Solved exercises from ELMA textbook v. 0.95

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You are getting your hands on a collection of **detailed** solved exercises from the Štoll: Electricity and Magnetism textbook.

Each of the exercises should more or less form a separate unit. It is often not necessary to read the preceding or following exercises to understand the solution of a given example (but not always...). If an example is an exception to this rule, it is usually referred to the relevant passage elsewhere in the book.

A solved exercise always consists of a problem, its solution, and an appendix if necessary. The appendix is not necessary to (understand) the solution; it often comments on, extends, or looks at the calculation from a different perspective.

The examples are arranged in logical units and are therefore often not in the order they appear in the original textbook. However, their number is retained for ease of reference. The table of contents is followed by a list of examples arranged in the same order as in the textbook, which makes it easy to quickly find where an exercise is in here.

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Chapter 1

Fundamentals of relativity

1.1 Time dilation and length contraction

1.1.1 1.1 Muon in the atmosphere

A muon in cosmic rays has been observed to travel the distance $d_0 = 5\,km$ in the atmosphere from its formation to its decay at speed $v = 0,99c$. How long did it exist in our frame of reference, how long in its own rest frame, and how thick a layer of atmosphere passed around it in its own frame?

Solution: We can determine the lifetime in our frame of reference simply from the kinematic relation

$$t = \frac{d_0}{v} = \frac{5\,km}{0,99c} = 1,68.10^{-5}\,s = 16,8\,\mu s \tag{1.1}$$

(we used the approximate value of the speed of light $c = 3.10^8\,m/s$). From our point of view, the time for muon passes more slowly due to time dilation. We therefore determine the proper time of the muon from the relation for time dilation:

$$\tau = \frac{t}{\gamma} = t\sqrt{1-\frac{v^2}{c^2}} = \frac{d_0}{v}\sqrt{1-\frac{v^2}{c^2}} = 2,375.10^{-6}\,s = 2,375\,\mu s. \tag{1.2}$$

From the muon's point of view, the atmosphere is whizzing around it at speed v, so it will be contracted in the direction of its motion; substituting in the relation for length contraction, we get the thickness of the atmosphere from the muon's point of view:

$$d = \frac{d_0}{\gamma} = d_0\sqrt{1-\frac{v^2}{c^2}} = 705\,m. \tag{1.3}$$

The same result is obtained by using the kinematic relation

$$d = \tau\, v = 705\,m, \tag{1.4}$$

where we have taken advantage of the fact that the atmosphere around the muon travels at v for its own lifetime τ.

Supplement: In this example, it is possible to come across a relativistic "paradox". From the muon's point of view, it is the observer on Earth who is moving, so the time dilation "affects" him. Thus, the muon measures the calculated time $\tau = 2,37\,\mu s$ on his watch during his lifetime, and according to the muon, the observer on Earth will only experience time

$$\tau' = \frac{\tau}{\gamma} = \tau\sqrt{1-\frac{v^2}{c^2}} = 0,335\,\mu s \quad \text{(instead of } t = 16,8\,\mu s\text{)}. \tag{1.5}$$

How is this possible? The problem here is with the relativity of the simultaneity. Two events that appear to have occurred simultaneously in one frame of reference may not be simultaneous in another frame of reference. In the system associated with the observer on Earth, we have two simultaneous events: the creation of the muon and "starting the stopwatch" by the observer on Earth, also these are simultaneous events: the extinction of the muon and the stopping of the stopwatch by the Earth observer. In the system associated with the muon, these events will generally not be simultaneous! From the point of view of the muon, therefore, the observer on Earth, quite incomprehensibly, starts and stops the stopwatch at a completely different times than the instants when the muon comes into existence and ceases to exist. Let us now quantify these considerations.

Let's write down the spatial and temporal coordinates of each event in the system associated with the Earth and transform them using the Lorentz transformation into the system associated with the muon. Consider a spatial coordinate x whose origin is at the muon point of creation and points downward toward the observer. Then the muon's point of creation P_V, the beginning of the measurement P_M, the muon's extinction P_Z, and the end of the measurement P_K have spatiotemporal coordinates (x, t):

$$P_V = (0,0), \quad P_M = (d_0, 0), \quad P_Z = (d_0, t), \quad P_K = (d_0, t). \tag{1.6}$$

(for simplicity we consider that the muon decays exactly at the observer's feet).

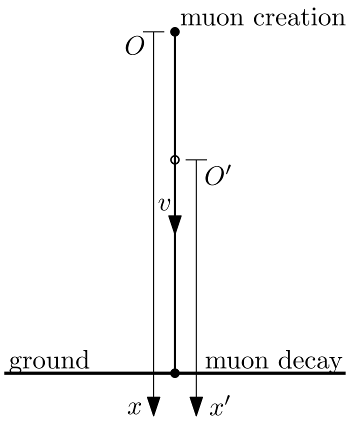

Figure 1.1: Coordinate x associated with an observer on Earth, with the origin at the muon's point of creation. Coordinate x' associated with a moving muon, the muon is located at the origin.

We introduce the coordinate x' pointing in the direction of the coordinate x, and its origin is associated with the moving muon. See also Figure 1.1 for the introduction of coordinates x and x'. We can go between coordinates (x, t) and (x', t') in the reference frames thus introduced by the following Lorentz transformations

$$x' = \gamma(x - vt) = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right) = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{1.7}$$

After substituting specific values of coordinates (x, t) for individual events, we obtain the coordinates (x', t') of events in the reference frame associated with the muon (in order: muon formation P'_V, start of measurement P'_M, muon extinction P'_Z, end of measurement P'_K):

$$P'_V = (0,0), \quad P'_M = \left(\gamma d_0, -\gamma\frac{v}{c^2}d_0\right), \quad P'_Z = \left(0, \frac{t}{\gamma}\right), \quad P'_K = \left(0, \frac{t}{\gamma}\right), \tag{1.8}$$

where we used the relation $d_0 = vt$ to obtain the final expressions. Thus, we see that in the system associated with the muon, the observer on Earth started the measurement long before the muon's creation in time $t'_M = -\gamma \frac{v}{c^2} d_0 = -117\,\mu s$ (the muon emerged in time $t'_V = 0\,s$). The end of the measurement occurred (simultaneously with the extinction of the muon) at time $t'_Z = t'_K = \frac{t}{\gamma} = 2,37\,\mu s$. Thus, in total, the observer in the system associated with the muon has been measuring for the time

$$t'_m = t'_M - t'_Z = \frac{t}{\gamma} + \gamma \frac{v}{c^2} d_0 = t\left(\frac{1}{\gamma} + \gamma \frac{v^2}{c^2}\right) = t\gamma = 119\mu s, \tag{1.9}$$

where we again used the relation $d_0 = vt$. The observer's measurement time is then $\frac{t'_m}{\gamma} = t = 16,8\,\mu s$, which is exactly the value we had at the beginning. The time $0,335\,\mu s$, which led us to this whole reasoning, simply represents a small fraction of the observer's proper time. So there is no paradox involved.

1.1.2 1.2 Protons flying through the galaxy

Protons of energy $E = 10^{10}\,GeV$ are found in cosmic rays. How long will it take for them to fly through our galaxy in our reference frame and in their own?

Solution: First, let's calculate the speeds of the protons passing through the galaxy. The relationship between energy and velocity is obtained from the famous relationship

$$E = mc^2 = m_0\gamma c^2 = E_0\gamma = \frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{1.10}$$

where we have introduced the factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ and the rest energy $E_0 = m_0 c^2$. By expressing the velocity we get

$$v = \sqrt{1 - \left(\frac{E_0}{E}\right)^2}\; c. \tag{1.11}$$

The rest energy of the proton is approximately $E_0 = 1\,GeV$. The value of the velocity is then

$$v = \sqrt{1 - 10^{-20}}\, c \approx (1 - 0,5 \cdot 10^{-20})c = 0,\underbrace{99\ldots99}_{20\text{x}}5c. \tag{1.12}$$

where we have used the approximation (Taylor expansion to first order) $\sqrt{1+x} \approx 1 + \frac{x}{2}$. From the point of view of an observer in the galaxy we can consider $v \approx c$. If we consider the length of our Galaxy $l_0 = 100\,000\,\text{ly}$ (light years), then the time it takes for protons to pass through the Galaxy from the point of view of an observer in the Galaxy is simply

$$t = \frac{l_0}{v} \approx \frac{l_0}{c} = 100\,000\,\text{years}. \tag{1.13}$$

The time elapsed from the point of view of the proton is obtained by using the relation for time dilation (here we must use the exact velocity v)

$$\tau = \frac{t}{\gamma} = t\sqrt{1 - \frac{v^2}{c^2}}. \tag{1.14}$$

From the relation (1.10) we see that $\frac{E_0}{E} = \sqrt{1 - \frac{v^2}{c^2}}$, for the proton's proper time, we get

$$\tau = t\,\frac{E_0}{E} = 10^5\,\text{years} \cdot 10^{-10} = 10^{-5}\,\text{years} = 315\,s. \tag{1.15}$$

Addendum: If we look at the situation from the point of view of the protons, the Galaxy whizzes past them at a tremendous speed v and is therefore extremely shortened due to length contraction. Let us calculate this length using the relation for length contraction (again we must use the exact velocity v):

$$l = \frac{l_0}{\gamma} = l_0\sqrt{1-\frac{v^2}{c^2}} = l_0\frac{E_0}{E} = 10^5\,\mathrm{ly}\cdot 10^{-10} = 10^{-5}\,\mathrm{ly} = 315\,\mathrm{ls}\ \text{(light seconds)}, \tag{1.16}$$

which is less than the distance from the Earth to the Sun. We can also get this distance using the kinematic relation

$$l = \tau v \approx \tau c, \tag{1.17}$$

where we have used the fact that the Galaxy flies around the proton at approximately c for its proper transit time τ.

1.1.3 1.5 Density

The body is moving with respect to the reference frame at a velocity $v = 0,8c$. Determine the ratio between its density in this frame and its rest density.

Solution: The density in the rest frame ρ_0 and the density in the moving frame ρ are given by the relations

$$\rho_0 = \frac{m_0}{V_0}, \qquad \rho = \frac{m}{V}, \tag{1.18}$$

where m_0 and V_0 are the mass and volume in the rest frame and m and V are the mass and volume in the moving frame.

The volume is transformed due to Lorentz length contraction as

$$V = \frac{V_0}{\gamma} = V_0\sqrt{1-\frac{v^2}{c^2}}. \tag{1.19}$$

This transformation relation follows from the fact that the dimension in the direction of motion is subject to contraction, see Figure 1.2.

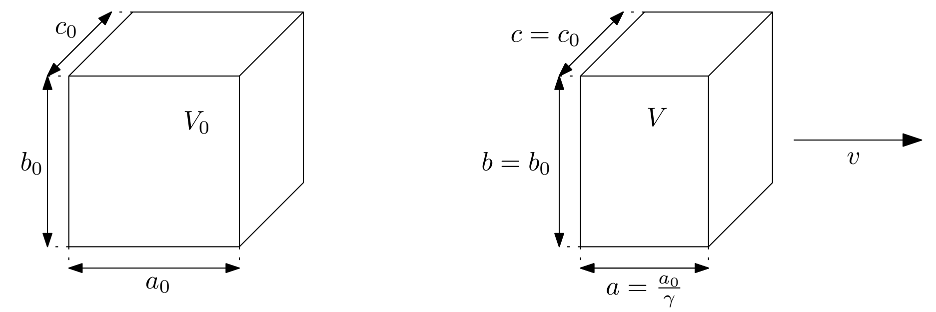

Figure 1.2: Rest volume and moving volume.

When a body moves, the mass of the body increases at the same time, $m = m_0\gamma$. Putting these relations together, we get the result:

$$\rho = \frac{m}{V} = \frac{m_0\gamma}{\frac{V_0}{\gamma}} = \frac{m_0}{V_0}\gamma^2 = \rho_0\gamma^2. \tag{1.20}$$

1.1.4 1.4 Doppler effect

A physicist gambler, who ran a red light with his car and was stopped by a police officer, defended himself by saying that he saw green instead of red as a result of the Doppler effect. However, the physics-literate police officer ticketed him anyway, for speeding. Determine this speed, assuming that red corresponds to spectral line $\lambda_0 = 700\,nm$ and green to $\lambda = 550\,nm$.

Solution: The relativistic relationship between the emitted frequency of the source f_0 and the observed frequency f_p, if the observer approaches the source at a speed of v (and hence with a factor of $\beta = \frac{v}{c}$), is

$$f_p = \sqrt{\frac{1+\beta}{1-\beta}}\, f_0. \tag{1.21}$$

We add the frequencies using the wavelengths, $f = \frac{c}{\lambda}$:

$$\frac{c}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}}\frac{c}{\lambda_0}. \tag{1.22}$$

Now we just express the factor β:

$$\beta = \frac{\lambda_0^2 - \lambda^2}{\lambda_0^2 + \lambda^2} = 0,237, \tag{1.23}$$

So the speed of the physicist gambler in the car was $v = \beta c = 71\,000\,km/s$.

Addendum: Let's derive the above relationship between frequencies f_0 and f_p. Consider a source that is at rest, and an observer approaching it at speed v.

What will be the period of the wave T for the observer? He travels at speed v towards the wavelets propagating at speed c. The period T indicating how long it takes for one wavelength to propagate around the observer is then given by the equation

$$vT + cT = \lambda_0, \tag{1.24}$$

where λ_0 is the wavelength of light emitted by the source, see also Figure 1.3. Thus $T = \frac{\lambda_0}{v+c}$.

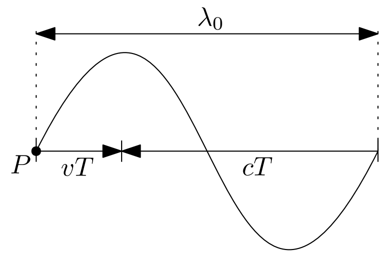

Figure 1.3: The observer P travels at speed v to meet the wave radiating towards him at speed c.

We must not forget, however, that the moving observer's clock runs slower. The period T we have determined so far is determined by the time passed in the system associated with the source of the waves. Meanwhile, the moving observer's proper time τ is given by the time dilation relation:

$$\tau = \frac{T}{\gamma} = T\sqrt{1 - \frac{v^2}{c^2}} = T\sqrt{1-\beta^2}. \tag{1.25}$$

So the observed frequency is $f_p = \frac{1}{\tau}$, successive substitutions give the resulting relation:

$$f_p = \frac{1}{\tau} = \frac{1}{\sqrt{1-\beta^2}}\frac{1}{T} = \frac{1}{\sqrt{1-\beta^2}}\frac{v+c}{\lambda_0} = \frac{1+\beta}{\sqrt{1-\beta^2}}\frac{c}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}}\, f_0. \tag{1.26}$$

1.2 Speed addition

1.2.1 Derivation of the formula for velocity addition using Lorentz transformations

Find the law of relativistic velocity folding in one direction by composing two Lorentz transformations. In particular, find the form of one Lorentz transformation that is equivalent to the composition of the two transformations mentioned above.

Solution: For two frames of reference (S) and (S') that have identically oriented axes, and the system (S') moving along the axis x (or x') with respect to (S) at a speed of V, the Lorentz transformation has the form[1]

$$x' = \gamma(x - Vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{V}{c^2}x\right), \qquad \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}. \tag{1.27}$$

Consider three reference frames (S), (S') and (S''). The system (S') is moving at speed V with respect to the system (S), the system (S'') is moving at speed W w.r.t. (S'), and finally the (S'') is moving at searched-for speed Y w.r.t (S).

Between the systems (S) and (S') we pass at the given speed V, between (S') and (S'') at the given speed W, and between (S) straight to (S'') we pass at the search speed Y. See also Figure 1.4.

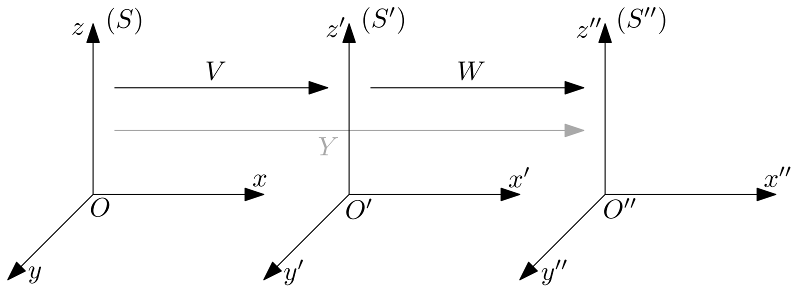

Figure 1.4: Three reference frames (S), (S') and (S'').

Let us denote the gamma factors with by the corresponding velocity index:

$$\gamma = \gamma_V = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}, \qquad \gamma_W = \left(1 - \frac{W^2}{c^2}\right)^{-1/2}, \qquad \gamma_Y = \left(1 - \frac{Y^2}{c^2}\right)^{-1/2}. \tag{1.28}$$

Then the Lorentz transformations for the coordinates x, x', x'' and t, t', t'' between the systems (S) and (S'), (S') and (S''), (S) and (S'') have the form

$$x' = \gamma_V (x - Vt), \qquad t' = \gamma_V \left(t - \frac{V}{c^2}x\right), \tag{1.29}$$

$$x'' = \gamma_W (x' - Wt'), \qquad t'' = \gamma_W \left(t' - \frac{W}{c^2}x'\right), \tag{1.30}$$

$$x'' = \gamma_Y (x - Yt), \qquad t'' = \gamma_Y \left(t - \frac{Y}{c^2}x\right). \tag{1.31}$$

We now perform the composition of the Lorentz transformations (1.30) and (1.29) and compare the result with (1.31). Take the relation for the transformation x'' (1.30) and substitute for x'

[1] For the origins of systems passing through each other in time $t = t' = 0$.

and t' from $(1.29)^2$:

$$x'' = \gamma_W \left(\gamma_V (x - Vt) - W\gamma_V \left(t - \frac{V}{c^2} x \right) \right). \tag{1.32}$$

We want to modify the above expression to the form $x'' = \gamma_Y(x - Yt)$ (1.31) – we then read the expression for the compound velocity Y standing next to the variable t. Let us then modify (1.32):

$$\begin{aligned} x'' &= \gamma_W \gamma_V \left(\left(1 + \frac{VW}{c^2} \right) x - (V + W)\, t \right) \\ &= \left[\gamma_W \gamma_V \left(1 + \frac{VW}{c^2} \right) \right] \left(x - \frac{V + W}{1 + \frac{VW}{c^2}}\, t \right) = \gamma_Y (x - Yt)\,. \end{aligned} \tag{1.33}$$

From the last equality in the previous expression, we get the relation for the compound velocity Y by comparison:

$$Y = \frac{V + W}{1 + \frac{VW}{c^2}}. \tag{1.34}$$

At the same time, we obtain the relationship between the gamma coefficients for each velocity,

$$\gamma_W \gamma_V \left(1 + \frac{VW}{c^2} \right) = \gamma_Y; \tag{1.35}$$

its validity can be proved by simply breaking down all the gamma factors (and inserting a specific value for the velocity Y) and then doing a tedious expression manipulations...

1.2.2 1.3 Spacecraft and rocket

A rocket was launched from a spacecraft moving relative to the Earth at $v_1 = 0,8c$ in the direction of its motion at $v_2 = 0,6c$ relative to the spacecraft. The proper length of the rocket is $l_0 = 10\,m$. What is the length of this rocket from the point of view of an observer in the ship and from the point of view of an observer on Earth?

Solution: The formula for relativistic velocity folding is as follows:

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}, \tag{1.36}$$

where v is the resulting velocity, v_1 and v_2 are the original velocities with the same positive direction. After substitution, we get the resultant velocity v:

$$v = \frac{(0,8 + 0,6)c}{1 + 0,6 \cdot 0,8} = 0,946c. \tag{1.37}$$

To obtain the observed rocket lengths, we just need to substitute the correct velocities into the formula for length contraction

$$l = \frac{l_0}{\gamma} = l_0 \sqrt{1 - \frac{v^2}{c^2}}, \tag{1.38}$$

where l_0 is the proper (rest) length, l the length after contraction, and v the corresponding velocity of motion.

The rocket, from the point of view of an astronaut on a spacecraft, has a velocity of v_2, and from the point of view of an observer on Earth it has just a compound velocity of v. The resulting contracted lengths are thus:

$$l_{v_2} = 10\sqrt{1 - 0,6^2} = 8\,m, \qquad l_v = 10\sqrt{1 - (0,946)^2} = 3,24\,m. \tag{1.39}$$

[2] We could also take the relation for the transformation t'', but the procedure would be essentially identical.

1.2.3 1.6 Astronaut on the Moon

An astronaut on the Moon observes two spacecrafts approaching him from opposite sides at $v_1 = 0,8c$ and $v_2 = 0,9c$. What is the velocity of one of the ships measured from the deck of the other?

Solution: Just for the sake of argument, note that the astronaut observes the spacecraft approaching each other by the simple sum of their velocities: $v_1 + v_2 = 1,7c$.

The situation is different for the observers on each spacecraft, from their perspective the spacecraft are approaching each other at the speed of

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{(0,8 + 0,9)c}{1 + 0,8 \cdot 0,9} = 0,988c. \tag{1.40}$$

1.3 Relativistic equation of motion

1.3.1 1.7 Hyperbolic motion

Determine the velocity and position of a relativistic particle subject to a constant force F. Compare with uniformly accelerated motion in nonrelativistic physics and show that the particle's velocity does not exceed the speed of light c.

Solution: Let's solve the relativistic equation of motion, or a one-dimensional version of it

$$\frac{d}{dt}(m_0\gamma\vec{v}) = \vec{F}, \qquad \frac{d}{dt}(m_0\gamma v) = F. \tag{1.41}$$

After substituting $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ and integrating by time (the force F is constant), we obtain

$$m_0 \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = Ft + C_1, \tag{1.42}$$

where we determine the integration constant C_1 from the initial conditions. If we consider zero velocity at time zero, $v(0) = 0$, we get $C_1 = 0$. From the previous equation, we express the velocity $v(t)$:

$$v(t) = \frac{Ftc}{\sqrt{m_0^2c^2 + F^2t^2}} = \frac{Ft}{m_0}\frac{1}{\sqrt{1 + \left(\frac{Ft}{m_0c}\right)^2}} = \frac{c}{\sqrt{1 + \left(\frac{m_0c}{Ft}\right)^2}}. \tag{1.43}$$

From the last expression, obviously $v(t) < c$ for arbitrary time t, moreover, $\lim_{t\to+\infty} v(t) = c$ holds. We obtain the position $x(t)$ by integrating the velocity with respect to time:

$$x(t) = \int v(t)\,dt = \int \frac{Ftc}{\sqrt{m_0^2c^2 + F^2t^2}}\,dt. \tag{1.44}$$

Using the substitution $u = m_0^2c^2 + F^2t^2$, $du = 2F^2t\,dt$ we obtain the result

$$x(t) = \frac{c}{F}\int \frac{du}{2\sqrt{u}} = \frac{c}{F}\sqrt{u} + C_2 = \frac{c}{F}\sqrt{m_0^2c^2 + F^2t^2} + C_2. \tag{1.45}$$

For the initial condition $x(0) = 0$, we have $C_2 = -\frac{c}{F}m_0c$. We can then write the result in the form:

$$x(t) = \frac{m_0c^2}{F}\left(\sqrt{1 + \left(\frac{Ft}{m_0c}\right)^2} - 1\right). \tag{1.46}$$

Finally, we show how the relativistic result differs from the non-relativistic one. We will use the Taylor polynomial of the function $(1+x)^\alpha$, where $\alpha \in \mathbb{R}$ and x is close to zero:

$$(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k = 1 + \alpha\, x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots \tag{1.47}$$

Let us now take the functions $v(t)$ in the middle expression (1.43) and $x(t)$ (1.46) and apply the Taylor expansion to the square roots in them:

$$\left(1 + \left(\frac{Ft}{m_0 c}\right)^2\right)^{\pm 1/2} = 1 \pm \frac{1}{2}\left(\frac{Ft}{m_0 c}\right)^2 + \frac{\pm\frac{1}{2}\left(\pm\frac{1}{2}-1\right)}{2}\left(\frac{Ft}{m_0 C}\right)^4 + \dots, \tag{1.48}$$

where we have considered $x = \left(\frac{Ft}{m_0 c}\right)^2$ and hence our expansion is valid only for the times t when x is small. Substituting these expansions into the functions $x(t)$ and $v(t)$ (for position we used the second order expansion, for velocity only the first order expansion) we get

$$x(t) = \frac{1}{2}\frac{F}{m_0} t^2 - \frac{1}{8}\frac{F^3}{m_0^3 c^2} t^4 + \dots, \tag{1.49}$$

$$v(t) = \frac{F}{m_0} t - \frac{1}{2}\frac{F^3}{m_0^3 c^2} t^3 + \dots. \tag{1.50}$$

Thus we see that the formulas for non-relativistic motion are a first approximation to relativistic motion. In the so-called non-relativistic limit $c \to +\infty$ we get the formulas for non-relativistic motion.

Addendum: Why is the example called hyperbolic motion? Because the trajectory of relativistic motion with a constant force applied is a hyperbola in the space-time diagram. To make it easier to show, it will be useful to change the initial condition of the initial position to $x(0) = \frac{m_0 c^2}{F} = \alpha$. Then the integration constant comes out $C_2 = 0$ and the resulting position as a function of time $x(t)$ is

$$x(t) = \alpha \sqrt{1 + \frac{c^2 t^2}{\alpha^2}}. \tag{1.51}$$

The manipulation yields the equation for the trajectory in the space-time diagram with variables (x, t) as

$$x^2 - c^2 t^2 = \alpha^2, \tag{1.52}$$

which is the hyperbola shown in Figure 1.5. The non-relativistic motion $x = \frac{1}{2}\frac{F}{m_0} t^2$ then represents the parabolic motion.

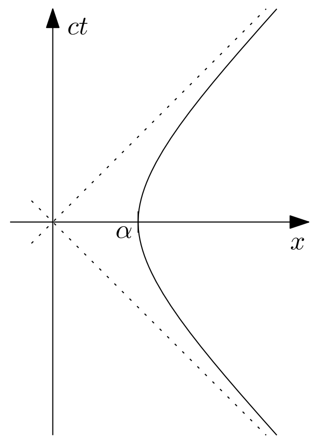

Figure 1.5: Hyperbolic motion plotted on a space-time diagram

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Addendum: In the scripts, the result is written using the constant $a = \frac{F}{m_0}$. This constant **does not represent** the acceleration of the particle. For relativistic motion, when a constant force is applied, the acceleration is non-constant. The actual value of the acceleration is obtained in the classical way from the kinematic relation:

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(\frac{Ftc}{\sqrt{m_0^2c^2 + F^2t^2}}\right) = \frac{m_0^2Fc^3}{\left(m_0^2c^2 + F^2t^2\right)^{3/2}}. \tag{1.53}$$

1.4 Relativistic energy and work

1.4.1 1.8 Accelerator

The accelerator gives energy to protons $E = 500\,GeV$. What speed do they reach? (The rest energy of a proton is $E_0 = 0,938\,GeV$.)

Solution: The relationship between energy and velocity is obtained from the famous relationship

$$E = mc^2 = m_0\gamma c^2 = E_0\gamma = \frac{E_0}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{1.54}$$

where we have introduced the notation of the factor $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$ and the rest energy of the proton $E_0 = m_0c^2$. By expressing the velocity, we obtain

$$v = \sqrt{1-\left(\frac{E_0}{E}\right)^2}\,c = 0,9999982c. \tag{1.55}$$

1.4.2 1.9 Work done on the electron

How much work is required to increase the velocity of an electron from $v_1 = 1,2.10^8\,m.s^{-1}$ to $v_2 = 2,4.10^8\,m.s^{-1}$ according to non-relativistic and relativistic mechanics? (The rest energy of the electron is $0,511\,MeV$.)

Solution: In non-relativistic mechanics, work is the difference in kinetic energies of an object

$$W = E_{K2} - E_{K1} = \frac{1}{2}m_0 v_2^2 - \frac{1}{2}m_0 v_1^2 = \frac{1}{2}m_0 c^2 \left(\frac{v_2^2}{c^2} - \frac{v_1^2}{c^2}\right) = \frac{1}{2}E_0\left(\beta_2^2 - \beta_1^2\right) = 0,24E_0 = 0,123\, MeV, \tag{1.56}$$

where we have written the relationship using "relativistic" quantities – rest energy E_0 and factor $\beta = \frac{v}{c}$; in our particular case we have $\beta_1 = \frac{2}{5}$ and $\beta_2 = \frac{4}{5}$ (using the approximate value of the speed of light $c = 3.10^8\, m/s$).

In the relativistic case, work is directly equal to the difference of the total energies of the bodies:

$$W = E_2 - E_1 = m_2c^2 - m_1c^2 = m_0c^2\left(\gamma_2 - \gamma_1\right) = E_0\left(\gamma_2 - \gamma_1\right) = 0.58E_0 = 0.296\, MeV, \tag{1.57}$$

where we have used the factor $\gamma = (1 - \beta^2)^{-1/2}$ in the expression.

The resulting relations for non-relativistic, or relativistic, work are

$$W = \frac{1}{2}E_0\left(\beta_2^2 - \beta_1^2\right), \quad \text{resp.} \quad W = E_0\left(\gamma_2 - \gamma_1\right). \tag{1.58}$$

We obtain the non-relativistic relation from the relativistic one by Taylor expansion of the factor γ to first order: $\gamma = \left(1 - \beta^2\right)^{-1/2} \approx 1 + \frac{1}{2}\beta^2$.

1.4.3 1.10 Meson decay

The π^--meson (with rest energy $E_{0\pi} = 139,6\, MeV$) decays at rest to the muon μ^- (rest energy $E_{0\mu} = 105,7\, MeV$) and the antineutrino $\bar{\nu}$. Determine the energy of the muon and antineutrino and the released kinetic energy.

Solution: First, we calculate the total kinetic energy released. This is given by the simple difference of the total rest energies before and after the decay – the mass loss must have been converted to kinetic energy:

$$E_K = E_{0\pi} - (E_{0\mu} + E_{0\bar{\nu}}) = E_{0\pi} - E_{0\mu} = 33,9\, MeV, \tag{1.59}$$

where we put the rest energy of the antineutrino $E_{0\bar{\nu}}$ equal to zero. Although present experiments show that the neutrino mass is non-zero, it is also on the order of unity eV or less (i.e., about 8 orders of magnitude less than the rest masses of the π-meson and muon).

How is the kinetic energy distributed among the particles produced? For this we must use the laws of conservation of momentum and energy:

$$p_\pi = p_\mu + p_{\bar{\nu}}, \qquad E_\pi = E_\mu + E_{\bar{\nu}}. \tag{1.60}$$

The π-meson was at rest at the beginning and so we have $p_\pi = 0$ and $E_\pi = E_{0\pi}$:

$$0 = p_\mu + p_{\bar{\nu}}, \qquad E_{0\pi} = E_\mu + E_{\bar{\nu}}. \tag{1.61}$$

The relativistic relationship between energy and momentum is

$$\frac{E_0^2}{c^2} = \frac{E^2}{c^2} - p^2 \quad \rightarrow \quad p^2 = \frac{1}{c^2}(E^2 - E_0^2). \tag{1.62}$$

The law of conservation of momentum implies the equality of the squares of the momenta, $p_\mu^2 = p_{\bar{\nu}}^2$, and after substituting from the above relation we get (again after putting $E_{0\bar{\nu}} = 0$)

$$\frac{1}{c^2}(E_\mu^2 - E_{0\mu}^2) = \frac{1}{c^2}(E_{\bar{\nu}}^2 - E_{0\bar{\nu}}^2) \quad \rightarrow \quad E_\mu^2 - E_{0\mu}^2 = E_{\bar{\nu}}^2. \tag{1.63}$$

Thus we arrive at the following system of two equations with unknowns E_μ and $E_{\bar{\nu}}$:

$$E_{0\pi} = E_\mu + E_{\bar{\nu}}, \qquad E_{0\mu}^2 = E_\mu^2 - E_{\bar{\nu}}^2. \tag{1.64}$$

After decomposing the right-hand side of the second equation into $(E_\mu - E_{\bar{\nu}})(E_\mu + E_{\bar{\nu}})$ and substituting from the first, we avoid solving the quadratic equation and obtain a set of linear equations:

$$E_{0\pi} = E_\mu + E_{\bar{\nu}}, \qquad \frac{E_{0\mu}^2}{E_{0\pi}} = E_\mu - E_{\bar{\nu}}. \tag{1.65}$$

Adding and subtracting these equations gives the results:

$$E_\mu = \frac{1}{2}\left(E_{0\pi} + \frac{E_{0\mu}^2}{E_{0\pi}}\right) = 109,8\,MeV, \qquad E_{\bar{\nu}} = \frac{1}{2}\left(E_{0\pi} - \frac{E_{0\mu}^2}{E_{0\pi}}\right) = 29,8\,MeV. \tag{1.66}$$

Finally, it is still worth explicitly calculating the kinetic energy of the decayed particles simply by subtracting the rest energies:

$$E_{K\mu} = E_\mu - E_{0\mu} = 4,1\,MeV, \qquad E_{K\bar{\nu}} = E_{\bar{\nu}} = 29,8\,MeV. \tag{1.67}$$

We see that the massless antineutrino takes away most of the kinetic energy produced!

1.4.4 1.11 Binding energy of the alpha particle

Determine the binding energy of the particle α in MeV if the rest masses of the proton, neutron and particle α are: $m_p = 1,67265.10^{-27}\,kg$, $m_n = 1,67495.10^{-27}\,kg$ and $m_\alpha = 6,644.10^{-27}\,kg$.

Solution: The binding energy is given by the difference of the rest energies of the product and its constituents:

$$E_V = m_\alpha c^2 - (2m_p + 2m_n)c^2 = -4,61.10^{-12}\,J = -28,8\,MeV, \tag{1.68}$$

where we used the approximate value of the speed of light $c = 3.10^8\,m/s$. The negative sign indicates that this energy is released when the alpha particle is created. The conversion between joules and electron volts is given by the following simple unit consideration

$$1\,eV = (1\,e).V = \frac{1\,e}{C}\,C.V = \frac{1\,e}{C}\,J = 1,602.10^{-19}J. \tag{1.69}$$

1.4.5 1.12 Sun

The energy of solar radiation incident per unit time per square metre at the boundary of the Earth's atmosphere is the so-called *solar constant* $K = 1327\,W.m^{-2}$, the mean distance of the Earth from the Sun being $d = 1,5.10^{11}\,m = 1\,AU$. The source of solar energy is the so-called hydrogen cycle, in which four hydrogen nuclei (protons) of relative atomic mass $m_{rp} = 1,008$ are converted into a helium nucleus ($m_{r\alpha} = 4,0039$). Determine the mass loss of the Sun and the amount of hydrogen burned per second. Estimate the time it would take to burn an amount of hydrogen equivalent to the mass of the Sun today $M_\odot = 2.10^{30}\,kg$.

Solution: First, we determine the total radiant power of the Sun $P_\odot$. If at a distance d the power K passes through one square meter, then the total power is obtained by multiplying by the area of a sphere of radius d:

$$P_\odot = 4\pi d^2 K = 3,752.10^{26}\,W = 3,752.10^{14}\,TW. \tag{1.70}$$

This energy is taken from nuclear reactions inside the Sun, where a small fraction of the mass is converted into energy (either kinetic energy subsequently radiated by electromagnetic radiation or straight gamma radiation). The amount of converted matter is determined from equation $E = mc^2$, or its time derivative:

$$P = \frac{dE}{dt} = \frac{dm}{dt}c^2 \quad \rightarrow \quad \frac{dm}{dt} = \frac{P}{c^2} = \frac{4\pi d^2 K}{c^2} = 4,169.10^9\,kg/s = 4,169\,Mt/s, \tag{1.71}$$

where we have used the approximate value of the speed of light $c = 3.10^8\,m/s$. This is therefore the total mass loss of the Sun per unit time due to the conversion of mass to radiated energy.

We now determine the amount of hydrogen burned per unit time. One reaction of the hydrogen cycle will provide the following amount of energy:

$$E_1 = \Delta m\, c^2 = (4m_p - m_\alpha)c^2. \tag{1.72}$$

The total number of reactions required in one second for a power $P_\odot$ is then

$$N = \frac{P_\odot}{E_1} = \frac{4\pi d^2 K}{(4m_p - m_\alpha)c^2}. \tag{1.73}$$

In one reaction, $4m_p$ hydrogen is burned, the total amount of hydrogen burned is

$$\frac{dm_H}{dt} = 4m_p N = 4m_p \cdot \frac{4\pi d^2 K}{(4m_p - m_\alpha)c^2} = \frac{4\pi d^2 K}{(1 - \frac{m_\alpha}{4m_p})c^2}. \tag{1.74}$$

In the last expression, only the mass ratio $\frac{m_\alpha}{m_p}$ comes out, and thus the masses can be given in any units, e.g. using relative atomic weights:

$$\frac{dm_H}{dt} = \frac{4\pi d^2 K}{(1 - \frac{m_{r\alpha}}{4m_{rp}})c^2} = 5,982.10^{11}\,kg/s = 598,2\,Mt/s. \tag{1.75}$$

If the Sun were made only of hydrogen, and nuclear reactions proceeded at the same rate the whole time, then the hydrogen would burn out in time

$$T = \frac{M_\odot}{\frac{dm_H}{dt}} = 3,343.10^{18}\,s = 106\,\text{bn. years}. \tag{1.76}$$

Stellar evolution is, of course, much more complicated, and current models predict a total burn time of about 10 billion years for hydrogen.

Chapter 2

Electrostatics

2.1 Formulae overview

- **Coulomb's law** The force $\vec{F}_C$ from a point charge Q acting on a point charge q is given by

$$\vec{F}_C = \frac{q\,Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}, \qquad F_C = \frac{1}{4\pi\varepsilon_0}\frac{q\,Q}{r^2}, \tag{2.1}$$

where $\vec{r}$ is the vector connecting the charge Q to the charge q (from Q to q), r is its magnitude, and $\varepsilon_0 \doteq 8,854.10^{-12}\,F.m^{-1}$ is the vacuum permittivity.

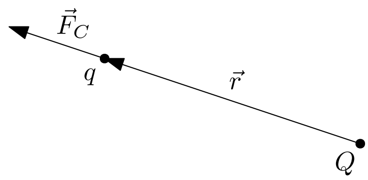


Figure 2.1: Coulomb's law (the direction of the force is plotted here for charges of the same polarity).

- **Electrostatic energy W of a system of point charges:**

$$W = \sum_{\alpha<\beta} \frac{1}{4\pi\varepsilon_0}\frac{q_\alpha q_\beta}{r_{\alpha\beta}}, \tag{2.2}$$

where q_α are the individual magnitudes of the point charges and $r_{\alpha\beta}$ are the distances between the individual charges.

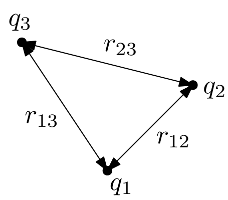


Figure 2.2: Electrostatic energy of a system of three point charges.

- **Gaussian law** relates the flow of an electric field $\vec{E}$ through a closed surface with a total charge Q inside the volume enclosed by the surface:

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}. \tag{2.3}$$

The surface element is $d\vec{S} = \vec{n}\, dS$, where $\vec{n}$ is a unit normal vector to the surface of area dS, see also Figure 2.3. The normal $\vec{n}$ is oriented to point outward from the closed surface.

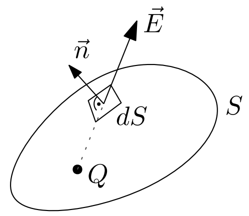


Figure 2.3: The closed surface S in Gaussian law with point charge Q and the surface element $d\vec{S} = \vec{n}\, dS$.

If the electric field strength vector $\vec{E}$ is tangent to the surface S at a given location, then it does not contribute to the integral, since $\vec{E} \perp \vec{n}$ and hence $\vec{E} \cdot d\vec{S} = 0$. If, on the other hand, the vector $\vec{E}$ is perpendicular to the surface S at a given location, then the scalar product reduces to $\vec{E} \cdot d\vec{S} = E\, dS$.

In the case that we are working with point charges and some of the charges lie directly on the surface, Gaussian's law does not apply (we cannot actually decide whether or not the charge lies inside the surface)[1]! This is because part of the electric field lines then "leak" out of the surface and are not included in the total electric field flux.

- **Electrostatic potential** φ for point charges:

$$\varphi = \frac{1}{4\pi\varepsilon_0} \sum_\alpha \frac{q_\alpha}{R_\alpha}, \tag{2.4}$$

q_α are the charges of individual particles. For a continuous charge distribution; in order for linear, surface, and volume charge distributions:

$$\varphi = \frac{1}{4\pi\varepsilon_0} \int_l \frac{\tau}{R}\, dl, \qquad \varphi = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma}{R}\, dS, \qquad \varphi = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho}{R}\, dV, \tag{2.5}$$

where τ, σ and ρ are, in order, the length, area, volume charge density functions. The distance R (for point charges R_α) is the distance between the chosen point for determination of electrostatic potential φ at position vector $\vec{r}$ and the integrating element (dl, dS, dV) at position vector $\vec{r}'$, i.e. $R = |\vec{r} - \vec{r}'|$, see Figure 2.4. For point charges, the role $\vec{r}'$ is played by the particle position vector $\vec{r}_\alpha$, i.e., $R_\alpha = |\vec{r} - \vec{r}_\alpha|$.

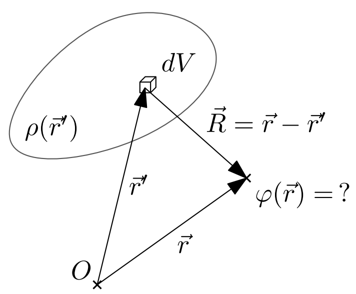


Figure 2.4: The vector shown $\vec{R} = \vec{r} - \vec{r}'$ for determining the electrostatic (scalar) potential φ. The vector $\vec{r}$ is the position vector of the point for determination $\varphi(\vec{r})$, and the vector $\vec{r}'$ is the position vector of the volume element dV.

[1]Similar problem happens with line and surface charges, if part of the curve or part of the surface with the charges lies on the surface S in Gaussian's law.

In detail, the formulas have the following forms. For point charges:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_\alpha \frac{q_\alpha}{|\vec{r} - \vec{r}_\alpha|}, \tag{2.6}$$

and for a continuous distribution of charges:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_l \frac{\tau(\vec{r}')}{|\vec{r} - \vec{r}'|}\, dl, \quad \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|}\, dS, \quad \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}\, dV. \tag{2.7}$$

- **Electric field $\vec{E}$ (in the static case):** is defined as the (Coulombic) force acting on a unit (test) charge:

$$\vec{E} = \frac{\vec{F}_C}{q}. \tag{2.8}$$

The electric field vector $\vec{E}$ can be obtained from the electrostatic potential φ:

$$\vec{E} = -\mathrm{grad}\, \varphi = \left(-\frac{\partial \varphi}{\partial x}, -\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial z} \right). \tag{2.9}$$

Direct calculation for point charges:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha \frac{\vec{R}_\alpha}{R_\alpha^3}, \tag{2.10}$$

and for a continuous charge distribution:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_l \tau \frac{\vec{R}}{R^3}\, dl, \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0} \int_S \sigma \frac{\vec{R}}{R^3}\, dS, \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0} \int_V \rho \frac{\vec{R}}{R^3}\, dV, \tag{2.11}$$

where τ, σ and ρ are in turn functions of the length, area, volume charge density. The vector $\vec{R}$ (for point charges $\vec{R}_\alpha$) is the distance between the electric field location $\vec{E}$ at the position vector $\vec{r}$ and the integrating element (dl, dS, dV) at the position vector $\vec{r}'$, i.e. $\vec{R} = \vec{r} - \vec{r}'$, see Figure 2.5. For point charges, the role $\vec{r}'$ is played by the particle position vector $\vec{r}_\alpha$, i.e., $\vec{R}_\alpha = \vec{r} - \vec{r}_\alpha$. The distance R is the magnitude of the vector $\vec{R}$, i.e. $R = |\vec{r} - \vec{r}'|$ (for point charges we have $R_\alpha = |\vec{r} - \vec{r}_\alpha|$).

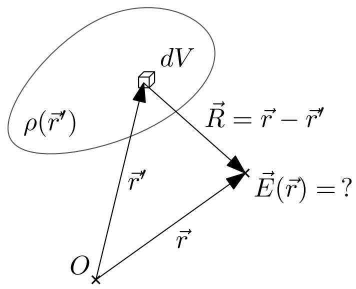


Figure 2.5: Showing the vector $\vec{R} = \vec{r} - \vec{r}'$ for determining the electric field $\vec{E}$. Vector $\vec{r}$ is the position vector of the point for determination of the value $\vec{E}(\vec{r})$, vector $\vec{r}'$ is the position vector of the volume element dV.

In detail, the formulas have the following forms. For point charges:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha \frac{\vec{r} - \vec{r}_\alpha}{|\vec{r} - \vec{r}_\alpha|^3} \tag{2.12}$$

and for a continuous distribution of charges:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_l \tau(\vec{r}')\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,dl,$$
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \sigma(\vec{r}')\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,dS,$$
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r}')\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,dV. \tag{2.13}$$

- **Multipole expansion of the electrostatic potential** is given by

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\frac{\sum_{i,j} Q_{ij}x_i x_j}{r^5} + \dots\right], \tag{2.14}$$

where r is the magnitude of the position vector $\vec{r}$, Q is the total charge, $\vec{p}$ is the dipole moment vector, and (Q_{ij}) is the quadrupole moment tensor. These quantities are determined by the following formulae (for point charges and continuous volume charge distributions[2], respectively):

  - **Total charge:**

$$Q = \sum_\alpha q_\alpha, \qquad Q = \int_V \rho(\vec{r})\,dV, \tag{2.15}$$

  - **Dipole moment:**

$$\vec{p} = \sum_\alpha q_\alpha \vec{r}_\alpha, \qquad \vec{p} = \int_V \rho(\vec{r})\,\vec{r}\,dV, \tag{2.16}$$

  - **Quadrupole moment:**

$$Q_{ij} = \sum_\alpha q_\alpha\left(3x_i x_j - \delta_{ij}r^2\right)_\alpha, \qquad Q_{ij} = \int_V \rho(\vec{r})\left(3x_i x_j - \delta_{ij}r^2\right)dV, \tag{2.17}$$

  where q_α are the charges of the individual particles, $\vec{r}_\alpha = (x_{1\alpha}, x_{2\alpha}, x_{3\alpha}) = (x_\alpha, y_\alpha, z_\alpha)$ their position vectors, to simplify the notation we use the notation $(x_\alpha, y_\alpha, z_\alpha) = (x, y, z)_\alpha$. The vector $\vec{r} = (x_1, x_2, x_3) = (x, y, z)$ is the position vector of the volume element dV, $\rho(\vec{r})$ is the volume charge density function.

- **Electrical voltage** is defined as the work of external forces $\vec{F}$ acting on a unit charge along a path l:

$$U = \frac{1}{q}\int_l \vec{F}\cdot d\vec{l}, \tag{2.18}$$

where $d\vec{l} = \vec{t}\,dl$, dl is the line element and $\vec{t}$ is a unit tangent vector to the curve l. In electrostatics, the only force acting on a charge is usually the electric force, $\vec{F} = q\vec{E}$, so the formula is concretized to

$$U = \int_l \vec{E}\cdot d\vec{l}. \tag{2.19}$$

[2] For brevity, the formulae for linear and surface charge distributions are not given. The difference is only in the substitution of the volume integral for the line or surface integral and in the substitution of the volume charge density function for the linear or surface density.

- **Capacitance of a capacitor** is defined as the charge on the capacitor per unit voltage:

$$C = \frac{Q}{U}. \tag{2.20}$$

  Units: capacitance C $[F] = [C.V^{-1}]$, charge Q $[C]$, voltage U $[V]$.

- **Capacitance addition:** The formulas for the series (left) and parallel (right) connections (see Figure 2.6) of capacitors are

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}, \qquad C = C_1 + C_2. \tag{2.21}$$

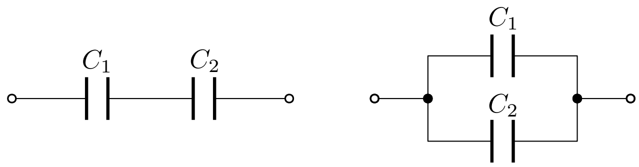


Figure 2.6: Capacitance addition of C_1 and C_2. On the left, there is the series connection, and on the right, there is the parallel connection.

2.2 Coulomb's Law

2.2.1 2.1 Balls on threads

Two identical small balls of masses $m = 1\,g$ hang from two threads of length $l = 1\,m$. If we charge them with a same polarity charge of the same magnitude q, they will spread apart so that the threads make a right angle. Determine the magnitude of the charge q.

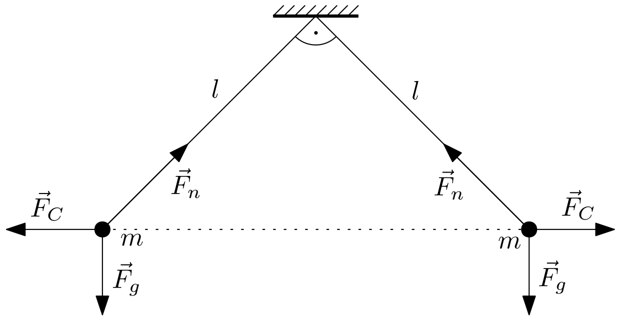


Figure 2.7: Balls on the threads.

Solution: For the balls to be in equilibrium, the resultant of the gravitational $\vec{F}_g$ and Coulomb $\vec{F}_C$ forces must point in the direction of the threads (then they can be balanced by the tension force $\vec{F}_n$). The magnitude of the Coulomb force between two charges is generally given by

$$F_C = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}, \tag{2.22}$$

where q_1, q_2 are the values of the individual charges and r is the distance between them. For threads forming right angles and thus for the deflection of the individual threads by an angle 45°, the magnitudes of the gravitational and Coulomb forces must be equal, $F_C = F_g$. The distance between the balls is from the Pythagorean theorem $r = \sqrt{2}l$ and the magnitudes of the individual forces are then:

$$F_C = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2l^2}, \qquad F_g = mg. \tag{2.23}$$

Expressing the charge q from the equation of equality of forces, we get

$$F_C = F_g \quad \longrightarrow \quad q = \pm\sqrt{8\pi\varepsilon_0 mg}\, l = \pm 1,48.10^{-6}\, C, \tag{2.24}$$

where we have used the value of the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\, F.m^{-1}$ and the gravitational acceleration $g = 9,81\, m.s^{-2}$ for the numerical result.

2.2.2 2.2 Charged drops

Two identical water droplets have one excess electron each, and the force of electrical repulsion is as big as the force of gravitational attraction. Determine the radius of the drops.

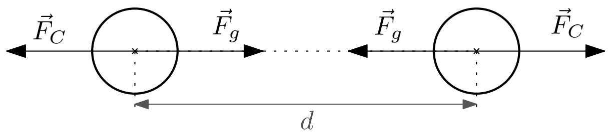


Figure 2.8: Droplets with an excess electron.

Solution: The spheres are repelled by the Coulomb force $\vec{F}_C$ and attracted by the gravitational force $\vec{F}_g$, the magnitudes of these forces are generally given by

$$F_C = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}, \qquad F_g = \kappa\frac{m_1 m_2}{r^2}, \tag{2.25}$$

where q_1, q_2 are the individual charges, m_1 and m_2 are the masses of the bodies, and r is their distance. These formulas are valid for spherically symmetrically distributed charge and mass (Gauss theorem) – r is the distance of their centers. We consider here that the only excess electron on the droplets is at their center. The formulas (2.25) are then specifically

$$F_C = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{d^2}, \qquad F_g = \kappa\frac{m^2}{d^2}, \tag{2.26}$$

where m is the masses of the individual droplets, e is the magnitude of the elementary electric charge, and d is the distance of the droplet centers.

We express the mass of the droplets in terms of their density ρ and radius r:

$$m = \rho V = \frac{4}{3}\pi r^3 \rho. \tag{2.27}$$

From the equality of the magnitudes of the forces $F_C = F_g$ we express the radius of the droplets:

$$r = \sqrt[6]{\frac{9e^2}{64\pi^3\varepsilon_0\kappa\rho^2}} = 7,63.10^{-5}\, m = 76,3\, \mu m, \tag{2.28}$$

where we have used the values of the elementary electric charge $e = 1,602.10^{-19}\, C$, the permittivity of the vacuum $\varepsilon_0 = 8,854.10^{-12}\, F.m^{-1}$, the gravitational constant $\kappa = 6,674.10^{-11}\, m^3.kg^{-1}.s^{-2}$ and the density of water $\rho = 1000\, kg.m^{-3}$ for the numerical result.

2.3 Electrostatic energy

2.3.1 2.3 Three charges

The three charges $-e$, e, $-e$ are placed at equal distances a in the order shown. Determine the forces acting on each charge and the electrostatic energy of the system.

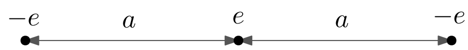


Figure 2.9: Three charges $-e$, e, $-e$ at distances a from each other.

Solution: The Coulomb force $\vec{F}_C$ from a point charge Q acting on a point charge q is given by

$$\vec{F}_C = \frac{q\,Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}, \tag{2.29}$$

where $\vec{r}$ is the vector connecting charge Q to charge q (from Q to q) and r is its magnitude. This notation then naturally expresses the property that charges of the same polarity repel and charges of opposite polarity attract. Let us denote the force acting from the i-th charge to the j-th charge as $\vec{F}_{ij}$. These forces are shown in Figure 2.10. Their magnitudes are

$$F_{12} = F_{21} = F_{23} = F_{31} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2}, \qquad F_{13} = F_{31} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4a^2}. \tag{2.30}$$

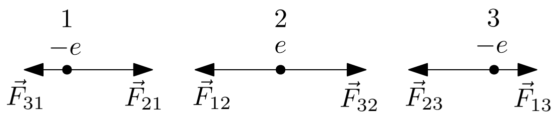


Figure 2.10: The individual forces acting on the charges. The force $\vec{F}_{ij}$ represents the force from the ith charge acting on the jth charge.

If we denote F_1, F_2 and F_3 the total magnitudes of the forces acting from the left on the individual charges we get (according to the magnitudes of the partial forces $\vec{F}_{ij}$ (2.30) and their directions in Figure 2.10):

$$F_1 = F_3 = \frac{1}{4\pi\varepsilon}\left(\frac{e^2}{a^2} - \frac{e^2}{(2a)^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{3e^2}{4a^2}, \qquad F_2 = 0. \tag{2.31}$$

The directions of these resultant forces are shown in Figure 2.11.

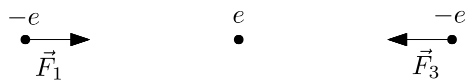


Figure 2.11: Total forces acting on the charges. The force $\vec{F}_2$ is zero.

The electrostatic energy of the system is given by the general relation

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{\alpha<\beta}\frac{q_\alpha q_\beta}{r_{\alpha\beta}}, \tag{2.32}$$

where q_α are the charges on the individual particles and $r_{\alpha\beta}$ are their mutual distances. Here specifically

$$W = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{-e^2}{a} + \frac{e^2}{2a} + \frac{-e^2}{a}\right) = -\frac{1}{4\pi\varepsilon_0}\frac{3e^2}{2a}. \tag{2.33}$$

2.3.2 2.4 Zero electrostatic energy

Find an arrangement of one proton and two electrons on a line such that the electrostatic energy of the system is zero.

Solution: There are two possible non-equivalent arrangements of one proton and two electrons on a straight line, see Figure 2.12. We have labeled the distances between the charges in general a and b.

Figure 2.12: Two arrangements of one proton and two electrons on a straight line.

The electrostatic energy for a system of three charges is

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{\alpha<\beta} \frac{q_\alpha q_\beta}{r_{\alpha\beta}} = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right), \tag{2.34}$$

where q_α are the charges on each particle and $r_{\alpha\beta}$ are their mutual distances.

For the electron-proton-electron arrangement we have the expression

$$W = \frac{1}{4\pi\varepsilon_0} \left(\frac{-e^2}{a} + \frac{e^2}{a+b} + \frac{-e^2}{b} \right) = -\frac{e^2}{4\pi\varepsilon_0} \frac{a^2+b^2+ab}{ab(a+b)}. \tag{2.35}$$

In the numerator we have the sum of all positive numbers (the distances a and b must be considered positive since we used them as values for r_{12}, r_{13} and r_{23}, which are always positive), so in the e-p-e arrangement the electrostatic energy W cannot be zero.

For the electron-electron-proton arrangement we get

$$W = \frac{1}{4\pi\varepsilon_0} \left(\frac{e^2}{a} + \frac{-e^2}{a+b} + \frac{-e^2}{b} \right) = \frac{e^2}{4\pi\varepsilon_0} \frac{b^2-a^2-ab}{ab(a+b)}. \tag{2.36}$$

The electrostatic energy is zero, $W = 0$ just when

$$b^2 - a^2 - ab = 0 \quad \longrightarrow \quad \left(\frac{b}{a}\right)^2 - \frac{b}{a} - 1 = 0 \quad \longrightarrow \quad \left(\frac{b}{a}\right)_{1,2} = \frac{1 \pm \sqrt{5}}{2}. \tag{2.37}$$

The ratio of positive distances must be positive, so the solution is the ratio of distances $\frac{b}{a} = \frac{1+\sqrt{5}}{2}$ when the electrostatic energy W of the e-e-p arrangement is zero.

2.3.3 2.5 Charged tetrahedron

Find the energy required to place four electrons at the vertices of a tetrahedron with edge $a = 10^{-10}\,m$, with a proton at its center.

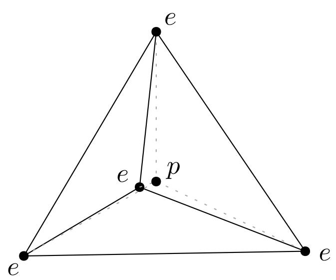


Figure 2.13: A tetrahedron with electrons at its vertices and a proton at its center.

Solution: The electrostatic energy of a system of charges W is given by

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{\alpha<\beta} \frac{q_\alpha q_\beta}{r_{\alpha\beta}}, \tag{2.38}$$

where q_α are the charges on each particle and $r_{\alpha\beta}$ are their mutual distances.

We will not list here all ten terms of this sum for 5 charges. We will note that due to the symmetry of the tetrahedron we have only two different kinds of interactions between the charges. It is the interaction of the electrons at the vertices of the tetrahedron – they are all equidistant along the length of the tetrahedron edge a, and there are six of these edges. The second interaction is a proton in the center of the tetrahedron interacting with four electrons at a distance r:

$$W = \frac{1}{4\pi\varepsilon_0} \left(6\,\frac{e^2}{a} + 4\,\frac{-e^2}{r} \right). \tag{2.39}$$

The distance of the center of the tetrahedron from its vertices is[3] $r = \frac{\sqrt{6}}{4} a$, where a is the length of its edge. After substitution:

$$W = \frac{e^2}{4\pi\varepsilon_0} \frac{6 - 8\sqrt{\frac{2}{3}}}{a} = 1,23.10^{-18}\, J, \tag{2.40}$$

we used the values of the elementary electric charge $e = 1,602.10^{-19}\, C$ and the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\, F.m^{-1}$ for the numerical result.

2.3.4 2.6 Nuclear decay

The atomic nuclei of heavy elements can be thought of as spheres charged with a volume charge density $\rho = \frac{4}{3} \cdot 10^{25}\, C.m^{-3}$. How does the electrostatic energy change when a uranium $_{92}$U nucleus symmetrically decays into two identical palladium $_{46}$Pd nuclei?

Solution: The electrostatic energy of a volume-charged sphere of radius R with constant charge density ρ, or total charge on the sphere Q, is

$$W = \frac{4\pi R^5 \rho^2}{15\varepsilon_0} = \frac{3}{5} \frac{1}{4\pi\varepsilon_0} \frac{Q^2}{R}. \tag{2.41}$$

The charge of the uranium nucleus is $Q_\mathrm{U} = 92e$ and the charge of the palladium nucleus is $Q_\mathrm{Pd} = 46e = \frac{1}{2} Q_\mathrm{U}$. Their radii are determined from the relation for the total charge Q of a homogeneously charged sphere:

$$Q = \rho V = \rho\, \frac{4}{3} \pi R^3 \quad \longrightarrow \quad R = \sqrt[3]{\frac{3Q}{4\pi\rho}}. \tag{2.42}$$

Thus, the radii of the uranium and palladium nuclei are

$$R_\mathrm{U} = \sqrt[3]{\frac{69e}{\pi\rho}}, \qquad R_\mathrm{Pd} = \sqrt[3]{\frac{69e}{2\pi\rho}} = R_\mathrm{U} \frac{1}{\sqrt[3]{2}}. \tag{2.43}$$

The change in electrostatic energy ΔW is

$$\begin{aligned} \Delta W = W_\mathrm{U} - 2W_\mathrm{Pd} &= \frac{3}{5} \frac{1}{4\pi\varepsilon_0} \left(\frac{Q_\mathrm{U}^2}{R_\mathrm{U}} - 2\,\frac{Q_\mathrm{Pd}^2}{R_\mathrm{Pd}} \right) = \frac{3}{5} \frac{1}{4\pi\varepsilon_0} \frac{Q_\mathrm{U}^2}{R_\mathrm{U}} \left(1 - 2 \frac{\frac{1}{2^2}}{\frac{1}{\sqrt[3]{2}}} \right) \\ &= \frac{3}{5} \frac{1}{4\pi\varepsilon_0} Q_\mathrm{U}^2 \sqrt[3]{\frac{\pi\rho}{69e}} \left(1 - \frac{1}{\sqrt[3]{4}} \right) = 7,36.10^{-11}\, J, \end{aligned} \tag{2.44}$$

[3] Derivation of this fact *deliberately* omitted...

where we have used the values of the elementary electric charge $e = 1,602.10^{-19}\,C$ and the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\,F.m^{-1}$ for the numerical result.

2.4 Gaussian law

2.4.1 2.7 Charge in a cube

A point charge is located a) in the center of a cube, b) in one of the corners of the cube. Determine the flux of electric field through each of the walls of the cube.

(a) Charge in the middle of the cube. (b) Charge in the corner of the cube.

Figure 2.14: Charges in cubes.

Solution: We will use Gauss's law to solve this problem. This states that the flux of electric field Φ through a closed surface S is proportional to the charge Q enclosed in that surface:

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}. \tag{2.45}$$

If we consider the charge in the middle of the cube, we can take the surface of the cube as the closed surface S in Gauss's law. Thus the total flux of electric field $\vec{E}$ through the cube is $\Phi = \frac{Q}{\varepsilon_0}$. The symmetry of the problem says that the same electric field flux must flow through each of the cube walls[4]. This flux is obviously equal to

$$\Phi_{1\,\mathrm{wall}} = \frac{1}{6}\Phi = \frac{Q}{6\varepsilon_0}. \tag{2.46}$$

For the charge at the corner of the cube, the situation is more complicated because Gauss's law does not hold if the point charge lies on the chosen surface S. But we can determine the flux through the three walls of the cube in which the charge lies directly from the definition of the electric field flux:

$$\Phi = \int_S \vec{E} \cdot d\vec{S}. \tag{2.47}$$

Electric field vectors $\vec{E}$ are radial – they always point directly away from or toward the charge. This means that the vectors $\vec{E}$ are tangential to the walls in which the charge lies, see Figure 2.15.

[4]The cube is symmetric when rotated by multiples of right angles about axes passing through the center of the cube (and the charge) perpendicular to the respective two cube walls.

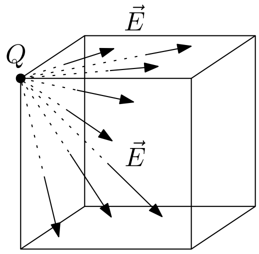


Figure 2.15: The electric intensity vectors $\vec{E}$ for the charge in the corner of the cube are tangent to the walls in which the charge lies.

When tangent to the walls, they are perpendicular to the normal vector of those walls, $\vec{E} \perp \vec{n}$. But then the flux through these walls is zero, $\Phi_1 = 0$, since the scalar products vanish in the flux definition, $\vec{E} \cdot d\vec{S} = (\vec{E} \cdot \vec{n})\, dS$.

To determine the flux of the remaining three faces, we surround the small cube with the charge in the corner with seven other cubes of equal size to form one large cube, see Figure 2.16. We use this supercube as the surface S in Gauss's law. The law is now valid; the charge lies inside (in the middle of) the supercube.

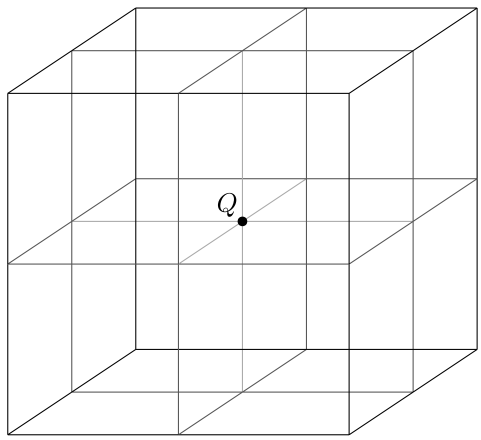


Figure 2.16: Eight cubes built around the charge.

The non-zero flux through one wall of the small cube will then be one quarter of the flux through the large wall, which in turn is one sixth of the total flux through the supercube:

$$\Phi_{1\,\text{malá stěna}} = \frac{1}{4}\Phi_{1\,\text{velká stěna}} = \frac{1}{4}\frac{1}{6}\frac{Q}{\varepsilon_0} = \frac{Q}{24\varepsilon_0}. \tag{2.48}$$

2.5 Electrostatic potential and electric field strength

2.5.1 2.8 Charged rod

A thin rod charged with linear charge density τ is located on the axis z between the points $z = a$, $z = -a$. Determine the potential φ at the points on the axis $x > 0$.

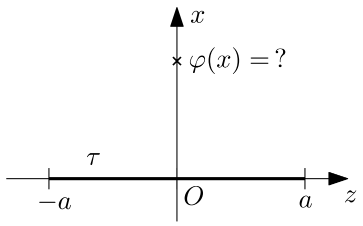


Figure 2.17: The charged rod on axis z and the electrostatic potential φ induced by it on axis x.

Solution: The electrostatic potential of a linear charge distribution is determined using the following relation:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int_l \frac{\tau}{R}\,dl, \tag{2.49}$$

where τ is the linear charge density and R is the distance between the potential location and the line element dl, $R = |\vec{r} - \vec{r}'|$ (we are determining the potential φ at the location $\vec{r}$, vector $\vec{r}'$ is the position vector of the line element dl).

The rod lies on the Cartesian axis z, the line element is $dl = dz$. The coordinates z, on which the rod lies, are $z \in \langle -a, a\rangle$. The distance R is $R = \sqrt{x^2 + z^2}$, see figure 2.18.

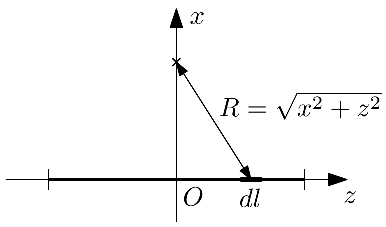


Figure 2.18: Line element dl and its distance R from the point where we determine the electrostatic potential φ

Substituting the above information into the formula for the electrostatic potential (2.49) gives the integral:

$$4\pi\varepsilon_0\,\varphi = \int_{-a}^{a} \frac{\tau}{\sqrt{x^2 + z^2}}\,dz = \int_{-a}^{a} \frac{\tau}{x\sqrt{1 + \left(\frac{z}{x}\right)^2}}\,dz = 2\int_0^a \frac{\tau}{x\sqrt{1 + \left(\frac{z}{x}\right)^2}}, \tag{2.50}$$

where in the last modification we used the evenness of the function in the integrand. After substitutions $u = \frac{z}{x}$, $du = \frac{dz}{x}$ and $u = \sinh v$, $du = \cosh v\,dv$ we arrive at the result:

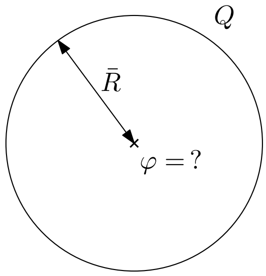
$$\begin{aligned} 4\pi\varepsilon_0\,\varphi &= 2\tau\int_0^{\frac{a}{x}} \frac{du}{\sqrt{1 + u^2}} = 2\tau\int_0^{\operatorname{argsinh}\frac{a}{x}} \frac{\cosh v\,dv}{\sqrt{1 + \sinh^2 v}} = 2\tau\int_0^{\operatorname{argsinh}\frac{a}{x}} dv \\ &= 2\tau\,[v]_0^{\operatorname{argsinh}\frac{a}{x}} = 2\tau \operatorname{argsinh}\frac{a}{x} = 2\tau\ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1}\right). \end{aligned} \tag{2.51}$$

The final result is therefore

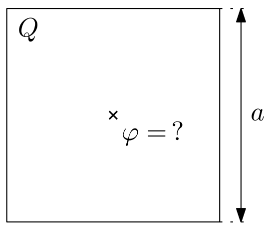
$$\varphi(x) = \frac{\tau}{2\pi\varepsilon_0}\ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1}\right). \tag{2.52}$$

2.5.2 2.9 Charged plates

Determine the potential φ at the centre of a charged plate Q if the plate is a) a circle of radius $\bar{R}$, b) a square of side a.



(a) Circular plate.



(b) Square plate.

Figure 2.19: Electrostatic potential in the middle of a circular and square plate.

Solution: The electrostatic potential of a planar body is determined by the following relation:

$$\varphi = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma}{R}\, dS, \tag{2.53}$$

where σ is the surface charge density and R is the distance between the potential location and the surface element dS, $R = |\vec{r} - \vec{r}'|$ (we are determining the potential φ at the location $\vec{r}$, vector $\vec{r}'$ is the position vector of the surface element dS).

Solution for a circular plate: Introduce the polar coordinates (r, α) starting at the center of the plate. The plate is then at coordinates $r \in \langle 0, \bar{R} \rangle$ and $\alpha \in \langle 0, 2\pi \rangle$.

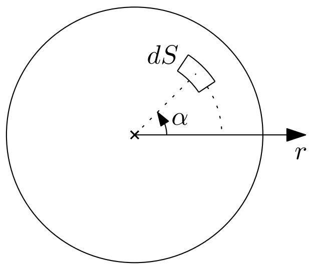


Figure 2.20: Polar coordinate (r, α) in the circular plate, the surface element is $dS = r\, dr\, d\alpha$.

The surface element in polar coordinates is $dS = r\, dr\, d\alpha$. The distance R here is simply $R = r$. The surface charge density here is constant $\sigma = \frac{Q}{\pi \bar{R}^2} = \text{const}$. After plugging all this information into the integral (2.53) for the electrostatic potential, we get:

$$4\pi\varepsilon_0\, \varphi = \int_0^{\bar{R}} \int_0^{2\pi} \frac{\sigma}{r}\, r\, dr\, d\alpha = \sigma \int_0^{\bar{R}} dr \int_0^{2\pi} d\alpha = \sigma \bar{R}\, 2\pi = \frac{2Q}{\bar{R}}. \tag{2.54}$$

The resulting expression for the electrostatic potential φ is

$$\varphi = \frac{\sigma \bar{R}}{2\varepsilon_0} = \frac{Q}{2\pi \bar{R}\, \varepsilon_0}. \tag{2.55}$$

Solution for the square plate: Here we use the result of the previous example 2.8 (Section

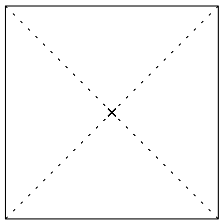
2.5.1) for the potential of a charged rod of length $2a$ at distance x from the rod on its axis:

$$\varphi(x) = \frac{\tau}{2\pi\varepsilon_0} \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1}\right). \tag{2.56}$$

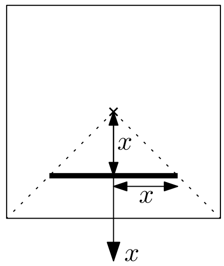
Let us divide the square plate diagonally into four parts, see Figure 2.21 on the left. Because of symmetry, each part will contribute equally by potential φ_{part} to the total potential φ:

$$\varphi = 4\varphi_{\text{part}}. \tag{2.57}$$

Then we divide each of the sections into thin strips of width dx, see figure 2.21 on the right.



(a) We divide the cube diagonally into quarters.



(b) We divide a quarter of the plate into thin strips of width dx. If their distance is x from the center of the wafer, their length is $2x$.

Figure 2.21: Division of a square plate.

Let the distance of the strip from the center of the plate be x, then the strips have length $2x$. The total charge on the strip is $dQ = \sigma\, dS = \sigma\, 2x\, dx$. "Linear charge density" is then $d\tau = \frac{dQ}{l} = \frac{dQ}{2x} = \sigma\, dx$. Substituting these data into the formula for the potential of a charged rod (2.56) gives the contribution to the potential

$$d\varphi(x) = \frac{\sigma\, dx}{2\pi\varepsilon_0} \ln\left(\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + 1}\right) = \frac{\sigma}{2\pi\varepsilon_0} \ln\left(1 + \sqrt{2}\right) dx. \tag{2.58}$$

We integrate this contribution $d\varphi$ to give the total potential of the section φ_{part}. We need to sum the contributions from the strips with distance $x \in \langle 0, \frac{a}{2} \rangle$:

$$\varphi_{\text{part}} = \int_0^{a/2} d\varphi(x) = \int_0^{a/2} \frac{\sigma}{2\pi\varepsilon_0} \ln\left(1 + \sqrt{2}\right) dx = \frac{\sigma}{2\pi\varepsilon_0} \ln\left(1 + \sqrt{2}\right) \frac{a}{2}. \tag{2.59}$$

The surface charge density is $\sigma = \frac{Q}{S} = \frac{Q}{a^2}$. After substitution, we get the result:

$$\varphi = 4\varphi_{\text{part}} = \frac{\sigma a}{\pi\varepsilon_0} \ln\left(1 + \sqrt{2}\right) = \frac{Q}{\pi a\, \varepsilon_0} \ln\left(1 + \sqrt{2}\right). \tag{2.60}$$

Addendum: Alternatively, we can calculate the potential at the center of a square plate as in the case of a circular plate, i.e. by integration in polar(!) coordinates. Coming soon.

2.5.3 2.10 Axis of a charged circular disc

Determine the potential φ and the magnitude of the electric field E on the axis of a circular disc of radius $\bar{R}$ charged with an surface charge density σ.

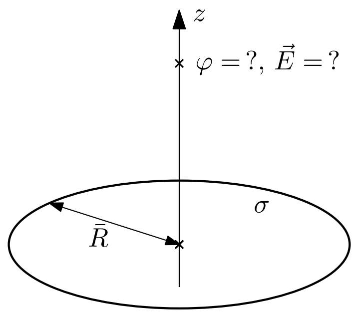


Figure 2.22: Electrostatic potential φ and electric field strength $\vec{E}$ on the axis of the circular disc.

Solution: The electrostatic potential of a planar body is determined using the following relation:

$$\varphi = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma}{R}\, dS, \tag{2.61}$$

where σ is the surface charge density and R is the distance between the potential location and the surface element dS, $R = |\vec{r} - \vec{r}'|$ (we are determining the potential φ at the location $\vec{r}$, vector $\vec{r}'$ is the position vector of the surface element dS).

We introduce the polar coordinate (r, α) with origin at the center of the disc (and introduce the Cartesian coordinate z along the disc axis). The disc is then at coordinates $r \in \langle 0, \bar{R} \rangle$ and $\alpha \in \langle 0, 2\pi \rangle$.

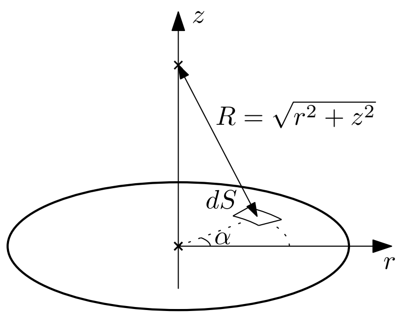


Figure 2.23: Distance R between the surface element dS and the the chosen point for determining the electrostatic potential φ.

The surface element in polar coordinates is $dS = r\,dr\,d\alpha$. The distance R here is $R = \sqrt{r^2 + z^2}$, see Figure 2.23. The surface charge density here is constant $\sigma = \frac{Q}{\pi \bar{R}^2} = \text{const}$. After substituting this information into the integral (2.61) for the electrostatic potential, we get:

$$4\pi\varepsilon_0\, \varphi = \int_0^{\bar{R}} \int_0^{2\pi} \frac{\sigma}{\sqrt{r^2 + z^2}}\, r\,dr\,d\alpha = \sigma \int_0^{\bar{R}} \frac{r}{\sqrt{r^2 + z^2}}\, dr \int_0^{2\pi} d\alpha = 2\pi\sigma \int_0^{\bar{R}} \frac{r}{\sqrt{r^2 + z^2}}\, dr. \tag{2.62}$$

Substituting $u = r^2 + z^2$, $du = 2r\,dr$, we have:

$$4\pi\varepsilon_0\, \varphi = 2\pi\sigma \int_{z^2}^{\bar{R}^2 + z^2} \frac{du}{2\sqrt{u}} = 2\pi\sigma \left[\sqrt{u}\right]_{z^2}^{\bar{R}^2 + z^2} = 2\pi\sigma \left(\sqrt{\bar{R}^2 + z^2} - |z| \right). \tag{2.63}$$

The resulting electrostatic potential is (after possibly substituting for the charge density σ):

$$\varphi(z) = \frac{\sigma}{2\varepsilon_0} \left(\sqrt{\bar{R}^2 + z^2} - |z| \right) = \frac{Q}{2\pi \bar{R}^2 \varepsilon_0} \left(\sqrt{\bar{R}^2 + z^2} - |z| \right). \tag{2.64}$$

The electric field $\vec{E}$ on the axis is obtained from the relation

$$\vec{E} = -\operatorname{grad}\varphi = -\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right). \tag{2.65}$$

We must note, however, that we have calculated the potential φ only at points on the axis of the disc – so we only know the functional values $\varphi(x = 0, y = 0, z)$ for the constant coordinate values x and y. Therefore, we cannot calculate the derivatives of $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$! We have to determine the electric field components E_x and E_y differently.

The rotational symmetry about the axis of the disc implies that these components must be zero, $E_x = E_y = 0$. The resulting vector $\vec{E}$ is therefore

$$\vec{E}(z) = \left(0, 0, -\frac{d\varphi}{dz}\right) \tag{2.66}$$

and the component E_z is given by the expression

$$E_z = -\frac{d\varphi}{dz} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{\bar{R}^2 + z^2}} - \operatorname{sgn} z\right). \tag{2.67}$$

2.5.4 2.11 Hemispherical shell

Determine the magnitude of the electric field E at the center of a spherical shell of radius $\bar{R}$ if one half of it is charged with surface density σ.

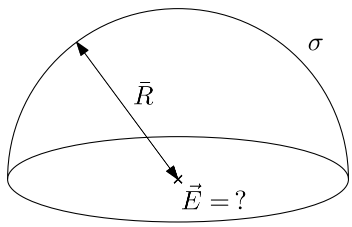


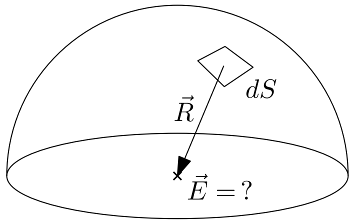
Figure 2.24: The electric field $\vec{E}$ at the centre of a charged hemispherical shell.

Solution: The electric field $\vec{E}$ of a planar body is determined using the following relation:

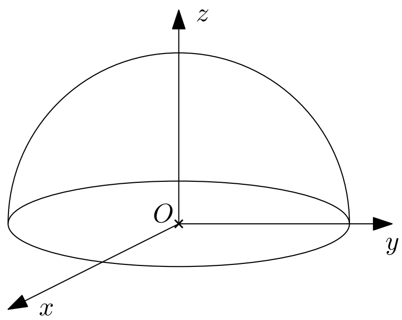
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_S \sigma\frac{\vec{R}}{R^3}\, dS, \tag{2.68}$$

where σ is the surface charge density and $\vec{R}$ is the vector connecting the potential location given by the position vector $\vec{r}$ and the position of the surface element dS given by the vector $\vec{r}'$, $\vec{R} = \vec{r} - \vec{r}'$ (R is the magnitude of this vector, $R = |\vec{r} - \vec{r}'|$).

Specifically, the vector $\vec{R} = (X, Y, Z)$ here has the form $\vec{R} = -\vec{r}'$, since the intensity $\vec{E}$ is determined at the origin of the coordinates, so $\vec{r} = 0$ holds, see figure 2.25 on the left. At the same time, rotational symmetry about the axis of the shell must imply that the resulting electric field vector $\vec{E}$ must lie on the axis of the hemispherical shell. If we introduce Cartesian coordinates as in Figure 2.25 on the right, then $E_x = E_y = 0$ will hold, hence $\vec{E} = (0, 0, E_z)$.



(a) Vector $\vec{R}$ connecting the surface element dS with the point of determination of the electric field $\vec{E}$.



(b) Cartesian coordinates (x, y, z) with origin at the center of the shell and axis z along the shell axis.

Figure 2.25: Cartesian coordinates and vector $\vec{R}$ on the hemispherical shell.

The formula for the component E_z then looks like the following according to (2.68):

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_S \sigma \frac{Z}{R^3}\, dS. \tag{2.69}$$

If we introduce spherical coordinates $(r,\, \theta,\, \varphi)$ (see also Figure 2.26)

$$x = r \sin\theta \cos\varphi, \qquad y = r \sin\theta \sin\varphi, \qquad z = r\cos\theta \tag{2.70}$$

with origin at the center of the spherical shell, the hemisphere will be at the coordinates

$$r = \bar{R}, \qquad \theta \in \left\langle 0, \frac{\pi}{2} \right\rangle, \qquad \varphi \in \langle 0, 2\pi \rangle. \tag{2.71}$$

The surface element in spherical coordinates is (here we have constant $r = \bar{R}$):

$$dS = r^2 \sin\theta\, d\theta\, d\varphi = \bar{R}^2 \sin\theta\, d\theta\, d\varphi. \tag{2.72}$$

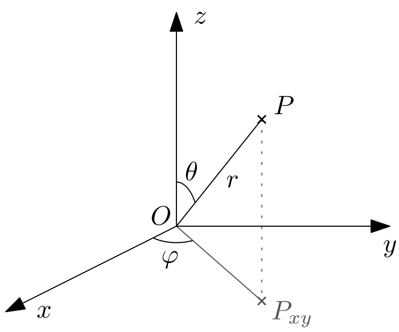


Figure 2.26: The spherical coordinates (r, θ, φ) of point P. The point P_{xy} represents the perpendicular projection of the point P into the plane xy.

The actual calculation then looks as follows. In the integral (2.69) we use (2.70)-(2.72) and $\vec{R} = -\vec{r}'$:

$$\begin{aligned} 4\pi\varepsilon_0\, E_z &= \int_0^{\pi/2} \int_0^{2\pi} \sigma \frac{-\bar{R}\cos\theta}{\bar{R}^3}\, \bar{R}^2 \sin\theta\, d\alpha\, d\theta = -\sigma \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\alpha \\ &= -2\pi\sigma \int_0^{\pi/2} \frac{1}{2}\sin 2\theta\, d\theta = 2\pi\sigma \left[\frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = -\pi\sigma. \end{aligned} \tag{2.73}$$

Thus, the value of E_z and the magnitude of $|\vec{E}|$ is

$$E_z = -\frac{\sigma}{4\varepsilon_0} \quad \longrightarrow \quad |\vec{E}| = \frac{\sigma}{4\varepsilon_0}. \tag{2.74}$$

2.5.5 2.13 Almost closed circle

A thin bar is bent into the shape of an almost closed circle of radius $r = 0,5\,m$. A gap of width $d = 2\,cm$ remains between the ends, and the rod carries a charge $q = 3,34.10^{-10}\,C$. Determine the magnitude and direction of the electric field at the center of the circle.

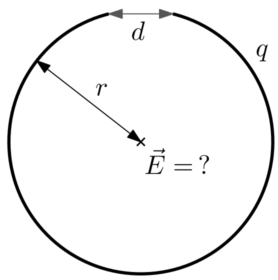


Figure 2.27: An almost closed circle.

Solution: The electric field of a linear charge distribution is determined using the following relation:

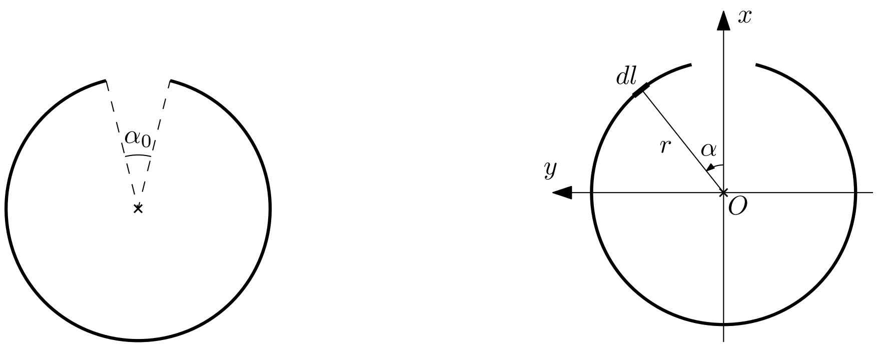
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_l \tau \frac{\vec{R}}{R^3}\, dl, \tag{2.75}$$

where τ is the linear charge density and $\vec{R}$ is the vector connecting the potential location given by the position vector $\vec{r}$ and the position of the line element dl given by the vector $\vec{r}'$, $\vec{R} = \vec{r} - \vec{r}'$ (R is the magnitude of this vector, $R = |\vec{r} - \vec{r}'|$).

Let us denote by α_0 the central angle that the ends of the bar are at, see figure 2.28 on the left. We can express this as $\sin\frac{\alpha_0}{2} = \frac{d/2}{r}$. The longitudinal charge density can then be expressed as

$$\tau = \frac{q}{(2\pi - \alpha_0)r}, \tag{2.76}$$

where $(2\pi - \alpha_0)r$ is the length of the rod.



(a) The central angle α_0 that marks the missing segment of the circle.

(b) Cartesian and polar coordinates on the circle and the corresponding line element.

Figure 2.28: Coordinates on an almost closed circle.

Specifically, the vector $\vec{R} = (X, Y, Z)$ here has the form $\vec{R} = -\vec{r}'$, since the intensity $\vec{E}$ is determined at the origin of the coordinates, so $\vec{r} = 0$ holds. At the same time, mirror symmetry about the plane that halves circle implies that the resulting electric field strength vector must lie in this plane (and also in the plane of the circle, since this is a planar problem). If we introduce Cartesian coordinates as in Figure 2.28 on the right it will be true that $E_y = E_z = 0$, hence $\vec{E} = (E_x, 0, 0)$. Then the formula for the component E_x according to (2.75) looks as follows:

$$E_x = \frac{1}{4\pi\varepsilon_0} \int_l \tau \, \frac{X}{R^3} \, dl. \tag{2.77}$$

In polar coordinates

$$x = r\cos\alpha, \qquad y = r\sin\alpha \tag{2.78}$$

the bar extends in coordinates $\alpha \in \langle \frac{\alpha_0}{2}, 2\pi - \frac{\alpha_0}{2} \rangle$ and the line element is $dl = r\, d\alpha$. The vector $\vec{r}' = (x, y, 0)$ in these coordinates has the expression $\vec{r}' = (r\cos\alpha, r\sin\alpha, 0)$ and hence $\vec{R} = (-r\cos\alpha, -r\sin\alpha, 0)$ and $R = r$. After substituting all the above expressions into the integral (2.77), we can calculate the value of E_x:

$$\begin{aligned} 4\pi\varepsilon_0 \, E_x &= \int_{\alpha_0/2}^{2\pi - \alpha_0/2} \tau \, \frac{-r\cos\alpha}{r^3} \, r \, d\alpha = -\frac{\tau}{r} \, [\sin\alpha]_{\alpha_0/2}^{2\pi - \alpha_0/2} \\ &= \frac{\tau}{r} \left(\sin\frac{\alpha_0}{2} - \sin\left(2\pi - \frac{\alpha_0}{2}\right) \right) = \frac{2\tau}{r} \sin\frac{\alpha_0}{2}, \end{aligned} \tag{2.79}$$

where we have used the relations $\sin(2\pi + x) = \sin x$ and $\sin(-x) = -\sin x$. After substituting in the charge density τ from (2.76) we get

$$E_x = \frac{1}{4\pi\varepsilon_0} \frac{2\sin\frac{\alpha_0}{2}}{(2\pi - \alpha_0)} \frac{q}{r^2}. \tag{2.80}$$

This is the exact result for an arbitrarily large angle α_0[5]. The exercise assignment emphasizes that this is a small gap, so let us consider $\alpha_0 \ll 2\pi$ and make the following approximations. We have $2\pi - \alpha_0 \approx 2\pi$, $\sin\frac{\alpha_0}{2} \approx \frac{\alpha_0}{2}$:

$$E_x \approx \frac{1}{4\pi\varepsilon_0} \frac{\frac{\alpha_0}{2\pi} q}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{\tilde{q}}{r^2}, \tag{2.81}$$

where we denote the charge $\tilde{q} = \frac{\alpha_0}{2\pi} q$, which represents the charge that would be on the missing segment of the circle. Thus, the approximated result for the magnitude of the electric field looks like the field from a point charge of magnitude $\tilde{q}$ located at the center of the missing circle segment. And this is no coincidence, see Addendum.

Addendum: We could also solve the exercise using the superposition principle. We could write the original electric field strength $\vec{E}$ as the sum of the electric fields of the full circle $\vec{E}_{\text{full}}$ and of the small **oppositely charged** complement $\vec{E}_{\text{gap}}$, $\vec{E} = \vec{E}_{\text{full}} + \vec{E}_{\text{gap}}$, see Figure 2.29.

[5] We could still substitute for α_0: $\sin\frac{\alpha_0}{2} = \frac{d}{2r}$ and $\alpha_0 = 2\text{arcsin}\frac{d}{2r}$ for $\alpha_0 < \pi$.

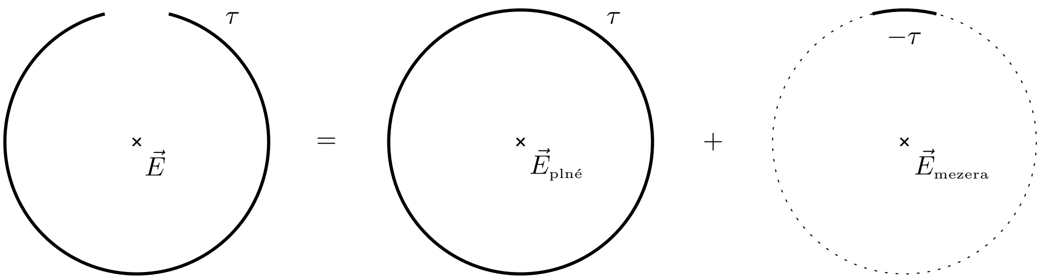


Figure 2.29: Original electric field $\vec{E}$ as the sum of the electric fields of the solid circle $\vec{E}_{\text{full}}$ and of the small oppositely charged complement $\vec{E}_{\text{gap}}$, $\vec{E} = \vec{E}_{\text{full}} + \vec{E}_{\text{gap}}$.

From symmetry, however, the electric field ofthe solid circle $\vec{E}_{\text{full}}$ is zero, $\vec{E}_{\text{full}} = 0$. Thus, the original electric field at the center of the circle is the same as the field from the oppositely charged complement, $\vec{E} = \vec{E}_{\text{gap}}$. The exact result (2.79), or (2.80), would be obtained by a very similar calculation as in (2.79), i.e., by calculating the integral of

$$4\pi\varepsilon_0 \, E_x = 4\pi\varepsilon_0 \, E_{x\text{mezera}} = \int_{-\alpha_0/2}^{\alpha_0/2} (-\tau) \, \frac{-r\cos\alpha}{r^3} \, r \, d\alpha. \tag{2.82}$$

We, however, make a straightforward approximation for the small gap. In this case, we can consider the small segment of the circle as a point charge, since the center of the circle is far enough away from the charges. The magnitude of this charge is $-\tilde{q} = (-q)\frac{\alpha_0}{2\pi}$ and hence the magnitude of the electric field is approximately

$$E_x \approx \frac{1}{4\pi\varepsilon_0} \frac{\tilde{q}}{r^2}. \tag{2.83}$$

2.5.6 2.14 Cut shell

Consider a spherical shell of radius $\bar{R}$ charged with areal density σ. Around a selected point on this surface, cut a small spherical cap of radius $a \ll \bar{R}$. Determine the magnitude of the electric field at the centre of the hole.

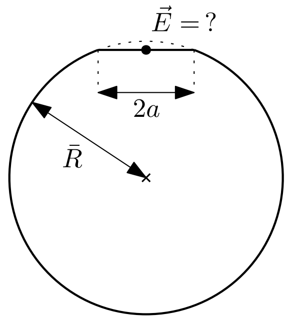


Figure 2.30: The electric field $\vec{E}$ at the center of the hole created by cutting the cap of the spherical shell.

Solution: The rotational symmetry about the axis of the spherical shell (and the cut cap) implies that the resulting electric field strength vector $\vec{E}$ will lie on this axis. If we introduce Cartesian coordinates as in Figure 2.31, then $\vec{E} = (0, 0, E_z)$ will hold.

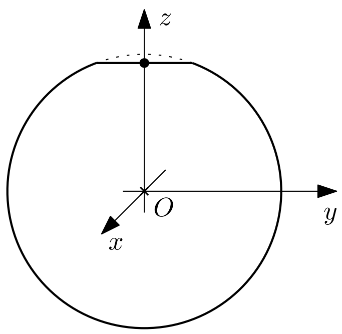


Figure 2.31: Cartesian coordinates (x, y, z) in the truncated spherical shell.

Let's determine the electric field component E_z by calculating the electrostatic potential φ at an arbitrary location on the axis of the spherical shell, $\varphi(z)$ and then deriving the electric field from this by taking the derivative:

$$\vec{E} = -\operatorname{grad}\varphi, \qquad E_z = -\frac{d\varphi}{dz}. \tag{2.84}$$

We start from the relation for the potential of a surface charged body

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma}{R}\, dS, \tag{2.85}$$

where σ is the surface charge density and R is the distance between the potential location and the surface element dS, $R = |\vec{r} - \vec{r}'|$ (we are determining the potential φ at the location $\vec{r}$, vector $\vec{r}'$ is the position vector of the surface element dS).

We introduce the spherical coordinates

$$x = r\sin\theta\cos\alpha, \qquad y = r\sin\theta\sin\alpha, \qquad z = r\cos\theta. \tag{2.86}$$

Using the cosine theorem, the distance R is given by $R = \sqrt{z^2 + \bar{R}^2 - 2z\bar{R}\cos\theta}$ (z here represents the position on the axis z, not the position of the surface element dS, so we do not substitute z for spherical coordinates), see Figure 2.32. The cut shell extends in coordinates $r = \bar{R}$, $\theta \in \langle\theta_0, \pi\rangle$, $\alpha \in \langle 0, 2\pi\rangle$, where θ_0 is half the central angle of the cut shell (again, see Figure 2.32). The surface element in spherical coordinates is $dS = r^2 \sin\theta\, d\theta\, d\alpha$.

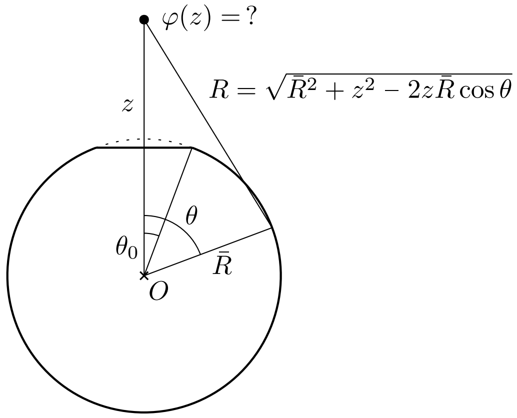


Figure 2.32: Distance R expressed in terms of shell radius $\bar{R}$, angle θ and position on axis z.

Substituting all of the above information into (2.85) yields the following double integral:

$$\varphi(z) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_{\theta_0}^{\pi} \frac{\sigma\bar{R}^2\sin\theta}{\sqrt{z^2 + \bar{R}^2 - 2z\bar{R}\cos\theta}}\, d\alpha\, d\theta. \tag{2.87}$$

After a relatively simple calculation

$$\varphi(z) = \frac{\sigma\bar{R}^2}{2\varepsilon_0}\int_{\theta_0}^{\pi}\frac{\sin\theta}{\sqrt{z^2+\bar{R}^2-2z\bar{R}\cos\theta}}\,d\theta = \left|\begin{array}{c} u=-\cos\theta \\ du=\sin\theta\,d\theta \end{array}\right| = \tag{2.88}$$

$$= \frac{\sigma\bar{R}^2}{2\varepsilon_0}\int_{-\cos\theta_0}^{1}\frac{du}{\sqrt{z^2+\bar{R}^2+2z\bar{R}u}} = \frac{\sigma\bar{R}^2}{2\varepsilon_0}\left[\frac{1}{z\bar{R}}\sqrt{z^2+\bar{R}^2+2z\bar{R}u}\right]_{-\cos\theta_0}^{1}, \tag{2.89}$$

we obtain an expression for the potential

$$\varphi(z) = \frac{\sigma\bar{R}}{2\varepsilon_0}\frac{|z+\bar{R}|-\sqrt{z^2+\bar{R}^2-2z\bar{R}\cos\theta_0}}{z} = \frac{\sigma\bar{R}^2}{\varepsilon_0}\frac{1+\cos\theta_0}{|z+\bar{R}|+\sqrt{z^2+\bar{R}^2-2z\bar{R}\cos\theta_0}}. \tag{2.90}$$

By differentiation and subsequent algebraic modifications we arrive at the expression for the electric field component E_z[6]:

$$E_z(z) = -\frac{d\varphi}{dz} = \ldots = \frac{\sigma}{2\varepsilon_0}\frac{\bar{R}^2}{z^2}\left[\operatorname{sgn}(z+\bar{R}) - \frac{\bar{R}-z\cos\theta_0}{\sqrt{z^2+\bar{R}^2-2z\bar{R}\cos\theta_0}}\right]. \tag{2.91}$$

This is the exact result for any large angle θ_0. For $z=\bar{R}\cos\theta_0$ (i.e., the point in the middle of the canopy), we have

$$E(\bar{R}\cos\theta_0) = \ldots = \frac{\sigma}{2\varepsilon_0}\frac{1}{1+\sin\theta_0}. \tag{2.92}$$

Using the radius of the cut canopy a, the angle θ_0 is given by $\sin\theta_0 = \frac{a}{\bar{R}}$. The result then takes the form

$$E(\bar{R}\cos\theta_0) = \frac{\sigma}{2\varepsilon_0}\frac{1}{1+\frac{a}{\bar{R}}}. \tag{2.93}$$

For small a, we can simplify the result a little further by approximating $(1+x)^{-1} = 1-x$ to

$$E(\bar{R}\cos\theta_0) \approx \frac{\sigma}{2\varepsilon_0}\left(1-\frac{a}{\bar{R}}\right). \tag{2.94}$$

Addendum: Other interesting places on the z axis are: at the top of the missing cap, i.e., for $z=\bar{R}$; in the middle of the spherical shell, i.e., for $z=0$. For these locations we obtain the following exact and approximate expressions:

$$E(R) = \frac{\sigma}{2\varepsilon_0}\left(1-\sin\frac{\theta_0}{2}\right) \approx \frac{\sigma}{2\varepsilon_0}\left(1-\frac{a}{2\bar{R}}\right), \tag{2.95}$$

$$E(0) = \frac{\sigma}{4\varepsilon_0}\sin^2\theta_0 \approx \frac{\sigma}{4\varepsilon_0}\frac{a^2}{\bar{R}^2}, \tag{2.96}$$

where we put $\sin x \approx x$ and hence $\theta_0 \approx \frac{a}{\bar{R}}$ to obtain the approximate expressions.

2.5.7 2.12 Soap bubble potential
2.15 Earth's electrostatic field
2.16 Dielectric strength of air

A conducting soap bubble of radius $R = 2\,cm$ charged to potential $\varphi_1 = 10^4\,V$ will form a water droplet of radius $r = 0{,}05\,cm$ when it bursts. Determine the potential of the drop.

[6]We differentiate the "left" expression for the potential φ using the identity $z\operatorname{sgn}(z+\bar{R}) = |z+\bar{R}| - \bar{R}\operatorname{sgn}(z+\bar{R})$ and convert the terms with square roots to the common denominator.

The electrostatic field strength near the surface of the Earth is $E = 100\,V.m^{-1}$ and is directed downward. Determine the charge and potential of the Earth.

What is the maximum charge that can be held on a metal sphere of radius $R = 10\,cm$ if the dielectric strength of air is $E_{max} = 30\,kV.cm^{-1}$?

Solution: The electrostatic potential φ and the magnitude of the electric field strength E from a spherically symmetrically distributed charge of total magnitude Q are given by

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}, \qquad E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}, \tag{2.97}$$

where r is the distance from the center of spherical symmetry. We use these relations to calculate all three examples, see the Addendum for their derivation.

Example 2.12 Denote by φ_2 the potential after the drop breaks. Thus we have

$$\varphi_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}, \qquad \varphi_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}, \tag{2.98}$$

where we assume that the charge is spherically symmetrically distributed on the droplet and is not lost when it bursts. After expressing Q we have

$$Q = 4\pi\varepsilon_0 R\varphi_1, \qquad Q = 4\pi\varepsilon_0 r\varphi_2, \tag{2.99}$$

and by comparing them we get the result

$$\varphi_2 = \frac{R}{r}\varphi_1 = 4.10^5\,V = 400\,kV. \tag{2.100}$$

Example 2.15 Expressing the charge Q from the formula (2.97) on the right, we have

$$Q = 4\pi\varepsilon_0 r^2 E(r). \tag{2.101}$$

Substituting in the radius of the Earth, $r = R_Z$, we get the result

$$Q = 4\pi\varepsilon_0 R_Z^2 E(R_Z) = 4,52.10^5\,C, \tag{2.102}$$

where we have used the value of the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\,F.m^{-1}$ and the Earth radius $R_Z = 6378\,km$. Since the assignment says that the vector $\vec{E}$ points towards the Earth, the total charge of the Earth must be negative, so in fact $Q = -4,52.10^5\,C$ (the formula for the magnitude E actually works with the absolute value of the charge $|Q|$). The potential at the surface of the Earth is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R_Z} = R_Z E(R_Z) = 6,38.10^8\,V, \tag{2.103}$$

or $\varphi = -6,38.10^8\,V$ taking into account the sign of the charge.

Example 2.16 The magnitude of the electric field E from the charged sphere is again given by the relation (2.97) on the right. We require $E(r) \leq E_{max}$ to hold. The strongest electric field is obviously for $r = R$, directly on the surface of the sphere. The resulting maximum charge Q_{max} is then

$$Q_{max} = 4\pi\varepsilon_0 R^2 E_{max} = 3,34.10^{-6}\,C. \tag{2.104}$$

Addendum: We now derive the formulas (2.97) for the potential φ and the electric field strength $\vec{E}$ outside a spherically symmetrically distributed charge Q. Let us start by determining the electric field $\vec{E}$ using Gauss's law

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}, \tag{2.105}$$

which relates the flow of electric field strength $\vec{E}$ through a closed surface S to the total charge Q that is enclosed in that surface.

Let us first see what constraints the symmetry of the problem places on the form of the electric field $\vec{E}$. Spherical symmetry ensures that the vector $\vec{E}$ can only depend on the distance from the center of spherical symmetry r – $\vec{E}(r)$ – the vectors $\vec{E}$ must be constant on spheres of a given radius. What will be the direction of the vector $\vec{E}$? Spherical symmetry allows only a radial direction – a vector pointing directly to/from the center of symmetry: consider an axis of symmetry connecting the center of spherical symmetry and the point at which we determine the electric field $\vec{E}$ – then rotational symmetry about this axis ensures that the vector $\vec{E}$ must lie on this axis, i.e. point in the radial direction.

We can now calculate the left-hand side of Gauss's law (2.105). We choose the surface S a sphere of radius r (where r is so large that all the charge already lies in this sphere) with a centre agreeing with the centre of the symmetry. The vector $\vec{E}$ then points in the direction $d\vec{S} = \vec{n}\, dS$ ($\vec{n}$ is a unit normal vector to the surface element dS), i.e. $\vec{E} \cdot d\vec{S} = E\, dS$:

$$\oint_S \vec{E} \cdot d\vec{S} = \oint_S E\, dS = E(r) \oint_S dS = 4\pi r^2 E(r), \tag{2.106}$$

where we have further exploited the fact that the magnitude of the electric field strength on the surface of the sphere S is constant and can be factored out of the integral, and also the integral of unity over the sphere is the area of the sphere $4\pi r^2$. If we plug this result into (2.105) and express $E(r)$, we get the result:

$$E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}. \tag{2.107}$$

The formula for the vector $\vec{E}$ is obtained simply by multiplying $E(r)$ by the unit vector pointing in the radial direction[7] – $\vec{E} = E(r)\frac{\vec{r}}{r}$:

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{\vec{r}}{r^3}. \tag{2.108}$$

The potential φ is obtained by "solving" the equation

$$\vec{E} = -\operatorname{grad} \varphi. \tag{2.109}$$

We know that the following general formula for $\alpha \in \mathbb{R}$ holds:

$$\operatorname{grad} r^\alpha = \alpha\, r^{\alpha-2}\, \vec{r}, \tag{2.110}$$

by putting $\alpha = -1$ we get

$$\operatorname{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}, \tag{2.111}$$

which is exactly the relation needed to solve (2.109). Therefore, the following holds

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}. \tag{2.112}$$

[7]We have somewhat swept under the rug whether the direction $\vec{E}$ is radially outward from the center or inward to the center. In Gauss's law, we always choose normal vectors to point outward from a closed surface. So here we have vectors $\vec{n}$ pointing away from the center of symmetry. For $Q > 0$ we need the vectors $\vec{E}$ and $\vec{n}$ to point in the same direction, then the scalar product $\vec{E} \cdot d\vec{S} = E\, dS$ is positive and the integral $\int \vec{E} \cdot d\vec{S}$ is also positive (and then Gauss' law gives two positive numbers in the equation). For $Q < 0$, the direction of $\vec{E}$ must be opposite to the direction of $\vec{n}$ (i.e., $\vec{E}$ now points to the center), and then the scalar product $\vec{E} \cdot d\vec{S} = -E\, dS$ is negative and Gaussian's law consistently gives two negative numbers in the equation.

These relations can easily be generalized to situations where we ask what the electric field will be not only outside but also inside a spherically symmetrically distributed charge. The only thing that changes in Gauss's law (2.105) is that the charge Q on the right hand side will now depend on r, $Q(r)$, and will represent the total charge enclosed in a sphere of radius r. The resulting magnitude of the electric field strength (and self intensity) is then:

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}, \qquad \vec{E} = \frac{Q(r)}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}. \tag{2.113}$$

The potential is again given by the solution of equation $\vec{E} = -\operatorname{grad}\varphi$, this time with the result

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\int_r^\infty \frac{Q(r')}{r'^2}dr'. \tag{2.114}$$

(For r large enough that all the charge is already contained in a sphere of radius r, i.e. $Q(r) = Q_{\text{total}}$, the formula (2.114) reduces to the simpler (2.112).)

Calculation of function $Q(r)$. We calculate the charge contained in a sphere of radius r (denote V_r and its surface as S_r) using the volume charge density function $\rho(\vec{r})$ as

$$Q(r) = \int_{V_r} \rho(\vec{r}')\, dV, \tag{2.115}$$

where $\vec{r}'$ is the position vector of the volume element dV. In the spherically symmetric case, however, the charge density must depend only on the distance from the center of symmetry, $\rho(r)$. Writing the volume element dV as $dV = dr\, dS$, where dr is the increment of the radial coordinate r and dS is the surface element on the sphere of radius r. Then the volume integral over the sphere can be decomposed as follows:

$$Q(r) = \int_{V_r} \rho(\vec{r}')\, dV = \int_{V_r} \rho(\vec{r}')\, dS\, dr' = \int_0^r \int_{S_{r'}} dS\, \rho(r')\, dr'. \tag{2.116}$$

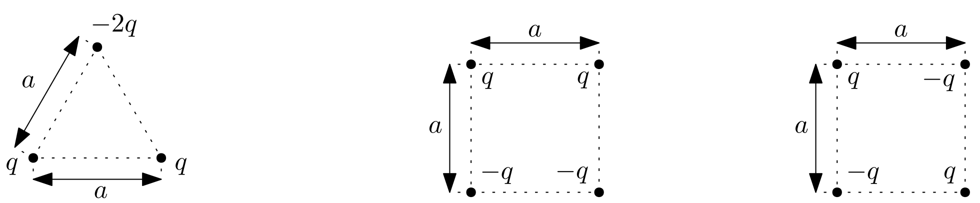
After integrating over the surface of the sphere, $\int_{S_{r'}} dS = 4\pi r'^2$, we get the relation

$$Q(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'. \tag{2.117}$$

2.6 Electric dipole and quadrupole moment

2.6.1 2.17 Point charges

The point charges are arranged a) at the vertices of an equilateral triangle with side a in the order q, q, $-2q$, b) in the vertices of a square of side a in the order $-q$, q, q, $-q$, c) in the order $-q$, q, $-q$, q. Determine the electric dipole moment of the system.



(a) Triangle. (b) Polarized square. (c) Symmetric square.

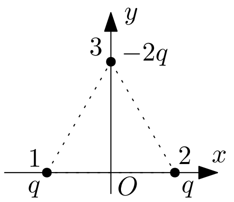
Figure 2.33: Dipole moment $\vec{p}$ of point charges.

Solution: The formula for the dipole moment of a discrete point charge distribution is as follows:

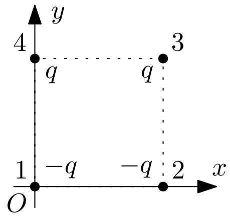
$$\vec{p} = \sum_{\alpha} q_\alpha \vec{r}_\alpha, \tag{2.118}$$

where sum is over all charges and q_α are the magnitudes and $\vec{r}_\alpha$ are the position vectors of the corresponding charges. It holds that if the total charge of the system Q is zero, then the dipole moment $\vec{p}$ does not depend on the choice of the origin of the coordinate system.

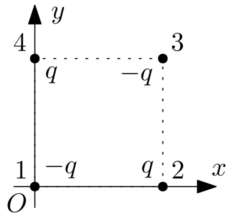
So let us introduce Cartesian coordinates so that we have the simplest possible calculation. One possibility is shown in Figure 2.34.



(a) Triangle.



(b) Polarized square.



(c) Symmetric square.

Figure 2.34: Cartesian coordinates (x, y, z) for individual point charge systems.

The dipole moment $\vec{p}$ for the triangle is then calculated as follows:

$$\vec{p} = q\,\vec{r}_1 + q\,\vec{r}_2 - 2q\,\vec{r}_3 = q\left(-\frac{a}{2}, 0, 0\right) + q\left(\frac{a}{2}, 0, 0\right) - 2q\left(0, \frac{\sqrt{3}a}{2}, 0\right) = \left(0, -\sqrt{3}qa, 0\right). \tag{2.119}$$

For the polarized square we have:

$$\vec{p} = -q\,\vec{r}_1 - q\,\vec{r}_2 + q\,\vec{r}_3 + q\,\vec{r}_4 = -q(0,0,0) - q(a,0,0) + q(a,a,0) + q(0,a,0) = (0, 2qa, 0). \tag{2.120}$$

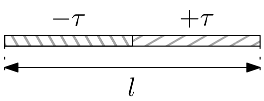
Finally, for the symmetric square:

$$\vec{p} = -q\,\vec{r}_1 + q\,\vec{r}_2 - q\,\vec{r}_3 + q\,\vec{r}_4 = -q(0,0,0) + q(a,0,0) - q(a,a,0) + q(0,a,0) = (0,0,0). \tag{2.121}$$

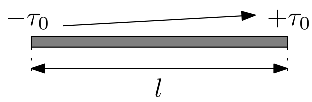
In this case we have a total charge Q and a dipole moment $\vec{p}$ zero. The non-zero moment will then only be the quadrupole moment, see Example 2.21 in Section 2.6.5.

2.6.2 2.18 Polarized rod

Determine the electric dipole moment of a thin rod of length l, a) one half of which is positively charged and the other negatively charged with a linear charge density τ, b) whose charge density increases linearly from $-\tau_0$ at one end to τ_0 at the other end.



(a) Constant charge density.



(b) Linear charge density.

Figure 2.35: Polarized rods.

Solution: The formula for the dipole moment of a continuous linear charge distribution is as follows:

$$\vec{p} = \int_l \tau\,\vec{r}\,dl, \tag{2.122}$$

where τ is the linear charge density and $\vec{r}$ is the position vector of the line element dl. It holds that if the total charge of the system Q is zero, then the dipole moment $\vec{p}$ does not depend on the choice of the origin of the coordinate system.

Let us introduce a Cartesian coordinate x with origin at the center of the rod, see Figure 2.36. The rod is then located at coordinates $x \in \langle -\frac{l}{2}, \frac{l}{2} \rangle$. The line element is then $dl = dx$ and its position vector is $\vec{r} = (x, 0, 0)$.

(a) Constant charge density.

(b) Linear charge density.

Figure 2.36: Polarized rods with an established Cartesian coordinate x.

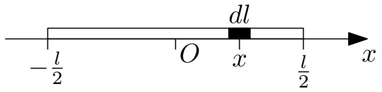


Figure 2.37: Line element dl at coordinate x in the rod.

Substituting the above information into the formula (2.122) for the dipole moment $\vec{p}$, we get

$$\vec{p} = \int_{-\frac{l}{2}}^{\frac{l}{2}} \tau(x)\, \vec{r}\, dx, \qquad \vec{p} = (p_x, 0, 0), \qquad p_x = \int_{-\frac{l}{2}}^{\frac{l}{2}} \tau(x)\, x\, dx, \tag{2.123}$$

(The only non-zero component is p_x, since the only non-zero component of $\vec{r} = (x, 0, 0)$ is x.)

For a polarized rod where each half is charged with opposite charge density, we have the following charge density function $\tau(x)$:

$$\tau(x) = \begin{cases} -\tau & \text{pro} \quad x \in \left\langle -\frac{l}{2}, 0 \right\rangle \\ +\tau & \text{pro} \quad x \in \left\langle 0, \frac{l}{2} \right\rangle \end{cases} \tag{2.124}$$

Substituting into the integral (2.123), we have:

$$p_x = \int_{-\frac{l}{2}}^{0} (-\tau)\, x\, dx + \int_{0}^{\frac{l}{2}} \tau\, x\, dx = -\tau \left[\frac{x^2}{2} \right]_{-\frac{l}{2}}^{0} + \tau \left[\frac{x^2}{2} \right]_{0}^{\frac{l}{2}} = 2\tau \frac{\frac{l^2}{4}}{2} = \frac{\tau l^2}{4}. \tag{2.125}$$

For a rod where the charge density varies linearly, the function $\tau(x)$ is generally of the form $\tau(x) = ax + b$, and $\tau(-\frac{l}{2}) = -\tau_0$ and $\tau(\frac{l}{2}) = +\tau_0$ must hold. These conditions lead to the function

$$\tau(x) = \frac{2\tau_0}{l} x. \tag{2.126}$$

When substituted into the integral (2.123):

$$p_x = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{2\tau_0}{l} x^2\, dx = 2 \frac{2\tau_0}{l} \int_{0}^{\frac{l}{2}} x^2\, dx = 2\, \frac{2\tau_0}{l} \frac{\frac{l^3}{8}}{3} = \frac{\tau_0 l^2}{6}. \tag{2.127}$$

2.6.3 2.19 Polarized sphere

The charge is distributed on the surface of a sphere of radius R such that one hemisphere has a positive charge with density σ, the other hemisphere has a negative charge with density

$-\sigma$. Determine the electric dipole moment of the sphere. What will this moment be if both hemispheres are charged volumetrically with opposite charges of the same volume density ρ?

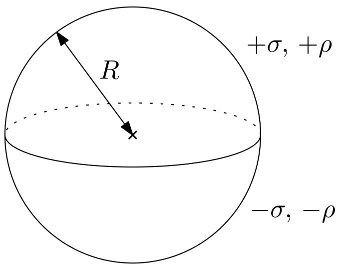


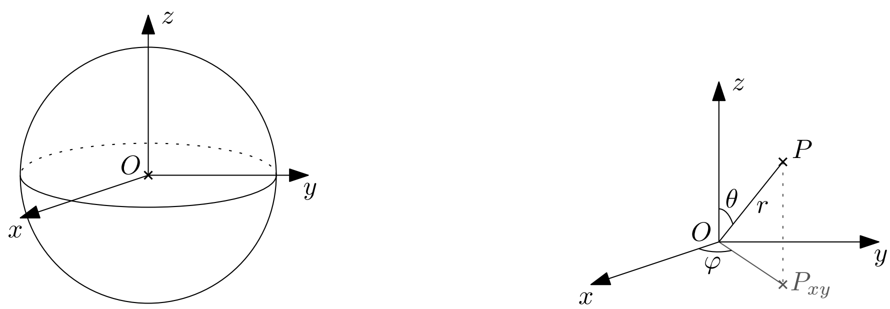
Figure 2.38: Dipole moment $\vec{p}$ of the surface and volume charged sphere.

Solution: The formulas for the dipole moment of a continuous area and volume charge distribution are as follows:

$$\vec{p} = \int_S \sigma\, \vec{r}\, dS, \qquad \vec{p} = \int_V \rho\, \vec{r}\, dV, \tag{2.128}$$

where σ and ρ are the area and volume charge densities, respectively, and $\vec{r}$ is the position vector of the surface element dS and volume element dV, respectively. It holds that if the total charge of the system Q is zero (which is obviously satisfied here from the symmetry of the charge distribution), then the dipole moment $\vec{p}$ does not depend on the choice of the origin of the coordinate system.

We introduce Cartesian coordinates as in Figure 2.39 on the left with the origin at the center of the sphere and the plane xy merging with the plane that divides the sphere into two charged halves.



(a) Cartesian coordinates (x, y, z) in a sphere.

(b) Spherical coordinates (r, θ, φ).

Figure 2.39: Dipole moment $\vec{p}$ of a surface and volume charged sphere.

From the symmetry of the problem (rotation about the z axis), the resulting dipole moment must lie on the z axis, i.e., $\vec{p} = (0, 0, p_z)$. For the dipole moment component p_z, according to (2.128) we have the formula

$$p_z = \int_S \sigma\, z\, dS, \qquad p_z = \int_V \rho\, z\, dV, \tag{2.129}$$

where z is the third component of the position vector $\vec{r} = (x, y, z)$. We further introduce spherical coordinates as in Figure 2.39 on the right, i.e., using the following prescriptions,

$$x = r\cos\varphi\sin\theta, \qquad y = r\sin\varphi\sin\theta, \qquad z = r\cos\theta. \tag{2.130}$$

The surface and volume elements are then of the following forms:

$$dS = r^2 \sin\theta \, d\theta \, d\varphi, \qquad dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi. \tag{2.131}$$

The surface-charged sphere is then layed out at coordinates $r = R$, $\theta \in \langle 0, \pi \rangle$ (hemisphere charged with $+\sigma$ at $\theta \in \langle 0, \frac{\pi}{2} \rangle$, hemisphere with $-\sigma$ at $\theta \in \langle \frac{\pi}{2}, \pi \rangle$), $\varphi \in \langle 0, 2\pi \rangle$. The actual computation of the integral (2.129) after all of the above information has been inserted is then as follows:

$$\begin{aligned} p_z &= \int_S \sigma(\vec{r}) \, z \, dS = \int_0^{\pi} \int_0^{2\pi} \sigma(\theta) \, R \cos\theta \, R^2 \sin\theta \, d\theta \, d\varphi \\ &= R^3 \int_0^{2\pi} d\varphi \left(\int_0^{\frac{\pi}{2}} \sigma \sin\theta \cos\theta \, d\theta + \int_{\frac{\pi}{2}}^{\pi} (-\sigma) \sin\theta \cos\theta \, d\theta \right) \\ &= 2\pi R^3 \sigma \left(\left[-\frac{1}{4} \cos 2\theta \right]_0^{\frac{\pi}{2}} - \left[-\frac{1}{4} \cos 2\theta \right]_{\frac{\pi}{2}}^{\pi} \right) \\ &= 2\pi R^3 \sigma \left(-\frac{1}{4} \right) \left((-1-1) - (1-(-1)) \right) = 2\pi\sigma R^3. \end{aligned} \tag{2.132}$$

For a volume-charged sphere we have the ranges $r \in \langle 0, R \rangle$, $\theta \in \langle 0, \pi \rangle$ (hemisphere charged $+\rho$ again to $\theta \in \langle 0, \frac{\pi}{2} \rangle$, hemisphere with $-\rho$ to $\theta \in \langle \frac{\pi}{2}, \pi \rangle$), $\varphi \in \langle 0, 2\pi \rangle$. Calculation of the integral (2.129):

$$\begin{aligned} p_z &= \int_V \rho(\vec{r}) \, z \, dV = \int_0^R \int_0^{\pi} \int_0^{2\pi} \rho(\theta) \, r \cos\theta \, r^2 \sin\theta \, dr \, d\theta \, d\varphi \\ &= \int_0^R r^3 \, dr \int_0^{2\pi} d\varphi \left(\int_0^{\frac{\pi}{2}} \rho \sin\theta \cos\theta \, d\theta + \int_{\frac{\pi}{2}}^{\pi} (-\rho) \sin\theta \cos\theta \, d\theta \right) \\ &= 2\pi \frac{R^4}{4} \rho \left(\left[-\frac{1}{4} \cos 2\theta \right]_0^{\frac{\pi}{2}} - \left[-\frac{1}{4} \cos 2\theta \right]_{\frac{\pi}{2}}^{\pi} \right) \\ &= \frac{1}{2} \pi R^4 \rho \left(-\frac{1}{4} \right) \left((-1-1) - (1-(-1)) \right) = \frac{1}{2} \pi \rho R^4. \end{aligned} \tag{2.133}$$

Addendum: The symmetry arguments allow us to say straightforwardly that the form of the dipole moment will be $\vec{p} = (0, 0, p_z)$. However, we could safely forget them and calculate the components p_x and p_y directly according to relations analogous to (2.129):

$$p_x = \int_S \sigma \, x \, dS, \qquad p_y = \int_S \sigma \, y \, dS, \tag{2.134}$$

(and the same for the volume-charged sphere).

2.6.4 2.20 Force on electric dipole

An electric dipole with the moment $\vec{p} = (0, p, 0)$ lies at a point $\vec{r} = (x, 0, 0)$ in the electric field of a point charge q located at the origin. Determine the force $\vec{F}$ and moment of the force $\vec{D}$ that will act on the dipole.

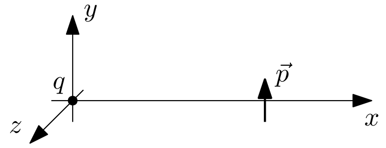


Figure 2.40: Point charge q and dipole $\vec{p}$.

Solution: If we have a small electric dipole with dipole moment $\vec{p}$ in an external electric field $\vec{E}$, then the force acting on the dipole due to this field is given by the following equation

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E}, \tag{2.135}$$

i.e., using the $\vec{p}$-grad operator, which is defined as follows:

$$\vec{p}\text{-grad}\,\vec{E} = (\vec{p} \cdot \nabla)\vec{E} = (\vec{p} \cdot \nabla E_x, \vec{p} \cdot \nabla E_y, \vec{p} \cdot \nabla E_z) = \left(\sum_{i=1}^{3} p_i \frac{\partial E_x}{\partial x_i}, \sum_{i=1}^{3} p_i \frac{\partial E_y}{\partial x_i}, \sum_{i=1}^{3} p_i \frac{\partial E_z}{\partial x_i} \right). \tag{2.136}$$

The electric field vector $\vec{E}$ from a point charge of magnitude q located at the origin (or its component expression) is:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}}{r^3}, \qquad E_i = \frac{q}{4\pi\varepsilon_0} \frac{x_i}{r^3} \tag{2.137}$$

Let us now calculate in general the derivatives $\frac{\partial E_i}{\partial x_j}$ for the field from the point charge (2.137), which we will need to calculate the force $\vec{F}$ according to the relation (2.136):

$$\frac{\partial E_i}{\partial x_j} = \frac{q}{4\pi\varepsilon_0} \frac{\partial}{\partial x_j} \left(\frac{x_i}{r^3} \right) = \frac{q}{4\pi\varepsilon_0} \left[\frac{\partial x_i}{\partial x_j} \frac{1}{r^3} + x_i \frac{\partial}{\partial x_j} \left(\frac{1}{r^3} \right) \right], \tag{2.138}$$

where we have used the product derivative rule. Let's calculate the derivatives of each term in the expression (2.138) above one by one:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \qquad \delta_{ij} = \begin{cases} 0 & \text{pro} \quad i \neq j, \\ 1 & \text{pro} \quad i = j \end{cases}, \tag{2.139}$$

where we introduced the Kronecker delta symbol δ_{ij}. We differentiate the term r^{-3} as a composite function:

$$\frac{\partial r^{-3}}{\partial x_j} = -3r^{-4} \frac{\partial r}{\partial x_j}, \tag{2.140}$$

where we compute the derivative of the magnitude of the position vector r as follows:

$$\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j} \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{1}{2\sqrt{x_1^2 + x_2^2 + x_3^2}} \cdot 2x_j = \frac{x_j}{r}. \tag{2.141}$$

After substituting these intermediate calculations into (2.138), we get

$$\frac{\partial E_i}{\partial x_j} = \frac{q}{4\pi\varepsilon_0} \left(\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5} \right). \tag{2.142}$$

Since we have $\vec{p} = (0, p, 0)$ from the assignment, the general formula (2.136) for the force $\vec{F}$ simplifies to

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E} = p \left(\frac{\partial E_x}{\partial y}, \frac{\partial E_y}{\partial y}, \frac{\partial E_z}{\partial y} \right). \tag{2.143}$$

After substituting from the general formula for the derivative of $\frac{\partial E_i}{\partial x_j}$ (2.142):

$$\vec{F} = \frac{pq}{4\pi\varepsilon_0} \left(-3\frac{xy}{r^5}, \frac{1}{r^3} - 3\frac{y^2}{r^5}, -3\frac{yz}{r^5} \right). \tag{2.144}$$

This result represents the force acting on the dipole $\vec{p} = (0, p, 0)$ at any location $\vec{r} = (x, y, z)$. For $\vec{r} = (x, 0, 0)$, i.e., $y = 0$ and $z = 0$, we get

$$\vec{F}(\vec{r}) = \frac{pq}{4\pi\varepsilon_0}\left(0, \frac{1}{r^3}, 0\right) = \frac{1}{4\pi\varepsilon_0}\frac{pq}{r^3}(0, 1, 0). \tag{2.145}$$

The moment of the force $\vec{D}$ acting on the dipole $\vec{p}$ in the electric field $\vec{E}$ is

$$\vec{D} = \vec{p} \times \vec{E}. \tag{2.146}$$

Inserting $\vec{p} = (0, p, 0)$ and $\vec{E} = (E_x, 0, 0)$, we have

$$\vec{D} = (0, 0, -pE_x) = \frac{pq}{4\pi\varepsilon_0}\frac{1}{x^2}(0, 0, -1), \tag{2.147}$$

where we have substituted $E_x = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2}$.

Addendum: The total force acting on a dipole can be illustrated by the composition of the Coulombic forces acting on the individual charges of a "model" dipole, see Figure 2.41. Coming soon.

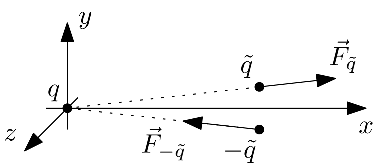


Figure 2.41: A point charge q and a model dipole $\vec{p}$ composed of charges $\tilde{q}$ and $-\tilde{q}$ separated by distances d so that $p = \tilde{q}d$.

2.6.5 2.21 Quadrupole moment of point charges

The four charges q, $-q$, q, $-q$ are respectively located in the corners of the square with side a. Determine the principal quadrupole moments of the system.

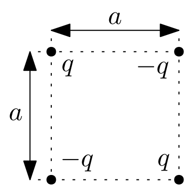


Figure 2.42: Quadrupole moment of a system of point charges.

Solution: The formula for the quadrupole moments of a system of point charges is as follows:

$$Q_{ij} = \sum_{\alpha} q_\alpha \left(3x_i x_j - \delta_{ij} r^2\right)_\alpha, \tag{2.148}$$

where q_α are the charges of the individual particles and $\vec{r}_\alpha = (x_{1\alpha}, x_{2\alpha}, x_{3\alpha}) = (x_\alpha, y_\alpha, z_\alpha)$ are their position vectors (purely to simplify the notation, we have factored out the index α after the bracket, $(\ldots)_\alpha$).

It holds that if the total charge of the system Q and the dipole moment $\vec{p}$ are zero, then the quadrupole moments Q_{ij} do not depend on the choice of origin of the coordinate system. (Here the dipole moment $\vec{p}$ is zero, see Example 2.17 in Section 2.6.1.)

The principal quadrupole moments are obtained if we choose the Cartesian axes so that the matrix Q_{ij} comes out diagonal. So we want the off-diagonal moments to be zero, the formulas for them look like this:

$$Q_{12} = Q_{21} = \sum_\alpha q_\alpha \left(3xy\right)_\alpha, \quad Q_{13} = Q_{31} = \sum_\alpha q_\alpha \left(3xz\right)_\alpha, \quad Q_{23} = Q_{32} = \sum_\alpha q_\alpha \left(3yz\right)_\alpha. \tag{2.149}$$

Since we have a planar problem, let us choose the charges in the plane $z = 0$, hence $Q_{13} = Q_{23} = 0$. If we now choose the axes x and y so that the charges lie alternately on these axes, i.e. as in Figure 2.43, we also get $Q_{12} = 0$ (see below).

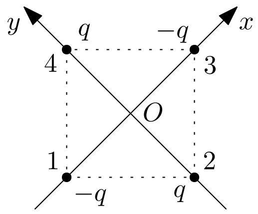


Figure 2.43: Coordinates for calculating the quadrupole moment of the system of point charges.

The position vectors of the individual particles have the following form,

$$\vec{r}_1 = \left(-\frac{\sqrt{2}}{2}\,a, 0, 0\right), \quad \vec{r}_2 = \left(0, -\frac{\sqrt{2}}{2}\,a, 0\right), \quad \vec{r}_3 = \left(\frac{\sqrt{2}}{2}\,a, 0, 0\right), \quad \vec{r}_4 = \left(0, \frac{\sqrt{2}}{2}\,a, 0\right), \tag{2.150}$$

and the products $x_\alpha y_\alpha$ are thus always zero.

The general formula for the moment Q_{11} is as follows

$$Q_{11} = \sum_\alpha q_\alpha \left(3x^2 - r^2\right)_\alpha = \sum_\alpha q_\alpha \left(2x^2 - y^2 - z^2\right)_\alpha; \tag{2.151}$$

after substituting the values of the charges and position vectors (2.150):

$$Q_{11} = -q\,(2\frac{a^2}{2}) + q\,(-\frac{a^2}{2}) - q\,(2\frac{a^2}{2}) + q\,(-\frac{a^2}{2}) = -3qa^2. \tag{2.152}$$

The same calculation leads to the value $Q_{22} = 3qa^2$. Since for diagonal quadrupole moments the following holds

$$\sum_{i=1}^{3} Q_{ii} = Q_{11} + Q_{22} + Q_{33} = 0, \tag{2.153}$$

it must be $Q_{33} = 0$. The resulting matrix of quadrupole moments Q_{ij} is therefore of the form

$$Q_{ij} = \begin{pmatrix} -3qa^2 & 0 & 0 \\ 0 & 3qa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.154}$$

where on the diagonal are the corresponding principal quadrupole moments.

2.6.6 2.22 Quadrupole moment of the ellipsoid

Determine the electric quadrupole moment of a rotational ellipsoid.

Solution: The formula for the quadrupole moments of a continuous volume charge distribution is as follows:

$$Q_{ij} = \int_V \rho(\vec{r})(3x_i x_j - \delta_{ij} r^2)\, dV, \tag{2.155}$$

where $\rho(\vec{r})$ is the volume charge density function and $\vec{r} = (x_1, x_2, x_3) = (x, y, z)$ is the position vector of the volume element dV. Let us first calculate the quadrupole moment Q_{11}, which is given by (2.155), from the formula

$$Q_{11} = \int_V \rho(\vec{r})(2x^2 - y^2 - z^2) dV. \tag{2.156}$$

The volume charge density ρ is here considered constant. We introduce Cartesian coordinates with origin at the center of the ellipsoid and axes oriented in the directions of the ellipsoid axes. Since the ellipsoid has a nonzero total charge, $Q \neq 0$, the resulting quadrupole moment Q_{ij} (and also the dipole moment $\vec{p}$) depends on the choice of the origin of the coordinates. It is natural to choose the origin at the center of the ellipsoid for the reason that then the dipole moment $\vec{p}$ of the ellipsoid comes out vanishing. The volume element is in Cartesian coordinates $dV = dx\, dy\, dz$ and the resulting relation for the moment Q_{11} is:

$$Q_{11} = \int \int_{\text{from}}^{\text{to}} \int \rho \left(2x^2 - y^2 - z^2\right)\, dx\, dy\, dz. \tag{2.157}$$

The integration limits are chosen to satisfy the inequality

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} \leq 1. \tag{2.158}$$

We now perform the substitution where "we make the ellipsoid a sphere":

$$x = a\tilde{x}, \qquad y = a\tilde{y}, \qquad z = b\tilde{z}, \qquad dx\, dy\, dz = a^2 b\, d\tilde{x}\, d\tilde{y}\, d\tilde{z}. \tag{2.159}$$

After substituting in the integral, we have

$$Q_{11} = \int \int_{\text{from'}}^{\text{to'}} \int \rho \left(2a^2\tilde{x}^2 - a^2\tilde{y}^2 - b^2\tilde{z}^2\right) a^2 b\, d\tilde{x}\, d\tilde{y}\, d\tilde{z}, \tag{2.160}$$

where the new limits in coordinates $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ satisfy the inequality

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 \leq 1, \tag{2.161}$$

i.e., by substitution we have made the ellipsoid into a unit sphere. Next, we perform the substitution into spherical coordinates

$$\tilde{x} = r \sin\theta \cos\varphi, \quad \tilde{y} = r \sin\theta \sin\varphi, \quad \tilde{z} = r\cos\theta, \quad d\tilde{x}\, d\tilde{y}\, d\tilde{z} = r^2 \sin\theta\, dr\, d\theta\, d\varphi, \tag{2.162}$$

where the ellipsoid (now a unit sphere in coordinates $\tilde{x}$, $\tilde{y}$, $\tilde{z}$) is located at coordinates $r \in \langle 0, 1\rangle$, $\theta \in \langle 0, \pi\rangle$, $\varphi \in \langle 0, 2\pi\rangle$. The integral now looks as follows:

$$Q_{11} = \int_0^1 \int_0^\pi \int_0^{2\pi} \rho \left(2a^2\tilde{x}^2 - a^2\tilde{y}^2 - b^2\tilde{z}^2\right) a^2 b\, r^2 \sin\theta\, dr\, d\theta\, d\varphi, \tag{2.163}$$

where we have not yet substituted for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ from (2.162). Now we are faced with the task of finding the value of

$$\int_{\text{sphere}} \tilde{x}^2\, dV = ?, \quad \int_{\text{sphere}} \tilde{y}^2\, dV = ?, \quad \int_{\text{sphere}} \tilde{z}^2\, dV = ?, \tag{2.164}$$

which could be calculated simply by substituting from (2.162) and integrating. But let's try to avoid this complicated computation by the following trick. The symmetry of the sphere must imply

$$\int_{\text{sphere}} \tilde{x}^2\, dV = \int_{\text{sphere}} \tilde{y}^2\, dV = \int_{\text{sphere}} \tilde{z}^2\, dV \tag{2.165}$$

and hence

$$\begin{aligned} \int_{\text{sphere}} \tilde{x}^2\, dV &= \frac{1}{3}\int_{\text{sphere}} \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2\, dV = \frac{1}{3}\int_{\text{sphere}} r^2\, dV = \frac{1}{3}\int_0^1\int_0^\pi\int_0^{2\pi} r^4 \sin\theta\, dr\, d\theta\, d\varphi \\ &= \frac{1}{3}\left[\frac{r^5}{5}\right]_0^1 [-\cos\theta]_0^\pi\ [\varphi]_0^{2\pi} = \frac{4\pi}{15}. \end{aligned} \tag{2.166}$$

Now we just plug this result into (2.163) and get

$$Q_{11} = \frac{4\pi}{15}\rho\left(2a^2 - a^2 - b^2\right)a^2 b = \frac{4\pi}{15}\rho\, a^2 b(a^2 - b^2). \tag{2.167}$$

We can still express the result in terms of the total charge on the ellipsoid Q, which is

$$Q = \rho V = \rho\,\frac{4}{3}\pi a^2 b, \tag{2.168}$$

and after plugging it into (2.167) we have

$$Q_{11} = \frac{1}{5}Q(a^2 - b^2). \tag{2.169}$$

Since the following holds for the diagonal quadrupole moments

$$\sum_{i=1}^{3} Q_{ii} = Q_{11} + Q_{22} + Q_{33} = 0, \tag{2.170}$$

and further from the symmetry of the ellipsoid we have $Q_{11} = Q_{22}$, it must be $Q_{33} = -2Q_{11}$:

$$Q_{33} = -2Q_{11} = -\frac{2}{5}(a^2 - b^2). \tag{2.171}$$

The off-diagonal quadrupole moments, which are given by

$$Q_{ij} = \int_V 3\rho(\vec{r})\, x_i x_j\, dV, \quad i \neq j, \tag{2.172}$$

come out to be zero. If we perform analogous calculations up to equation (2.163) we would get integrals in the variable φ of the form

$$\int_0^{2\pi} \sin\varphi\, d\varphi, \quad \text{resp.} \quad \int_0^{2\pi} \cos\varphi\, d\varphi, \quad \text{resp.} \quad \int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi, \tag{2.173}$$

which are vanishing.

2.7 Capacitors

2.7.1 2.26, 2.28 and 2.29 Plate capacitor

How many electrons make up the charge of a sphere of mass $m = 10^{-11}\,g$ if it is kept in equilibrium in a plate capacitor whose plates are spaced apart $d = 5\,mm$ and charged to a voltage $U = 76,5\,V$.

What area would the electrodes of a plate capacitor with a distance of $d = 1\,mm$ have to have for the capacitor to have a capacitance of $C = 1\,F$?

What is the force of attraction between the plates of the capacitor?

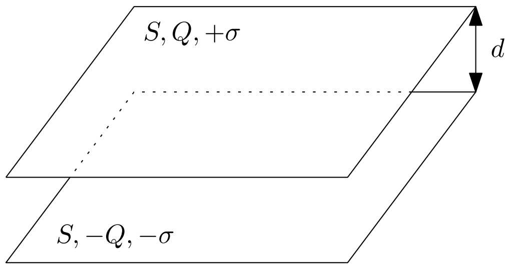


Figure 2.44: Plate capacitor.

Solution: The magnitudes of the electric field strength E inside the plate capacitor, the voltage U between the plates and the capacitance of this capacitor are given by the relations (see the Appendix to this exercise for their derivation):

$$E = \frac{\sigma}{\varepsilon_0}, \qquad U = E\,d = \frac{\sigma}{\varepsilon_0}d, \qquad C = \varepsilon_0\frac{S}{d}, \tag{2.174}$$

where σ is the surface charge density on the plates of the capacitor, S is the area of the plates (each of them), and d is the distance of these plates. The electric field vector $\vec{E}$ points perpendicular to the plates of the capacitor.

Example 2.26: If we place a charge of magnitude q between the plates of a capacitor, a force of magnitude $F_E = q\,E = q\frac{U}{d}$ will act on this charge. This force must be cancelled by the gravitational force $F_g = mg$, i.e.

$$F_E = F_g, \qquad q\frac{U}{d} = mg, \qquad q = \frac{mgd}{U}. \tag{2.175}$$

If we write the charge q as n-times the elementary electric charge e, $q = n\,e$, we can write the result as the number of elementary electric charges on the sphere n:

$$n = \frac{mgd}{Ue} \doteq 40, \tag{2.176}$$

where we have used the value of the gravitational acceleration $g = 9,81\,m.s^{-2}$ and the magnitude of the elementary electric charge $e = 1,602.10^{-19}\,C$.

Example 2.28: After expressing the area S from the relation (2.174) on the right, we have

$$S = \frac{Cd}{\varepsilon_0} = 1,13.10^{8}\,m^2 = 113\,km^2, \tag{2.177}$$

where we used the value of the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\,F.m^{-1}$.

Example 2.29: We can solve the example in two ways. Either we start from the energy W of the electrostatic field between the plates of the capacitor:

$$W = \int_V \frac{1}{2}\varepsilon_0 \vec{E}^2\, dV = \frac{1}{2}\varepsilon_0 E^2 \int_V dV = \frac{1}{2}\varepsilon_0 E^2 S d = \frac{1}{2}\left(\varepsilon_0 \frac{S}{d}\right)(Ed)^2 = \frac{1}{2} C U^2. \tag{2.178}$$

We will use the relation $W = \frac{1}{2}\varepsilon_0 E^2 S d$. When the distance between the plates changes by dx, the electrostatic field energy changes by $dW = \frac{1}{2}\varepsilon_0 E^2 S\, dx$ (E and S are constant and $d(d) = dx$). The change in this energy must come from the work done by moving the plates of the capacitor apart (closer), i.e., $dW = F\, dx$ and thus the relation for the force is

$$F = \frac{1}{2}\varepsilon_0 E^2 S = \frac{1}{2}\frac{\sigma^2 S}{\varepsilon_0} = \frac{1}{2}\frac{Q^2}{\varepsilon_0 S}. \tag{2.179}$$

Another way to get the result is to calculate the force $dF = f\, dS$ acting on a small area dS of one plate of the capacitor from the entire (infinitely) large other plate (f is called the force density). The total force will then be

$$F = \int_S dF = \int_S f\, dS = f \int_S dS = fS, \tag{2.180}$$

since the force density is constant everywhere. Force $dF = E\, dq$, where $E = \frac{\sigma}{2\varepsilon_0}$ (We are directly in the plane of one of the plates where the electric field strength is half! See the Addendum.) and $dq = \sigma\, dS$, hence $dF = \frac{\sigma^2}{2\varepsilon_0} dS$. Then $f = \frac{\sigma^2}{2\varepsilon_0}$ and hence $F = fS = \frac{\sigma^2}{2\varepsilon_0} S$, which is the same result as via the energy calculation.

Addendum: Let us now derive the relations (2.174) for electric field $\vec{E}$, voltage U and capacitance C. We begin by determining the electric field $\vec{E}$ from a single **infinitely large** charged plane from Gauss's law

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}, \tag{2.181}$$

which relates the flux of electric field $\vec{E}$ through a closed surface S to the total charge Q that is enclosed in that surface.

Let us see what constraints on the electric field $\vec{E}$ are imposed by the symmetry of the problem. If we have an infinite charged plane, the electric field can only depend on the distance z from this plane, $\vec{E} = \vec{E}(z)$ – we have a translational symmetry in any direction along the charged plane, which ensures that the vector $\vec{E}$ must be constant on planes parallel to the charged plane. At the same time, the electric field must always point perpendicular to the charged plane, since we have rotational symmetry about an axis perpendicular to the plane and passing through the point where we determine the field $\vec{E}$ – the only direction of the vector $\vec{E}$ that is conserved in this rotation is the direction perpendicular to the plane. Reflection symmetry through the charged plane provides the relation $\vec{E}(-z) = -\vec{E}(z)$, i.e., the vector $\vec{E}$ on the opposite side of the plane is opposite.

To apply Gauss's law, we choose the surface S to be a cylindrical surface whose axis is perpendicular to the charged plane, and the bases have the same distance z from this plane, see Figure 2.45.

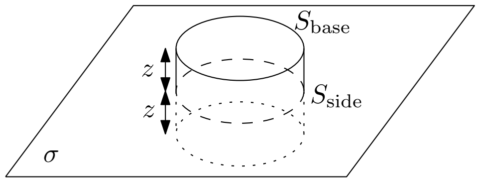


Figure 2.45: Closed cylindrical surface S in Gaussian law to determine the magnitude of the electric field E around an infinite charged plane. We have denoted the area of the cylinder side as S_{side} and the area of each of the bases as S_{base}.

Let us now split the integral on the left-hand side (2.181) separately into integration over the side and over the bases:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_{S_{\text{bases}}} \vec{E} \cdot d\vec{S} + \int_{S_{\text{side}}} \vec{E} \cdot d\vec{S}. \tag{2.182}$$

Since the vectors $\vec{E}$ point perpendicular to the charged plane, the total flux through the cylinder side is zero, since $\vec{E} \cdot d\vec{S} = \vec{E} \cdot \vec{n}\, dS = 0$, where $\vec{n}$ is the unit normal vector to the surface dS. On the other hand, $\vec{E} \cdot d\vec{S} = E\, dS$ holds for the bases since the vector $\vec{E}$ points in the direction $d\vec{S}$, see Figure 2.46.

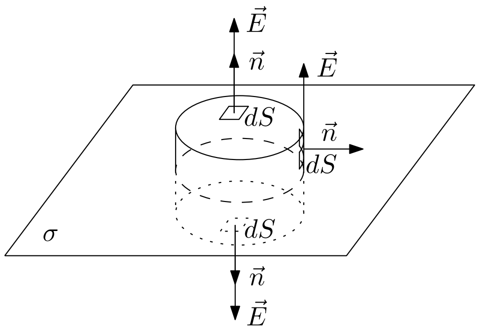


Figure 2.46: Directions of the vectors $\vec{E}$ and $d\vec{S} = \vec{n}\, dS$ for each base and cylinder side. The vector $\vec{n}$ is the unit normal vector to the element dS.

We continue with the manipulations (2.182):

$$\oint_S \vec{E} \cdot d\vec{S} = \int_{S_{\text{bases}}} \vec{E} \cdot d\vec{S} + \int_{S_{\text{side}}} \vec{E} \cdot d\vec{S} = \int_{S_{\text{bases}}} E\, dS = E(z) \int_{S_{\text{bases}}} dS = 2S_{\text{base}} E(z), \tag{2.183}$$

where we have taken advantage of the fact that on the substrate (at equal distances from the charged plane) the magnitude of the electric field strength $E(z)$ is constant, and we can therefore put it in front of the integral, and the integral of unity over the surface S is the area of that surface $2S_{\text{base}}$.

The charge Q enclosed in the surface S on the right-hand side of Gauss's law (2.181) is simply $Q = \sigma S_{\text{base}}$. Gauss's law then gives:

$$2E(z)S_{\text{base}} = \frac{\sigma S_{\text{base}}}{\varepsilon_0} \quad \longrightarrow \quad E = \frac{\sigma}{2\varepsilon_0}, \tag{2.184}$$

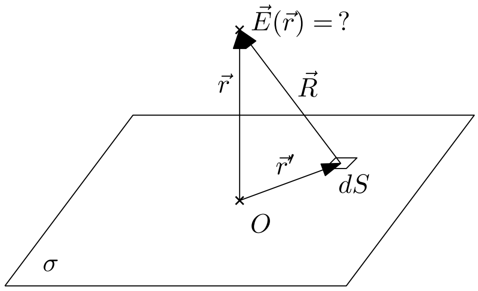
where we have expressed the magnitude of the electric field strength E and shown that it does not depend on the distance from the plane z.

Remark: We can achieve this result without using Gauss's law by directly integrating the contributions $d\vec{E}$ to the electric field from each part of the charged plane:

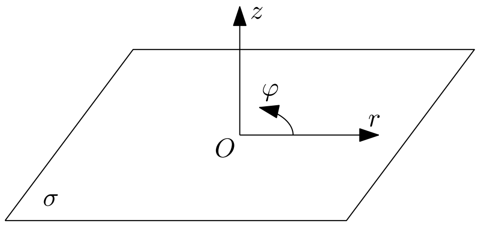
$$\vec{E} = \int_S d\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_S \sigma \frac{\vec{R}}{R^3}\, dS, \tag{2.185}$$

where $\vec{R}$ is the vector connecting the electric field location $\vec{E}$ given by the position vector $\vec{r}$ and the position of the surface element dS given by the vector $\vec{r}'$, $\vec{R} = \vec{r} - \vec{r}'$ (R is the magnitude of this vector, $R = |\vec{r} - \vec{r}'|$), see figure 2.47 on the left.

Let us introduce cylindrical coordinates (r, φ, z) as in figure 2.47 on the right, i.e., the origin located in the charged plane and the axis z pointing perpendicular to this plane. Then the charged plane is at coordinates $z = 0$, $r \in \langle 0, +\infty \rangle$, $\varphi \in \langle 0, 2\pi \rangle$. The surface element in polar coordinates is $dS = r\, dr\, d\varphi$. For the symmetry reasons discussed above, the vector $\vec{E}$ must have the form $\vec{E} = (0, 0, E_z)$. The distance R between the surface element dS and the electric field location $\vec{r} = (0, 0, z)$ is $R = \sqrt{r^2 + z^2}$. The vector $\vec{R}$ expressed in cylindrical coordinates has the form $\vec{R} = (-r\cos\varphi, -r\sin\varphi, z)$.



(a) Vector $\vec{R} = \vec{r} - \vec{r}'$ as a vector connecting the surface element dS and the point where we determine the electric field $\vec{E}$.



(b) Cylindrical coordinates (r, φ, z) introduced as polar coordinates (r, φ) in the charged plane and Cartesian coordinate z perpendicular to this plane.

Figure 2.47: Charged plane – vector $\vec{R}$ and coordinate (r, φ, z).

Substituting all the above information into (2.185) and calculating the single non-zero component E_z we get:

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_0^{+\infty} \int_0^{2\pi} \frac{\sigma\, z}{(r^2 + z^2)^{3/2}}\, r\, dr\, d\varphi = \frac{z\sigma}{2\varepsilon_0} \int_0^{+\infty} \frac{r\, dr}{(r^2 + z^2)^{3/2}} \tag{2.186}$$

After substituting $u = r^2 + z^2$, $du = 2r\, dr$ we have

$$E_z = \frac{z\sigma}{2\varepsilon_0} \int_{z^2}^{+\infty} \frac{du}{u^{3/2}} = \frac{z\sigma}{4\varepsilon_0} \left[-2 \frac{1}{\sqrt{u}} \right]_{z^2}^{+\infty} = \frac{z\sigma}{2\varepsilon_0} \frac{1}{|z|} = \frac{\sigma}{2\varepsilon}\, \mathrm{sgn}\, z, \tag{2.187}$$

i.e., the same result as from Gauss's law (now with a sign).

We must now add the electric fields from two oppositely charged infinite plates. The situation is illustrated and described in Figure 2.48. For our purposes, we need the result that the field inside the capacitor is twice as strong compared to the situation with only one charged plate, $E = \frac{\sigma}{\varepsilon_0}$.

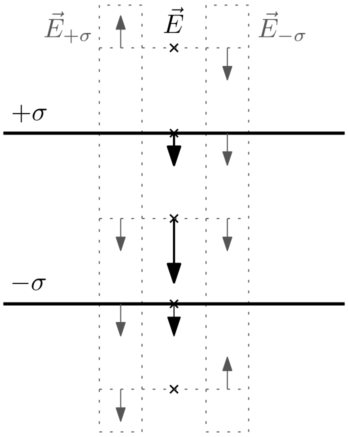


Figure 2.48: Vectors of electric field strengths from each oppositely charged plane (in gray, $\vec{E}_{+\sigma}$ from the positively charged plane and $\vec{E}_{-\sigma}$ from the negatively charged plane) and their resulting superposition (in black, $\vec{E}$). The result is a zero field outside the capacitor, $\vec{E}_{\text{outside}} = 0$; a twice strong field inside the capacitor, $E_{\text{inside}} = \frac{\sigma}{\varepsilon_0}$; directly on the plates is the field $E_{\text{plate}} = \frac{\sigma}{2\varepsilon_0}$!

Now we must integrate this electric field to obtain the voltage between the electrodes. The definition of the voltage between two points along the path l is

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l}, \tag{2.188}$$

where $\vec{F}$ is the force acting on the charge q and $d\vec{l} = \vec{t}\, dl$ is the line element pointing tangentially to the curve ($\vec{t}$ is the unit tangent vector). Thus, the voltage U is the work done along the path l to move a unit charge.

The only force acting inside the capacitor is from the electric field, $\vec{F} = q\vec{E}$. We choose a line perpendicular to the electrodes as the curve l, see Figure 2.49. Then the vectors $\vec{E}$ and $d\vec{l}$ point in the same direction (again, see Figure 2.49), and $\vec{E} \cdot d\vec{l} = E\, dl$ holds. Thus, we have

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l} = \int_l \vec{E} \cdot d\vec{l} = \int_l E\, dl = E \int_l dl = E\, d = \frac{\sigma}{\varepsilon_0}\, d, \tag{2.189}$$

where, in addition, we have taken advantage of the fact that the electric field E is constant and the integral of unity $\int_l dl$ is the length of the curve, i.e. the distance of the plates, d.

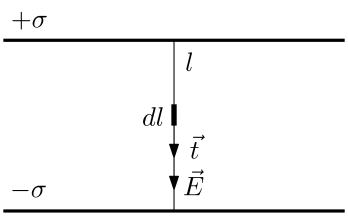


Figure 2.49: The section l connecting the plates of the capacitor for calculating the voltage U. Vectors $\vec{E}$ and $d\vec{l} = \vec{t}\, dl$ (where $\vec{t}$ is the unit tangent vector to curve l) point in the same direction.

The capacitance C of a plate capacitor is by definition $C = \frac{Q}{U}$:

$$C = \frac{Q}{U} = \frac{\sigma S}{\frac{\sigma}{\varepsilon_0} d} = \varepsilon_0 \frac{S}{d}. \tag{2.190}$$

2.7.2 2.30 and 2.33 Cylindrical capacitor and Geiger-Müller counter

Consider a cylindrical capacitor with electrode radii $R_1 = 3\,cm$, $R_2 = 10\,cm$ charged to a voltage $U = 450\,V$. Determine the charge per unit length, the surface charge density on each cylinder, and the electrostatic field strength at the center of the distance between the cylinders.

The capacitor (Geiger-Müller counter) consists of a wire of radius $R_1 = 5\,mm$ and a coaxial cylinder of radius $R_2 = 5\,cm$. To what maximum voltage can we charge the capacitor if the breakdown voltage is air $E_{max} = 3\,kV.cm^{-1}$? How will the voltage between the electrodes change if we decrease the radius of the inner electrode?

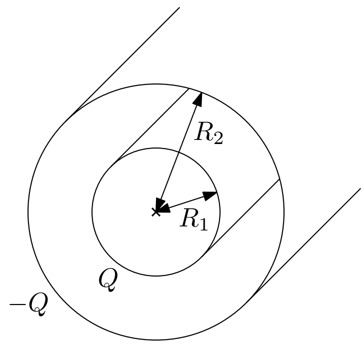


Figure 2.50: Cylindrical capacitor.

Solution: We first determine the electric field $\vec{E}$ and voltage U between the electrodes of **infinitely** long cylindrical capacitor and then its capacitance C for its length section l with radii of inner and outer electrodes R_1 and R_2. We then apply the resulting formulas to the specific problems in exercises 2.30 and 2.33.

Let a segment of length l of the inner electrode be charged to charge Q and the outer electrode to charge $-Q$. We can then define the linear charge density at the electrodes as $\tau = \frac{Q}{l}$, that is, the charge per unit length of the electrode; and also the surface charge densities

$$\sigma_{\text{inner}} = \frac{Q}{S_{\text{inner}}} = \frac{Q}{2\pi R_1 l} = \frac{\tau}{2\pi R_1}, \qquad \sigma_{\text{outer}} = \frac{Q}{S_{\text{outer}}} = \frac{Q}{2\pi R_2 l} = \frac{\tau}{2\pi R_2} \tag{2.191}$$

on the inner and outer electrodes of areas S_{inner} and S_{outer} (these are the areas of the length segment l).

We start by determining the electric field $\vec{E}$ between the electrodes (and then integrate this to determine the voltage U between the electrodes) from Gauss's law

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}, \tag{2.192}$$

which relates the flux of electric field strength $\vec{E}$ through a closed surface S to the total charge Q that is enclosed in that surface.

Let us see what constraints on the electric field $\vec{E}$ the symmetry of the problem imposes. If we introduce naturally cylindrical coordinates (r, φ, z) (as in figure 2.51), then rotational symmetry about the z axis (the axis of the capacitor) forbids the dependence of the magnitude E on the angle φ and translational symmetry along the axis of the capacitor (axis z) precludes dependence on the coordinate z (this is true only for an infinite capacitor). Thus, the magnitude of the electric field can depend only on the distance from the axis of symmetry, $E(r)$.

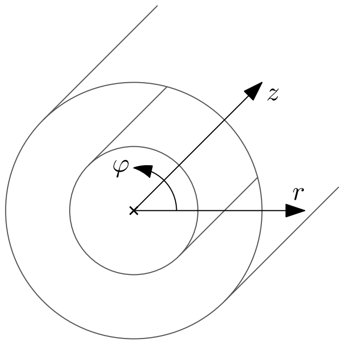
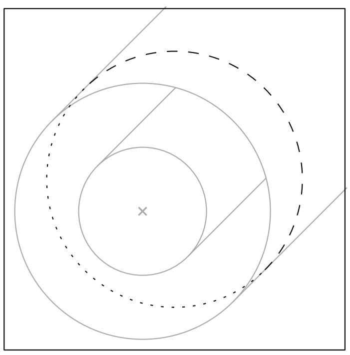
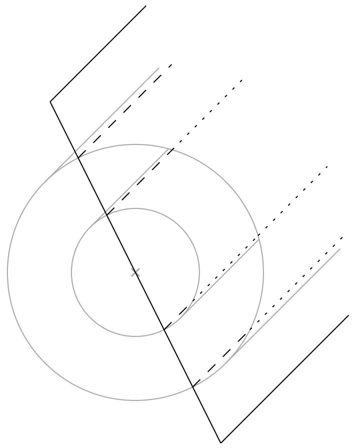


Figure 2.51: Cylindrical coordinates (r, φ, z) in a cylindrical capacitor.

Mirror symmetry about planes perpendicular to the z axis implies that the vector $\vec{E}$ must lie in these planes. At the same time, mirror symmetry about the planes in which the axis z lies implies again that the vectors $\vec{E}$ must lie in these planes. For the planes of symmetry, see Figure 2.52. This leads to the only admissible direction, which is the radial direction – in the direction "axis r".



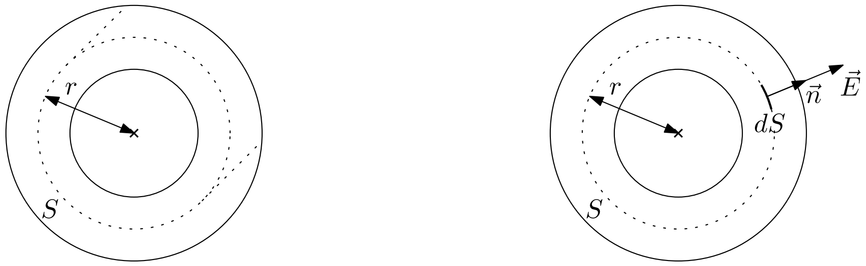
(a) A plane of symmetry perpendicular to the axis z.



(b) The z axis lies in the plane of symmetry.

Figure 2.52: Role of mirror symmetry to determine the direction of the electric field vector $\vec{E}$.

Next, we need a surface S for Gaussian law. Consider a cylindrical surface of height l of general radius r, $R_1 < r < R_2$, concentric with the electrodes of the capacitor, see figure 2.53 on the left.



(a) Cylindrical surface S of radius r for Gaussian law.

(b) Directions of the electric field $\vec{E}$ and the normal vector $\vec{n}$ to the element dS on the cylinder side of radius r.

Figure 2.53: Use of Gauss's law to determine the electric field $\vec{E}$.

Now we can start to manipulate the left-hand side of Gauss's law (2.192). First, we split the integral over the cylinder S into an integration over the side and the cylinder's bases:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_{S_{\text{side}}} \vec{E} \cdot d\vec{S} + \int_{S_{\text{bases}}} \vec{E} \cdot d\vec{S}, \tag{2.193}$$

where the surface element is $d\vec{S} = \vec{n}\, dS$; $\vec{n}$ is the unit normal vector. Since the vector $\vec{E}$ is radial, i.e., it lies in the cylinder bases S, the scalar product vanishes under this integral, $\vec{E} \cdot d\vec{S} = 0$:

$$\int_{S_{\text{bases}}} \vec{E} \cdot d\vec{S} = 0. \tag{2.194}$$

The situation on the cylinder side S is shown in the figure 2.53 on the right. Here the vector $\vec{E}$ points in the direction $d\vec{S}$ (in the direction $\vec{n}$) and hence $\vec{E} \cdot d\vec{S} = E\, dS$ holds. At the same time, the magnitude of the electric field strength E depends only on the coordinate r, $E(r)$, so it is constant on the cylinder side S and can be factored out from the integral:

$$\int_{S_{\text{side}}} \vec{E} \cdot d\vec{S} = \int_{S_{\text{side}}} E\, dS = E(r) \int_{S_{\text{side}}} dS = E(r)\, 2\pi r l. \tag{2.195}$$

(We have used the formula for the cylinder's side area, $S = \int dS = 2\pi r l$.) The total charge enclosed in the cylindrical surface S is just Q (it is the charge at length l, and the cylinder S also has length l). Substituting the result (2.195) into Gauss's law (2.192) and expressing the magnitude of the electric field $E(r)$, we get

$$E(r) = \frac{Q}{2\pi r l \varepsilon_0} = \frac{\tau}{2\pi r \varepsilon_0}. \tag{2.196}$$

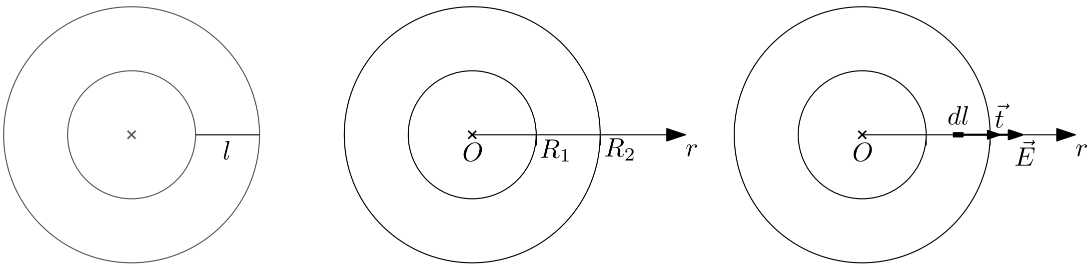
Now we must integrate this electric field to obtain the voltage between the electrodes. The definition of the voltage between two points along the path l is

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l}, \tag{2.197}$$

where $\vec{F}$ is the force acting on the charge q and $d\vec{l} = \vec{t}\, dl$ is the line element pointing tangentially to the curve ($\vec{t}$ is the unit tangent vector). Thus, the voltage U is the work done along the path l to move a unit charge.

The only force acting inside the capacitor is from the electric field, $\vec{F} = q\vec{E}$. We choose the curve l as the radial line joining the inner and outer electrodes (see figure 2.54 on the left), and when the radial coordinate r is introduced, the line is at $r \in \langle R_1, R_2 \rangle$ (see figure 2.54 in the middle) and the line element is $dl = dr$. Then the vectors $\vec{E}$ and $\vec{dl}$ point in the same direction (see figure 2.54 on the right) and $\vec{E} \cdot \vec{dl} = E\, dl$ holds. Thus, we have

$$U = \frac{1}{q} \int_l \vec{F} \cdot \vec{dl} = \int_l \vec{E} \cdot \vec{dl} = \int_l E\, dl = \int_{R_1}^{R_2} E(r)\, dr. \tag{2.198}$$



(a) The line l connecting the inner and outer electrode.

(b) Radial coordinate r.

(c) Vectors $\vec{E}$ and $\vec{dl} = \vec{t}\, dl$ point in the same direction.

Figure 2.54: The curve l, its parameterization and the vectors $\vec{E}$ and $\vec{dl}$ shown.

After substituting for $E(r)$ from (2.196), we can easily calculate the voltage U:

$$U = \int_{R_1}^{R_2} \frac{Q}{2\pi r l \varepsilon_0}\, dr = \frac{Q}{2\pi l \varepsilon_0} [\ln r]_{R_1}^{R_2} = \frac{Q}{2\pi l \varepsilon_0} \ln \frac{R_2}{R_1} = \frac{\tau}{2\pi \varepsilon_0} \ln \frac{R_2}{R_1}. \tag{2.199}$$

The capacitance of a section of length l of a cylindrical capacitor can be obtained trivially from the definition $C = \frac{Q}{U}$, i.e.

$$C = \frac{Q}{U} = \frac{2\pi l \varepsilon_0}{\ln \frac{R_2}{R_1}}, \qquad \frac{C}{l} = \frac{2\pi \varepsilon_0}{\ln \frac{R_2}{R_1}}, \tag{2.200}$$

where on the right we have given the capacity C/l per unit length.

And now to the actual exercises. In the *exercise 2.30* we simply express the linear charge density τ from the relation (2.199):

$$\tau = 2\pi \varepsilon_0 \frac{U}{\ln \frac{R_2}{R_1}} = 2,08.10^{-8}\, C.m^{-1}, \tag{2.201}$$

where we have used the value of the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\, F.m^{-1}$. The surface charge densities according to (2.191) are

$$\sigma_{\text{inner}} = \frac{\tau}{2\pi R_1} = 1,10.10^{-7}\, C.m^{-2}, \qquad \sigma_{\text{outer}} = \frac{\tau}{2\pi R_2} = 3,31.10^{-8}\, C.m^{-2}. \tag{2.202}$$

The electric field strength for $r = \frac{R_1 + R_2}{2}$ is according to (2.196)

$$E\left(\frac{R_1 + R_2}{2}\right) = \frac{\tau}{\pi (R_1 + R_2) \varepsilon_0} = 5,75\, kV.m^{-1}. \tag{2.203}$$

In the *exercise 2.33*, we need to express the electric field of $E(r)$ in terms of the voltage across the capacitor, i.e. we combine the relations (2.196) and (2.199):

$$E(r) = \frac{U}{\ln \frac{R_2}{R_1}} \frac{1}{r}. \tag{2.204}$$

We see that the strongest electric field is at the inner electrode for $r = R_1$. Our limit given by the breakdown voltage of air $E_{max} = 30\,kV.cm^{-1}$ will therefore be $E(R_1) \leq E_{max}$, giving the maximum voltage

$$U_{max} = R_1 \ln \frac{R_2}{R_1} E_{max} = 34,5\,kV. \tag{2.205}$$

2.7.3 2.31 Spherical capacitor

Determine the voltage between two concentric spheres of radii $R_1 < R_2$ and charges Q_1, Q_2.

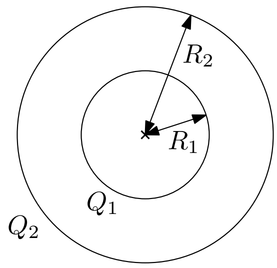


Figure 2.55: Spherical electrodes.

Solution: The magnitude of the electric field of a spherically symmetric charge distribution is

$$E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q(r)}{r^2}, \tag{2.206}$$

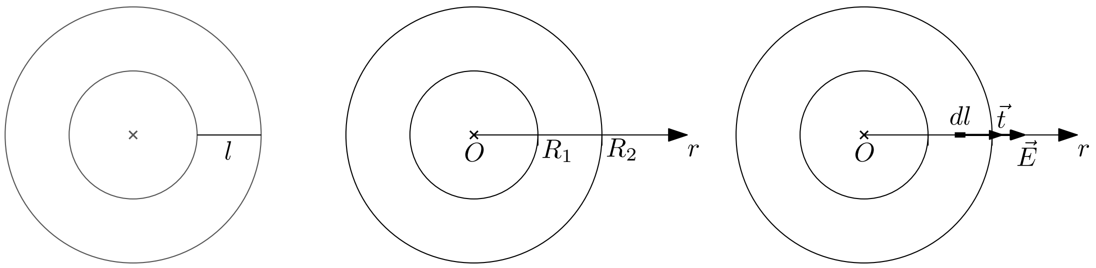
where $Q(r)$ is the total charge enclosed in a sphere of radius r. The directions of the electric field vectors are radial – pointing from/to the center of spherical symmetry. See the Addendum in section 2.5.7 for the derivation. We now integrate this electric field to obtain the voltage between the spherical electrodes. The definition of the voltage between two points along the path l is

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l}, \tag{2.207}$$

where $\vec{F}$ is the force acting on the charge q and $d\vec{l} = \vec{t}\,dl$ is the line element pointing tangentially to the curve ($\vec{t}$ is a unit tangent vector). Thus, the voltage U is the work done along the path l to move a unit charge.

The only force acting inside the capacitor is from the electric field, $\vec{F} = q\vec{E}$. We choose the curve l as a radial line joining the inner and outer electrodes (see figure 2.56 on the left), and when the radial coordinate r is introduced, the line extends over $r \in \langle R_1, R_2 \rangle$ (see figure 2.56 in the middle) and the line element is $dl = dr$. Then the vectors $\vec{E}$ and $d\vec{l}$ point in the same direction (see figure 2.56 on the right) and $\vec{E} \cdot d\vec{l} = E\,dl$ holds. Thus, we have

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l} = \int_l \vec{E} \cdot d\vec{l} = \int_l E\,dl = \int_{R_1}^{R_2} E(r)\,dr. \tag{2.208}$$



(a) The path l connecting the inner and outer electrode.

(b) Radial coordinate r.

(c) Vectors $\vec{E}$ and $d\vec{l} = \vec{t}\,dl$ point in the same direction.

Figure 2.56: The curve l, its parametrization and the vectors $\vec{E}$ and $d\vec{l}$ shown.

Substituting for $E(r)$ from (2.206), where we consider $Q(r) = Q_1$ (since for $r \in \langle R_1, R_2 \rangle$ the charge enclosed in a sphere of radius r is simply the charge on the inner electrode Q_1), we easily calculate the voltage U:

$$U = \int_l \vec{E} \cdot d\vec{l} = \int_l E\,dl = \int_{R_1}^{R_2} E(r)\,dr = \frac{Q_1}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q_1}{4\pi\varepsilon_0} \left[-\frac{1}{r} \right]_{R_1}^{R_2} = \frac{Q_1}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right). \tag{2.209}$$

Addendum: If we want to know the capacitance of a spherical capacitor, we put $Q_1 = Q$ (and $Q_2 = -Q$) and knowing the voltage U (2.208) we can easily determine the capacitance $C = \frac{Q}{U}$ from its definition:

$$C = \frac{Q}{U} = \frac{4\pi\varepsilon_0}{\frac{1}{R_1} - \frac{1}{R_2}} = \frac{4\pi R_1 R_2 \varepsilon_0}{R_2 - R_1}. \tag{2.210}$$

2.7.4 2.32 Line capacitance

Determine the capacitance of a line formed by two parallel wires of length $l = 9\,km$, radius $r = 1\,mm$ and mutual distance $d = 15\,cm$.

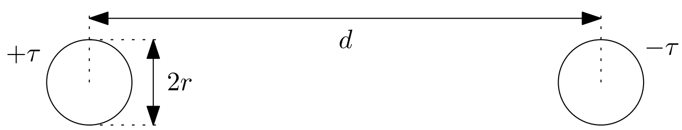


Figure 2.57: Capacity of power lines.

Solution: The magnitude of the electric field E from a charged cylindrical conductor (see section 2.7.2 for derivation) is given by

$$E = \frac{\tau}{2\pi r \varepsilon_0}, \tag{2.211}$$

where τ is the linear charge density on each conductor, r is the radius of the conductor. The direction of the vector $\vec{E}$ is always radial from/to the axis/e of the cylindrical conductor. To determine the total capacitance of the conductor C, we need to find the voltage U between the conductors, which is obtained by integrating the electric field $\vec{E}$ between the conductors.

Let the conductors be charged with a constant linear charge density $+\tau$ or $-\tau$, see Figure 2.57. Let's denote the electric fields from these conductors by $\vec{E}_1$ (from conductor $+\tau$) and $\vec{E}_2$ (from conductor $-\tau$). The definition of the voltage between two points along the path l is

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l}, \tag{2.212}$$

where $\vec{F}$ is the force acting on the charge q and $d\vec{l} = \vec{t}\,dl$ is the line element pointing tangentially to the curve ($\vec{t}$ is a unit tangent vector). Thus, the voltage U is the work done along the path l to move a unit charge.

The only force acting inside the capacitor is from the electric field, $\vec{F} = q\vec{E}$, where $\vec{E} = \vec{E}_1 + \vec{E}_2$ is the total electric field from the conductors. We choose the shortest line segment connecting the left and right wires as the curve l, see Figure 2.58.

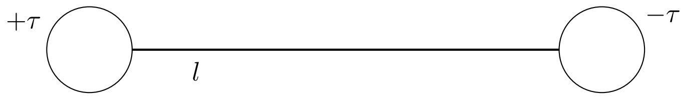


Figure 2.58: Curve l for integrating the voltage U between the wires.

Introducing the Cartesian coordinate x as in Figure 2.59, the line l extends at $x \in \langle r, d-r\rangle$. The line element is $dl = dx$. The vectors $\vec{E} = \vec{E}_1 + \vec{E}_2$ and $d\vec{l}$ point in the same direction (since $\vec{E}_1$ points away from the positively charged conductor, i.e., in the positive direction of the x axis, and $\vec{E}_2$ points toward the negatively charged conductor, i.e., also in the positive direction of the x axis, see Figure 2.60), and thus $\vec{E} \cdot d\vec{l} = E\,dl$ holds. Thus, we have

$$U = \frac{1}{q}\int_l \vec{F} \cdot d\vec{l} = \int_l \vec{E} \cdot d\vec{l} = \int_l E\,dl = \int_r^{d-r} E(x)\,dx. \tag{2.213}$$

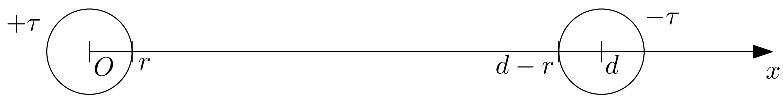


Figure 2.59: Cartesian coordinate x for the line capacitance calculation.

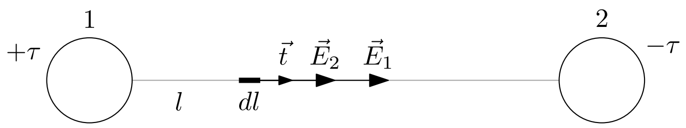


Figure 2.60: The line element dl and its associated vectors $d\vec{l} = \vec{t}\,dl$, $\vec{E}_1$ and $\vec{E}_2$. The electric field $\vec{E}_1$ is from the left (positively charged) conductor, the field $\vec{E}_2$ is from the right (negatively charged) conductor.

The magnitudes of the electric fields $E_1(x)$ and $E_2(x)$ as a function of the position x between the conductors are obtained by substituting the correct distance r into the relation (2.211). For these distances, $r_1 = x$ and $r_2 = d - x$ apply, see Figure 2.61. The individual magnitudes of the electric fields E_1 and E_2 and their total magnitude $E = E_1 + E_2$ are then:

$$E_1(x) = \frac{\tau}{2\pi x\varepsilon_0}, \qquad E_2(x) = \frac{\tau}{2\pi(d-x)\varepsilon_0}, \qquad E(x) = \frac{\tau}{2\pi\varepsilon_0}\left(\frac{1}{x} + \frac{1}{d-x}\right). \tag{2.214}$$

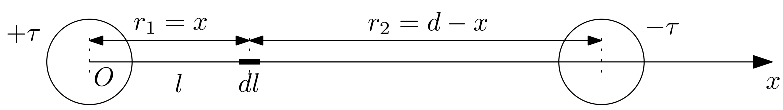


Figure 2.61: Distances r_1 and r_2 of the line element dl from the centres of the individual conductors.

After substituting for $E(x)$ from (2.214) into (2.213) we can easily calculate the voltage U:

$$\begin{aligned} U &= \frac{\tau}{2\pi\varepsilon_0}\int_r^{d-r} \frac{1}{x} + \frac{1}{d-x}\,dx = \frac{\tau}{2\pi\varepsilon_0}\left[\ln x - \ln(d-x)\right]_r^{d-r} \\ &= \frac{\tau}{2\pi\varepsilon_0}\left(\ln\frac{d-r}{r} - \ln\frac{r}{d-r}\right) = \frac{\tau}{\pi\varepsilon_0}\ln\frac{d-r}{r}. \end{aligned} \tag{2.215}$$

The capacitance C of a line of length l is determined simply from the definition, $C = \frac{Q}{U}$, where we substitute the total charge on the line for the charge Q, i.e., $Q = \tau\, l$:

$$C = \frac{Q}{U} = \frac{\pi\varepsilon_0 l}{\ln\frac{d-r}{r}} = 5,0.10^{-8}\,F = 50\,nF, \tag{2.216}$$

where we have used the value of the vacuum permittivity $\varepsilon_0 = 8,854.10^{-12}\,F.m^{-1}$.

2.7.5 2.34 Capacitance addition

Determine the capacitance between points A, B of the capacitor system in Figure 2.62. All capacitors have the same capacitance C.

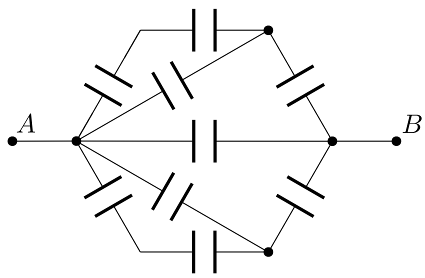


Figure 2.62: Capacitor capacitance addition.

Solution: We determine the total capacitance C_{AB} by sequentially adding the respective capacitors in series and parallel. The formulas for the total capacitance C of the series and parallel connection of capacitors with capacitances C_1 and C_2 are

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}, \quad \text{resp.} \quad C = C_1 + C_2. \tag{2.217}$$

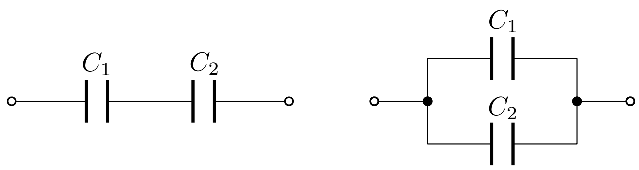


Figure 2.63: Capacitance addition of C_1 and C_2. On the left is the series connection and on the right is the parallel connection.

In the capacitor network, we mark the capacitance value of the given groups of resistors as C_a, C_b and C_c, see Figure 2.64.

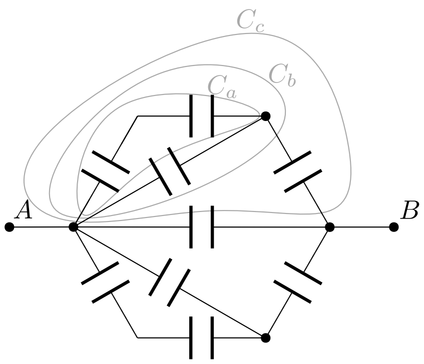


Figure 2.64: Capacitor capacitance addition.

The capacitors in group C_a are connected in series; group C_b is then formed by a capacitor connected in parallel to them, and C_c is obtained by connecting another capacitor in series. Using the formulas for capacitance addition, we arrive at the relations:

$$C_a = \frac{C}{2}, \qquad C_b = C_a + C, \qquad \frac{1}{C_c} = \frac{1}{C_b} + \frac{1}{C}. \tag{2.218}$$

The total capacitance C_{AB} is given by connecting two capacitors with capacitances C_c and one with capacitance C in parallel:

$$C_{AB} = C + 2C_c. \tag{2.219}$$

After substitution and some manipulation, we get the result

$$C_{AB} = \frac{11}{5}C. \tag{2.220}$$

2.7.6 2.35 Capacitor half-filled with dielectric

A plate capacitor is half filled with a dielectric of relative permittivity ε_r a) parallel to the plates, b) perpendicular to the plates (see Figure 2.65). How does its capacitance change?

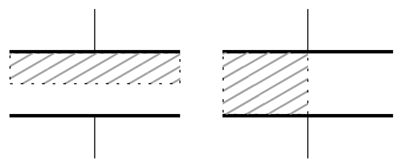


Figure 2.65: Stacking capacitances of capacitors with dielectrics.

Solution: The capacitance of a plate capacitor with a dielectric of relative permittivity ε_r is given by

$$C = \varepsilon_r \frac{\varepsilon_0 S}{d}, \tag{2.221}$$

where S is the area of each plate and d is the distance between the plates.

Capacitor with dielectric half longitudinally. Think of a capacitor as two capacitors in series, see Figure 2.66.

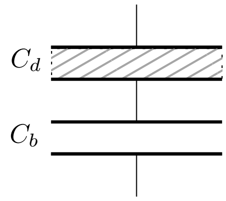


Figure 2.66: Series connection of half capacitors with capacitances C_d and C_b, of which capacitor C_d contains the dielectric.

The relation for the series composition of capacitors C_d and C_b is

$$\frac{1}{C} = \frac{1}{C_d} + \frac{1}{C_b}, \tag{2.222}$$

where the individual capacitances are according to (2.221)

$$C_d = \varepsilon_r \frac{\varepsilon_0 S}{\frac{d}{2}} = 2\varepsilon_r \frac{\varepsilon_0 S}{d} = 2\varepsilon_r C_0, \qquad C_b = \frac{\varepsilon_0 S}{\frac{d}{2}} = 2C_0, \tag{2.223}$$

since the area S of the capacitors has remained the same and the plate spacing has been halved $\frac{d}{2}$. After substitution, we obtain the result:

$$\frac{1}{C} = \frac{1}{2\varepsilon_r C_0} + \frac{1}{2C_0} = \frac{1+\varepsilon_r}{2\varepsilon_r C_0}, \qquad C = \frac{2\varepsilon_r}{1+\varepsilon_r} C_0. \tag{2.224}$$

Capacitor with dielectric half across. We imagine the capacitor divided into two capacitors connected in parallel, see Figure 2.67.

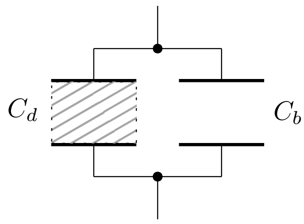


Figure 2.67: Parallel connection of half capacitors with capacitances C_d and C_b, of which capacitor C_d contains the dielectric.

The formula for the parallel addition of capacitors C_d and C_b is

$$C = C_d + C_b, \tag{2.225}$$

where the individual capacitances are according to (2.221)

$$C_d = \varepsilon_r \frac{\varepsilon_0 \frac{S}{2}}{d} = \frac{\varepsilon_r}{2} \frac{\varepsilon_0 S}{d} = \frac{\varepsilon_r}{2} C_0, \qquad C_b = \frac{\varepsilon_0 \frac{S}{2}}{d} = \frac{1}{2} C_0, \tag{2.226}$$

since the plate spacing of the capacitors d remained the same and the area of the plates was halved $\frac{S}{2}$. After the substitution we get the result:

$$C = \frac{\varepsilon_r}{2} C_0 + \frac{1}{2} C_0 = \frac{\varepsilon_r + 1}{2} C_0. \tag{2.227}$$

Addendum: The example can also be computed "more by definition" by determining the magnitude of the electric field E in the capacitor, integrating the voltage U and from the definition of the capacitance C (similar to Example 2.36 in Section 2.7.7). Coming soon.

2.7.7 2.36 Inhomogeneous dielectric capacitor

The space between the plates of a capacitor is filled with a dielectric whose permittivity varies linearly from ε_1 on one plate to ε_2 on the other plate. Determine its capacitance.

Solution: Let's work with relative permittivities instead of absolute permittivities, i.e., we will use constants ε_{r1} and ε_{r2}, where $\varepsilon_1 = \varepsilon_{r1}\varepsilon_0$, $\varepsilon_2 = \varepsilon_{r2}\varepsilon_0$. The relative permittivity function $\varepsilon_r(x)$ between the plates of the capacitor in the Cartesian coordinate x introduced as in Figure 2.68 looks as follows:

$$\varepsilon_r(x) = \varepsilon_{r1} + \frac{\varepsilon_{r2} - \varepsilon_{r1}}{d} x, \tag{2.228}$$

see Figure 2.69. We obtain it by (either guessing and/or) solving equations $\varepsilon_r(0) = \varepsilon_{r1}$ and $\varepsilon_r(d) = \varepsilon_{r2}$ for a general linear function $\varepsilon_r(x) = ax + b$.

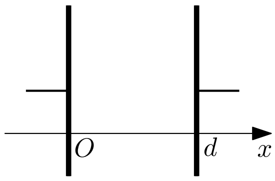


Figure 2.68: Cartesian coordinate x between the plates of the capacitor.

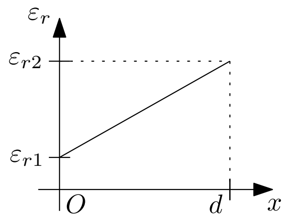


Figure 2.69: Function of relative permittivity $\varepsilon_r(x)$.

The magnitude of the electric field E is reduced in a dielectric from field E_0 in a vacuum as $E = \frac{1}{\varepsilon_r} E_0$. The voltage between the plates of a capacitor is obtained by definition as the work of forces in the capacitor per unit charge:

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l} = \int_l \vec{E} \cdot d\vec{l} = \int_l E \, dl, \tag{2.229}$$

In making the manipulations we have taken advantage of the fact that the only force acting in the capacitor is the electric force $\vec{F} = q\vec{E}$ and if we consider the path l along which we integrate as a line segment perpendicularly connecting the plates of the capacitor, then the direction of the electric field $\vec{E}$ and the direction of the line element $d\vec{l}$ are the same and hence the scalar product reduces to the product of the magnitudes of these vectors, $\vec{E} \cdot d\vec{l} = E \, dl$.

If we consider again the Cartesian coordinate as in Figure 2.68, then the line segment l is given by the coordinate range $x \in \langle 0, d \rangle$ and the line element is $dl = dx$. The actual calculation of the voltage U is then as follows:

$$\begin{aligned} U &= \int_0^d \frac{1}{\varepsilon_r(x)} E_0 \, dx = \int_0^d \frac{E_0}{\varepsilon_{r1} + \frac{\varepsilon_{r2} - \varepsilon_{r1}}{d} x} \, dx = E_0 \left[\frac{d}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \left(\varepsilon_{r1} + \frac{\varepsilon_{r2} - \varepsilon_{r1}}{d} x \right) \right]_0^d \\ &= \frac{E_0 d}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \frac{\varepsilon_{r2}}{\varepsilon_{r1}} = \frac{U_0}{\varepsilon_{r2} - \varepsilon_{r1}} \ln \frac{\varepsilon_{r2}}{\varepsilon_{r1}}, \end{aligned} \tag{2.230}$$

where we have denoted the original voltage on the plate capacitor without dielectric as $U_0 = E_0 d$. When substituted into the formula for defining the capacitance C, we get

$$C = \frac{Q}{U} = \frac{Q}{U_0} \frac{\varepsilon_{r2} - \varepsilon_{r1}}{\ln \frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \frac{\varepsilon_{r2} - \varepsilon_{r1}}{\ln \frac{\varepsilon_{r2}}{\varepsilon_{r1}}} C_0, \tag{2.231}$$

where we have denoted the original capacitance of the capacitor without dielectric as $C_0 = \frac{Q}{U_0}$.

Addendum: The example can also be computed as a series composition of infinite capacitors, i.e., similarly (but more complicated and by integration) to Example 2.35 in Section 2.7.6. Coming soon.

2.7.8 2.37 Energy of the capacitor

A plate capacitor filled with air has a capacitance of C_0. It is connected to a voltage source U_0 and has energy stored on it W_0. It is then immersed in oil of relative permittivity ε_r while remaining connected to the voltage source. Its energy changes to W_1. Finally, we disconnect it from the source and remove it from the oil. It will have a voltage of U_2 and an energy of W_2. Determine W_1, U_2, W_2.

Solution: The energy W and charge Q on the capacitor are given by the following formulae

$$W = \frac{1}{2} C U^2, \qquad Q = CU, \tag{2.232}$$

where C is the capacitance of the capacitor and U is the voltage across the capacitor. The initial energy and charge stored on the capacitor is

$$W_0 = \frac{1}{2} C_0 U_0^2, \qquad Q_0 = C_0 U_0. \tag{2.233}$$

When immersed in oil with a relative permittivity of ε_r, the capacitance changes to $C_1 = \varepsilon_r C_0$, the voltage remains the same as the capacitor is still connected to the source, i.e. $U_1 = U_0$. Thus, the energy W_1 and charge Q_1 is

$$W_1 = \frac{1}{2} C_1 U_1^2 = \frac{1}{2} \varepsilon_r C_0 U_0^2 = \varepsilon_r W_0, \qquad Q_1 = C_1 U_1 = \varepsilon_r C_0 U_0 = \varepsilon_r Q_0. \tag{2.234}$$

When disconnected from the source, the charge on the capacitor must remain constant, i.e. $Q_2 = Q_1$. When removed from the oil, its capacitance returns to its original value, $C_2 = C_0$. The resulting energy W_2 and voltage on the capacitor U_2 is

$$W_2 = \frac{1}{2} C_2 U_2^2 = \frac{1}{2} \varepsilon_r^2 C_0 U_0^2 = \varepsilon_r^2 W_0, \qquad U_2 = \frac{Q_2}{C_2} = \frac{Q_1}{C_0} = \frac{\varepsilon_r C_0 U_0}{C_0} = \varepsilon_r U_0. \tag{2.235}$$

Chapter 3

Stationary electric field

3.1 Formulae overview

- **Resistance addition:** The total resistance R $[\Omega]$ of series and parallel connected resistors R_1 $[\Omega]$ and R_2 $[\Omega]$ is given by the following relations:

$$R = R_1 + R_2, \quad \text{resp.} \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{3.1}$$

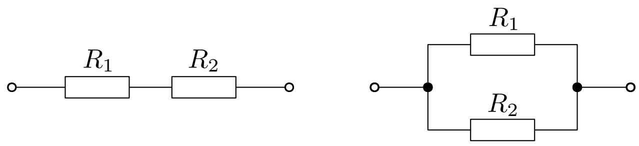


Figure 3.1: Resistors of resistances R_1 and R_2 connected in series (left) and parallel (right) respectively.

- **Resistance of "cylindrical" conductor:**

$$R = \rho\,\frac{l}{S}, \tag{3.2}$$

where ρ $[\Omega.m]$ is the resistivity of the conductor material, l $[m]$ is its length, and S $[m^2]$ is its cross section.

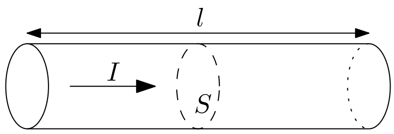


Figure 3.2: The current through a cylindrical conductor of length l and cross-section S.

- **Ohm's law:**

$$U = RI, \tag{3.3}$$

where R $[\Omega]$ is the resistance (of the resistor, appliance, circuit, etc.), I $[A]$ is the current (flowing through the resistor, etc.), and U $[V]$ is the voltage (applied to the resistor, etc.; or voltage drop).

- **Battery internal resistance:**

$$U = \mathcal{E} - R_i I, \tag{3.4}$$

where $\mathcal{E}$ $[V]$ is the electromotive voltage of the battery, R_i $[\Omega]$ is the internal resistance of the battery, U $[V]$ is the terminal voltage across the battery, and I $[A]$ is the current through the battery. As current passes through the battery, there is a voltage drop across the internal resistance and therefore the terminal voltage measured across the battery is reduced by this drop compared to the electromotive voltage.

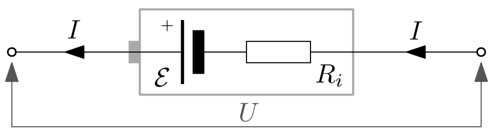


Figure 3.3: Battery with electromotive voltage $\mathcal{E}$ with internal resistance R_i and current flow I.

- **Joule heating:** The heat output generated at a resistor (appliance, etc.) of resistance R $[\Omega]$ with current flowing through it I $[A]$ is

$$P_{heat} = RI^2. \tag{3.5}$$

- **Kirchhoff's laws:** Kirchhoff's first law states that the sum of the currents flowing into and out of a node is zero. Kirchhoff's second law states that the sum of the voltages on the sources along a loop must equal the sum of the voltage drops across the resistors in the same loop.

$$\sum_\alpha I_\alpha = 0, \qquad \sum_\alpha U_\alpha = \sum_\beta R_\beta I_\beta. \tag{3.6}$$

For more details (mainly due to sign convention), see Section 3.6.

- **Definition of current:** Current is the charge flowing through a given location (a given area) per unit time:

$$I = \frac{dQ}{dt}. \tag{3.7}$$

Units: current I $[A] = [C.s^{-1}]$, charge Q $[C]$, time t $[s]$.

3.2 Resistance addition

3.2.1 3.4 Resistance addition I

In the circuit shown in Figure 3.4, a resistor R_0 is given. Determine the resistance R_1 so that the input resistance between points A, B is again R_0.

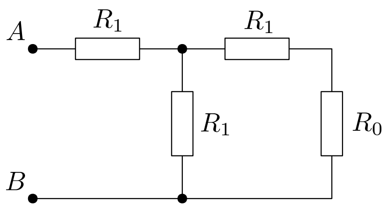


Figure 3.4: Circuit with resistors R_0 and R_1.

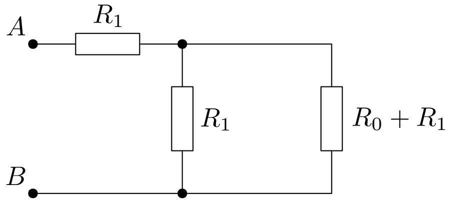
Solution: We are going to use the formulas for series and parallel connection of resistors with resistance values R_1 and R_2. The total resistance R is then

$$R = R_1 + R_2, \quad \text{resp.} \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{3.8}$$

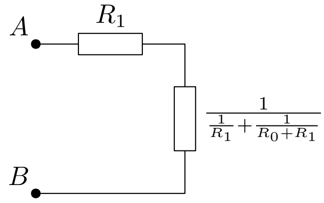
The total resistance between points A and B is

$$R_{AB} = R_1 + \frac{1}{\frac{1}{R_1} + \frac{1}{R_1+R_0}} = \frac{3R_1^2 + 2R_1R_0}{2R_1 + R_0}, \tag{3.9}$$

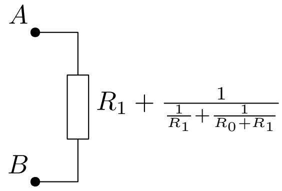
where Figure 3.5 shows the sequential stacking of resistors.



(a) First serial composition.



(b) Next parallel composition.



(c) Final composition.

Figure 3.5: Sequential compounding of connected resistors in series and in parallel.

The requirement in the specification $R_{AB} = R_0$ implies

$$R_1 = \frac{R_0}{\sqrt{3}}. \tag{3.10}$$

3.2.2 3.5 Resistance addition II

Determine the resistance between points A, B of the network in Figure 3.6. All resistances have the same magnitude R.

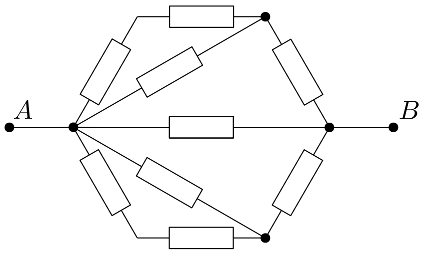


Figure 3.6: All resistors have the same resistance R.

Solution: We determine the total resistance R_{AB} by successively adding the series and parallel connected resistors. The formulas for the total resistance R of the series and parallel connection of resistors with resistance values R_1 and R_2, respectively, are

$$R = R_1 + R_2, \quad \text{resp.} \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{3.11}$$

In a resistor network, we denote the resistance values of specific groups of resistors as R_a, R_b and R_c, see Figure 3.7.

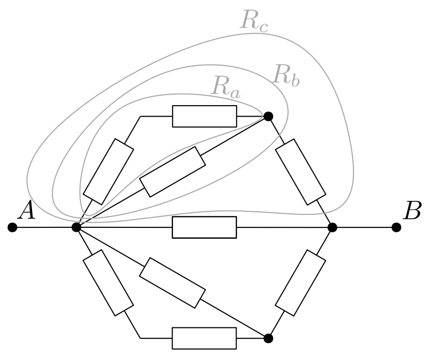


Figure 3.7: We denote the total resistance of the circled resistors by R_a, R_b, and R_c

The resistors in group R_a are connected in series; group R_b is then formed by a resistor connected in parallel to them, and R_c is obtained by connecting another resistor in series. Using the formulas for resistance addition, we arrive at the relations:

$$R_a = 2R, \qquad \frac{1}{R_b} = \frac{1}{R_a} + \frac{1}{R}, \qquad R_c = R_b + R. \tag{3.12}$$

The total resistance R_{AB} is given by the parallel connection of two resistors of the values R_c and one of the size R:

$$\frac{1}{R_{AB}} = \frac{1}{R} + \frac{1}{R_c} + \frac{1}{R_c} = \frac{1}{R} + \frac{2}{R_c}. \tag{3.13}$$

After substitution and some manipulation, we get the result

$$R_{AB} = \frac{5}{11}R. \tag{3.14}$$

3.3 Resistance of conductors

3.3.1 3.1 Proportional conductors

On three equal length sections, the conductor cross section changes in the ratio $S_1 : S_2 : S_3 = 1 : 2 : 3$. What will be the voltage drop on these sections?

Solution: The resistance R of a cylindrical conductor is given by

$$R = \rho\frac{l}{S}, \tag{3.15}$$

where ρ is the resistivity of the conductor material, l is its length, and S is its cross section. Thus, the resistances of the individual sections will be in inverse ratio than the ratios of the cross sections: $R_1 : R_2 : R_3 = 6 : 3 : 2$. The current through one conductor must be the same everywhere, so from Ohm's law we have the relation for the voltage drop across the sections

$$U_{1,2,3} = R_{1,2,3}I, \tag{3.16}$$

The voltage ratios are therefore the same as the resistance ratios, i.e. $U_1 : U_2 : U_3 = 6 : 3 : 2$.

3.3.2 3.2 Tensioned wire

How does the resistance of a copper wire change if we stretch it so that it is extended by $\alpha = 0,1\%$?

Solution: The resistance R of a cylindrical wire is given by

$$R = \rho \frac{l}{S}, \tag{3.17}$$

where ρ is the resistivity of the conductor material, l is its length, and S is its cross section. If we stretch the wire by $\alpha = 0,1\%$, its length increases to $l' = (1+\alpha)l$. Since the volume of the material from which the wire is made must remain the same, its cross-section must be reduced:

$$V = Sl = S'l' \quad \longrightarrow \quad S' = S\frac{l}{l'} = \frac{S}{1+\alpha}. \tag{3.18}$$

The resistance of the conductor is then changed to

$$R' = \rho \frac{l'}{S'} = \rho \frac{l}{S}(1+\alpha)^2 = R(1+\alpha)^2 \approx R(1+2\alpha), \tag{3.19}$$

where we have neglected the term α^2 in the last equation. That is, the resistance will change by approximately $2\alpha = 0,2\%$.

3.3.3 3.3 Resistive cube

A cube with edge length a is positioned such that one corner lies at the origin of the coordinate system and the whole cube lies in the octant determined by the positive directions of the axes. The resistivity of the material varies linearly in the direction of axis x as $\rho = \rho_0(1 + x/x_0)$. Determine the resistance between the walls of the cube parallel to axes y, z and axes x, z.

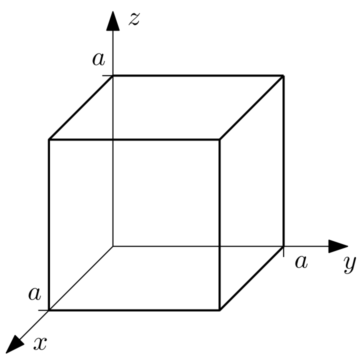


Figure 3.8: Resistance cube.

Solution: We would like to use the formula for the resistance R of a cylindrical conductor:

$$R = \rho \frac{l}{S}, \tag{3.20}$$

where ρ is the resistivity of the conductor material, l is its length, and S is its cross section. However, the resistivity changes throughout of the conductor. So we must divide the cube into suitably chosen parts and add their resistance using the relations for resistance addition.

Let us first consider the case of the resistance between the walls of the cube parallel to the axes y, z. Here the current flows between the back and front walls, see Figure 3.9, and the resistivity therefore varies along the conductor.

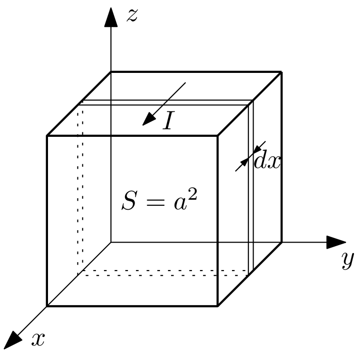


Figure 3.9: Resistance cube.

We therefore divide the cube into thin plates of thickness dx perpendicular to the direction of the current. In these plates, the resistivity is constant and we can calculate their small resistance dR as

$$dR(x) = \rho(x)\frac{dl}{S} = \rho(x)\frac{dx}{a^2} = \rho_0\left(1 + \frac{x}{x_0}\right)\frac{dx}{a^2}, \tag{3.21}$$

where we have substituted for the small conductor length $dl = dx$ and the conductor cross section $S = a^2$. The plates are then all connected in series and we can use the formula for series connection of resistors, which we generalize to the continuous case:

$$R = R_1 + R_2 \quad \longrightarrow \quad R = \sum_i R_i \quad \longrightarrow \quad R = \int dR. \tag{3.22}$$

The actual calculation then consists of integrating the resistances of all the plates that are at coordinates $x \in \langle 0, a \rangle$:

$$R = \int_{\text{plates}} dR = \int_0^a dR(x) = \frac{\rho_0}{a^2}\int_0^a 1 + \frac{x}{x_0}\,dx = \frac{\rho_0}{a^2}\left(a + \frac{a^2}{2x_0}\right). \tag{3.23}$$

The procedure for determining the resistance of a cube between walls parallel to the axes x and z will be similar. The current now flows between the left and right walls, see Figure 3.10, and the resistivity varies across the conductor.

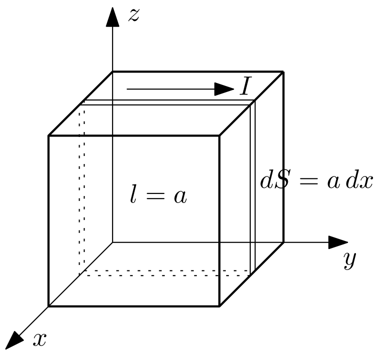


Figure 3.10: Resistance cube.

So we divide the cube into thin plates parallel to the direction of the current. The plates are placed parallel to each other and their total resistance will be given by the relation for parallel connection of resistors:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \longrightarrow \quad \frac{1}{R} = \sum_i \frac{1}{R_i} \quad \longrightarrow \quad \frac{1}{R} = \int d\left(\frac{1}{R}\right), \tag{3.24}$$

where we have again generalized the standard formula for two resistors connected in parallel to the continuous case[1]. The inverse of the resistance of a cylindrical conductor is

$$\frac{1}{R} = \frac{1}{\rho}\frac{S}{l} \tag{3.25}$$

and its infinitesimal version is then

$$d\left(\frac{1}{R}\right)(x) = \frac{1}{\rho(x)}\frac{dS}{l} = \frac{1}{\rho(x)}\frac{a\,dx}{a} = \frac{dx}{\rho_0\left(1+\frac{x}{x_0}\right)}, \tag{3.26}$$

where the length of the conductor is now $l = a$ and its small cross section is $dS = a\,dx$. Now all that is left is to integrate over all plates as in the previous case:

$$\frac{1}{R} = \int_{\text{plates}} d\left(\frac{1}{R}\right) = \int_0^a d\left(\frac{1}{R}\right)(x) = \frac{1}{\rho_0}\int_0^a \frac{dx}{1+\frac{x}{x_0}} = \frac{x_0}{\rho_0}\ln\left(1+\frac{a}{x_0}\right). \tag{3.27}$$

The inverted value is then the result we are looking for:

$$R = \frac{\rho_0}{x_0}\frac{1}{\ln\left(1+\frac{a}{x_0}\right)}. \tag{3.28}$$

3.3.4 3.8 Insulation in coaxial cable

A shielded coaxial cable of length $l = 10\,m$ has a conductor radius $R_1 = 1\,mm$ and a shield $R_2 = 10\,mm$. The insulation is made of polystyrene with resistivity $\rho = 10^{17}\,\Omega.cm$ and dielectric strength $E_{max} = 250\,kV.cm^{-1}$. Determine the maximum voltage between the conductor and the shield, the leakage resistance and the current at this voltage.

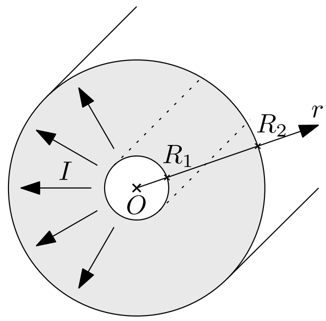


Figure 3.11: Cylindrical insulation cross section.

Solution: Let us first calculate the leakage resistance of the coaxial cable. This will be the total resistance of the conductor in the shape of a hollow cylinder, but where the current does not flow along the cable but across between the inner and outer cylindrical surfaces. We would like to use the formula for the resistance R of a cylindrical conductor according to the relations

$$R = \rho\frac{l}{S}, \tag{3.29}$$

[1]In the case of continuous connection of parallel resistors, the resistance of each resistor needs to go to infinity in the limit, i.e. its inverse is an infinitesimal quantity. Here we have a plate whose cross-section is infinitesimal, so the resistance of this plate is infinitely large and its inverse is therefore an infinitesimal quantity suitable for integration.

where ρ is the resistivity of the conductor material, l is its length, and S is its cross section. Here, however, the cross section of the conductor varies with the distance r from the axis of the coaxial cable, $S(r) = 2\pi r\, l$. We must therefore divide the conductor into suitably chosen parts whose resistivity, length and cross section will be constant, and then add resistances of these parts.

Here, it is natural to divide the cylindrical insulation into thin cylindrical shells of radius r thickness dr, where the cross-section of the conductor changes only negligibly[2], see Figure 3.12.

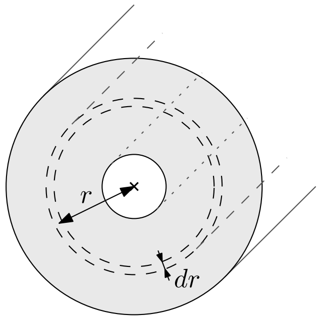


Figure 3.12: A thin cylindrical shell of thickness dr and resistance dR.

The small resistance dR of this cylindrical skin is

$$dR(r) = \rho \frac{dl}{S(r)} = \rho \frac{dr}{2\pi r l}. \tag{3.30}$$

The individual small cylindrical resistors are then connected in series and we can use the formula for series connection of resistors, which we generalize to the continuous case:

$$R = R_1 + R_2 \quad \longrightarrow \quad R = \sum_i R_i \quad \longrightarrow \quad R = \int dR. \tag{3.31}$$

The concrete calculation then consists of integrating the resistances of all the shells that are at the coordinates $r \in \langle R_1, R_2 \rangle$:

$$R = \int_{R_1}^{R_2} dR(r) = \frac{\rho}{2\pi l} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{\rho}{2\pi l} \ln \frac{R_2}{R_1} = 3,66.10^{13}\,\Omega. \tag{3.32}$$

The dielectric strength E_{max} indicates to what maximum value of the electric field the material will retain its insulating properties. In Example 2.30 (in Section 2.7.2), we derived the magnitude of the electric field around a charged cylindrical conductor and the voltage between radii R_1 and R_2 as:

$$E(r) = \frac{Q}{2\pi r l \varepsilon_0}, \qquad U = \frac{Q}{2\pi l \varepsilon_0} \ln \frac{R_2}{R_1}, \tag{3.33}$$

where Q is the total charge on the conductor and ε_0 is the permittivity of the vacuum. But now the capacitor is filled with polystyrene and so we must replace the permittivity of the vacuum ε_0 with the permittivity of the polystyrene ε (but this does not change the result at all). We see that the electric field is inversely proportional to the distance from the cylinder axis, so the largest magnitude will be in the immediate vicinity of the conductor and this must be less than

[2]To be precise, this will be a 2nd order change which will not manifest itself in the subsequent integration, and we will therefore obtain an exact result.

the dielectric strength $E(R_1) \leq E_{max}$. Now just substitute the voltage U for the charge Q using the voltage U from (3.33),

$$E_{max} \geq E(R_1) = \frac{Q}{2\pi R_1 l \varepsilon} = \frac{U}{R_1 \ln \frac{R_2}{R_1}}, \tag{3.34}$$

and by expressing the voltage U we get the resulting maximum voltage at which the insulation still insulates:

$$U_{max} = E_{max} R_1 \ln \frac{R_2}{R_1} \doteq 57,6\,kV. \tag{3.35}$$

The current through the insulation at the maximum voltage U_{max} is simply from Ohm's law:

$$I = \frac{U_{max}}{R} = 1,57.10^{-9}\,A = 1,57\,nA. \tag{3.36}$$

3.3.5 3.9 Leakage resistance of a spherical capacitor

Determine the leakage resistance of a spherical capacitor ($R_1 = 10\,cm$, $R_2 = 20\,cm$) when the space between the electrodes is filled with oil with specific resistance $\rho = 1,0.10^{16}\,\Omega.cm$.

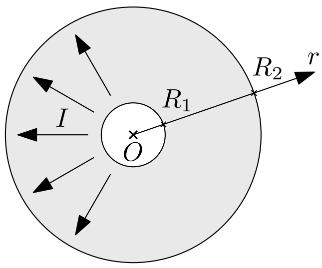


Figure 3.13: Through a spherical capacitor.

Solution: The solution is very similar to the first part of Example 3.8 in the previous section 3.3.4. Here we want to determine the total resistance of a hollow sphere shaped conductor where the leakage current flows between the inner and outer spherical electrodes of the capacitor. We would like to use the formula for the resistance R of a cylindrical conductor according to

$$R = \rho \frac{l}{S}, \tag{3.37}$$

where ρ is the resistivity of the conductor material, l is its length, and S is its cross section. Here, however, the cross section of the conductor varies with the distance r from the common center of the spherical conductors, $S(r) = 4\pi r^2$. We must therefore divide the conductor into suitably chosen parts whose resistivity, length and cross section will be constant, and then add the resistances of these parts.

We divide the spherical insulation into thin spherical shells of radius r and thickness dr, where the cross-section of the conductor varies only negligibly[3], see Figure 3.14.

[3]Same remark as in the previous example. This will be a 2nd order change in cross-section, which will not show up in the subsequent integration, so we get the exact result.

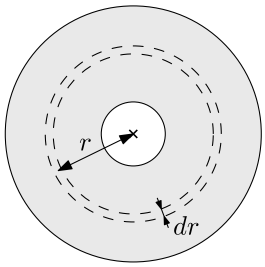


Figure 3.14: A thin spherical shell of thickness dr and resistance dR.

The small resistance dR of this spherical shell is

$$dR = \rho \frac{dl}{S} = \rho \frac{dr}{4\pi r^2}. \tag{3.38}$$

The individual small spherical resistors are then connected in series and we can use the formula for series connection of resistors, which we generalize to the continuous case:

$$R = R_1 + R_2 \quad \longrightarrow \quad R = \sum_i R_i \quad \longrightarrow \quad R = \int dR. \tag{3.39}$$

The concrete calculation then consists of integrating the resistances of all the shells that are at coordinates $r \in \langle R_1, R_2 \rangle$:

$$R = \int_{R_1}^{R_2} dR(r) = \frac{\rho}{4\pi} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{\rho}{4\pi} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \doteq 3,0.10^{13}\,\Omega. \tag{3.40}$$

3.4 Ohm's Law

3.4.1 3.6 Resistor cube

There is a resistor R in each edge of the cube. Determine the resulting resistance between two opposite vertices of the cube.

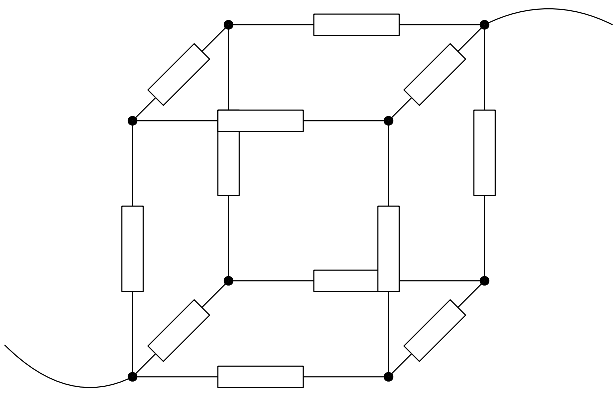


Figure 3.15: Cube from the resistors at the edges.

Solution: In this example, we can no longer use the formulas for series and parallel resistor connections. A cube with resistors in the edges cannot be decomposed into groups of resistors connected in parallel or in series, which we then just put together (as we did, for example, in

Example 3.5 in Section 3.2.2). To illustrate, consider the circuit diagram redrawn to be planar in Figure 3.16.

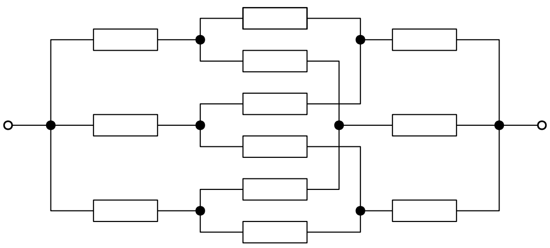
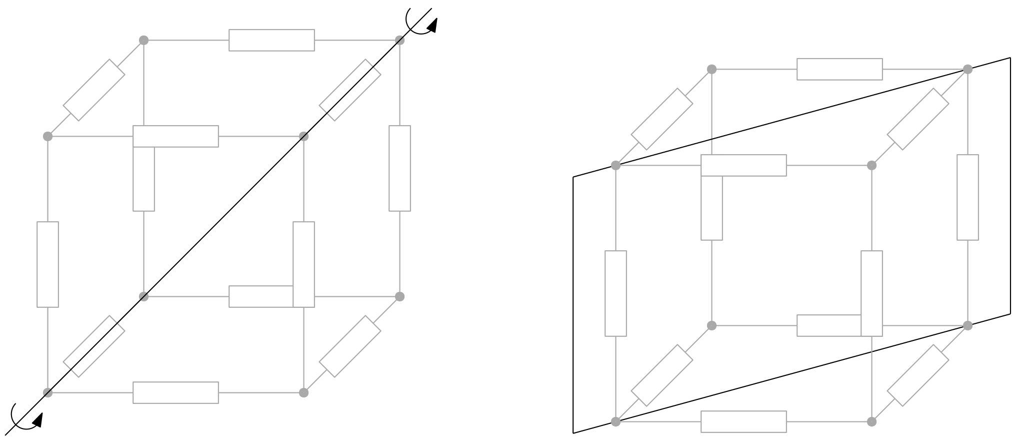


Figure 3.16: Resistors in the edges of the cube drawn in the plane.

The strategy for determining the resistance will be as follows. From the symmetry we determine the currents flowing on each resistor. We then use these to calculate the total voltage drop U after passing through the resistor network using Ohm's law. The total resistance of the cube R_c will then be given again by Ohm's law:

$$R_c = \frac{U}{I}. \tag{3.41}$$

Due to the symmetry, the current I flowing into the "input" node of the cube is divided into thirds $\frac{I}{3}$ and likewise the current I flowing out of the "output" node must be composed of equal currents $\frac{I}{3}$. At the remaining resistors, it must also be further divided in half at $\frac{I}{6}$ because of the symmetry. The resulting distribution of currents through each edge is shown in Figure 3.18. The symmetries used are shown in Figure 3.17.



(a) Symmetry with respect to a discrete rotation by multiples of the angle 120° ($\frac{2\pi}{3}$ rad) about the solid diagonal.

(b) Symmetry of mirroring with respect to a plane passing through the wall diagonal and perpendicular to the given wall.

Figure 3.17: Symmetry of the cube justifying the division of the currents into thirds $\frac{I}{3}$ (left) and then further into halves $\frac{I}{6}$ (right).

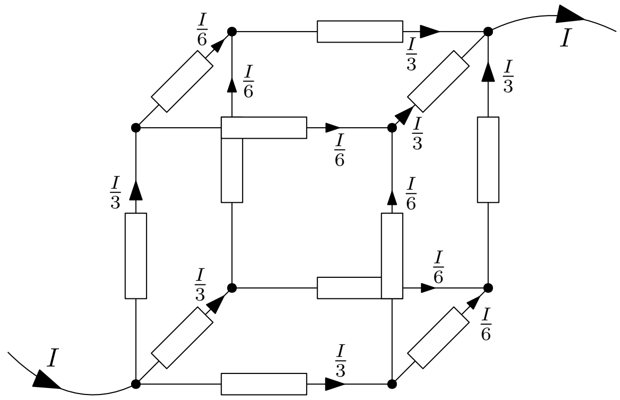


Figure 3.18: Currents in the individual edges of the cube.

Now we just choose a path to get from the input node of the cube to the output node. We calculate the voltage drops at the individual resistors and sum them[4]. The path can be arbitrary, but in practice we choose the simplest possible path, e.g. as in Figure 3.19.

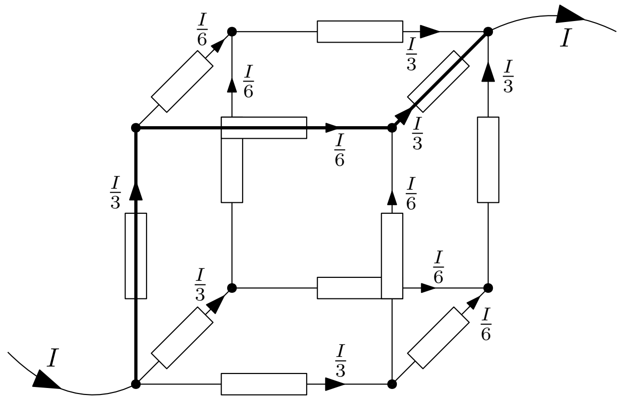


Figure 3.19: Path through the cube from the input node to the output node.

Then the total voltage drop is

$$U = U_1 + U_2 + U_3 = R\frac{I}{3} + R\frac{I}{6} + R\frac{I}{3} = \frac{5R}{6}I = R_c I \tag{3.42}$$

So the total resistance is $R_c = \frac{5}{6}R$.

Addendum: After we have determined the currents through the resistors from the symmetry, we can also use the following "electrical trick" to calculate the total resistance. If we have places in the circuit with the same potential level, we can connect them with a conductor without any current flowing through that conductor, and thus there is no change in the currents anywhere in the circuit. We have this situation after the "first" resistors at the input node and before the "last" resistors at the output node. Let us connect these locations with two conductors as in Figure 3.20.

[4] If the path were such that we were going against the direction of the flowing current, then we would subtract the corresponding drop.

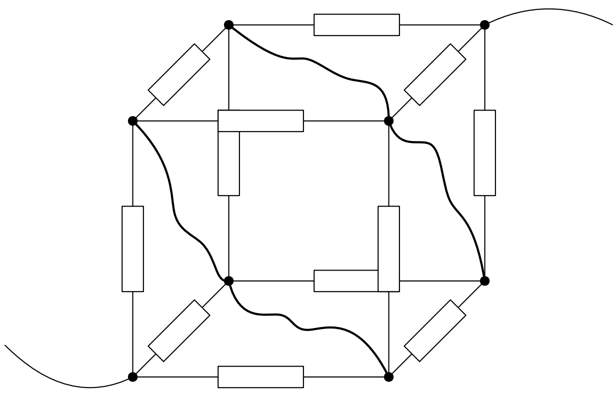


Figure 3.20: Cubes with resistors with connected circuit sites of the same potential level

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Then we will have a circuit like the one in Figure 3.21, which is nothing but a series connection of groups of resistors connected in parallel.

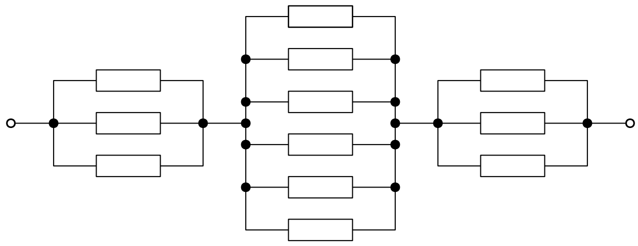


Figure 3.21: Circuit with points of the same potential level connected.

The total resistance is then calculated simply

$$R_c = \frac{R}{3} + \frac{R}{6} + \frac{R}{3} = \frac{5R}{6}. \tag{3.43}$$

Addendum: What about the case where the symmetry arguments cannot be used? For example, if we had general resistances $R_1, \ldots, R_{12}$ in the edges of the cube? Then we have to use Kirchhoff's laws to determine the currents through each branch, see Section 3.6. We have twelve unknown currents through each edge of the cube $I_1, \ldots, I_{12}$. For each vertex of the cube, we get the first Kirchhoff's law of conservation of the inflowing and outflowing currents – so there are eight equations in total, 7 of which are independent. The remaining five equations is provided by the second Kirchhoff's law for circuit loops – here, for example, the loops forming the five walls of the cube (the sixth wall would give the dependent equation).

3.4.2 3.11 Voltage drops in the circuit

To what voltage does the capacitor C in Figure 3.22 charge if the terminal voltage between A, B is equal to U_{AB}?

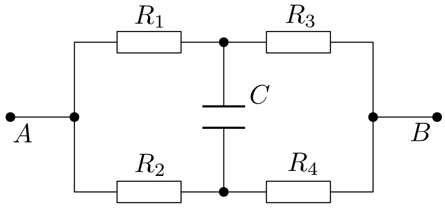


Figure 3.22: What is the voltage across the capacitor C?

Solution: The voltage across the capacitor U_C will be given by the potential difference across the upper and lower capacitor leads, $U_C = \varphi_h - \varphi_d$. These potentials are obtained by subtracting the voltage drops U_1, respectively U_2, on the resistors R_1, respectively R_2, from the potential φ_A at point A:

$$\varphi_h = \varphi_A - U_1, \qquad \varphi_d = \varphi_A - U_2. \tag{3.44}$$

Then the voltage across the capacitor

$$U_C = \varphi_h - \varphi_d = (\varphi_A - U_1) - (\varphi_A - U_2) = U_2 - U_1. \tag{3.45}$$

The drops on the resistors are obtained simply from Ohm's law, $U = RI$:

$$U_1 = R_1 I_h, \qquad U_2 = R_2 I_d, \tag{3.46}$$

where I_h and I_d denote the current through the upper and lower branches of the circuit, respectively. The currents through each branch are again calculated from Ohm's law, $I = \frac{U}{R}$. If there is a voltage on a branch U_{AB}, then the current through that branch is simply given by the total resistance of the branch. Here:

$$I_h = \frac{U_{AB}}{R_1 + R_3}, \qquad I_d = \frac{U_{AB}}{R_2 + R_4}. \tag{3.47}$$

Substituting (3.46) and (3.47) into (3.45) we get the result:

$$U_C = R_2 \frac{U_{AB}}{R_2 + R_4} - R_1 \frac{U_{AB}}{R_1 + R_3} = \frac{R_2 R_3 - R_1 R_4}{(R_1 + R_3)(R_2 + R_4)} U_{AB}. \tag{3.48}$$

3.4.3 3.10 Damaged telegraph lines

A homogeneous telegraph line is damaged by being grounded by resistance R. Prove that the current on the receiving apparatus side will be the smallest if the fault is in the middle of the line (neglect the apparatus resistance).

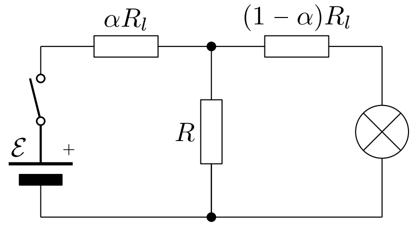


Figure 3.23: A telegraph line of total resistance R_l is impaired by leakage resistance R.

Solution: The circuit diagram of a telegraph is shown in Figure 3.23. On the transmitting side it is a source with a switch (telegraph key) and on the receiving side it is some form of signaling – a light bulb, buzzer, electromagnet, etc. A telegraph line of total resistance R_l is damaged by leakage resistance R at $\alpha = \frac{x}{l} \in \langle 0, 1\rangle$, where x is the distance of the fault location x from the transmitter and l is the total length of the line.

Let us denote the currents through each branch as follows: I_{tot} – total current flowing through the battery, I_{leak} – current flowing through the leakage resistor, I_{rec} – current flowing through the receiving device, see Figure 3.24.

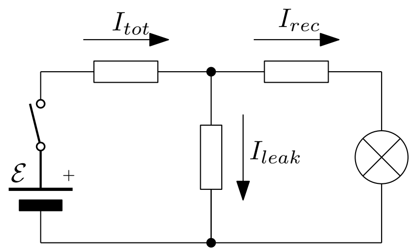


Figure 3.24: The currents in the various parts of the telegraph – I_{tot}, I_{leak} and I_{rec}.

We now calculate the current through the receiving part of the telegraph I_{rec} depending on the position of the fault α. The total current I_{tot} is easily found from Ohm's law, $\mathcal{E} = R_{tot} I_{tot}$ where R_{tot} is the total resistance connected to the telegraph source. We find the total resistance by the rules for adding series and parallel resistors as

$$R_{tot} = \alpha R_l + \frac{1}{\frac{1}{(1-\alpha)R_l} + \frac{1}{R}} = \frac{\alpha(1-\alpha)R_l^2 + RR_l}{(1-\alpha)R_l + R}. \tag{3.49}$$

The total current I_{tot} is divided into the leakage current I_{leak} and the receiver current I_{rec}:

$$I_{tot} = I_{rec} + I_{leak}, \quad \text{where} \quad I_{tot} = \frac{\mathcal{E}}{R_{tot}} = \frac{(1-\alpha)R_l + R}{\alpha(1-\alpha)R_l^2 + RR_l}\,\mathcal{E}. \tag{3.50}$$

We relate the currents I_{rec} and I_{leak} to each other again using Ohm's law, $U = RI$: the voltage drops across the leakage resistor R and the line resistance at the receiver $(1-\alpha)R_l$ must be equal:

$$U = R\,I_{leak} = (1-\alpha)R_l\,I_{rec} \quad \longrightarrow \quad I_{leak} = \frac{(1-\alpha)R_l}{R} I_{rec}. \tag{3.51}$$

Substituting (3.51) into (3.50) gives the following equation for the current I_{rec}:

$$\frac{(1-\alpha)R_l + R}{\alpha(1-\alpha)R_l^2 + RR_l}\,\mathcal{E} = I_{rec} + \frac{(1-\alpha)R_l}{R} I_{rec}. \tag{3.52}$$

By expressing I_{rec} we have

$$I_{rec} = \frac{R}{\alpha(1-\alpha)R_l^2 + RR_l}\,\mathcal{E}. \tag{3.53}$$

We differentiate this result

$$\frac{dI_{rec}}{d\alpha} = -\frac{(1-2\alpha)\,R_l^2 R}{\left[\alpha(1-\alpha)R_l^2 + RR_l\right]^2}\,\mathcal{E} \tag{3.54}$$

and look for when the derivative is zero:

$$\frac{dI_{rec}}{d\alpha} = 0 \quad \Leftrightarrow \quad \alpha = \frac{1}{2}. \tag{3.55}$$

Thus, the current through the receiver is extremal if the leakage resistance is at half of the line. The signs of the first derivative around the extreme tell us that it is a minimum.

3.4.4 3.13 Branching current

The current I_0 branches between the parallel resistors R_1, R_2 and then reconnects (Figure 3.25). Determine the currents I_1, I_2 flowing across these resistors and show that the current distribution corresponds to the minimum of the dissipated heat power.

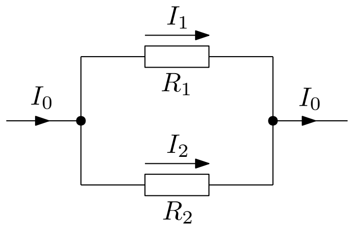


Figure 3.25: The current I_0 branches into currents I_1 and I_2.

Solution: From Ohm's law, $U = RI$, we can easily obtain the current ratios for the individual branches. There must be the same voltage drop across both resistors:

$$U = R_1 I_1 = R_2 I_2 \quad \longrightarrow \quad \frac{I_1}{I_2} = \frac{R_2}{R_1}, \tag{3.56}$$

that is, the ratio of the currents in the branches is the inverse of the ratio of the resistances of the individual branches. The sum of the currents I_1 and I_2 must give the total current I_0:

$$I_0 = I_1 + I_2. \tag{3.57}$$

Equations (3.56) and (3.57) give the following expressions for the currents I_1 and I_2:

$$I_1 = \frac{R_2}{R_1 + R_2} I_0, \qquad I_2 = \frac{R_1}{R_1 + R_2} I_0. \tag{3.58}$$

We show that the dissipated heat power dissipated at the resistors is smallest with this current distribution. Consider the current split in a general way:

$$I_1 = \alpha I_0, \qquad I_2 = (1 - \alpha) I_0, \qquad \alpha \in \langle 0, 1 \rangle. \tag{3.59}$$

The Joule heat is given by $P_{heat} = RI^2$ and thus the total dissipated power would be

$$P_{heat}(\alpha) = P_{heat1} + P_{heat2} = R_1 I_1^2 + R_2 I_2^2 = R_1 \alpha^2 I_0^2 + R_2 (1 - \alpha)^2 I_0^2, \tag{3.60}$$

Let's look for an extremum in the variable α:

$$\frac{dP_{heat}}{d\alpha} = 2 \left[\alpha R_1 - R_2 (1 - \alpha) \right] I_0^2 = 2 \left[(R_1 + R_2) \alpha - R_2 \right] I_0^2 = 0 \quad \Leftrightarrow \quad \alpha = \frac{R_2}{R_1 + R_2} \tag{3.61}$$

Thus, the currents where the power dissipation is extremal (the signs of the first derivative tell us that it is a minimum) are

$$I_1 = \alpha I_0 = \frac{R_2}{R_1 + R_2} I_0, \qquad I_2 = (1 - \alpha) I_0 = \frac{R_1}{R_1 + R_2} I_0. \tag{3.62}$$

These are the same values as for the actual currents determined from Ohm's law.

3.4.5 3.12 Battery internal resistance I

The internal resistance of the galvanic cell R_i is five times ($k = 5$) smaller than the external resistance R, which closes the circuit. How many times will the terminal voltage U be less than the electromotive voltage of the cell?

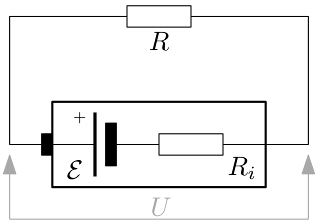


Figure 3.26: The internal resistance R_i of the galvanic cell causes the terminal voltage U to be smaller than the electromotive cell voltage $\mathcal{E}$.

Solution: The terminal voltage U is reduced by the voltage drop across the internal resistance of the cell compared to the electromotive voltage $\mathcal{E}$:

$$U = \mathcal{E} - R_i I. \tag{3.63}$$

The current through the circuit I is then given by Ohm's law,

$$I = \frac{\mathcal{E}}{R + R_i}, \tag{3.64}$$

where the denominator is the total resistance in the circuit. Substituting the expression for the current I into the equation for the terminal voltage, we get

$$U = \mathcal{E}\left(1 - \frac{R_i}{R + R_i}\right) = \mathcal{E}\frac{R}{R + R_i}. \tag{3.65}$$

If the relationship between the resistance of the appliance and the internal resistance is $R_i = \frac{1}{k}R$, the relationship between U and $\mathcal{E}$ is

$$U = \frac{R}{R + R_i}\mathcal{E} = \frac{R}{R + \frac{R}{k}}\mathcal{E} = \frac{k}{1 + k}\mathcal{E} = \frac{5}{6}\mathcal{E}. \tag{3.66}$$

3.4.6 3.18 Battery internal resistance II

We have a battery of unknown electromotive voltage $\mathcal{E}$ and internal resistance R_i. If we connect a resistor $R_1 = 30\,\Omega$ to it, current $I_1 = 125\,mA$ will flow; if we connect a resistor $R_2 = 40\,\Omega$, current $I_2 = 100\,mA$ will flow. Determine the $\mathcal{E}$ and R_i of the battery.

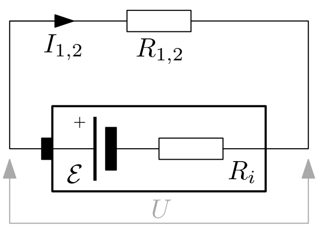


Figure 3.27: The internal resistance R_i of the battery causes the terminal voltage U to decrease compared to the electromotive cell voltage $\mathcal{E}$.

Solution: The terminal voltage U is reduced by the voltage drop across the internal resistance of the cell compared to the electromotive voltage $\mathcal{E}$:

$$U = \mathcal{E} - R_i I. \tag{3.67}$$

This voltage is "consumed" at the connected resistors $R_{1,2}$:

$$\mathcal{E} - R_i I_1 = R_1 I_1, \qquad \mathcal{E} - R_i I_2 = R_2 I_2. \tag{3.68}$$

We can also say that the electromotive voltage $\mathcal{E}$ is dropped at the total resistance in the circuit $R_i + R_{1,2}$, which of course leads to the same equations:

$$\mathcal{E} = (R_i + R_1) I_1, \qquad \mathcal{E} = (R_i + R_2) I_2. \tag{3.69}$$

Solving these equations for $\mathcal{E}$ and R_i, we arrive at the result:

$$\mathcal{E} = (R_2 - R_1) \frac{I_1 I_2}{I_1 - I_2} = 5\,V, \qquad R_i = \frac{R_2 I_2 - R_1 I_1}{I_1 - I_2} = 10\,\Omega. \tag{3.70}$$

3.4.7 3.16 Voltmeter and ammeter

The instrument has a scale with $N = 100$ divisions and an internal resistance of $R_i = 100\,\Omega$. When current $I_1 = 10\,\mu A$ passes through, it will indicate one notch on the scale. What arrangement must we choose if we want to use the instrument as a voltmeter with a range up to $U_0 = 100\,V$ and as an ammeter for currents up to $I_0 = 1\,A$.

Solution: The maximum current through the instrument is apparently $I_{max} = N I_1 = 1\,mA$. We must now choose such circuits that when measuring $U_0 = 100\,V$ or $I_0 = 1\,A$, the current flowing through the instrument is I_{max}.

Let us start with the voltmeter. Here we need to put a resistor R_s in front of the instrument to limit the current flowing due to the connected voltage U_0, see figure 3.28.

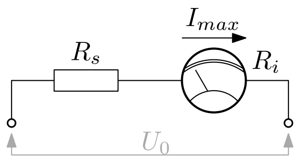


Figure 3.28: Connecting a measuring instrument as a voltmeter.

We can easily calculate the required resistance from Ohm's law:

$$U_0 = (R_s + R_i) I_{max} \quad \longrightarrow \quad R_s = \frac{U_0}{I_{max}} - R_i \approx \frac{U_0}{I_{max}} = 100\,k\Omega, \tag{3.71}$$

where in the conclusion we have neglected the internal resistance of the instrument R_i (its magnitude is three orders of magnitude less than the magnitude of the series resistance R_s).

Let us now proceed to the ammeter. We must now connect a shunt resistor in parallel to the instrument so that most of the current I_0 is led outside the instrument, see Figure 3.29.

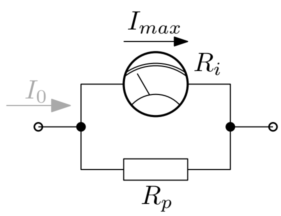


Figure 3.29: Connecting the measuring instrument as an ammeter.

Current flowing through the instrument is

$$I_{max} = \frac{R_p}{R_p + R_i} I_0 \tag{3.72}$$

(for derivation using Ohm's law, see the first half of the solution to Example 3.13 in Section 3.4.4). The required shunt resistance is then

$$R_p = \frac{I_{max}}{I_0 - I_{max}} R_i \approx \frac{I_{max}}{I_0} R_i = 0,1\,\Omega, \tag{3.73}$$

where in the denominator we have neglected the current through the device I_{max} relative to the measured current I_0 (again, it is three orders of magnitude smaller).

3.5 Joule heat

3.5.1 3.14 Resistor sizing

For the network in Figure 3.30, all resistors are individually sized to $P_1 = 0,5\,W$ with values $R_1 = 100\,\Omega$ and $R_2 = 200\,\Omega$. Determine the total resistance and the maximum allowable voltage between points A, B.

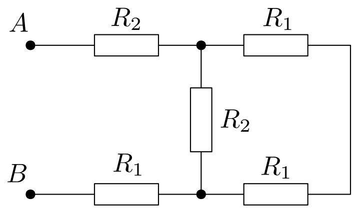


Figure 3.30: Resistor network with resistance values $R_1 = 100\,\Omega$ and $R_2 = 200\,\Omega$.

Solution: We find the total resistance using the formulas for series and parallel connection of resistors (see examples in section 3.2 for details):

$$R_{AB} = R_2 + \frac{1}{\frac{1}{R_2} + \frac{1}{2R_1}} + R_1 = 400\,\Omega. \tag{3.74}$$

The power dissipation generated on a resistor of size R through which current I flows is $P_{heat} = RI^2$. Now let's see which resistor in the circuit will be the most thermally stressed – it will limit the maximum allowable voltage. The currents through the circuit are shown in Figure 3.31.

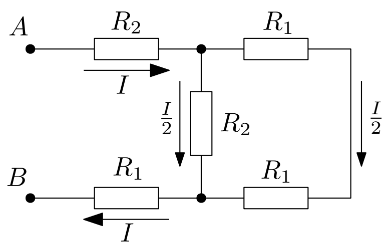


Figure 3.31: Currents flowing through the circuit are shown.

Since the resistances on the two right branches are the same ($R_2 = 2R_1$), the current splits in half in those branches. We can see that the resistor on the top left experiences the greatest thermal stress – it has the greater resistance R_2 and more current flowing through it I. This stress is therefore $P_{heat} = R_2 I^2$. We can easily determine the current I from the connected voltage U and the total resistance R_{AB} using Ohm's law:

$$I = \frac{U}{R_{AB}}. \tag{3.75}$$

When substituted into the formula for Joule's heat for a given resistor:

$$P_{heat} = R_2 I^2 = R_2 \frac{U^2}{R_{AB}^2}. \tag{3.76}$$

This thermal stress must be less than the specified maximum value P_{max}; from this condition we simply express the maximum connected voltage:

$$P_{heat} \leq P_{max} \quad \longrightarrow \quad U_{max} = R_{AB}\sqrt{\frac{P_{max}}{R_2}} = 20\,V. \tag{3.77}$$

3.5.2 3.15 Losses in powerline

A source of voltage $U = 110\,V$ is to supply power $P = 5\,kW$ to a distance $l = 5\,km$. What must be the diameter of the copper wire so that the power losses in the network do not exceed $\alpha = 10\%$ of the transmitted power?

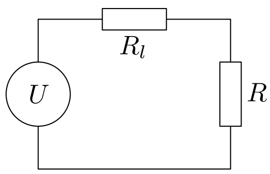


Figure 3.32: Losses in a line of resistance R_l with an appliance of resistance R connected.

Solution: The circuit is shown in Figure 3.32. Here we have a voltage source U followed by a series connected line resistor R_l and an appliance of resistance R. The heat output generated on the line P_l and on the appliance P are given by the Joule heating formula:

$$P = RI^2, \qquad P_l = R_l I^2. \tag{3.78}$$

Our condition is that the losses in the line are less than a specified fraction of the transmitted power (power at the appliance): $P_l \leq \alpha P$. Current I flowing through the line and the appliance is given by Ohm's law:

$$I = \frac{U}{R + R_l}. \tag{3.79}$$

The fraction of power on the line and appliance at the limiting losses is given by (3.78):

$$\frac{1}{\alpha} = \frac{P}{P_l} = \frac{RI^2}{R_l I^2} = \frac{R}{R_l} \quad \longrightarrow \quad R = \frac{R_l}{\alpha}, \tag{3.80}$$

where we have expressed the condition on the resistance of the appliance. Substituting for R from (3.80) and for I from (3.79) into the expression for the power at the appliance P (3.78) we get:

$$P = RI^2 = R\frac{U^2}{(R + R_l)^2} = \frac{R_l}{\alpha}\frac{U^2}{R_l^2\left(1 + \frac{1}{\alpha}\right)^2} = \alpha\frac{U^2}{R_l(1+\alpha)^2}. \tag{3.81}$$

Now we just express the maximum allowable line resistance R_l from the previous relation:

$$R_v = \frac{\alpha U^2}{P(1+\alpha)^2}. \tag{3.82}$$

Next, we express the resistance of the line R_l in terms of its dimensional parameters using the formula for the resistance of a cylindrical conductor:

$$R_l = \rho \frac{2l}{S} = \rho \frac{2l}{\pi \frac{d^2}{4}} \quad \longrightarrow \quad d = \sqrt{\frac{8\rho\, l}{\pi R_v}}, \tag{3.83}$$

where ρ is the resistivity of the conductor material, the length of the conductor is $2l$ since the line is made up of two wires of length l, and we have expressed the cross-section of the conductor using its diameter d, $S = \frac{d^2}{4}$. Substituting for R_l from (3.82) we get the condition on the diameter of the conductor as

$$d \geq \sqrt{\frac{8\rho\, lP}{\pi\alpha}}\, \frac{1+\alpha}{U} = 3,27\, cm, \tag{3.84}$$

where we have substituted the given values and the value of copper resistivity $\rho_{Cu} = 1,68.10^{-8}\, \Omega.m$ to obtain the numerical result.

3.6 Kirchhoff's laws

We use Kirchhoff's laws to find currents in more complex circuits with sources and resistors, where Ohm's law and resistance addition rules can no longer be simply applied.

Circuits consist of branches connected at nodes (where at least three branches meet). To describe the currents in each branch of the circuit, we need to introduce **a fundamental direction of current** in each branch, i.e. an arbitrarily chosen imaginary direction that allows us to interpret at the end of the calculation where the actual current flows through the branch. If the calculation gives us a positive current, it flows in the chosen positive direction; if it gives us a negative current, it flows against the chosen positive direction.

The branches can form a closed loop. In this case, we introduce the so-called **circling direction**, an arbitrarily chosen imaginary direction of passage through the loop. This direction defines along the entire loop the "positive voltage direction", against which we then define the corresponding voltage gains and losses at the sources and resistors.

The first **Kirchhoff's law of nodes** is the current continuity equation: the sum of the currents flowing in and out of a node must be zero:

$$\sum_\alpha I_\alpha = 0. \tag{3.85}$$

In general, if we have n nodes, then the equations constructed for $n-1$ arbitrarily chosen nodes will be independent. The **sign convention** is as follows: if the positive direction of the current on a given branch points into a node, we write it with a plus sign; if it points away from a node, we write it with a minus sign, see also Figure 3.33.

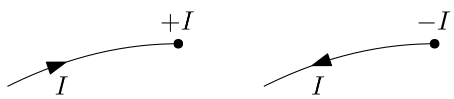
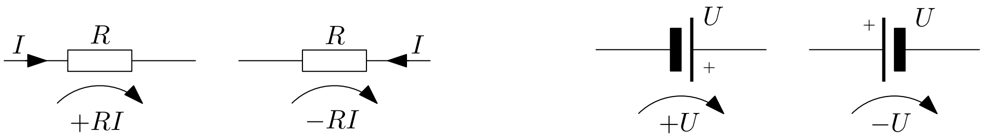


Figure 3.33: The sign convention for Kirchhoff's first law for nodes.

The second **Kirchhoff's law of loops** states that potential gains and losses along a loop in a circuit must add up to zero. That is, the sum of the electromotive voltages at the sources must equal the sum of the voltage drops at the resistors along the loop:

$$\sum_{\alpha} U_\alpha = \sum_{\beta} R_\beta I_\beta. \tag{3.86}$$

From Kirchhoff's first law we have $n-1$ equations, and if we have m branches, i.e. m unknown currents through these branches, then we are left with $m-(n-1)$ equations from Kirchhoff's second law. The **sign convention** is as follows: For voltage drops across resistors, if the direction of circulation in a given loop agrees with the direction of current in a given branch, then we write the corresponding drop with a positive sign, otherwise with a negative sign. For electromotive source voltages, if the source is connected in such a way that the positive pole points in the direction of circulation, then we write the voltage with a positive sign, otherwise with a negative sign[5]. See Figure 3.34 for an illustration.



(a) Voltage drop across resistors. (b) Electromotive voltage sources.

Figure 3.34: The sign convention in Kirchhoff's second law of loops.

3.6.1 3.7 Two-loop circuit

What current flows between points A, B in figure 3.35?

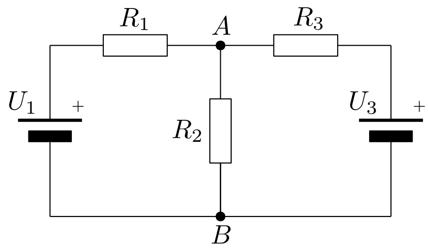


Figure 3.35: A circuit with two nodes, three branches, two sources, and three resistors.

Solution: Introduce the positive directions of the currents in each branch and the directions of the circulating currents for the left and right loops in the circuit, for example, as in Figure 3.36. At the same time, we have named the currents in the left, middle, and right branches as I_1, I_2 and I_3, respectively.

[5]Warning! These conventions, of course, apply only to the second Kirchhoff's law of the form (3.86), i.e. the form where the sources are on one side of the equation and the losses on the other!

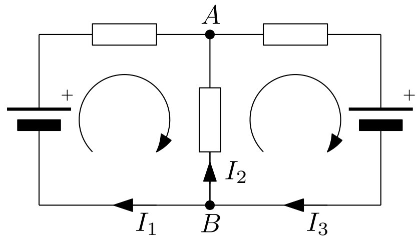


Figure 3.36: Selected positive cuurent directions and circling directions.

Component values are given. Thus, the currents in each branch I_1, I_2, I_3 are the unknowns. Thus, we need to find three equations for these currents. The first Kirchhoff's law for the nodes A and B gives

$$I_1 + I_2 - I_3 = 0, \qquad -I_1 - I_2 + I_3 = 0. \tag{3.87}$$

Here trivially the dependent equations come out, so we discard one of them. From the second Kirchhoff's law for loops, we get for the left and right loops

$$U_1 = R_1 I_1 - R_2 I_2, \qquad -U_3 = R_2 I_2 + R_3 I_3. \tag{3.88}$$

If we wrote a third equation for the "outer loop" passing through the sources U_1 and U_3 and the resistors R_1 and R_3 (where we would again choose the clockwise circling direction as in the other loops):

$$U_1 - U_3 = R_1 I_1 + R_3 I_3, \tag{3.89}$$

we see that it is of the form of the sum of the equations for the left and right loops, so it is not independent (and we discard it).

Solving equations (3.87) and (3.88) we get

$$I_2 = -\frac{R_3 U_1 + R_1 U_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}, \tag{3.90}$$

so the current actually flows from point A to point B. After inserting the specific values in the assignment $R_1 = 50\,\Omega$, $R_2 = 100\,\Omega$, $R_3 = 80\,\Omega$, $U_1 = 3\,V$ and $U_3 = 2\,V$ we get

$$I_2 = -20\,mA. \tag{3.91}$$

3.6.2 3.17 A moron plugging in batteries

Two lead-acid batteries have $\mathcal{E}_1 = 12\,V$, $R_{i1} = 0,04\,\Omega$, $\mathcal{E}_2 = 6\,V$, $R_{i2} = 0,02\,\Omega$. Some jerk accidentally plugged them in side by side. What current will flow through the batteries and what voltage will be at their terminals?

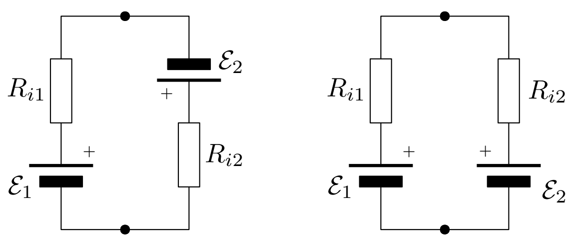


Figure 3.37: Some idiot accidentally wired the batteries side by side. One way or the other. Either way, he's a moron.

Solution: Introduce a positive direction of current and circling in the loop as in Figure 3.38.

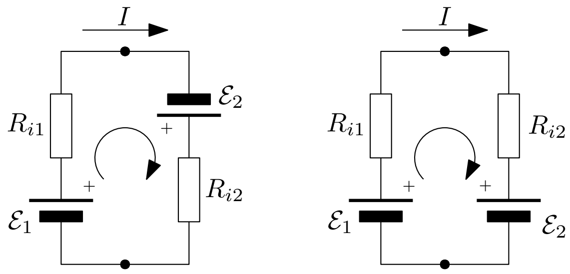


Figure 3.38: Positive current and voltage directions (circling direction).

Then, according to Kirchhoff's second law, for the left-hand circuit (batteries connected "in agreement") we have:

$$\mathcal{E}_1 + \mathcal{E}_2 = (R_{i1} + R_{i2})I \quad \longrightarrow \quad I = \frac{\mathcal{E}_1 + \mathcal{E}_2}{R_{i1} + R_{i2}} = 300\,A. \tag{3.92}$$

And for the right-hand wiring (batteries wired to "disagree"):

$$\mathcal{E}_1 - \mathcal{E}_2 = (R_{i1} + R_{i2})I \quad \longrightarrow \quad I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_{i1} + R_{i2}} = 100\,A. \tag{3.93}$$

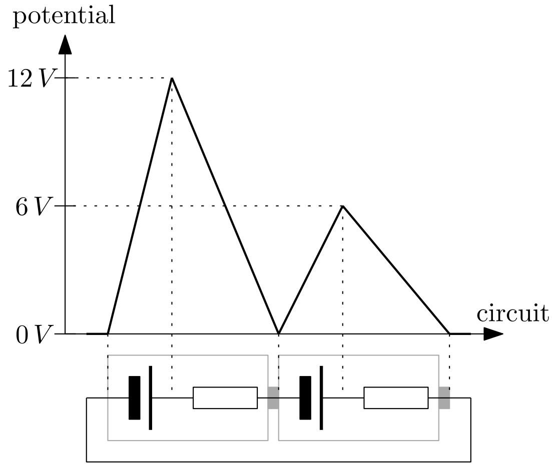
To get the terminal voltages on batteries U_1 and U_2, we just rearrange the terms in equations (3.92) and (3.93) appropriately – we always add the electromotive voltage and the voltage drop across the internal resistance. Then for the left-hand circuit we have:

$$\underbrace{\mathcal{E}_1 - R_{i1}I}_{U_1} = -(\underbrace{\mathcal{E}_2 - R_{i2}I}_{U_2}) = 0\,V. \tag{3.94}$$

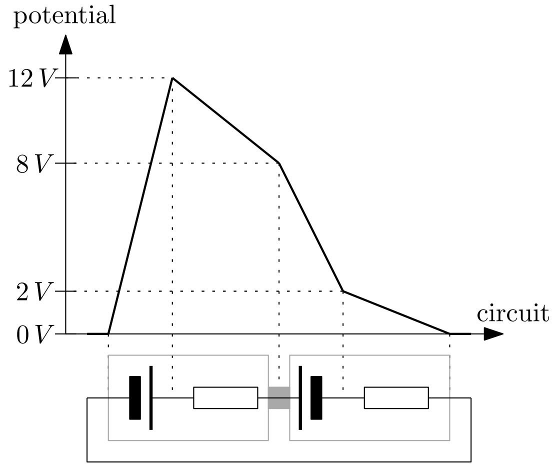
For the right-hand circuit:

$$\underbrace{\mathcal{E}_1 - R_{i1}I}_{U_1} = -(\underbrace{-\mathcal{E}_2 - R_{i2}I}_{U_2}) = 8\,V. \tag{3.95}$$

Figure 3.39 plots the potentials in the circuits. The potential always decreases as it passes through the resistor (in the direction of current), the battery causing either an increase or decrease in potential depending on the orientation.



(a) Batteries connected "consistently". Zero voltage at battery terminals.



(b) Batteries connected "discordantly". There is a voltage of $8\,V$ at the battery terminals.

Figure 3.39: Potential plot in the circuits with differently wired batteries. The voltage across the batteries is given by the potential difference across their terminals.

3.7 Current definition

3.7.1 3.19 Electron velocity in a wire

A current $I = 1\,A$ flows through a wire of cross-section $S = 10\,mm^2$. The electron concentration is $n = 2{,}5.10^{28}\,m^{-3}$. Determine the current density and the mean ordered electron velocity.

Solution: Electric current is defined as the charge flowing through a given area per unit time:

$$I = \frac{dQ}{dt}. \tag{3.96}$$

The amount of charge flowing through a surface S in a small amount of time dt is given by

$$dQ = \rho\, dV, \tag{3.97}$$

where ρ is the bulk charge density of electrons in the conductor and dV is the volume of the conductor from which electrons pass through a given imaginary cross section in time dt, see Figure 3.40.

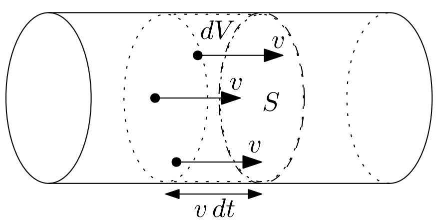


Figure 3.40: Only the charges in volume dV will pass through the surface S in time dt.

Only charges at a distance less than $ds = v\,dt$ will pass through surface S, so the volume is $dV = S\,v\,dt$. For the charge density, $\rho = ne$ holds, where e is the elementary electric charge. The charge dQ then comes out as

$$dQ = \rho\, dV = ne\, Sv\, dt, \tag{3.98}$$

and the current I is

$$I = \frac{dQ}{dt} = \frac{ne\,Sv\,dt}{dt} = neSv. \tag{3.99}$$

Expressing the velocity v gives the result:

$$v = \frac{I}{neS} = 2,5.10^{-5}\,m.s^{-1}. \tag{3.100}$$

The current density is defined as the current per unit area, i.e.

$$j = \frac{I}{S} = 10^5\,A.m^{-2}. \tag{3.101}$$

3.7.2 3.20 Electrons in an accelerator

In an electron synchrotron, electrons orbit in a circular path of length $o = 240\,m$. There are a total of $N = 10^{11}$ electrons in the path, whose speed is practically equal to the speed of light. What current flows through the accelerating path?

Solution: Electric current is defined as the charge flowing through a given location per unit time:

$$I = \frac{dQ}{dt}. \tag{3.102}$$

Let's calculate the charge dQ passing through a given spot in the accelerator in a small amount of time dt:

$$dQ = \tau\,dl = \tau c\,dt, \tag{3.103}$$

where we have denoted τ the longitudinal charge density of the electrons in the accelerator and $dl = c\,dt$ is the distance from which the electrons manage to pass through a given accelerator site in time dt (c denotes the speed of light, i.e. the speed of the electrons). The charge density τ is

$$\tau = ne = \frac{N}{o}\,e, \tag{3.104}$$

where $n = \frac{N}{o}$ denotes the (number) length density of electrons in the accelerator and e is the elementary electric charge. After substitution, the charge is

$$dQ = \frac{N}{o} e\,c\,dt \tag{3.105}$$

and the current from the definition

$$I = \frac{dQ}{dt} = \frac{N}{o} ec \doteq 20\,mA, \tag{3.106}$$

where we used the value of the speed of light $c = 3.10^8\,m.s^{-1}$ and the magnitude of elementary electric charge $e = 1,602.10^{-19}\,C$.

3.7.3 3.21 Van der Graaff current

In a van der Graaff accelerator, a belt with width $s = 20\,cm$ moves at $v = 15\,m/s$. The surface charge of the belt induces a field of intensity $E = 12\,kV.cm^{-1}$ on both sides. What is the current carried by the belt?

Solution: The electric field around a charged plane is given by the following relation (the derivation of which can be found in Example 2.26 in Section 2.7.1):

$$E = \frac{\sigma}{2\varepsilon_0} \quad \longrightarrow \quad \sigma = 2\varepsilon_0 E, \tag{3.107}$$

where σ is the areal charge density and ε_0 is the permittivity of the vacuum. Electric current is defined as the charge flowing through a given location per unit time:

$$I = \frac{dQ}{dt}. \tag{3.108}$$

The charge dQ carried by a strip through a given location in a small amount of time dt is

$$dQ = \sigma\, dS = \sigma\, sv\, dt = 2\varepsilon_0 E\, sv\, dt, \tag{3.109}$$

where $dS = s\, v dt$ denotes the area of the belt that will pass through that location in time dt, see Figure 3.41.

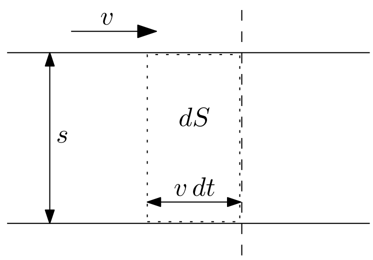


Figure 3.41: The moving charged belt and the area dS that passes through that location in time dt.

The current is then by definition

$$I = \frac{dQ}{dt} = 2\varepsilon_0 Esv \doteq 63,7\,\mu A, \tag{3.110}$$

where we have used the value of the permittivity of the vacuum $\varepsilon_0 = 8,854.10^{-12}\, F.m^{-1}$.

Chapter 4

Stationary magnetic field

4.1 Relativity

4.1.1 4.1 Moving capacitor

How does the voltage U_0 between the plates of a charged capacitor change, measured in a laboratory system, if the capacitor starts moving at $V = 0,8c$ in a direction a) perpendicular to the plates, b) parallel to the plates.

Solution: A capacitor in its rest frame (S) has the area of the plates S at a mutual distance d and they are charged to a total charge Q and $-Q$, respectively – so the surface charge density on the plates is $\sigma = \pm Q/S$.

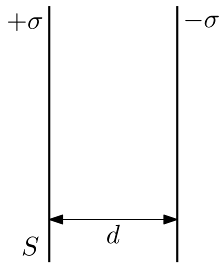


Figure 4.1: Capacitor at rest.

The electric field between the plates of the capacitor is

$$E = \frac{\sigma}{\varepsilon_0}, \tag{4.1}$$

this expression is determined using Gauss's law – see Appendix in Section 2.7.1 – this law is valid for any state of motion of the charges, so the expression (4.1) remains valid for flying capacitors with appropriately changed charge density σ, see below.

The voltage between the plates is then by definition

$$U = \frac{1}{q}\int_l \vec{F}\cdot d\vec{l} = \int_l \vec{E}\cdot d\vec{l} = \int_l E\,dl = Ed, \tag{4.2}$$

see again the appendix in section 2.7.1 for details.

If the capacitor in the system (S') moves at a speed V perpendicular to its plates (see Figure 4.2),

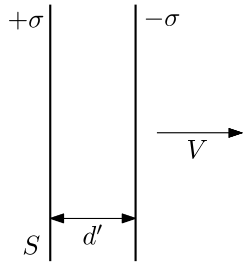


Figure 4.2: Capacitor moving in a direction perpendicular to its plates.

the Lorentz length contraction causes the plates to be at a new distance

$$d' = \frac{d}{\gamma} = \sqrt{1 - \frac{V^2}{c^2}}\, d. \tag{4.3}$$

The area of the plates remains the same, $S' = S$. The charge as a relativistic invariant also remains the same $Q' = Q$ and consequently so does the charge density $\sigma' = Q'/S' = Q/S = \sigma$. The electric field is therefore also the same $E' = \sigma'/\varepsilon_0 = \sigma/\varepsilon_0 = E$ and the voltage changes due to the change in distance between the plates of the capacitor:

$$U' = E'd' = E\frac{d}{\gamma} = \frac{U}{\gamma} = \sqrt{1 - \frac{V^2}{c^2}}\, U. \tag{4.4}$$

When the capacitor moves parallel to the plates at speed V (in the system (S'')), the plates shorten due to length contraction.

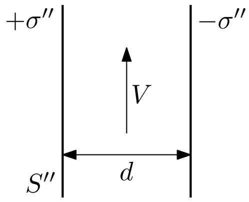


Figure 4.3: Capacitor moving in the direction parallel to the plates.

Thus, their area $S'' = a''b''$ changes from the original $S = ab$, where a, b are the original dimensions of the plates of the capacitor and a'', b'' are the dimensions of the plates of the moving capacitor. Due to length contraction, $a'' = a/\gamma$ (dimension in the direction of motion) and $b'' = b$ (dimension perpendicular to the direction of motion) hold. For the surface S'' we get:

$$S'' = a''b'' = \frac{a}{\gamma}\, b = \frac{S}{\gamma} = \sqrt{1 - \frac{V^2}{c^2}}\, S. \tag{4.5}$$

As a consequence, the charge density σ'' and therefore the electric field strength between the plates E'' changes:

$$\sigma'' = \frac{Q}{S''} = \frac{Q}{S}\,\gamma = \sigma\gamma = \frac{\sigma}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad E'' = \frac{\sigma''}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}\gamma = E\gamma = \frac{E}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{4.6}$$

The resulting voltage across the capacitor is

$$U'' = E''d'' = Ed\gamma = U\gamma = \frac{U}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{4.7}$$

4.1.2 4.2 Current density

In an accelerator, charges with proper (rest) density $\rho' = 10^{-4}\,C \cdot m^{-3}$ move at a velocity $v = 0,8c$ in the direction of the axis x. What is the current density measured in the laboratory system?

Solution: The charge density ρ is given by the amount of charge Q in a given volume V:

$$\rho = \frac{Q}{V}. \tag{4.8}$$

Charge is a relativistic invariant, it is conserved during the transition between systems, $Q' = Q$. In contrast, the volume transforms due to Lorentz length contraction as

$$V = \frac{V'}{\gamma} = V'\sqrt{1 - \frac{v^2}{c^2}}, \tag{4.9}$$

where V' is the intrinsic volume (in the rest frame) and V is the volume in the laboratory frame. This transformation follows from the fact that the dimension in the direction of motion is subject to contraction, see the following figure 4.4.

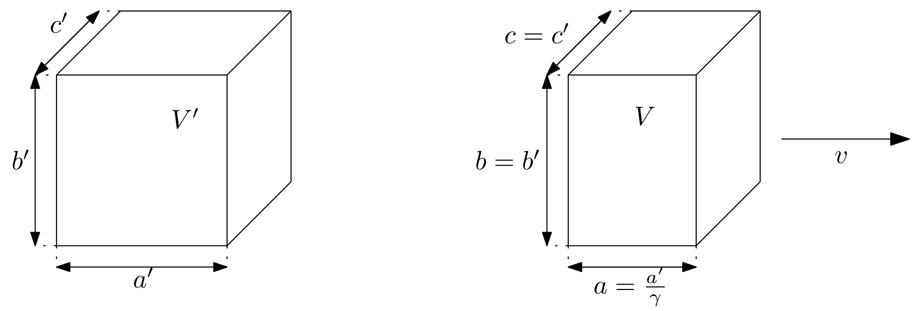


Figure 4.4: Volume and moving volume.

Thus, the charge density in a laboratory system ρ is

$$\rho = \frac{Q}{V} = \frac{Q'}{V'}\gamma = \rho'\gamma = \frac{\rho'}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.10}$$

The current density is $\vec{j} = \rho\,\vec{v}$ or its magnitude $j = \rho\,v$. After substituting from (4.10) we have

$$j = \rho\,v = \rho'\gamma\,v = \frac{\rho' v}{\sqrt{1 - \frac{v^2}{c^2}}} = 10^4\,A.m^{-2}. \tag{4.11}$$

4.1.3 4.4 and 4.5 Electric and magnetic field transformations

The electric and magnetic fields are given as $E_x = E_y = E_z = 3 \cdot 10^4\,V \cdot m^{-1}$, $B_x = 0$, $B_y = -B_z = 5 \cdot 10^{-5}\,T$. Find the coordinate system in which $B = 0$.

A current $I = 100\,A$ flows through a straight conductor. Determine the electric and magnetic fields $\vec{E}$, $\vec{B}$ as they appear at a distance $r = 10\,cm$ from the conductor in a coordinate system moving parallel to the conductor at a velocity $V = 0,8c$.

Solution: Consider an inertial frame of reference (S) and let the frame of reference (S') move with respect to it uniformly with velocity V in the direction of the axis x. Let the coordinate axes be oriented parallel to each other – not rotated with respect to each other.

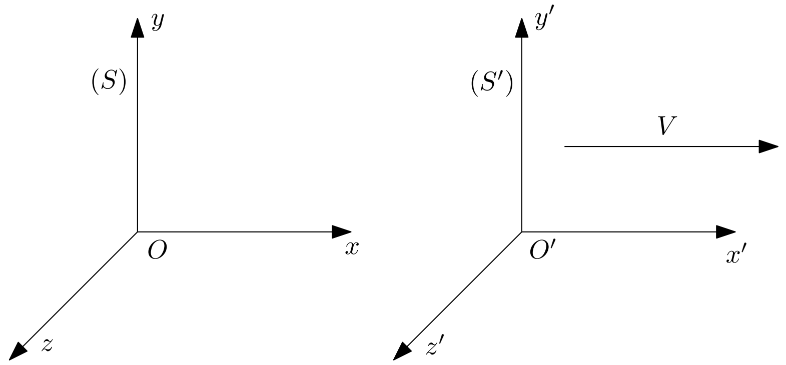


Figure 4.5: Inertial reference frames (S) and (S').

Denote the electric field and magnetic field in (S) as $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$ and in the system (S') as $\vec{E}' = (E'_x, E'_y, E'_z)$ and $\vec{B}' = (B'_x, B'_y, B'_z)$. The relations for the transformation of the components of the vectors $\vec{E}$, $\vec{B}$ and $\vec{E}'$, $\vec{B}'$ are then:

$$\begin{aligned} E'_x &= E_x, & E'_y &= \gamma\left(E_y - \beta c B_z\right), & E'_z &= \gamma\left(E_z + \beta c B_y\right), \\ B'_x &= B_x, & B'_y &= \gamma\left(B_y + \frac{\beta}{c} E_z\right), & B'_z &= \gamma\left(B_z - \frac{\beta}{c} E_y\right), \end{aligned} \tag{4.12}$$

where

$$\beta = \frac{V}{c}, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \left(1 - \beta^2\right)^{-1/2}. \tag{4.13}$$

In **first example** we have $\vec{E} = (E, E, E)$, where we denote $E = E_x = E_y = E_z$ (note that here E is not the magnitude of the vector $\vec{E}$, that is $|\vec{E}| = \sqrt{3}E$), and $\vec{B} = (0, B, -B)$, where we denote $B = B_y = -B_z$ (again, B is not the magnitude of the vector $\vec{B}$ but $|\vec{B}| = \sqrt{2}B$). Substituting in the transformation formulas for magnetic field $\vec{B}'$ (4.12), we get

$$B'_x = 0, \qquad B'_y = \gamma\left(B + \frac{\beta}{c} E\right), \qquad B'_z = \gamma\left(-B - \frac{\beta}{c} E\right). \tag{4.14}$$

Equations $B'_y = B'_z = 0$ lead to the same condition on the velocity V and β, respectively:

$$\beta = -\frac{B}{E}\, c \doteq -0{,}5, \qquad V = -\frac{B}{E}\, c^2 \doteq -0{,}5\, c \doteq 1{,}5 \cdot 10^8 m.s^{-1}, \tag{4.15}$$

where we have used the approximate value $c = 3 \cdot 10^8\, m.s^{-1}$ for the speed of light.

In **second example** we have a (neutral) infinite conductor with a current I that generates a magnetic field around it of magnitude

$$B = \frac{\mu_0 I}{2\pi r}, \tag{4.16}$$

where r is the perpendicular distance from the conductor. The direction of the magnetic field $\vec{B}$ is given by the right-hand rule. The magnetic field vectors are tangent to concentric circles around the conductor, and if the thumb points in the direction of the current in the conductor, the fingers point in the direction of the magnetic field vector.

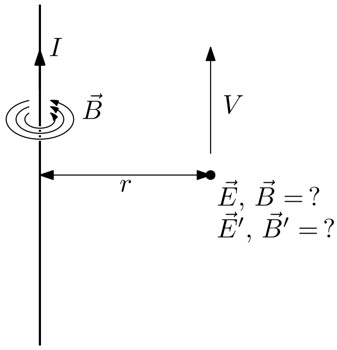


Figure 4.6: The electric and magnetic fields around a (moving) conductor with current.

We now introduce the reference frames (S) and (S') and the corresponding Cartesian coordinates (x, y, z) and (x', y', z') so that we can use the electric and magnetic field transformation formulas (4.12). The system (S') will move with velocity V in the direction of the axis x of the system (S). We will put the axes x and x' in the conductor and point the axes z and z' to the point where we want to know the electric and magnetic fields $\vec{E}'$ and $\vec{B}'$ in the system (S'). See Figure 4.7 (or also Figure 4.6).

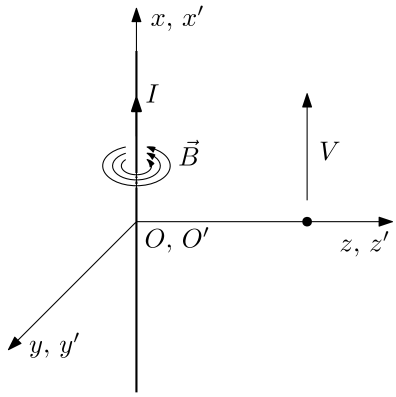


Figure 4.7: Introduction of reference frames (S) and (S').

In the reference system (S) thus introduced, we have the following component expressions for the electric field $\vec{E}$ and magnetic field $\vec{B}$:

$$\vec{E} = 0, \qquad \vec{B} = (0, -B, 0) = \left(0, -\frac{\mu_0 I}{2\pi r}, 0\right). \tag{4.17}$$

We simply plug these into the formulas (4.12):

$$\begin{aligned} &E'_x = 0, \qquad &&E'_y = 0, \qquad &&E'_z = \gamma\beta c\, B_y = -\gamma\beta c\,\frac{\mu_0 I}{2\pi r}, \\ &B'_x = 0, \qquad &&B'_y = \gamma B_y = -\gamma\,\frac{\mu_0 I}{2\pi r}, \qquad &&B'_z = 0. \end{aligned} \tag{4.18}$$

The result is that in a moving system (S') we measure a non-zero electric field $E' = \gamma\beta c\,\frac{\mu_0 I}{2\pi r}$ directed radially towards the conductor (or away from the conductor, for $I < 0$) in addition to a stronger magnetic field of magnitude $B' = \gamma\,\frac{\mu_0 I}{2\pi r}$ ($\vec{B}' = \gamma\vec{B}$). See Figure 4.8. Substituting the given numerical values (the speed of light is considered $c = 3.10^8\, m.s^{-1}$), we obtain $B' = 3.33.10^{-4}\, T$ and $E' = 8.10^4 V.m^{-1}$.

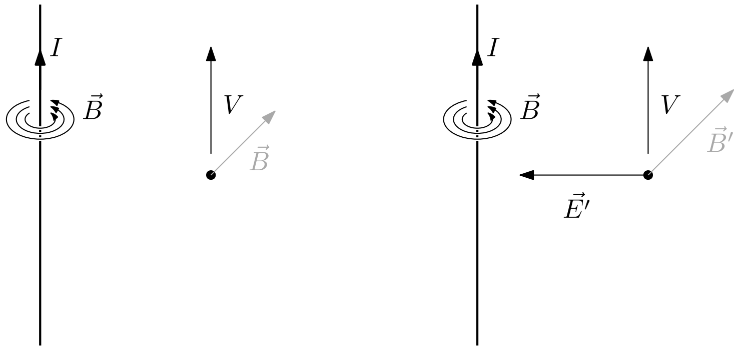


Figure 4.8: Electric and magnetic fields in systems (S), respectively (S') – $\vec{E}$, $\vec{B}$, respectively $\vec{E}'$, $\vec{B}'$.

4.2 Force acting on conductor with current

4.2.1 4.3 Rectangular loop in magnetic field

Determine the total force that a long straight conductor flowing with current $I_1 = 10\,A$ will exert on a rectangular loop as shown in Figure 4.9 through which current $I_2 = 5\,A$ is flowing.

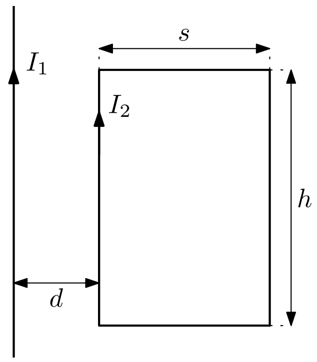


Figure 4.9: What force does an infinite conductor exert on a rectangular loop?

Solution: The small force $d\vec{F}$ acting on a small section dl of a conductor with current I that lies in a given magnetic field $\vec{B}$ is given by the Ampere formula

$$d\vec{F} = I d\vec{l} \times \vec{B}, \tag{4.19}$$

where $d\vec{l} = \vec{t}\,dl$ and $\vec{t}$ is the unit tangent vector to the conductor at the location of the line element dl. See Figure 4.10.

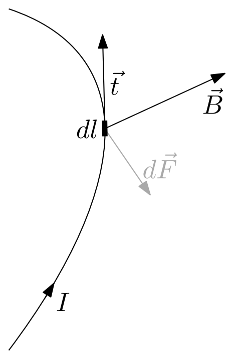


Figure 4.10: The small force $d\vec{F}$ according to Ampere's formula is $d\vec{F} = Id\vec{l} \times \vec{B}$.

The magnetic field $\vec{B}$ from an infinitely long straight conductor with current I has the magnitude of

$$B = \frac{\mu_0 I}{2\pi r}, \tag{4.20}$$

where r is the distance from the conductor. The direction of the magnetic field vectors $\vec{B}$ is given by the right hand rule – the thumb points in the direction of the current, the fingers point in the direction of the magnetic field. In our case, the magnetic field points perpendicular to the rectangular loop and into the paper, see Figure 4.11.

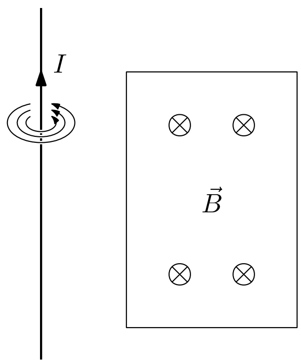
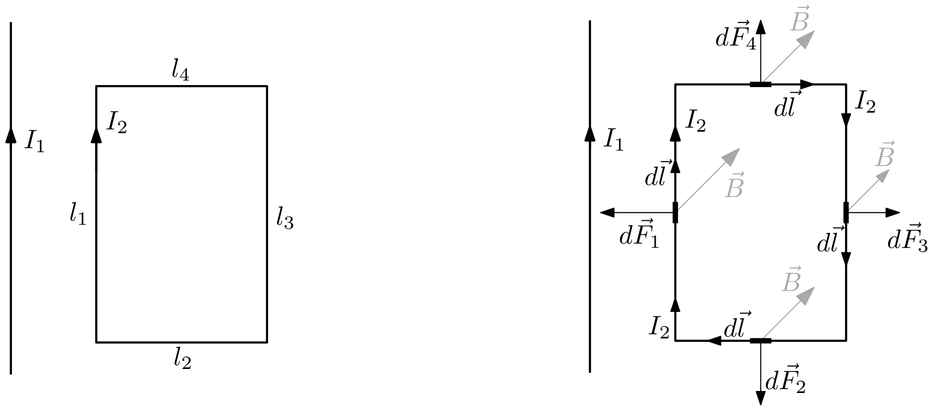


Figure 4.11: Magnetic field from an infinite straight conductor with current.

We will divide the rectangular loop into individual sides, labeled l_1, l_2, l_3, and l_4 – see Figure 4.12 (a) – in each individual side the tangent vector $\vec{t}$ points in the same direction along the whole segment, which will be useful for simplifying the integration of the force elements $d\vec{F}$ (4.19).



(a) Divide a rectangular loop into four segments $l_1, \ldots, l_4$.

(b) Directions of forces $d\vec{F}_i$ in each section of the rectangular loop.

Figure 4.12: Divide the rectangular loop into individual sections l_i and the resulting directions of force contributions $d\vec{F}_i$.

Figure 4.12 (b) shows the directions of the length elements $d\vec{l}$ (or tangent vectors $\vec{t}$, $d\vec{l} = \vec{t}\,dl$), the magnetic field vectors $\vec{B}$ and the resulting directions of the force elements $d\vec{F}$ according to Ampère's formula (4.19) in each section of the rectangular loop.

The division of the loop into its individual parts ensured that in each of them the force $d\vec{F}$ is always pointing in the same direction, and to find the magnitude of the total force we need integrate only the magnitudes of these contributions, i.e.

$$F_i = \int_{l_i} dF_i, \tag{4.21}$$

where F_i denotes the total force acting on each part of the loop[1].

We now proceed to the actual calculation of the forces F_i. The magnitude of the force dF_i from the Ampère formula because of the perpendicularity of the vectors $d\vec{l}$ and $\vec{B}$ is equal to

$$dF_i = I_2 B\, dl. \tag{4.22}$$

Next, we introduce the Cartesian coordinates r and z as shown in Figure 4.13. We choose the origin of the coordinate r with respect to the form of the magnetic field (4.20).

[1]Warning, it is generally not true that

$$\vec{F} = \int_l d\vec{F} \quad \Rightarrow \quad F = \int_l dF.$$

Thus, the magnitude of the vector $\vec{F}$ can only be computed if all the contributions $d\vec{F}$ point in the same direction (which is satisfied in this case). In the general case, the vector integral cannot be avoided and the following holds

$$F = \left| \int_l d\vec{F} \right|.$$

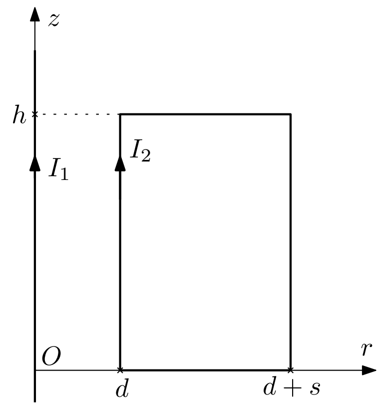


Figure 4.13: We introduce the Cartesian coordinates z and r and the coordinate position of the rectangular loop.

In the coordinates thus introduced, the individual segments l_i are described as follows:

$$
\begin{aligned}
l_1 \quad &\leftrightarrow \quad r = d, \qquad\qquad z \in \langle 0, h\rangle, \\
l_2 \quad &\leftrightarrow \quad r \in \langle d, d+s\rangle, \; z = 0, \\
l_3 \quad &\leftrightarrow \quad r = d+s, \qquad z \in \langle 0, h\rangle, \\
l_4 \quad &\leftrightarrow \quad r \in \langle d, d+s\rangle, \; z = h,
\end{aligned}
\tag{4.23}
$$

We use all of the above ingredients to construct concrete integrals (4.21) with the following result:

$$
\begin{aligned}
F_1 &= \int_0^h I_2 B dz = I_2 \frac{\mu_0 I_1}{2\pi d} \int_0^h dz = \frac{\mu_0 I_1 I_2}{2\pi d} h, \\
F_2 = F_4 &= \int_d^{d+s} I_2 B dr = I_2 \frac{\mu_0 I_1}{2\pi} \int_d^{d+s} \frac{dr}{r} = \frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{d+s}{d}, \\
F_3 &= \int_0^h I_2 B dz = I_2 \frac{\mu_0 I_1}{2\pi (d+s)} \int_0^h dz = \frac{\mu_0 I_1 I_2}{2\pi (d+s)} h.
\end{aligned}
\tag{4.24}
$$

The directions of the $\vec{F}_i$ forces can be seen in Figure 4.12 (b). We see that the forces $\vec{F}_2$ and $\vec{F}_4$ cancel each other out and the total force acting on the loop will be of magnitude $F = F_1 - F_3$ in the direction towards the wire with current I_1 (since $F_1 > F_3$).

4.3 Biot-Savart Law

4.3.1 4.7 Magnetic field of circular and polygonal loops

Determine the magnetic field at the center of a loop carrying a current I in the shape of a circle, equilateral triangle, square, rectangle, hexagon.

Solution: We determine the magnetic field $\vec{B}$ of a conductor with current I using the Biot-Savart law:

$$
\vec{B} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{l} \times \vec{R}}{R^3},
\tag{4.25}
$$

where $d\vec{l} = \vec{t}\, dl$ is the line element with the unit tangent vector $\vec{t}$ pointing in the direction of the current I and $\vec{R}$ is the vector pointing from the conductor element dl to the point where we are asking about the magnetic field $\vec{B}$; also see Figure 4.14.

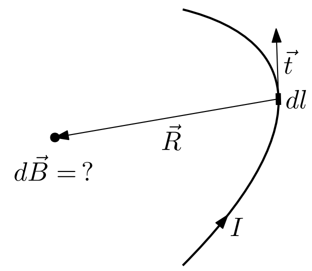


Figure 4.14: Vectors $d\vec{l} = \vec{t}\,dl$ and $\vec{R}$ in the Biot-Savart law giving the magnetic field $\vec{B}$ from the current-carrying conductor.

We can imagine that each section of the conductor dl contributes to the total magnetic field $\vec{B}$ by a small contribution $d\vec{B}$ of the form

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{R}}{R^3}. \tag{4.26}$$

The direction of this contribution is given by the direction of the vector product $d\vec{l} \times \vec{R}$ – this is perpendicular to the plane formed by the vectors $d\vec{l}$ and $\vec{R}$. If our problem is planar, i.e., if the entire current loop and the point where we calculate the magnetic field $\vec{B}$ lie in the same plane, then all contributions $d\vec{B}$ will be directed in the same direction (namely, perpendicular to this common plane)[2] – see Figure 4.16.

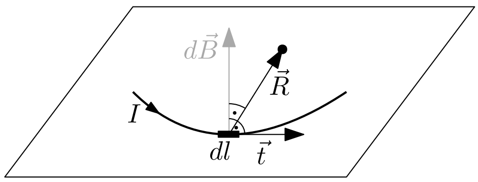


Figure 4.16: Planar problem.

This allows us to calculate the magnitude of the total magnetic field B as the integral of the magnitudes of the individual contributions dB:

$$B = \int_l dB = \frac{\mu_0 I}{4\pi} \int_l \frac{|d\vec{l} \times \vec{R}|}{R^3} = \frac{\mu_0 I}{4\pi} \int_l \frac{\sin\alpha\, dl}{R^2}, \tag{4.27}$$

[2]The sense of wrapping the current conductor around the point where we calculate the magnetic field $\vec{B}$ must not change. This is because then the direction of the contribution $d\vec{B}$ will change to the opposite. It is then necessary to divide the current loop into sections with the same sense of circling and subtract the resulting partial magnetic fields accordingly, see schematic figure 4.15. Alternatively, the "full" vector integral (4.25) can be computed directly.

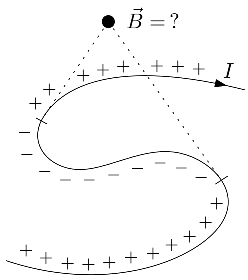


Figure 4.15: Changing the sense of the circling changes the sign of the contribution dB to the total magnetic field B (or switches the direction of the vector contribution $d\vec{B}$).

where α is the (generally variable) angle between the vectors $\vec{dl}$ and $\vec{R}$.

Let us now use the formula (4.27) to calculate the magnitude of the magnetic field B at the centre of a circular loop carrying current I.

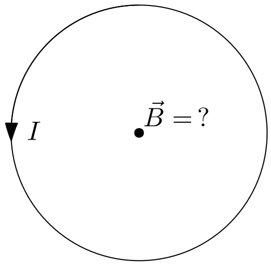


Figure 4.17: Magnetic field $\vec{B}$ in the middle of a circular loop.

The vectors $\vec{R}$, $\vec{dl} = \vec{t}\,dl$ and the angle α between them (and the resulting vector $d\vec{B}$) are plotted in Figure 4.18. The angle $\alpha = \frac{\pi}{2}$ is constant. The magnitude of the vector $\vec{R}$ is also constant and equal to the radius of the circle $R = r$.

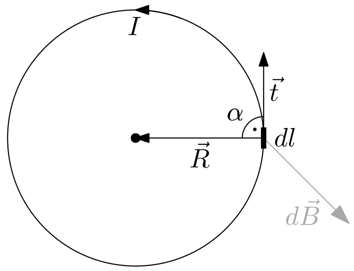


Figure 4.18: Vectors $\vec{R}$, $\vec{dl} = \vec{t}\,dl$ (and the angle α between them) and $d\vec{B}$.

Substituting these values into the formula (4.27) gives

$$B = \frac{\mu_0 I}{4\pi} \int_l \frac{\sin\alpha\, dl}{R^2} = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \int_l dl = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} 2\pi r = \frac{\mu_0 I}{2r}, \tag{4.28}$$

where we have used the relation for the circumference of a circle $\int_l 1 dl = 2\pi r$.

To calculate the magnetic field $\vec{B}$ inside a polygon, we first determine the magnetic field generated by a piece of straight wire (this must then be connected to a closed circuit to allow a constant current I to flow) – see figure 4.19 with the situation and dimensions shown.

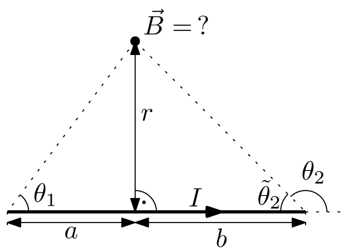


Figure 4.19: Magnetic field $\vec{B}$ from a finite straight conductor.

The problem is again planar, so we can use the formula (4.27). This time it will be better to write directly the vector expressions for $\vec{dl}$ and $\vec{R}$ and use them to calculate the magnitude of the vector product $|\vec{dl} \times \vec{R}|$ – so we will not work with the angle α. The vectors $\vec{dl} = \vec{t}\, dl$ and $\vec{R}$ (and also the angle α) for the current line are shown in Figure 4.20.

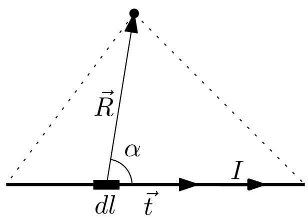


Figure 4.20: Vectors $\vec{R}$ and $\vec{dl} = \vec{t}\, dl$ in the Biot-Savart law for a finite straight conductor.

Introduce Cartesian coordinates as in Figure 4.21 – the x axis is placed in the conductor, the magnetic field location will lie on the y axis, and the z axis is chosen to give a right-handed coordinate system.

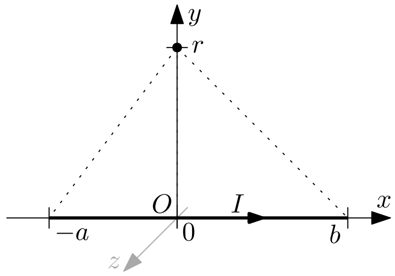


Figure 4.21: Cartesian coordinates around a conductor line segment.

In these coordinates we obviously have $\vec{t} = (1, 0, 0)$, $dl = dx$, $\vec{R} = (-x, r, 0)$. The vector product $\vec{dl} \times \vec{R}$ and its magnitude is

$$\vec{dl} \times \vec{R} = \vec{t} \times \vec{R}\, dx = (0, 0, r)dx \quad \rightarrow \quad |\vec{dl} \times \vec{R}| = r dx. \tag{4.29}$$

The magnitude of R is $|\vec{R}| = \sqrt{x^2 + r^2}$ and the range of coordinates for integration over the line segment is $x \in \langle -a, b \rangle$. The Biot-Savart law in this particular case then takes the form

$$B = \frac{\mu_0 I}{4\pi} \int_l \frac{|\vec{dl} \times \vec{R}|}{R^3} = \frac{\mu_0 I}{4\pi} \int_{-a}^{b} \frac{r\, dx}{(r^2 + x^2)^{3/2}}. \tag{4.30}$$

We now use the familiar formula[3]

$$\int \frac{dx}{(r^2 + x^2)^{3/2}} = \frac{x}{r^2 \sqrt{r^2 + x^2}} \tag{4.31}$$

in the relation (4.30) and we arrive at the result:

$$B = \frac{\mu_0 I}{4\pi} \left[\frac{x}{r\sqrt{r^2 + x^2}} \right]_{-a}^{b} = \frac{\mu_0 I}{4\pi} \frac{1}{r} \left(\frac{a}{\sqrt{r^2 + a^2}} + \frac{b}{\sqrt{r^2 + b^2}} \right). \tag{4.32}$$

[3]This integral can be calculated in a number of ways. One is to use the substitution $\cos\alpha = -\frac{x}{\sqrt{r^2+x^2}}$ – that is, to actually go to the angle variable α from Figure 4.20. Then we have (after some manipulation) $\sin\alpha\, d\alpha = \frac{r^2}{(r^2+x^2)^{3/2}} dx$ and after substitution into the integral: $\int \frac{1}{r^2} \sin\alpha\, d\alpha = -\frac{1}{r^2}\cos\alpha = \frac{x}{r^2\sqrt{r^2+x^2}}$. And now it is familiar.

We can still introduce the angles θ_1 and θ_2 (and $\tilde{\theta}_2$, respectively) as in Figure 4.19 to write the formula for magnetic field B using these angles:

$$\boxed{B = \frac{\mu_0 I}{4\pi}\frac{1}{r}\left(\cos\theta_1 - \cos\theta_2\right) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\left(\cos\theta_1 + \cos\tilde{\theta}_2\right)}. \tag{4.33}$$

From this formula we can already easily calculate magnetic field of all sorts of current formations:

- **Infinitely long conductor:** The limiting value is $\theta_1 = \tilde{\theta}_2 = 0$, substituting into (4.33) the magnitude of the magnetic field at a distance r from the conductor is $B = \frac{\mu_0 I}{2\pi r}$.
- **Semi-infinite conductor:** Here the angles are $\theta_1 = 0$, $\tilde{\theta}_2 = \theta_2 = \pi/2$ and hence $B = \frac{\mu_0 I}{4\pi r}$.

We calculate the magnetic field from the following n-gon loops by always determining the magnetic field from one side and multiplying by the number of sides (except for the rectangle, where there are two non-equivalent types of sides):

- **Rectangle:**

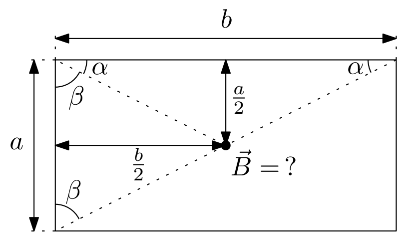


Figure 4.22: Magnetic field $\vec{B}$ at the centre of the rectangle.

We express the cosines of the angles in the rectangle using the side lengths as $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos\beta = \frac{b}{\sqrt{a^2+b^2}}$:

$$\begin{aligned} B_{\text{rectangle}} &= 2B_a + 2B_b = 2\,\frac{\mu_0 I}{4\pi}\left[\frac{2}{a}2\left(\cos\alpha\right) + \frac{2}{b}2\left(\cos\beta\right)\right] \\ &= \frac{2\mu_0 I}{\pi}\left[\frac{1}{a}\frac{b}{\sqrt{a^2+b^2}} + \frac{1}{b}\frac{a}{\sqrt{a^2+b^2}}\right] = \frac{2\mu_0 I\sqrt{a^2+b^2}}{\pi ab}. \end{aligned} \tag{4.34}$$

- **Square:**

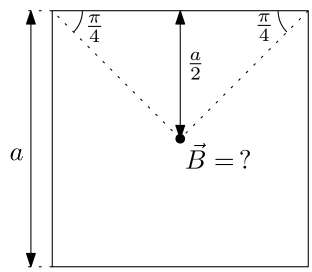


Figure 4.23: Magnetic field $\vec{B}$ at the centre of the square.

Either we consider the square as a special case of a rectangle, $b = a$ or easily determine the necessary distances from the figure 4.23:

$$B_{\text{square}} = 4B_1 = 4\,\frac{\mu_0 I}{4\pi}\frac{2}{a}2\left(\cos\frac{\pi}{4}\right) = \frac{2\sqrt{2}\mu_0 I}{\pi a}. \tag{4.35}$$

- **Equilateral triangle:**

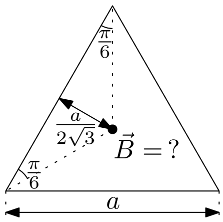


Figure 4.24: Magnetic field $\vec{B}$ at the centre of the triangle.

The main thing to determine is the distance of the center of the triangle from its side – it is one-third of the height, $r = \frac{1}{3}\frac{\sqrt{3}a}{2}$.

$$B_{\text{triangle}} = 3B_1 = 3\,\frac{\mu_0 I}{4\pi}\frac{2\sqrt{3}}{a}2\left(\cos\frac{\pi}{6}\right) = \frac{9\mu_0 I}{2\pi a}. \tag{4.36}$$

- **Hexagon:**

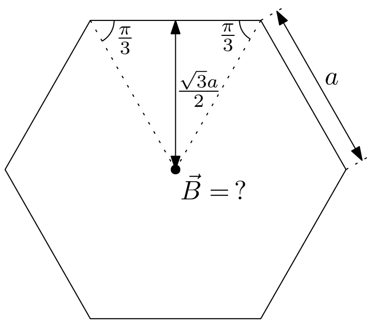


Figure 4.25: Magnetic field $\vec{B}$ at the center of the hexagon.

In a hexagon, the distance r is just equal to the height of one of the triangles forming the hexagon, $r = \frac{\sqrt{3}a}{2}$.

$$B_{\text{hexagon}} = 6B_1 = 6\,\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{3}a}2\left(\cos\frac{\pi}{3}\right) = \frac{\sqrt{3}\mu_0 I}{\pi a}. \tag{4.37}$$

4.3.2 4.11 Bent wire

The infinite wire is bent according to the figure 4.26. Determine the magnetic field at the centre of the semicircle.

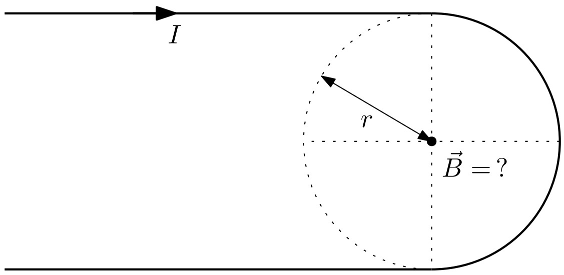


Figure 4.26: Bent wire with curren I.

Solution: The magnetic field at the center of the semicircle will be half that of the whole circle (see the first half of Example 4.7 (Section 4.3.1)):

$$B_{\text{halfcircle}} = \frac{1}{2} B_{\text{circle}} = \frac{\mu_0 I}{4r}. \tag{4.38}$$

The magnetic field at a distance r "at the edge" of the semi-infinite wire is half that of the magnetic field of the entire infinite wire (see the second half of Example 4.7 (Section 4.3.1)):

$$B_{\text{halfwire}} = \frac{1}{2} B_{\text{wire}} = \frac{\mu_0 I}{4\pi r}. \tag{4.39}$$

The whole problem is planar and hence all contributions to the magnetic field will be in the same direction, namely perpendicular to the plane of the wire. For the total magnitude of the magnetic field B then

$$B = 2B_{\text{halfwire}} + B_{\text{halfcircle}} = 2\,\frac{\mu_0 I}{4\pi r} + \frac{\mu_0 I}{4r} = \frac{\mu_0 I}{4r}\left(\frac{2}{\pi} + 1\right). \tag{4.40}$$

We determine the direction of the magnetic field from the right hand rule (the thumb points in the direction of the current, the fingers point in the "direction of wrapping" of the magnetic field lines around the wire). For the direction of current drawn here, the magnetic field is directed into the paper.

Addendum: The example can also be solved by "trick", by laying two equally shaped wires across each other to effectively form a whole circle with current and two (whole) infinite conductors with current. See the following figure 4.27.

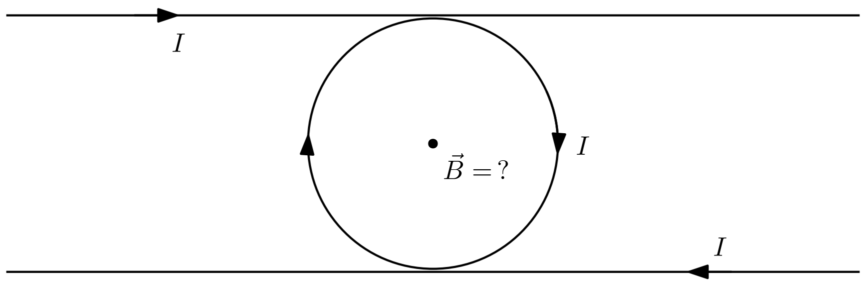


Figure 4.27: Doubled bent wire forming two infinite straight conductors and a circle with current.

The resulting magnetic field is then twice the original one, and we just need to know the expressions for the whole circle and the infinite wire:

$$B = \frac{1}{2}\left(2B_{\text{wire}} + B_{\text{circle}}\right) = \frac{1}{2}\left(2\,\frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2r}\right) = \frac{\mu_0 I}{4r}\left(\frac{2}{\pi} + 1\right). \tag{4.41}$$

4.3.3 4.10 Magnetic field on the axis of a square loop

A current $I = 10\,A$ flows through a square loop with side $a = 6\,m$. Determine the magnetic field at a point on the axis of the loop at a height $h = 4\,m$ above the plane of the loop.

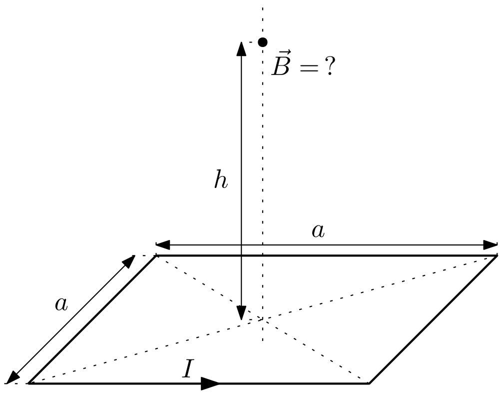


Figure 4.28: Magnetic field on the axis of a square loop with current.

Solution: In this case, we do not have a planar problem – the current loop and the magnetic field determination point $\vec{B}$ do not lie in the same plane. We can, however, determine the partial magnetic fields from each side of the square loop and then add these together.

From the symmetry of the problem (rotational symmetry about the square axis by multiples of the angle $\frac{\pi}{2}$), we know that the resulting magnetic field $\vec{B}$ will point in the direction of the square loop axis. We will therefore want to calculate the projections of the magnetic field from each side of the square in the direction of its axis. Symmetry also says that these projections are the same for each side of the square. The magnetic fields, their projections, and the corresponding angles are shown in Figure 4.29.

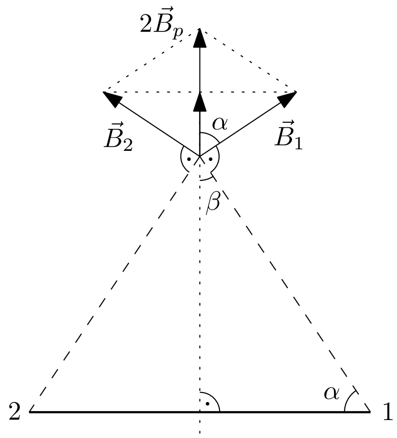


Figure 4.29: Side view of a square loop with current. The magnetic fields from opposite sides of the square $\vec{B}_1$ and $\vec{B}_2$ and their projections in the direction of the loop axis $\vec{B}_p$ are shown. The corresponding sides of the square are marked with numbers (these are the sides that are perpendicular to the paper in this figure). Side one has current flowing into the paper, side two out of the paper. The magnetic fields $\vec{B}_1$ and $\vec{B}_2$ are perpendicular to the planes formed by the sides and a point on the axis of the square. The specific direction is given by the right-hand rule. Angle $\beta = \frac{\pi}{2} - \alpha$.

The magnitude of the total magnetic field B will therefore be $B = 4B_p$, where B_p is the magnitude of the projection of the magnetic field contribution from one side of the square. We now denotes the important distances and angles as: x, y, θ and α. These are marked in Figure 4.30. Next, we determine their expression using the given distances a and h. In the course of the subsequent calculation of the total magnetic field $\vec{B}$, we will gradually see where these quantities are needed.

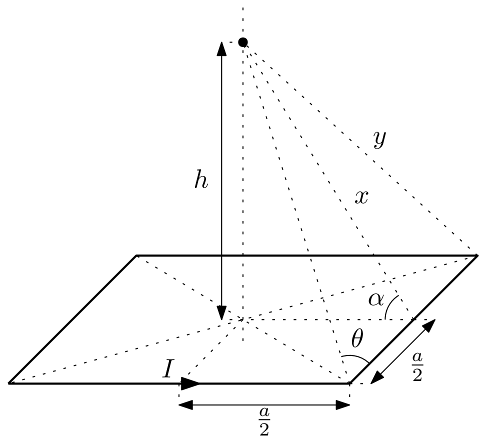


Figure 4.30: The important distances x and y and angles θ and α shown.

From the Pythagorean theorem, we can easily determine

$$x = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} = \sqrt{h^2 + \frac{a^2}{4}}, \qquad y = \sqrt{x^2 + \left(\frac{a}{2}\right)^2} = \sqrt{h^2 + 2\left(\frac{a}{2}\right)^2} = \sqrt{h^2 + \frac{a^2}{2}}. \tag{4.42}$$

We express the angles θ and α from the corresponding triangles as

$$\cos\theta = \frac{\frac{a}{2}}{y} = \frac{a}{2\sqrt{h^2 + \frac{a^2}{2}}}, \quad \cos\alpha = \frac{\frac{a}{2}}{x} = \frac{a}{2\sqrt{h^2 + \frac{a^2}{4}}}. \tag{4.43}$$

To calculate the magnetic field from one side of the square B_1, we use the formula derived in Example 4.7 (Section 4.3.1) for the magnetic field of a line segment:

$$B = \frac{\mu_0 I}{4\pi}\frac{1}{r}\left(\cos\theta_1 - \cos\theta_2\right) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\left(\cos\theta_1 + \cos\tilde{\theta}_2\right), \tag{4.44}$$

where r is the perpendicular distance from the line segment, and see Figure 4.31 for the introduction of angles θ_1 and θ_2 (respectively $\tilde{\theta}_2$).

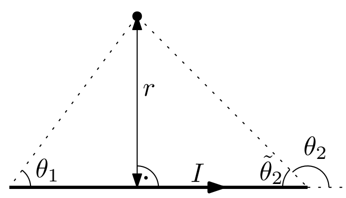


Figure 4.31: Distance r and angles θ_1 and θ_2 (respectively $\tilde{\theta}_2$) to determine the magnetic field from a line segment.

Here, $r = x$, $\theta_1 = \tilde{\theta}_2 = \theta$, after substitution (from (4.42) and (4.43)) it holds:

$$B_1 = \frac{\mu_0 I}{4\pi}\frac{2}{x}\cos\theta = \frac{\mu_0 I}{2\pi}\cdot\frac{1}{\sqrt{h^2 + \frac{a^2}{4}}}\cdot\frac{a}{2\sqrt{h^2 + \frac{a^2}{2}}}.$$

The magnitude of the projection is $B_p = B_1\cos\alpha$ (see Figure 4.29). For the expression for the angle α, see (4.43). Now we put all the above information together:

$$B = 4B_p = 4B_1\frac{a}{2\sqrt{h^2 + \frac{a^2}{4}}} = 4\frac{\mu_0 I}{2\pi}\cdot\frac{1}{\sqrt{h^2 + \frac{a^2}{4}}}\cdot\frac{a}{2\sqrt{h^2 + \frac{a^2}{2}}}\cdot\frac{a}{2\sqrt{h^2 + \frac{a^2}{4}}}. \tag{4.45}$$

After some manipulation, we get the result

$$B = \frac{\mu_0 I}{2\pi} \frac{a^2}{\left(h^2 + \frac{a^2}{4}\right)\sqrt{h^2 + \frac{a^2}{2}}} = \frac{\mu_0 I}{\pi} \frac{2\sqrt{2}\,a^2}{(4h^2 + a^2)\sqrt{2h^2 + a^2}}. \tag{4.46}$$

After plugging in the numerical values, we have $B = 4,94.10^{-7}\,T$.

4.3.4 4.8 Triangle of wire

An equilateral triangle is spliced from a homogeneous wire. An electromotive voltage is applied to the two vertices of the triangle. What will be the magnetic field at the centre of the triangle?

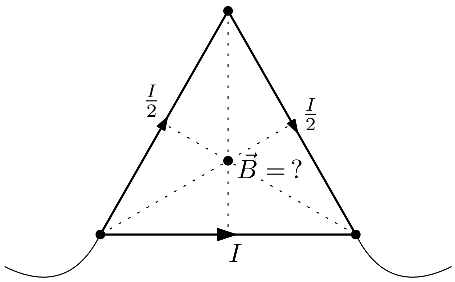


Figure 4.32: Magnetic field at the center of the triangle.

Solution: The current flowing through each section of the triangle is indicated in Figure 4.32. The current flowing through two sides of the triangle must be half the current flowing through one side because of the two sides having twice the total length and therefore twice the resistance.

The magnetic field at the centre of the triangle could be found by calculating the contributions from each side exactly using the Biot-Savart law and then summing these. Indeed, we already know the specific expression for the magnetic field from one side of the triangle from Example 4.7 (Section 4.3.1). But let us look at the calculation a little more generally. Biot-Savart law

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{l} \times \vec{R}}{R^3} \tag{4.47}$$

can be rewritten in a very simple form:

$$\vec{B} = I \left(\frac{\mu_0}{4\pi} \int_l \frac{d\vec{l} \times \vec{R}}{R^3} \right) = I\,\vec{\mathcal{B}}(\text{geometry}). \tag{4.48}$$

The resulting magnetic field is always linearly dependent on the current I flowing through it and is further determined only by the geometry of the problem – the position and shape of the current carrying conductor and the location of the magnetic field. This particular geometry results in a constant coefficient $\vec{\mathcal{B}}$, which gives the value of the magnetic field per unit current.

Here the geometry of each side of the triangle is exactly the same – hence the coefficient $\vec{\mathcal{B}}$ – if we consider the current flowing in the same sense. We determine the actual direction of magnetic field by the right-hand rule – see Figure 4.33; we express this direction by choosing the appropriate sign for the value of the current.

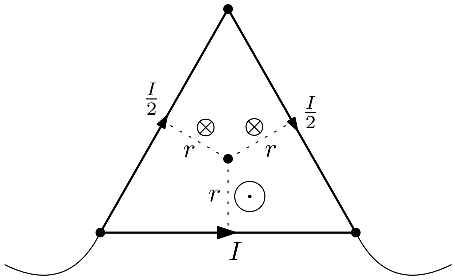


Figure 4.33: Directions and magnitudes of magnetic field from each side of the triangle are shown.

Now just add up the individual contributions (once again, we repeat that the opposite direction of magnetic field is expressed by the sign of the current):

$$\vec{B} = \left(I - \frac{I}{2} - \frac{I}{2}\right)\vec{\mathcal{B}} = 0. \tag{4.49}$$

4.3.5 4.9 Wire cubes

The cube is made of equal sections of wire. We connect voltage to two opposite vertices of the cube. What will be the magnetic field at the centre of the cube?

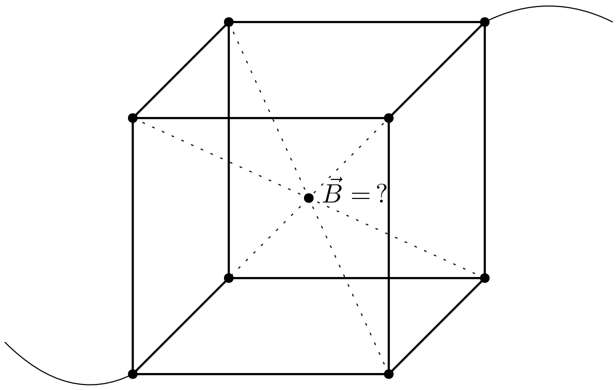


Figure 4.34: Magnetic field at the center of the cube.

Solution: This example is essentially a three-dimensional analogue of the previous example 4.8. The procedure will be very similar. We will determine the currents flowing through each edge of the cube from symmetry – the currents are split into thirds or halves at the nodes (and then joined again); see Example 3.6 (Section 3.4.1) for details. The resulting currents are shown in Figure 4.35.

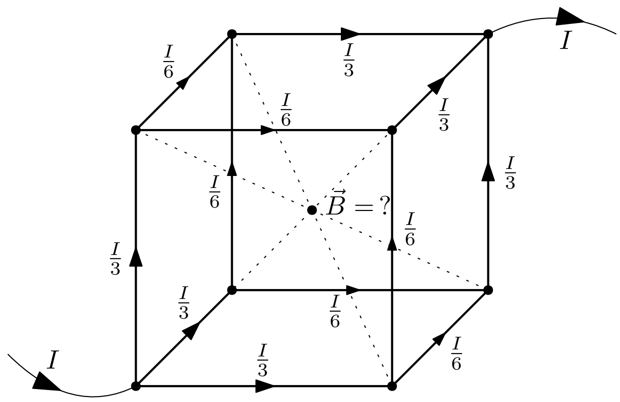


Figure 4.35: Currents flowing through each edge of the cube – $\frac{I}{3}$ and $\frac{I}{6}$.

Consider four parallel edges, the contributions to the magnetic field from these edges then lie in one plane – the situation is shown in Figure 4.36. We determine the directions of the contributions from the right-hand rule – the direction of magnetic field is perpendicular to the plane formed by the edge and the center of the cube, and if the thumb points in the direction of the current, then the fingers point in the direction of magnetic field.

To determine the magnitude of magnetic field specifically from the Biot-Savart law, we would again only need the information that magnetic field is linearly dependent on current:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{l} \times \vec{R}}{R^3} = I \left(\frac{\mu_0}{4\pi} \int_l \frac{d\vec{l} \times \vec{R}}{R^3} \right) = I\vec{\mathcal{B}}(\text{geometry}) \quad \rightarrow \quad B = I\mathcal{B}, \tag{4.50}$$

where the coefficient $\vec{\mathcal{B}}$ gives the value of the magnetic field per unit current and is determined purely by the geometry of the conductor and the location of the magnetic field. For all edges of the cube the geometry is equivalent in the sense that the coefficient $\mathcal{B}$ is the same for all of them (only its magnitude, the directions of the contributions are different and we had to determine them in advance).

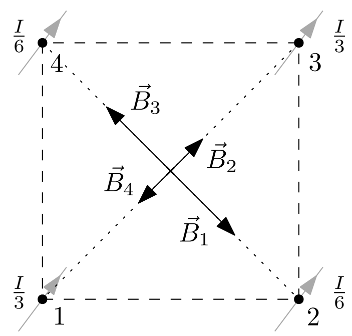
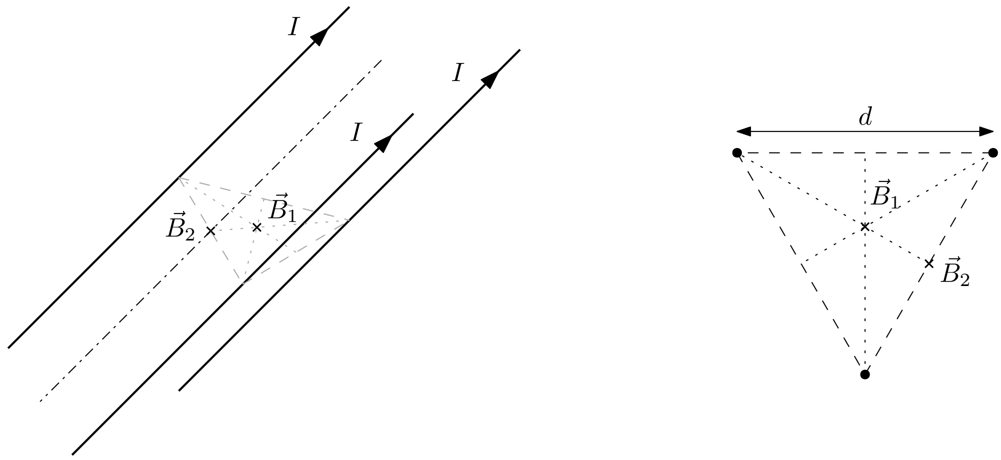


Figure 4.36: Magnetic fields from the four edges of the cube. Shows a section perpendicular to the selected edges passing through the center of the cube. Currents flow through the edges "into the paper". The edges are numbered and have vectors $\vec{B}_i$ associated with them.

Here, specifically, the magnitudes of the magnetic field contributions B_i from the edges with current $\frac{I}{3}$ are the same, $B_1 = B_3$, and so are the magnitudes of the contributions from the edges with current $\frac{I}{6}$, $B_2 = B_4$, (again, see Figure 4.36). The contributions then cancel each other out due to them pointing in the opposite directions. Thus, the total magnetic field from parallel edges is zero, and the magnetic field from all edges (consisting of three sets of parallel edges) in the center of the cube is also zero.

4.3.6 4.14 Three wires

Three parallel straight wires form the edges of a triangular equilateral prism, they are spaced $d = 10\,cm$ apart, and each carries current $I = 20\,A$ flowing in the same direction. Determine the direction and magnitude of the magnetic field on the axis of the prism and on the axis of one of the walls of the prism.



(a) Perspective view. (b) Front view.

Figure 4.37: Magnetic field from three parallel wires.

Solution: The situation on the axis of the prism is simple. The prism has a discrete rotational symmetry by multiples of $120° = \frac{2\pi}{3}$ – rotating it by these angles does not change the physical situation (the position of the conductors and the currents in them), and therefore the generated magnetic field must not change. Infinite straight conductors generate magnetic fields lying in planes perpendicular to the conductor, the resultant magnetic fields from the three conductors forming the prism must again lie in a plane perpendicular to the conductors. However, the only vector perpendicular to the axis of the prism that does not change when rotated by multiples of 120° is the zero vector. The magnetic field on the axis of the prism is therefore zero. In Figure 4.38 we have shown the contributions to the magnetic field from the individual conductors, but for the symmetry argument used above all that was needed was that the vectors $\vec{B}_A$, $\vec{B}_B$, $\vec{B}_C$ lie in a plane perpendicular to the axis of the prism and not their particular direction and magnitude.

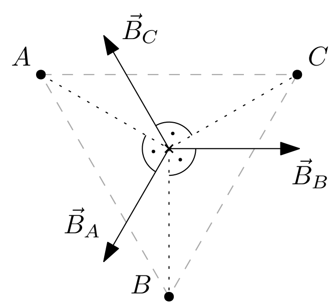


Figure 4.38: Magnetic fields on the axis of the prism from individual conductors in the plane perpendicular to the current-carrying conductors (denoted here by A, B and C and their corresponding magnetic fields $\vec{B}_A$, $\vec{B}_B$ and $\vec{B}_C$). The magnetic field directions correspond to the current flowing through the conductor in the direction “into the paper”.

The situation on the wall axis of the prism is more complicated, see Figure 4.39. The contributions from the wires marked as B and C are cancelled. Their magnitude is the same since the wall axis lies at the same distance from the conductors forming this wall. Their direction is given by the right-hand rule – if the thumb points in the direction of the current flowing through the conductor, then the fingers point in the direction of the magnetic field. It remains, then, to determine the contribution from the conductor A.

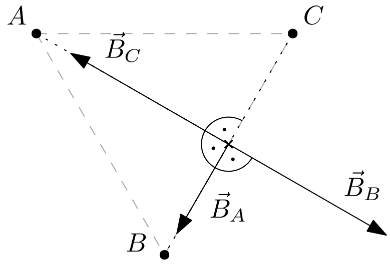


Figure 4.39: Magnetic fields on the wall axis from each conductor in the plane perpendicular to the current-carrying conductors (here labeled A, B, and C and the corresponding magnetic fields $\vec{B}_A$, $\vec{B}_B$, and $\vec{B}_C$). The magnetic field directions correspond to currents flowing through the conductor in the direction "into the paper".

The direction of this contribution is again given by the right-hand rule. The magnitude is given by the formula for the magnetic field from an infinitely long straight conductor:

$$B = \frac{\mu_0 I}{2\pi r}, \tag{4.51}$$

where r is the distance from the conductor. In our case, $r = \sqrt{d^2 - \left(\frac{d}{2}\right)^2} = \frac{\sqrt{3}}{2} d$ holds and thus

$$B = \frac{\mu_0 I}{\sqrt{3}\,\pi d}. \tag{4.52}$$

After substituting the numerical values, we have $B = 4,62.10^{-5}\,T$.

4.4 Ampere's Law

4.4.1 4.13 Pipe with electrical current

An electric current I flows through the walls of a hollow metal tube of inner and outer radii R_1 and R_2. What will be the magnetic field in the walls of the pipe?

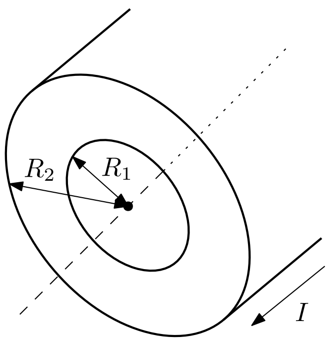


Figure 4.40: Infinitely long pipe with current I.

Solution: Let us first see what constraints are imposed by the symmetries of the problem on the form of the magnetic field generated by the current in the pipe. These are primarily rotational symmetry about the axis of the pipe and translational symmetry in the direction of the pipe. We introduce cylindrical coordinates (r, φ, z) such that the axis z passes through the pipe axis, see Figure 4.41. Then, due to rotational symmetry, the magnitude of the magnetic field B must not depend on the coordinate φ. Due to translational symmetry, the magnitude does not depend on the coordinate z. Thus, we have $B = B(r)$.

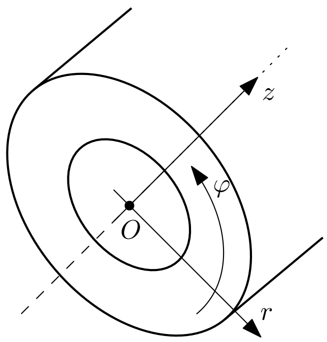


Figure 4.41: Cylindrical coordinate (r, φ, z) in the pipe.

What will be the direction of magnetic field $\vec{B}$? Imagine the pipe with current as a composition of infinitely many infinitely long infinitely thin conductors. The magnetic field generated individually by these conductors is known – the magnetic field vectors always point tangentially and the magnitude depends inversely proportional to the distance ($B \propto \frac{1}{r}$). Consider a fixed but arbitrary point P in space. This point uniquely defines a plane bisecting the pipe longitudinally and passing through the point P. Now we can always take the two conductors in the pipe to be opposite to this plane – see figure 4.42 – the conductors labeled A and B. For arbitrarily chosen opposite pairs of conductors, the resulting contribution to the total magnetic field is always in the tangential direction. Thus, even the total magnetic field at any point P is tangential.

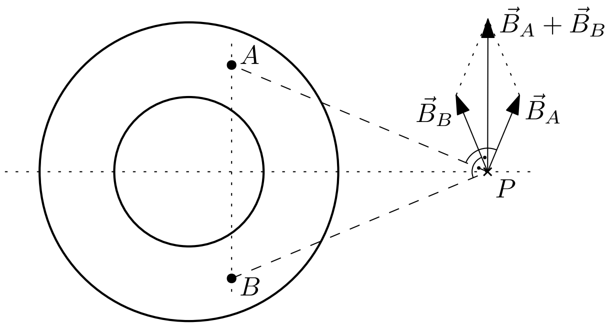


Figure 4.42: The direction of the contribution $\vec{B}_A + \vec{B}_B$ at point P to the total magnetic field $\vec{B}$ from the opposite parts of the pipe (the conductors A and B and their magnetic fields $\vec{B}_A$ and $\vec{B}_B$) is tangential.

We now know that the magnitude of the magnetic field depends only on the distance r from the axis of the pipe, $B = B(r)$, and the direction of this field is purely tangential. To calculate the specific form of the magnitude of the magnetic field, $B(r)$, we use Ampere's law:

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 I_{in}, \tag{4.53}$$

which relates the circulation of magnetic field $\vec{B}$ along a closed curve l with the total current I_{in} enclosed in that curve. We choose the curve l as a circle of general radius r concentric with the axis of the pipe (axis z). A vector element of length $d\vec{l} = \vec{t}\,dl$, where $\vec{t}$ is a unit tangent vector to the curve l, points in the direction of magnetic field $\vec{B}$, see Figure 4.43.

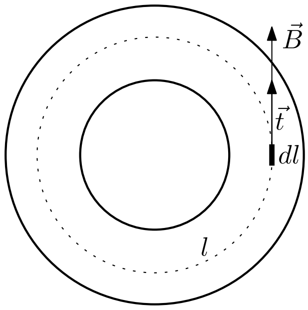


Figure 4.43: Curve l for Ampere's law. The line element $d\vec{l}$ points in the direction of the magnetic field $\vec{B}$.

For the left-hand side of Ampere's law we get:

$$\oint_l \vec{B} \cdot d\vec{l} = \oint_l B\,dl = B(r) \oint_l dl = 2\pi r B(r), \tag{4.54}$$

where we have used the following facts: for the collinear vectors $\vec{B}$ and $d\vec{l}$, $\vec{B} \cdot d\vec{l} = B\,dl$ holds. The circle has a constant distance r from the axis of the pipe and hence the magnetic field $B(r)$ is constant along the integration curve and can be drawn from the integral. Finally, we used the relation for the length of the curve (here the circumference of the circle) $\int_l 1 dl = 2\pi r$.

The right-hand side of Ampere's law requires the calculation of the current I_{in} enclosed inside the curve l. We distinguish three cases. For $r \leq R_1$ the curve lies inside the pipe and thus does not enclose any current, $I_{in} = 0$. For $r \geq R_2$ the whole pipe lies inside the curve and the enveloping current is $I_{in} = I$. Finally, consider the case of $R_1 < r < R_2$. Let us calculate the current density j in the pipe from the formula $I = jS$, where S is the cross section of the conductor:

$$j = \frac{I}{S} = \frac{I}{\pi(R_2^2 - R_1^2)}. \tag{4.55}$$

The current I_{in} enclosed in the loop is obtained by multiplying the current density j calculated above by the conductor cross section in the loop l: $S(r) = \pi(r^2 - R_1^2)$, see figure 4.44, i.e.

$$I_{in}(r) = jS(r) = I\frac{r^2 - R_1^2}{R_2^2 - R_1^2}. \tag{4.56}$$

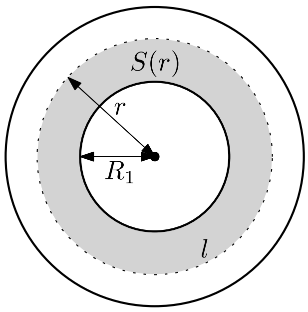


Figure 4.44: A section of the conductor $S(r)$ enclosed in a curve l – a circle of radius r.

The current $I_{in}(r)$ is shown in Figure 4.45.

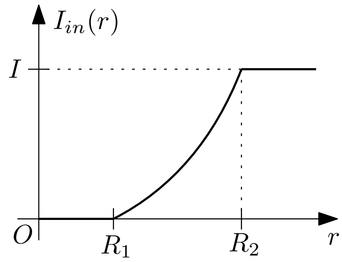


Figure 4.45: Current magnitude $I_{in}(r)$ enclosed in a circle l of radius r.

By comparing the left and right sides of Ampere's law (4.54) and (4.56) (and the "trivial" cases I_{in} for $r < R_1$ and $r > R_2$), we obtain the result (shown graphically in Figure 4.46):

$$B(r) = \begin{cases} 0 & r \leq R_1, \\ \frac{\mu_0 I}{2\pi r} \frac{r^2 - R_1^2}{R_2^2 - R_1^2} & R_1 \leq r \leq R_2, \\ \frac{\mu_0 I}{2\pi r} & R_2 \leq r. \end{cases} \tag{4.57}$$

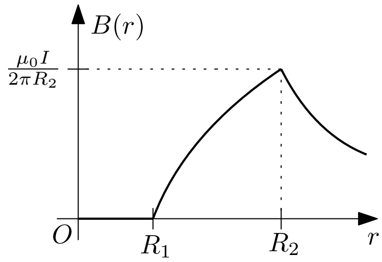


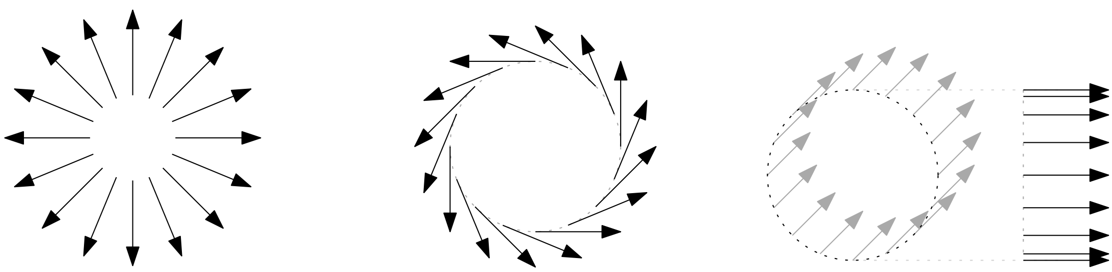
Figure 4.46: The magnitude of the magnetic field $B(r)$ versus the distance r from the axis of the pipe.

Addendum: For the magnetic field of a solid conductor of radius R, we just need to put $R_1 = 0$ and $R_2 = R$ and get the result

$$B(r) = \begin{cases} \frac{\mu_0 I}{2\pi} \frac{r}{R^2} & r \leq R \\ \frac{\mu_0 I}{2\pi r} & r \geq R \end{cases} . \tag{4.58}$$

This will come in handy in Example 4.12 (see the following section 4.4.2).

Addendum: Eventually there will be another justification of the direction of the magnetic field in the pipe purely by symmetries and by studying what type of field symmetry the problem admits at all. The fields that are rotationally symmetric are shown in Figure 4.47 (all linear combinations of these fields are also rotationally symmetric).



(a) Radial vector field. (b) Tangential vector field. (c) Longitudinal vector field.

Figure 4.47: Vector fields respecting rotational symmetry about the axis z.

4.4.2 4.12 Drilled hole

Inside a long conductor of circular cross-section of radius $R = 5\,mm$, a cylindrical cavity of radius $a = 0,5\,mm$ is drilled, the axis of which passes parallel to the axis of the conductor at a distance $b = 3\,mm$. A current $I = 1\,A$ flows through the conductor. What will be the magnetic field in the cavity?

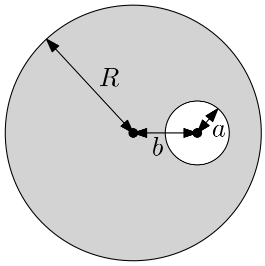


Figure 4.48: A conductor with a drilled hole.

Solution: First, let's use the principle of superposition. In a conductor with a drilled hole, the current density is equal to

$$j = \frac{I}{S} = \frac{I}{\pi(R^2 - a^2)}, \tag{4.59}$$

where $S = \pi(R^2 - a^2)$ is the cross section of the conductor. We decompose this situation into the following two "current configurations". The first is a solid conductor without a drilled hole with a total current of $I_c = jS_c = j\pi R^2$. The second is a current flowing **in the opposite direction** through only a drilled hole of magnitude $I_d = jS_d = j\pi a^2$. We consider the current densities in both cases identical to (4.59). If we "interleave" (superpose) these two situations, we obtain the original current configuration of the conductor with a current where no current flows through the drilled hole. See Figure 4.49.

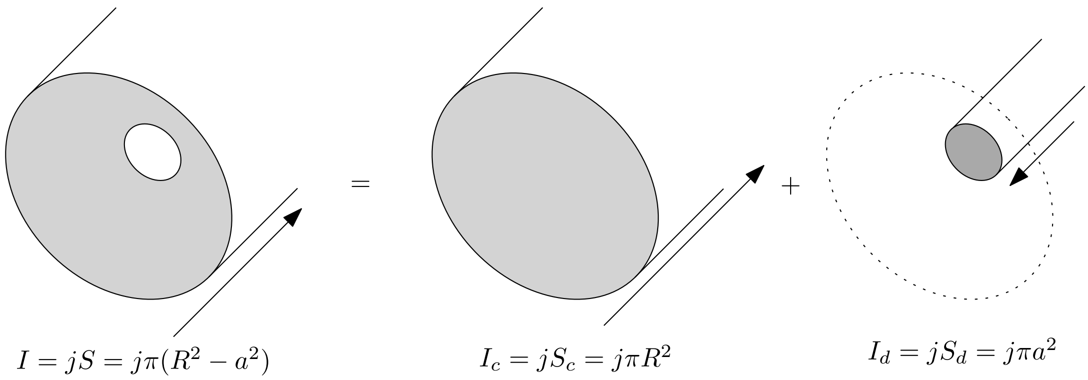
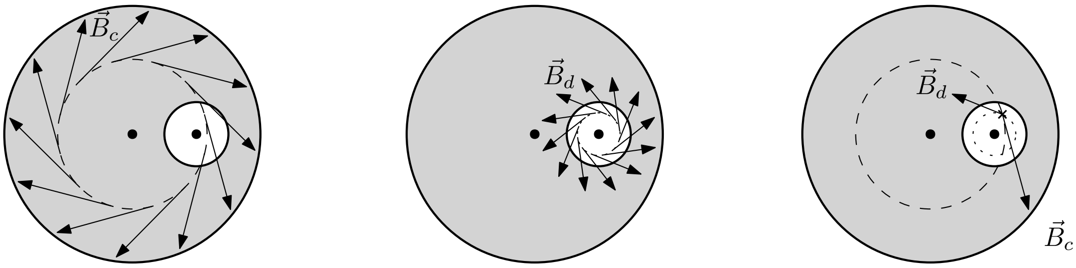


Figure 4.49: A drilled-hole conductor can be thought of as a superposition of a solid conductor and a conductor at the location of the drilled hole with current flowing in the opposite direction. The total currents through the formations I, I_c and I_d are given by the product of the constant current density j and the cross sections of the respective formations S, S_c and S_d.

We now determine the magnetic fields from these two configurations, $\vec{B}_c$ and $\vec{B}_d$, and add their values to obtain the resulting magnetic field from the original configuration, $\vec{B} = \vec{B}_c + \vec{B}_d$. We will use the result (4.58) in the appendix of Example 4.13 (see Section 4.4.1), which gives the result for the magnitude of the magnetic field inside the cylindrical conductor:

$$B(r) = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}, \tag{4.60}$$

where I is the total current flowing through the conductor, R is its radius, and r is the distance from its axis ($r < R$, considering only points inside the conductor). We, however, need the "full" vector expression $\vec{B}(r)$, since the contributions $\vec{B}_c$ and $\vec{B}_d$ from each current configuration do not generally point in the same direction, see Figure 4.50 (c).



(a) Field $\vec{B}_c$. (b) Field $\vec{B}_d$. (c) Field $\vec{B}_c$ and $\vec{B}_d$.

Figure 4.50: Contributions to the total magnetic field from the solid conductor $\vec{B}_c$ (left) and from "reverse current drilled hole" $\vec{B}_d$ (middle). On the right, the sum of $\vec{B}_c + \vec{B}_d$ in one place

.

We know that the magnetic field vectors $\vec{B}$ from the solid conductor are tangential (see Example 4.13 in Section 4.4.1), i.e., lying in a plane perpendicular to the cylinder and tangent to the imaginary circles centered on the conductor axis (illustrated in the left and middle figures 4.50). The full vector expression for the magnitude and direction of the magnetic field can then be written as follows:

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{\vec{n}\times\vec{r}}{R^2}, \tag{4.61}$$

where $\vec{n}$ is the unit vector pointing in the direction of the current, and $\vec{r}$ is the vector connecting the axis of a given conductor to the magnetic field location (verify that the magnitude of $|\vec{B}|$ is the same as (4.60) and the vector product by the right hand rule gives the correct direction).

Thus, the vectors $\vec{B}_c$ and $\vec{B}_d$ have the expression

$$\vec{B}_c = \frac{\mu_0 j\pi R^2}{2\pi}\frac{\vec{n}_c\times\vec{r}_c}{R^2} = \frac{\mu_0 j}{2}\,\vec{n}_c\times\vec{r}_c, \qquad \vec{B}_d = \frac{\mu_0 j\pi a^2}{2\pi}\frac{\vec{n}_d\times\vec{r}_d}{a^2} = \frac{\mu_0 j}{2}\,\vec{n}_d\times\vec{r}_d, \tag{4.62}$$

where we have given specific values for the currents I_c and I_d using the current density j. Furthermore, the vectors $\vec{n}_c$ and $\vec{n}_d$ give the directions of the currents, so for example $\vec{n}_c = -\vec{n}_d$ holds. Finally, the vector $\vec{r}_c$ connects the solid conductor axis, and the vector $\vec{r}_d$ connects the "drilled conductor" axis, to the magnetic field location, see Figure 4.51.

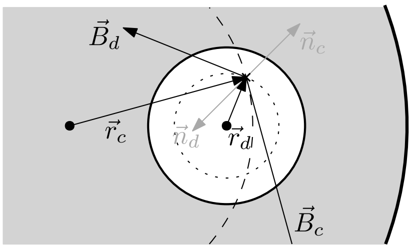


Figure 4.51: Vectors $\vec{r}_c$, $\vec{r}_d$, $\vec{n}_c$ and $\vec{n}_d$ for the determination of the magnetic field vectors $\vec{B}_c$ and $\vec{B}_d$. The vector $\vec{n}_c$ points in the direction "into the paper", the vector $\vec{n}_d$ points "out of the paper".

We now introduce Cartesian coordinates as in Figure 4.52.

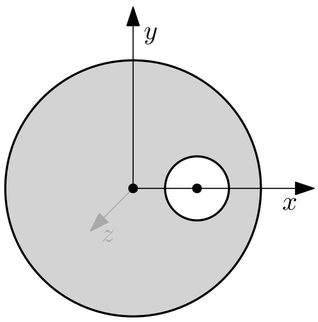


Figure 4.52: Cartesian coordinates (x, y, z) in the wire with the hole. The z axis points in the direction "out of the paper".

We write the vectors $\vec{n}_c$, $\vec{n}_d$, $\vec{r}_c$ and $\vec{r}_d$ in these coordinates. The direction of the current I_c is in the negative direction of the axis z and hence $\vec{n}_c = (0, 0, -1)$. The direction of current I_d is the opposite, so $\vec{n}_d = (0, 0, 1)$. Next, consider an arbitrary point inside the drilled hole with position vector $\vec{r} = (x, y, z)$. Then $\vec{r}_c = \vec{r} = (x, y, z)$ and $\vec{r_d} = (x - b, y, z)$ hold. Substituting these expressions into (4.62) we obtain:

$$\vec{B}_c = \frac{\mu_0 j}{2}(y, -x, 0), \qquad \vec{B}_d = \frac{\mu_0 j}{2}(-y, x - b, 0). \tag{4.63}$$

Adding them together gives the result:

$$\vec{B} = \vec{B}_c + \vec{B}_d = \frac{\mu_0 j}{2}(0, -b, 0) = \frac{\mu_0 j b}{2}(0, -1, 0). \tag{4.64}$$

Thus, the magnetic field in the whole space of the cavity is constant! Its magnitude is $B = \frac{\mu_0 j b}{2}$, where we could still substitute $j = \frac{I}{\pi(R^2 - a^2)}$ for the current density. After substituting specific numerical values, we have $B = 2,42.10^{-5}\,T$.

4.4.3 4.15 Solenoid

The solenoid has length $L = 30\,cm$ and diameter $d = 6\,cm$. There are 5 turns wound on $1\,cm$ ($n = 5\,t./cm$), the wire has a resistance of $\Omega_m = 0,01\,\Omega \cdot m^{-1}$ and is connected to $\mathcal{E} = 24\,V$. What will be the magnetic field inside the solenoid, the pressure on the wall and the power consumed?

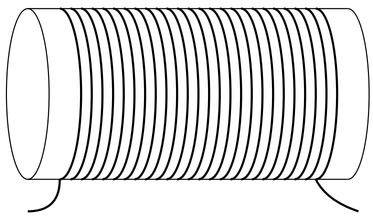


Figure 4.53: Solenoid.

Solution: If we know the formula for the magnitude of the magnetic field inside an infinitely long solenoid, $B = \mu_0 n I$ (n is the density of turns, I is the current flowing through the solenoid), the exercise is trivial. The total resistance of the wire is likely to be $R = \Omega_m \pi d n L$, where

$l = (\pi d)N$ is the total length of the wire and $N = nL$ is the number of turns of the solenoid. The current flowing through the solenoid is obtained from Ohm's law $I = \frac{\mathcal{E}}{R}$. The magnetic field is therefore

$$B = \mu_0 n I = \mu_0 n \frac{\mathcal{E}}{\Omega_m \pi d n L} = \frac{\mu_0 \mathcal{E}}{\Omega_m \pi d L} = 5,33\,.\,10^{-2}\,T. \tag{4.65}$$

The power consumed is given by Joule's heat $P = RI^2 = \frac{\mathcal{E}^2}{R}$:

$$P = \frac{\mathcal{E}^2}{\Omega_m \pi d n L} = 2037\,W. \tag{4.66}$$

The wall pressure is given by the relation for the pressure of a surface current (current sheet), where the magnetic fields from one side and the other are B_1 and B_2:

$$p = \frac{1}{2\mu_0}\left(B_1^2 - B_2^2\right). \tag{4.67}$$

In the solenoid we have $B_1 = B$ inside and $B_2 = 0$ outside. So the result is:

$$p = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0 \mathcal{E}}{\Omega_m \pi d L}\right)^2 = \frac{\mu_0 \mathcal{E}^2}{2(\Omega_m \pi d L)^2} = 1132\,Pa. \tag{4.68}$$

It can be shown that the pressure is exerted from inside the solenoid – so it tends to stretch the coil, using, for example, Ampère's formula for the force on a conductor with current $d\vec{F} = Id\vec{l} \times \vec{B}$.

Now what follows is the derivation of the formula $B = \mu_0 n I$. Let us first see what constraints the cylindrical symmetry problem places on the possible shapes of the magnetic field $\vec{B}$ around the solenoid. Let us introduce cylindrical coordinates (z, r, φ) such that the axis z is identical to the axis of the solenoid, see Figure 4.54.

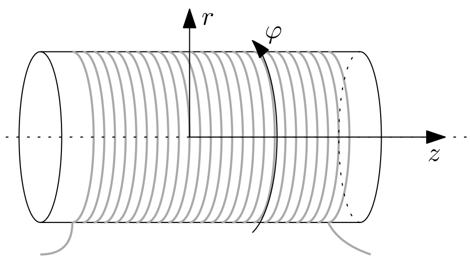


Figure 4.54: Cylindrical coordinates (z, r, φ) in the solenoid.

In general, the magnitude of the vector field of magnetic field $B(z, r, \varphi)$ could depend on all spatial variables. However, due to rotational symmetry about the axis, it cannot depend on φ and due to translational symmetry along the axis, it cannot depend on z. Thus we are left with only radial dependence, i.e. dependence on the distance from the solenoid axis, $B(r)$.

Where will the vector $\vec{B}$ point? We show that the magnetic field is so-called longitudinal – pointing in the direction of the solenoid axis. Let us take an arbitrary location and examine the contributions to the magnetic field from two symmetrically placed coil turns, see Figure 4.55. Although we do not know the exact form of the magnetic field from the current loop, the reflection symmetry about the plane of the circular loop implies that the magnetic fields from one and the other current loop, $\vec{B}_1$ and $\vec{B}_2$, add up to a longitudinal (longitudinal, pointing in the direction of the z axis) vector.

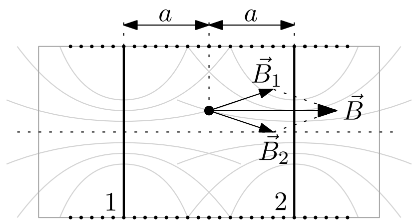


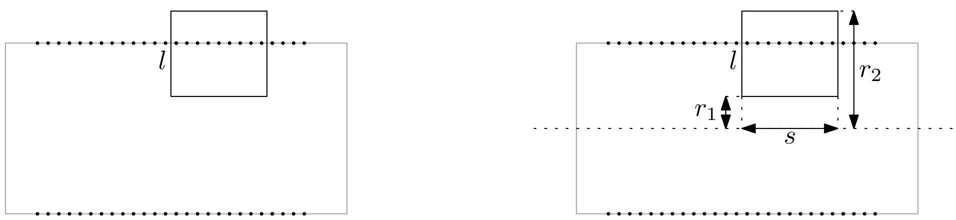
Figure 4.55: Contributions $\vec{B}_1$ and $\vec{B}_2$ to the magnetic field from two symmetrically placed coil turns. The approximate magnetic field lines from these individual coils are shown in grey.

We can now proceed to the actual determination of the magnetic field around the solenoid. To do this, we use Ampere's law

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 I_{in}, \tag{4.69}$$

which relates the circulation of magnetic field $\vec{B}$ along a closed curve l with the current I_{in} encircled by this curve.

To determine the magnetic field of the solenoid, we choose a rectangular loop whose two sides are perpendicular to the solenoid wall and whose other two sides are parallel to the solenoid axis. Let "width" of the rectangle be s and the distances of the lower and upper sides from the solenoid axis be r_1 and r_2. See Figures 4.56.



(a) Curve l in the solenoid for Ampere's law.

(b) Curve dimensions.

Figure 4.56: Curve l in Ampere's law.

There are $N = ns$ turns trapped inside the curve l and hence the total current flowing through the curve is $I_{in} = Ins$. We divide the integral over the whole rectangular loop into four integrals over each side:

$$\oint_l \vec{B} \cdot d\vec{l} = \int_{\text{left}} \vec{B} \cdot d\vec{l} + \int_{\text{bottom}} \vec{B} \cdot d\vec{l} + \int_{\text{right}} \vec{B} \cdot d\vec{l} + \int_{\text{top}} \vec{B} \cdot d\vec{l}. \tag{4.70}$$

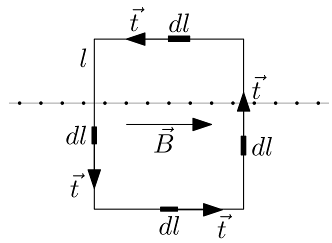


Figure 4.57: Length elements dl and tangent vectors $\vec{t}$ in each side of the rectangular loop l.

The directions of the unit tangent vectors $\vec{t}$ from the elements $d\vec{l} = \vec{t}\,dl$ in each part of the curve l are shown in Figure 4.57. The integrals over the left and right sides are zero due to the perpendicularity of the magnetic field vector $\vec{B}$ and the tangent vector to the curve $\vec{t}$, $\vec{B}\cdot d\vec{l}=0$, on the contrary, the scalar products under the integrals over the upper and lower sides give $\vec{B}\cdot d\vec{l}=-B\,dl$ and $B\,dl$, respectively:

$$\oint_l \vec{B}\cdot d\vec{l} = \int_{\text{bottom}} B\,dl - \int_{\text{top}} B\,dl. \tag{4.71}$$

The magnetic field depends only on the distance from the axis of the solenoid, $B(r)$. The segments of the upper and lower faces lie at constant values of the coordinate r. Thus the terms under the integrals are constant and can be factored out:

$$\oint_l \vec{B}\cdot d\vec{l} = B(r_1)\int_{\text{bottom}} dl - B(r_2)\int_{\text{top}} dl = [B(r_1)-B(r_2)]\,s; \tag{4.72}$$

we have used the relation $\int dl = s$ (the length of the upper and lower segments is s). Substituting the computed expressions for the circulation of magnetic field and the current enclosed in the loop into Ampere's law, we get:

$$[B(r_1)-B(r_2)]\,s = \mu_0 Ins \quad \to \quad B(r_1)-B(r_2)=\mu_0 In. \tag{4.73}$$

This equation implies that the magnetic field is constant inside and outside the solenoid and varies by a constant value $\mu_0 nI$. This fact is easiest to see if (4.73) is differentiated with respect to r_1, i.e., $\frac{\partial}{\partial r_1}$(4.73), or by r_2, $\frac{\partial}{\partial r_2}$(4.73), with the result[4]:

$$\frac{dB(r_1)}{r_1}=0 \quad \to \quad B(r_1)=\text{const.}, \qquad \frac{dB(r_2)}{r_2}=0 \quad \to \quad B(r_2)=\text{const.} \tag{4.74}$$

Let us now introduce the labels $B_{\text{inside}}=B(r_1)$ and $B_{\text{outside}}=B(r_2)$. Thus, it is valid

$$B_{\text{inside}}-B_{\text{outside}}=\mu_0 nI. \tag{4.75}$$

The last step is to show that $B_{\text{outside}}=0$. Let's leave this for the addendum. Then it will obviously be

$$B_{\text{inside}}=\mu_0 nI. \tag{4.76}$$

Addendum: Let's show that $B_{\text{outside}}=0$. The general strategy will be as follows. Since it is very difficult to calculate exactly the magnetic field outside the solenoid using the Biot-Savart law[5], we will try to get some upper bound for the magnitude of this field, $B_{\text{outside}}\le B_{\text{bound}}$. If $B_{\text{bound}}\le \frac{C}{(r-r_0)^\alpha}$ holds for this estimate, where $C>0$, $\alpha>0$, r_0 are constants and r is the distance from the solenoid axis, combining these inequalities gives the result that the exact field must decrease with distance from the solenoid axis: $B_{\text{outside}}\le\frac{C}{(r-r_0)^\alpha}$. But since we have already established that the field outside the solenoid is constant, the only value of this field consistent with its decay is zero[6], $B_{\text{outside}}=0$.

Let us find this estimate. We first estimate the magnitude of the magnetic field of a single coil turn, B_{turn}. Let us take the Biot-Savart law and estimate:

$$B_{\text{turn}}=\left|\frac{\mu_0 I}{4\pi}\int_l \frac{d\vec{l}\times\vec{R}}{R^3}\right| \le \frac{\mu_0 I}{4\pi}\int_l \frac{|d\vec{l}\times\vec{R}|}{R^3} \le \frac{\mu_0 I}{4\pi}\int_l \frac{dl}{R^2}. \tag{4.77}$$

[4]The same result can, of course, be obtained without derivation. The values r_1 and r_2 are completely independent of each other. For example, we can hold r_2 fixed, then $B(r_1)=B(r_2)+\mu_0 nI=\text{const.}$ and vice versa

[5]Also, this is why we have used Ampere's law in this example.

[6]If the field outside the solenoid were not zero $B_{\text{outside}}=B_0\neq 0$, we would always find a distance r_v where our estimate $B_{\text{outside}}\le\frac{C}{(r-r_0)^\alpha}$ would be violated, specifically for $r_v>r_0+\left(\frac{C}{B_0}\right)^{1/\alpha}$.

We used the estimates $\left|\int d\vec{B}\right| \leq \int dB$ and $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\alpha \leq |\vec{a}||\vec{b}|$. Next, we need to estimate the distance R between the parts of a coil turn and the magnetic field location, see Figure 4.58.

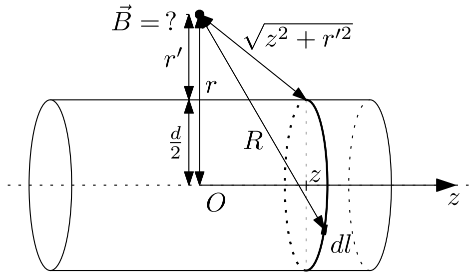


Figure 4.58: Distances required to estimate the magnetic field from a single turn.

Let the coil be located on the z-axis at a general coordinate value z, then we can estimate the distance as follows: $R \geq \sqrt{z^2 + \left(r - \frac{d}{2}\right)^2}$ – that is, we have reduced all distances to the distance of the closest piece of the coil turn, again see Figure 4.58. Purely for simplicity of notation, we introduce the distance $r' = r - \frac{d}{2}$ as the distance from the solenoid wall: $R \geq \sqrt{z^2 + r'^2}$. Putting all this info into our estimate and further manipulating:

$$B_{\text{turn}} \leq \frac{\mu_0 I}{4\pi} \int_l \frac{dl}{R^2} \leq \frac{\mu_0 I}{4\pi} \int_l \frac{dl}{z^2 + r'^2} = \frac{\mu_0 I}{4\pi} \frac{1}{z^2 + r'^2} \int_l dl = \frac{\mu_0 I d}{4} \frac{1}{z^2 + r'^2}, \tag{4.78}$$

where we have removed all constants from the integral and integrated $\int dl = \pi d$. The total magnetic field B_{outside} is obtained by integration over all turns:

$$\vec{B}_{\text{outside}} = \int_{\text{turns}} \vec{B}_{\text{turn}}\, dN \quad \to \quad B_{\text{outside}} \leq \int_{\text{turns}} B_{\text{turn}}\, dN = \int_{-\infty}^{\infty} B_{\text{turn}}(z)\, n\, dz, \tag{4.79}$$

where $dN = n\, dz$ is the number of turns of a piece of coil of width dz. Again, we used the estimate $\left|\int d\vec{B}\right| \leq \int dB$. After substituting from (4.78):

$$B_{\text{outside}} \leq \int_{-\infty}^{\infty} \frac{\mu_0 I n d}{4} \frac{1}{z^2 + r'^2} dz = \frac{\mu_0 I n d}{4} \frac{1}{r'^2} \int_{-\infty}^{\infty} \frac{dz}{1 + \left(\frac{z}{r'}\right)^2}. \tag{4.80}$$

By calculating the integral

$$\int_{-\infty}^{\infty} \frac{dz}{1 + \left(\frac{z}{r'}\right)^2} = \left[r' \operatorname{arctg}\left(\frac{z}{r'}\right)\right]_{-\infty}^{\infty} = \pi r' \tag{4.81}$$

we get the resulting estimate (upper bound):

$$B_{\text{outside}} \leq \frac{\mu_0 \pi I n d}{4} \frac{1}{r'} = \frac{C}{r'} = \frac{C}{r - \frac{d}{2}}, \tag{4.82}$$

where $C = \frac{\mu_0 \pi I n d}{4}$. This implies, as mentioned at the beginning, that $B_{\text{outside}} = 0$.

4.5 Magnetic dipole

4.5.1 4.16 Earth's magnetic dipole

The Earth's magnetic field at the North Pole has an induction of magnitude $B = 6{,}20 \cdot 10^{-5}\,T$ and its vector points perpendicularly towards the Earth. Determine the magnitude of the

Earth's magnetic dipole moment and the current that would have to flow along the equator to induce such a moment.

Solution: The magnetic field $\vec{B}$ from the magnetic dipole moment is given by the following formula:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right). \tag{4.83}$$

This relation describes the magnetic field far from the flowing currents, where the dipole contribution to the field dominates and the higher multipole moments are already negligible. Thus, we imagine that the Earth's magnetic field originates somewhere in the middle at its core, and the location at the north pole is already far enough away for the dipole approximation to be sufficiently accurate.

We now introduce Cartesian coordinates with the origin at the center of the Earth and orient the axis z in the direction of the North Pole P, see Figure 4.59. The pole P thus has a position vector $\vec{r}_P = (0, 0, R_Z)$.

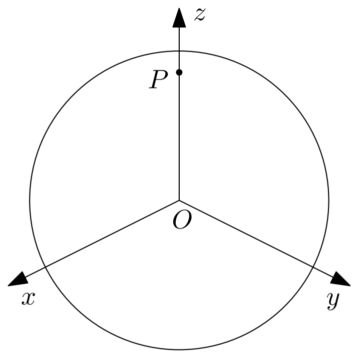


Figure 4.59: Cartesian coordinates with origin at the center of the Earth and axis z pointing in the direction of the North Pole.

Thus, a magnetic field pointing perpendicular to the Earth is directed against the direction of the position vector $\vec{r}$. The formula (4.83) consists of two parts – the first part points in the direction of the vector $\vec{r}$ and the second in the direction of $\vec{m}$. For the resulting magnetic field $\vec{B}$ (which is of the form $\vec{B} = \alpha\vec{r} + \beta\vec{m}$) to point in/against the direction $\vec{r}$, the condition $\vec{m} \parallel \vec{r}$ is needed. Thus, for $\vec{r} = \vec{r}_P$ we have $\vec{m} = (0, 0, m)$. At the pole, the magnetic field $\vec{B}$ is then equal to

$$\vec{B}(\vec{r}_P) = \frac{\mu_0}{4\pi}\left(\frac{3mR_Z}{R_Z^5}(0, 0, R_Z) - \frac{1}{R_Z^3}(0, 0, m)\right) = \frac{2\mu_0 m}{4\pi R_Z^3}(0, 0, 1). \tag{4.84}$$

For $m > 0$, the vector $\vec{B}$ points away from the Earth and so $\vec{m} = (0, 0, -m)$ must be considered. The value of the magnetic dipole moment m is obtained from the vector equation (4.84) by calculating the magnitudes of the left and right sides:

$$B = \frac{\mu_0 m}{2\pi R_Z^3} \quad \rightarrow \quad m = \frac{2\pi R_Z^3 B}{\mu_0}. \tag{4.85}$$

The additional question in the assignment, how much current I would have to flow along the equator to induce the dipole moment m calculated above, makes little sense. For a circle with a current of radius R_Z, the formula (4.83) will not give the correct result, since the size of the region with currents is on the order of the distance of that region to the north pole – the dipole approximation will not be accurate. But we can calculate the exact result for a current flowing along the equator inducing a specified magnetic field, and compare it to the value of the field that comes out if we plug the same current into the dipole field formula.

The magnitude of the magnetic field on the axis of a circular loop with a current I of radius r and height h is

$$B_{\text{circle}} = \frac{\mu_0 I}{2} \frac{r^2}{(r^2+h^2)^{3/2}}; \tag{4.86}$$

(can be determined from the Biot-Savart law). After inserting $r = h = R_Z$ we have

$$B_{\text{circle}} = \frac{\mu_0 I}{4\sqrt{2}R_Z}. \tag{4.87}$$

Thus, the current for a given magnetic field B is (for $R_Z = 6378\,km$)

$$I = \frac{1}{\mu_0} 4\sqrt{2}\,BR_Z = 1,78.10^9\,A. \tag{4.88}$$

A planar loop with current I encircling a surface S has a magnetic dipole moment $m = IS$; in our case, $S = \pi R_Z^2$. Substituting this relation into the left-hand formula in (4.85) and the current calculated above, we obtain for the magnetic field at the pole in the dipole approximation

$$B_{\text{dip}} = \frac{\mu_0 IS}{2\pi R_Z^3} = \frac{\mu_0 I}{2R_Z} = 1,75.10^{-3}\,T, \tag{4.89}$$

which is substantially different from that given, thus confirming our suspicion that the second part of the assignment makes little sense.

4.6 Lorentz force

4.6.1 4.6 Perpendicular fields

What resultant force acts on a charged particle moving at velocity $v = E/B$ in mutually perpendicular electric and magnetic fields such that the vectors $\vec{E}$, $\vec{B}$, and $\vec{v}$ form an orthogonal right-handed system?

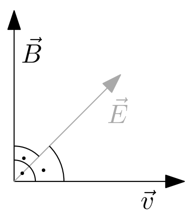


Figure 4.60: A charged particle in mutually perpendicular fields.

Solution: The formula for the Lorentz force acting on a charged particle with charge q in electric and magnetic fields $\vec{E}$ and $\vec{B}$ is as follows:

$$\vec{F}_L = q\left(\vec{E} + \vec{v}\times\vec{B}\right) = \vec{F}_E + \vec{F}_B. \tag{4.90}$$

The magnitudes of the electric and magnetic forces F_E and F_B are equal because, due to the perpendicularity of the vectors $\vec{v}$ and $\vec{B}$, the following holds

$$F_B = qvB = q\frac{E}{B}B = qE = F_E. \tag{4.91}$$

For $q > 0$, the force $\vec{F}_E$ points in the direction of the vector $\vec{E}$. According to the right-hand rule, we determine that the force $\vec{F}_B = q\,\vec{v} \times \vec{B}$ is directed against the direction of the vector $\vec{E}$. For $q < 0$ ditto just the other way around. The resulting force on the charged particle is then zero, $\vec{F}_L = 0$.

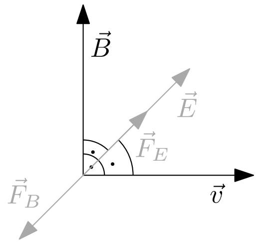


Figure 4.61: Directions of the electric and magnetic forces $\vec{F}_E$ and $\vec{F}_B$ for $q > 0$.

4.6.2 4.21 Circular motion in a magnetic field

A deuteron moves along a circle of radius $r = 40\,cm$ in a magnetic field $B = 1,5\,T$. Determine the velocity, energy, and orbital period of the deuteron.

Solution: We have "choice" to calculate the example relativistically or non-relativistically. Let's do both and compare the procedure and results. The magnetic part of the Lorentz force acting on the particle is

$$\vec{F}_B = q\,\vec{v} \times \vec{B}. \tag{4.92}$$

This force acts on the right-hand side of the relativistic, or non-relativistic, equation of motion:

$$\frac{d}{dt}\left(m_0\gamma\vec{v}\right) = \vec{F}_B, \qquad \frac{d}{dt}(m_0\vec{v}) = \vec{F}_B; \tag{4.93}$$

where we introduces factor $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$ and put $c \to +\infty$ for the non-relativistic equation (effectively $\gamma = 1$). The magnetic force plays the role of a centripetal force causing the circular motion. What does this force look like in the relativistic case? The centripetal acceleration is a purely kinematic quantity, so its expression does not change in relativity – $a_d = \frac{v^2}{r}$. The force $\vec{F}_B$ acting at any instant perpendicular to the velocity $\vec{v}$ does not change its magnitude, i.e. $v =$ konst., and hence $\gamma =$ konst., and we can manipulate the relativistic equation of motion in the following way:

$$\frac{d}{dt}(m_0\gamma\vec{v}) = m_0\gamma\frac{d}{dt}\vec{v} = m_0\gamma\vec{a} = \vec{F}. \tag{4.94}$$

The acceleration in this equation is just the centripetal acceleration causing the circular motion, so we see that the relativistic centripetal force is $\vec{F}_d = m_0\gamma\vec{a}_d$. Now we can write the equations relating the magnetic and centripetal forces:

$$qvB = m_0\gamma\frac{v^2}{r} = \frac{m_0 v^2}{r\sqrt{1 - \frac{v^2}{c^2}}}, \qquad qvB = m_0\frac{v^2}{r}. \tag{4.95}$$

From these equations we need only express the velocity v, and after some simple manipulations we obtain

$$v_r = \frac{1}{\sqrt{\frac{1}{c^2} + \frac{m_0^2}{r^2q^2B^2}}} = \frac{c}{\sqrt{1 + \left(\frac{m_0 c}{qBr}\right)^2}}, \qquad v_{nr} = \frac{qBr}{m_0}, \tag{4.96}$$

where we have introduced the notation v_r and v_{nr}, respectively, for the relativistic and non-relativistic results. For $c \to +\infty$, i.e., in the non-relativistic limit, the expression for v_r transitions to v_{nr}. Substituting the numerical values (mass of the deuteron $m_d = 3,343.10^{-27}\,kg$, speed of light $c = 3.10^8\,m.s^{-1}$, elementary electric charge $e = 1,607.10^{-19}\,C$) we get

$$v_r = 2,871.10^7\,m.s^{-1}, \qquad v_{nr} = 2,884.10^7\,m.s^{-1}. \tag{4.97}$$

We get essentially identical results – so the non-relativistic approximation is fine in this case (factor $\beta = \frac{v}{c} < 0,1$ and $\gamma \approx 1$).

The kinetic energy is then relativistically and non-relativistically

$$E_{Kr} = (\gamma-1)m_0c^2 = 1,39.10^{-12}\,J = 8,63\,MeV, \quad E_{Knr} = \frac{1}{2}m_0v_{nr}^2 = 1,39.10^{-12}\,J = 8,65\,MeV. \tag{4.98}$$

The orbit time is obtained from the simple kinematic relation

$$T_r = \frac{2\pi r}{v_r} = 8,75.10^{-8}\,s, \qquad T_{nr} = \frac{2\pi r}{v_{nr}} = 8,71.10^{-8}\,s. \tag{4.99}$$

4.6.3 4.20 Motion in magnetic field along a helix

An electron flies into a homogeneous magnetic field with a velocity of $v = 5.10^6\,m.s^{-1}$ and starts moving along a helix of radius $r = 5\,cm$ and pitch $s = 30\,cm$. Determine the magnitude of the magnetic field.

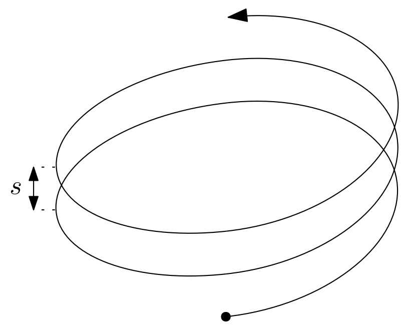


Figure 4.62: Motion of a particle in a magnetic field along a helix.

Solution: The magnetic part of the Lorentz force acting on the particle is

$$\vec{F}_B = q\,\vec{v} \times \vec{B}. \tag{4.100}$$

If we split the particle's velocity vector $\vec{v}$ into a component perpendicular and parallel to the magnetic field, $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$ (see Figure 4.63), we obtain an expression for the force $\vec{F}_B$ and its magnitude

$$\vec{F}_B = q(\vec{v}_\perp + \vec{v}_\parallel) \times \vec{B} = q\,\vec{v}_\perp \times \vec{B}, \qquad F_B = qv_\perp B. \tag{4.101}$$

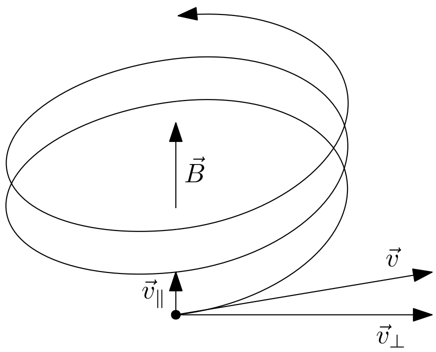


Figure 4.63: Decomposition of the velocity $\vec{v} = \vec{v}_{\perp} + \vec{v}_{\|}$ into a direction perpendicular and parallel to the magnetic field $\vec{B}$.

The motion along the helix can be decomposed into circular motion at orbital velocity $v_{\perp}$ and upward motion at velocity $v_{\|}$. The magnetic force always acts perpendicular to the velocity $\vec{v}_{\perp}$ and therefore plays the role of centripetal force for the circular motion – hence:

$$qv_{\perp}B = m\frac{v_{\perp}^2}{r}. \tag{4.102}$$

Expressing the magnetic field from the equation B we get

$$B = \frac{mv_{\perp}}{qr}. \tag{4.103}$$

Now we just need to find the relation for $v_{\perp}$ in terms of v, s and r (the pitch s represents the distance the particle travels in the "longitudinal" direction in one revolution of the circular motion). Take one revolution of the helix and "unroll it" into the plane. This produces a right-angled triangle with legs of sizes s and $2\pi r$, see Figure 4.64.

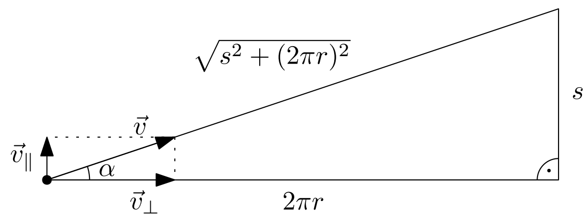


Figure 4.64: Motion of a particle in a magnetic field along a helix.

From the similarity of the triangles, we get

$$\frac{v_{\perp}}{v} = \frac{2\pi r}{\sqrt{s^2 + (2\pi r)^2}} \quad \rightarrow \quad v_{\perp} = \frac{v}{\sqrt{1 + \left(\frac{s}{2\pi r}\right)^2}}. \tag{4.104}$$

Substituting into (4.103) gives the result

$$B = \frac{m}{qr}\frac{v}{\sqrt{1 + \left(\frac{s}{2\pi r}\right)^2}} = \frac{mv}{q\sqrt{r^2 + \left(\frac{s}{2\pi}\right)^2}}. \tag{4.105}$$

For specific numerical values (electron mass $m_e = 9,109.10^{-31}\,kg$, elementary electric charge $e = 1,602.10^{-19}\,C$) we have $B = 4,11.10^{-4}\,T$.

Chapter 5

Electromagnetic field

5.1 Electromagnetic induction

5.1.1 5.2 Induction on rails

Two long perfectly conducting rails are spaced apart $d = 0,5\,m$ and connected by a resistor $R = 0,2\,\Omega$. A perfectly conducting rod slides along them at speed $v = 4\,m.s^{-1}$. A magnetic field $B = 0,5\,T$ is applied perpendicular to the plane of the rails. Determine the induced voltage, the force required to maintain a constant velocity, and the mechanical and thermal power generated in this device.

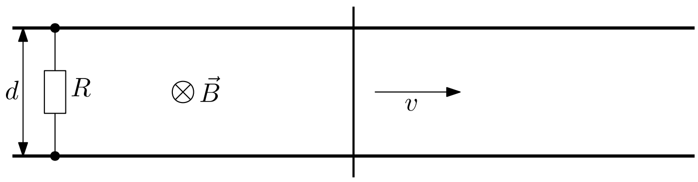


Figure 5.1: Induction on rails.

Solution: Let's use Faraday's law of electromagnetic induction to determine the induced voltage $\mathcal{E}_{ind}$:

$$\mathcal{E}_{ind} = -\frac{d\Phi}{dt}. \tag{5.1}$$

We need to determine the magnetic field flux Φ, which is given by

$$\Phi = \int_S \vec{B} \cdot d\vec{S}, \tag{5.2}$$

where S is the area bounded by the loop in which we want to determine the induced voltage. In our case, we choose a rectangle bounded by rails, a bar, and a resistor; see Figure 5.2.

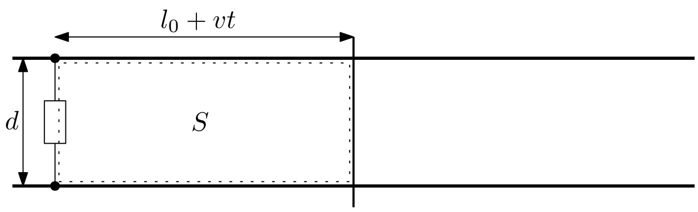


Figure 5.2: Surface S for calculating the magnetic field flux Φ.

The normal vector $\vec{n}$ to this rectangle points to the paper (ground) and is therefore parallel to the specified magnetic field $\vec{B}$. Given $d\vec{S} = \vec{n}\, dS$, we get for the scalar product $\vec{B} \cdot d\vec{S} = BdS$. Moreover, the magnetic field $\vec{B}$ is constant everywhere in space and can be factored out from the integral (5.2). Thus, the flux Φ calculation is

$$\Phi = \int_S \vec{B} \cdot d\vec{S} = \int_S B\, dS = B \int_S dS = BS, \tag{5.3}$$

where S on the right-hand side denotes the area of the rectangle in Figure 5.2. Obviously $S = S(t) = (l_0 + vt)d$ where l_0 denotes the distance of the rod from the resistor at time $t = 0\, s$. We now differentiate the flux Φ with respect to time according to (5.1) and obtain the induced voltage:

$$|\mathcal{E}_{ind}| = \frac{d\Phi}{dt} = B\frac{dS}{dt} = Bvd. \tag{5.4}$$

After substituting the given numerical values, we have $\mathcal{E}_{ind} = 1\, V$.

The heat output P_{heat} is given by

$$P_{heat} = \frac{\mathcal{E}_{ind}^2}{R} = \frac{(Bvd)^2}{R} = 5\, W. \tag{5.5}$$

This, by the law of conservation of energy, must equal the mechanical power P_{mech} supplied to "equipment". Thus $P_{mech} = P_{heat} = 5\, W$. This mechanical power is due to the force acting on the rod, which is required to maintain a constant rod velocity. Valid $P_{mech} = Fv$, and hence $F = \frac{(Bd)^2 v}{R} = 1,25\, N$.

Addendum: The induced voltage and the force required to maintain a constant velocity can also be calculated here using the Lorentz force (this procedure cannot be used if the magnetic field is time-varying)

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \stackrel{\vec{E}=0}{=} q\, \vec{v} \times \vec{B}. \tag{5.6}$$

The free charges in the rod are forced to move at a velocity $\vec{v}$. According to the right-hand rule, we determine the direction of the Lorentz force $\vec{F}_L$ acting on the charges in the bar (we consider that the positive charges $q > 0$ form the conductivity of the bar, so as not to complicate the discussion with additional signs):

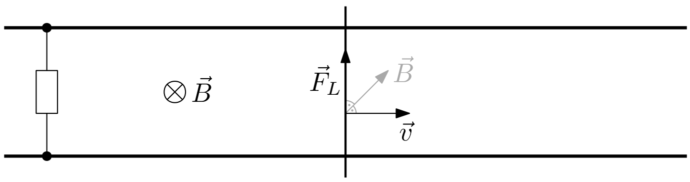


Figure 5.3: Lorentz force acting on the charges in the rod moving at $\vec{v}$.

We now use the definition of voltage

$$U = \frac{1}{q} \int_l \vec{F} \cdot d\vec{l}. \tag{5.7}$$

The voltage U in the circuit here is generated just by the effect of the Lorentz force $\vec{F}_L$ acting on the charges in the moving rod. We will therefore integrate along the rod from one rail to the other. The Lorentz force $\vec{F}_L$ points in the direction of the tangent vector $\vec{t}$ to the bar (see Figure 5.4), so $\vec{F}_L$ has the same direction as $d\vec{l} = \vec{t}\, dl$ and $\vec{F}_L \cdot d\vec{l} = F_L dl$ holds.

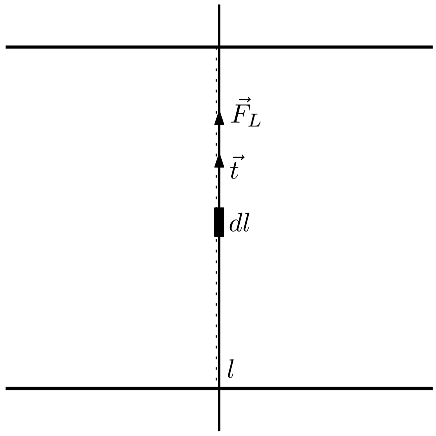


Figure 5.4: Integration curve l to determine the voltage U with the vectors $\vec{F}_L$ and $\vec{t}$ marked.

Since the force $F_L = qvB$ (from (5.6) and the directions of vectors $\vec{v}$ and $\vec{B}$ in Figure 5.3) is constant along the bar, we get

$$U = \frac{1}{q}\int_l \vec{F}_L \cdot d\vec{l} = \frac{1}{q}\int_l F_L dl = \frac{1}{q}F_L \int_l dl = \frac{1}{q}F_L d = Bvd. \tag{5.8}$$

Due to the induced voltage U, a current I starts to flow through the circuit, causing additional movement of charges in the rod. Thus, the charges also move at a velocity $\vec{v}_d$ in the direction of the rod, which induces an additional Lorentz force acting against the direction of the rod motion – according to the right hand rule, see Figure 5.5

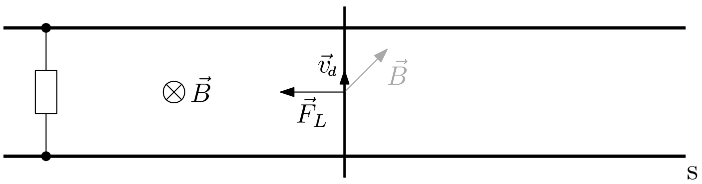


Figure 5.5: Additional Lorentz force due to the current I decelerating the rod.

The magnitude of this force is obtained using Ampère's formula

$$d\vec{F} = I d\vec{l} \times \vec{B}, \tag{5.9}$$

where $d\vec{F}$ denotes the force on a small element of the rod dl. The formula (5.9) is derived from the formula for the Lorentz force, so the direction of the force is the same as in the figure above – the role of $d\vec{l}$ is played by the velocity $\vec{v}_d$. Along the entire bar between the rails, the contribution to the total force $d\vec{F}$ is constant – $d\vec{l}$ and $\vec{B}$ are always the same direction and magnitude. The magnitude of this contribution is $dF = IBdl$ (from the perpendicularity of vectors $d\vec{l}$ and $\vec{B}$ and formula (5.9)). The total magnitude of the force F is then

$$F = \int_l dF = \int_l IBdl = IB\int_l dl = IBd = \frac{U}{R}Bd = \frac{(Bd)^2 v}{R}. \tag{5.10}$$

A force of the same magnitude but opposite orientation must then be applied to the bar to keep it from slowing down.

Note that it is generally not true that

$$F = \int dF. \tag{5.11}$$

The total magnitude of the vector $\vec{F}$ can only be calculated in this way if all the contributions $d\vec{F}$ point in the same direction (which is satisfied in our case). In the general case, the vector integral cannot be avoided and the following holds

$$F = \left| \int d\vec{F} \right|. \tag{5.12}$$

5.1.2 5.1 Moving loop

A current flows through a long straight conductor I. Determine the magnetic flux through the rectangular loop located as shown. If the loop moves away from the conductor at a speed of v determine the induced voltage.

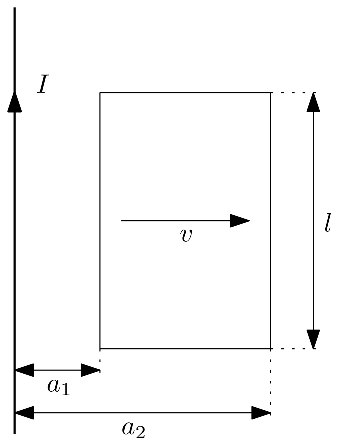


Figure 5.6: Rectangular loop moving in a magnetic field.

Solution: The magnetic field flux is given by

$$\Phi = \int_S \vec{B} \cdot d\vec{S}. \tag{5.13}$$

The magnetic field $\vec{B}$ in this example is generated by an (infinitely) long straight conductor with current I and its magnitude is

$$B = \frac{\mu_0 I}{2\pi r}, \tag{5.14}$$

where r is the distance from the conductor (this relation can be obtained from the integral Ampere's law or by using the Biot-Savart law). We determine the direction of the magnetic field vectors $\vec{B}$ from the right hand rule – the thumb points in the direction of the current, the fingers point in the direction of the magnetic field, i.e. $\vec{B}$ is perpendicular to the paper and points into it, see Figure 5.7.

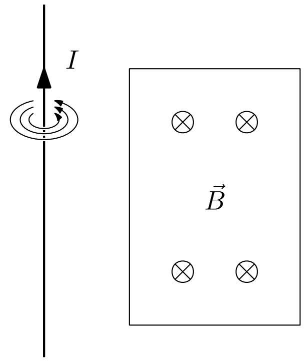


Figure 5.7: Direction of magnetic field $\vec{B}$ from an infinitely long straight conductor in the plane of the loop.

The surface for the integral (5.13) is naturally chosen to be the rectangle bounded by a rectangular loop. We choose the normal vector $\vec{n}$ to point to the paper. It holds that the vectors $\vec{B}$ and $d\vec{S} = \vec{n}\,dS$ are parallel, and hence $\vec{B} \cdot d\vec{S} = BdS$. We introduce the Cartesian coordinates r and z as shown in Figure 5.8.

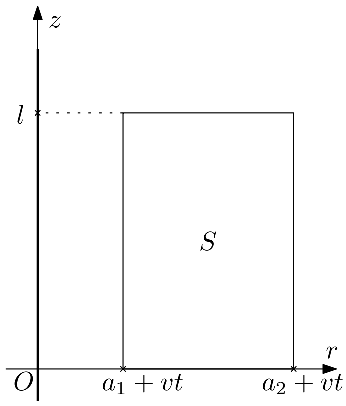


Figure 5.8: Cartesian coordinates r and z.

We see that the rectangular loop is parameterized by the following coordinate ranges: $r \in \langle a_1 + vt, a_2 + vt \rangle$ (the loop moves to the right at constant speed v) and $z \in \langle 0, h \rangle$. The surface element in Cartesian coordinates is simply $dS = drdz$. We have introduced the origin of the coordinates and the coordinate name r so that the magnitude of the magnetic field expressed in (r, z) has the same form as in (5.14). We can now construct the given surface integral and calculate it:

$$\Phi(t) = \int_S BdS = \int_{a_1+vt}^{a_2+vt} \int_0^l \frac{\mu_0 I}{2\pi r}\, dzdr = \frac{\mu_0 I}{2\pi} \int_{a_1+vt}^{a_2+vt} \frac{dr}{r} \int_0^l dz = \frac{\mu_0 I}{2\pi} \ln\left(\frac{a_2 + vt}{a_1 + vt}\right) l \tag{5.15}$$

The flux Φ in time $t = 0\,s$ is obtained by a simple substitution:

$$\Phi(t = 0) = \frac{\mu_0 I l}{2\pi} \ln \frac{a_2}{a_1}. \tag{5.16}$$

We calculate the induced voltage $\mathcal{E}_{ind}$ from Faraday's law of electromagnetic induction as the time derivative of the magnetic flux Φ:

$$\mathcal{E}_{ind}(t) = -\frac{d\Phi}{dt} = -\frac{\mu_0 I l}{2\pi} \cdot \frac{a_1 + vt}{a_2 + vt} \cdot \frac{v(a_1 + vt) - (a_2 + vt)v}{(a_1 + vt)^2} = \frac{\mu_0 I l}{2\pi} \frac{v(a_2 - a_1)}{(a_1 + vt)(a_2 + vt)}. \tag{5.17}$$

And the voltage $\mathcal{E}_{ind}$ at time zero:

$$\mathcal{E}_{ind}(t=0) = \frac{\mu_0 I l}{2\pi}\frac{v(a_2-a_1)}{a_1 a_2}. \tag{5.18}$$

Addendum: Just for illustration, we calculate the flux at time $t = 0\,s$ for the origin of the coordinates located not on the wire but in the lower left corner of the loop:

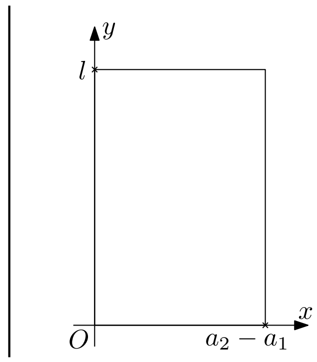


Figure 5.9: Introduce coordinates x and y with the origin in the lower left corner of the loop at time $t = 0\,s$.

In this case, the rectangle is at coordinates $x \in \langle 0, a_2 - a_1\rangle$ and $y \in \langle 0, l\rangle$. The distance from the wire is $r = a_1 + x$. The integral for the flow Φ then looks like this:

$$\Phi(t=0) = \int_S BdS = \int_0^{a_2-a_1}\int_0^l \frac{\mu_0 I}{2\pi(a_1+x)}dxdy. \tag{5.19}$$

The result is of course the same as in the previous coordinates (substitution $r = a_1 + x$). We would calculate the flux Φ at arbitrary time by changing the integration limits: $x \in \langle vt, a_2 - a_1 + vt\rangle$.

5.1.3 5.7 and 5.8 Rotating coils

A square loop with side $a = 10\,cm$ rotates in a homogeneous magnetic field $B = 0,2\,T$ about an axis parallel to the plane of the square and perpendicular to the field with frequency $50\,Hz$. At instant $t = 0$ the loop lies in a plane perpendicular to the field. Determine the time dependence of the induced voltage.

What is the maximum voltage that can be induced in a coil with $N = 4000$ turns of mean radius $R = 12\,cm$ rotating with frequency $f = 30\,Hz$ in an earth magnetic field of induction $B = 5 \cdot 10^{-5}\,T$?

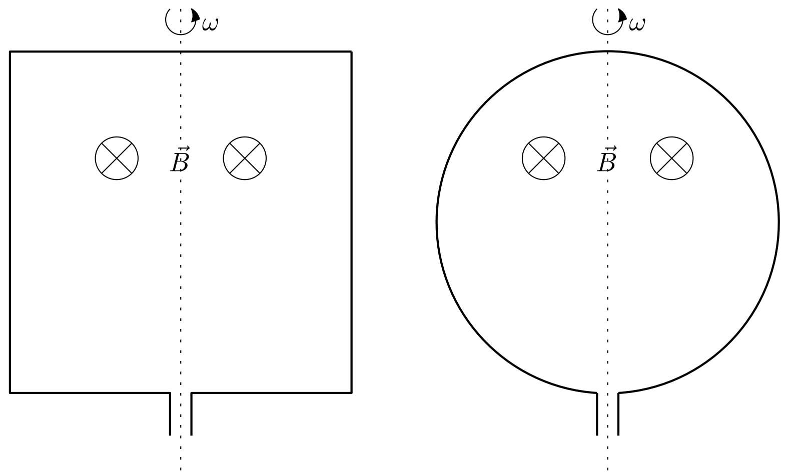


Figure 5.10: Rotating square loop and circular coil.

Solution: These exercises are almost identical and we will solve them both at once. First we calculate the voltage induced on the simple loop, we will show how the situation changes for N turns later.

We will use Faraday's law of electromagnetic induction

$$\mathcal{E}_{ind} = -\frac{d\Phi}{dt}. \tag{5.20}$$

Thus we need to calculate the magnetic field flux:

$$\Phi = \int_S \vec{B} \cdot d\vec{S}. \tag{5.21}$$

Let us now look at the situation with the rotating loop in top view in Figure 5.11.

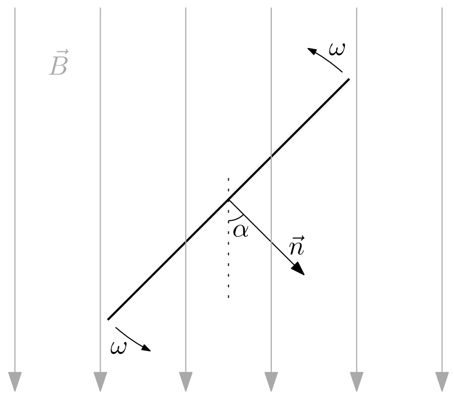


Figure 5.11: View of the rotating loop in top view.

We choose the surface S for integration as a square, or circle, whose boundary is the square loop, or the circle loop. The normal vector then rotates with the loop and there is a time-varying angle $\alpha(t) = \alpha_0 + \omega t$ between the magnetic field and the normal vector (for $t = 0$ we have $\alpha = \alpha_0$ and hence the requirement of perpendicularity of the coil plane to the magnetic field direction in the assignment leads to $\alpha_0 = 0$). The scalar product under the integral takes the form $\vec{B} \cdot d\vec{S} = \vec{B} \cdot \vec{n}\, dS = B \cos\alpha\, dS = B \cos\omega t\, dS$. The magnetic field is constant everywhere and we can factor it out from the integral. Similarly, the angle α is constant everywhere on the surface of integration. The flux calculation is therefore:

$$\Phi = \int_S \vec{B} \cdot d\vec{S} = \int_S B \cos \omega t \, dS = B \cos \omega t \int_S dS = BS \cos \omega t. \tag{5.22}$$

The induced voltage $\mathcal{E}_{ind}$ is obtained by differentiation:

$$\mathcal{E}_{ind} = -\frac{d\Phi}{dt} = BS \, \omega \sin \omega t. \tag{5.23}$$

For the square loop case we have $S = a^2$, for the circular loop case $S = \pi R^2$. How does the result change if the coil consists of N turns? The individual turns of a coil winding are connected in series, see illustration 5.12.

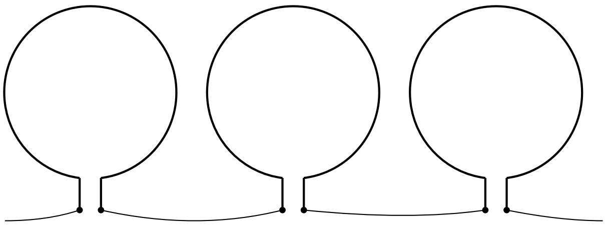


Figure 5.12: The turns wound on a coil are connected in series.

Thus, the voltages induced on each of the turns, $\mathcal{E}_{ind}^{(i)} = -d\Phi^{(i)}/dt$ (where $\Phi^{(i)}$ is the flux through i-th turn), add up:

$$\mathcal{E}_{ind}^{(celk)} = \sum_{i=1}^{N} \mathcal{E}_{ind}^{(i)} = -\sum_{i=1}^{N} \frac{d\Phi^{(i)}}{dt}. \tag{5.24}$$

However, the turns are stacked on top of each other, so they share the same flux,

$$\Phi^{(1)} = \Phi^{(2)} = \ldots = \Phi^{(N)} = \Phi. \tag{5.25}$$

The total voltage is then N-times the voltage on one turn:

$$\mathcal{E}_{ind}^{(celk)} = N\mathcal{E}_{ind}^{(1)} = NBS \, \omega \sin \omega t. \tag{5.26}$$

The induced voltage has a harmonic waveform and its maximum value is given by the amplitude of this oscillatory wave:

$$\mathcal{E}_{ind}^{(max)} = NBS \, \omega. \tag{5.27}$$

5.1.4 5.6 Homopolar generator

A metal disc of radius $R = 10 \, cm$ rotates with frequency $f = 60 \, Hz$ about its axis in a homogeneous magnetic field $B = 0,2 \, T$ perpendicular to the plane of the disc. Find the potential difference between the centre and the edge of the disc. What will this difference be without the magnetic field?

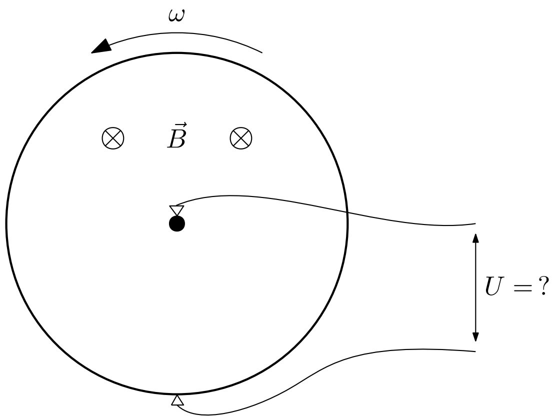


Figure 5.13: Homopolar generator.

Solution: In this case Faraday's law of electromagnetic induction is **not applicable** . This law, although very universal and encompassing several principles of voltage induction, can only be used for circuits that consist of so-called thin conductors (they do not have to be physically thin, only their length dimension must predominate over the other dimensions). It is necessary to have a clearly defined boundary (formed by the circuit) of the surface through which we calculate the magnetic flux. In our case, part of the circuit is formed by a metal disc, where the boundary of the surface is not clearly given. In any case, whichever way we would choose the surface, Faraday's law would give here an erroneous result $0\,V$.

We have to start directly from the definition of voltage

$$U = \frac{1}{q}\int_l \vec{F}\cdot d\vec{l} \tag{5.28}$$

and integrate the forces acting on the charges in the rotating disk. These are twofold – the centrifugal force and the magnetic Lorentz force:

$$|\vec{F}_o| = m\omega^2 r, \qquad \vec{F}_L = q\vec{v}\times\vec{B}. \tag{5.29}$$

The velocities of the charges and the directions of the forces (the magnetic force for $q > 0$) are shown in Figure 5.14:

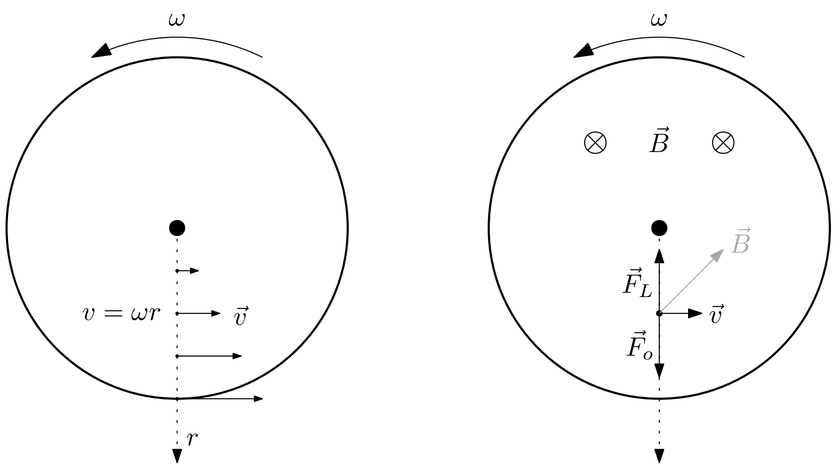


Figure 5.14: The left figure shows the speed of charges in the rotating disc. The right figure show the directions of the forces acting on the charges.

We choose the integration curve to be as simple as possible with one end at the center of the disc and the other at the edge of the disc at the feed wire. Naturally, it suggesteds itself to take a radial line from the center to the edge of the disc. If we introduce a radial coordinate r – the distance from the center of the disc, then the curve is characterized by the coordinate range $r \in \langle 0, R\rangle$ and the line element is $dl = dr$. We assign a tangent vector $\vec{t}$ pointing radially from the center to the line element, i.e., $d\vec{l} = \vec{t}\,dl$. See Figure 5.15 for the situation.

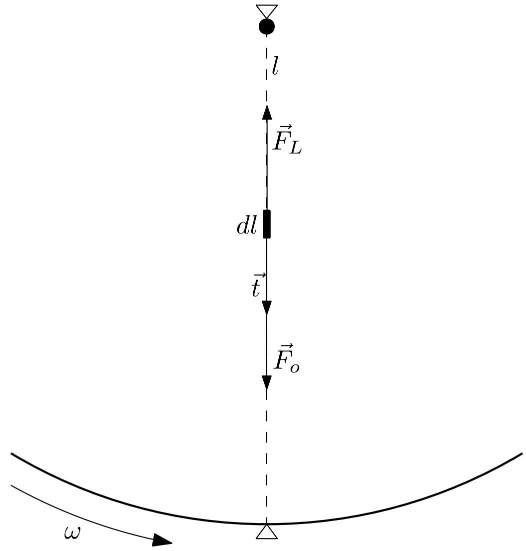


Figure 5.15: Integration curve l with the line element $d\vec{l}$ and the directions of the forces $\vec{F}_L$ and $\vec{F}_o$.

Then the scalar product is $\vec{F} \cdot d\vec{l} = \pm F dl$ – the positive sign for the centrifugal force and the negative sign for the centripetal magnetic force. The voltage is then calculated as follows

$$U = \frac{1}{q}\int_l \vec{F} \cdot d\vec{l} = \pm\frac{1}{q}\int_l F dl = \pm\frac{1}{q}\int_0^R F(r)\,dr \tag{5.30}$$

and we get the specific expressions for the specific forces:

$$U_o = \frac{1}{q}\int_0^R m\omega^2 r\,dr = \frac{m\omega^2}{q}\frac{R^2}{2}, \qquad U_L = -\frac{1}{q}\int_0^R q\omega B r\,dr = -\omega B\frac{R^2}{2}. \tag{5.31}$$

Thus, for our chosen magnetic field direction $\vec{B}$, the voltages induced by the centrifugal and magnetic forces act against each other. Substituting the given numerical values and the mass and charge of the electron m_e and e for m and q we have

$$|U_o| = 4,04 \cdot 10^{-9}\,V, \qquad |U_L| = 0,377\,V. \tag{5.32}$$

Addendum: The voltage between the center and the outside of the disk does not depend on the path along which we integrate. We show this by finding the potentials for the centrifugal and magnetic force fields in the disk. These forces depend only on the distance from the center, $F(r)$, and therefore the potential will be a function of r, $\mathcal{U}(r)$ only (we use the letter $\mathcal{U}$ for the potential to distinguish it from the voltage U). We choose the positive direction of the force as pointing away from the centre of the disc. From condition $F = -\frac{d\mathcal{U}}{dr}$ we can easily find the resulting potentials:

$$\mathcal{U}_o = -\frac{1}{2}m\omega^2 r^2, \qquad \mathcal{U}_L = \frac{1}{2}q\omega B r^2. \tag{5.33}$$

Since the voltage is defined as work per unit charge and the work is given by the difference of the potentials at the starting and ending points, we have

$$U_o = \frac{1}{q}\left(\mathcal{U}_o(0) - \mathcal{U}_o(R)\right), \qquad U_L = \frac{1}{q}\left(\mathcal{U}_L(0) - \mathcal{U}_L(R)\right); \tag{5.34}$$

these lead to the already calculated voltage values.

5.2 Inductance and mutual inductance

5.2.1 5.3 and 5.4 Inductance of a cylindrical coil

Determine the inductance and magnetic energy of a solenoid of radius $R = 1\,cm$ and length $l = 50\,cm$ with $n = 6$ turns per 1 cm of length if current $I = 1\,A$ flows through the turns.

Determine the inductance of a toroidal coil of small cross-section $S = 1\,cm^2$ with a radius of the central circle $r = 5\,cm$, with a total number of turns $N = 100$.

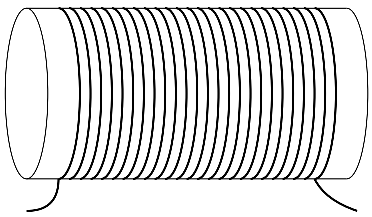


Figure 5.16: Cylindrical coil.

Solution:

The magnetic field inside an (infinitely) long solenoidal coil is $B = \mu_0 n I$, where n is the density of turns, $n = N/l$. The magnetic field vector points in the direction of the coil axis according to the right-hand rule – the thumb points in the direction of the current flowing through the coil, and the fingers point in the direction of the magnetic field (see Example 4.15 – section (4.4.3)).

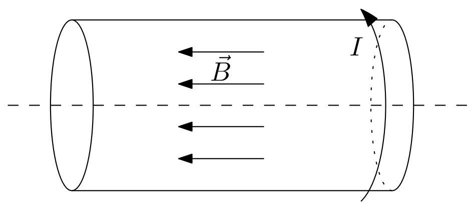


Figure 5.17: The direction of magnetic field is given by the right hand rule according to the direction of current I.

The magnetic field flux Φ through one coil turn is

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \int_S B\,dS = B\int_S dS = BS = \mu_0 n I S, \tag{5.35}$$

where S is the disk surface bounded by the coil turn and we have taken advantage of the fact that the magnetic field $\vec{B}$ inside the coil points perpendicular to the plane of the turn (i.e., it is parallel to the normal vector $\vec{n}$ to the surface S: $\vec{B}\cdot d\vec{S} = \vec{B}\cdot\vec{n}\,dS = BdS$) and is also homogeneous, so that it can be factored out from the integral. Also see Figure 5.18.

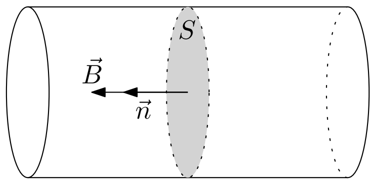


Figure 5.18: Surface S, normal vector $\vec{n}$ and magnetic field vector $\vec{B}$.

The inductance is then by definition $N\Phi = LI$[1]:

$$L = \frac{N\Phi}{I} = \mu_0 nNS = \mu_0 n^2 (lS) = \mu_0 n^2 V; \tag{5.36}$$

in the result we used the volume of the coil $V = lS$.

For a toroidal coil of small cross-section we make the approximation that the magnetic field inside the coil is homogeneous (in the following example we calculate the field inside the toroidal coil exactly) and that its volume is approximately $V = 2\pi r S$.

5.2.2 5.5 Inductance of the toroidal coil

Determine the inductance of a toroidal coil of rectangular cross-section with inner radius $R_1 = 10\,cm$, outer radius $R_2 = 20\,cm$ and height $h = 5\,cm$ when $N = 1000$ turns are wound on it.

[1]This relationship is called the static definition of inductance. It follows easily from the dynamic definition of inductance – relating the change in current through a coil to the induced voltage across it:

$$\mathcal{E} = L\dot{I}.$$

The total induced voltage $\mathcal{E}$ is given by the sum of the induced voltages in the individual coil turns

$$\mathcal{E} = \sum_{i=1}^{N} \mathcal{E}_i = \sum_{i=1}^{N} \frac{d\Phi^{(i)}}{dt},$$

where we have omitted the sign for Faraday's law, which is anyway given only by the choice of positive directions. If the magnetic field fluxes through the individual coils are identical, $\Phi^{(i)} = \Phi$ we get a dynamic definition of inductance of the form

$$N\frac{d\Phi}{dt} = L\frac{dI}{dt}.$$

Integrating by time, we arrive at a static definition of inductance

$$N\Phi = LI.$$

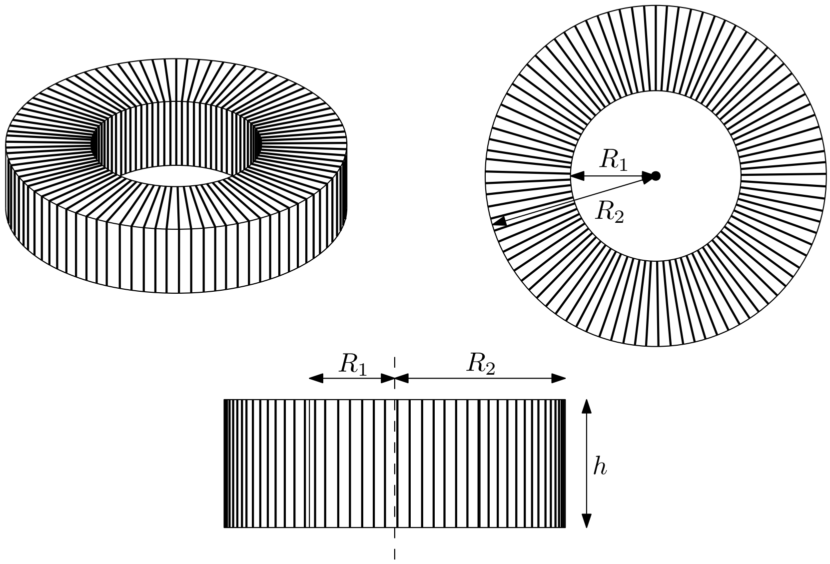


Figure 5.19: Toroidal coil of rectangular cross-section.

Solution: The calculation will consist of three main parts. First, from the symmetry of the problem we will determine the possible magnetic fields respecting this symmetry, then we will determine the magnetic field inside the toroidal coil using Ampere's law, and finally we will calculate the magnetic flux through one turn. From the definition of inductance, $N\Phi = LI$ (see footnote to the previous example for the origin of this relation), we can then easily write the resulting inductance.

Let us now introduce cylindrical coordinates (r, φ, z) with the origin at the center of the toroid and the axis z pointing in the direction of the toroid axis – see Figure 5.20.

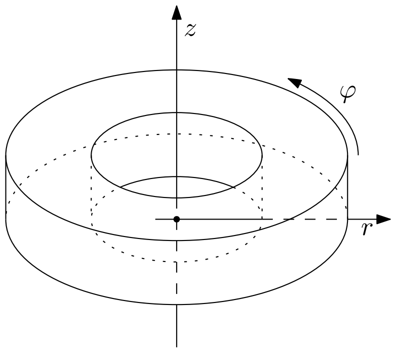


Figure 5.20: Cylindrical coordinates (r, φ, z) in the toroidal coil.

The toroidal coil has continuous rotational symmetry about the z axis – we imagine that the coils are very densely wound, and thus rotating them by any angle will not change the physical situation. Thus, we replace the current flowing through the individual wires on the toroid shell by a uniformly distributed sheet current.

Note: In reality, the toroidal coil would have only a discrete rotational symmetry of angle $2\pi/N$ (when wound uniformly) – and thus the magnetic field would have small variations as the coordinate φ changes. Thus we use "an approximation" of very dense coil winding. From now on, everything will be without any approximation.

A consequence of rotational symmetry is that the magnitude of the magnetic field does not depend on the polar angle φ, i.e. $B = B(r, z)$.

Ampere's law (in integral form)

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot d\vec{S} = \mu_0 I_{in} \tag{5.37}$$

relates the circulation Γ of magnetic field along a closed curve and the total current I_{in} that is encircled by this loop. It is very convenient to use it in cases where we have a symmetric physical situation. We choose the curve for the left-hand side of Ampere's law to be a circle of radius r whose axis is the same as the axis of the toroid. The circle is placed at an arbitrary height z inside the toroid, see Figure 5.21:

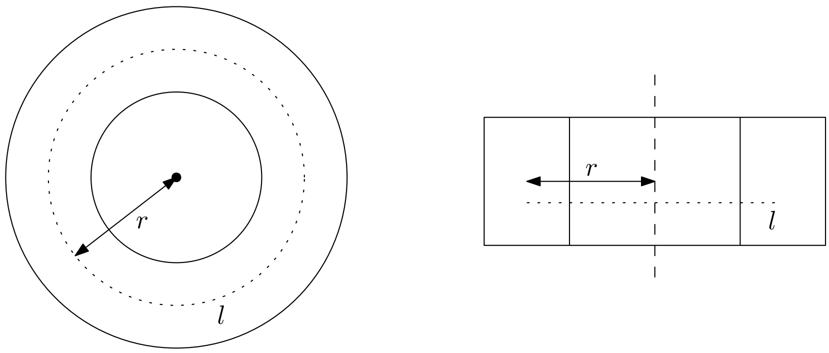


Figure 5.21: Curve l for Ampere's law.

Ampere's law allows us to determine only the tangential component $\vec{B}_t$ of the total magnetic field, since it holds (see Figure 5.22 for the definitions of $\vec{B}_t$, $\vec{B}_\perp$, $d\vec{l}$ and $\vec{t}$)

$$\vec{B} \cdot d\vec{l} = (\vec{B}_\perp + \vec{B}_t) \cdot d\vec{l} = \vec{B}_t \cdot d\vec{l} = B_t \, dl. \tag{5.38}$$

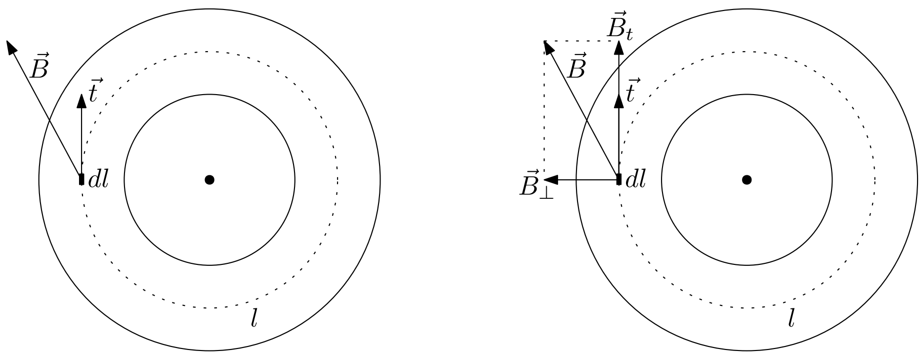


Figure 5.22: Magnetic field vector $\vec{B}$ and its decomposition into tangential and normal directions. Line element $d\vec{l} = \vec{t}\, dl$, where $\vec{t}$ is the unit tangent vector to the curve l.

Note: In fact, the field in the toroid is purely tangential, i.e., $\vec{B}_\perp = 0$, so using Ampere's law we determine the total magnetic field inside the toroid $\vec{B} = \vec{B}_t$ – see the Addendum for why the field is only tangential. At the same time, however, we are not interested in the normal component of the magnetic field $\vec{B}_\perp$ for calculating the magnetic flux, as will be seen in the following.

The left-hand side of Ampere's law gives:

$$\int_l \vec{B} \cdot d\vec{l} = \int_l B_t \, dl = B_t \int_l dl = 2\pi r B_t, \tag{5.39}$$

since due to rotational symmetry the magnetic field is constant at a constant distance r from the coil axis and at a constant height z. The loop l encircles all the turns of the toroid, the total current enclosed in it is thus $I_{in} = NI$. The result is an expression for the tangential component of the magnetic field as a function of position in the toroid:

$$2\pi r B_t = \mu_0 N I \quad \rightarrow \quad B_t = \frac{\mu_0 N I}{2\pi r} \tag{5.40}$$

The magnetic field depends only on the distance from the axis of the toroid (coordinate r) and does not depend on the vertical position in the toroid (coordinate z).

We now calculate the magnetic flux Φ through one turn of the coil:

$$\Phi = \int_S \vec{B} \cdot d\vec{S}. \tag{5.41}$$

The surface of integration S is the rectangle forming the cross section of the turn. We manipulate the scalar product $\vec{B} \cdot d\vec{S}$, where $d\vec{S} = \vec{n}\, dS$, $\vec{n}$ is the unit normal vector to the plate dS, similarly to Ampère's law, i.e. we decompose the magnetic field in perpendicular directions $\vec{B} = \vec{B}_t + \vec{B}_\perp$, and obtain

$$\vec{B} \cdot d\vec{S} = \vec{B} \cdot \vec{n}\, dS = (\vec{B}_t + \vec{B}_\perp) \cdot \vec{n}\, dS = \vec{B}_t \cdot \vec{n}\, dS = B_t\, dS \tag{5.42}$$

(we keep the notation $\vec{B}_\perp$ and $\vec{B}_t$ as shown in Figure 5.22, i.e. the vector $\vec{B}_t$ is parallel to the normal vector $\vec{n}$. For the relationships between the integration surface S, the normal vector $\vec{n}$ and the decomposition $\vec{B} = \vec{B}_t + \vec{B}_\perp$, see Figure 5.23).

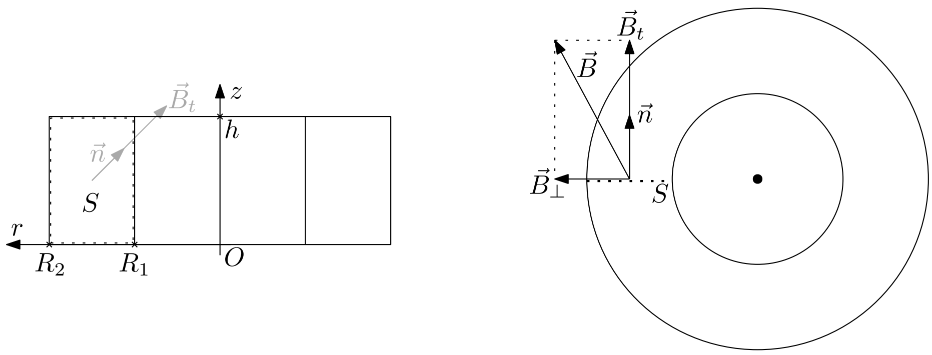


Figure 5.23: Integration surface S and vectors $\vec{n}$ and $\vec{B}_t$ perpendicular to it.

The rectangle whose boundary is one coil turn is given by the coordinate ranges $r \in \langle R_1, R_2 \rangle$ and $z \in \langle 0, h \rangle$. The surface element is in Cartesian coordinates $dS = drdz$. The actual integration is straightforward after giving the expression for B_t obtained from Ampere's law:

$$\Phi = \int_S B_t\, dS = \int_0^h \int_{R_1}^{R_2} \frac{\mu_0 N I}{2\pi r}\, dr\, dz = \frac{\mu_0 N I}{2\pi} \int_0^h dz \int_{R_1}^{R_2} \frac{dr}{r} = \frac{\mu_0 N I}{2\pi} h \ln \frac{R_2}{R_1}. \tag{5.43}$$

The inductance of the toroid is, by definition

$$L = \frac{N\Phi}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1}. \tag{5.44}$$

Addendum: Magnetic field direction. Or $\vec{B}_\perp = 0$. Coming soon.

5.2.3 5.9 Mutual inductance I

Two coils are inductively coupled by mutual inductance $L_{mut} = 5\,H$. How must the current in the primary coil be varied to induce a constant voltage $\mathcal{E} = 1\,V$ in the secondary coil? Can it be induced permanently in this way?

Solution: We start from the formulas defining inductance and mutual inductance relating the changes in currents through each coil and the voltages induced on them:

$$\mathcal{E}_1 = L_1\dot{I}_1 + L_{mut}\dot{I}_2, \qquad \mathcal{E}_2 = L_{mut}\dot{I}_1 + L_2\dot{I}_2. \tag{5.45}$$

The voltage induced on the secondary coil is given by the second of the equations, specifically the voltage induced due to the change in current in the primary coil is given by the first term of the right hand side:

$$\mathcal{E} = L_{mut}\dot{I}_1. \tag{5.46}$$

We solve this simple differential equation.

$$\dot{I}_1 = \frac{\mathcal{E}}{L_{mut}} = \text{const.} \quad \xrightarrow{\int dt} \quad I_1 = \frac{\mathcal{E}}{L_{mut}}\,t + I_0, \tag{5.47}$$

where I_0 is the integration constant representing the current at time $t = 0\,s$. The current in the coil increases linearly without stopping, so we certainly cannot hold the voltage $\mathcal{E}$ constant forever.

5.2.4 5.10 Mutual inductance II

The two coils have inductances $L_1 = 0,2\,H$, $L_2 = 0,3\,H$ and mutual inductance $L_{mut} = 0,1\,H$. What will be the resulting inductance when these coils are connected in series?

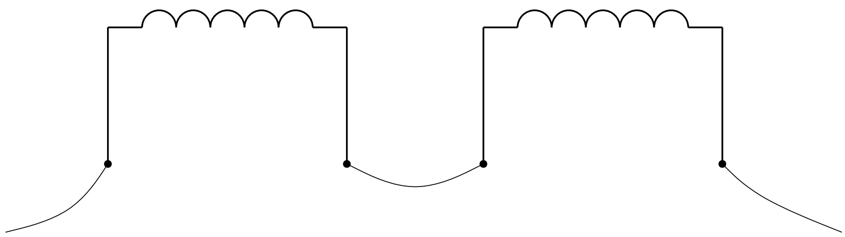


Figure 5.24: Coils connected in series.

Solution: First, we introduce the positive directions. Let's say the current flowing through the coil is said to be positive if it flows from left to right. Additionally, we call the voltage induced in the coil positive if we measure a positive pole on the left terminal of the coil and a negative pole on the right. See Figure 5.25.

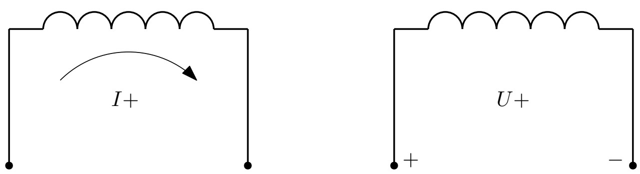


Figure 5.25: Positive direction of current and polarity of positively induced voltage.

If, at the moment when a positive voltage is induced on the coil, we connect a load to the coil, the current flowing through the coil would be negative (according to the selected positive current direction) – Figure 5.26.

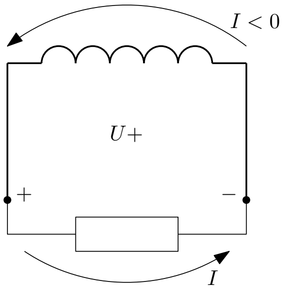


Figure 5.26: The positive induced voltage induces a negative current in the coil.

With the positive directions thus introduced, we obtain a dynamic definition of inductance relating the change in current in the coil to the voltage induced in it of the form

$$\mathcal{E} = L\dot{I}, \tag{5.48}$$

where the positive sign follows from Lenz's law. It states that the induced voltage acts against the change that caused it. Thus, if we have an increasing current flowing through a coil, a positive voltage must be induced on the coil, which induces an additional current flowing in the negative direction, thus weakening the increasing current that caused the induced voltage.

For the mutual inductance we have the relations (positive signs remain due to the previous argument)

$$\mathcal{E}_1 = L_1\dot{I}_1 + L_{mut}\dot{I}_2, \qquad \mathcal{E}_2 = L_{mut}\dot{I}_1 + L_2\dot{I}_2; \tag{5.49}$$

these relate the changes in currents through the individual coils to the voltages induced on the individual coils.

The coils can be connected in two ways as shown in Figure 5.27. It shows the **true** directions of the currents and only the **selected positive** directions of the induced voltages (i.e., the polarity of the voltages in the figure serves only to indicate the positive direction, not to indicate the true polarity of the induced voltages – the latter comes directly from the formulas for defining mutual inductance, where we have carefully used Lenz's law to determine the correct sign). In the first case, the voltages are added, in the second, they are subtracted.

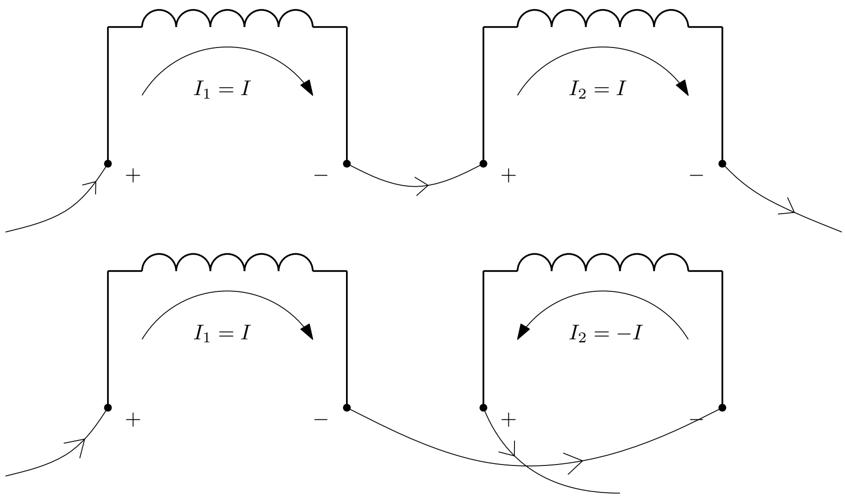


Figure 5.27: Two ways of connecting coils in series.

The connected coils now behave as one large coil with current I. We can then determine the total inductance L_{tot} of the connected coils using the relationship

$$\mathcal{E}_{tot} = L_{tot}\dot{I}. \tag{5.50}$$

In the first case, the voltages add $\mathcal{E}_{tot} = \mathcal{E}_1 + \mathcal{E}_2$ and the currents flow in a uniform direction $I_1 = I_2 = I$; in the second case, we have a voltage difference $\mathcal{E}_{tot} = \mathcal{E}_1 - \mathcal{E}_2$ and opposing currents flowing $I_1 = -I_2 = I$, see the previous figure 5.27. Substituting from the formulae for mutual inductance (5.49) for the voltages $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\begin{aligned} \mathcal{E}_{tot}^{(1)} &= \mathcal{E}_1 + \mathcal{E}_2 = (L_1\dot{I} + L_{mut}\dot{I}) + (L_{mut}\dot{I} + L_2\dot{I}) = (L_1 + L_2 + 2L_{mut})\dot{I} = L_{tot}^{(1)}\dot{I}, \\ \mathcal{E}_{tot}^{(2)} &= \mathcal{E}_1 - \mathcal{E}_2 = (L_1\dot{I} - L_{mut}\dot{I}) - (L_{mut}\dot{I} - L_2\dot{I}) = (L_1 + L_2 - 2L_{mut})\dot{I} = L_{tot}^{(2)}\dot{I}. \end{aligned} \tag{5.51}$$

The results for the total inductance by wiring method are then

$$L_{tot}^{(1)} = L_1 + L_2 + 2L_{mut}, \qquad L_{tot}^{(2)} = L_1 + L_2 - 2L_{mut}. \tag{5.52}$$

5.3 LR and RC circuits

5.3.1 5.11 and 5.12 RC circuit

A capacitor of capacitance $C = 0,1\,\mu F$ with initial voltage $U_0 = 1000\,V$ is discharged through resistor $R = 10\,\Omega$. In what time will the magnitude of the charge on the capacitor drop to the level of one elementary charge?

A capacitor of capacitance $C = 100\,\mu F$ is charged to a voltage $U_0 = 10\,000\,V$. We discharge it through a resistor $R = 1\,k\Omega$. In what time can we touch the capacitor without danger?

Solution: The voltage across the capacitor and the current in the RC circuit as functions of time are as follows:

$$U(t) = U_0 e^{-\frac{t}{RC}}, \qquad I(t) = I_0 e^{-\frac{t}{RC}}. \tag{5.53}$$

The magnitude of the charge on the capacitor is from the definition of capacitance $Q = CU$ and hence is

$$Q(t) = CU_0 e^{-\frac{t}{RC}} (= Q_0 e^{-\frac{t}{RC}}). \tag{5.54}$$

Expressing the time from the relations for voltage (5.53) and charge (5.54) we have

$$t = -RC \ln \frac{U(t)}{U_0} = RC \ln \frac{U_0}{U(t)}, \qquad t = -RC \ln \frac{Q(t)}{CU_0} = RC \ln \frac{CU_0}{Q(t)}. \tag{5.55}$$

In the first example, we have $Q(t_e) = e$ and the resulting time is

$$t_e = RC \ln \frac{CU_0}{e}. \tag{5.56}$$

In the second example, we need to choose the value of the (un)safe voltage U_{death} to get a particular result...

$$t_{death} = RC \ln \frac{U_0}{U_{death}}. \tag{5.57}$$

5.3.2 5.13 Energy of the capacitor

Prove that the energy dissipated on the resistor during the discharge of the capacitor is just equal to the energy that has been stored in the capacitor.

Solution: The voltage across the capacitor and the current in the RC circuit as functions of time are as follows:

$$U(t) = U_0 e^{-\frac{t}{RC}}, \qquad I(t) = I_0 e^{-\frac{t}{RC}}. \tag{5.58}$$

The initial conditions are $U(0) = U_0$ and $I(0) = I_0$. The thermal power P_{heat} generated at the resistor is given by the Joule heat $P_{heat} = RI^2$. The power is the change in energy over time, $P = dE/dt$ and hence the energy is obtained by integrating the power with respect to time. The discharge starts at time $t = 0\,s$ and formally, according to the relations for voltage and current, never finishes. We therefore integrate for $t \in \langle 0, +\infty \rangle$:

$$W = \int_0^{+\infty} P_{heat}\, dt = \int_0^{+\infty} RI(t)^2 dt = RI_0^2 \int_0^{+\infty} e^{-\frac{2t}{RC}} = RI_0^2 \left[-\frac{RC}{2} e^{-\frac{2t}{RC}} \right]_0^{+\infty}$$
$$= \frac{1}{2} C (RI_0)^2 = \frac{1}{2} C U_0^2. \tag{5.59}$$

The result is indeed the energy originally stored in the electrostatic field in the capacitor.

5.3.3 5.14 LR circuit

The coil has resistance $R = 100\,\Omega$. If the leads of the coil are short-circuited while a steady current is flowing through the coil, the current in the coil drops to one-tenth of its original value in $T = 0,01\,s$. What is the inductance of the coil?

Solution: The current in the RL circuit and the voltage induced on the coil as functions of time are as follows:

$$I(t) = I_0 e^{-\frac{R}{L} t}, \qquad U(t) = U_0 e^{-\frac{R}{L} t}. \tag{5.60}$$

From the equation for the current, we express L:

$$L = -\frac{R\,t}{\ln \frac{I(t)}{I_0}} = \frac{R\,t}{\ln \frac{I_0}{I(t)}}. \tag{5.61}$$

According to the assignment $I(T) = \alpha I_0$, where $\alpha = \frac{1}{10}$. The result is therefore

$$L = -\frac{R\,T}{\ln \alpha}. \tag{5.62}$$

5.4 AC circuits

5.4.1 5.15 Battery charging

It takes $Q = 20Ah$ (ampere hours) of steady current to charge a battery. How long will it take to charge the battery with an alternating current of effective value $I_{ef} = 1\,A$, which we rectify with a full-wave rectifier?

Solution: The full-wave rectified AC current is obtained by flipping the negative half waves to positive values (see Figure 5.28).

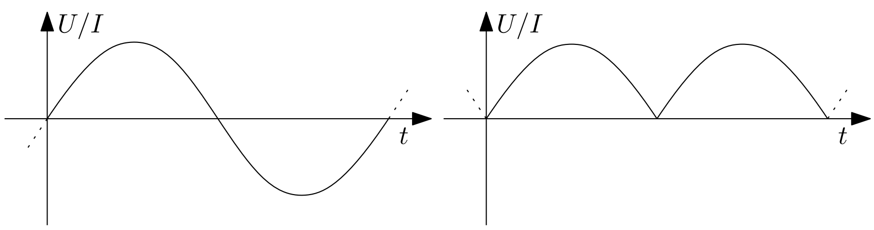


Figure 5.28: AC current and full-wave rectified current (voltage).

A DC current of $I_{DC} = 1\,A$ would obviously charge the battery in $t_{DC} = Q/I_{DC} = 20\,h$. An alternating current with an effective value of $I_{ef} = 1\,A$ has an amplitude of $I_{AC} = \sqrt{2}\,A$ (see the Addendum for an explanation). Thus, it is sufficient to compare the magnitude of the area (charge) under a single half-wave of full-wave rectified current with the area under a direct current of the same duration (see Figure 5.29), and calculate the result using a "rule of three".

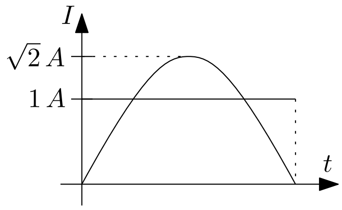


Figure 5.29: Areas under direct current $I_{DC} = 1\,A$ and alternating current $I_{AC} = \sqrt{2}\,A$.

The area under one half wave of a sine wave is

$$
\begin{aligned}
Q_{AC} &= \int_0^{T/2} I_{AC} \sin \omega t \, dt = I_{AC} \left[-\frac{1}{\omega} \cos \omega t \right]_0^{T/2} = \frac{I_{AC}}{\omega} \left(1 - \cos \frac{\omega T}{2} \right) \\
&= \frac{I_{AC}}{\omega} \left(1 - \cos \pi \right) = \frac{2 I_{AC}}{\omega},
\end{aligned} \tag{5.63}
$$

We integrate over one half-wave, i.e., for times $t \in \langle 0, T/2 \rangle$, using $\omega = 2\pi f = \frac{2\pi}{T}$ to get the result. The area (charge) under constant current in time $T/2$ is

$$
Q_{DC} = I_{DC} \frac{T}{2} = \frac{\pi I_{DC}}{\omega}. \tag{5.64}
$$

The inverse relationship between the charging time and the charge transferred per half-period holds:

$$
\frac{Q_{DC}}{Q_{AC}} = \frac{t_{AC}}{t_{DC}}, \tag{5.65}
$$

i.e. after the

$$
t_{AC} = \frac{Q_{DC}}{Q_{AC}} t_{DC} = \frac{I_{DC}}{I_{AC}} \frac{\pi}{2} t_{DC}. \tag{5.66}
$$

For specific values of current $I_{DC} = 1\,A$ and $I_{AC} = \sqrt{2}\,A$ (i.e. $I_{ef} = 1\,A$) we have

$$
t_{AC} = \frac{\pi}{2\sqrt{2}} t_{DC} > t_{DC}. \tag{5.67}
$$

Addendum: The effective value of the voltage and current is defined so that the mean value of the AC voltage and current output can be calculated as the simple product of the effective values, i.e., $\langle P \rangle = U_{ef} I_{ef}$.

The instantaneous power is given by the instantaneous values of voltage and current at the load $P(t) = U(t)I(t)$. The time-averaged power is defined as (for periodic voltage and current waveforms with period T)

$$
\langle P \rangle = \frac{1}{T} \int_0^T P(t) \, dt. \tag{5.68}
$$

For harmonic voltage and current waveforms,

$$
U(t) = U_0 \cos \omega t, \qquad I(t) = I_0 \cos \omega t, \tag{5.69}
$$

(U_0 and I_0 are the amplitudes of this voltage and current), we get the time-averaged power $\langle P\rangle$

$$\langle P\rangle = \frac{1}{T}\int_0^T U_0 I_0 \cos^2 \omega t\, dt = \frac{1}{2} U_0 I_0. \tag{5.70}$$

If we now define the effective values of the voltage and current

$$U_{ef} = \frac{U_0}{\sqrt{2}}, \qquad I_{ef} = \frac{I_0}{\sqrt{2}}, \tag{5.71}$$

we can write $\langle P\rangle = U_{ef} I_{ef} = R\, I_{ef}^2 = U_{ef}^2/R$.

If the voltage and current are phase shifted with respect to each other, i.e. have, for example, the prescriptions

$$U(t) = U_0 \cos\omega t, \qquad I(t) = I_0\cos(\omega t + \varphi_0), \tag{5.72}$$

then the mean power value comes out

$$\langle P\rangle = \frac{1}{T}\int_0^T U_0 I_0 \cos\omega t \cos(\omega t + \varphi_0) dt = \frac{1}{2} U_0 I_0 \cos\varphi_0 = U_{ef} I_{ef}\cos\varphi_0. \tag{5.73}$$

5.4.2 5.20 Appliance

Consider an appliance of real impedance R that consumes power $P = 60\,W$ at an effective voltage $U_{ef}^{(0)} = 120\,V$. We want to operate this appliance at the same power at the effective voltage $U_{ef} = 240\,V$ in the network $f = 50\,Hz$. What inductance or capacitance would we need to put in series?

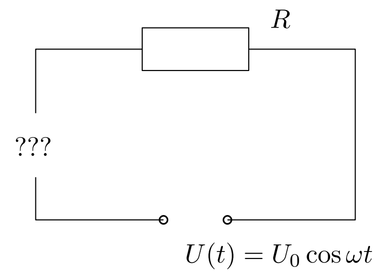


Figure 5.30: What inductance or capacitance do we need to put in series?

Solution: Here we apply the method of so-called phasors to the solution. We assign a complex number (phasor) $\hat{A} = A_0 e^{i\varphi_0}$ to a quantity that has a harmonic waveform in time $A_0\cos(\omega t + \varphi_0)$. In the circuit, we consider the voltage $U(t) = U_0\cos\omega t$ and, in general, the phase-shifted current $I(t) = I_0\cos(\omega t + \varphi_0)$ and the associated phasors $\hat{U} = U_0$ and $\hat{I} = I_0 e^{i\varphi_0}$. We can now define **impedance** Z as a complex number

$$Z = \frac{\hat{U}}{\hat{I}}. \tag{5.74}$$

For a resistor in series with resistance R, a capacitor with capacitance C, and a coil with inductance L, the impedance is equal to

$$Z = R + i\omega L + \frac{1}{i\omega C} \tag{5.75}$$

(in the case of an omitted capacitor, the term $\frac{1}{i\omega C}$ is omitted).

For the instantaneous power over time t we have the relation $P(t) = U(t)I(t)$. Averaging this power over time of one period we get the relation for the mean power

$$\langle P\rangle = \frac{1}{2}U_0 I_0 \cos\varphi_0 = U_{ef} I_{ef} \cos\varphi_0, \tag{5.76}$$

(see the appendix for Example 5.15 – Section 5.4.1). We can further modify this expression in the language of phasors

$$\langle P\rangle = U_{ef} I_{ef} \cos\varphi_0 = \mathrm{Re}\,(\hat{U}_{ef}\hat{I}_{ef}) = \mathrm{Re}\,\frac{U_{ef}^2}{Z} = U_{ef}^2\,\mathrm{Re}\,\frac{1}{Z}, \tag{5.77}$$

where $\hat{U}_{ef} = U_{ef}$ and $\hat{I}_{ef} = I_{ef}e^{i\varphi_0}$. Thus, we need to calculate the real part of the inverse of the impedance (which is also called the admittance). In the case of a series capacitor, we have

$$\frac{1}{Z} = \frac{1}{R + \frac{1}{i\omega C}} = \frac{i\omega C}{i\omega RC + 1}\cdot\frac{1 - i\omega RC}{1 - i\omega RC} = \frac{1}{1 + \omega^2 R^2 C^2}(\omega^2 RC^2 + i\omega C), \tag{5.78}$$

The power on the load is therefore

$$\langle P\rangle = U_{ef}^2 \frac{\omega^2 RC^2}{1 + \omega^2 R^2 C^2}. \tag{5.79}$$

For a series inductance, we get

$$\frac{1}{Z} = \frac{1}{R + i\omega L}\cdot\frac{R - i\omega L}{R - i\omega L} = \frac{1}{R^2 + \omega^2 L^2}(R - i\omega L) \tag{5.80}$$

and for the power we have the relation

$$\langle P\rangle = U_{ef}^2 \frac{R}{R^2 + \omega^2 L^2}. \tag{5.81}$$

For the case where there is only an ohmic load in the circuit, we get “ordinary” expression for the power

$$\langle P\rangle = \frac{U_{ef}^2}{R}. \tag{5.82}$$

From this expression we can easily express the resistance of the appliance R using the originally connected effective voltage $U_{ef}^{(0)}$:

$$R = \frac{U_{ef}^{(0)2}}{\langle P\rangle}. \tag{5.83}$$

Substituting this expression for resistance R into the formulas for power with a capacitor (5.79) or inductor (5.81), we can express the required capacitance or inductance. After a bit of calculation, we get the results:

$$\begin{aligned} C &= \frac{\langle P\rangle}{\omega}\frac{1}{U_{ef}^{(0)}\sqrt{U_{ef}^2 - U_{ef}^{(0)\,2}}} = 7,68\,\mu F, \\ L &= \frac{1}{\omega\langle P\rangle}U_{ef}^{(0)}\sqrt{U_{ef}^2 - U_{ef}^{(0)\,2}} = 1,32\,H. \end{aligned} \tag{5.84}$$