Waves exercises

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1 Mean values

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function of one variable, $f = f(x)$. Its mean value in the interval $\langle x_1, x_2 \rangle$ is defined as the integral

$$\langle f \rangle_{\langle x_1, x_2 \rangle} := \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x) dx. \tag{1}$$

The mean value over the entire $\mathbb{R}$ is defined by the limit transition

$$\langle f \rangle \equiv \langle f \rangle_{\langle -\infty, \infty \rangle} = \lim_{x' \to \infty} \langle f \rangle_{\langle -x', x' \rangle} \equiv \lim_{x' \to \infty} \frac{1}{2x'} \int_{-x'}^{x'} f(x) dx. \tag{2}$$

If f is periodic with period L, we can calculate its mean value $\langle f \rangle$ as an integral over any interval of length L, so for any $x' \in \mathbb{R}$ we have

$$\langle f \rangle = \langle f \rangle_{\langle x', x'+L \rangle} = \frac{1}{L} \int_{x'}^{x'+L} f(x) dx. \tag{3}$$

x' is typically chosen to make the calculation as simple as possible.

Exercise 1.1. Calculate $\langle \cos(\omega t) \rangle$, $\langle \sin(\omega t) \rangle$, $\langle \cos^2(\omega t) \rangle$, and $\langle \sin^2(\omega t) \rangle$.

Solution: Consider first the function $f(t) = \cos(\omega t)$. It is a periodic function with period $L = \frac{2\pi}{\omega}$. Its mean value is thus calculated as

$$\langle \cos(\omega t) \rangle = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \cos(\omega t) dt = \frac{1}{2\pi} [\sin(\omega t)]_0^{\frac{2\pi}{\omega}} = 0. \tag{4}$$

The case $\langle \sin(\omega t) \rangle$ is very similar:

$$\langle \sin(\omega t) \rangle = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \sin(\omega t) dt = \frac{1}{2\pi} [-\cos(\omega t)]_0^{\frac{2\pi}{\omega}} = 0. \tag{5}$$

For the next case, using the trick for calculating the square of cosine, we get

$$\begin{aligned} \langle \cos^2(\omega t) \rangle &= \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t) dt = \frac{\omega}{4\pi} \int_0^{\frac{2\pi}{\omega}} (1 + \cos(2\omega t)) dt \\ &= \frac{\omega}{4\pi} [t - \frac{1}{2\omega} \sin(2\omega t)]_0^{\frac{2\pi}{\omega}} = \frac{1}{2}. \end{aligned} \tag{6}$$

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This is not so surprising, since we are actually calculating the mean value of the constant function $\frac{1}{2}$ plus the mean value of the function $\cos(2\omega t)$ over two periods, which is zero from the previous part.

In other words, generally $\langle f+g\rangle = \langle f\rangle + \langle g\rangle$ (if all mean values are finite). Therefore, it suffices to take the equality $\cos^2(\omega t) + \sin^2(\omega t) = 1$, from which it immediately follows

$$\langle\cos^2(\omega t)\rangle + \langle\sin^2(\omega t)\rangle = \langle 1\rangle = 1. \tag{7}$$

From the previous result, we easily obtain $\langle\sin^2(\omega t)\rangle = \frac{1}{2}$.

2 Complex Numbers

A complex number $z \in \mathbb{C}$ is understood to be an object in the form $z = a + ib$, where $a, b \in \mathbb{R}$. The symbol i is called the **complex unit**. We define $i^2 = -1$. If $z' = c + id$, we define addition and multiplication intuitively and in agreement with the usual rules:

$$z + z' = (a+ib) + (c+id) := (a+c) + i(b+d), \quad zz' = (a+ib)(c+id) := (ac-bd) + i(ad+bc).$$

If $z = a + ib$, we write $a = \operatorname{Re} z$ (**real part**) and $b = \operatorname{Im} z$ (**imaginary part**). If $\operatorname{Re} z = 0$, we say that z is **purely imaginary**. We write $0 \equiv 0 + i0$.

The complex conjugate $\bar{z}$ of a number z is defined as $\bar{z} = a - ib$. It holds that $\overline{zz'} = \bar{z}\bar{z}'$. **The magnitude** $|z|$ **of a complex number** is defined as $|z| = \sqrt{a^2+b^2}$. It holds that $|z| = \sqrt{z\bar{z}}$. How do we define division of complex numbers? Let $z, z' \in \mathbb{C}$ and $z' \neq 0$. We define it using the formal "fraction expansion".

$$\frac{z}{z'} = \frac{z}{z'}\frac{\bar{z}'}{\bar{z}'} := \frac{z\bar{z}'}{|z'|^2}. \tag{8}$$

The operations on the right-hand side make sense because for $z' \neq 0$, $|z'| > 0$ and we simply multiply the complex number $z\bar{z}'$ by the real number $|z'|^{-2}$. The equations hold

$$\operatorname{Re}(z) = \frac{z+\bar{z}}{2}, \quad \operatorname{Im}(z) = \frac{z-\bar{z}}{2i}. \tag{9}$$

Now we would like to define the **complex exponential**, i.e., the complex number e^z for every $z \in \mathbb{C}$. If $z = a + ib$, we define

$$e^z = e^{a+ib} := e^a(\cos b + i\sin b) \tag{10}$$

The complex exponential can (see mathematical analysis) be defined by a power series. For $a = 0$, this relationship is called the **Euler's formula**. It holds that $|e^z| = e^a$.

Every $z \in \mathbb{C}$ can be written in the **polar form** $z = |z|e^{i\varphi}$, where $\varphi \in \mathbb{R}$ is the solution of the equations

$$\cos(\varphi) = \frac{\operatorname{Re} z}{|z|}, \quad \sin(\varphi) = \frac{\operatorname{Im} z}{|z|}. \tag{11}$$

φ is called the **argument of the complex number** and is uniquely determined up to the addition of an integer multiple of 2π. The geometric meaning of this notation can easily be obtained by representing complex numbers in the **Gaussian plane**, where the real and imaginary parts are plotted on the Cartesian axes: Geometrically, the addition of complex numbers corresponds to the addition of vectors in the plane. Multiplication can also be easily interpreted. If $z = |z|e^{i\varphi}$

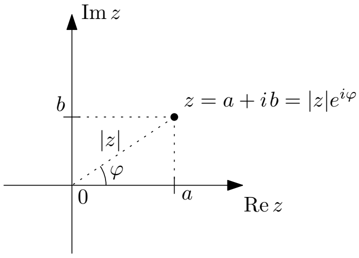


Figure 1: Gaussian plane

and $z' = |z'|e^{i\varphi'}$, we get $zz' = |z||z'|e^{i(\varphi+\varphi')}$, thus a number whose magnitude is the product of magnitudes and the argument is the sum of angles.

Multiplying by $e^{i\varphi}$, for example, corresponds to rotating the complex number in the Gaussian plane

by an angle φ.

It is often useful to write trigonometric functions using complex exponentials. From Euler's formula, we obtain the relationships

$$\cos(\varphi) = \mathrm{Re}(e^{i\varphi}) = \frac{\cos(\varphi) + i\sin(\varphi)}{2}, \quad \sin(\varphi) = \mathrm{Im}(e^{i\varphi}) = \frac{\cos(\varphi) - i\sin(\varphi)}{2i} \tag{12}$$

Using these relationships, we can define $\cos(z)$ and $\sin(z)$ for any $z \in \mathbb{C}$.

Exercise 2.1. Find the real and imaginary part of the number

$$w = \frac{a + ib}{c + id} \tag{13}$$

Solution: Dividing by a complex number is carried out by formally expanding the fraction and subsequent adjustment:

$$w = \frac{a + ib}{c + id}\frac{c - id}{c - id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}. \tag{14}$$

From here it is easy to see that

$$\mathrm{Re}(w) = \frac{ac + bd}{c^2 + d^2}, \quad \mathrm{Im}(w) = \frac{bc - ad}{c^2 + d^2}. \tag{15}$$

Exercise 2.2. Show that $\overline{e^z} = e^{\bar{z}}$, specifically $\overline{e^{ib}} = e^{-ib}$.

Solution: For a complex number $z = a + ib$, we have $e^z = e^{a+ib} = e^a(\cos b + i\sin b)$ and thus

$$\overline{e^z} = e^a(\cos b - i\sin b) = e^a e^{-ib} = e^{a-ib} = e^{\bar{z}}. \tag{16}$$

Choosing $a = 0$ we obtain the given special case.

Exercise 2.3. Calculate $\mathrm{Re}[(C - iD)e^{i\Omega t}]$, where $C, D, \Omega t \in \mathbb{R}$.

Solution: Using Euler's formula

$$\begin{aligned}(C - iD)e^{i\Omega t} &= (C - iD)(\cos(\Omega t) + i\sin(\Omega t))\\ &= (C\cos(\Omega t) + D\sin(\Omega t)) + i(C\sin(\Omega t) - D\cos(\Omega t)).\end{aligned} \tag{17}$$

Hence, $\mathrm{Re}[(C - iD)e^{i\Omega t}] = C\cos(\Omega t) + D\sin(\Omega t)$.

***Exercise 2.4.** Prove the validity of the relations $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ and $\mathrm{Im}(iz) = \mathrm{Re}(z)$ for each $z \in \mathbb{C}$.

Using them, prove the validity of the identity $\cos(x) = \sin(x + \frac{\pi}{2})$ for all $x \in \mathbb{R}$.

Solution: Considering $z = a + ib$, then $\mathrm{Re}\, z = a$ and $\mathrm{Im}\, z = b$. For $iz = -b + ia$ it follows that $\mathrm{Re}(iz) = -b$ and $\mathrm{Im}(iz) = a$. Clearly, then

$$\mathrm{Re}\, z = a = \mathrm{Im}(iz), \qquad \mathrm{Im}\, z = b = -\mathrm{Re}(iz). \tag{18}$$

The trigonometric identity is then obtained as

$$\cos x = \mathrm{Re}(e^{ix}) = \mathrm{Im}(ie^{ix}) = \mathrm{Im}(e^{i\frac{\pi}{2}}e^{ix}) = \mathrm{Im}(e^{i(x+\frac{\pi}{2})}) = \sin\left(x + \frac{\pi}{2}\right), \tag{19}$$

where we wrote $i = e^{i\frac{\pi}{2}}$.

Exercise 2.5. Derive the formulas for sines and cosines of sum and difference of angles using the trivial identity

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}. \tag{20}$$

Solution: Rewrite the left side using Euler's formula and expand:

$$\begin{aligned}e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned} \tag{21}$$

By comparing the real and imaginary part with the right side $\cos(\alpha+\beta) + i\sin(\alpha+\beta)$ we obtain the desired formulas. The formula for the difference is easily obtained by substituting $-\beta$ for β.

***Exercise 2.6.** Derive the formulas for products of sines and cosines by modifying the expression

$$e^{i\alpha} + e^{i\beta} = e^{i\frac{\alpha}{2}}e^{i\frac{\beta}{2}}(e^{i\frac{\alpha-\beta}{2}} + e^{i\frac{\beta-\alpha}{2}}). \tag{22}$$

Solution: This time we start by modifying the right side. We easily notice that it can be written as

$$2e^{i\frac{\alpha+\beta}{2}}\cos\frac{\alpha-\beta}{2} = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + i2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \tag{23}$$

Comparing with the left side $(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)$ we obtain the result.

***Exercise 2.7.** Prove the validity of the relations

$$\sin(ix) = i\sinh(x), \;\; \cos(ix) = \cosh(x), \;\; \sinh(ix) = i\sin(x), \;\; \cosh(ix) = \cos(x). \tag{24}$$

Solution: All relations are proved in the same way, we will prove just two of them. We have (from the definition of complex sines and cosines)

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i}\frac{i}{i} = i\frac{e^{x} - e^{-x}}{2} = i\sinh x. \tag{25}$$

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x. \tag{26}$$

Here we used the definition of the complex hyperbolic cosine $\cosh(z) = \frac{e^z+e^{-z}}{2}$. Notice that generally $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i\sin(z)$.

Exercise 2.8. Consider the expression $c_1 e^{i\omega t} + c_2 e^{-i\omega t}$, where $c_1, c_2 \in \mathbb{C}$ and $\omega t \in \mathbb{R}$. What are the conditions on the constants c_1 and c_2 for the expression to be real for all $t \in \mathbb{R}$.

Solution: It must hold that $\mathrm{Im}(c_1 e^{i\omega t} + c_2 e^{-i\omega t}) = 0$. This occurs if the expression is equal to its complex conjugate. Thus, we get

$$c_1 e^{i\omega t} + c_2 e^{-i\omega t} = \bar{c}_1 e^{-i\omega t} + \bar{c}_2 e^{i\omega t}. \tag{27}$$

This can be rearranged into the equation

$$(c_1 - \bar{c}_2)e^{i\omega t} = (c_2 - \bar{c}_1)e^{-i\omega t}. \tag{28}$$

By choosing $t = 0$ we get $c_1 - \bar{c}_2 = c_2 - \bar{c}_1$ and by choosing $t = \frac{\pi}{2\omega}$ we get the equation $c_1 - \bar{c}_2 = -(c_2 - \bar{c}_1)$. This immediately implies that both sides of the equation must be zero and thus necessarily $c_2 = \bar{c}_1$. It is easy to see that this is also a sufficient condition, because then

$$c_1 e^{i\omega t} + \bar{c}_1 e^{-i\omega t} = 2\,\mathrm{Re}(c_1 e^{i\omega t}) \in \mathbb{R}. \tag{29}$$

Exercise 2.9. The solution to the harmonic oscillator equation can be written in equivalent forms as

$$x(t) = A\cos(\omega t + \varphi) = A\sin(\omega t + \phi) = a\cos(\omega t) + b\sin(\omega t) = ce^{i\omega t} + \bar{c}e^{-i\omega t}. \tag{30}$$

Find the relationship between the constants A, ω, φ, ϕ, a, b, and c.

Solution: The relationship between φ and ϕ is obtained easily from the already proven identity $\cos(x) = \sin(x + \frac{\pi}{2})$. Hence $\phi = \varphi + \frac{\pi}{2}$. Using sum-to-product formulas, we get

$$A\cos(\omega t + \varphi) = A(\cos(\omega t)\cos(\varphi) - \sin(\omega t)\sin(\varphi)) = A\cos(\varphi)\cos(\omega t) - A\sin(\varphi)\sin(\omega t). \tag{31}$$

Thus, we have $a = A\cos(\varphi)$ and $b = -A\sin(\varphi)$. Notice that for each $a, b \in \mathbb{R}$, we can find A and φ satisfying this relationship. Finally, using the representation of sines and cosines through complex exponentials:

$$\begin{aligned} a\cos(\omega t) + b\sin(\omega t) &= a\frac{e^{i\omega t} + e^{-i\omega t}}{2} + b\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \\ &= \frac{1}{2}(a - ib)e^{i\omega t} + \frac{1}{2}(a + ib)e^{-i\omega t}. \end{aligned} \tag{32}$$

We see that $c = \frac{1}{2}(a - ib)$. For each complex number $c \in \mathbb{C}$, we can find $a, b \in \mathbb{R}$ satisfying this relationship. And we are done.

***Exercise 2.10.** "Prove" Euler's formula using the differential identity

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x} \tag{33}$$

Solution: Consider the complex function of real variable $f(x) = \cos(x) + i\sin(x)$. Its derivative by components $f'(x) = -\sin(x) + i\cos(x) = if(x)$. It holds $f(0) = 1$. But the same ordinary first-order differential equation with the same initial conditions is solved by the function e^{ix}. From the uniqueness

$$e^{ix} = f(x) = \cos(x) + i\sin(x) \tag{34}$$

for all $x \in \mathbb{R}$, and we have proved.

***Exercise 2.11.** Write the functions $\cos^2(x)$, $\cos^3(x)$, and generally $\cos^n(x)$, $n \in \mathbb{N}$, using only the functions $\cos(kx)$, $k \in \mathbb{N}_0$.

Solution: Using Euler's formula, we get

$$\cos^2(x) = \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^2 = \frac{1}{4}(e^{2ix} + 2e^{ix}e^{-ix} + e^{-2ix}) = \frac{1}{4}(2 + 2\cos(2x)) = \frac{1+\cos(2x)}{2}. \tag{35}$$

For the third power, it is very similar, we get

$$\begin{aligned}\cos^3(x) &= \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^3 = \frac{1}{8}(e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}) \\ &= \frac{1}{8}(2\cos(3x) + 6\cos(x)) = \frac{3\cos(x)+\cos(3x)}{4}\end{aligned} \tag{36}$$

For a general $n \in \mathbb{N}$, using the binomial theorem, we get

$$\cos^n(x) = \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(e^{ix})^k(e^{-ix})^{n-k} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(2k-n)x} \tag{37}$$

Now it is advantageous to distinguish between odd and even n. For odd n, the sum has an even number of terms, and we can split it into two sums:

$$\frac{1}{2^n}\Big(\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{k}e^{i(2k-n)x} + \sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n}\binom{n}{k}e^{i(2k-n)x}\Big) \tag{38}$$

In the second sum, we perform a substitution of the summation index to $q = n-k$ and use the symmetry of binomial coefficients $\binom{n}{k} = \binom{n}{n-k}$. The second sum can thus be rewritten as

$$\sum_{q=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{q}e^{-i(2q-n)x} \tag{39}$$

We see that it differs from the first sum only by the sign in the exponent. Using Euler's formulas, we thus obtain the formula

$$\cos^n(x) = \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{q}\cos((n-2k)x) \tag{40}$$

For even n, the situation is similar except that the sum splits into three terms:

$$\frac{1}{2^n}\Big(\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}e^{i(2k-n)x} + \binom{n}{n/2} + \sum_{k=\frac{n}{2}+1}^{n}\binom{n}{k}e^{i(2k-n)x}\Big) \tag{41}$$

By substituting the summation index in the third term and using Euler's formula, we finally get the formula

$$\cos^n(x) = \frac{1}{2^{n-1}}\Big(\frac{1}{2}\binom{n}{n/2} + \sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos((n-2k)x)\Big). \tag{42}$$

Notice that the last term is half compared to substituting $k = n/2$ into the sum on the right.

***Exercise 2.12.** Sum the series

$$\sum_{m=0}^{N} \cos mx \tag{43}$$

Solution: Using the linearity of the function Re, we can write

$$\sum_{m=0}^{N} \cos mx = \mathrm{Re}\left[\sum_{m=0}^{n} e^{imx}\right] = \mathrm{Re}\left[\sum_{m=0}^{N} (e^{ix})^m\right] \tag{44}$$

Now, simply use the well-known formula for the sum of a geometric series $\sum_{m=0}^{N} a^m = \frac{a^{N+1}-1}{a-1}$ for $a = e^{ix}$. This expression can further be modified as

$$\frac{a^{N+1}-1}{a-1} = a^{\frac{N}{2}}\frac{a^{\frac{N+1}{2}} - a^{-\frac{N+1}{2}}}{a^{1/2} - a^{-1/2}} = e^{i\frac{N}{2}x}\frac{e^{i\frac{N+1}{2}x} - e^{-i\frac{N+1}{2}x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} = e^{i\frac{N}{2}x}\frac{\sin(\frac{N+1}{2}x)}{\sin\frac{x}{2}}. \tag{45}$$

After substituting into the formula above, we thus obtain

$$\sum_{m=0}^{N} \cos mx = \frac{\sin(\frac{N+1}{2}x)}{\sin\frac{x}{2}}\,\mathrm{Re}\left[e^{i\frac{N}{2}x}\right] = \frac{\sin(\frac{N+1}{2}x)}{\sin\frac{x}{2}}\cos\frac{N}{2}x. \tag{46}$$

Exercise 2.13. Calculate the definite integrals

$$\int_0^{+\infty} e^{-ax}\cos bx\,dx, \qquad \int_0^{+\infty} e^{-ax}\sin bx\,dx. \tag{47}$$

*Calculate also the corresponding indefinite integrals (primitive functions).

Solution: The second integral is multiplied by the complex unit and added to the first one. From the linearity of the integral, we then get one integral of the complex exponential:

$$\int e^{-ax}\cos bx\,dx + i\int e^{-ax}\sin bx\,dx = \int e^{-ax}(\cos bx + i\sin bx)dx = \int e^{-(a-ib)x}dx. \tag{48}$$

This can be easily calculated using the standard formula.

$$\int e^{-(a-ib)x}dx = -\frac{1}{a-ib}e^{-(a-ib)x} + C, \tag{49}$$

where $C \in \mathbb{C}$ is some complex constant. For the definite integral, we get

$$\int_0^{+\infty} e^{-(a-ib)x}dx = \left[-\frac{1}{a-ib}e^{-(a-ib)x}\right]_0^{+\infty} = -\frac{1}{a-ib} = -\frac{a+ib}{a^2+b^2}. \tag{50}$$

By comparing the real and imaginary parts, we obtain the sought integrals:

$$\int_0^{+\infty} e^{-ax}\cos bx\,dx = -\frac{a}{a^2+b^2}, \qquad \int_0^{+\infty} e^{-ax}\sin bx\,dx = -\frac{b}{a^2+b^2}. \tag{51}$$

For indefinite integrals, we proceed with the modifications

$$\int e^{-(a-ib)x}dx = -\frac{1}{a-ib}e^{-(a-ib)x} + C = -\frac{a+ib}{a^2+b^2}e^{-ax}(\cos(bx)+i\sin(bx)) + C. \tag{52}$$

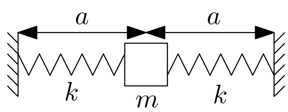
Writing $C = C_1 + iC_2$, by comparing the real and imaginary parts, we get the sought integrals:

$$\int e^{-ax}\cos bx\,dx = -\frac{e^{-ax}}{a^2+b^2}(a\cos(bx) - b\sin(bx)) + C_1, \tag{53}$$

$$\int e^{-ax}\sin bx\,dx = -\frac{e^{-ax}}{a^2+b^2}(a\sin(bx) + b\cos(bx)) + C_2. \tag{54}$$

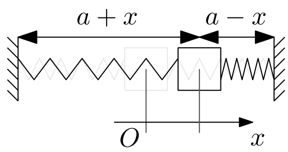
3 Small Oscillations and the Mode Method

Exercise 3.1. Construct the potential for longitudinal and transverse oscillations of weights on springs as in the figure. The length of the unstretched springs is a_0.



Find the forms of these potentials in the approximation of small oscillations.

Solution: First, consider the longitudinal oscillations of the weight. I introduce a coordinate x describing the displacement of the weight from the equilibrium position to the right:



The potential energy of a spring always has the form $\frac{1}{2}$stiffness(length−rest length)^2. The potential of longitudinal oscillations is the sum of the potential energies (as functions of displacement x) of both springs: $U(x) = U_1(x) = U_2(x)$. Here $U_1(x) = \frac{1}{2}k(a+x-a_0)^2$ and $U_2 = \frac{1}{2}k(a-x-a_0)^2$ thus

$$U(x) = \frac{1}{2}k(a+x-a_0)^2 + \frac{1}{2}k(a-x-a_0)^2. \tag{55}$$

Let us recall what is meant by "the approximation of small oscillations". Generally, for a system with n degrees of freedom, we introduce coordinates $(x_1,\ldots,x_n)$ describing the displacement from the *equilibrium position.* Writing $\vec{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, we find the potential function $U = U(\vec{x})$. The potential in the approximation of small oscillations is

$$U_{s.o.}(\vec{x}) = \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{U}_{ij}x_ix_j, \;\; \mathbb{U}_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j}\Big|_{\vec{x}=0}. \tag{56}$$

Here we have $n = 1$ and $x_1 \equiv x$. The matrix $\mathbb{U}$ is of size 1×1 and its only element is given by the second derivative of the function $U = U(x)$ at the point $x = 0$. We have

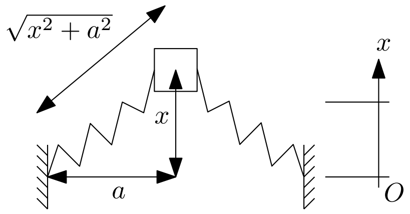
$$U'(x) = k(a + x - a_0) - k(a - x + a_0), \;\; U''(x) = 2k, \;\; U''(0) = 2k. \tag{57}$$

It is always advantageous to verify that the first partial derivatives of U at $\vec{x} = 0$ are zero and therefore, that the point $\vec{x} = 0$ is indeed an equilibrium position! Here $U'(0) = k(a - a_0) - k(a - a_0) = 0$. The matrix $\mathbb{U}$ thus has the form

$$\mathbb{U} = \left(U''(0)\right) = \left(2k\right). \tag{58}$$

Substituting into the relation for the potential of small oscillations, we get $U_{s.o.}(\vec{x}) = kx^2$.

Now consider the transverse oscillations:

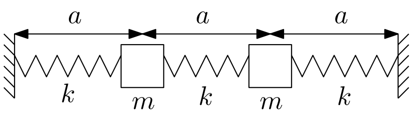


The lengths of both springs for a given displacement are identical, thus $U_1(x) = U_2(x) = \frac{1}{2}k(\sqrt{x^2 + a^2} - a_0)^2$. Thus, we have $U(x) = k(\sqrt{x^2 + a^2} - a_0)^2 = k(x^2 + a^2 + a_0^2 - 2a_0\sqrt{x^2 + a^2})$. Therefore

$$U'(x) = k\left(2x - 2a_0\frac{x}{\sqrt{x^2 + a^2}}\right), \;\; U''(x) = 2k\left[1 - a_0\left(\frac{1}{\sqrt{x^2 + a^2}} + x\left(\frac{1}{\sqrt{x^2 + a^2}}\right)'\right)\right] \tag{59}$$

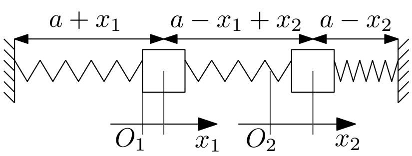
After substitution, $U''(0) = 2k(1 - \frac{a_0}{a})$ and $U_{s.o.}(x) = k(1 - \frac{a_0}{a})x^2$. Note that for $a = a_0$, $U''(0) = 0$, and thus $U_{s.o.}(x) \equiv 0$. Despite this, $U = U(x)$ at $x = 0$ has a sharp local minimum! The approximation of small oscillations has its limits.

Exercise 3.2. Construct the equations of motion for longitudinal oscillations of the system in the figure. The length of the unstretched springs is a_0.



Find their solution by the mode method.

Solution: We introduce displacement coordinates (x_1, x_2) as in the figure:



The potential is thus given by the equation

$$U(\vec{x}) = \frac{1}{2}k(a + x_1 - a_0)^2 + \frac{1}{2}k(a - x_1 + x_2 - a_0)^2 + \frac{1}{2}k(a - x_2 - a_0)^2. \tag{60}$$

Partial derivatives give:

$$\frac{\partial U}{\partial x_1} = k(a + x_1 - a_0) - k(a - x_1 + x_2 - a_0) = 2kx_1 - kx_2, \tag{61}$$

$$\frac{\partial U}{\partial x_2} = k(a - x_1 + x_2 - a_0) - k(a - x_2 - a_0) = -kx_1 + 2kx_2. \tag{62}$$

From here, we easily construct the matrix of second derivatives at the point $\vec{x} = 0$:

$$\mathbb{U} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}. \tag{63}$$

Both weights have mass m and thus we easily see that the matrix of kinetic energy $\mathbb{T}$ has the form

$$\mathbb{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. \tag{64}$$

The equations of motion (small oscillations problem) for the displacement vector $\vec{x} = \vec{x}(t)$ are given by the matrix equation $\mathbb{T}\ddot{\vec{x}} + \mathbb{U}\vec{x} = 0$. Substitution gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m\ddot{x}_1 + 2kx_1 - kx_2 \\ m\ddot{x}_2 - kx_1 + 2kx_2 \end{pmatrix}. \tag{65}$$

The mode method works as follows. It asserts that the general solution to the equations of motion is a superposition of harmonic motions (modes) in the

form $\vec{x}(t) = A\vec{a}\cos(\omega t + \varphi)$, where ω (the natural frequency of the mode) is one of the solutions to the *secular equation*

$$\det(\mathbb{U} - \omega^2\mathbb{T}) = 0 \tag{66}$$

And the vector $\vec{a}$ corresponding to the natural frequency ω is obtained by solving the system of linear equations $(\mathbb{U} - \omega^2\mathbb{T})\vec{a} = 0$. In our case, we get the secular equation in the form

$$0 = \det \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} = (2k - m\omega^2)^2 - k^2 = m^2\omega^4 - 4km\omega^2 + 3k^2. \tag{67}$$

This is a quadratic equation for ω^2, which has two solutions:

$$\omega^2 = \frac{k(2 \pm 1)}{m} \tag{68}$$

The natural frequencies of the modes are thus

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}. \tag{69}$$

Vectors of amplitude ratios $\vec{a} = (a_1, a_2)^T$ are obtained by solving linear equations $(\mathbb{U} - \omega^2\mathbb{T})\vec{a} = 0$. For the first mode:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\mathbb{U} - \omega_1^2\mathbb{T})\vec{a} = \begin{pmatrix} 2k - m\frac{k}{m} & -k \\ -k & 2k - m\frac{k}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{70}$$

We are thus looking for the kernel vector and by equivalent modifications, we get

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}. \tag{71}$$

The solution to this system is any nonzero vector satisfying $a_1 = a_2$, it is advantageous to choose the simplest, for example, $\vec{a} = (1,1)^T$. The first mode thus corresponds to the weights oscillating in phase!

For the second mode, we similarly get the system

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\mathbb{U} - \omega_2^2 \mathbb{T})\vec{a} = \begin{pmatrix} 2k - m\frac{3k}{m} & -k \\ -k & 2k - m\frac{3k}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{72}$$

Again, by equivalent modifications

$$\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \sim \begin{pmatrix} k & k \\ k & k \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{73}$$

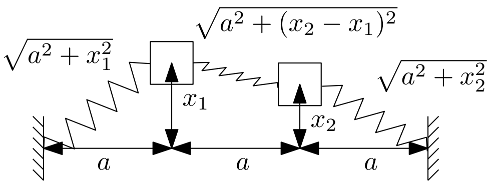
Its solution is any vector satisfying $a_1 = -a_2$, so we choose $\vec{a} = (1,-1)^T$. The second mode is thus counter-phase oscillation of the two weights. In conclusion, the most general solution to the problem is given by the superposition of modes, i.e., the sum

$$\vec{x}(t) = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\frac{k}{m}}t + \varphi_1\right) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\frac{3k}{m}}t + \varphi_2\right), \tag{74}$$

where the constants $A_1, A_2, \varphi_1, \varphi_2 \in \mathbb{R}$ must be obtained from the initial conditions.

Exercise 3.3. Consider the same case as above, only for transverse oscillations.

Solution: We introduce coordinates $\vec{x} = (x_1, x_2)$ as in the figure:



We find the potential from the known lengths of the springs:

$$U(\vec{x}) = \frac{1}{2}k\left(\sqrt{a^2 + x_1^2} - a_0\right)^2 + \frac{1}{2}k\left(\sqrt{a^2 + (x_2 - x_1)^2} - a_0\right)^2 + \frac{1}{2}k\left(\sqrt{a^2 + x_2^2} - a_0\right)^2. \tag{75}$$

Before calculating the partial derivatives, it is advantageous to regroup the terms slightly:

$$U(\vec{x}) = \frac{1}{2}k\big(x_1^2 + (x_2 - x_1)^2 + x_2^2 - 2a_0\{\sqrt{a^2 + x_1^2} + \sqrt{a^2 + (x_2 - x_1)^2} + \sqrt{a^2 + x_2^2}\} + \text{constants}\big). \tag{76}$$

From here, we relatively easily get

$$\frac{\partial U}{\partial x_1} = k\left[x_1 - (x_2 - x_1) - a_0\left(\frac{x_1}{\sqrt{a^2 + x_1^2}} + \frac{-(x_2 - x_1)}{\sqrt{a^2 + (x_2 - x_1)^2}}\right)\right], \tag{77}$$

$$\frac{\partial U}{\partial x_2} = k\left[(x_2 - x_1) + 2x_2 - a_0\left(\frac{(x_2 - x_1)}{\sqrt{a^2 + (x_2 - x_1)^2}} + \frac{2x_2}{\sqrt{a^2 + x_2^2}}\right)\right]. \tag{78}$$

Now we need to derive cleverly and directly substitute $\vec{x} = 0$, which ensures that we do not have to derive the square roots again. We get

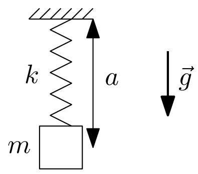
$$\mathbb{U}_{11} = \frac{\partial^2 U}{\partial x_1^2}\Big|_{\vec{x}=0} = 2k\left(1 - \frac{a_0}{a}\right), \quad \mathbb{U}_{12} = -k\left(1 - \frac{a_0}{a}\right). \tag{79}$$

From the symmetry of the partial derivatives $\mathbb{U}_{21} = \mathbb{U}_{12}$ and from the symmetry of the problem $\mathbb{U}_{22} = \mathbb{U}_{11}$, thus

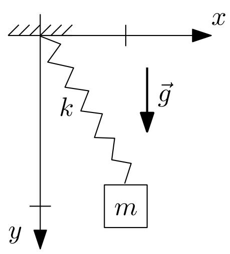
$$\mathbb{U} = \begin{pmatrix} 2k' & -k' \\ -k' & 2k' \end{pmatrix}, \quad k' = k\left(1 - \frac{a_0}{a}\right). \tag{80}$$

The matrix $\mathbb{U}$ is thus exactly the same as in the previous example, just replace k with k'. Since the matrix of kinetic energy $\mathbb{T}$ is exactly the same, we can confidently use the result of the previous exercise.

Exercise 3.4. Find the potential of a spring pendulum (see figure) in the approximation of small oscillations. The pendulum can perform two-dimensional motion in the vertical plane.



Solution: First, let us show that a, the length of the spring in the equilibrium position, can be found and expressed using constants g, k, and m. Let the rest length of the spring be a_0 and introduce coordinates x, y relative to the pendulum's suspension:



The potential energy has the form $U(x, y) = \frac{1}{2}k\left(\sqrt{x^2 + y^2} - a_0\right)^2 - mgy$. We seek the minimum (x_0, y_0):

$$0 = \frac{\partial U}{\partial x}\Big|_{(x_0,y_0)} = k\left(\sqrt{x_0^2 + y_0^2} - a_0\right)\frac{x_0}{\sqrt{x_0^2 + y_0^2}} \tag{81}$$

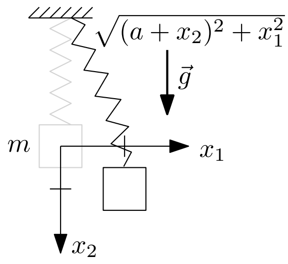
$$0 = \frac{\partial U}{\partial y}\Big|_{(x_0,y_0)} = k\left(\sqrt{x_0^2 + y_0^2} - a_0\right)\frac{y_0}{\sqrt{x_0^2 + y_0^2}} - mg \tag{82}$$

$$\tag{83}$$

The first equation can occur for $\sqrt{x_0^2 + y_0^2} = a_0$, but that excludes the validity of the second equation. Thus, it must be $x_0 = 0$ and the second equation gives us

$$k(y_0 - a_0) - mg = 0. \tag{84}$$

That is, of course, the condition of the balance of elastic and gravitational forces, from where $a \equiv y_0 = \frac{m}{k}g + a_0$. Now back to the problem of small oscillations. We introduce coordinates (x_1, x_2) as in the figure:



The potential is given by the sum of the elastic and gravitational potential energy:

$$\begin{aligned} U(\vec{x}) &= \frac{1}{2}k\left(\sqrt{(a+x_2)^2+x_1^2} - a_0\right)^2 - mg(a+x_2) \\ &= \frac{1}{2}\left(k(x_2^2 + x_1^2 - 2a_0\sqrt{(a+x_2)^2+x_1^2}\right) + (ka-mg)x_2 + \text{constants} \end{aligned} \tag{85}$$

Partial derivatives are thus

$$\frac{\partial U}{\partial x_1} = k\left(x_1 - a_0\frac{x_1}{\sqrt{(a+x_2)^2+x_1^2}}\right), \tag{86}$$

$$\frac{\partial U}{\partial x_2} = k\left(x_2 - a_0\frac{a+x_2}{\sqrt{(a+x_2)^2+x_1^2}}\right) + ka - mg. \tag{87}$$

The second derivatives are

a bit more complicated because we cannot avoid further derivation of the square root by x_2. We obtain

$$\mathbb{U}_{11} = \frac{\partial^2 U}{\partial x_1^2}\Big|_{\vec{x}=0} = k\left(1 - \frac{a_0}{a}\right), \tag{88}$$

$$\mathbb{U}_{12} = \frac{\partial^2 U}{\partial x_1 \partial x_2}\Big|_{\vec{x}=0} = 0, \tag{89}$$

$$\mathbb{U}_{22} = \frac{\partial^2 U}{\partial x_2^2}\Big|_{\vec{x}=0} = k. \tag{90}$$

We see that the term from the gravitational force completely disappeared. The matrix of potential energy is diagonal. The resulting potential of small oscillations has the form

$$U_{s.o.}(\vec{x}) = \frac{1}{2}k\left(1 - \frac{a_0}{a}\right)x_1^2 + \frac{1}{2}kx_2^2. \tag{91}$$

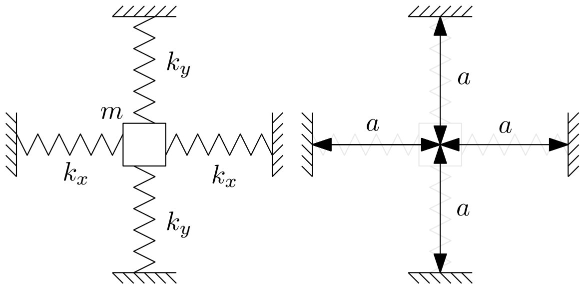
Thanks to the diagonality of the matrix $\mathbb{U}$, the equations of motion are independent:

$$m\ddot{x}_1 + k\left(1 - \frac{a_0}{a}\right)x_1 = 0, \qquad m\ddot{x}_2 + kx_2 = 0. \tag{92}$$

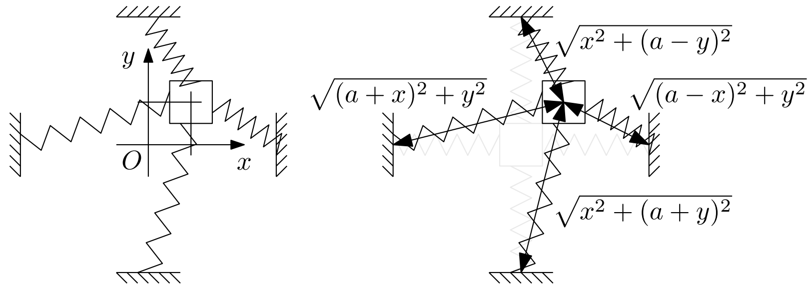
Thus, this system has two modes – one corresponds to oscillations in the horizontal direction x_1 with frequency $\sqrt{\frac{k}{m}(1 - \frac{a_0}{a})}$ and the other to oscillations in the vertical direction x_2 with frequency

$\sqrt{\frac{k}{m}}$. Horizontal oscillation corresponds to transverse oscillations relative to the spring and vertical corresponds to longitudinal oscillations. This explains the form of the presence (effective) stiffness $k' = k(1-\frac{a_0}{a})$ and k in the angular frequencies.

***Exercise 3.5.** Find the potential of a spring pendulum (see figure) in the approximation of small oscillations. The pendulum can perform two-dimensional motion in the vertical plane.



Solution: We introduce coordinates (x, y) as in the figure:



The potential energy is obtained from a bit of Pythagorean theorems:

$$
\begin{aligned}
U(\vec{x}) &= \frac{1}{2}k_x\left(\left(\sqrt{(a+x)^2+y^2}-a_{0x}\right)^2+\left(\sqrt{(a-x)^2+y^2}-a_{0x}\right)^2\right)\\
&+\frac{1}{2}k_y\left(\left(\sqrt{x^2+(a-y)^2}-a_{0y}\right)^2+\left(\sqrt{x+(a+y)^2}-a_{0y}\right)^2\right)\\
&=\frac{1}{2}k_x\left(2x^2+2y^2-2a_{0x}\left(\sqrt{(a+x)^2+y^2}+\sqrt{(a-x)^2+y^2}\right)\right)\\
&+\frac{1}{2}k_y\left(2y^2+2x^2-2a_{0y}\left(\sqrt{(a+y)^2+x^2}+\sqrt{(a-y)^2+x^2}\right)\right)+\text{constants}.
\end{aligned}
\tag{93}
$$

Deriving U with respect to x gives us

$$
\begin{aligned}
\frac{\partial U}{\partial x} &= k_x\left[2x-a_{0x}\left(\frac{a+x}{\sqrt{(a+x)^2+y^2}}+\frac{a-x}{\sqrt{(a-x)^2+y^2}}\right)\right]\\
&+k_y\left[2x-a_{0y}\left(\frac{x}{\sqrt{(a+y)^2+x^2}}+\frac{x}{\sqrt{((a-y)^2+x^2}}\right)\right].
\end{aligned}
\tag{94}
$$

Now we need to proceed cleverly. When further deriving partially with respect to x, we notice that the ugly terms in the first row are composite functions that differ only by swapping x for $-x$.

When deriving and substituting $\vec{x}=0$, they necessarily cancel out. In the second row, we do not need to derive the square roots, because we are anyway substituting $x=0$. The calculation is thus not as terrible:

$$\mathbb{U}_{11}=\frac{\partial^2 U}{\partial x^2}\bigg|_{\vec{x}=0}=2k_x+2k_y\left(1-\frac{a_{0x}}{a}\right). \tag{95}$$

From the symmetry of the problem, $\mathbb{U}_{22}$ is obtained by swapping x and y:

$$\mathbb{U}_{22}=\frac{\partial^2 U}{\partial y^2}\bigg|_{\vec{x}=0}=2k_y+2k_x\left(1-\frac{a_{0y}}{a}\right). \tag{96}$$

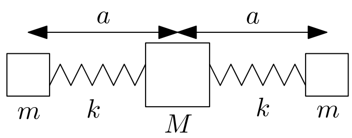
The remaining task is to calculate the mixed term. The partial derivatives of the ugly terms in the first row give zero, because the result will be proportional to y, and in the second row, they cancel out, because they are again composite functions, differing only by swapping y and $-y$. Thus we get $\mathbb{U}_{12}=0$.

The resulting matrix $\mathbb{U}$ is again diagonal, and the equations of motion are thus independent harmonic oscillator equations in the horizontal and vertical directions:

$$m\ddot{x}+\left[2k_x+2k_y\left(1-\frac{a_{0x}}{a}\right)\right]x=0,\qquad m\ddot{y}+\left[2k_y+2k_x\left(1-\frac{a_{0y}}{a}\right)\right]y=0. \tag{97}$$

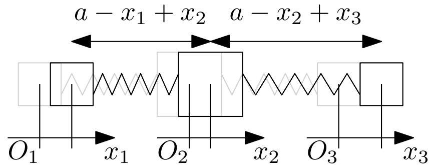
From the forms of the angular frequencies of individual modes, we also see that for horizontal oscillations (equation for x), the horizontal springs are longitudinally oscillating and the vertical ones are transversely oscillating, for vertical oscillations (equation for y) it is vice versa.

Exercise 3.6. Find the solution of the equations of motion of the following system by the mode method. Only longitudinal motion is allowed. Assume that a is the rest length of the spring.



Is the found solution complete? "Where did the error occur"?

Solution: We introduce coordinates as in the figure:



Now we have $\vec{x}=(x_1,x_2,x_3)$ and the potential has the form

$$\begin{aligned}U(\vec{x})&=\frac{1}{2}k(a-x_1+x_2-a)^2+\frac{1}{2}k(a-x_2+x_3-a)^2\\&=\frac{1}{2}k(x_1^2+2x_2^2+x_3^2-2x_1x_2-2x_2x_3)+\text{constants}.\end{aligned} \tag{98}$$

The potential in the approximation of small oscillations is obtained by second partial derivatives and can easily be considered to exactly match the quadratic form above, thus

$$U_{s.o.}(\vec{x})=\frac{1}{2}\sum_{i,j=1}^{3}\mathbb{U}_{ij}x_ix_j=\frac{1}{2}k(x_1^2+2x_2^2+x_3^2-2x_1x_2-2x_2x_3). \tag{99}$$

We can also write $U_{s.o.}(\vec{x}) = \frac{1}{2}\vec{x}^T \mathbb{U}\vec{x}$, where $\vec{x} = (I_1, I_2)$ and

$$\mathbb{U} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}. \tag{100}$$

The masses of the weights are m, M, and m thus the matrix $\mathbb{T}$ is

$$\mathbb{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}. \tag{101}$$

Now, we need to solve the secular equation $\det(\mathbb{U} - \omega^2\mathbb{T}) = 0$, which gives

$$\begin{aligned} 0 = \det \begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} &= (k - m\omega^2)^2(2k - M\omega^2) - 2k^2(k - m\omega^2) \\ = (k - m\omega^2)\left[(k - m\omega^2)(2k - M\omega^2) - 2k^2\right] &= \omega^2(k - m\omega^2)(mM\omega^2 - k(M + 2m)). \end{aligned} \tag{102}$$

This equation has three solutions for ω^2, which are easy to find, let's denote them:

$$\omega_0 = 0, \;\; \omega_1 = \sqrt{\frac{k}{m}}, \;\; \omega_2 = \sqrt{\frac{k(M + 2m)}{mM}}. \tag{103}$$

Now we must solve the equations $(\mathbb{U} - \omega^2\mathbb{T})\vec{a} = 0$ to get the amplitude ratio vectors.

(i) $\omega = 0$. We seek the vector $\vec{a} = (a_1, a_2, a_3)$ solving the system of equations

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} k(a_1 - a_2) \\ k(2a_1 - a_1 - a_3) \\ k(a_3 - a_2) \end{pmatrix}, \tag{104}$$

by equivalent modifications

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{105}$$

We obtain the condition $a_1 = a_2 = a_3$ and a suitable candidate is thus $\vec{a} = (1, 1, 1)^T$. In this mode, the weights do not oscillate at all.

(ii) $\omega = \sqrt{k/m}$. We solve the system

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -k & 0 \\ -k & k(2 - \frac{M}{m}) & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -ka_2 \\ -ka_1 + k(2 - \frac{M}{m})a_2 - ka_3 \\ -ka_2 \end{pmatrix}, \tag{106}$$

by equivalent modifications

$$\begin{pmatrix} 0 & -k & 0 \\ -k & k(2 - \frac{M}{m}) & -k \\ 0 & -k & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 - \frac{M}{m} & -1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{107}$$

The solution is $a_2 = 0$ and $a_1 = -a_3$. We choose, for example, $\vec{a} = (1, 0, -1)$. The middle weight does not oscillate and the outer weights oscillate in opposite directions with angular frequency $\sqrt{\frac{k}{m}}$.

(iii) $\omega = \sqrt{\frac{k(M+2m)}{mM}}$. We solve the system

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix} -\frac{2m}{M}k & -k & 0 \\ -k & -\frac{M}{m}k & -k \\ 0 & -k & -\frac{2m}{M}k \end{pmatrix} \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix} -k(\frac{2m}{M}a_1 + a_2) \\ -k(a_1 + \frac{M}{m}a_2 + a_3) \\ -k(\frac{2m}{M}a_3 + a_2), \end{pmatrix}, \tag{108}$$

by equivalent modifications

$$\begin{pmatrix} -\frac{2m}{M}k & -k & 0 \\ -k & -\frac{M}{m}k & -k \\ 0 & -k & -\frac{2m}{M}k \end{pmatrix} \sim \begin{pmatrix} \frac{2m}{M} & 1 & 0 \\ 1 & \frac{M}{m} & 1 \\ 0 & 1 & \frac{2m}{M} \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{M}{2m} & 0 \\ 0 & \frac{M}{2m} & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{109}$$

Thus, it must be $a_1 = -\frac{M}{2m}a_2$ and $a_3 = -\frac{M}{2m}a_2$. We may choose $a_2 = 1$ and thus $\vec{a} = (-\frac{M}{2m}, 1, -\frac{M}{2m})^T$. The middle weight oscillates and the outer weights oscillate in the same direction opposite to the middle one.

Is

the solution complete? The root $\omega = 0$ is in fact double and thus admits one more linearly independent solution (see general theory of differential equations) in the form $\vec{x}(t) := A\vec{a}t\cos(\omega t + \varphi)$. Here we have $\omega = 0$ and thus we get $\vec{x}(t) = A\cos(\varphi)\vec{a}t$, where we already found $\vec{a} = (1,1,1)^T$. But this corresponds to the simultaneous uniform linear motion of all three weights!

The error thus occurred because the matrix $\mathbb{U}$ is not positively definite – the point $(0,0,0)$ is not a stable equilibrium position. Strictly speaking, the method of small oscillations cannot be used.

Exercise 3.7. Consider the general solution of the motion of the system in the form

$$\vec{x}(t) = \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = A_1\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_1 t + \varphi_1) + A_2\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_2 t + \varphi_2). \tag{110}$$

Find the specific solution for the initial conditions

$$x_1(0) = A \neq 0, \ \ x_2(0) = 0, \ \ \dot{x}_1(0) = 0, \ \ \dot{x}_2(0) = 0. \tag{111}$$

Solution: Deriving $\vec{x}$, we get

$$\dot{\vec{x}}(t) = \begin{pmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\end{pmatrix} = A_1\omega_1\begin{pmatrix}-1\\-1\end{pmatrix}\sin(\omega_1 t + \varphi_1) + A_2\omega_2\begin{pmatrix}-1\\1\end{pmatrix}\sin(\omega_2 t + \varphi_2). \tag{112}$$

Substituting the initial conditions then gives us equations

$$\begin{pmatrix}A\\0\end{pmatrix} = \begin{pmatrix}A_1\cos\varphi_1 + A_2\cos\varphi_2\\A_1\cos\varphi_1 - A_2\cos\varphi_2\end{pmatrix}, \ \ \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-A_1\omega_1\sin\varphi_1 - A_2\omega_2\sin\varphi_2\\-A_1\omega_1\sin\varphi_1 + A_2\omega_2\sin\varphi_2\end{pmatrix}. \tag{113}$$

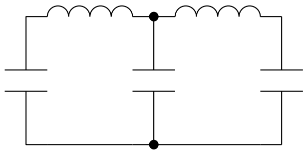
This is a system of four equations for four unknowns. By adding and subtracting equations, we get

$$A_1\sin\varphi_1 = A_2\sin\varphi_2 = 0, \qquad A_1\cos\varphi_1 = A_2\cos\varphi_2 = \frac{A}{2}. \tag{114}$$

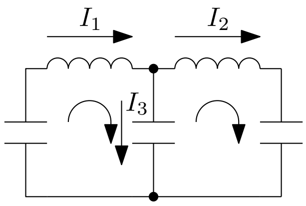
Since $A \neq 0$, the second set of equations immediately ensures that $A_1, A_2 \neq 0$ and thus from the first set of equations, $\varphi_1, \varphi_2 \in \{0, \pi\}$. From the second set of equations, for $\varphi_i = 0$, we have $A_i = \frac{A}{2}$, for $\varphi_i = \pi$, then $A_i = -\frac{A}{2}$. Since $\cos(x) = -\cos(x+\pi)$, these solutions are equivalent, and we can choose $\varphi_1 = \varphi_2 = 0$ and $A_1 = A_2 = A/2$. The unique solution satisfying the initial conditions is thus

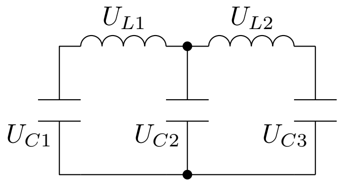
$$\vec{x}(t) = A/2\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_1 t) + A/2\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_2 t). \tag{115}$$

***Exercise 3.8.** Find the current flow in the circuit in the figure:



Solution: We label the currents and their positive directions in the branches, the directions of circulation in the left and right loops, and the voltages across the elements as in the figures:





The voltage across the capacitor and the inductor is given by the equations

$$U_C = \frac{Q}{C}, \qquad U_L = L\dot{I}. \tag{116}$$

By deriving the equation for the voltage across the capacitor with respect to time, we get $\dot{U}_C = \frac{I}{C}$. Viewing capacitors and inductors as voltage sources in the circuit, then the sign convention is as follows: if the direction of circulation of a given loop agrees with the direction of the current in the respective branch, then we add a minus to the formulas for the voltages (if it disagrees, we leave a plus). Thus, for the left, respectively, right loop, we get from the second Kirchhoff's law:

$$-U_{C1} - U_{C2} - U_{L1} = 0, \qquad U_{C2} - U_{C3} - U_{L2} = 0. \tag{117}$$

After deriving these equations with respect to time and substituting for the individual voltages (and multiplying by minus one):

$$\frac{1}{C}I_1 - \frac{1}{C}I_3 - L\ddot{I}_1 = 0, \qquad -\frac{1}{C}I_3 + \frac{1}{C}I_2 + L\ddot{I}_2 = 0. \tag{118}$$

After substituting for I_3 from the first Kirchhoff's law for currents, $I_1 = I_2 + I_3$, we get the final set of differential equations for the currents flowing through each inductor:

$$0 = L\ddot{I}_1 + \frac{2}{C}I_1 - \frac{1}{C}I_2 \tag{119}$$

$$0 = L\ddot{I}_2 - \frac{1}{C}I_1 + \frac{2}{C}I_2. \tag{120}$$

We can write this system of equations in matrix form as $\mathbb{T}\ddot{\vec{I}} + \mathbb{U}\vec{I} = 0$, where $\vec{I} = (I_1, I_2)$ and

$$\mathbb{T} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \quad \mathbb{U} = \frac{1}{C}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{121}$$

But this is the same system of equations as in Exercise 3.2, only here $m = L$ and $k = \frac{1}{C}$. We already know that the general solution is in the form

$$\vec{I}(t) = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\frac{1}{LC}}t + \varphi_1\right) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\frac{3}{LC}}t + \varphi_2\right). \tag{122}$$

The inductance of the inductors here plays the role of inertia mass – resistance of

the inductors to changes in current; the reciprocal value of the capacitance plays the role of stiffness – the smaller the value of the capacitor's capacitance, the faster its voltage changes for a given current and thus faster it causes changes in current in the circuit.

4 String Vibrations and Fourier Series

Exercise 4.1. If we shorten a string by $\Delta l = 10cm$, its frequency increases to $\alpha = 1.5$ times its original value. Calculate the length of the string L. Assume that the tension in the string remains the same.

Solution: A string with fixed ends of length L at points $z = 0$ and $z = L$ has a solution in the form of a superposition of modes:

$$\psi(z,t) = \sum_{m=1}^{\infty} A_m \sin(k_m z)\sin(\omega_m t + \varphi_m). \tag{123}$$

The relation between k and ω is given by the **dispersion relation** $\omega = \sqrt{\frac{T_0}{\rho_0}}k$ and the m-th wave number satisfies $k_m = \frac{\pi m}{L}$. ρ_0 is the linear density of the string and T_0 its tension. The frequency f is related to the angular frequency as $\omega = 2\pi f$.

Let f' be the new frequency and L' the new length of the string. So we have $f' = \alpha f$ and $L' = L - \Delta l$. According to the dispersion relation (if we do not change the tension or the material of the string), the ratio of ω and k (in any but the same mode) must remain constant: $\frac{\omega}{k} = \frac{\omega'}{k'}$. Substituting, we get the equation

$$fL = f'L' = \alpha f \cdot (L - \Delta l). \tag{124}$$

From here, we can easily express L as $L = \frac{\alpha}{\alpha-1}\Delta l = 3\Delta l = 30cm$.

Exercise 4.2. A piano string $L = 1m$ long with a diameter $d = 0.5mm$ emits the fundamental tone C with a frequency $f = 256Hz$. The volumetric density of this string is $\rho = 9g/cm^3$. What is the tension T_0 in the string?

Solution: The wave number of the fundamental tone is $k_1 = \frac{\pi}{L}$. The linear density is obtained by multiplying the volumetric density by the cross section of the string, i.e., $\rho_0 = \frac{1}{4}\pi d^2 \rho$. From the dispersion relation $\omega = \sqrt{\frac{T_0}{\rho_0}}k$ thus

$$T_0 = \frac{\omega^2}{k^2}\rho_0 = 4\rho_0 f^2 L^2 = \pi d^2 \rho f^2 L^2 = 3.14 \cdot (5 \cdot 10^{-4})^2 \cdot 9000 \cdot 256^2 \approx 459.2N. \tag{125}$$

This is therefore the force exerted by a weight of approximately 46kg! There are about 230 strings in a piano.

Exercise 4.3. Find the forms of modes for a string of length L (stretched from $z \in [0, L]$) for free ends. Assume a solution in the form of a mode (standing wave) $\psi(z,t) = X(z)\cos(\omega t + \varphi)$. Write the general solution as a superposition of these modes. Is there something missing in the solution?

Solution: The function ψ must satisfy the *wave equation*:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{T_0}{\rho_0} \frac{\partial^2 \psi}{\partial z^2} \tag{126}$$

The free end condition is defined as

$$\frac{\partial \psi}{\partial z}(0,t) = \frac{\partial \psi}{\partial z}(L,t) = 0. \tag{127}$$

Substituting the ansatz into the wave equation immediately yields

$$\left(X''(z) + \frac{\rho_0}{T_0}\omega^2 X(z) \right) \cos(\omega t + \varphi_0) = 0. \tag{128}$$

Let the wave number $k > 0$ be $k^2 = \frac{\rho_0}{T_0}\omega^2$ (thus obtaining the dispersion relation); requiring the previous equation to be satisfied at all times, we get an ordinary differential equation

$$X''(z) + k^2 X(z) = 0. \tag{129}$$

This is the equation of a harmonic oscillator (in variable z). Write its solution, for example, in the form

$$X(z) = a\cos kz + b \sin kz. \tag{130}$$

The resulting function $\psi(z,t) = X(z)\cos(\omega t + \varphi)$ must be substituted into the boundary condition. Easily $\frac{\partial \psi}{\partial z} = X'(z)\cos(\omega t + \varphi)$. The initial conditions thus give equations $X'(0) = 0$ and $X'(L) = 0$. We have $X'(z) = -ak\sin kz + bk\cos kz$.

The condition $X'(0) = 0$ gives $bk = 0$ and thus $b = 0$. The condition $X'(L) = 0$ then gives $a \sin kL = 0$ and for a nontrivial solution thus $kL \in \{m\pi\}_{m\in\mathbb{N}}$ (we only consider natural number multiples, since the constant $kL > 0$). The wave number must satisfy $k = k_m = \frac{m\pi}{L}$, $m \in \mathbb{N}$. The resulting form of the m-th mode is thus $X_m(z) = A_m \cos k_m z$.

The resulting function $\psi(z,t)$ is given by the superposition of these modes (do not forget that ω is different for each value of the wave number and given by the dispersion relation):

$$\psi(z,t) = \sum_{m=1}^{\infty} A_m \cos\left(\frac{m\pi}{L}z\right) \cos\left(\sqrt{\frac{T_0}{\rho_0}}\frac{m\pi}{L}t + \varphi_m\right). \tag{131}$$

Which solution did we forget? Since the string has both ends free, it can, in addition to vibrations, perform uniform linear motion as a whole: $\psi(z,t) = x_0 + v_0 t$. This solution is not in the form of the assumed solution form, so it could not come out. If we took the method of separation of variables more generally, where we assume a solution of the form $\psi(z,t) = X(z)T(t)$ (thus generalizing the form of the time function), we would get a solution including uniform motion. The complete solution (and now truly complete) of the wave equation with the given boundary conditions is thus

$$\psi(z,t) = x_0 + v_0 t + \sum_{m=1}^{\infty} A_m \cos\left(\frac{m\pi}{L}z\right) \cos\left(\sqrt{\frac{T_0}{\rho_0}}\frac{m\pi}{L}t + \varphi_m\right). \tag{132}$$

***Exercise 4.4.** The same assignment as the previous example with the difference that now consider one end fixed and the other free.

Solution: The procedure is completely analogous to the previous example. Only the boundary conditions and thus the requirements for the form of the function $X(z) = a\cos kz + b\sin kz$ differ. WLOG (without loss of generality) consider the left end (at $z = 0$) fixed and the right end (at $z = L$) free, i.e.,

$$\psi(0,t) = 0, \qquad \frac{\partial\psi(L,t)}{\partial z} = 0, \tag{133}$$

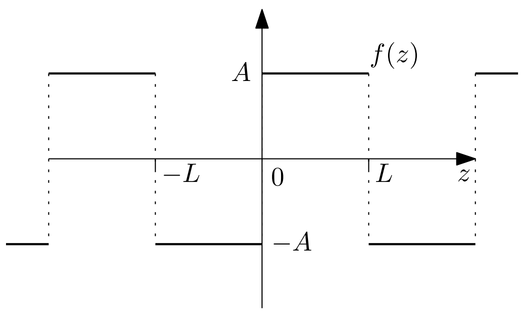
that is, for the function $X(z)$: $X(0) = 0$ and $\frac{\partial X(L)}{\partial z} = 0$. The condition $X(0) = 0$ gives $a = 0$ and then from $\frac{\partial X(L)}{\partial z} = 0$ we have $b\cos kL = 0$. If we require a non-trivial solution, $b \neq 0$ and $\cos kL = 0$, hence $kL = \frac{\pi}{2} + m\pi$, $m \in \mathbb{N}_0$ ($kL > 0$ hence $m \geq 0$). The permissible wave numbers are thus of the form $k_m = (\frac{\pi}{2} + m\pi)\frac{1}{L}$. The resulting solution is again given by the superposition of individual modes:

$$\psi(z,t) = \sum_{m=0}^{+\infty} A_m \sin k_m z \cos(\omega_m t + \varphi_m), \tag{134}$$

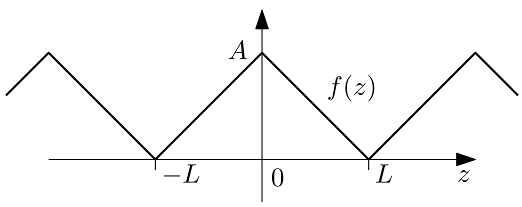
where $k_m = (\frac{\pi}{2} + m\pi)\frac{1}{L}$ and $\omega_m = \sqrt{\frac{T_0}{\rho_0}} k_m$.

Exercise 4.5. Calculate the Fourier series of the following functions f with period $2L$:

a) Square wave



b) *Sawtooth wave



Solution: If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with period $2L$, its Fourier series is a function f_F given by the relation

$$f_F(z) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi z}{L}\right) + b_m \sin\left(\frac{m\pi z}{L}\right), \tag{135}$$

where the coefficients a_m and b_m are given by the relations

$$a_m = \frac{1}{L}\int_{-L}^{L} f(z)\cos\left(\frac{m\pi z}{L}\right) dz, \quad m \in \mathbb{N}_0, \quad b_m = \frac{1}{L}\int_{-L}^{L} f(z)\sin\left(\frac{m\pi z}{L}\right) dz, \quad m \in \mathbb{N}. \tag{136}$$

If the function f is even, $b_m = 0$ and if it is odd, $a_m = 0$.

a) Square wave: the function f is obviously odd, so it suffices to calculate the coefficients b_m. We have

$$\begin{aligned} b_m &= \frac{1}{L}\int_{-L}^{L} f(z)\sin\left(\frac{m\pi z}{L}\right)dz = \frac{2}{L}\int_0^L A\sin\left(\frac{m\pi z}{L}\right)dz = \frac{2A}{L}\left[-\frac{L}{m\pi}\cos\left(\frac{m\pi z}{L}\right)\right]_0^L \\ &= \frac{2A}{m\pi}\left[-\cos\left(\frac{m\pi z}{L}\right)\right]_0^L = \frac{2A}{m\pi}(1-\cos m\pi). \end{aligned} \tag{137}$$

Finally, we can distinguish between even and odd m. For even m, $1-\cos m\pi = 0$ and thus $b_m = 0$. It suffices to consider odd m, thus $m = 2k-1$, $k \in \mathbb{N}$, then $1-\cos m\pi = 2$. We get

$$b_{2k-1} = \frac{4A}{(2k-1)\pi}. \tag{138}$$

The resulting Fourier series of function f is thus

$$f_F(z) = \sum_{k=1}^{\infty} \frac{4A}{(2k-1)\pi}\sin\left(\frac{(2k-1)\pi z}{L}\right). \tag{139}$$

b) Sawtooth wave: the function f is even. It suffices to calculate the coefficients a_m. For $m > 0$ we get

$$\begin{aligned} a_m &= \frac{1}{L}\int_{-L}^{L} f(z)\cos\left(\frac{m\pi z}{L}\right)dz = \frac{2}{L}\int_0^L f(z)\cos\left(\frac{m\pi z}{L}\right)dz = \frac{2}{L}\int_0^L A\left(1-\frac{z}{L}\right)\cos\left(\frac{m\pi z}{L}\right)dz \\ &= \frac{2}{L}\left[A\left(1-\frac{z}{L}\right)\frac{L}{m\pi}\sin\left(\frac{m\pi z}{L}\right)\right]_0^L + \frac{2}{L}\int_0^L \frac{A}{L}\frac{L}{m\pi}\sin\left(\frac{m\pi z}{L}\right)dz \\ &= \frac{2A}{m\pi L}\int_0^L \sin\left(\frac{m\pi z}{L}\right)dz = \frac{2A}{(m\pi)^2}\left[-\cos\left(\frac{m\pi z}{L}\right)\right]_0^L. \end{aligned} \tag{140}$$

We are in the same situation as in the previous example – only odd $m = 2k-1$ contribute and thus

$$a_{2k-1} = \frac{4A}{(2k-1)^2\pi^2}. \tag{141}$$

We must not forget about a_0, which is obtained by the integral

$$a_0 = \frac{2}{L}\int_0^L A\left(1-\frac{z}{L}\right)dz = \frac{2}{L}\left[A\left(z-\frac{z^2}{2L}\right)\right]_0^L = \frac{2A}{L}\left(L-\frac{L^2}{2L}\right) = A. \tag{142}$$

The Fourier series of the sawtooth wave is thus

$$f_F(z) = \frac{A}{2} + \sum_{k=1}^{\infty}\frac{4A}{(2k-1)^2\pi^2}\cos\left(\frac{(2k-1)\pi z}{L}\right). \tag{143}$$

Exercise 4.6. Consider a string with fixed ends. Find a specific solution for its motion if you make it vibrate so that at time $t = 0$ it is at rest and has the form[1] $\psi(z,0) = A$, where A is a constant.

[1] Strictly speaking, ψ at time $t = 0$ does not satisfy the boundary conditions. One can imagine that at both ends the function describing the string drops very sharply to 0.

Solution: The solution of a standing wave with fixed ends has the form

$$\psi(z,t) = \sum_{m=1}^{\infty} A_m \sin(k_m z)\sin(\omega_m t + \varphi_m). \tag{144}$$

Let $f(z) = \psi(z,0)$ be the function $f : [0,L] \to \mathbb{R}$ specifying the initial shape of the string, and $g(z) = \frac{\partial \psi}{\partial t}(z,0)$ the initial speed of the string. Substituting the solution, we get equations:

$$f(z) = \sum_{m=1}^{\infty} A_m \sin\varphi_m \sin\left(\frac{m\pi z}{L}\right), \tag{145}$$

$$g(z) = \sum_{m=1}^{\infty} A_m \omega_m \cos\varphi_m \sin\left(\frac{m\pi z}{L}\right). \tag{146}$$

To solve these conditions, it is necessary to find the constants A_m and φ_m. We see that the right sides resemble the Fourier series of an odd periodic function. It suffices to find a unique **odd extension of the function** f, a function $\bar{f} : \mathbb{R} \to \mathbb{R}$ satisfying:

(i) $\bar{f}$ is periodic with period $2L$;

(ii) $\bar{f}$ is odd;

(iii) $\bar{f}$ restricted to the interval $[0,L]$ gives the function f.

If we find the coefficients f_m of the Fourier development of the function $\bar{f}(z) = \sum_{m=1}^{+\infty} f_m \sin\left(\frac{m\pi z}{L}\right)$, by comparing coefficients we get

$$f_m = A_m \sin\varphi_m, \quad m \in \mathbb{N}. \tag{147}$$

Similarly, we find the odd extension $\bar{g}$ of the function g and if we denote g_m the coefficients of its Fourier development, we obtain relations

$$g_m = A_m \omega_m \cos\varphi_m, \quad m \in \mathbb{N}. \tag{148}$$

Let's solve this system in this case. According to the task, we have $f(z) = A$ for all $z \in [0,L]$ and $g(z) = 0$ for all $z \in L$. We see that as an odd extension $\bar{f}$ we get a rectangular wave from the previous example and $g \equiv 0$ (and thus $g_m = 0$). We thus obtain a system of equations:

$$0 = A_{2k} \sin\varphi_{2k}, \quad k \in \mathbb{N}, \tag{149}$$

$$\frac{4A}{(2k-1)\pi} = A_{2k-1} \sin\varphi_{2k-1}, \quad k \in \mathbb{N}, \tag{150}$$

$$0 = A_m \omega_m \cos\varphi_m, \quad m \in \mathbb{N}. \tag{151}$$

Therefore, I can choose $A_{2k} = 0$, $k \in \mathbb{N}$ and φ_{2k} arbitrarily. Since necessarily $A_{2k-1} \neq 0$, I get from the last set of equations $\cos\varphi_{2k-1} = 0$. Hence, $\varphi_{2k-1} \in \{\frac{\pi}{2} + n\pi\}_{n\in\mathbb{Z}}$. We can choose $\varphi_{2k-1} = \frac{\pi}{2}$, because in any case $\sin\varphi_{2k-1} \in \{-1, 1\}$ and we would just have to hide the sign in the amplitude. From the remaining equation, thus

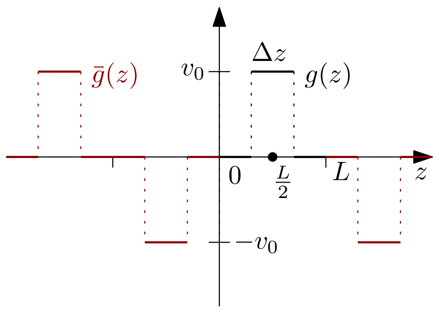
$$A_{2k-1} = \frac{4A}{(2k-1)\pi}. \tag{152}$$

The resulting solution of the wave equation with this initial condition is thus

$$\psi(z,t) = \sum_{k=1}^{\infty} \frac{4A}{(2k-1)\pi} \sin\left(\frac{(2k-1)\pi z}{L}\right)\sin(\omega_{2k-1} t). \tag{153}$$

Exercise 4.7. Consider a string with fixed ends. Find a specific solution for its motion if at time $t = 0$ it is in equilibrium position and you strike it with a hammer so that a segment of the string of length Δz centered at $L/2$ is given a speed v_0.

Solution: Using the notation from the previous example, we have $f(z) \equiv 0$ and $g(z)$ (and its odd extension $\bar{g}(z)$) has the form



We must therefore find the Fourier series of the odd extension $\bar{g}$ of the function g. The development coefficients g_m are

$$g_m = \frac{2}{L}\int_0^L g(z)\sin\left(\frac{m\pi z}{L}\right). \tag{154}$$

Into the integral, obviously, only the section $[\frac{L-\Delta z}{2}, \frac{L+\Delta z}{2}]$ will contribute. We get the integral

$$\begin{aligned} g_m &= \frac{2v_0}{L}\int_{\frac{L-\Delta z}{2}}^{\frac{L+\Delta z}{2}} \sin\left(\frac{m\pi z}{L}\right)\,dz = \frac{2v_0}{L}\left[-\frac{L}{m\pi}\cos\left(\frac{m\pi z}{L}\right)\right]_{\frac{L-\Delta z}{2}}^{\frac{L+\Delta z}{2}} \\ &= \frac{2v_0}{m\pi}\left(\cos\left(\frac{m\pi}{2} - \frac{m\pi\Delta z}{2L}\right) - \cos\left(\frac{m\pi}{2} + \frac{m\pi\Delta z}{2L}\right)\right). \end{aligned} \tag{155}$$

Now it is still advantageous to use the sum formula in the form:

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin(\alpha)\sin(\beta), \quad \alpha = \frac{m\pi}{2}, \quad \beta = \frac{m\pi\Delta z}{2L}. \tag{156}$$

Hence, we get a simplified expression for g_m:

$$g_m = \frac{4v_0}{m\pi}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{m\pi\Delta z}{2L}\right). \tag{157}$$

We see that for even m we again get $g_{2k} = 0$. For odd $m = 2k-1$, we must solve what $\sin\frac{(2k-1)\pi}{2}$ gives. For odd $k \in \{1,3,5,\dots\}$ we get $\sin\frac{\pi}{2} = 1$ and for even k we get $\sin\frac{3\pi}{2} = -1$. We can therefore write $\sin\frac{(2k-1)\pi}{2} = (-1)^{k-1}$. Hence

$$g_{2k-1} = \frac{4v_0}{(2k-1)\pi}(-1)^{k-1}\sin\left(\frac{(2k-1)\pi\Delta z}{2L}\right). \tag{158}$$

By comparing coefficients, we therefore obtain a system of equations

$$0 = A_m\sin\varphi_m, \tag{159}$$

$$0 = A_{2k}\omega_{2k}\cos\varphi_{2k}, \tag{160}$$

$$g_{2k-1} = A_{2k-1}\omega_{2k-1}\cos\varphi_{2k-1}. \tag{161}$$

I can therefore choose $A_{2k} = 0$ and φ_{2k} arbitrarily. We can choose $\varphi_{2k-1} = 0$, which ensures $\cos\varphi_{2k-1} = 1$ and the remaining set of equations then determines $A_{2k-1} = \frac{g_{2k-1}}{\omega_{2k-1}}$. We thus get the resulting function

$$\psi(z,t) = \sum_{k=1}^{\infty} \frac{4v_0(-1)^{k-1}}{(2k-1)\pi\omega_{2k-1}} \sin\left(\frac{(2k-1)\pi\Delta z}{2L}\right) \sin\left(\frac{(2k-1)\pi z}{L}\right) \sin\omega_{2k-1}t. \tag{162}$$

***Exercise 4.8.** *Initial problem for a string with free ends.* Modify the procedure for finding a specific solution from given initial conditions for a string of length L with free ends. The general solution from the method of separation of variables might, for example, take the form

$$\psi(z,t) = z_0 + v_0 t + \sum_{m=1}^{+\infty} A_m \cos k_m z \, \sin(\omega_m t + \varphi_m), \quad \text{where} \quad k_m = \frac{m\pi}{L} \quad \text{and} \quad \omega_m = \sqrt{\frac{T_0}{\rho_0}} k_m.$$

Solution: Substitute the above solution into the initial conditions $\psi(z,0) = f(z)$ and $\frac{\partial\psi(z,0)}{\partial t} = g(z)$:

$$\begin{aligned} \psi(z,0) &= z_0 + \sum_{m=1}^{+\infty} (A_m \sin\varphi_m) \cos k_m z = f(z), \\ \frac{\partial\psi(z,0)}{\partial t} &= v_0 + \sum_{m=1}^{+\infty} (A_m \omega_m \cos\varphi_m) \cos k_m z = g(z). \end{aligned} \tag{163}$$

These are the equations for the unknowns A_m, φ_m, z_0, and v_0. We see that we would need to decompose the functions f and g into a superposition of cosines and a constant term. But exactly this looks like the Fourier series of an even function! So, it suffices to consider even extensions of functions f, g, let's denote them again $\bar{f}$, $\bar{g}$, with properties:

(i) $\bar{f}$, $\bar{g}$ are periodic with period $2L$;

(ii) $\bar{f}$, $\bar{g}$ are even;

(iii) $\bar{f}$, $\bar{g}$ restricted to the interval $[0, L]$ give the function f, g.

Their Fourier series are thus of the form

$$f(z) = \frac{f_0}{2} + \sum_{m=1}^{+\infty} f_m \cos\frac{m\pi z}{L}, \qquad g(z) = \frac{g_0}{2} + \sum_{m=1}^{+\infty} g_m \cos\frac{m\pi z}{L}, \tag{164}$$

where

$$f_m = \frac{2}{L}\int_0^L f(z)\cos\frac{m\pi z}{L} dz, \qquad g_m = \frac{2}{L}\int_0^L g(z)\cos\frac{m\pi z}{L} dz, \qquad m \in \mathbb{N}_0. \tag{165}$$

Substituting these developments into the initial conditions and comparing the series term by term, we get equations

$$z_0 = \frac{f_0}{2}, \qquad A_m \sin\varphi_m = f_m \quad (m \in \mathbb{N}), \qquad v_0 = \frac{g_0}{2}, \qquad A_m\omega_m \cos\varphi_m = g_m \quad (m \in \mathbb{N}). \tag{166}$$

Solving for A_m and φ_m, we get

$$A_m = \sqrt{f_m^2 + \frac{g_m^2}{\omega_m^2}}, \qquad \sin\varphi_m = \frac{f_m}{A_m}, \quad \cos\varphi_m = \frac{g_m}{A_m\omega_m}. \tag{167}$$

The angle $\varphi_m \in [0, 2\pi)$ is uniquely determined by its sine and cosine values. The resulting specific solution for motion is then in the form

$$\psi(z,t) = \frac{f_0}{2} + \frac{g_0}{2}t + \sum_{m=1}^{+\infty} \sqrt{f_m^2 + \frac{g_m^2}{\omega_m^2}} \cos k_m z \, \sin(\omega_m t + \varphi_m). \tag{168}$$

5 traveling and Standing Waves

Exercise 5.1. Two tuning forks emit 20 beats in 10 seconds. One tuning fork has a frequency $f = 256Hz$. What is the frequency of the second tuning fork?

Solution: Our ear hears the superposition of two harmonic waves. For simplicity, assume they have the same amplitude. Thus, $x_1(t) = A\cos(\omega_1 t + \varphi_1)$ and $x_2(t) = A\cos(\omega_2 t + \varphi_2)$. Then

$$\begin{aligned} x(t) = x_1(t) + x_2(t) &= A\left(\cos(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi)\right) \\ &= 2A\cos\left(\frac{(\omega_1+\omega_2)t + \varphi_1 + \varphi_2}{2}\right)\cos\left(\frac{(\omega_1-\omega_2)t + \varphi_1 - \varphi_2}{2}\right). \end{aligned} \tag{169}$$

The result is thus the product of two functions – oscillation with the average frequency $f_p = \frac{f_1+f_2}{2}$ and oscillation with the frequency $f_m = \frac{f_1-f_2}{2}$. This "slow oscillation" modulates the amplitude of the "fast oscillations" twice per its period, see the figure. The frequency of beats f_r is therefore double compared to f_m! $f_r = f_1 - f_2$.

Since we do not know which tuning fork is tuned to a higher frequency, we have two possibilities:

$$f' = f \pm f_r. \tag{170}$$

We have $f_r = 2\,Hz$, and the second tuning fork therefore has 254 or 258 Hertz.

Exercise 5.2. What is the amplitude, period, phase velocity, and wavelength of a wave, expressed in SI units by the equation

$$\psi(z,t) = 4\cdot 10^{-2}\sin(2\pi(8t + 5z)). \tag{171}$$

Solution: The amplitude is the numerical factor before the harmonic function, thus $A = 4\cdot 10^{-2}\,m = 4\,cm$. The period is the time it takes for a complete wave to pass a given point ($z = const$). It can thus be directly obtained from the relation $2\pi 8T = 2\pi$. Hence, $T = 1/8\,s$. Of course, we also have $2\pi f = \omega = 2\pi 8\,s^{-1}$ and $T = 1/f$.

To determine the phase velocity, let's fix the phase value $\varphi(z,t) = 2\pi(8t + 5z) = \varphi_0 = \text{const}$. I see that z can be expressed as a function of time: $z(t) = \frac{1}{2\pi 5}(\varphi_0 - 2\pi 8t)$. A place with a constant phase thus moves uniformly linearly (in this case in the opposite direction of the z axis) with a phase velocity $v = \frac{\omega}{k} = \frac{2\pi 8}{2\pi 5}\,m\cdot s^{-1} = \frac{8}{5}\,m\cdot s^{-1}$.

The wavelength is the distance traveled by any place with a constant phase over a period, thus $\lambda = v\cdot T = \frac{\omega}{k}\cdot\frac{2\pi}{\omega} = \frac{2\pi}{k}$. Here the wave number is $k = 2\pi 5\,m^{-1}$, from which $\lambda = \frac{1}{5}\,m = 20\,cm$.

Exercise 5.3. *The superposition of two traveling waves traveling in the same direction is a traveling wave.* Show that the sum

$$A_1\cos(\omega t - kz + \varphi_1) + A_2\cos(\omega t - kz + \varphi_2) \tag{172}$$

can be written as $A\cos(\omega t - kz + \varphi)$. Determine the values of A and φ.

Solution: Using the linearity of the Re function, we can rewrite the sum as

$$A_1 \cos(\omega t - kz + \varphi_1) + A_2 \cos(\omega t - kz + \varphi_2) = \mathrm{Re}\left[(A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}) e^{i(\omega t - kz)}\right]. \tag{173}$$

The complex number $A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}$ thus needs to be written in polar form $Ae^{i\varphi}$. Then we get

$$\mathrm{Re}\left[Ae^{i(\omega t - kz + \varphi)}\right] = A \cos(\omega t - kz + \varphi). \tag{174}$$

Determine the constants A and φ. We have

$$\begin{aligned} A^2 &= |A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}|^2 = (A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2})(A_1 e^{-i\varphi_1} + A_2 e^{-i\varphi_2}) \\ &= A_1^2 + A_2^2 + A_1 A_2 (e^{i(\varphi_1 - \varphi_2)} + e^{-i(\varphi_1 - \varphi_2)}) \\ &= A_1^2 + A_2^2 + 2A_1 A_2 \cos(\varphi_1 - \varphi_2). \end{aligned} \tag{175}$$

The argument φ is then determined by solving equations

$$\cos\varphi = \frac{\mathrm{Re}[Ae^{i\varphi}]}{|Ae^{i\varphi}|} = \frac{\mathrm{Re}\left[A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}\right]}{A} = \frac{A_1 \cos\varphi_1 + A_2 \cos\varphi_2}{A}, \tag{176}$$

$$\sin\varphi = \frac{\mathrm{Im}[Ae^{i\varphi}]}{|Ae^{i\varphi}|} = \frac{\mathrm{Im}\left[A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}\right]}{A} = \frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A}. \tag{177}$$

Exercise 5.4. *The superposition of two oppositely traveling traveling waves is a standing wave.* Show that the sum

$$A \cos(\omega t - kz + \varphi_1) + A \cos(\omega t + kz + \varphi_2) \tag{178}$$

is of the form $X(z) \cos(\omega t + \varphi)$. Determine the form of function $X(z)$ and the value of constant φ.

Solution: This is a simple application of the sum formula

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right). \tag{179}$$

Here we have $\alpha = \omega t + kz + \varphi_2$, $\beta = \omega t - kz + \varphi_1$, hence we get

$$2A\cos\left(kz + \frac{\varphi_2 - \varphi_1}{2}\right)\cos\left(\omega t + \frac{\varphi_2 + \varphi_1}{2}\right). \tag{180}$$

Hence, we get $\varphi = (\varphi_1 + \varphi_2)/2$ and the function $X(z)$ has the form

$$X(z) = 2A\cos\left(kz + \frac{\varphi_2 - \varphi_1}{2}\right). \tag{181}$$

Exercise 5.5. Two sources on the z axis at $z = -d$ and $z = d$ oscillate according to the law $x_1(t) = x_2(t) = A\cos(\omega t)$ and emit waves in both directions. Determine the traveling waves from each source and discuss the character of their superposition.

Solution: The beginning of the string can thus be considered as a source that oscillates with the time dependence $x(t) = A\cos(\omega t + \varphi)$. A traveling wave $\psi(z,t) = x(t - \frac{z}{v})$ will thus be created on the string. Here thus

$$\psi(z,t) = A\cos\left(\omega\left(t - \frac{z}{v}\right) + \varphi\right) = A\cos(\omega t - kz + \varphi), \tag{182}$$

where $k = \frac{\omega}{v}$. The energy flux S is given by the relation

$$S = -T\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z} = T\omega k A^2 \sin^2(\omega t - kz + \varphi). \tag{183}$$

Substituting for the wave number from the dispersion relation, we get $S = \sqrt{T\rho}\cdot\omega^2 A^2 \sin^2(\omega t - kz + \varphi)$. The quantity $Z = \sqrt{T\rho}$ is called **impedance**. Hence,

$$\langle S\rangle = Z\omega^2 A^2 \langle\sin^2(\omega t - kz + \varphi)\rangle. \tag{184}$$

We have calculated that $\langle\sin^2(\omega t)\rangle = \frac{1}{2}$. The time average of a periodic function over its period cannot depend (by definition) on the phase shift. If $g(t) = f(t+\varphi)$, we have

$$\langle g\rangle = \frac{1}{T}\int_a^{a+T} f(t+\varphi)dt = \frac{1}{T}\int_{a+\varphi}^{a+\varphi+T} f(t)dt = \langle f\rangle. \tag{185}$$

Hence,

$$\langle S\rangle = \frac{1}{2}Z\omega^2 A^2 = 2\pi^2 f^2 A^2 Z = 2\cdot 9,85\cdot 10^4\cdot 10^{-4}\cdot\sqrt{400\cdot 10^{-2}} \approx 39,5W. \tag{186}$$

Exercise 5.6. Show that the energy flux vector on a string over which two oppositely traveling traveling waves propagate is equal to the sum of the fluxes corresponding to the individual waves. Hint: Consider d'Alembert's solution and show that the interference term in this case vanishes.

Solution: We need to calculate the energy flux for a wave of the form $\psi(z,t) = \psi_1(z,t) + \psi_2(z,t)$, where $\psi_1(z,t) = F(z - vt)$, $\psi_2(z,t) = G(z + vt)$, and $v = \frac{\omega}{k} = \sqrt{\frac{T}{\rho}}$ is the phase velocity given by the material and tension of the string. The calculation by substitution into the definition of flux then gives

$$\begin{aligned}
S(z,t) &= -T\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z} = -T\frac{\partial(\psi_1+\psi_2)}{\partial t}\frac{\partial(\psi_1+\psi_2)}{\partial z} = \\
&= -T\frac{\partial\psi_1}{\partial t}\frac{\partial\psi_1}{\partial z} - T\frac{\partial\psi_2}{\partial t}\frac{\partial\psi_2}{\partial z} - T\left(\frac{\partial\psi_1}{\partial t}\frac{\partial\psi_2}{\partial z} + \frac{\partial\psi_2}{\partial t}\frac{\partial\psi_1}{\partial z}\right) \\
&= S_1(z,t) + S_2(z,t) - T\left(\frac{\partial\psi_1}{\partial t}\frac{\partial\psi_2}{\partial z} + \frac{\partial\psi_2}{\partial t}\frac{\partial\psi_1}{\partial z}\right).
\end{aligned} \tag{187}$$

Direct substitution verifies that the interference term vanishes:

$$\frac{\partial\psi_1}{\partial t}\frac{\partial\psi_2}{\partial z} + \frac{\partial\psi_2}{\partial t}\frac{\partial\psi_1}{\partial z} = -vF'(z-vt)G'(z+vt) + vG'(z+vt)F'(z-vt) = 0. \tag{188}$$

Therefore, $S(z,t) = S_1(z,t) + S_2(z,t)$.

Exercise 5.7. Two harmonic traveling waves travel in the same direction on a string in superposition. They have the same wavelength and angular frequency. If the intensity (time-averaged energy flux) of each wave is I, what must be the phase difference between these waves for the resulting intensity to be 0, I, $2I$, $4I$?

Solution: Above, we calculated that the intensity $I = \langle S \rangle$ for a harmonic traveling wave $\psi(z,t) = A\cos(\omega t - kz + \varphi)$ comes out as $I = \frac{1}{2} Z\omega^2 A^2$.

Thus, we have two traveling waves $\psi_1(z,t) = A_1 \cos(\omega t - kz + \varphi_1)$ and $\psi_2(z,t) = A_2 \cos(\omega t - kz + \varphi_2)$. The condition is $I_1 = I_2 = I$, from which immediately $A_1 = A_2$. Their superposition is again a traveling wave. Using sum formulas, it turns out

$$\psi(z,t) = 2A \cos\frac{\varphi_2 - \varphi_1}{2} \cos\left(\omega t - kz + \frac{\varphi_1 + \varphi_2}{2}\right). \tag{189}$$

The resulting intensity is to be $\alpha \cdot I$, hence we get the equation

$$\alpha \cdot \frac{1}{2} Z\omega^2 A^2 = \frac{1}{2} Z\omega^2 \left(2A \cos\frac{\varphi_1 - \varphi_2}{2}\right)^2. \tag{190}$$

A lot of terms immediately cancel out, and we get the relation

$$\alpha = 4\cos^2 \frac{\varphi_1 - \varphi_2}{2}. \tag{191}$$

Let $\Delta\varphi = \varphi_1 - \varphi_2$. Now we just need to find the individual solutions. The phase shift $\Delta\varphi$ suffices to search in the interval $[0, 2\pi)$ (and thus $\frac{\Delta\varphi}{2} \in [0,\pi)$). We obtain successively:

(i) $\alpha = 0$. *Destructive interference.* We solve $0 = \cos^2(\Delta\varphi/2)$. From here, $\Delta\varphi = \pi$.

(ii) $\alpha = 1$. We solve $1/4 = \cos^2(\Delta\varphi/2)$. Thus, we need to satisfy $\cos(\Delta\varphi/2) = \pm 1/2$. This happens for $\Delta\varphi/2 \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}$, hence $\Delta\varphi \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$.

(iii) $\alpha = 2$. We solve $1/2 = \cos^2(\Delta\varphi/2)$ and hence the equation $\cos(\Delta\varphi/2) = \pm\frac{\sqrt{2}}{2}$. This happens for $\Delta\varphi/2 \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$. Thus, $\Delta\varphi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

(iv) $\alpha = 4$. *Constructive interference.* We solve $1 = \cos^2(\Delta\varphi/2)$, i.e., $\cos(\Delta\varphi/2) = \pm 1$, which gives $\Delta\varphi = 0$.

6 Wave packets, uncertainty relations, group velocity

Exercise 6.1. Find the form of the wave packet $f(t)$ for a spectrum shaped $B(\omega) = 0$ and

$$A(\omega) = \begin{cases} A_0 & \text{for } \omega \in [\omega_0 - \frac{\Delta\omega}{2}, \omega_0 + \frac{\Delta\omega}{2}], \\ 0 & \text{otherwise}. \end{cases} \tag{192}$$

Show how the spectrum width $\Delta\omega$ is related to the duration of the packet Δt defined here as the distance between the first zero points of the amplitude envelope of the wave packet.

Solution: The source of the wave packet is given by its spectral functions $A(\omega)$ and $B(\omega)$ through a continuous Fourier transform:

$$f(t) = \int_0^\infty A(\omega)\cos\omega t + B(\omega)\sin\omega t \, d\omega. \tag{193}$$

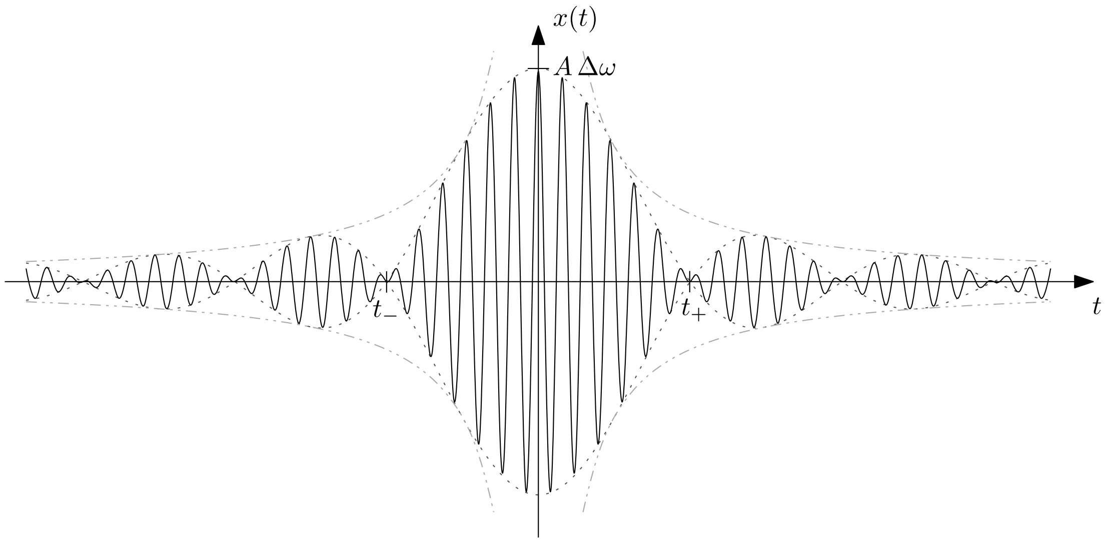
Functions $A(\omega)$ and $B(\omega)$ can be recovered from the function f by

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt. \tag{194}$$

Here we compute the integral

$$\begin{aligned} f(t) &= \int_0^\infty A(\omega)\cos\omega t\, d\omega = \int_{\omega_0-\frac{\Delta\omega}{2}}^{\omega_0+\frac{\Delta\omega}{2}} A_0 \cos\omega t\, d\omega \\ &= A_0 \frac{1}{t}[\sin\omega t]_{\omega_0-\frac{\Delta\omega}{2}}^{\omega_0+\frac{\Delta\omega}{2}} = \frac{A_0}{t}\left[\sin\left(\omega_0+\frac{\Delta\omega}{2}\right)t - \sin\left(\omega_0-\frac{\Delta\omega}{2}\right)t\right] \\ &= \frac{2A_0}{t}\cos\omega_0 t \sin\frac{\Delta\omega\cdot t}{2} \\ &= A_0\Delta\omega \frac{\sin\frac{\Delta\omega\cdot t}{2}}{\frac{\Delta\omega\cdot t}{2}}\cos\omega_0 t. \end{aligned} \tag{195}$$

Resulting time evolution of the signal see figure.



The resulting traveling wave in a non-dispersive medium would then be $\psi(z,t) = f(t-\frac{z}{v})$. The width of the wave packet Δt is obtained as the distance of the first zeros of the amplitude envelope $A_0\Delta\omega\frac{\sin\frac{\Delta\omega\cdot t}{2}}{\frac{\Delta\omega\cdot t}{2}}$, thus $\Delta t = t_+ - t_-$, where $t_\pm$ are the solutions of the equation $\sin\frac{\Delta\omega\cdot t_\pm}{2} = 0$, i.e., $\frac{\Delta\omega\cdot t_\pm}{2} = \pm\pi$. Hence, $t_\pm = \pm\frac{2\pi}{\Delta\omega}$ and from this

$$\Delta\omega\cdot\Delta t = 4\pi. \tag{196}$$

Exercise 6.2. Consider a rectangular pulse $f(t)$ of the form

$$f(t) = \begin{cases} A_0 & \text{for } \omega \in [-\frac{\Delta t}{2}, \frac{\Delta t}{2}], \\ 0 & \text{otherwise .} \end{cases} \tag{197}$$

Find its spectrum. Show how the pulse duration Δt is related to the width of its frequency spectrum $\Delta\omega$, here defined as the first zero of the frequency spectrum.

Solution: Similar to Fourier series, it is easy to see that for even functions $B(\omega) = 0$ and for odd $A(\omega) = 0$. The given function f is even, so it suffices to calculate $A(\omega)$:

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = \frac{2}{\pi}\int_{0}^{\infty} f(t)\cos\omega t\,dt = \frac{2}{\pi}\int_{0}^{\frac{\Delta t}{2}} A_0 cos\omega t\,dt \\ &= \frac{2A_0}{\pi\omega}[\sin\omega t]_0^{\frac{\Delta t}{2}} = \frac{A_0}{\pi}\frac{\sin\frac{\Delta t\,\omega}{2}}{\frac{\Delta t\,\omega}{2}}. \end{aligned} \tag{198}$$

The first zero of the spectral function thus occurs at point ω_0, where $\sin\frac{\Delta t\,\omega_0}{2} = 0$, i.e., for $\frac{\Delta t\,\omega_0}{2} = \pi$. Since here $\Delta\omega = \omega_0$, we obtain the relation

$$\Delta t \cdot \Delta\omega = 2\pi. \tag{199}$$

***Exercise 6.3.** Consider damped oscillation $f(t)$ in the form

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ e^{-\alpha t}\cos(\omega_0 t) & \text{otherwise}. \end{cases} \tag{200}$$

Find its spectrum.

Solution: The result is obtained by direct calculation, i.e.,

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = \frac{1}{\pi}\int_{0}^{\infty} e^{-\alpha t}\cos\omega_0 t\cos\omega t\,dt \\ &= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\alpha t}\big(\cos(\omega+\omega_0)t + \cos(\omega-\omega_0)t\big)\,dt \end{aligned} \tag{201}$$

Now we use the results of exercise 2.13, where we found

$$\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}. \tag{202}$$

Substituting into this formula, we get

$$A(\omega) = \frac{1}{2\pi}\left(\frac{\alpha}{\alpha^2+(\omega+\omega_0)^2} + \frac{\alpha}{\alpha^2+(\omega-\omega_0)^2}\right). \tag{203}$$

For the second spectral function, a similar calculation yields

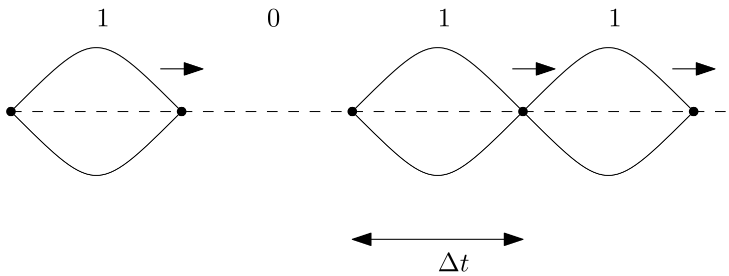
$$\begin{aligned} B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt = \frac{1}{\pi}\int_{0}^{\infty} e^{-\alpha t}\sin\omega t\cos\omega_0 t\,dt \\ &= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\alpha t}\big(\sin(\omega+\omega_0)t - \sin(\omega-\omega_0)t\big)\,dt \\ &= \frac{1}{2\pi}\left(\frac{\omega+\omega_0}{\alpha^2+(\omega+\omega_0)^2} - \frac{\omega-\omega_0}{\alpha^2+(\omega-\omega_0)^2}\right). \end{aligned} \tag{204}$$

Exercise 6.4. Wi-Fi covers a frequency range of 20 MHz (channel width). Estimate its transmission speed. Use the uncertainty relation.

Solution: We imagine the Wi-Fi signal as sending wave packets, which we can generate with the given frequency range. Thus, we know the spectrum width $\Delta\omega = 2\pi\Delta f$. The duration of the wave packet Δt satisfies the **uncertainty relation** $\Delta\omega \cdot \Delta t \geq \pi$. This gives us a lower estimate for Δt, i.e.,

$$\Delta t \geq \frac{\pi}{\Delta\omega}. \tag{205}$$

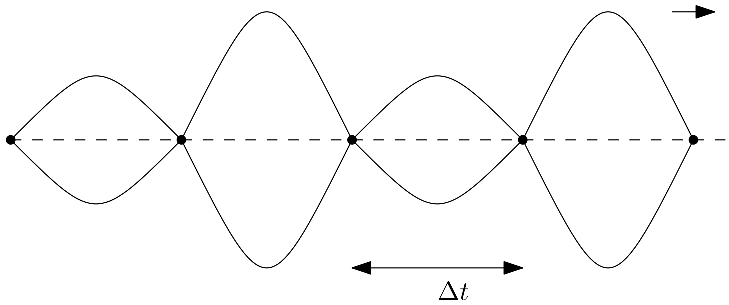
Sending data using the Wi-Fi signal is imagined as sending packets at regular time intervals, where sending = 1 and not sending = 0. To be clearly distinguishable, the shortest interval with which we can transmit them is Δt. See figure:



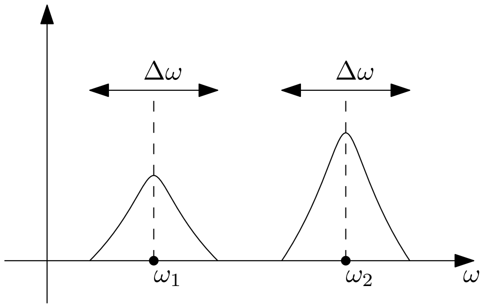
In one second, therefore, we can transmit at most $N = \frac{1}{\Delta t}$ bits. Thus, we get an upper estimate of the transmission speed $N = \frac{1}{\Delta t} \leq \frac{\Delta\omega}{\pi} = 2\Delta f = 40 \cdot 10^6\,\text{b/s} = 40\,\text{Mbit/s}$.

***Exercise 6.5.** Estimate the maximum frequency of trill f_{tr} of two tones separated by a semitone depending on the frequency of one of the tones in the trill f. Use the uncertainty relation. Why is trilling not performed on a tuba?

Solution: Trilling, i.e., the rapid alternating playing of two close tones, at a frequency f_{tr} can be imagined as alternately sending two wave packets. Assume that both tones sound for the same duration. The time width of both packets will then be $\Delta t = \frac{1}{2f_{tr}}$. See figure:



The spectrum of each packet will have a maximum around the respective angular frequencies of the tones ω_1 and ω_2. Both spectra will have a minimum width given by the uncertainty relations $\Delta\omega \geq \frac{\pi}{\Delta t}$. The resulting spectral function is their superposition:



The condition for distinguishing both tones will be that the spectral "peaks" do not overlap, which gives us the condition $\omega_2 - \omega_1 \geq \Delta\omega$. Overall, we get an estimate $\omega_2 - \omega_1 \geq 2\pi f_{tr}$. In terms of frequencies, then $f_2 - f_1 \geq f_{tr}$. The higher frequency is a semitone higher. Considering well-tempered tuning, then $f_2 = \sqrt[12]{2} \cdot f_1$. The resulting estimate is thus

$$f_{tr} \leq (\sqrt[12]{2} - 1) f_1. \tag{206}$$

For information, we have $\sqrt[12]{2} \approx 1.06$ and thus approximately $f_{tr} \leq 0.06 \cdot f_1$. The fundamental tone of the tuba is typically around 32Hz. The maximum frequency of trill on a tuba is then approximately 1.9 Hz!

Exercise 6.6. A linear dispersion relation is of the form $\omega = vk$, where $v = const$. Such a medium is called non-dispersive. Determine the phase and group velocity.

Solution: The phase velocity v_φ is obtained from the dispersion relation

$\omega = \omega(k)$ by $v_\varphi(k) = \frac{\omega(k)}{k}$. The group velocity v_g then by the derivative with respect to the wave number $v_g(k) = \frac{d\omega}{dk}(k)$. Here, thus $v_\varphi = v_g = v$.

Exercise 6.7. Determine the phase and group velocity for electromagnetic waves in plasma. This medium is described by the dispersion relation $\omega^2 = \omega_{min}^2 + c^2k^2$. Is the phase or group velocity greater than the speed of light? What does this mean?

Solution: We have

$$v_\varphi(k) = \frac{\omega}{k} = \frac{1}{k}\sqrt{\omega_{min}^2 + c^2k^2} = \sqrt{c^2 + \left(\frac{\omega_{min}}{k}\right)^2} = c \cdot \sqrt{1 + \left(\frac{\omega_{min}}{ck}\right)^2} > c. \tag{207}$$

The group velocity then

$$v_g(k) = \frac{d\omega}{dk} = \frac{c^2k}{\sqrt{\omega_{\min}^2 + c^2k^2}} = c \cdot \frac{1}{\sqrt{1 + (\frac{\omega_{min}}{ck})^2}} < c. \tag{208}$$

The magnitude of the phase velocity can be greater than c without any problems – this would correspond to the propagation speed of a monochromatic wave with constant amplitude – which does not carry any information. Conversely, the group velocity – the speed of propagation of wave packets – is less than the speed of light.

Exercise 6.8. Consider light in a material with a refractive index n, which is defined as $n = \frac{c}{v_\varphi}$. The refractive index in the material for a simple electron model is described as

$$n(\omega) = 1 + \frac{\alpha}{\omega_0^2 - \omega^2},$$

where $\alpha > 0$ and we consider only $\omega < \omega_0$. Determine the group velocity and show that it is less than the speed of light.

Solution: From the definition of the refractive index, we get the phase velocity as

$$v_\varphi(\omega) = \frac{c}{n(\omega)}. \tag{209}$$

Also, we know that the phase velocity is given by $v_\varphi = \omega/k$. We can thus easily express the dispersion relation $k = k(\omega)$ as

$$k(\omega) = \frac{\omega}{v_\varphi} = \frac{n(\omega)}{c}\omega. \tag{210}$$

This inverse form (as opposed to $\omega = \omega(k)$) is preferred since we have the function of refractive index $n(\omega)$ as a function of angular frequency ω. Then, we need to calculate the inverse of the group velocity (to be able to derive the inverse function):

$$\frac{1}{v_g(\omega)} = \frac{1}{\frac{d\omega}{dk}} = \frac{dk(\omega)}{d\omega}. \tag{211}$$

From this, we get

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left(n(\omega) + \omega\frac{dn(\omega)}{d\omega}\right). \tag{212}$$

In our specific case, $\frac{dn}{d\omega} = \frac{2\alpha\omega}{(\omega_0^2-\omega^2)^2}$, thus

$$\frac{dk}{d\omega}(\omega) = \frac{1}{c}\left(1 + \frac{\alpha(\omega_0^2+\omega^2)}{(\omega_0^2-\omega^2)^2}\right) > \frac{1}{c}. \tag{213}$$

Substituting, we obtain the final expression for the group velocity v_g as a function of ω:

$$v_g(\omega) = \frac{c}{1 + \frac{\alpha(\omega_0^2+\omega^2)}{(\omega_0^2-\omega^2)^2}} < c. \tag{214}$$

Exercise 6.9. Show that for light in a medium with refractive index $n(\lambda_0)$, where λ_0 is the wavelength of light *in vacuum*, it holds

$$\frac{1}{v_g} = \frac{1}{v_\varphi} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}. \tag{215}$$

Solution: The wavelength of light in vacuum is $\lambda_0 = \frac{2\pi}{k_0}$, where k_0 is the wave number in vacuum given by the dispersion relation $\omega = ck_0$. We can thus express the wavelength using the angular frequency ω as $\lambda_0 = \frac{2\pi c}{\omega}$. With this substitution, we get the refractive index function from variable ω: $n(\lambda_0(\omega)) = n(\frac{2\pi c}{\omega})$.

From the previous example, we know

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left(n(\omega) + \omega\frac{dn(\omega)}{d\omega}\right). \tag{216}$$

The first term on the right side is $\frac{1}{v_\varphi} = \frac{n}{c}$. In the second term, we express ω as a function of vacuum wavelength $\omega = \frac{2\pi c}{\lambda_0}$ and must now derive the refractive index function as a composite function:

$$\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega} = \frac{dn}{d\lambda_0}\frac{d}{d\omega}\left(\frac{2\pi c}{\omega}\right) = -\frac{2\pi c}{\omega^2}\frac{dn}{d\lambda_0}. \tag{217}$$

Substituting these results, we obtain the sought relation

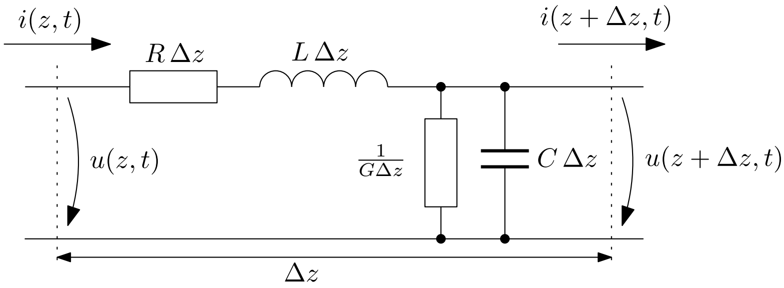
$$\frac{1}{v_g} = \frac{1}{v_\varphi} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}. \tag{218}$$

7 Reflections

***Exercise 7.1.** Derive the telegraph equations for the voltage and current waves $u(z,t)$ and $i(z,t)$ on a homogeneous transmission line in the form

$$-\frac{\partial u}{\partial z} = Ri + L\frac{\partial i}{\partial t}, \quad -\frac{\partial i}{\partial z} = Gu + C\frac{\partial u}{\partial t}, \tag{219}$$

where L is the **inductance per unit length** of the line, [L] = H.m^{-1}, C is the **capacitance**, [C] = F.m^{-1}, R is the resistance, [R] = Ω.m^{-1}, and G is the **conductance per unit length**, [G] = Ω^{-1}.m^{-1}. Derive the equations by analyzing the equivalent circuit of a segment of line length Δz:



Solution: When the chosen direction of current leads to a voltage drop across the resistor and inductor, we thus obtain the following equation:

$$u(z+\Delta z,t) = u(z,t) - R\Delta z \cdot i(z,t) - L\Delta z \cdot \frac{\partial i}{\partial t}(z,t). \tag{220}$$

Dividing by Δz and rearranging, we obtain

$$\frac{u(z+\Delta z,t) - u(z,t)}{\Delta z} = -R \cdot i(z,t) - L \cdot \frac{\partial i}{\partial t}(z,t). \tag{221}$$

Taking the limit as $\Delta z \to 0$ gives us the desired equation. Similarly, a decrease in current occurs, where part $G\Delta z \cdot u(z+\Delta z,t)$ leaks through the leakage resistance and $C\Delta z \cdot \frac{\partial u}{\partial t}(z+\Delta z,t)$ charges the capacitor. Thus, we obtain the equation

$$i(z+\Delta z,t) = i(z,t) - G\Delta z \cdot u(z+\Delta z,t) - C\Delta z \cdot \frac{\partial u}{\partial t}(z+\Delta z,t). \tag{222}$$

Dividing by Δz, we get

$$\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G \cdot u(z+\Delta z,t) - C \cdot \frac{\partial u}{\partial t}(z+\Delta z,t). \tag{223}$$

Thus, taking the limit as $\Delta z \to 0$ gives us the second equation.

Exercise 7.2. Consider an ideal homogeneous line, where $R = G = 0$. Show that the telegraph equations yield wave equations for the functions $u(z,t)$ and $i(z,t)$. Find the d'Alembert solution satisfying the original telegraph equations.

Hint #1: Consider an ansatz in the form of d'Alembert solutions

$$u(z,t) = F(z-vt) + G(z+vt), \quad i(z,t) = \alpha_1 F(z-vt) + \alpha_2 G(z+vt). \tag{224}$$

Hint #2: Substitute the d'Alembert solution for u into the telegraph equations and solve for i.

Note: The proportionality coefficient between the voltage and current wave is called the impedance Z.

Solution: The telegraph equations for an ideal line now take the form

$$-\frac{\partial u}{\partial z} = L\frac{\partial i}{\partial t}, \quad -\frac{\partial i}{\partial z} = C\frac{\partial u}{\partial t}. \tag{225}$$

Partially differentiate the first equation with respect to z and substitute from the second equation:

$$\frac{\partial^2 u}{\partial z^2} = -L\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial z}\right) = LC\frac{\partial^2 u}{\partial t^2}. \tag{226}$$

This is indeed the wave equation for u in the form

$$\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial z^2}, \tag{227}$$

where $v = \frac{1}{\sqrt{LC}}$. By a similar method, we obtain the wave equation for current i: differentiate the second equation with respect to z and substitute from the first:

$$\frac{\partial^2 i}{\partial z^2} = -C\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}\right) = LC\frac{\partial^2 i}{\partial t^2} \quad \rightarrow \quad \frac{\partial^2 i}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 i}{\partial z^2}. \tag{228}$$

Hint #1: These equations have a general solution in the d'Alembert form, for voltage $u(z,t) = F(z-vt) + G(z+vt)$ for any twice differentiable functions $F, G : \mathbb{R} \to \mathbb{R}$. The resulting wave equations turned out independent for functions u and i, but the original telegraph equations u and i bind them together! Therefore, we cannot simply take any d'Alembert solution for i! The solution for i is found by solving the telegraph equations after substituting the found form of u:

$$\frac{\partial i}{\partial t} = -\frac{1}{L}\big(F'(z-vt) + G'(z+vt)\big), \tag{229}$$

$$\frac{\partial i}{\partial z} = \sqrt{\frac{C}{L}}\big(F'(z-vt) - G'(z+vt)\big). \tag{230}$$

The second equation is easily solved by integrating with respect to z, yielding

$$i(z,t) = \sqrt{\frac{C}{L}}\big(F(z-vt) - G(z+vt)\big) + i_0(t). \tag{231}$$

Upon substituting into the first equation, we obtain $\frac{d}{dt}i_0(t) = 0$, from which we see that except for a constant current value $i_0 = \text{const.}$, which is uninteresting, we therefore choose $i_0 = 0$, i must be in the form

$$i(z,t) = \sqrt{\frac{C}{L}}\big(F(z-vt) - G(z+vt)\big). \tag{232}$$

We see that the impedance is given by the relationship $Z = \frac{U}{I} = \sqrt{\frac{L}{C}}$. Note, the left-traveling current wave has the opposite sign! The resulting forms of voltage and current waves on the telegraphic line are thus

$$u(z,t) = F(z-vt)+G(z+vt), \quad i(z,t) = \frac{1}{Z}F(z-vt)-\frac{1}{Z}G(z+vt), \quad v = \frac{1}{\sqrt{LC}}, \quad Z = \sqrt{\frac{L}{C}}. \tag{233}$$

Hint #2: If we substitute the prescribed ansatze into the telegraph equations, we get:

$$\begin{aligned} -F'(z-vt) - G'(z+vt) &= L\big(\alpha_1(-v)F'(z-vt) + \alpha_2 v G'(z+vt)\big), \\ -\big(\alpha_1 F'(z-vt) + \alpha_2 G'(z+vt)\big) &= C\big((-v)F'(z-vt) + vG'(z+vt)\big). \end{aligned} \tag{234}$$

After adjusting

$$0 = \big(1 - \alpha_1 vL\big)F'(z - vt) + \big(1 + \alpha_2 vL\big)G'(z + vt),$$
$$0 = \big(\alpha_1 - vC\big)F'(z - vt) + \big(\alpha_2 + vC\big)G'(z + vt). \tag{235}$$

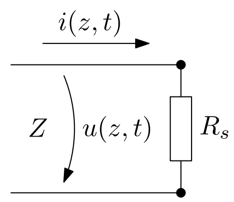
For general waves, $F'(x)$ and $G'(x)$ are linearly independent functions, so for their linear combination to equal zero, the corresponding coefficients standing by them must equal zero, leading to conditions for the constants α_1 and α_2:

$$\frac{1}{vL} = \alpha_1 = vC, \quad -\frac{1}{vL} = \alpha_2 = -vC. \tag{236}$$

After substituting $v = \frac{1}{\sqrt{LC}}$, we consistently get $\alpha_1 = \frac{1}{Z} = \sqrt{\frac{C}{L}} = -\alpha_2$. Thus, the same result as in the first guide.

Exercise 7.3. A homogeneous transmission line with impedance Z is terminated with a shunt resistor of size R_s. Find the reflection coefficient R for the voltage waves arriving along the line. Discuss the special cases $R_s = 0$ (short circuit), $R_s = +\infty$ (disconnected resistor) and $R = 0$ (nothing is reflected). Use harmonic traveling waves.

Solution: The transmission line is therefore terminated as follows:



We consider the harmonic incident u_d and reflected u_r voltage waves of forms:

$$u_d(z,t) = e^{i(\omega t - kz)}, \quad u_r(z,t) = Re^{i(\omega t + kz)}, \tag{237}$$

where $R \in \mathbb{C}$ is the reflection coefficient encoding the change in amplitude of the reflected wave (and possibly phase shift, if it comes out complex; $R = |R|e^{i\varphi}$). The corresponding incident i_d and reflected i_r current waves according to the results of the previous exercise are

$$i_d(z,t) = \frac{u_d(z,t)}{Z} = \frac{1}{Z}e^{i(\omega t - kz)}, \quad i_r(z,t) = -\frac{u_r(z,t)}{Z} = -\frac{1}{Z}R\, e^{i(\omega t + kz)}. \tag{238}$$

The function of the total voltage and current on the line then is $u(z,t) = u_d(z,t) + u_r(z,t)$ and $i(z,t) = i_d(z,t) + i_r(z,t)$. Consider the termination of the line at $z = 0$. The boundary condition of this termination is simply given by Ohm's law – the voltage drop on the terminating resistor is given by the product of its resistance R_s and the current flowing through it: $u(0,t) = R_s\, i(0,t),\ \forall t \in \mathbb{R}$. After substituting the forms of individual waves:

$$e^{i\omega t} + Re^{i\omega t} = R_s\left(\frac{1}{Z}e^{i\omega t} - \frac{1}{Z}R\, e^{i\omega t}\right). \tag{239}$$

We can cancel out the exponentials and have $1 + R = \frac{R_s}{Z}(1 - R)$. From this equation, we easily express the resulting reflection coefficient

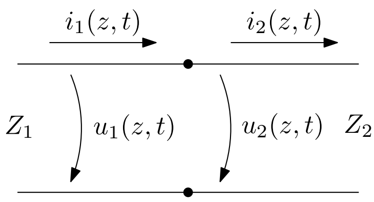
$$R = \frac{R_s - Z}{R_s + Z}. \tag{240}$$

For a short circuit ($R_s = 0$), we immediately get $R = -1$, and for a disconnected resistor ($R_s = +\infty$), it makes sense to write the reflection coefficient in the form $R = \frac{1-\frac{Z}{R_s}}{1+\frac{Z}{R_s}}$ and thus then $R = 1$. The condition for $R = 0$ is $R_s = Z$ – thus, it is necessary to terminate the line with a shunt resistor of the same size as the impedance of the given line.

Exercise 7.4. A homogeneous transmission line with impedance $Z_1 = 50\,\Omega$ is connected to a line with impedance $Z_2 = 100\,\Omega$. Find the transmission and reflection coefficients for voltage and current waves coming from the first line to the second. If a pulse with an amplitude of $15\,V$ hits the interface, what will be the amplitude of the transmitted and reflected waves?

Guide: Set up the appropriate connection conditions. Use harmonic traveling waves.

Solution: We proceed similarly as in the previous example. Now, however, we need to describe the voltage and current on two different lines, let's denote them $u_{1,2}$ and $i_{1,2}$.



Then on the left line, we have the incident and reflected wave, hence $u_1 = u_d + u_r$ and $i_1 = i_d + i_r$, on the right line we have the transmitted wave, $u_2 = u_p$ and $i_2 = i_p$. The forms of individual voltage and current waves are then

$$u_d(z,t) = e^{i(\omega t - k_1 z)}, \quad u_r(z,t) = R\, e^{i(\omega t + k_1 z)}, \quad u_p(z,t) = P\, e^{i(\omega t - k_2 z)}, \tag{241}$$

$$i_d(z,t) = \frac{1}{Z_1} e^{i(\omega t - k_1 z)}, \quad i_r(z,t) = -\frac{1}{Z_1} R\, e^{i(\omega t + k_1 z)}, \quad i_p(z,t) = \frac{1}{Z_2} P\, e^{i(\omega t - k_2 z)}, \tag{242}$$

where we introduced the reflection coefficient R and the transmission coefficient P and further the wave numbers on individual lines k_1 and k_2. The coefficients P and R are determined from the conditions of connection at the interface, let's place it at $z = 0$. At the interface, nothing special happens. The current and voltage at $z = 0$ must therefore continuously follow:

$$u_1(0,t) = u_2(0,t), \quad i_1(0,t) = i_2(0,t). \tag{243}$$

After substituting the forms of individual waves, we obtain equations

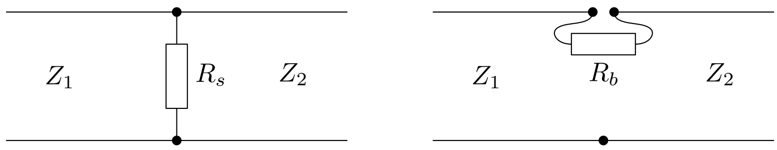
$$(1+R)e^{i\omega t} = P e^{i\omega t}, \quad \frac{1}{Z_1}(1-R)e^{i\omega t} = \frac{1}{Z_2} P e^{i\omega t}. \tag{244}$$

From the continuity of voltage, we have $1 + R = P$, from the equation for currents $1 - R = \frac{Z_1}{Z_2} P$. Solving these equations, we get

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad P = \frac{2Z_2}{Z_2 + Z_1}. \tag{245}$$

For the given values, we therefore have $R = (100 - 50)/(100 + 50) = 1/3$ and thus $P = 4/3$. The amplitudes of the transmitted and reflected waves will therefore be $20V$ and $5V$.

Exercise 7.5. A homogeneous transmission line with impedance $Z_1 = 50\,\Omega$ is connected to a line with impedance $Z_2 = 100\,\Omega$ in the following two ways:



Find the transmission and reflection coefficients for voltage waves for these two situations. Under what conditions does no reflection occur? Use harmonic traveling waves. Write down the connection conditions and solve them.

Solution: The difference compared to

the previous task is only in the conditions of connection. Here, a step change in current or voltage may occur. The voltage and current waves will look exactly the same as in the previous task.

Consider the first of the cases. Here, the voltage at the interface is continuous, but part of the current leaks through the shunt resistor:

$$u_1(0,t) = u_2(0,t), \quad i_1(0,t) = i_2(0,t) + i_s(t), \tag{246}$$

where the shunt current $i_s(t)$ is given by Ohm's law $i_s(t) = \frac{u_{1,2}(0,t)}{R_s}$ – in the fraction we can choose either u_1 or u_2, since these are equal at the point of connection; and since u_2 has a simpler expression than u_1, let's choose the shunt current in the form $i_s = \frac{u_2}{R_s}$. After substituting the forms of waves from the previous example, we get equations

$$1 + R = P, \quad \frac{1}{Z_1}(1 - R) = \frac{1}{Z_2}P + \frac{1}{R_s}P. \tag{247}$$

We adjust the second equation to the form $1 - R = Z_1(\frac{1}{Z_2} + \frac{1}{R_s})P$. Solving them gives us the coefficients

$$R = \frac{R_s(Z_2 - Z_1) - Z_1 Z_2}{Z_1 Z_2 + R_s Z_1 + R_s Z_2}, \quad P = \frac{2 Z_2 R_s}{Z_1 Z_2 + R_s Z_1 + R_s Z_2}. \tag{248}$$

If we require $R = 0$, then we get the condition $R_s(Z_2 - Z_1) = Z_1 Z_2$, i.e., $R_s = \frac{Z_1 Z_2}{Z_2 - Z_1}$. Physically only $R_s > 0$ is possible, and thus, to eliminate reflections in this connection of two lines, it is necessary that $Z_2 > Z_1$. Therefore, to eliminate reflections for the given impedance values, we must choose $R_s = \frac{50 \cdot 100}{100 - 50} = 100\,\Omega$.

Now consider the second case. Here, the currents are continuous, but the voltage has a jump due to the voltage drop on the lateral resistor R_b:

$$u_1(0,t) = u_2(0,t) + u_b(t), \quad i_1(0,t) = i_2(0,t), \tag{249}$$

where the voltage drop u_b is expressed using Ohm's law $u_b(t) = R_b\, i_{1,2}(0,t)$ – from the continuity of current at the point of connection, it again does not matter whether we use function i_1 or i_2, we choose again for a simpler form i_2. Substitute into the conditions of connection the forms of voltage and current waves:

$$1 + R = P + R_b \frac{1}{Z_2} P, \quad \frac{1}{Z_1} - \frac{1}{Z_1} R = \frac{1}{Z_2} P. \tag{250}$$

After adjustment, we have equations $1 + R = (1 + \frac{R_b}{Z_2})P$, $1 - R = \frac{Z_1}{Z_2}P$. Solving them gives us the coefficients

$$R = \frac{Z_2 - Z_1 + R_b}{Z_1 + Z_2 + R_b}, \quad P = \frac{2Z_2}{Z_1 + Z_2 + R_b}. \tag{251}$$

Conditions for the disappearance of reflections, $R = 0$, leads to the size of the lateral resistor $R_b = Z_1 - Z_2$. Again, from the requirement $R_b > 0$, it follows that in this configuration, we can eliminate reflections only for lines satisfying $Z_1 > Z_2$. For the given impedance values, therefore, this configuration cannot be used.

Exercise 7.6. Consider three media interconnected through two interfaces, one at $z = 0$ and the other at $z = L$. Let's denote the amplitude transmission and reflection coefficients as T_{ij} and R_{ij} representing the transmission and reflection coefficients when transitioning from the i-th to the j-th medium. The wave numbers in the individual media are k_1, k_2, k_3. Consider a harmonic incident wave of the form $Ae^{i(\omega t - k_1 z)}$. Find the total reflection coefficient $R \in \mathbb{C}$, i.e., the total reflected wave of the form $ARe^{i(\omega t + k_1 z)} = A|R|e^{i(\omega t + k_1 z + \varphi)}$ resulting from an infinite superposition of reflected waves between two interfaces. Require the continuity of phase functions of individual waves at the interfaces.

At the end, specialize the result, considering the relations $1 + R_{ij} = T_{ij}$ and $R_{ij} = -R_{ji}$.

Solution: Here we have directly given the amplitude transmission and reflection coefficients at individual interfaces. What remains is to consider what continuity of phase functions at the interface means. For simplicity, consider one interface at $z = L$ between media with wave numbers k_1 and k_2. Considering an incident, reflected, and transmitted wave of the forms

$$\psi_d(z,t) = e^{i(\omega t - k_1 z + \phi_d)}, \qquad \psi_r(z,t) = R_{12}e^{i(\omega t + k_1 z + \phi_r)}, \qquad \psi_p(z,t) = T_{12}e^{i(\omega t - k_2 z + \phi_p)}, \tag{252}$$

where we consider general phase shifts ϕ_d, ϕ_r, and ϕ_p in individual waves. The phase functions of individual waves are thus in the form

$$\varphi_d(z,t) = \omega t - k_1 z + \phi_d, \qquad \varphi_r(z,t) = \omega t + k_1 z + \phi_r, \qquad \varphi_p(z,t) = \omega t - k_2 z + \phi_p. \tag{253}$$

The requirement of continuity of these phases upon reflection and transmission at coordinate $z = L$ leads to requirements

$$\varphi_d(L,t) = \varphi_r(L,t), \qquad \varphi_d(L,t) = \varphi_p(L,t), \tag{254}$$

$$\omega t - k_1 L + \phi_d = \omega t + k_1 L + \phi_r, \qquad \omega t - k_1 L + \phi_d = \omega t - k_2 L + \phi_p. \tag{255}$$

From these relations, we can easily express the phase shifts of the reflected and transmitted waves:

$$\phi_r = \phi_d - 2k_1 L, \qquad \phi_p = \phi_d + (k_2 - k_1)L. \tag{256}$$

Upon reflection away from $z = 0$, there are phase shifts in the reflected and transmitted waves! These need to be added at each reflection or transmission, as we will see later. On the other hand, for $z = 0$, nothing needs to be resolved, as $\phi_r = \phi_d$ and $\phi_p = \phi_d$ – the phase shifts remain the same as for the incident wave.

We could also have started from the fact that nothing special happens for the interface at $z = 0$ (i.e., we do not need to deal with phase) and introduce a substitution $z' = z + L$ to move the interface to $z' = L$, then we would have

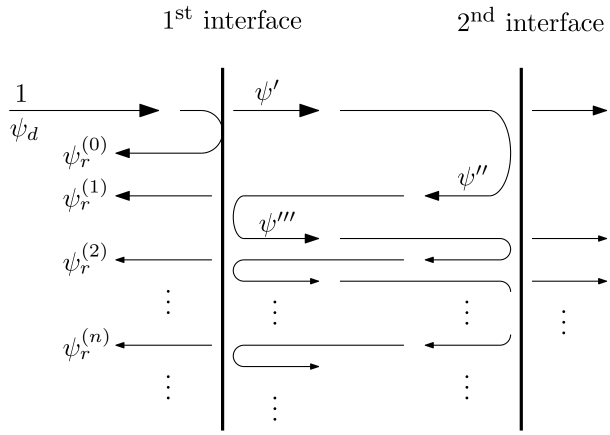
$$\psi_d(z,t) = e^{i(\omega t - k_1 z)} = e^{i(\omega t - k_1 z' + k_1 L)}, \tag{257}$$

$$\psi_r(z,t) = R_{12}\, e^{i(\omega t + k_1 z)} = R_{12}\, e^{i(\omega t + k_1 z' - k_1 L)} = R_{12}\, e^{-2ik_2 L} e^{i(\omega t - k_1 z' + k_1 L)}, \tag{258}$$

$$\psi_p(z,t) = T_{12}\, e^{i(\omega t - k_2 z)} = T_{12}\, e^{i(\omega t - k_2 z' - k_2 L)} = T_{12}\, e^{i(k_2 - k_1)L} e^{i(\omega t - k_1 z' + k_1 L)}, \tag{259}$$

where in the last adjustment for the reflected and transmitted wave, we factored out the phase shift compared to the incident wave.

Proceed to solving the task with two interfaces. Let $A = 1$ for simplicity, saving us writing. Now denote $\psi_r^{(n)}(z,t)$ the wave that reflected back into the first medium and reflected exactly n times at the second interface. See the figure:



The individual waves are obtained by accounting for the respective amplitude coefficients at individual interfaces and adding a phase $\Delta\varphi = -2k_2L$ for each reflection at the second interface. We have

$$\psi_r^{(0)}(z,t) = R_{12}\, e^{i(\omega t + k_1 z)}. \tag{260}$$

The wave $\psi_r^{(1)}(z,t)$, which reflects exactly once at the second interface, is obtained from the wave transmitted through the first interface $\psi'(z,t) = T_{12}\, e^{i(\omega t - k_2 z)}$, by reflection from the second interface $\psi''(z,t) = R_{23}T_{12}\, e^{i(\omega t + k_2 z - 2k_2 L)}$ and finally by passing back through the first interface, thus

$$\psi_r^{(1)}(z,t) = T_{21}R_{23}T_{12}\, e^{i(\omega t + k_1 z - 2k_2 L)}. \tag{261}$$

Continuing similarly, to obtain $\psi_r^{(2)}(z,t)$, we take the wave $\psi''(z,t)$, reflect it to the right from the first interface (getting wave $\psi'''(z,t) = R_{21}R_{23}T_{12}e^{i(\omega t - k_2 z - 2k_2 L)}$), reflect it from the second interface (adding $R_{23}e^{-2ik_2L}$), and let it pass back into the first medium (adding T_{21}). We get

$$\psi_r^{(2)}(z,t) = T_{21}R_{21}R_{23}^2T_{12}\, e^{i(\omega t + k_1 z - 4k_2 L)}. \tag{262}$$

From here, we can deduce a general formula – for each "inner reflection," an additional factor $R_{21}R_{23}e^{-2ik_2L}$ is added. We get

$$\psi_r^{(n)}(z,t) = T_{21}R_{21}^{n-1}R_{23}^{n}T_{12}\, e^{i(\omega t + k_1 z - 2nk_2 L)}. \tag{263}$$

To obtain the total reflected wave $\psi_r(z,t)$, we must sum the individual contributions, $\psi_r(z,t) = \sum_{k=0}^{+\infty} \psi_r^{(k)}(z,t)$. We thus get a series

$$\psi_r(z,t) = \psi_r^{(0)}(z,t) + \sum_{k=1}^{+\infty} \psi_r^{(k)}(z,t) = \left[R_{12} + \frac{T_{21}T_{12}}{R_{21}} \sum_{k=1}^{\infty} \left(R_{21}R_{23}e^{-2ik_2L} \right)^k \right] e^{i(\omega t + k_1 z)}. \tag{264}$$

The expression in square brackets is the sought total reflection coefficient R. We greatly simplify the expression by summing the geometric series using the formula $\sum_{k=1}^{+\infty} x^k = \left(\sum_{k=0}^{+\infty} x^k\right) - 1 =$

$\frac{1}{1-x} - 1 = \frac{x}{1-x}$ for $x = R_{21}R_{23}e^{-2ik_2L}$:

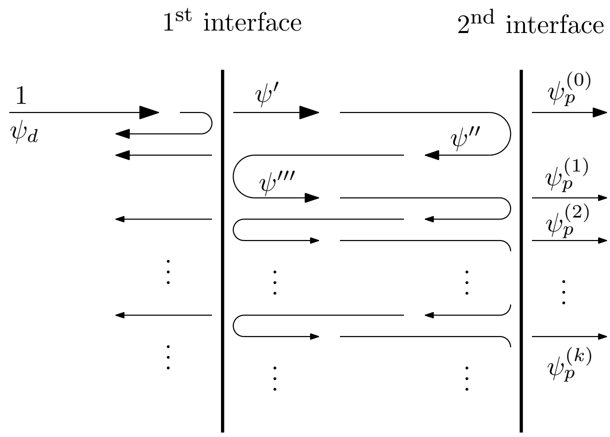
$$R = R_{12} + \frac{T_{21}T_{12}}{R_{21}} \frac{R_{21}R_{23}e^{-2ik_2L}}{1 - R_{21}R_{23}e^{-2ik_2L}} = R_{12} + \frac{T_{21}T_{12}R_{23}e^{-2ik_2L}}{1 - R_{21}R_{23}e^{-2ik_2L}}. \tag{265}$$

Considering at the end the relations $R_{21} = -R_{12}$, $T_{12} = 1+R_{12}$, $T_{21} = 1-R_{12}$ we get an expression for the total reflection coefficient R only dependent on the reflectivities at the individual interfaces R_{12} and R_{23}:

$$R = R_{12} + \frac{(1 - R_{12}^2)R_{23}e^{-2ik_2L}}{1 + R_{12}R_{23}e^{-2ik_2L}} = \frac{R_{12} + R_{23}e^{-2ik_2L}}{1 + R_{12}R_{23}e^{-2ik_2L}}. \tag{266}$$

***Exercise 7.7.** Find the total transmission coefficient $T \in \mathbb{C}$, i.e., the total transmitted wave $ATe^{i(\omega t - k_3 z)}$ for the situation described in the previous exercise.

Solution: The solution process will be very similar to the previous example. We already know how phase behaves during transmission and reflection at the interface at $z = L$. Now denote $\psi_p^{(n)}$ the wave that transmitted into the third medium, but on its way reflected exactly n times at the second interface, see the figure:



The wave $\psi_p^{(0)}(z,t)$ is created from the wave $\psi'(z,t) = T_{12}\, e^{i(\omega t - k_2 z)}$ (see previous example) by passing through the second interface:

$$\psi_p^{(0)}(z,t) = T_{23}T_{12}e^{i(k_3-k_2)L}e^{i(\omega t - k_3 z)}, \tag{267}$$

where, in addition to the amplitude coefficient T_{23}, we also added the phase shift $\Delta\varphi = (k_3 - k_2)L$ for passing through the interface at $z = L$ (see the first part of the previous example). Similarly, we obtain the wave $\psi_p^{(1)}(z,t)$ from the wave $\psi'''(z,t) = R_{21}R_{23}T_{12}e^{i(\omega t - k_2 z - 2k_2 L)}$:

$$\psi_p^{(1)}(z,t) = T_{23}R_{21}R_{23}T_{12}e^{-2ik_2L}e^{i(k_3-k_2)L}e^{i(\omega t - k_3 z)}. \tag{268}$$

Each subsequent transmitted wave will have one more reflection on the internal sides of the interfaces, hence

$$\psi_p^{(2)}(z,t) = T_{23}R_{21}^2R_{23}^2T_{12}(e^{-2ik_2L})^2e^{i(k_3-k_2)L}e^{i(\omega t - k_3 z)}. \tag{269}$$

We can easily deduce the expression for the k-th transmitted wave

$$\psi_p^{(k)}(z,t) = T_{23}R_{21}^kR_{23}^kT_{12}(e^{-2ik_2L})^ke^{i(k_3-k_2)L}e^{i(\omega t - k_3 z)}. \tag{270}$$

Now, to obtain the total transmitted wave $\psi_p(z,t)$,

we must sum the individual contributions:

$$\psi_p(z,t) = \sum_{k=0}^{+\infty} \psi_p^{(k)}(z,t) = \left[T_{12} T_{23} e^{i(k_3-k_2)L} \sum_{k=0}^{+\infty} \left(R_{21} R_{23} e^{-2ik_2 L} \right)^k \right] e^{i(\omega t - k_3 z)}. \tag{271}$$

The expression in square brackets is our sought total transmission coefficient T. We simplify the expression again by summing an infinite geometric series, $\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x}$ for $x = R_{21} R_{23} e^{-2ik_2 L}$, hence

$$T = \frac{T_{12} T_{23} e^{i(k_3-k_2)L}}{1 - R_{21} R_{23} e^{-2ik_2 L}}. \tag{272}$$

Considering at the end $R_{21} = -R_{12}$, $T_{12} = 1 + R_{12}$, and $T_{23} = 1 + R_{23}$ we get

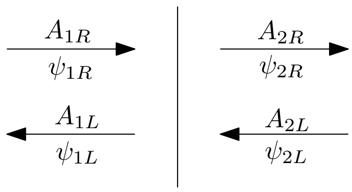
$$T = \frac{(1+R_{12})(1+R_{23}) e^{i(k_3-k_2)L}}{1 + R_{12} R_{23} e^{-2ik_2 L}}. \tag{273}$$

Exercise 7.8. A transition matrix $\mathbb{D}$ is given. Find the transmission P and reflection R coefficients for a wave coming from the first (left) medium into the second (right) one.

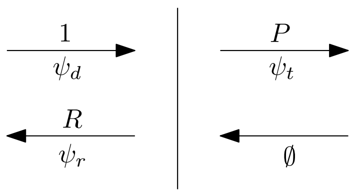
Solution: The transfer matrix $\mathbb{D} \in \mathbb{C}^{2,2}$ is defined by the equation

$$\begin{pmatrix} A_{1R} \\ A_{1L} \end{pmatrix} = \mathbb{D} \begin{pmatrix} A_{2R} \\ A_{2L} \end{pmatrix}, \tag{274}$$

where A_{1R} and A_{1L} are the amplitudes of harmonic traveling waves moving to the right (respectively, to the left) in the medium to the left of the interface, A_{2R} and A_{2L} similarly to the right of the interface, see the figure.



The matrix $\mathbb{D}$ would be obtained by solving the conditions of wave function connection at the interface. To find P and R, consider $A_{1R} = 1$, $A_{1L} = R$, $A_{2R} = P$, and $A_{2L} = 0$:



Thus, we get

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D} \begin{pmatrix} P \\ 0 \end{pmatrix}, \tag{275}$$

Let's denote the known elements of the matrix $\mathbb{D}$ as

$$\mathbb{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}. \tag{276}$$

By substituting into the matrix equation, we thus get a system of equations

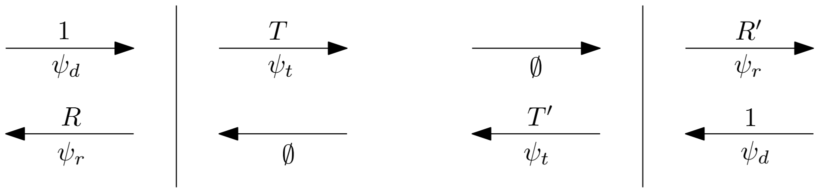
$$1 = d_{11}P, \quad R = d_{21}P. \tag{277}$$

From here, $P = 1/d_{11}$ and $R = d_{21}/d_{11}$.

Exercise 7.9. Given are the transmission and reflection coefficients, T and R, for a wave coming from the first to the second medium, and coefficients T' and R' for a wave coming from the second medium to the first. Find the corresponding form of the transfer matrix $\mathbb{D}$. Specialize the form of this matrix assuming that

$R' = -R$ and $1 + R = T$ (and $1 + R' = T'$).

Solution: The coefficients T, R, T', and R' determine the amplitudes of waves in the following two situations:



That is, from the definition of the transfer matrix $\mathbb{D}$ (see the previous example), we have matrix equations

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \mathbb{D} \begin{pmatrix} T \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ T' \end{pmatrix} = \mathbb{D} \begin{pmatrix} R' \\ 1 \end{pmatrix}. \tag{278}$$

Expanding into components (with the same notation for matrix elements as in the previous example), we get

$$\begin{aligned} 1 &= d_{11}T & \qquad 0 &= d_{11}R' + d_{12} \\ R &= d_{21}T & \qquad T' &= d_{21}R' + d_{22}. \end{aligned} \tag{279}$$

From the left set of equations, we easily find $d_{11} = \frac{1}{T}$, $d_{21} = \frac{R}{T}$, by substituting into the right equations we immediately have $d_{12} = -\frac{R'}{T}$ and $d_{22} = T' - \frac{RR'}{T} = \frac{TT'-RR'}{T}$. Overall,

$$\mathbb{D} = \frac{1}{T} \begin{pmatrix} 1 & -R' \\ R & TT' - RR' \end{pmatrix}. \tag{280}$$

For relations $R' = -R$, $T = 1 + R$ and $T' = 1 + R' = 1 - R$, we get

$$\mathbb{D} = \frac{1}{1+R} \begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}. \tag{281}$$

Exercise 7.10. Consider the interface defined in exercise 7.6. Write the transfer matrices for each interface by analyzing the reflections of harmonic waves at each interface using the result of exercise 7.9. Assemble these matrices and using the result of example 7.8 verify that the total reflection coefficient R for two interfaces comes out the same as in exercise 7.6.

Solution: Finding the transfer matrix $\mathbb{D}_1$ for the interface at $z = 0$ is easy. Here, there are no phase shifts, and it suffices to directly use the result of the previous exercise and just substitute the correct amplitude coefficients:

$$\mathbb{D}_1 = \frac{1}{T_{12}} \begin{pmatrix} 1 & -R_{21} \\ R_{12} & T_{12}T_{21} - R_{12}R_{21} \end{pmatrix}. \tag{282}$$

For the interface at $z = L$, we must remember that the requirement for phase continuity implies an additional phase shift upon reflection or passage through this interface. In the first part of exercise 7.6, we derived that for an incoming wave from the left on the second interface, the reflection adds a phase e^{-2ik_2L} and the passage $e^{i(k_3-k_2)L}$. Thus, the coefficients R and T here take the form

$$R = R_{23}e^{-2ik_2L}, \qquad T = T_{23}e^{i(k_3-k_2)L}. \tag{283}$$

We still need to determine the forms of coefficients R' and T' for a wave incoming on the second interface from the right. Again, these will not be merely the amplitude coefficients R_{32} and T_{32}, but it is necessary to add phases for reflection, respectively, passage, at the interface $z = L$. Let us briefly perform the same analysis as in the first part of example 7.6. Considering an incoming wave from the right on the second interface and reflected and passed wave forms

$$\psi_d(z,t) = e^{i(\omega t + k_3 z)}, \qquad \psi_r(z,t) = R_{32}e^{i(\omega t - k_3 z + \phi_r)}, \qquad \psi_p(z,t) = T_{32}e^{i(\omega t + k_2 z + \phi_p)}, \tag{284}$$

The requirement for phase function continuity of these waves upon reflection and passage at the coordinate $z = L$ leads to the requirements

$$\varphi_d(L,t) = \varphi_r(L,t), \qquad \varphi_d(L,t) = \varphi_p(L,t), \tag{285}$$

$$\omega t + k_3 L = \omega t - k_3 L + \phi_r, \qquad \omega t + k_3 L = \omega t + k_2 L + \phi_p. \tag{286}$$

From these relations, we easily express the phase shifts of the reflected and passed wave:

$$\phi_r = 2k_3L, \qquad \phi_p = (k_3 - k_2)L. \tag{287}$$

The reflection and transmission coefficients for a wave incoming from the right thus are

$$R' = R_{32}e^{2ik_3L}, \qquad T' = T_{32}e^{i(k_3-k_2)L}. \tag{288}$$

The transfer matrix for the second interface then has the form (again according to the result of exercise 7.9):

$$\mathbb{D}_2 = \frac{1}{T_{23}e^{i(k_3-k_2)L}} \begin{pmatrix} 1 & -R_{32}e^{2ik_3L} \\ R_{23}e^{-2ik_2L} & (T_{23}T_{32} - R_{23}R_{32})e^{2i(k_3-k_2)L} \end{pmatrix}. \tag{289}$$

We obtain the overall transfer matrix by multiplying the individual matrices, $\mathbb{D} = \mathbb{D}_1\mathbb{D}_2$

and according to the result of exercise 7.8 the total reflection coefficient $R = \frac{d_{21}}{d_{11}}$. Thus, we do not need to calculate all elements of the matrix $\mathbb{D}$, but only two:

$$d_{11} = \frac{1}{T_{12}T_{23}e^{i(k_3-k_2)L}} \left[1 - R_{21}R_{23}e^{-2ik_2L}\right],$$

$$d_{21} = \frac{1}{T_{12}T_{23}e^{i(k_3-k_2)L}} \left[R_{12} + R_{23}e^{-2ik_2L}(T_{12}T_{21} - R_{12}R_{21})\right]. \tag{290}$$

Upon substituting into the relationship for R and making a minor adjustment:

$$R = \frac{d_{21}}{d_{11}} = \frac{R_{12}(1 - R_{21}R_{23}e^{-2ik_2L}) + R_{23}T_{12}T_{21}e^{-2ik_2L}}{1 - R_{21}R_{23}e^{-2ik_2L}} = R_{12} + \frac{R_{23}T_{12}T_{21}e^{-2ik_2L}}{1 - R_{21}R_{23}e^{-2ik_2L}}. \tag{291}$$

And that is precisely the result of example 7.6.

***Exercise 7.11.** The same as in the previous exercise but for the overall transmission coefficient T.

Solution: We have already found the transfer matrices for individual interfaces in the previous exercise. According to exercise 7.8, the transmission coefficient is given by the relation $T = \frac{1}{d_{11}}$. We have already calculated this matrix element in the previous example. Thus, we have

$$T = \frac{1}{d_{11}} = \frac{T_{12}T_{23}e^{i(k_3-k_2)L}}{1 - R_{21}R_{23}e^{-2ik_2L}}. \tag{292}$$

And that is precisely the result of exercise 7.7.

***Exercise 7.12.** Consider connecting two strings with the same tension at $z = L$. The transfer matrix is

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} (1+\frac{k_2}{k_1})e^{i(k_1-k_2)L} & (1-\frac{k_2}{k_1})e^{i(k_1+k_2)L} \\ (1-\frac{k_2}{k_1})e^{-i(k_1+k_2)L} & (1+\frac{k_2}{k_1})e^{-i(k_1-k_2)L} \end{pmatrix}. \tag{293}$$

Find the transition matrix for two interfaces of three strings. The interfaces are at $z = 0$ and $z = L$. Find the overall reflection coefficient R.

Solution: If $\mathbb{D}_1$ and $\mathbb{D}_2$ are the transfer matrices at the two interfaces, we obtain the transfer matrix from the first to the third interface by simply multiplying, thus $\mathbb{D} = \mathbb{D}_1\mathbb{D}_2$. Here, we have

$$\mathbb{D}_1 = \frac{1}{2}\begin{pmatrix} 1+\frac{k_2}{k_1} & 1-\frac{k_2}{k_1} \\ 1-\frac{k_2}{k_1} & 1+\frac{k_2}{k_1} \end{pmatrix}, \quad \mathbb{D}_2 = \frac{1}{2}\begin{pmatrix} (1+\frac{k_3}{k_2})e^{i(k_2-k_3)L} & (1-\frac{k_3}{k_2})e^{i(k_2+k_3)L} \\ (1-\frac{k_3}{k_2})e^{-i(k_2+k_3)L} & (1+\frac{k_3}{k_2})e^{-i(k_2-k_3)L} \end{pmatrix}. \tag{294}$$

To compute R, we need to know only d_{11} and d_{21}. We get

$$\begin{aligned} d_{11} &= \frac{1}{4}\left[\left(1+\frac{k_2}{k_1}\right)\left(1+\frac{k_3}{k_2}\right)e^{i(k_2-k_3)L} + \left(1-\frac{k_2}{k_1}\right)\left(1-\frac{k_3}{k_2}\right)e^{-i(k_2+k_3)L}\right] \\ &= \frac{1}{4}e^{i(k_2-k_3)L}\frac{k_1+k_2}{k_1}\frac{k_2+k_3}{k_2}\left(1+\frac{k_1-k_2}{k_1+k_2}\frac{k_2-k_3}{k_2+k_3}e^{-2ik_2L}\right). \end{aligned} \tag{295}$$

For the coefficient d_{21}, we get

$$\begin{aligned} d_{21} &= \frac{1}{4}\left[\left(1-\frac{k_2}{k_1}\right)\left(1+\frac{k_3}{k_2}\right)e^{i(k_2-k_3)L} + \left(1+\frac{k_2}{k_1}\right)\left(1-\frac{k_3}{k_2}\right)e^{-i(k_2+k_3)L}\right] \\ &= \frac{1}{4}e^{i(k_2-k_3)L}\frac{k_1+k_2}{k_1}\frac{k_2+k_3}{k_2}\left(\frac{k_1-k_2}{k_1+k_2}+\frac{k_2-k_3}{k_2+k_3}e^{-2ik_2L}\right). \end{aligned} \tag{296}$$

Dividing both expressions thus provides a formula in the form

$$R = \frac{d_{21}}{d_{11}} = \frac{\frac{k_1-k_2}{k_1+k_2}+\frac{k_2-k_3}{k_2+k_3}e^{-2ik_2L}}{1+\frac{k_1-k_2}{k_1+k_2}\frac{k_2-k_3}{k_2+k_3}e^{-2ik_2L}}. \tag{297}$$

Noticing that if we denote $R_{ij} = \frac{k_i-k_j}{k_i+k_j}$ (which exactly comes out for connecting strings with the same tension), we get the result of example 7.6.

8 Waves in Space

Exercise 8.1. Show that a harmonic traveling plane wave of form $\psi(\vec{r},t) = Ae^{i(\omega t-\vec{k}\cdot\vec{r})}$, where $A \in \mathbb{C}$, $\vec{k} = k\vec{n}$, $|\vec{n}| = 1$ and $k > 0$, satisfies the three-dimensional wave equation assuming a certain dispersion relation is met. Find this relation.

Solution: The three-dimensional wave equation for the function $\psi = \psi(\vec{r}, t)$ is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \Delta \psi \equiv v^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right), \tag{298}$$

where $v = \text{const}$. Substituting into the left side, we get

$$\frac{\partial^2 \psi(\vec{r}, t)}{\partial t^2} = -\omega^2 A e^{i(\omega t - \vec{k} \cdot \vec{r})} = -\omega^2 \psi(\vec{r}, t). \tag{299}$$

On the right side, we have the sum of three second derivatives. It suffices to calculate one of them, the rest we can guess[2]:

$$\frac{\partial^2 \psi(\vec{r}, t)}{\partial x^2} = -k_x^2 A e^{i(\omega t - \vec{k} \cdot \vec{r})}, \quad \text{thus} \quad \frac{\partial^2 \psi(\vec{r}, t)}{\partial x_i^2} = -k_i^2 A e^{i(\omega t - \vec{k} \cdot \vec{r})} = -k_i^2 \psi(\vec{r}, t). \tag{300}$$

Substituting into the wave equation thus leads to

$$\omega^2 = v^2 (k_x^2 + k_y^2 + k_z^2) = v^2 |\vec{k}|^2 = v^2 k^2. \tag{301}$$

We obtain a dispersion relation in the form $\omega = vk$. Therefore, if ω and $|\vec{k}|$ satisfy the resulting dispersion relation, the given wave is a solution to the 3D wave equation.

Exercise 8.2. Find the dispersion relation of the wave equation forms

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \Delta \psi - \omega_0^2 \psi, \qquad \frac{\partial^2 \psi}{\partial t^2} = v^2 \Delta \psi - \alpha \Delta(\Delta \psi). \tag{302}$$

for a

harmonic traveling plane wave.

Solution: We proceed as in the previous example. In the first case, the right side after substitution modifies to $-(v^2 k^2 + \omega_0^2) A e^{i(\omega t - \vec{k} \cdot \vec{r})}$. The dispersion relation is thus

$$\omega^2 = v^2 k^2 + \omega_0^2. \tag{303}$$

In the second case, on the right side, we get $-(v^2 k^2 + \alpha k^4) A e^{i(\omega t - \vec{k} \cdot \vec{r})}$, since $\Delta(\Delta \psi) = -k^2 \Delta \psi = (-k^2)^2 \psi$. The dispersion relation gives

$$\omega^2 = v^2 k^2 + \alpha k^4. \tag{304}$$

Exercise 8.3. Show that a traveling plane wave of form $\psi(\vec{r}, t) = F(\vec{n} \cdot \vec{r} - vt)$, where $|\vec{n}| = 1$ and $F : \mathbb{R} \to \mathbb{R}$ is an arbitrary twice differentiable function, satisfies the three-dimensional wave equation.

[2] Or we can calculate it properly:

$$\frac{\partial \psi}{\partial x_i} = \frac{\partial(-i\vec{k} \cdot \vec{r})}{\partial x_i} A e^{i(\omega t - \vec{k} \cdot \vec{r})} = -i k_i A e^{i(\omega t - \vec{k} \cdot \vec{r})}, \qquad \frac{\partial \vec{k} \cdot \vec{r}}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial k_j x_j}{\partial x_i} = \sum_{j=1}^{3} k_j \delta_{ji} = k_i.$$

Then

$$\Delta \psi = \sum_{i=1}^{3} \frac{\partial^2 \psi}{\partial x_i^2} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{\partial \psi}{\partial x_i} \right) = \sum_{i=1}^{3} (-i k_i)^2 A e^{i(\omega t - \vec{k} \cdot \vec{r})} = -|\vec{k}|^2 \psi.$$

Solution: We again simply verify by substitution into the wave equation. The left side of the equation gives:

$$\frac{\partial^2\psi}{\partial t^2}(\vec{r},t) = v^2 F''(\vec{n}\cdot\vec{r} - vt). \tag{305}$$

On the right side, analogously as in example 8.1 (thus using $\frac{\partial \vec{n}\cdot\vec{r}}{\partial x_i} = n_i$):

$$v^2\Delta\psi(\vec{r},t) = v^2(n_x^2 + n_y^2 + n_z^2)F''(\vec{n}\cdot\vec{r} - vt). \tag{306}$$

But $|\vec{n}| = 1$, which gives the sought result.

***Exercise 8.4.** Show that a spherical wave of form $\psi(\vec{r},t) = \frac{1}{r}e^{i(\omega t - kr)}$ satisfies the wave equation assuming a linear dispersion relation $\omega = vk$.

Solution: In the lecture, you wrote down Laplace's operator for a function that depends only on the distance r from the origin (this simpler approach is at the end of this solution). We'll try the "infantry method" in Cartesian coordinates. The most difficult task is to calculate the Laplacian applied to a scalar function $\varphi(\vec{r}) = \frac{1}{r}e^{-ikr}$. This function is the product of functions $f(\vec{r}) = \frac{1}{r}$ and $g(\vec{r}) = e^{-ikr}$. Generally, the rule applies

$$\Delta(fg) = \operatorname{div}\operatorname{grad}(fg) = \operatorname{div}(f\operatorname{grad} g + g\operatorname{grad} f) = f\Delta g + g\Delta f + 2(\operatorname{grad} f)\cdot(\operatorname{grad} g). \tag{307}$$

In exercises on electricity and magnetism, you calculated that $\operatorname{grad}(r^\alpha) = \alpha\, r^{\alpha-2}\vec{r}$. Hence easily

$$\operatorname{grad} f = -r^{-3}\vec{r}. \tag{308}$$

Substituting into $\Delta = \operatorname{div}\operatorname{grad}$ and using the formula for the divergence of a product of a scalar and a vector field ($\operatorname{div}(f\vec{F}) = f\operatorname{div}\vec{F} + (\operatorname{grad} f)\cdot\vec{F}$), we get

$$\Delta f = -\operatorname{div}(r^{-3}\vec{r}) = -(\operatorname{grad} r^{-3})\cdot\vec{r} - r^{-3}\operatorname{div}\vec{r} = \frac{3}{r^5}\vec{r}\cdot\vec{r} - \frac{3}{r^3} = 0. \tag{309}$$

The calculation for

g is similar, resulting in

$$\operatorname{grad} g = -ike^{-ikr}\operatorname{grad} r = -ik\frac{1}{r}e^{-ikr}\vec{r}. \tag{310}$$

Applying divergence then

$$\begin{aligned}\Delta g =& -ik\operatorname{grad}\left(\frac{1}{r}e^{-ikr}\right)\cdot\vec{r} - ik\frac{1}{r}e^{-ikr}\operatorname{div}\vec{r}\\ =& -ik\left(-r^{-3}\vec{r}e^{-ikr} + r^{-1}(-ik)\frac{1}{r}e^{-ikr}\vec{r}\right)\cdot\vec{r} - ik\frac{3}{r}e^{-ikr}\\ =& -k^2e^{-ikr} - \frac{2ik}{r}e^{-ikr}.\end{aligned} \tag{311}$$

After substitution, we get in total

$$\begin{aligned}\Delta\varphi = f\Delta g + 2(\operatorname{grad} f)\cdot(\operatorname{grad} g) + g\Delta f = \frac{1}{r}\left(-k^2e^{-ikr} - \frac{2ik}{r}e^{-ikr}\right)\\ + 2(-r^{-3}\vec{r})\cdot(-ik\frac{1}{r}e^{-ikr}\vec{r}) = -k^2\varphi(\vec{r}).\end{aligned} \tag{312}$$

The wave function is of form $\psi(\vec{r},t) = e^{i\omega t}\varphi(\vec{r})$. Substituting into the wave equation, we get

$$-\omega^2 e^{i\omega t}\varphi(\vec{r}) = -v^2 k^2 e^{i\omega t}\varphi(\vec{r}). \tag{313}$$

Using the dispersion relation, we thus see that the given spherical wave satisfies the wave equation.

If we start from knowing the form of Laplace's operator for a function dependent only on the radial coordinate $\varphi(r)$:

$$\Delta\varphi(r) = \frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr}, \tag{314}$$

then it suffices to simply calculate

$$\frac{d\varphi}{dr} = -\frac{1}{r^2}e^{-ikr} - ik\frac{1}{r}e^{-ikr}, \qquad \frac{d^2\varphi}{dr^2} = \frac{2}{r^3}e^{-ikr} + ik\frac{2}{r^2}e^{-ikr} - k^2\frac{1}{r}e^{-ikr}. \tag{315}$$

Then after substituting, it easily comes out

$$\Delta\varphi = -k^2\frac{1}{r}e^{-ikr} = -k^2\varphi(r). \tag{316}$$

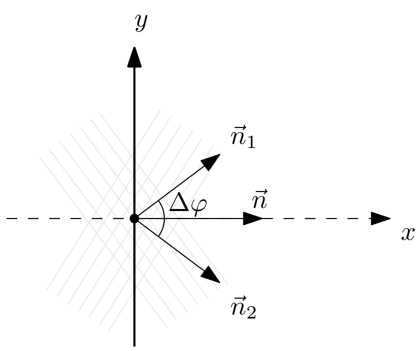
Exercise 8.5. *Superposition of spatial waves.* Consider two plane traveling harmonic waves with the same wavelength λ and different amplitudes, between whose directions of propagation there is an angle $\Delta\varphi$. Consider a plane screen that is perpendicular to the "average direction" of propagation of these waves. Find the intensity profile (i.e., the temporal mean value of the square) of the resulting superposition on the screen. Determine the distance Δy between interference maxima.

Solution: We thus consider two waves of form

$$\psi_1(\vec{r},t) = A_1\cos(\omega t - k\,\vec{n}_1\cdot\vec{r}), \quad \psi_2(\vec{r},t) = A_2\cos(\omega t - k\,\vec{n}_2\cdot\vec{r}), \tag{317}$$

where $\vec{n}_1\cdot\vec{n}_2 = \cos\Delta\varphi$. It is advantageous to solve the problem

in coordinates where both directional vectors lie in the plane $z=0$, and the screen is the plane $x=0$. See the figure:



In these coordinates, we have

$$\vec{n}_1 = \left(\cos\frac{\Delta\varphi}{2}, \sin\frac{\Delta\varphi}{2}, 0\right), \qquad \vec{n}_2 = \left(\cos\frac{\Delta\varphi}{2}, -\sin\frac{\Delta\varphi}{2}, 0\right). \tag{318}$$

We are also interested in the value of superposition only on the screen, thus for $\vec{r} = (0, y, z)$. We get

$$\begin{aligned}\psi(\vec{r} = (0, y, z), t) &= \psi_1(\vec{r} = (0, y, z), t) + \psi_2(\vec{r} = (0, y, z), t) \\ &= A_1 \cos\left(\omega t - ky \sin\frac{\Delta\varphi}{2}\right) + A_2 \cos\left(\omega t + ky \sin\frac{\Delta\varphi}{2}\right). \end{aligned} \tag{319}$$

The intensity of the wave is given by the temporal mean value of the square of the wave, $I = \langle\psi^2\rangle$. After substituting

$$\begin{aligned} I &= A_1^2 \left\langle\cos^2(\omega t + \ldots)\right\rangle + A_2^2 \left\langle\cos^2(\omega t + \ldots)\right\rangle + 2A_1A_2 \left\langle \cos\left(\omega t - ky \sin\frac{\Delta\varphi}{2}\right)\cos\left(\omega t + ky \sin\frac{\Delta\varphi}{2}\right)\right\rangle \\ &= \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + A_1A_2 \left\langle \cos(2\omega t) + \cos\left(2ky \sin\frac{\Delta\varphi}{2}\right)\right\rangle = \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + A_1A_2 \cos\left(2ky \sin\frac{\Delta\varphi}{2}\right), \end{aligned} \tag{320}$$

where we used the sum formula $\cos a \cos b = \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right)$. Thus, the intensity changes "harmonically" along the y axis and remains constant along the z axis – we have thus obtained a series of interference fringes on the screen. The distance of these fringes is determined by finding the positions of individual interference maxima y_m and then $\Delta y = y_{m+1} - y_m$. The positions of interference maxima y_m are given by

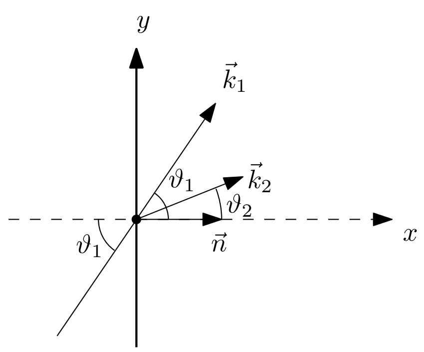
$$\cos\left(2ky_m \sin\frac{\Delta\varphi}{2}\right) = 1 \quad \Leftrightarrow \quad 2ky_m \sin\frac{\Delta\varphi}{2} = 2\pi m, \quad \Leftrightarrow \quad y_m = \frac{\pi m}{k \sin\frac{\Delta\varphi}{2}} = \frac{m\lambda}{2\sin\frac{\Delta\varphi}{2}}, \quad m \in \mathbb{Z}. \tag{321}$$

The distance between adjacent maxima on the screen is thus $\Delta y = \frac{\lambda}{2\sin\frac{\Delta\varphi}{2}}$. For a small angle $\Delta\varphi$, we can write $\Delta y \approx \frac{\lambda}{\Delta\varphi}$.

Exercise 8.6. Consider a plane interface between two transparent media with refractive indices n_1 and n_2. Consider an incident and a transmitted traveling harmonic wave. The wave vectors $\vec{k}_1$ and $\vec{k}_2$ lie in the plane perpendicular to the plane of the interface

and form an angle ϑ_1, resp. ϑ_2 with the normal vector. Based on the condition $\vec{k}_{1\|} = \vec{k}_{2\|}$ (this condition results from the continuity condition of the tangential components of the electric field at the interface, $\vec{E}_{1\|} = \vec{E}_{2\|}$), derive Snell's law of refraction.

Solution: For the purpose of this task, we draw a similar figure as before, hence



We can thus express the wave vectors using the introduced angles as

$$\vec{k}_1 = k_1(\cos\vartheta_1, \sin\vartheta_1, 0), \qquad \vec{k}_2 = k_2(\cos\vartheta_2, \sin\vartheta_2, 0). \tag{322}$$

The components of the wave vectors parallel to the interface are thus very simple, $\vec{k}_{1\|} = (0, k_1 \sin\vartheta_1, 0)$ and $\vec{k}_{2\|} = (0, k_2 \sin\vartheta_2, 0)$. Hence, we get the condition $k_1 \sin\vartheta_1 = k_2 \sin\vartheta_2$. The refractive index is given by the ratio of the speed of light and phase velocity, thus $n_i = \frac{c}{v_i} = \frac{ck_i}{\omega}$. From here, we get the well-known **Snell's law of refraction**:

$$n_1 \sin\vartheta_1 = n_2 \sin\vartheta_2. \tag{323}$$

Considering $n_1 > n_2$ (the second medium is optically less dense), we get the relation for the second angle $\sin\vartheta_2 = \frac{n_1}{n_2}\sin\vartheta_1$. For a sufficiently large angle ϑ_1, a situation occurs where the right side is greater than one and ϑ_2 cannot be found. This means that no traveling wave propagates into the second medium (total internal reflection occurs). The critical angle ϑ_c is thus given by $\sin\vartheta_c = \frac{n_2}{n_1}$.

Exercise 8.7. We have the same setting as in the previous task. Now consider the interface of these two media: a transparent medium with refractive index n and the ionosphere with plasma frequency ω_p. Derive the corresponding law of refraction.

Solution: We slightly modify the approach from the previous example. There we derived the condition $k_1 \sin\vartheta_1 = k_2 \sin\vartheta_2$. Now, however, we have the ionosphere on one side, which we consider as plasma; its dispersion relation is thus $\omega^2 = \omega_p^2 + c^2k^2$. For simplicity, let's consider only $\omega > \omega_p$, so that the transmitted wave can propagate in the ionosphere. The wave numbers in the respective media can thus be expressed as

$$k_1 = \frac{\omega}{c}n, \qquad k_2 = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}. \tag{324}$$

After substitution, we get the law of refraction in the form

$$n \sin\vartheta_1 = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}\sin\vartheta_2. \tag{325}$$

The refractive index is always $n \geq 1$ and the square root on the right side $\sqrt{1-(\omega_p/\omega)^2} \leq 1$. Expressing the sine of the second angle: $\sin\vartheta_2 = \frac{n}{\sqrt{1-(\omega_p/\omega)^2}}\sin\vartheta_1$, we can again explore the limit of the angle ϑ_1, at which $\vartheta_2 = \frac{\pi}{2}$;

which is given by

$$\sin\vartheta_c = \frac{1}{n}\sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}. \tag{326}$$

Exercise 8.8. Show that an electromagnetic standing wave of form

$$\vec{E} = (A\cos\omega t\cos kz, 0, 0), \;\; \vec{B} = \left(0, \frac{1}{c}A\sin\omega t\sin kz, 0\right), \tag{327}$$

where $\omega = ck$, satisfies Maxwell's equations in vacuum. Determine the density of electric and magnetic energy and the Poynting vector.

Solution: Vacuum Maxwell's equations (without charges and currents) are

$$\operatorname{div} \vec{E} = 0, \qquad \operatorname{div} \vec{B} = 0, \qquad \operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \operatorname{rot} \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}. \tag{328}$$

The first two equations are trivially satisfied:

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} = 0, \qquad \operatorname{div} \vec{B} = \frac{\partial B_y}{\partial y} = 0. \tag{329}$$

In the second set of equations, we get

$$\operatorname{rot} \vec{E} = \left(0, \frac{\partial E_x}{\partial z}, -\frac{\partial E_x}{\partial y}\right) = (0, -Ak\cos\omega t \sin kz, 0), \tag{330}$$

$$\operatorname{rot} \vec{B} = \left(-\frac{\partial B_y}{\partial z}, 0, \frac{\partial B_y}{\partial x}\right) = \left(-\frac{1}{c}kA\sin\omega t\cos kz, 0, 0\right), \tag{331}$$

$$-\frac{\partial \vec{B}}{\partial t} = \left(0, -\frac{1}{c}\omega A\cos\omega t\sin kz, 0\right), \tag{332}$$

$$\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \left(-\frac{1}{c^2}\omega A\sin\omega t\cos kz, 0, 0\right). \tag{333}$$

We see that Maxwell's equations are indeed satisfied, provided that $\omega = ck$. The densities of electric and magnetic fields are given by

$$w_E = \frac{1}{2}\epsilon_0 E^2, \;\; w_B = \frac{1}{2\mu_0}B^2. \tag{334}$$

After substitution, we get

$$w_E = \frac{1}{2}\epsilon_0 A^2\cos^2\omega t\cos^2 kz, \;\; w_B = \frac{1}{2\mu_0 c^2}A\sin^2\omega t\sin^2 kz = \frac{1}{2}\epsilon_0 A\sin^2\omega t\sin^2 kz. \tag{335}$$

We can easily substitute into the formula for the Poynting vector:

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = \left(0, 0, \frac{1}{\mu_0 c}A^2\cos\omega t\sin\omega t\cos kz\sin kz\right) \\ &= \left(0, 0, \frac{A^2}{4\mu_0 c}\sin 2\omega t\sin 2kz\right).\end{aligned} \tag{336}$$

Exercise 8.9. *Larmor formula.* Show that by integrating the Poynting vector $\vec{S}$ of the radiation field $\vec{E}_{rad}$ from an accelerated charge,

$$\vec{E}_{\mathrm{rad}}(\vec{r}, t) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}\frac{\vec{a}_\perp(t_r)}{r},$$

over a sphere of radius r you get Larmor's formula for the total radiated power P of the electromagnetic wave,

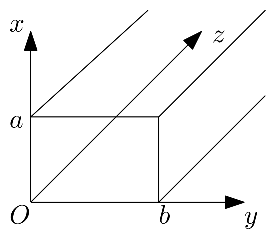
$$P(t, r) = \frac{\mu_0 q^2}{6\pi c}a^2(t_r).$$

The retarded time t_r is $t_r = t - \frac{r}{c}$. Poynting's vector is $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \sqrt{\frac{\varepsilon_0}{\mu_0}}E^2\,\vec{n}$, where $\vec{n}$ is the direction of propagation perpendicular to the imagined sphere.

Solution: See lecture notes section 6.3.6.

Exercise 8.10. Consider a waveguide of rectangular cross-section $a = 5\,\text{cm}$ and $b = 10\,\text{cm}$. What is the lowest frequency f_0 that an electromagnetic wave can have to pass through the waveguide without attenuation. Calculate the phase and group velocity (as a multiple of c), whose frequency is $f = \frac{5}{4}f_0$. What is the highest mode m_0 that can be excited for the propagating wave of this frequency? For a wave with frequency $f = \frac{4}{5}f_0$, determine the distance at which the amplitude of the wave decreases by a factor of e.

Solution: A rectangular waveguide is an infinite tube of rectangular cross-section as in the figure:



The walls are made of perfectly conducting material. Solving Maxwell's equations with boundary conditions, where we assume $\vec{E}$ depends only on (y, z, t), the electric field inside $\vec{E} = (E_x, 0, 0)$, where $E_x = E_x(y, z, t)$ is a superposition of modes in the form

$$E_x(y, z, t) = E_0 \sin\left(\frac{m\pi y}{b}\right) e^{i(\omega t - kz)}, \tag{337}$$

where constants ω, k, and $m \in \mathbb{N}$ satisfy the dispersion relation

$$\omega^2 = \left(\frac{m\pi c}{b}\right)^2 + c^2k^2. \tag{338}$$

We see that for a given m, this equation has a real solution for wave number k only for $\omega > \omega_{\min(m)}$, where $\omega_{\min(m)} = \frac{m\pi c}{b}$. The lowest value is obtained for $m = 1$, which is the lowest of the angular frequencies that anything can propagate through the waveguide without attenuation, $\omega_0 = \frac{\pi c}{b}$. From there $f_0 = \frac{\omega_0}{2\pi} = \frac{c}{2b}$. In our case,

$$f_0 = \frac{3.10^8}{2 \cdot 0,1}\text{Hz} = 1,5\,\text{GHz}. \tag{339}$$

The phase velocity is given by the ratio $v_\varphi = \omega/k$. Hence,

$$v_\varphi(k) = c\sqrt{1 + \frac{m^2\omega_0^2}{k^2c^2}}. \tag{340}$$

We need to express $v_\varphi = v_\varphi(\omega)$, so we substitute k from the dispersion relation on the right side:

$$v_\varphi(\omega) = c\sqrt{1 + \frac{m^2\omega_0^2}{\omega^2 - m^2\omega_0^2}} = c\sqrt{\frac{\omega^2}{\omega^2 - m^2\omega_0^2}} = \frac{c}{\sqrt{1 - \left(m\frac{\omega_0}{\omega}\right)^2}}. \tag{341}$$

We need to find the value of phase velocity for $\omega = \frac{5}{4}\omega_0$, thus $f = 1,875\,\text{GHz}$. Since $f_{\min(m)} = m \cdot 1,5\,\text{GHz}$, only the lowest mode $m = 1$ can be excited and we get

$$v_\varphi\left(\frac{5}{4}\omega_0\right) = \frac{5}{3}c. \tag{342}$$

The group velocity as a function of k is obtained by differentiating the dispersion relation with respect to k, resulting in

$$v_g(k) = \frac{c}{\sqrt{1 + \frac{\omega_0^2 m^2}{c^2 k^2}}}. \tag{343}$$

After substituting k from the dispersion relation, we get the group velocity as a function of ω with exactly the inverse dependence as the phase velocity:

$$v_g(\omega) = c\sqrt{\frac{\omega^2 - m^2\omega_0^2}{\omega^2}} = c\sqrt{1 - \left(m\frac{\omega_0}{\omega}\right)^2} \overset{m=1}{=} \frac{3}{5}c. \tag{344}$$

We have also answered the question of the highest mode that can be excited – only the first one. If the frequency $\omega < \omega_{\min(m)}$, then no real k satisfying the dispersion relation can be found. The ansatz $k = -i\kappa$ leads to $\omega^2 = m^2\omega_0^2 - c^2\kappa^2$, where κ then appears in the exponentially damped standing wave not propagating through the waveguide:

$$E_x(y, z, t) = E_0 \sin\left(\frac{m\pi y}{b}\right) e^{i\omega t} e^{-\kappa z}. \tag{345}$$

After expressing from the dispersion relation, the coefficient κ comes out as

$$\kappa = \frac{1}{c}\sqrt{m^2\omega_0^2 - \omega^2}. \tag{346}$$

The least attenuation occurs for $m = 1$, here we have $\omega = \frac{3}{5}\omega_0$ and thus $\kappa = \frac{3\omega_0}{5c} = \frac{6\pi f_0}{5c}$. We solve the task $e^{-\kappa z} = e^{-1}$, hence $\kappa z = 1$. From which $z = \kappa^{-1} = \frac{5c}{6\pi f_0} \approx \frac{5}{3\pi} \cdot 10^{-1}\mathrm{m} \approx 5\,\mathrm{cm}$.

9 Polarization

Exercise 9.1. How does the intensity of circularly polarized light change after passing through a polarizer?

Solution: A traveling electromagnetic wave propagating in the direction of the z axis generally has an electric component in complexified form (at a given location $z = z_0$) of the form

$$\vec{E}(t) = E_{x0}\,\vec{x}\,e^{i(\omega t + \varphi_1)} + E_{y0}\,\vec{y}\,e^{i(\omega t + \varphi_2)} = \begin{pmatrix} E_{x0}e^{i\varphi_1} \\ E_{y0}e^{i\varphi_2} \end{pmatrix} e^{i\omega t} = \hat{\vec{E}}\,e^{i\omega t}, \tag{347}$$

where $\vec{x}$ and $\vec{y}$ denote unit vectors in the directions of axes x and y.

The orientation of the polarizer is determined by the axis of transmission described by the unit direction vector $\vec{n}$. If $\vec{E}_{in}$ is the incident wave and $\vec{E}_{out}$ is the transmitted wave, the waves are related as

$$\vec{E}_{out} = (\vec{E}_{in} \cdot \vec{n})\,\vec{n}. \tag{348}$$

This can be simply rewritten in the language of the corresponding complex vectors $\hat{\vec{E}}_{in}$ and $\hat{\vec{E}}_{out}$, we get

$$\hat{\vec{E}}_{out} = \mathbb{P}_{\vec{n}}\hat{\vec{E}}_{in}, \tag{349}$$

where $\mathbb{P}_{\vec{n}}$ is the matrix of the projector onto the axis in the direction $\vec{n}$. Explicitly for $\vec{n} = (n_x, n_y)$ we get

$$\mathbb{P}_{\vec{n}} = \begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix}. \tag{350}$$

The intensity of the electric field is given by the formula

$$I = \sqrt{\frac{\varepsilon}{\mu}}\,\langle \vec{E}^2\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,(E_{x0}^2 + E_{y0}^2) = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(|\hat{E}_1|^2 + |\hat{E}_2|^2\right), \tag{351}$$

where $\hat{\vec{E}} = (\hat{E}_1, \hat{E}_2)^T$. Typically when calculating intensity, we will not write the factor $\sqrt{\frac{\varepsilon}{\mu}}$, so we will consider the relationship $I = \langle \vec{E}^2\rangle$, which corresponds only to a change in the units in which intensity is measured.

Here, the incoming light is circularly polarized, characterized by the conditions $E_{x0} = E_{y0} = E_0$ and $\varphi_1 - \varphi_2 = \pm\frac{\pi}{2}$. Now, let's divide the calculation depending on whether we want to calculate the example "vectorially" or "matrix-wise":

Vectorially: Circularly polarized light can be written

$$\vec{E}_{in} = E_0\vec{x}\cos(\omega t + \varphi) + E_0\vec{y}\cos(\omega t + \varphi \pm \tfrac{\pi}{2}) = E_0\vec{x}\cos(\omega t + \varphi) \mp E_0\vec{y}\sin(\omega t + \varphi). \tag{352}$$

The input intensity then is

$$I_{in} = \langle \vec{E}^2\rangle = \frac{1}{2}E_0^2 + \frac{1}{2}E_0^2 = E_0^2. \tag{353}$$

BÚNO (without loss of generality) we can consider the direction of transmission of the polarizer in the direction $\vec{n} = \vec{x} = (1,0)^T$. Thus, the action of the linear polarizer will be

$$\vec{E}_{out} = (\vec{E}_{in}\cdot\vec{x})\,\vec{x} = E_0\,\vec{x}\cos(\omega t + \varphi). \tag{354}$$

We easily compute the output intensity

$$I_{out} = \langle \vec{E}_{out}^2\rangle = \frac{1}{2}E_0^2 = \frac{1}{2}I_{in}. \tag{355}$$

Matrix-wise: For circularly polarized light, the vector $\hat{\vec{E}}_{in}$ has the form

$$\hat{\vec{E}}_{in} = E_0 e^{i\varphi}\begin{pmatrix}1\\ e^{\pm i\frac{\pi}{2}}\end{pmatrix} = E_0 e^{i\varphi}\begin{pmatrix}1\\ \pm i\end{pmatrix} \sim E_0\begin{pmatrix}1\\ \pm i\end{pmatrix}, \tag{356}$$

where in the last modification we eliminated the common phase, which does not affect the polarization state. We see that the intensity of the incoming wave is simply

$$I_{in} = \frac{1}{2}\left(|\hat{E}_1|^2 + |\hat{E}_2|^2\right) = \frac{1}{2}(E_0^2 + E_0^2) = E_0^2. \tag{357}$$

BÚNO we can consider the direction of transmission of the polarizer in the direction $\vec{n} = \vec{x} = (1,0)^T$. Then the projector takes the form

$$\mathbb{P}_{\vec{n}} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{358}$$

The light behind the polarizer $\hat{\vec{E}}_{out}$ thus

$$\hat{\vec{E}}_{out} = \mathbb{P}_{\vec{n}}\hat{\vec{E}}_{in} = E_0\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{359}$$

Therefore, the output intensity is

$$I_{out} = \frac{1}{2}\left(|\hat{E}_1|^2 + |\hat{E}_2|^2\right) = \frac{1}{2}E_0^2 = \frac{1}{2}I_{in}. \tag{360}$$

Exercise 9.2. How does the intensity of unpolarized light change after passing through a linear polarizer?

Solution: Completely unpolarized light of intensity I_{in} can be imagined, for example, as a linearly polarized wave, the direction of which $\vec{n} = \vec{n}(t)$ changes completely randomly in time. Let $\theta(t)$ be the angle formed by the direction $\vec{n}$ with the axis of transmission of the linear polarizer. According to Malus's law, the instantaneous intensity of the transmitted wave is

$$I_{out}(t) = I_{in} \cos^2 \theta(t). \tag{361}$$

The device measuring the intensity $I_{out}(t)$, however, cannot measure instantaneous intensity (light changes its polarization too quickly), but only its average value over the measurement time of the device t_{roz}:

$$I_{out} = I_{in} \langle \cos^2 \theta(t) \rangle_{t_{roz}} \tag{362}$$

Since $\theta(t)$ changes completely randomly, every angle is represented completely uniformly. The time average can be replaced by the average over angles:

$$I_{out} = I_{in} \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{1}{2} I_{in}. \tag{363}$$

Exercise 9.3. Consider linearly polarized light $\vec{E} = E_0 \, \vec{x} \cos \omega t$. Place N polarizers in its path, each with the axis of transmission rotated by $\frac{\pi}{2N}$ compared to the previous one (and the first compared to the plane of the incident light). What will be the intensity of the transmitted light for $N = 1$, $N = 2$, and general $N \in \mathbb{N}$? What is the limit for $N \to +\infty$?

Solution: In the lecture, you derived that if light linearly polarized in the direction $\vec{n}_1$ hits a linear polarizer with the axis of transmission $\vec{n}_2$, the input and output intensity are related by *Malus's law*:

$$I_{out} = I_{in} \cos^2 \theta, \tag{364}$$

where θ is the angle formed by vectors $\vec{n}_1$ and $\vec{n}_2$. The intensity of the original wave is $I_0 = \langle \vec{E}^2 \rangle = \frac{1}{2} E_0^2$.

(i) For $N = 1$ there is one rotation by an angle $\frac{\pi}{2}$. The resulting intensity is $I_{out}^{(1)} = I_0 \cos^2 \frac{\pi}{2} = 0$.

(ii) For $N = 2$ there are two rotations by an angle $\frac{\pi}{4}$. The resulting intensity is given by

$$I_{out}^{(2)} = \left(I_0 \cos^2 \tfrac{\pi}{4} \right) \cos^2 \tfrac{\pi}{4} = \frac{1}{4} I_0. \tag{365}$$

(iii) For general $N \in \mathbb{N}$ there are N rotations by an angle $\frac{\pi}{2N}$. The resulting intensity is thus

$$I_{out}^{(N)} = I_0 \cos^{2N} \left(\tfrac{\pi}{2N} \right). \tag{366}$$

It holds[3] $\lim_{k \to +\infty} \cos^k \left(\frac{\pi}{k} \right) = 1$ and therefore $\lim_{N \to +\infty} I_{out}^{(N)} = I_0$.

[3] This limit can be calculated for example as follows:

$$\lim_{k \to +\infty} \cos^k \left(\tfrac{\pi}{k} \right) = \lim_{k \to +\infty} \exp \left(\pi \frac{\ln \cos \frac{\pi}{k}}{\frac{\pi}{k}} . \right)$$

Now it suffices to show that $\lim_{x \to 0} \frac{\ln \cos x}{x} = 0$. Using l'Hospital's rule

$$\lim_{x \to 0} \frac{\ln \cos x}{x} = \lim_{x \to 0} -\frac{\sin x}{\cos x} = 0.$$

Exercise 9.4. Consider generally elliptically polarized light. Place in its path a polarizer with the axis of transmission $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$. Show that for the intensity of the transmitted light the following holds

$$I_{out} = \frac{1}{2}(I_x + I_y) + I_{xy}, \tag{367}$$

where $I_{out} = \langle E_{out}^2 \rangle,\ \ I_x = \langle E_x^2 \rangle,\ I_y = \langle E_y^2 \rangle,\ \ I_{xy} = \langle E_x E_y \rangle$.

Solution: Due to the definitions of the individual intensities, it pays off to work with the general expression for the electric

field in the xy plane and not to detail it into the respective harmonic waves.

Vectorially: Generally, elliptically polarized light has the form $\vec{E} = E_x\vec{x} + E_y\vec{y}$. After passing through the polarizer, we get

$$\vec{E}_{out} = (\vec{E} \cdot \vec{n})\,\vec{n} = \left(\frac{E_x}{\sqrt{2}} + \frac{E_y}{\sqrt{2}}\right)\vec{n}, \tag{368}$$

since $\vec{x} \cdot \vec{n} = \vec{y} \cdot \vec{n} = \frac{1}{2}$. The resulting intensity is then

$$I_{out} = \langle \vec{E}_{out}^2 \rangle = \left\langle \left(\frac{E_x + E_y}{\sqrt{2}}\right)^2 \vec{n}^2 \right\rangle = \frac{1}{2}\langle E_x^2 + E_y^2 + 2E_xE_y \rangle = \frac{1}{2}(I_x + I_y) + I_{xy}. \tag{369}$$

Matrix-wise: Generally, elliptically polarized light has the form

$$\vec{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{370}$$

The projector onto the axis $\vec{n}$ has the form

$$\mathbb{P}_{\vec{n}} = \begin{pmatrix} n_x^2 & n_xn_y \\ n_xn_y & n_y^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{371}$$

After passing through the polarizer, we therefore get

$$\vec{E}_{out} = \mathbb{P}_{\vec{n}}\vec{E} = \frac{1}{2}\begin{pmatrix} E_x + E_y \\ E_x + E_y \end{pmatrix}. \tag{372}$$

The resulting intensity is then

$$I_{out} = \langle \vec{E}_{out}^2 \rangle = \left\langle 2\frac{1}{4}(E_x + E_y)^2 \right\rangle = \frac{1}{2}\langle E_x^2 + E_y^2 + 2E_xE_y \rangle = \frac{1}{2}(I_x + I_y) + I_{xy}. \tag{373}$$

Exercise 9.5. The refractive indices of crystalline quartz for light with a vacuum wavelength $\lambda_0 = 500\,\text{nm}$ are $n_1 = 1.544$ and $n_2 = 1.553$. Determine the minimum thickness of a quarter-wave plate made of this material.

Solution: If we have two perpendicular directions $\vec{n}_1$ and $\vec{n}_2$ such that light in the crystal propagates with refractive index n_1 in the first direction and with n_2 in the second, we can write the incoming wave in the form

$$\vec{E}_{in} = E_1\vec{n}_1 e^{i(\omega t + \varphi_1)} + E_2\vec{n}_2 e^{i(\omega t + \varphi_2)}. \tag{374}$$

Through a plate of thickness d, the phases in the individual waves change by $k_1 d$, resp. $k_2 d$, to a wave of the form

$$\vec{E}_{out} = E_1 \vec{n}_1 e^{i(\omega t + \varphi_1 - k_1 d)} + E_2 \vec{n}_2 e^{i(\omega t + \varphi_2 - k_2 d)}. \tag{375}$$

The phase difference $\delta\varphi$ changes upon passage to $\varphi_1 - \varphi_2 + \Delta\varphi$, where $\Delta\varphi = (k_2 - k_1)d$. For the study of polarization, the specific values of phases are irrelevant, so the electric field vector $\vec{E}_{out}$ can be written as

$$\vec{E}_{out} = E_1 \vec{n}_1 e^{i(\omega t + \varphi_1 + \Delta\varphi)} + E_2 \vec{n}_2 e^{i(\omega t + \varphi_2)} \tag{376}$$

(we added the phase $+k_2 d$ to both exponentials). We have the dispersion relation $k = \frac{n}{c}\omega$ and for vacuum $k_0 = \frac{\omega}{c}$ and $k_0 = \frac{2\pi}{\lambda_0}$ and thus $\Delta\varphi = (n_2 - n_1)\frac{\omega}{c} d = \frac{2\pi}{\lambda_0}(n_2 - n_1)d$. Choosing directions so that $n_2 > n_1$, we thus get $\Delta\varphi > 0$ and it is added to the wave in the direction $\vec{n}_1$.

A quarter-wave plate is supposed to shift the phase by a quarter of a wavelength, hence $\Delta\varphi = \frac{\pi}{2}$. We thus solve the equation

$$\frac{\pi}{2} = \frac{2\pi}{\lambda_0}(n_2 - n_1)d. \tag{377}$$

From this, $d = \frac{\lambda_0}{4(n_2 - n_1)} = \frac{5\cdot 10^{-7}}{4\cdot 9\cdot 10^{-3}} \approx 0.014$ mm.

Exercise 9.6. Write the matrix $\mathbb{D}_{\Delta\varphi}$ for a wave plate with axis $\vec{n}_1 = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$ (and perpendicular direction $\vec{n}_2 = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$).

Solution: Since the wave plate only adds a phase $\Delta\varphi$ to the component of the wave in the direction $\vec{n}_1$ (i.e., the axis corresponding to the smaller refractive index), the matrix of the wave plate is given by

$$\mathbb{D}_{\Delta\varphi} = e^{i\Delta\varphi}\mathbb{P}_{\vec{n}_1} + \mathbb{P}_{\vec{n}_2}. \tag{378}$$

The matrix of the projector onto a general axis $\vec{n} = (n_x, n_y)^T$ is of the form

$$\mathbb{P}_{\vec{n}} = \begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix} \tag{379}$$

and thus for directions $\vec{n}_1$ and $\vec{n}_2$ we specifically get

$$\mathbb{P}_{\vec{n}_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbb{P}_{\vec{n}_2} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{380}$$

Substituting into the above-mentioned relationship thus gives us

$$\mathbb{D}_{\Delta\varphi} = \frac{1}{2}\begin{pmatrix} e^{i\Delta\varphi} + 1 & e^{i\Delta\varphi} - 1 \\ e^{i\Delta\varphi} - 1 & e^{i\Delta\varphi} + 1 \end{pmatrix} = e^{i\frac{\Delta\varphi}{2}}\begin{pmatrix} \cos\frac{\Delta\varphi}{2} & i\sin\frac{\Delta\varphi}{2} \\ i\sin\frac{\Delta\ varphi}{2} & \cos\frac{\Delta\varphi}{2} \end{pmatrix}. \tag{381}$$

Note that in terms of polarization, we can forget about the complex unit in front of the matrix – it changes the phase of both components equally, hence it is irrelevant, so the resulting $\mathbb{D}_{\Delta\varphi}$ can be written as:

$$\mathbb{D}_{\Delta\varphi} \sim \begin{pmatrix} \cos\frac{\Delta\varphi}{2} & i\sin\frac{\Delta\varphi}{2} \\ i\sin\frac{\Delta\varphi}{2} & \cos\frac{\Delta\varphi}{2} \end{pmatrix}. \tag{382}$$

Exercise 9.7. Consider linearly polarized light $\vec{E} = E_0\vec{x}\cos(\omega t)$. Place in its path a half-wave plate with axis oriented in the direction $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$. What will be the polarization state of the light after passing through the plate? How does the intensity change?

Solution: **Vectorially**: Since the electric field of the incoming light is not decomposed in the direction of the wave plate's axis $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$ (and perpendicular $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$), we cannot simply add a phase π as the action of the wave plate. First, we must decompose the incoming light into these directions, i.e., we would like to find the following coefficients of the linear combination α and β:

$$\vec{x} = \alpha\frac{\vec{x}+\vec{y}}{\sqrt{2}} + \beta\frac{\vec{x}-\vec{y}}{\sqrt{2}}. \tag{383}$$

After rewriting

$$\left(1 - \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right)\vec{x} + \left(\frac{\beta}{\sqrt{2}} - \frac{\alpha}{\sqrt{2}}\right)\vec{y} = 0 \tag{384}$$

Vectors $\vec{x}$ and $\vec{y}$ are linearly independent, so the brackets must be zero, thus $\alpha = \beta = \frac{1}{\sqrt{2}}$. Alternatively, we can rewrite without calculation: $\vec{x} = \frac{1}{2}(\vec{x}+\vec{x}+\vec{y}-\vec{y}) = \frac{1}{\sqrt{2}}\left(\frac{\vec{x}+\vec{y}}{\sqrt{2}} + \frac{\vec{x}-\vec{y}}{\sqrt{2}}\right)$. The incoming light can thus be written in the form

$$\vec{E} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t. \tag{385}$$

The action of the wave plate is now trivially performed:

$$\vec{E}_{out} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos(\omega t+\pi) + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t, \tag{386}$$

adding a phase π to the electric field in the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. Using the formula $\cos(x+\pi) = -\cos x$, we can write

$$\vec{E}_{out} = -\frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t = -E_0\vec{y}\cos\omega t \sim E_0\vec{y}\cos\omega t. \tag{387}$$

(after the last modification, we shifted the overall phase by π to get rid of the minus sign, this modification does not change the polarization state). After passing through the wave plate, you get linearly polarized light with the plane of polarization in the direction of the $\vec{y}$ axis, thus rotated by 90° compared to the original! Moreover, it has the same amplitude as the original light, so there was no loss of intensity (unlike in example 9.3)!

Matrix-wise[4]: It suffices to substitute $\Delta\varphi = \pi$ from the previous example to get the matrix

$$\mathbb{D}_\pi = \begin{pmatrix} \cos\frac{\pi}{2} & i\sin\frac{\pi}{2} \\ i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{388}$$

where in the last equation, we multiplied the matrix by the complex unit $e^{i\pi} = -i$ (overall phase does not change the polarization state) to find the simplest form of the matrix $\mathbb{D}_\pi$. Our input light has the polarization vector $\hat{\vec{E}} = (E_0, 0)^T$. The transmitted wave thus

$$\hat{\vec{E}}_{out} = \mathbb{D}_\pi \begin{pmatrix} E_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ E_0 \end{pmatrix} \quad \rightarrow \quad \vec{E}_{out} = E_0\vec{y}\cos\omega t. \tag{389}$$

We found out that the transmitted wave is linearly polarized with the plane of polarization in the direction of the $\vec{y}$ axis, thus rotated by 90° compared to the original! Moreover, it has the same amplitude as the original light, so there was no loss of intensity (unlike in example 9.3)!

[4]Here, the solution seems much shorter than the vectorial approach, but it's because we have already pre-calculated everything in example 9.6.

Exercise 9.8. A circular polarizer is a linear polarizer followed by a quarter-wave plate with axes rotated by $\pi/4$ relative to the axis of transmission of the linear polarizer. Show that depending on the choice of axes in the wave plate, we obtain either a left-handed or a right-handed circular polarizer, which converts any light coming from the side of the linear polarizer into the corresponding circularly polarized light.

Show that left-handed polarized light propagating from the side of the wave plate is absorbed in the right-handed polarizer.

Solution: Let's choose, BÚNO, the axis of the linear polarizer in the direction $\vec{x}$. The quarter-wave plate can then be oriented either with its axis in the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$ (and perpendicular $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$) or vice versa, i.e., with its axis in the direction $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$ (and perpendicular $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$). Depending on the choice of orientation, therefore, the axis of the quarter-wave plate forms an angle $\pm\frac{\pi}{4}$ with the x axis.

Vectorially: From example 9.7, we know that the light entering the wave plate can be written in the form

$$\vec{E}_{in} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t. \tag{390}$$

The quarter

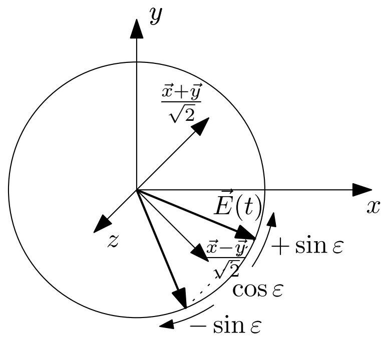
-wave plate adds a phase shift $\frac{\pi}{4}$ to the wave in the corresponding direction. Adding a phase in the component $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$ can also be written as subtracting a phase in the component $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. We can thus write the action of the wave plate for both options as

$$\vec{E}_{out} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos(\omega t \pm \tfrac{\pi}{2}) + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t, \tag{391}$$

where the signs in the phase shift $\pm\frac{\pi}{2}$ correspond to the sign of the angle of the wave plate's axis $\pm\frac{\pi}{2}$. Using the formula $\cos(x \pm \frac{\pi}{2}) = \mp\sin x$, we can write

$$\vec{E}_{out} = \mp\frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\sin\omega t + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t, \tag{392}$$

which is clearly circularly polarized light (has the same amplitude in both directions and one has a sine and the other a cosine). Which of them is left-handed and which is right-handed? Plotting the electric field vector $\vec{E}_{out}$ in the xy plane at a small positive time $\omega t = +\varepsilon$:



In a small positive time, the electric field vector almost entirely points in the direction $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$ and depending on the sign at $\sin\omega t$, points slightly towards/away from the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. We thus see that for a positive sign, $\vec{E}$ rotates counter-clockwise, thus it is left-handed polarized light, and for a negative sign, it rotates clockwise, thus it is right-handed polarized light. The right-handed polarizer is thus the one where the wave plate had its axis oriented in the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$, and the left-handed one in the direction $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$.

It remains to send left-handed polarized light into the right-handed polarizer from the other side. Take as the input light

$$\vec{E}_{in} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\sin\omega t + \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t \tag{393}$$

and send it into the reversed right-handed polarizer, so first it hits the quarter-wave plate and then the linear polarizer. It is important to realize that reversing the wave plate changes the direction of its axis! Thus, the reversed right-handed polarizer has a wave plate with its axis in the direction $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$. Light after passing through this plate will be

$$\vec{E}_{out} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\sin\omega t - \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\sin\omega t = E_0\vec{y}\sin\omega t \tag{394}$$

(using again $\cos(x+\frac{\pi}{2}) = -\sin x$), which is linearly polarized light in the direction $\vec{y}$ and it will be fully absorbed in the linear polarizer with the axis of transmission $\vec{x}$.

Matrix-wise: For the quarter-wave plate with axis $\vec{n}_+ = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$, we have already derived the corresponding matrix $\mathbb{D}_{\Delta\varphi}$ in exercise 9.6. Just substitute $\Delta\varphi = \frac{\pi}{2}$:

$$\mathbb{D}_{\frac{\pi}{2},\vec{n}_+} = \begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{395}$$

If the wave plate has its axis oriented in the direction $\vec{n}_- = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$ (which is perpendicular to the direction $\vec{n}_+ = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$), we must swap the matrices $\mathbb{P}_{\vec{n}_1}$ and $\mathbb{P}_{\vec{n}_2}$ in the solution of example 9.6, thus:

$$\mathbb{D}_{\frac{\pi}{2},\vec{n}_-} = \frac{1}{2}\begin{pmatrix} 1+e^{i\frac{\pi}{2}} & 1-e^{i\frac{\pi}{2}} \\ 1-e^{i\frac{\pi}{2}} & 1+e^{i\frac{\pi}{2}} \end{pmatrix} = \ldots = e^{i\frac{\pi}{4}}\begin{pmatrix} \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{396}$$

We can thus generally write

$$\mathbb{D}_{\frac{\pi}{2},\vec{n}_\pm} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}. \tag{397}$$

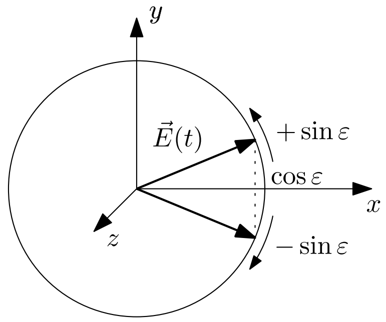
The light entering the plate has the polarization vector $\hat{\vec{E}}_{in} = (E_0, 0)^T$. Behind the wave plate, we then get light

$$\hat{\vec{E}}_{out} = \mathbb{D}_{\frac{\pi}{2},\vec{n}_\pm}\hat{\vec{E}}_{in} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}\begin{pmatrix} E_0 \\ 0 \end{pmatrix} = \frac{E_0}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \frac{E_0}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{\pm i\frac{\pi}{2}} \end{pmatrix}. \tag{398}$$

Both components have the same amplitude and are phase-shifted by $\pm\frac{\pi}{2}$ – thus, in both cases, it is circularly polarized light. Left-handedness/right-handedness is decided by rewriting in vector form

$$\vec{E}_{out} = \frac{E_0}{\sqrt{2}}\left(\vec{x}\,e^{i\omega t} + \vec{y}\,e^{i(\omega t\pm\frac{\pi}{2})}\right) \overset{Re}{=} \frac{E_0}{\sqrt{2}}\Big(\vec{x}\,\cos\omega t + \vec{y}\,\underbrace{\cos(\omega t\pm\tfrac{\pi}{2})}_{\mp\sin\omega t}\Big). \tag{399}$$

The direction of rotation of the electric field is illustrated in the following figure:



For $+\sin\omega t$ (and thus the axis of the wave plate $\vec{n}_- = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$), we have rotation counter-clockwise, thus it is left-handed polarized light. For $-\sin\omega t$ (axis $\vec{n}_+ = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$), it is rotation clockwise – right-handed polarized light.

Now, send left-handed polarized light, i.e., with the input vector

$$\hat{\vec{E}}_{in} = E_0 \begin{pmatrix} 1 \\ e^{-i\frac{\pi}{2}} \end{pmatrix} = E_0 \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{400}$$

into the reversed right-handed polarizer (i.e., the one where the wave plate had its axis in the direction $\vec{n}_+$). Reversing, of course, changes the order of optical elements – first, light hits the wave plate and then the linear polarizer. Reversing also means that the axis of the wave plate changes direction to $\vec{n}_-$! The action of the reversed right-handed polarizer is thus

$$\hat{\vec{E}}_{out} = \mathbb{P}_{\vec{x}}\,\mathbb{D}_{\frac{\pi}{2},\vec{n}_-}\hat{\vec{E}}_{in} = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{401}$$

Nothing thus passes through.

Exercise 9.9. Linearly polarized light with intensity I_0 enters an optical device in the direction $\vec{x}$. Determine the electric field (and name the respective polarization states) and the intensity of light after each of the optical elements in the following device, consisting of the following optical elements in sequence:

1.) polarizer with axis $\vec{n} = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$;

2.) half-wave plate with axis $\vec{n} = \vec{y}$;

3.) polarizer with axis $\vec{y}$;

4.) quarter-wave plate with axis $\vec{n} = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$.

Solution: Vectorially: The incoming light has an electric field of the form $\vec{E}_0 = E_0\vec{x}\cos\omega t$, its intensity is $I_0 = \langle \vec{E}_0^2 \rangle = \frac{1}{2}E_0^2$. After passing through the linear polarizer, we obtain the field

$$\vec{E}_1 = (\vec{E}\cdot\vec{n})\,\vec{n} = E_0 \left(\vec{x}\cdot\frac{\vec{x}+\vec{y}}{\sqrt{2}}\right)\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t = \frac{E_0}{\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t, \tag{402}$$

thus linearly polarized light with the plane of polarization in the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. The intensity is then $I_1 = \langle \vec{E}_1^2 \rangle = \frac{E_0^2}{2}\frac{1}{2} = \frac{1}{2}I_0$. Next, the light encounters a wave plate with axis $\vec{n} = \vec{y}$, the field $\vec{E}_1$ is expanded as

$$\vec{E}_1 = \frac{E_0}{2}\vec{x}\cos\omega t + \frac{E_0}{2}\vec{y}\cos\omega t \tag{403}$$

and then add a phase π (half-wave plate) to the component in the direction $\vec{y}$:

$$\vec{E}_2 = \frac{E_0}{2}\vec{x}\cos\omega t + \frac{E_0}{2}\vec{y}\underbrace{\cos(\omega t+\pi)}_{-\cos\omega t} = \frac{E_0}{\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t. \tag{404}$$

After the wave plate, we have linearly polarized light with the plane of polarization given by the directional vector $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$. Its intensity is $I_2 = \langle \vec{E}_2^2 \rangle = \frac{E_0^2}{2}\frac{1}{2} = I_1$ – the wave plate does not change the intensity. The polarizer with transmission axis $\vec{n} = \vec{y}$ simply annihilates the electric field component in the direction $\vec{x}$:

$$\vec{E}_3 = (\vec{E}_2 \cdot \vec{y})\,\vec{y} = -\frac{E_0}{2}\vec{y}\cos\omega t \sim \frac{E_0}{2}\vec{y}\cos\omega t \tag{405}$$

(we removed the minus sign because it represents just a total phase shift of π). The intensity is $I_3 = \langle \vec{E}_3^2 \rangle = \frac{E_0^2}{4}\frac{1}{2} = \frac{1}{4}I_0$. Finally, the light passes through a quarter-wave plate with axis $\vec{n} = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$. We must first express the field $\vec{E}_3$ in

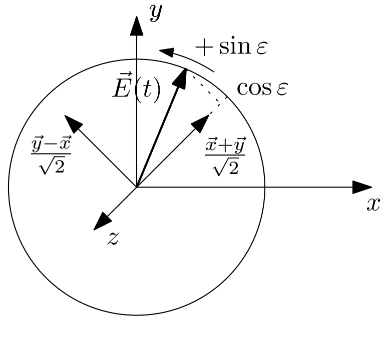
directions $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$ and (perpendicular to it) $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. We perform a similar calculation as in example 9.7 and get $\vec{y} = \frac{1}{2}(\vec{y}+\vec{y}+\vec{x}-\vec{x}) = \frac{1}{\sqrt{2}}\left(\frac{\vec{x}+\vec{y}}{\sqrt{2}} - \frac{\vec{x}-\vec{y}}{\sqrt{2}}\right)$, thus

$$\vec{E}_3 = \frac{E_0}{2\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t - \frac{E_0}{2\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\cos\omega t. \tag{406}$$

The quarter-wave plate adds a phase $\frac{\pi}{2}$ to the part of the wave in the direction $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$:

$$\vec{E}_4 = \frac{E_0}{2\sqrt{2}}\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t - \frac{E_0}{2\sqrt{2}}\frac{\vec{x}-\vec{y}}{\sqrt{2}}\underbrace{\cos\left(\omega t+\tfrac{\pi}{2}\right)}_{-\sin\omega t} = \frac{E_0}{2\sqrt{2}}\left(\frac{\vec{x}+\vec{y}}{\sqrt{2}}\cos\omega t + \frac{\vec{y}-\vec{x}}{\sqrt{2}}\sin\omega t\right). \tag{407}$$

Clearly, it is circularly polarized light. The direction of rotation (clockwise/anticlockwise) is determined by the direction of rotation of the electric field:



The electric field vector rotates counterclockwise, thus it is left-hand circularly polarized light. Its intensity is $I_4 = \langle \vec{E}_4^2 \rangle = \frac{E_0^2}{8}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{4}I_0$ (again, the wave plate does not change the intensity).

Matrix-wise: The incoming light has a polarization vector of the form

$$\hat{\vec{E}}_0 = E_0\begin{pmatrix}1\\0\end{pmatrix} \tag{408}$$

and the incoming intensity is $I_0 = \frac{1}{2}\left(E_0^2 + 0^2\right) = \frac{1}{2}E_0^2$. The polarizer with the transmission axis in the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$ acts through a projector

$$\hat{\vec{E}}_1 = \mathbb{P}_{\frac{\vec{x}+\vec{y}}{\sqrt{2}}}\hat{\vec{E}}_0 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}E_0\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}E_0\\ E_0\end{pmatrix} = \frac{E_0}{2\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}. \tag{409}$$

After the polarizer, the light is linearly polarized with the plane of polarization in the direction $\frac{\vec{x}+\vec{y}}{\sqrt{2}}$. The intensity is $I_1 = \frac{1}{2}\frac{1}{4}\left(E_0^2 + E_0^2\right) = \frac{1}{2}I_0$. The half-wave plate acts further as

$$\hat{\vec{E}}_2 = \mathbb{D}_{\pi,\vec{y}}\hat{\vec{E}}_1 = \left[e^{i\pi}\mathbb{P}_{\vec{y}} + \mathbb{P}_{\vec{x}}\right]\hat{\vec{E}}_1 = \left[e^{i\pi}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]\begin{pmatrix}\frac{1}{2}E_0\\ \frac{1}{2}E_0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2}E_0\\ \frac{1}{2}E_0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}E_0\\ -E_0\end{pmatrix}. \tag{410}$$

The intensity after passing through the plate $I_2 = \frac{1}{2}\left(\frac{1}{4}E_0^2 + \frac{1}{4}E_0^2\right) = \frac{1}{2}I_0$ – the wave plate does not change the intensity. Again, there is no phase shift between the components of the polarization vector $\hat{\vec{E}}_2$, thus it is linearly polarized light this time with the plane of polarization in the direction $\frac{\vec{x}-\vec{y}}{\sqrt{2}}$. (we can write $\hat{\vec{E}}_2 = \frac{E_0}{2\sqrt{2}}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T$).

The polarizer with transmission axis $\vec{n} = \vec{y}$ acts as follows

$$\hat{\vec{E}}_3 = \mathbb{P}_{\vec{y}}\hat{\vec{E}}_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{E_0}{2}\\ -\frac{E_0}{2}\end{pmatrix} = \begin{pmatrix}0\\ -\frac{E_0}{2}\end{pmatrix} \sim \frac{E_0}{2}\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{411}$$

Again, it is linearly polarized light with the plane of polarization in the direction $\vec{y}$ with intensity $I_3 = \frac{1}{2}\left(0^2 + \frac{E_0^2}{4}\right) = \frac{1}{4}I_0$.

In the last step, we have a quarter-wave plate with axis $\vec{n} = \frac{\vec{x}-\vec{y}}{\sqrt{2}}$. The complementary direction is thus $\vec{n}' = \frac{\vec{x}+\vec{y}}{\sqrt{2}}$. Its matrix is given by the relation $\mathbb{D}_{\frac{\pi}{2}} = e^{i\frac{\pi}{2}}\mathbb{P}_{\vec{n}} + \mathbb{P}_{\vec{n}'}$. Explicitly

$$\begin{aligned}\mathbb{D}_{\frac{\pi}{2},\vec{n}} &= e^{i\frac{\pi}{2}}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i & 1-i\\ 1-i & 1+i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1+i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}}\\ \frac{1-i}{\sqrt{2}} & \frac{1+i}{\sqrt{2}}\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}}\\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}}\end{pmatrix} = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix} \sim \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}.\end{aligned} \tag{412}$$

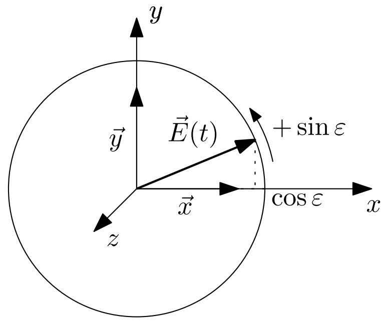
The resulting light then is

$$\hat{\vec{E}}_4 = \mathbb{D}_{\frac{\pi}{2},\vec{n}}\hat{\vec{E}}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\begin{pmatrix}0\\ \frac{E_0}{2}\end{pmatrix} = \frac{E_0}{2\sqrt{2}}\begin{pmatrix}-i\\ 1\end{pmatrix} = \frac{E_0}{2\sqrt{2}}\begin{pmatrix}e^{-i\frac{\pi}{2}}\\ 1\end{pmatrix}. \tag{413}$$

There is a phase difference of $\frac{\pi}{2}$ between the components of the electric field (with the same amplitude), thus it is circularly polarized light. By expressing in vector notation

$$\vec{E}_4 = \frac{E_0}{2\sqrt{2}}\left(\vec{x}\,e^{i(\omega t-\frac{\pi}{2})} + \vec{y}\,e^{i\omega t}\right) \stackrel{Re}{=} \frac{E_0}{2\sqrt{2}}\Big(\vec{x}\underbrace{\cos(\omega t-\tfrac{\pi}{2})}_{\sin\omega t} + \vec{y}\cos\omega t\Big) \tag{414}$$

we can easily determine the sense of rotation of the electric field:



It is left-hand circularly polarized light, as the electric field vector rotates counterclockwise in the xy plane. The resulting intensity is $I_4 = \frac{1}{2}\frac{E_0^2}{8}(1+1) = \frac{1}{4}I_0$ – the wave plate again does not change the intensity.

***Exercise 9.10.** What values of the Stokes parameters P_1, P_2, and P_3 correspond to linearly, respectively, circularly polarized light. Plot the results.

Solution: The Stokes parameters for the electric field $\vec{E}(t) = (E_x(t), E_y(t))$ are given by the following expressions:

$$P_1 = \frac{\langle E_x^2\rangle - \langle E_y^2\rangle}{\langle E_x^2\rangle + \langle E_y^2\rangle}, \qquad P_2 = \frac{\langle 2E_xE_y\rangle}{\langle E_x^2\rangle + \langle E_y^2\rangle}, \qquad P_3 = \frac{\langle 2E_x(\omega t - \frac{\pi}{2})E_y\rangle}{\langle E_x^2\rangle + \langle E_y^2\rangle}. \tag{415}$$

Consider light linearly polarized in the direction $\vec{n}$, thus $\vec{E}(t) = E_0\vec{n}\cos\omega t$. From this

$$E_x(t) = E_0 n_x \cos\omega t, \qquad E_y(t) = E_0 n_y \cos\omega t. \tag{416}$$

From this, we

find that

$$\langle E_x^2\rangle = E_0^2 n_x^2 \langle\cos^2\omega t\rangle = \frac{1}{2}E_0^2 n_x^2, \qquad \langle E_y^2\rangle = \frac{1}{2}E_0^2 n_y^2. \tag{417}$$

For calculating P_2, we need the mean value

$$\langle 2E_xE_y\rangle = \langle 2E_0^2 n_x n_y \cos^2\omega t\rangle = E_0^2 n_x n_y. \tag{418}$$

Finally, for calculating P_3, we need the mean value

$$\begin{aligned}\left\langle 2E_x(\omega t - \tfrac{\pi}{2})E_y\right\rangle &= 2\left\langle E_0^2 n_x n_y \cos\omega t\cos(\omega t - \tfrac{\pi}{2})\right\rangle \\ &= 2E_0^2 n_x n_y \langle\cos\omega t\sin\omega t\rangle \\ &= E_0^2 n_x n_y \langle\sin 2\omega t\rangle = 0.\end{aligned} \tag{419}$$

Substituting into the Stokes parameters, we get

$$P_1 = n_x^2 - n_y^2, \quad P_2 = 2n_xn_y, \quad P_3 = 0. \tag{420}$$

If we let $n_x = \cos\theta$ and $n_y = \sin\theta$, then $P_1 = \cos 2\theta$, $P_2 = \sin 2\theta$, $P_3 = 0$. In the space of Stokes parameters (P_1, P_2, P_3), we thus get a unit circle in the plane $P_3 = 0$. Note that a given point on the circle corresponds to exactly two directions $\vec{n}$ and $-\vec{n}$, so the Stokes parameters uniquely determine the plane of polarization of the given light!

Circularly polarized light is given, for example, by the vector $\vec{E}(t) = E_0(\cos\omega t, \pm\sin\omega t)$. Hence

$$\langle E_x^2\rangle = \langle E_0^2\cos^2\omega t\rangle = \frac{1}{2}E_0^2 = \langle E_0^2\sin^2\omega t\rangle = \langle E_y^2\rangle. \tag{421}$$

For calculating P_2, we need the mean value

$$\langle 2E_xE_y\rangle = \pm E_0^2\langle\cos\omega t\sin\omega t\rangle = 0. \tag{422}$$

Finally, for calculating P_3, we need to compute

$$\left\langle 2E_x(\omega t - \tfrac{\pi}{2})E_y\right\rangle = \pm 2E_0^2\langle\sin^2\omega t\rangle = \pm E_0^2. \tag{423}$$

Substituting into the Stokes parameters, we thus get $P_1 = 0$, $P_2 = 0$, and $P_3 = \pm 1$. Circularly polarized light thus corresponds to the poles of a sphere with radius 1.

***Exercise 9.11.** The light hitting a linear polarizer is a mixture of linearly polarized and unpolarized light. If you rotate the polarizer by 60° compared to the orientation with the maximum transmitted intensity, you get half the intensity. Determine the ratio of intensities of unpolarized and linearly polarized light in the mixture.

Solution: The

intensity of the incoming light I_d is the sum $I_d = I_p + I_n$, where I_p is the intensity of linearly polarized light and I_n is the intensity of unpolarized light. The interference term can be neglected since it is a superposition of incoherent waves. Depending on the angle θ to the plane of linearly polarized light, the transmitted light intensity $I_o(\theta) = I_p\cos^2\theta + \frac{1}{2}I_n$ (according to Malus's law and example 9.2). The transmitted intensity is greatest for $\theta = 0$. From the condition, we have the equation $I_o(0) = 2I_o(\pm\frac{\pi}{3})$. Substituting

$$I_p + \frac{1}{2}I_n = 2\left(\frac{1}{4}I_p + \frac{1}{2}I_n\right) = \frac{1}{2}I_p + I_n. \tag{424}$$

From this $I_p = I_n$ – the mixture contains equal proportions of linearly polarized and unpolarized light.

***Exercise 9.12.** The direction of polarization of linearly polarized light changes rapidly (much faster than the resolving time of the measuring device) between the following two states: $\vec{n} = (\cos\theta_0, \pm\sin\theta_0)$, where $\theta_0 < \frac{\pi}{2}$. Calculate the Stokes parameters. Determine the degree of polarization $|\vec{P}| = |(P_1, P_2, P_3)|$ depending on θ_0.

Solution: We thus have an electric field, in which the following two states $\vec{E}^+$ and $\vec{E}^-$ rapidly alternate:

$$\vec{E}^{\pm}(t) = E_0\begin{pmatrix}\cos\theta_0\\ \pm\sin\theta_0\end{pmatrix}\cos\omega t. \tag{425}$$

The mean values of the electric field components over the resolution time of the device will be given by the arithmetic mean of the mean values over the period of the two states mentioned above, thus schematically

$$\langle E^2\rangle_{t_{res}} = \frac{1}{2}\left(\langle E^{+2}\rangle_T + \langle E^{-2}\rangle_T\right). \tag{426}$$

Specifically then

$$\langle E_x^2\rangle_{t_{res}} = \frac{1}{2}\left(\langle E_x^{+2}\rangle_T + \langle E_x^{-2}\rangle_T\right) = \frac{1}{2}\left(\cos^2\theta_0\frac{1}{2} + \cos^2\theta_0\frac{1}{2}\right) = \frac{1}{2}\cos^2\theta_0. \tag{427}$$

A completely identical calculation gives $\langle E_y^2\rangle = \frac{1}{2}\sin^2\theta_0$. For the mixed terms, we get

$$\begin{aligned}\langle 2E_xE_y\rangle_{t_{res}} &= \frac{1}{2}\left(\langle 2E_x^+E_y^+\rangle_T + \langle 2E_x^-E_y^-\rangle_T\right) \\ &= \frac{1}{2}\left(2E_0^2\cos\theta_0\sin\theta_0\langle\cos^2\omega t\rangle_T - 2E_0^2\cos\theta_0\sin\theta_0\langle\cos^2\omega t\rangle_T\right) = 0.\end{aligned} \tag{428}$$

The last mean value needed for calculating P_3 is

$$\langle 2E_x(\omega t - \tfrac{\pi}{2})E_y\rangle = \frac{1}{2}\left(2E_0^2\cos\theta_0\sin\theta_0\langle\sin\omega t\cos\omega t\rangle_T - 2E_0^2\cos\theta_0\sin\theta_0\langle\sin\omega t\cos\omega t\rangle_T\right) = 0. \tag{429}$$

Overall, we get

$$P_1 = \cos^2\theta_0 - \sin^2\theta_0 = \cos 2\theta_0, \qquad P_2 = 0, \qquad P_3 = 0. \tag{430}$$

We see that the Stokes parameters come out similar to light polarized in the direction of the $\vec{x}$ or $\vec{y}$ axis, but the vector $\vec{P} = (P_1, P_2, P_3)$ does not lie on the unit sphere because $|\vec{P}| = |\cos 2\theta_0|$. The degree of polarization thus decreases with increasing angle θ_0, until for $\theta_0 = \frac{\pi}{4}$, we get $|\vec{P}| = 0$, corresponding to unpolarized light (then it increases again, because for $\theta_0 = \frac{\pi}{2}$ we get linearly polarized light in the direction of the $\vec{y}$ axis).

10 Interference

***Exercise 10.1** (*Fabry-Pérot etalon*)**.** Consider the result of exercise 7.5, i.e., the total reflection coefficient at two interfaces

$$R = \frac{R_{12} + R_{23}e^{-2ik_2L}}{1 + R_{12}R_{23}e^{-2ik_2L}}, \tag{431}$$

where R_{12} and R_{23} are the reflection coefficients of individual interfaces, k_2 is the wave number in the medium between the interfaces, and L is the distance between the interfaces. Now consider that the interfaces are formed by the same semi-transparent mirrors, i.e., $R_{12} = R_{23} = r$. Find the relationship between the wavelength λ and the distance between mirrors L, at which the total reflectivity $\mathcal{R} = |R^2|$ is zero.

Solution: After substitution, we solve the equation

$$0 = \frac{r(1 + e^{-2ik_2L})}{1 + r^2e^{-2ik_2L}}. \tag{432}$$

Since the denominator is always different from zero and considering $r \neq 0$, we obtain a simple condition $1 + e^{-2ik_2L} = 0$. From this, we get the equation

$$-2k_2L \in \{\pi + 2n\pi \mid n \in \mathbb{Z}\} \tag{433}$$

This gives us the condition $k_2L \in \{-\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\}$. Since we have $k_2L > 0$, we obtain

$$k_2L \in \{\frac{\pi}{2} + n\pi \mid n \in \mathbb{N}_0\} \tag{434}$$

Now, we just use the relationship between the wave number and wavelength, thus $k_2 = \frac{2\pi}{\lambda}$. After substitution, therefore

$$L\frac{2\pi}{\lambda} \in \{\frac{\pi}{2} + n\pi \mid n \in \mathbb{N}_0\} \tag{435}$$

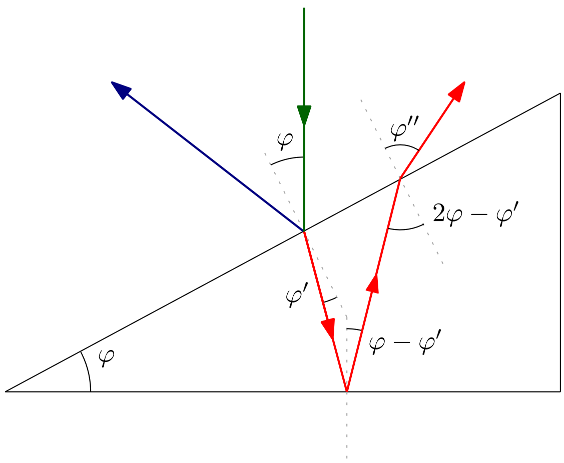
After dividing the equation by 2π, we get the final relationship

$$\frac{L}{\lambda} \in \{\frac{2n-1}{4} \mid n \in \mathbb{N}\}. \tag{436}$$

Exercise 10.2 (*Glass wedge*)**.** Flat surfaces of a glass wedge with a refractive index $n = 1.5$ form a very small angle $\varphi = 0.1'$. Light of wavelength $\lambda = 500$nm falls perpendicularly on the wedge. Calculate the distance between interference fringes.

Guide: Find the angle between the emerging rays and use the result of example 8.5.

Solution: The light falls perpendicularly on the base. We draw a picture:



We obtain the angles of deviation from the normals with a bit of trigonometry. For example, the angle $\bar{\varphi}$ of deviation from the perpendicular to the base is obtained by the sum of angles in the triangle:

$$\pi = \varphi + (\frac{\pi}{2} - \varphi') + (\frac{\pi}{2} - \bar{\varphi}). \tag{437}$$

From this, $\bar{\varphi} = \varphi - \varphi'$. Similarly for the second angle $2\varphi - \varphi'$. Since we assume all angles are very small, we approximately use Snell's law of refraction:

$$\sin(\varphi) = n\sin(\varphi'), \ \ \sin(\varphi'') = n\sin(2\varphi - \varphi'). \tag{438}$$

Since all angles are very small, we can use the approximation $\sin(x) \approx x$ everywhere. Thus, we get

$$\varphi = n\varphi', \ \ \varphi'' = n(2\varphi - \varphi'). \tag{439}$$

From this, $\varphi'' = (2n-1)\varphi$. The total angle formed by the reflected rays is then approximately

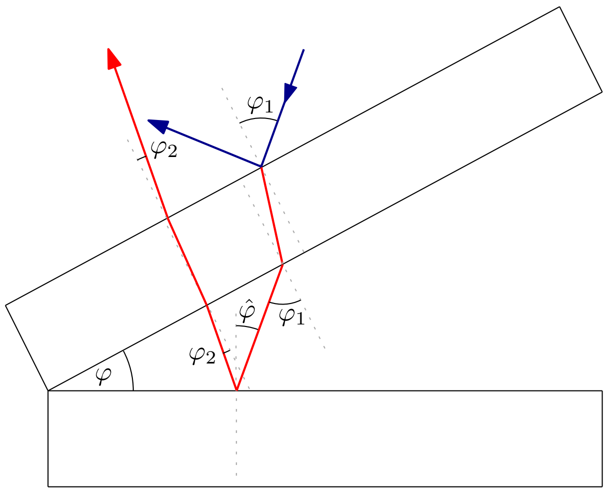
$$\Delta\varphi = \varphi + \varphi'' = 2n\varphi. \tag{440}$$

In example 8.5, we found that the distance between interference maxima on the screen (perpendicular to the direction of both rays) is given by

$$\Delta y = \frac{\lambda}{\Delta\varphi} = \frac{\lambda}{2n\varphi} = \frac{500 \cdot 10^{-9}}{2 \cdot \frac{3}{2} \cdot \frac{1}{600} \cdot \frac{\pi}{180}} = \frac{18 \cdot 10^{-3}}{\pi}\text{m} \approx 5.73\text{mm}. \tag{441}$$

***Exercise 10.3** (*Air wedge*)**.** An air wedge is bounded by two perfectly flat glass plates with a refractive index $n = 1.5$, which form a very small angle φ. This angle is given by the fact that a strip of aluminum foil of thickness $d = 0.02$mm was inserted at a distance $L = 10cm$ from their touching edges. Sodium light ($\lambda = 589nm$) falls perpendicularly on the wedge layer. Determine the distance between the interference fringes in a) reflected and b) transmitted light.

Solution: For situation a) we have the following picture:



Consider a more general situation of angle φ_1. Notice that the twice-refracted light is parallel to the original one. We only need to determine the angle φ_2. This is again a bit of trigonometry. The angle $\hat{\varphi}$ which the ray falling on the lower prism forms with the perpendicular satisfies the equation

$$\varphi + (\frac{\pi}{2} - \varphi_1) + (\frac{\pi}{2} + \hat{\varphi}) = \pi, \tag{442}$$

from which $\hat{\varphi} = \varphi_1 - \varphi$. The angle φ_2 then satisfies the equation

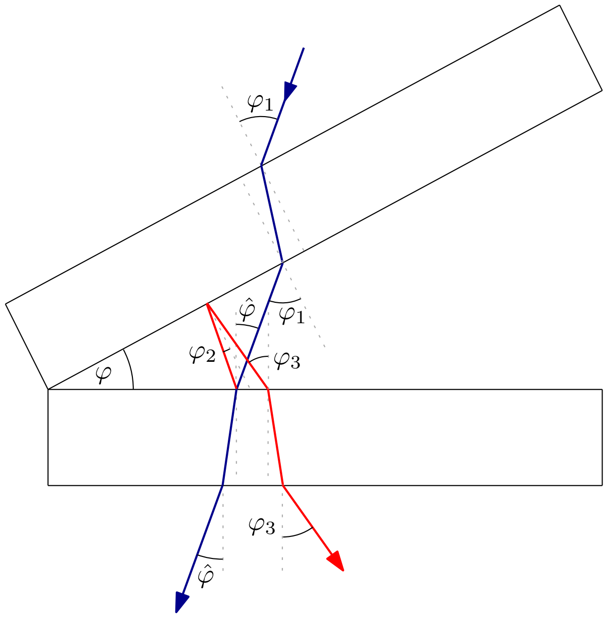
$$\varphi + (\frac{\pi}{2} - \varphi_2) + (\frac{\pi}{2} - \hat{\varphi}) = \pi. \tag{443}$$

Thus, $\varphi_2 = \varphi - \hat{\varphi} = 2\varphi - \varphi_1$. The total angle between both rays is therefore

$$\Delta\varphi = \varphi_1 + \varphi_2 = 2\varphi. \tag{444}$$

We see that the distance between interference maxima $\Delta y = \frac{\lambda}{2\varphi}$ does not depend on n nor on the angle of incidence ϑ_1 at all!

For situation b) we have a slightly modified picture:



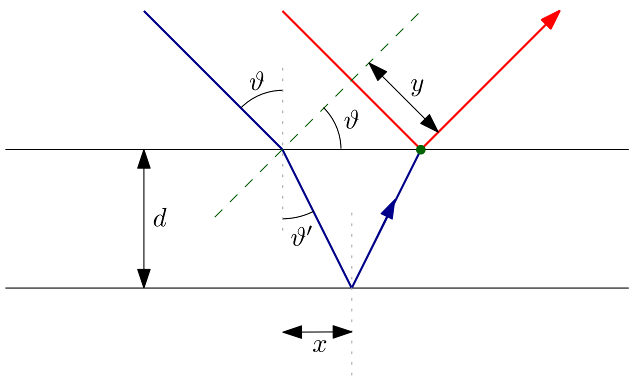
So, we just need to determine the angle φ_3 under which the twice-reflected ray falls on the lower prism. We easily obtain it from the equation

$$\varphi + (\frac{\pi}{2} + \varphi_2) + (\frac{\pi}{2} - \varphi_3) = \pi, \tag{445}$$

thus $\varphi_3 = \varphi + \varphi_2 = 3\varphi - \varphi_1$. The resulting angle between two rays is $\Delta\varphi = \hat{\varphi} + \varphi_3 = 2\varphi$.

Exercise 10.4 (*Soap Film Alias Interference on a Thin Layer*)**.** You have a planar soap film of thickness d with a refractive index n. If you observe the reflection of light at an angle ϑ on the soap film, due to constructive interference for a certain wavelength of light λ, you see the film colored. Find the condition for constructive interference for the parameters $(d, \vartheta, \lambda, n)$.

Solution: We must calculate the difference in so-called optical paths traveled by both rays. The optical path is the phase change along the actual traveled distance ℓ. For a planar traveling wave, this is simply $k\ell$, where k is the wave number of the given medium. In our case, we compare two rays in the figure:



Therefore, we compare the optical paths from the location marked by the dashed line to the location marked by the dot. The wave number for propagation in air is given by $k = \frac{2\pi}{\lambda}$, the wave number in a medium with refractive index n is its multiple nk.

The actual trajectory traveled by one reflected ray is denoted as y. This can be calculated from the right-angled triangle, with y being the opposite side:

$$y = 2x\sin(\vartheta). \tag{446}$$

The distance x can be obtained from the right-angled triangle with opposite and adjacent sides d and x, thus

$$x = d\,\mathrm{tg}(\vartheta') = d\frac{\sin(\vartheta')}{\sqrt{1-\sin^2(\vartheta')}} = d\frac{\frac{1}{n}\sin(\vartheta)}{\sqrt{1-\frac{1}{n^2}\sin^2(\vartheta)}} = d\frac{\sin(\vartheta)}{\sqrt{n^2-\sin^2(\vartheta)}} \tag{447}$$

Together, we obtain the expression for y in the form

$$y = \frac{2d\sin^2(\vartheta)}{\sqrt{n^2-\sin^2(\vartheta)}} \tag{448}$$

The optical path for the first ray is then

$$\varphi_1 = \frac{2\pi}{\lambda}y + \pi. \tag{449}$$

It's easy to forget about the π term. The first wave reflects at the air-soap interface, where the reflection coefficient is $R = \frac{1-n}{1+n} < 0$. The reflected wave thus gains an additional phase of π simply by reflecting.

On the other hand, the distance y' traveled by the second of the rays is given by twice the hypotenuse of both triangles:

$$y' = 2\frac{d}{\cos(\vartheta')} = \frac{2dn}{\sqrt{n^2-\sin^2(\vartheta)}} \tag{450}$$

The optical path traveled by the second ray is thus

$$\varphi_2 = nky' = \frac{2dn^2}{n^2-\sin^2(\vartheta)} \tag{451}$$

The sought difference in optical paths $\Delta\varphi = \varphi_2 - \varphi_1$ is then

$$\Delta\varphi = \frac{4\pi d}{\lambda}\frac{(n^2-\sin^2(\vartheta))}{\sqrt{n^2-\sin^2(\vartheta))}} - \pi = \frac{4\pi d}{\lambda}\sqrt{n^2-\sin^2(\vartheta)} - \pi. \tag{452}$$

Constructive interference of both lights occurs for $\Delta\varphi = 2m\pi$, $m \in \mathbb{Z}$. We thus get the condition

$$\frac{4\pi d}{\lambda}\sqrt{n^2-\sin^2(\vartheta)} = (2m+1)\pi, \quad m \in \mathbb{Z}. \tag{453}$$

Why does the bubble appear colored due to constructive interference? If n, d, and ϑ are given (we look at the plane of the bubble at some angle), in the light, components with wavelength λ given by the relation predominate:

$$\lambda = \frac{4d}{2m+1}\sqrt{n^2-\sin^2(\vartheta)}. \tag{454}$$

11 Diffraction

Exercise 11.1. What is the highest order maximum you can observe in green light with a wavelength of $\lambda = 550$nm for a diffraction grating with 5000 grooves per 1cm?

Solution: Let d be the distance between adjacent grooves on the grating. The angle θ_m, under which we observe the m-th order maximum on the screen, is given by the relationship

$$\sin\theta_m = m\frac{\lambda}{d}. \tag{455}$$

From this, we get the condition for the maximum order of the maximum in the form $m\frac{\lambda}{d} < 1$, thus

$$m < \frac{d}{\lambda}. \tag{456}$$

The groove density in our case $n = 5.10^5\,\mathrm{m}^{-1}$. The distance between adjacent grooves is then $d = 1/n$, and we obtain the condition

$$m < \frac{1}{n\lambda} = \frac{1}{550 \cdot 5 \cdot 10^{-4}} = \frac{1}{2{,}75 \cdot 10^{-1}} \approx 3{,}6. \tag{457}$$

From this, we see that we observe maxima of at most the third order.

Exercise 11.2. Can the spectra of the 1. and 2. order and the spectra of the 2. and 3. order, generated on a diffraction grating when illuminated with white light consisting of wavelengths 400–700 nm, overlap?

Solution: The distance of the m-th maximum from the axis of the diffraction grating depends on the wavelength by the relationship $y_m(\lambda) = m\frac{L\lambda}{d}$. We first address the condition whether the situation $y_1(\lambda_1) \geq y_2(\lambda_0)$, where $\lambda_0 = 400\,\mathrm{nm}$ and $\lambda_1 = 700\,\mathrm{nm}$, can occur. After substitution, we get the requirement

$$1\frac{L\lambda_1}{d} \geq 2\frac{L\lambda_0}{d}, \tag{458}$$

from which the condition $\lambda_1 \geq 2\lambda_0$ arises. For the given values, this cannot occur, and the first and second spectrum never overlap. For the second and third maximum, we get the inequality

$$\lambda_1 \geq \frac{3}{2}\lambda_0, \tag{459}$$

which the mentioned wavelengths satisfy. Thus, the 2nd and 3rd order spectra can overlap. Whether they actually will depends on the parameters of the diffraction grating (the 2nd and 3rd order spectrum may not be visible at all).

***Exercise 11.3.** A diffraction grating has 500 grooves per 1 mm. Calculate the so-called dispersion, i.e., the quantity $\frac{d\theta}{d\lambda}$, near green light ($\lambda = 500$ nm) for the first and second order.

Solution: For the grating, the angular dependence of the m-th order maximum is given by the relationship $\sin\theta_m = m\frac{\lambda}{d}$. Hence, $\theta_m(\lambda) = \arcsin(m\frac{\lambda}{d})$. The dispersion for the m-th order is obtained by differentiation:

$$\frac{d\theta_m}{d\lambda}(\lambda) = \frac{1}{\sqrt{1 - (\frac{m\lambda}{d})^2}} \cdot \frac{m}{d} = \frac{m}{\sqrt{d^2 - m^2\lambda^2}}. \tag{460}$$

The groove density n in this case is $n = 5{\cdot}10^5\,\mathrm{m}^{-1}$. Hence $d = 20{\cdot}10^{-7}\,\mathrm{m}$. The wavelength is $\lambda = 5{\cdot}10^{-7}\,\mathrm{m}$. The number under the square root is thus

$$d^2 - m^2\lambda^2 = (400 - 25m^2) \cdot 10^{-14}. \tag{461}$$

For $m = 1$ and $m = 2$, we thus get

$$\frac{d\theta_1}{d\lambda}(\lambda) = \frac{1}{\sqrt{375}} \cdot 10^7\,\mathrm{m}^{-1} \approx 5{,}16{\cdot}10^{-5}\,\mathrm{m}^{-1}, \quad \frac{d\theta_2}{d\lambda}(\lambda) = \frac{2}{\sqrt{300}} \cdot 10^7\,\mathrm{m}^{-1} \approx 11{,}5{\cdot}10^{-5}\,\mathrm{m}^{-1}. \tag{462}$$

Exercise 11.4. Yellow light emitted by sodium atoms is dominated by the so-called sodium doublet, whose wavelengths are $\lambda_1 = 589{,}0$ nm and $\lambda_2 = 589{,}6$ nm. What is the minimum number of grooves/slots on the grating required to distinguish these two wavelengths in the first-order spectrum?

Solution: Let us denote the angles under which we observe the first-order maxima for both wavelengths as θ_1 and θ_2. The approximate relationship holds

$$\theta_1 = \frac{\lambda_1}{d}, \qquad \theta_2 = \frac{\lambda_2}{d}. \tag{463}$$

For a diffraction grating, the widths of diffraction maxima (the distance between the first zeros of intensity around the maximum) are given by

$$\delta\theta = \frac{2\lambda}{Nd}, \tag{464}$$

where N is the total number of grooves on the diffraction grating. To be able to distinguish the spectra of both wavelengths, both halves of the widths must "fit" between the two maxima. We thus get the relationship

$$\frac{1}{d}(\lambda_2 - \lambda_1) = \theta_2 - \theta_1 \geq \frac{1}{2}\big(\delta\theta(\lambda_1) + \delta\theta(\lambda_2)\big) = \frac{1}{Nd}(\lambda_2 + \lambda_1). \tag{465}$$

From this, we can express the constant N as

$$N \geq \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1} = \frac{589{,}6 + 589{,}0}{0{,}6} \approx 1964{,}3. \tag{466}$$

Thus, the grating must contain at least 1965 grooves.

Exercise 11.5. Place a hair with diameter d in the path of a laser beam with a wavelength of $\lambda = 632{,}8$ nm. On a screen at a distance of $L = 6$ m, you observe diffraction maxima at a distance of $\Delta l = 3$ cm. What is the diameter of the hair?

Solution: According to Babinet's principle, the interference pattern for a hair is the same as for a finite-sized slit of width d. For a slit of finite width d, it turns out the same as for two thin slits d apart. The distance between adjacent maxima is thus

$$\Delta l \approx L \cdot \Delta\theta = L\frac{\lambda}{d}. \tag{467}$$

From this, we can express d as a function of the remaining variables and get

$$d = \frac{L\lambda}{\Delta l} = \frac{6 \cdot 632,8 \cdot 10^{-9}}{3 \cdot 10^{-2}} m = 2 \cdot 632{,}8 \cdot 10^{-7}\,\mathrm{m} = 126\,\mu\mathrm{m}. \tag{468}$$

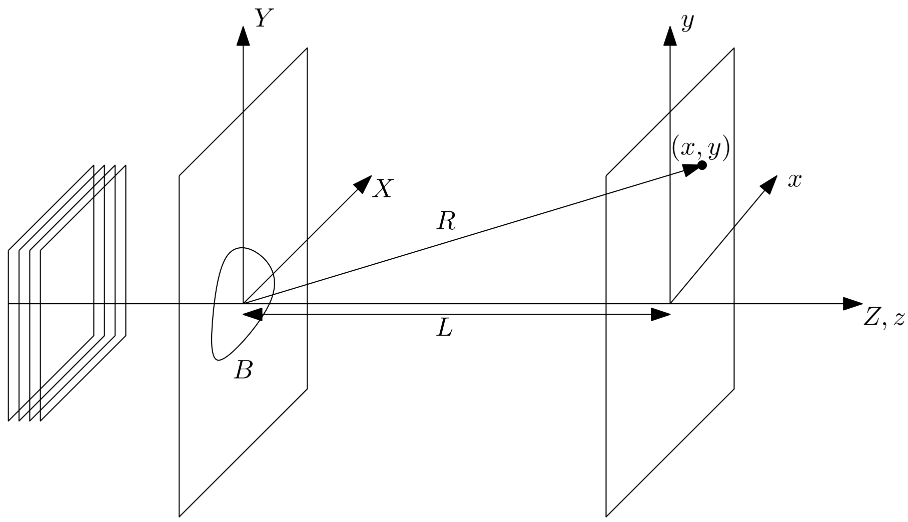
Exercise 11.6 (*Diffraction pattern of a rectangular slit*)**.** Find the diffraction pattern (intensity distribution on the screen)

of a rectangular slit of dimensions a, b.

Solution: When a plane wave of the electric field (in the direction of the z axis) hits a barrier with an opening B, the intensity of the electric field $\vec{E} = \vec{E}(x, y)$ on the parallel screen at a perpendicular distance L is given by the Fraunhofer integral

$$\vec{E}(x,y) = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \int_B e^{i\frac{k}{R}(xX+yY)} dX dY, \tag{469}$$

where to the right is the area integral over the area of the opening B and $R = R(x, y) = \sqrt{L^2 + x^2 + y^2}$. See figure:



In this example, B is a rectangle with the center at $(X, Y) = (0, 0)$ with sides a and b. The area integral in this case is very simple, we have to calculate

$$\int_B e^{i\frac{k}{R}(xX+yY)} dX dY = \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{i\frac{kx}{R}X} dX \right) \left(\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i\frac{ky}{R}Y} dY \right). \tag{470}$$

Both integrals give a similar result, let's compute just one of them:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{i\frac{kx}{R}X} dX = \frac{R}{ikx} \left(e^{i\frac{kx}{R}\frac{a}{2}} - e^{-i\frac{kx}{R}\frac{a}{2}} \right) = \frac{2R}{kx} \sin\left(\frac{kx}{R}\frac{a}{2} \right). \tag{471}$$

The resulting electric field is then after substitution

$$\vec{E}(x,y) = \vec{E}_0 e^{i(\omega t - kR)} \frac{ab}{R} \frac{\sin\left(\frac{kxa}{2R}\right)}{\frac{kxa}{2R}} \frac{\sin\left(\frac{kyb}{2R}\right)}{\frac{kyb}{2R}}. \tag{472}$$

The intensity $I = I(x, y)$ is the time average of the square (real part) of this field:

$$I(x,y) = \left\langle \mathrm{Re}[\vec{E}(x,y)]^2 \right\rangle = \frac{E_0^2 a^2 b^2}{2R^2} \left(\frac{\sin(\frac{kxa}{2R})}{\frac{kxa}{2R}} \right)^2 \left(\frac{\sin(\frac{kyb}{2R})}{\frac{kyb}{2R}} \right)^2. \tag{473}$$

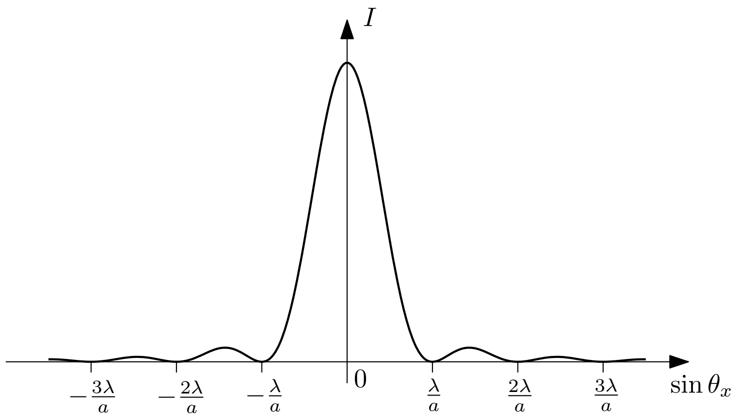
The result can be expressed using two angles defined by the relationships $\sin\theta_x = \frac{x}{R}$ and $\sin\theta_y = \frac{y}{R}$. Then we get

$$I(x,y) = \frac{E_0^2 a^2 b^2}{2R^2} \left(\frac{\sin(\frac{ka}{2}\sin\theta_x)}{\frac{ka}{2}\sin\theta_x} \right)^2 \left(\frac{\sin(\frac{kb}{2}\sin\theta_y)}{\frac{kb}{2}\sin\theta_y} \right)^2. \tag{474}$$

For the intensity distribution on the x axis, we can substitute $y = 0$ (or calculate $\lim_{y\to 0}$) with the result

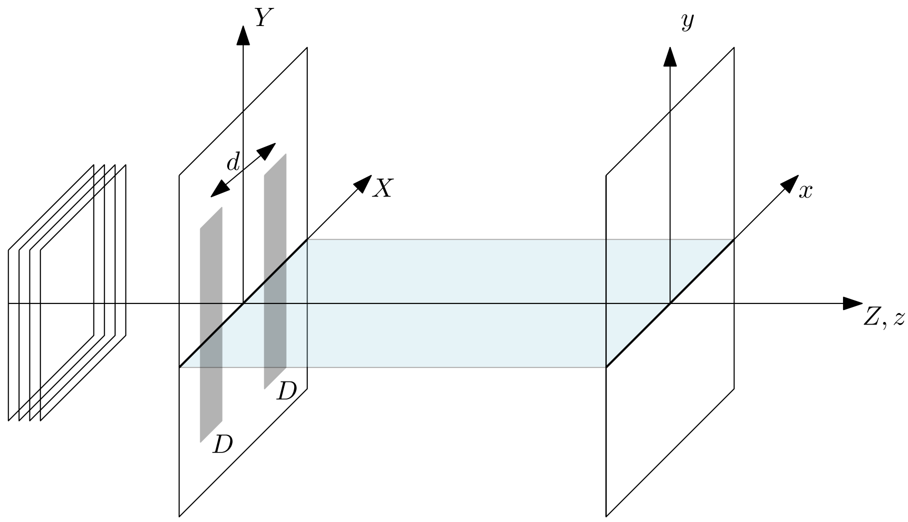
$$I(x) = \frac{E_0^2 a^2 b^2}{2R^2} \left(\frac{\sin(\frac{ka}{2} \sin\theta_x)}{\frac{ka}{2} \sin\theta_x} \right)^2 . \tag{475}$$

The distribution of this intensity (in the variable $\sin\theta_x$) is illustrated in the following figure:



Exercise 11.7 (*Diffraction pattern of two slits*)**.** Find the diffraction pattern of two slits of width D, whose centers are at a distance d.

Solution: We examine a one-dimensional problem, i.e., the intensity distribution depending on x on the slice $y = 0$. We get the result by modifying the previous calculation.



Let's first compute separately the electric fields from individual slits $\vec{E}_{\pm}(x)$. These differ from the electric field of one slit by shifting the limits in the integral over X. Compared to the previous example, we thus calculate:

$$\int_{\pm\frac{d}{2}-\frac{D}{2}}^{\pm\frac{d}{2}+\frac{D}{2}} e^{i\frac{kx}{R}X} dX = \frac{R}{ikx} e^{\pm i\frac{kx}{R}\frac{d}{2}} \left(e^{i\frac{kx}{R}\frac{D}{2}} - e^{-i\frac{kx}{R}\frac{D}{2}} \right) = \frac{D}{2} e^{\pm ik\frac{d}{2}\sin\theta} \frac{\sin(\frac{1}{2}kD\sin\theta)}{\frac{1}{2}kD\sin\theta}, \tag{476}$$

where now $R(x) = \sqrt{L^2 + x^2}$ and $\sin\theta = \frac{x}{R}$. The second integral for $y = 0$ yields $\int_{-\frac{b}{2}}^{\frac{b}{2}} dY = b$. Thus, we have

$$\vec{E}_{\pm}(x) = \vec{E}_0 \frac{bD}{2R} e^{i(\omega t - kR)} e^{\pm i \frac{1}{2} kd \sin\theta} \frac{\sin(\frac{1}{2}kD\sin\theta)}{\frac{1}{2}kD\sin\theta}. \tag{477}$$

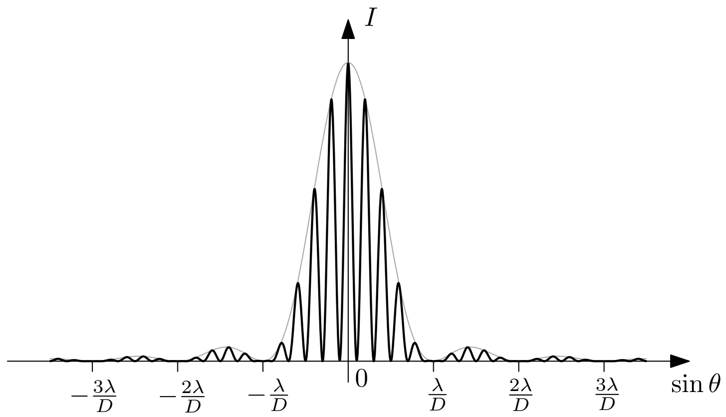
The resulting electric field is the superposition of these two, we get

$$\vec{E}(x) = \vec{E}_+(x) + \vec{E}_-(x) = \vec{E}_0 \frac{bD}{R} e^{i(\omega t - kR)} \cos\left(\frac{1}{2}kd\sin\theta\right) \frac{\sin(\frac{1}{2}kD\sin\theta)}{\frac{1}{2}kD\sin\theta}. \tag{478}$$

The intensity distribution is easily calculated as

$$I(x) = \left\langle \mathrm{Re}[\vec{E}(x,y)]^2 \right\rangle = \frac{E_0^2 b^2 D^2}{2R^2} \cos^2\left(\frac{1}{2}kd\sin\theta\right) \cdot \left(\frac{\sin(\frac{1}{2}kD\sin\theta)}{\frac{1}{2}kD\sin\theta}\right)^2. \tag{479}$$

The intensity distribution on the x axis for $d = \frac{D}{5}$ has this form (in gray is for comparison the intensity of one slit of width D):



***Exercise 11.8** (*Diffraction pattern of a circular aperture*)**.** Compose the diffraction integral for a circular aperture of diameter D. Write the result using the Bessel function $J_n(x)$, whose integral definition is

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x\sin(u) - nu)} du. \tag{480}$$

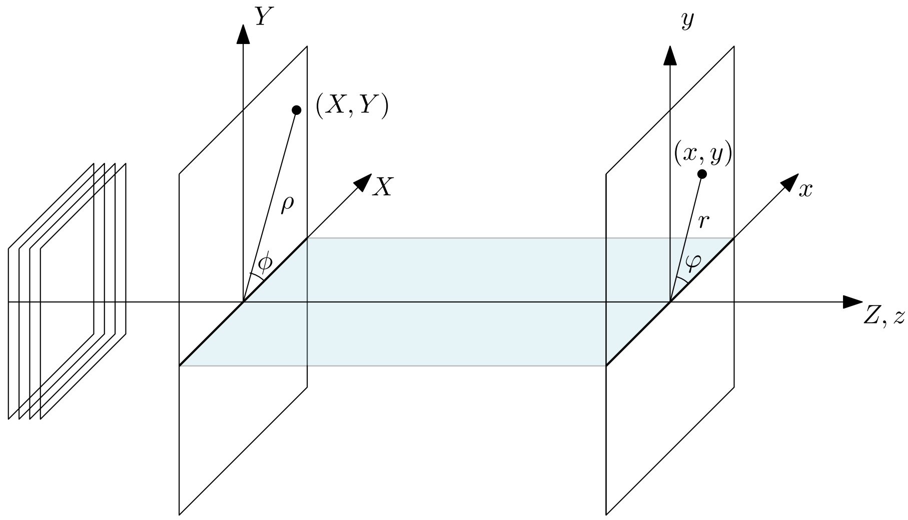
Hint: Introduce polar coordinates in both the screen and barrier planes. Realize that the result cannot depend on the value of the polar angle in the screen plane and set it to a suitable constant. Integrate first over the angular variable. Use the recursive relation

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \tag{481}$$

for $n = 1$.

Solution: Let's denote (r, φ) polar coordinates in the screen and (ρ, ϕ) in the barrier plane. We have

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad X = \rho\cos\phi, \qquad Y = \rho\sin\phi. \tag{482}$$



Notice that $R = \sqrt{L^2 + x^2 + y^2} = \sqrt{L^2 + r^2}$. We must not forget about the change in the area element, we have $dS = dX dY = \rho\, d\rho\, d\phi$. We set $\varphi = \frac{\pi}{2}$ and thus substitute $x = 0$ and $y = r$, $Y = \rho \sin\phi$. The diffraction integral then has the form

$$\vec{E}(r) = \frac{\vec{E}}{R} e^{i(\omega t - kR)} \underbrace{\int_B e^{i\frac{k}{R} r\rho \sin\phi} \rho\, d\rho\, d\phi}_{f(r)} . \tag{483}$$

The integral over the aperture $f(r)$ itself is specifically written as:

$$f(r) := \int_0^{\frac{D}{2}} \rho \int_{-\pi}^{\pi} e^{i\frac{k}{R} r\rho \sin\phi} d\phi\, d\rho. \tag{484}$$

The inner angular integral is, except for a factor of 2π, precisely the Bessel function J_0 evaluated at $\frac{kr\rho}{R}$, thus we get

$$f(r) = 2\pi \int_0^{\frac{D}{2}} \rho\, J_0\left(\frac{kr\rho}{R}\right) d\rho. \tag{485}$$

Now we introduce the substitution $u = \frac{kr\rho}{R}$ and get

$$f(r) = \frac{2\pi R^2}{k^2 r^2} \int_0^{\frac{krD}{2R}} u\, J_0(u)\, du. \tag{486}$$

From the recursive relation above, however, we know that $u\, J_0(u) = \frac{d}{du}[u\, J_1(u)]$. Then easily

$$f(r) = \frac{2\pi R^2}{k^2 r^2} [u J_1(u)]_0^{\frac{krD}{2R}} = \frac{2\pi R^2}{k^2 r^2} \frac{krD}{2R} J_1\left(\frac{krD}{2R}\right) = \frac{\pi R D}{kr} J_1\left(\frac{krD}{2R}\right). \tag{487}$$

Notice that the function $f(r)$ is real. Therefore, the resulting intensity is trivially obtained as

$$\begin{aligned} I(r) = \left\langle \mathrm{Re}[\vec{E}(r)]^2 \right\rangle = \frac{E_0^2}{2R^2} f(r)^2 = \frac{E_0^2 \pi^2 D^2}{2k^2 r^2} J_1\left(\frac{1}{2} kD \sin\theta\right)^2 = \frac{E_0^2 \pi^2 D^4}{8R^2} \left(\frac{J_1(\frac{1}{2}kD\sin\theta)}{\frac{1}{2}kD\sin\theta}\right)^2 \\ = \frac{2E_0^2 S^2}{R^2} \left(\frac{J_1(\frac{1}{2}kD\sin\theta)}{\frac{1}{2}kD\sin\theta}\right)^2 , \end{aligned} \tag{488}$$

where we have again defined angle θ this time by the relationship $\sin\theta = \frac{r}{R}$ and $S = \frac{1}{4}\pi D^2$ is the area of the circular aperture. The intensity distribution then has the following form (in gray is for comparison the intensity for a slit of width D), $\alpha \approx 1,22$: