretched from $z = 0$ to $z = +\infty$. The string has linear density $\rho = 0,1\,\mathrm{g.cm}^{-1}$ and is tensioned by force $T = 400\,\mathrm{N}$. The origin of the string $z = 0$ performs a harmonic motion of frequency $f = 100\,\mathrm{Hz}$ with amplitude $A = 1\,\mathrm{cm}$. What is the time-mean value of the energy flux in watts?

Exercise 5.7 There are waves on a string propagating in opposite directions. Show that the energy flux vector on the string is equal to the sum of the energy fluxes corresponding to each wave.

Instructions: Consider the d'Alembert solution and use the relation between the derivatives of z and t for travelling waves.

Exercise 5.8 Two harmonic travelling waves propagate in the same direction on a string in superposition. They have the same wavelength and angular frequency. If the intensity (time-mean value of energy flux) of each wave is I, what must be the phase shift of these waves for the resulting intensity to be 0, I, $2I$, $4I$?

6 Wave packets, uncertainty relations, group velocity

Fourier transform

$$f(t) = \int_0^{+\infty} A(\omega) \cos \omega t + B(\omega) \sin \omega t \, d\omega.$$

Spectral functions

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \cos \omega t \, dt, \qquad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \sin \omega t \, dt.$$

Uncertainty relation

$$\Delta\omega \, \Delta t \geq \pi.$$

Group velocity and phase velocity

$$v_g = \frac{d\omega}{dk}, \qquad v_\varphi = \frac{\omega}{k}.$$

Exercise 6.1 Find the form of the wave packet $f(t)$ for a spectrum of the form $B(\omega) = 0$ and

$$A(\omega) = \begin{cases} A_0 & \text{for } \omega_0 - \frac{\Delta\omega}{2} \leq \omega \leq \omega_0 + \frac{\Delta\omega}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Show how the width of the spectrum $\Delta\omega$ is related to the duration of the wave packet Δt, defined here as the distance of the first zero points of the amplitude envelope of the wave packet.

Exercise 6.2 Have a rectangular pulse $f(t)$ of the form

$$f(t) = \begin{cases} A_0 & \text{for } -\frac{\Delta t}{2} \leq t \leq \frac{\Delta t}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Determine its spectrum. Show how the duration of the pulse Δt is related to the width of its frequency spectrum $\Delta\omega$ defined here as the first zero of the frequency spectrum.

Exercise 6.3* Consider a damped oscillation $f(t)$ of the form

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ e^{-\alpha t} \cos \omega_0 t & \text{otherwise.} \end{cases}$$

Find its spectrum. Use the results of exercise 2.13.

Exercise 6.4 WiFi occupies the frequency range 20 MHz (channel width) with its transmission. Estimate what the transmission rate will be. Use the uncertainty relations.

Exercise 6.5* Estimate the maximum trill frequency[3] f_{trylek} of two tones a semitone apart[4] depending on the frequency of one of the notes in the trill f. Take advantage of the uncertainty relations. Why is the tuba not used for trilling?

Exercise 6.6 A linear dispersion relation is a relation of the form $\omega = vk$, where $v = \text{const.}$ Such an environment is called a nondispersive environment. Determine the phase and group velocity.

[3] A rapid alternation of two close tones.

[4] The octave divides into twelve semitones. A shift of an octave means frequency changes of twice or half. A shift of a semitone then means a change in frequency by a factor of $\sqrt[12]{2}$.

Exercise 6.7 Determine the phase and group velocity for electromagnetic waves in a plasma. This medium is described by the dispersion relation

$$\omega^2 = \omega_{min}^2 + c^2k^2.$$

Is the phase and group velocity greater or less than the speed of light? What does this mean?

Exercise 6.8 Consider light in a substance with a refractive index n, defined as $n = \frac{c}{v_\varphi}$. The refractive index in a substance is described for a simple model of electrons as

$$n(\omega) = 1 + \frac{\alpha}{\omega_0^2 - \omega^2},$$

where $\alpha > 0$ and we consider only $\omega < \omega_0$. Determine the group velocity and show that it is less than the speed of light.

Exercise 6.9 Show that for light in a medium with refractive index $n(\lambda_0)$, where λ_0 is the wavelength of light *in vacuum*, the following holds

$$\frac{1}{v_g} = \frac{1}{v_\varphi} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}.$$

7 Reflections

When studying reflections using harmonic waves, we consider an incident wave of the form

$$\psi_{inc} = Ae^{i(\omega t - k_1 z)}$$

and we look for reflected and transmitted wave forms

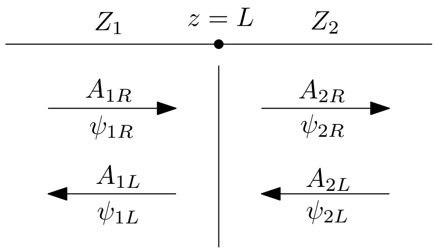
$$\psi_{ref} = ARe^{i(\omega t + k_1 z)}, \qquad \psi_{tr} = ATe^{i(\omega t - k_2 z)}.$$

The coefficients R and T are determined by the respective junction conditions on the interface.

The transmission matrix $\mathbb{D} \in \mathbb{C}^{2,2}$ is defined by the following equation:

$$\begin{pmatrix} A_{1R} \\ A_{1L} \end{pmatrix} = \mathbb{D} \begin{pmatrix} A_{2R} \\ A_{2L} \end{pmatrix}, \tag{15}$$

where A_{iLR} are the amplitudes of waves ψ_{iLR} as in the following figure:



The matrix $\mathbb{D}$ is given by the respective junction conditions at the interface.

To study the reflection at one interface for a given transmission matrix, we solve Eqs.

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D} \begin{pmatrix} T \\ 0 \end{pmatrix},$$

where R is the reflection coefficient and T is the transmission coefficient.

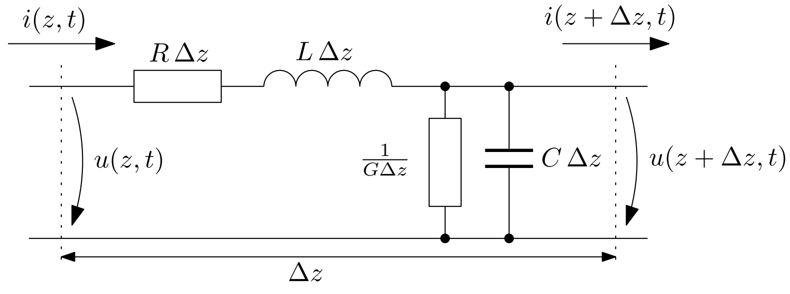
For two interfaces (where each interface has a transition matrix $\mathbb{D}_1$ and $\mathbb{D}_2$), the equation for the transmission and reflection coefficients is as follows:

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D}_1 \mathbb{D}_2 \begin{pmatrix} T \\ 0 \end{pmatrix}.$$

Exercise 7.1* Derive the telegraph equations for voltage and current waves $u(z,t)$ and $i(z,t)$ on a homogeneous line

$$-\frac{\partial u}{\partial z} = Ri + L\frac{\partial i}{\partial t}, \qquad -\frac{\partial i}{\partial z} = Gu + C\frac{\partial u}{\partial t},$$

where L is the line inductance per unit length, $[L] = \mathrm{H.m^{-1}}$, C is the capacitance, $[C] = \mathrm{F.m^{-1}}$, R is the resistance, $[R] = \Omega.\mathrm{m}^{-1}$ and $G = \frac{1}{R'}$ is the leakage conductance, $[G] = \Omega^{-1}.\mathrm{m}^{-1}$. Recall the equations by analyzing a substitute circuit of line length Δz:



Exercise 7.2 Consider an ideal homogeneous line, where $R = 0$ and $G = 0$. Show that the telegraph equations (see the previous exercise) yield the wave equations for the functions $u(z, t)$ and $i(z, t)$. Find a d'Alembert solution satisfying the original telegraph equations.

Instruction #1: Consider the ansatz in the form of d'Alembert solutions

$$u(z,t) = F(z - vt) + G(z + vt), \quad i(z,t) = \alpha_1 F(z - vt) + \alpha_2 G(z + vt). \tag{16}$$

Instruction #2: Substitute d'Alembert solution for u into the telegraph equations and solve for i.

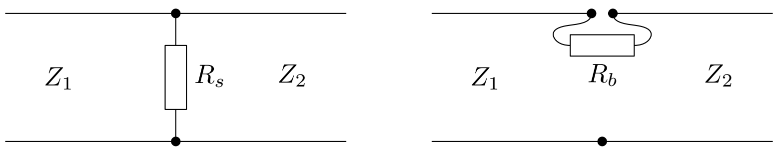
Note: The coefficient of proportionality between the voltage and current waves is called the impedance Z ("generalization of Ohm's law").

Exercise 7.3 A homogeneous line of impedance Z is terminated by a termination resistor of resistance R_s. Find the reflection coefficient R for voltage waves coming down the line. Discuss the special cases $R_s = 0$ (short circuit), $R_s = +\infty$ (unconnected resistor), and $R = 0$ (no reflection). Use harmonic travelling waves.

Exercise 7.4 A homogeneous line of impedance $Z_1 = 50\,\Omega$ is connected to a line of impedance $Z_2 = 100\,\Omega$. Find the coefficients of transmission P and reflection R for voltage waves passing from the first line to the second. If a pulse of amplitude 15 V is incident on the interface, what will be the amplitude of the transmitted and reflected waves?

Instructions: Use harmonic travelling waves.

Exercise 7.5 A homogeneous line of impedance $Z_1 = 50\,\Omega$ is connected to a line of impedance $Z_2 = 100\,\Omega$ in the following two ways:



Find the transmission and reflection coefficients for the voltage waves for these two situations. Under what conditions is there no reflection? Use harmonic travelling waves. Write down the junction conditions and solve for the coefficients.

Exercise 7.6 Consider three interacting environments through two interfaces, one in $z = 0$ and the other in $z = L$. Let us denote the amplitude coefficients of transmission and reflection as T_{ij} and R_{ij} representing the transmission and reflection coefficients at the interface from the i-th to the j-th environment. The wave numbers in each medium are k_1, k_2, k_3. Consider a harmonic incident wave of the form $Ae^{i(\omega t - k_1 z)}$. Find the total reflection coefficient $R \in \mathbb{C}$, i.e., the total reflected wave of the form $ARe^{i(\omega t + k_1 z)} = A|R|e^{i(\omega t + k_1 z + \varphi)}$ resulting from the infinite superposition of reflected waves between two interfaces. Require continuity of the phase functions of the individual waves at the interfaces.

Finally, specialize the result by considering the relations $1 + R_{ij} = T_{ij}$ and $R_{ij} = -R_{ji}$.

Exercise 7.7* Find the total transmission coefficient $T \in \mathbb{C}$, i.e., the total transmitted wave $ATe^{i(\omega t - k_3 z)} = A|T|e^{i(\omega t - k_3 z + \varphi)}$ for the situation described in the previous exercise.

Exercise 7.8 The transmission matrix $\mathbb{D}$ is given. Find the transmission T and reflection R coefficients for a wave incident from the first (left) medium to the second (right) medium.

Exercise 7.9 The coefficients of transmission and reflection, T and R, for a wave incident from the first to the second medium, and the coefficients T' and R' for a wave incident from the second medium to the first medium are given. Find the form of the transmission matrix $\mathbb{D}$. Specialize the matrix assuming $R' = -R$ and $1 + R = T$ (and $1 + R' = T'$).

Exercise 7.10 Consider the interfaces defined in Exercise 7.6. Write the transmission matrices for each interface by analyzing the harmonic wave reflections at each interface using the result of Exercise 7.9. Compose these matrices and use the result of the example 7.8 to verify that the total reflection coefficient R for the two interfaces comes out the same as in exercise 7.6.

Exercise 7.11* Do the same as in the previous exercise but for the total transmission coefficient T.

Exercise 7.12* Consider the connection of two strings at $z = L$ with the same tension. The transmission matrix is of the form

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} \left(1+\frac{k_2}{k_1}\right)e^{i(k_1-k_2)L} & \left(1-\frac{k_2}{k_1}\right)e^{i(k_1+k_2)L} \\ \left(1-\frac{k_2}{k_1}\right)e^{-i(k_1+k_2)L} & \left(1+\frac{k_2}{k_1}\right)e^{-i(k_1-k_2)L} \end{pmatrix}.$$

Find the total transmission matrix for two interfaces of three strings. The interfaces are at $z = 0$ and $z = L$. Using the total transmission matrix, find the total reflection coefficient R.

8 Waves in space

The 3D wave equation is an equation of the form

$$\frac{\partial^2\psi}{\partial t^2} = v^2\Delta\psi = v^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right).$$

Δ is the Laplace operator, v is the constant phase velocity.

Exercise 8.1 Show that a harmonic travelling plane wave of the form $\psi(\vec{r},t) = Ae^{i(\omega t-\vec{k}\cdot\vec{r})}$ satisfies the 3D wave equation provided the dispersion relation is satisfied. Find it.

Exercise 8.2 Find the dispersion relation of the following wave equations:

$$\frac{\partial^2\psi}{\partial t^2} = v^2\Delta\psi - \omega_0^2\psi, \qquad \frac{\partial^2\psi}{\partial t^2} = v^2\Delta\psi - \alpha\Delta(\Delta\psi).$$

Instructions: Substitute a harmonic travelling plane wave.

Exercise 8.3 Show that a travelling plane wave of the form $\psi(\vec{r},t) = F(\vec{n}\cdot\vec{r}-vt)$, where $|\vec{n}| = 1$ and $F : \mathbb{R} \to \mathbb{R}$ are arbitrary twice differentiable functions, satisfies the 3D wave equation.

Exercise 8.4* Show that a spherical wave of the form $\psi(\vec{r},t) = \frac{1}{r}e^{i(\omega t-kr)}$ satisfies the 3D wave equation provided the dispersion relation $\omega = vk$ is satisfied.

Exercise 8.5 *Superposition of spatial waves.* Consider two travelling harmonic plane waves with the same wavelength λ and different amplitudes, between whose directions of propagation there is an angle $\Delta\varphi$. Consider a planar screen that is perpendicular to the "average direction" of the propagation of these waves. Find the intensity waveform (i.e., the time-mean of the square of the waves) of the resulting superposition on the screen. Determine the distance Δy of the interference maxima.

Exercise 8.6 Consider the planar interface of two transparent media with refractive indices n_1 and n_2. Consider an incident and a transmitted harmonic travelling plane wave. The wave vectors $\vec{k}_1$ and $\vec{k}_2$ lie in a plane perpendicular to the plane of the interface and make angles ϑ_1 and ϑ_2, respectively, with the normal vector. Based on the condition $\vec{k}_{1\parallel} = \vec{k}_{2\parallel}$ (this condition follows from the condition of continuity of tangential components of the electric field at the interface, $\vec{E}_{1\parallel} = \vec{E}_{2\parallel}$), derive Snell's law of refraction.

Exercise 8.7 Let us have the same problem as in the previous problem. Consider now the interface of the following two environments: a transparent environment with refractive index n and an ionosphere with plasma frequency ω_p. Derive the relevant law of refraction.

Exercise 8.8 Show that an electromagnetic standing wave of the form

$$\vec{E} = (A\cos\omega t\cos kz, 0, 0), \qquad \vec{B} = (0, \tfrac{1}{c}A\sin\omega t\sin kz, 0),$$

where $\omega = ck$, satisfies Maxwell's equations in vacuum. Determine the electric and magnetic energy densities and the Poynting vector.

Exercise 8.9* *Larmore's formula.* Show that by integrating the Poynting vector $\vec{S}$ of the radiation field $\vec{E}_{\mathrm{rad}}$ from the accelerated charge,

$$\vec{E}_{\mathrm{rad}}(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}\frac{\vec{a}_\perp(t_r)}{r},$$

over a sphere of radius r, you get Larmor's formula for the total radiated power P of an emitted electromagnetic wave,

$$P(t,r) = \frac{\mu_0 q^2}{6\pi c} a^2(t_r).$$

Retarded time t_r is $t_r = t - \frac{r}{c}$. The Poynting vector has the form $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E^2\, \vec{n}$, where $\vec{n}$ is the direction of propagation perpendicular to the given sphere.

Exercise 8.10 Consider a waveguide of a rectangular cross-section with dimensions $a = 5\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$. What is the lowest frequency f_0 an electromagnetic wave can have to pass through the waveguide without damping? Calculate the phase and group velocity (as a multiple of c) of a wave whose frequency is $f = \frac{5}{4} f_0$. What highest mode m_0 can be excited for a propagating wave of this frequency? For a wave with frequency $f = \frac{4}{5} f_0$, determine the distance over which the amplitude of the wave decreases e-fold.

9 Polarization

A travelling harmonic electromagnetic plane wave propagating in the z axis direction generally has an electric component in the complexified form (for $z = z_0$):

$$\vec{E}(t) = E_{x0}\,\vec{x}\,e^{i(\omega t+\varphi_1)} + E_{y0}\,\vec{y}\,e^{i(\omega t+\varphi_2)} = \begin{pmatrix} E_{x0}e^{i\varphi_1} \\ E_{y0}e^{i\varphi_2} \end{pmatrix} e^{i\omega t} = \hat{\vec{E}} e^{i\omega t},$$

where $\vec{x}$ and $\vec{y}$ are unit vectors in the direction of the x and y axis, respectively. In general, we speak of elliptically polarized light.

A polarizer (linear polarizer) is defined by its transmittance axis $\vec{n}$ and its action is given by

$$\vec{E}_{\mathrm{out}} = (\vec{E}_{\mathrm{in}} \cdot \vec{n})\,\vec{n}, \qquad \hat{\vec{E}}_{\mathrm{out}} = \mathbb{P}_{\vec{n}} \hat{\vec{E}}_{\mathrm{in}},$$

where $\mathbb{P}_{\vec{n}}$ is the projector on the axis $\vec{n}$,

$$\mathbb{P}_{\vec{n}} = \begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix}.$$

The waveplate is characterized by the phase shift and the axis $\vec{n}_1$ (and the axis perpendicular to it $\vec{n}_2$). If the incoming light is in the form

$$\vec{E}_{\mathrm{in}}(t) = E_1\,\vec{n}_1\,e^{i(\omega t+\varphi_1)} + E_2\,\vec{n}_2\,e^{i(\omega t+\varphi_2)},$$

then the output light is given by

$$\vec{E}_{\mathrm{out}}(t) = E_1\,\vec{n}_1\,e^{i(\omega t+\varphi_1+\Delta\varphi)} + E_2\,\vec{n}_2\,e^{i(\omega t+\varphi_2)}.$$

The operator $\mathbb{D}_{\Delta\varphi}$ defined by $\hat{\vec{E}}_{\mathrm{out}} = \mathbb{D}_{\Delta\varphi} \hat{\vec{E}}_{\mathrm{in}}$ is of the form

$$\mathbb{D}_{\Delta\varphi} = e^{i\Delta\varphi}\,\mathbb{P}_{\vec{n}_1} + \mathbb{P}_{\vec{n}_2}.$$

The light intensity is given by

$$I = \langle \vec{E}^2 \rangle = \frac{1}{2}\big(E_{x0}^2 + E_{y0}^2\big).$$

Stokes parameters are given by the formulae

$$P_1 = \frac{\langle E_x^2 \rangle - \langle E_y^2 \rangle}{\langle E_x^2 \rangle + \langle E_y^2 \rangle}, \qquad P_2 = \frac{\langle 2E_x E_y \rangle}{\langle E_x^2 \rangle + \langle E_y^2 \rangle}, \qquad P_3 = \frac{\langle 2E_x(\omega t - \frac{\pi}{2})E_y \rangle}{\langle E_x^2 \rangle + \langle E_y^2 \rangle}.$$

Exercise 9.1 How does the intensity of circularly polarized light change after passing through a polarizer?

Exercise 9.2 How does the intensity of unpolarized light change after passing through a linear polarizer?

Exercise 9.3 *Rotation of the plane of linearly polarized light by 90°.* Consider linearly polarized light $\vec{E} = E_0 \vec{x} \cos(\omega t)$. Put in its path N polarizers, each with its axis of transmittance rotated by $\frac{\pi}{2N}$ relative to the previous one (and the first one relative to the plane of the incident light). What will be the intensity of the transmitted light for $N = 1$, $N = 2$ and the general $N \in \mathbb{N}$? *What is the limit for $N \to +\infty$?

Exercise 9.4 Consider generally elliptically polarized light. You put a polarizer with a $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$ transmittance axis in its path. Show that for the intensity of the output light it holds

$$I_{\text{out}} = \frac{1}{2}(I_x + I_y) + I_{xy},$$

where

$$I_{\text{out}} = \langle E_{\text{out}}^2 \rangle, \qquad I_x = \langle E_x^2 \rangle, \qquad I_y = \langle E_y^2 \rangle, \qquad I_{xy} = \langle E_x E_y \rangle,$$

E_x and E_y are the electric field components in the directions $\vec{x}$ and $\vec{y}$ for the incoming light.

Exercise 9.5 The refractive indices of crystalline quartz for light of wavelength $\lambda_0 = 500\,\text{nm}$ in vacuum are $n_1 = 1,544$ and $n_2 = 1,553$. Determine the smallest thickness of a quarter-wavelength waveplate made of this material.

Exercise 9.6 Write the matrix $\mathbb{D}_{\Delta\varphi}$ for a waveplate with axes $\vec{n}_1 = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$ and $\vec{n}_2 = \frac{\vec{y}-\vec{x}}{\sqrt{2}}$. Specialize the result for $\Delta\varphi = \frac{\pi}{2}$ and $\Delta\varphi = \pi$.

Exercise 9.7 *Rotation of the plane of linearly polarized light by 90°, the second time.* Consider linearly polarized light $\vec{E} = E_0\vec{x}\cos(\omega t)$. Put a half-wave waveplate in its path with its axis oriented in the $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$ direction. What polarization state will the light have after passing through the plate? How will the intensity change?

Exercise 9.8 A circular polarizer is a (linear) polarizer followed by a quarter-wave plate with axes rotated 45° relative to the transmittance axis of the linear polarizer.

Show that, depending on the choice of the axes of the wave plate, we get a left- or right-handed circular polarizer that converts any light coming from the side of the linear polarizer into corresponding circularly polarized light.

Show that left-handed polarized light coming from the waveplate side is absorbed in a right-handed polarizer.

Exercise 9.9 Linearly polarized light polarized in the direction of the axis $\vec{x}$ with intensity I_0 enters the optical instrument. Determine the electric field and the intensity of light after each of the optical elements in the following apparatus consisting of the optical elements in the following order:

- Polarizer with axis $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$.
- Half-wave waveplate with axis $\vec{n} = \vec{y}$.
- Polarizer with axis $\vec{y}$.
- Quarter-wave waveplate with axis $\vec{n} = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$.

Exercise 9.10* What values of the Stokes parameters P_1, P_2 and P_3 correspond to linearly polarized light and circularly polarized light, respectively? Plot the results.

Exercise 9.11* Light incident on a linear polarizer is a mixture of linearly polarized light and unpolarized light. If you rotate the polarizer by 60° compared to the rotation with the maximum transmitted intensity you get half the intensity. Determine the ratio of the intensities of unpolarized and linearly polarized light in the mixture.

Exercise 9.12* The direction of polarization of linearly polarized light changes rapidly (much faster than the resolving time of your measuring instrument) between the following two states: $\vec{n} = (\cos\theta_0, \pm\sin\theta_0)$ where $\theta_0 < \frac{\pi}{2}$. Calculate the Stokes parameters. Determine $|\vec{P}| = |(P_1, P_2, P_3)|$.

10 Interference

Exercise 10.1* *Fabry-Pérot etalon.* Consider the result of Exercise 7.6, i.e., the total reflection coefficient at the two interfaces,

$$R = \frac{R_{12} + R_{23}e^{-2ik_2L}}{1 + R_{12}R_{23}e^{-2ik_2L}},$$

where R_{12} and R_{23} are the reflection coefficients of each interface, k_2 is the wavenumber in the medium between the interfaces, and L is the distance between the interfaces. Now consider that the interfaces are composed of identical semi-transparent mirrors, i.e., $R_{12} = R_{23} = r$. Find the relationship between the wavelength λ and the distance between the mirrors L at which the total reflectance $\mathcal{R} = |R^2|$ is zero.

Exercise 10.2 *Glass wedge.* The planar surfaces of a glass wedge of refractive index $n = 1,5$ form a very small angle $\varphi = 0,1'$. Light of wavelength $\lambda = 500\,\mathrm{nm}$ is incident perpendicularly on the wedge. Calculate the distance of the interference fringes produced by the reflected light.

Instructions: Find the angle between the incident rays and use the result of Exercise 8.5.

Exercise 10.3* *Air wedge.* The air wedge is bounded by two perfectly planar glass plates with refractive index $n = 1,5$, which are at a very small angle φ. This angle is due to the fact that a strip of foil of thickness $d = 0,02\,\mathrm{mm}$ has been inserted between the glass plates at a distance of $L = 10\,\mathrm{cm}$ from their touching edges. Sodium light with a wavelength of $\lambda = 589\,nm$ is incident perpendicularly on the wedge layer. Determine the distance of the interference fringes in (a) reflected and (b) transmitted light.

Instructions: Find the angle between the incident rays and use the result of Exercise 8.5.

Exercise 10.4 *Soap bubble a.k.a. thin film interference.* You have a planar soap bubble of thickness d with refractive index n. If you observe the reflection of light at an angle ϑ on the soap film, due to constructive interference for a certain wavelength of light λ you see the film colored. Find the condition for constructive interference for the parameters of the thickness of the membrane d, the angle of incidence (and reflection) ϑ, the wavelength of light λ (and the index of refraction n).

11 Diffraction

The position of the interference maxima on the two thin slits, the diffraction grating and the slit of a finite width, is given by

$$\sin\theta_m = m\frac{\lambda}{d}, \qquad y_m = mL\frac{\lambda}{d},$$

where d is the distance of the slits, or the distance of adjacent slits in the diffraction grating, or the slit width. The angle θ_m denotes the angle at which the interference maximum is observed. Distance y_m then represents the distance from the origin on the screen. The value m is called the order of the maximum. The distance between adjacent maxima is then (for small m)

$$\Delta\theta = \frac{\lambda}{d}, \qquad \Delta y = L\frac{\lambda}{d}.$$

For a diffraction grating, the width of the diffraction maxima (the distance between the first intensity zeros around the maximum) is

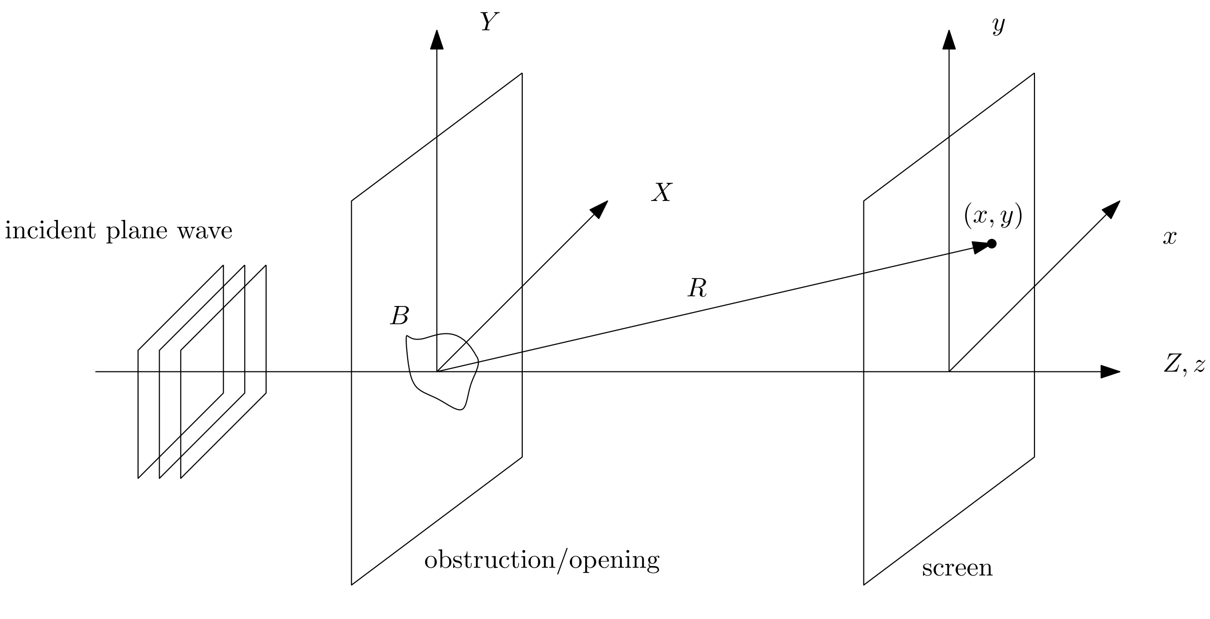
$$\delta\theta = \frac{2\lambda}{Nd}, \qquad \delta y = \frac{2L\lambda}{Nd},$$

where N is the number of notches/slits of the diffraction grating.

Fraunhofer diffraction integral

$$E(x,y) = \frac{E_0}{R}e^{i(\omega t - kR)}\int_B e^{i\frac{k}{R}(xX+yY)}\,dX\,dY.$$

The plane of the obstruction has Cartesian coordinates X, Y. The plane of the screen has coordinates x, y. B represents the obstacle/opening in the obstruction (from Babinet's principle, these situations are equivalent), R is the distance from the origin in the obstruction/opening to the point (x, y) on the screen, $R = \sqrt{L^2 + x^2 + y^2}$, where L is the perpendicular distance of the screen and the obstruction/opening.



Exercise 11.1 The maximum of what largest order can you observe in green light of wavelength $\lambda = 550\,\mathrm{nm}$ for a diffraction grating with 5000 indentations per 1 cm?

Exercise 11.2 Can the 1st- and 2nd-order spectra and the 2nd- and 3rd-order spectra produced on a diffraction grating overlap if it is illuminated with white light composed of wavelengths of $400 - 700\,\mathrm{nm}$?

Exercise 11.3* A diffraction grating has 500 indentations per 1 mm. Calculate its so-called dispersion, i.e., the magnitude $\frac{d\theta}{d\lambda}$, in the vicinity of green light ($\lambda = 500\,\mathrm{nm}$) for the first and second orders.

Exercise 11.4 The yellow light emitted by sodium atoms is dominated by the so-called sodium doublet, whose wavelengths are $\lambda_1 = 589,0\,\mathrm{nm}$ and $\lambda_2 = 589,6\,\mathrm{nm}$. How many notches must a diffraction grating have to distinguish these two wavelengths in a first-order spectrum?

Exercise 11.5 You place a hair of diameter d in the path of a laser beam of wavelength $\lambda = 632,8\,\mathrm{nm}$. On a screen at a distance of $L = 6\,\mathrm{m}$, you observe diffraction maxima at a distance of $\Delta l = 3\,\mathrm{cm}$. What is the diameter of the hair?

Exercise 11.6 *Diffraction pattern of a rectangular slit.* Find the diffraction pattern (i.e., find the intensity distribution on the screen) of a rectangular slit of dimensions a, b.

Exercise 11.7 *Diffraction pattern of two slits.* Find the diffraction pattern of two slits of width D whose centers are at distance d, for simplicity only for $y = 0$.

Instructions: Use the result of the previous example for $y = 0$. Show how the diffraction integral changes if you move the slit by $\pm\frac{d}{2}$ along the X axis. Add the fields from the two slits so shifted.

Exercise 11.8* *Diffraction pattern of a circular hole.* Construct the diffraction integral for a circular hole of diameter D. Write the result using the Bessel function $J_n(x)$, whose integral definition is

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin u - nu)\, du = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin u - nu)} du.$$

Instructions: Establish polar coordinates in the plane of both the screen and the obstruction. Notice that the result cannot depend on the value of the polar angle in the plane of the screen and set it equal to a suitable constant. Integrate first according to the angle variable. Use the recurrent relation

$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x)$$

for $m = 1$.

12 Exercise results

12.1 Complex numbers

12.2 Mean values

12.3 Small oscillations

Exercise 3.9 **Equations of motion**

a)

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \qquad \mathbb{U} = \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}$$

b)

Exercise 3.10 **Method of modes**

a) $\omega_1 = \sqrt{\frac{2k}{m}}$, $\omega_2 = \sqrt{\frac{7k}{m}}$, $\vec{a}_1 = (2, 1)$, $\vec{a}_2 = (-1, 2)$

b) $\omega_1 = \sqrt{\frac{7k}{6m}}$, $\omega_2 = \sqrt{\frac{3k}{m}}$, $\vec{a}_1 = (3, 4)$, $\vec{a}_2 = (-2, 1)$

c) $\omega_1 = \sqrt{\frac{(3-\sqrt{2})k}{m}}$, $\omega_2 = \sqrt{\frac{(3+\sqrt{2})k}{m}}$, $\vec{a}_1 = (1 + \sqrt{2}, 1)$, $\vec{a}_2 = (1 - \sqrt{2}, 1)$

d) $\omega_1 = \sqrt{\frac{(3-\sqrt{2})k}{m}}$, $\omega_2 = \sqrt{\frac{(3+\sqrt{2})k}{m}}$, $\vec{a}_1 = (\sqrt{2}, 1)$, $\vec{a}_2 = (-\sqrt{2}, 1)$

e) $\omega_1 = \sqrt{\frac{k}{m}}$, $\omega_2 = \sqrt{\frac{4k}{m}}$, $\omega_3 = \sqrt{\frac{7k}{m}}$, $\vec{a}_1 = (2, 2, 1)$, $\vec{a}_2 = (-2, 1, 2)$, $\vec{a}_3 = (1, -2, 2)$

f) $\omega_1 = \sqrt{\frac{(2-\sqrt{2})k}{m}}$, $\omega_2 = \sqrt{\frac{2k}{m}}$, $\omega_3 = \sqrt{\frac{(2+\sqrt{2})k}{m}}$, $\vec{a}_1 = (1, \sqrt{2}, 1)$, $\vec{a}_2 = (-1, 0, 1)$, $\vec{a}_3 = (1, -\sqrt{2}, 1)$

g) $\omega_1 = \sqrt{\frac{k}{2m}}$, $\omega_2 = \sqrt{\frac{2k}{m}}$, $\omega_3 = \sqrt{\frac{5k}{2m}}$, $\vec{a}_1 = (2, 3, 1)$, $\vec{a}_2 = (-1, 0, 1)$, $\vec{a}_3 = (2, -1, 1)$

h) $\omega_1 = \sqrt{\frac{(3-\sqrt{5})k}{m}}$, $\omega_2 = \sqrt{\frac{2k}{m}}$, $\omega_3 = \sqrt{\frac{(3+\sqrt{5})k}{m}}$, $\vec{a}_1 = (1, \sqrt{5} - 1, 1)$, $\vec{a}_2 = (-3, 0, 1)$, $\vec{a}_3 = (1, -\sqrt{5} - 1, 1)$

Exercise 3.11 **Small oscillations**

a)

$$\mathbb{U} = \begin{pmatrix} 3k & -2k \\ -2k & 6k \end{pmatrix}$$

b)

$$\mathbb{U} = \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}$$

c)

$$\mathbb{U} = \left(2k\left(1 - \frac{a_0}{a}\right)\right)$$

d)

$$\mathbb{U} = \begin{pmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{pmatrix}$$

References

[1] J. Tolar, J. Koníček, *Sbírka řešených příkladů z fyziky, Vlnění*, Vydavatelství ČVUT, Praha, 2005