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Quantum aspects of beyond-Standard-model theories with extended gauge symmetries

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Abstrakt: Standardní model částicové fyziky je úspěšný v mnoha ohledech, v rámci této teorie však nelze vysvětlit například nenulové hmoty neutrin anebo pozorování tzv. temné hmoty. V této práci studujeme různá rozšíření standardního modelu, která zahrnují výše zmíněné jevy, a hledáme možnosti, jak tyto teorie experimentálně testovat. Je-li standarní model doplněn o další skalární pole, lze vysvětlit malé hmoty neutrin a rovněž zavést částici, která by mohla tvořit temnou hmotu, tzv. axion. Tuto třídu modelů je možné testovat mimo jiné na urychlovačích. Dále studujeme specifické modely sjednocení kalibračních interakcí s kalibrační grupou $SU(5) \times U(1)$ a $SO(10)$, které předpovídají rozpad protonu, a naším hlavním výsledkem je výpočet příslušných větvících poměrů.

Abstract:

Although the Standard Model of particle interactions is successful in many respects, there are several observations concerning, e.g., non-zero neutrino masses or the presence of the so-called dark matter, which suggest that this scheme has to be extended. In this thesis we study different beyond-Standard-Model theories which could help us to understand the above mentioned phenomena and we also try to find the way in which these models can be experimentally tested. First, Standard Model is extended by scalar fields in order to accommodate small Majorana neutrino masses together with a dark matter candidate called axion. These scenarios can be probed, for instance, by collider experiments. Second, particular unified models based on the $SU(5) \times U(1)$ and the $SO(10)$ gauge groups are studied, and the potentially measurable partial proton decay rates are computed.

Contents

Chapter 1

Introduction

At the beginning of this chapter, the goals of this thesis are set, the scope of the subsequent part is then to help the reader to orient himself in this text. The content of the individual chapters is described in Section 1.2 and our notation and conventions are presented in Section 1.3.

1.1 Motivation and goals

The description of the elementary particles and their interactions in the context of the so-called Standard Model (SM) of particle physics was well established and, recently, the discovery of the Higgs boson at the Large Hadron Collider (LHC) in CERN confirmed experimentally the existence of the last missing piece of this model. Nevertheless, there are several reasons why to believe that the SM is not a complete description of the world at the microscopic scales. The observation of the so-called dark matter or the issue of the baryon-antibaryon asymmetry in our Universe indisputably call for explanation, besides, physicists would like to find the order in the setting of the SM parameters and to explain, e.g., the smallness of neutrino masses.

This thesis is based on the original research papers [1–4], attached to this work as Appendices B-E. The common goal of all these articles is to provide realistic extensions of the SM which address the deficiencies of this model. To that end, two qualitatively different strategies are followed, hence, this thesis will be divided into two parts. In the articles [1, 2] the bottom-up approach is used when we try to explain the smallness of neutrino masses together with the origin of the dark matter by adding minimum number of extra fields to the SM. On the other hand, [3,4] use the top-down approach of the so-called unified theories where the extended gauge symmetry dictates the structure of the extra fields and predicts relations between the SM parameters as well as new interactions among the SM fields.

Another goal of this thesis is to find experimental predictions of the proposed models, which allow to confirm or disprove these scenarios in future. This concerns, e.g., possible collider signatures of the extra fields or the process of proton decay in case of the unified models.

A link between the two parts of this thesis is provided also by the following model-independent way in which the effects beyond the SM can be looked for. The Appelquist-Carazzone theorem [5] says that at low energies the effects of heavy fields with mass $\sim \Lambda$ should be negligible and go to zero when $\Lambda \to \infty$. The decoupling of these heavy fields then leads to higher-dimensional operators suppressed by a certain power of Λ and the expansion

$$\mathcal{L} = \mathcal{L}^{d=4} + \frac{1}{\Lambda} \sum_k C_k^{d=5} \mathcal{O}_k^{d=5} + \frac{1}{\Lambda^2} \sum_k C_k^{d=6} \mathcal{O}_k^{d=6} + \dots \qquad (1.1)$$

can be considered. Since the SM as the low-energy theory has to be recovered, the first term includes the renormalizable SM Lagrangian. The subsequent higher-dimensional terms contain all possible operators $\mathcal{O}_k$ that can be build from the SM fields respecting the SM gauge symmetries. The dimensionless parameters C_k are usually called Wilson coefficients.

Let us note that there is no reason why the accidental global symmetries of the renormalizable SM Lagrangian corresponding to the baryon and lepton number conservation should be respected by the higher-dimensional terms in (1.1). Indeed, we will see that there is exactly one independent $d = 5$ term in (1.1) and this operator violates the lepton number. It also provides the Majorana neutrino masses and the possible renormalizable realizations of this operator will lead us to SM extensions studied in [1] and [2].

Similarly, some of the $d = 6$ operators induce baryon number violation and trigger the process of proton decay. We will see that such operators arise within the framework of unified theories which are considered in [3] and [4].

1.2 Organization of the text

As suggested above, the results of this thesis can be divided into two parts and this logic is followed by the organization of the text. Part I sets the scene for the description of the results in [1] and [2], whereas Part II introduces the works [3] and [4].

As our starting point, the Standard Model of particle physics is briefly described in Chapter 2. Part I is then opened by Chapter 3 where some of the shortcomings of the SM are mentioned. Furthermore, in Chapter 4 different solutions to the problem of the small neutrino masses are given, some of them are later incorporated in the particular models studied in articles [1] and [2] as described in Chapter 5. In this chapter also the axion as the well-established candidate for the dark matter particle is introduced since it is another important ingredient for the studies [1] and [2].

The second part of this thesis is devoted to unified theories. The study of the theories with baryon number violation is motivated in Chapter 6 where also the experimental search for the proton decay is briefly described. Furthermore, an overview of the existing unification models based on the $SU(5)$ and the $SO(10)$ gauge group is given in Chapter 7. Finally, specific models are introduced in Chapter 8 where also the second part of our results is presented.

The methods for studying the proposed models are mentioned for each case separately in Sections 5.2, 5.3, 8.1 and 8.2. Our main results and their possible applications are summarized in Chapter 9 which also contains the outlook for the future work.

Technical issues were left to Appendix A and the Appendices B-E consist of the attached articles [1–4]. In Appendix F we include the co-authorship statement by Stefano Bertolini, the senior author of [1] and [2]. Similar statement by Michal Malinský, the senior author of [3] and [4], will be included in the supervisor's evaluation of this thesis.

1.3 Conventions and notation

Unless stated otherwise, the natural system of units with $\hbar = c = 1$ is used. Numerical values of the observable quantities can be converted into ordinary units by means of

$$1\,\mathrm{MeV}^{-1} \approx 197\,\mathrm{fm} \tag{1.2}$$

or

$$1\,\mathrm{MeV}^{-1} \approx 6.58 \times 10^{-22}\,\mathrm{s}. \tag{1.3}$$

Furthermore, the flat metric is defined by

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.4}$$

For the reader's convenience we recall here also the definition of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

Unless specified otherwise, summation over repeated Greek and Latin indices is always understood.

Other conventions are explained straight in the text, in particular, the notation concerning the Standard Model fields and couplings is set in Chapter 2. Following acronyms are used throughout this thesis.

SM	Standard Model
EW	electro-weak
VEV	vacuum expectation value
LHC	Large Hadron Collider
C	discrete symmetry corresponding to particle-antiparticle conjugation
P	discrete symmetry corresponding to the change of sign in the spatial coordinates
B	baryon number
L	lepton number
Q	electric charge
T_3^L	third component of the weak isospin
Y	hypercharge (the normalization where $Q = T_3^L + Y$ is used)
$V_{\rm PMNS}$	Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix (assigned as V in Section 3.1, $V_{\rm PMNS}^D$ and $V_{\rm PMNS}^M$ distinguished in Section 3.1.4)
$V_{\rm CKM}$	Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix
Ω_x	fraction of the total energy density corresponding to x (see Appendix A.3)

Chapter 2

Standard Model in a nutshell

To set the notation we recall here the field content and the Lagrangian of the SM corresponding to the original formulation of Glashow, Weinberg and Salam [6–8]. In Section 2.4 we mention several experimental tests and theoretical consistency checks which indicate the validity of this theory for a wide range of energy scales.

2.1 The field content

The gauge group of the SM is $SU(3)_c \times SU(2)_L \times U(1)_Y$ where c, L and Y stand for color, left and hypercharge, respectively, and the matter fields are accommodated in the following multiplets with respect to individual gauge group factors:

$$\begin{aligned} L &= \begin{pmatrix} \nu \\ e \end{pmatrix}_L = (1,2,-\tfrac{1}{2}), \qquad e_R = (1,1,-1), \\ Q &= \begin{pmatrix} u \\ d \end{pmatrix}_L = (3,2,+\tfrac{1}{6}), \qquad u_R = (3,1,+\tfrac{2}{3}), \qquad d_R = (3,1,-\tfrac{1}{3}). \end{aligned} \tag{2.1}$$

The first and second row correspond to leptons (charged leptons e and neutrinos ν) and quarks (up-type u and down-type d), respectively, and all these fields occur in three copies in the nature, forming three generations (families) of fermions. As shown in Appendix A.1, instead of the right-handed fields, one can work with the left-handed conjugated fields:

$$(e^c)_L = (1,1,+1), \qquad (u^c)_L = (\bar{3},1,-\tfrac{2}{3}), \qquad (d^c)_L = (\bar{3},1,+\tfrac{1}{3}). \tag{2.2}$$

The gauge bosons corresponding to the three gauge group factors, respectively, will be denoted as

$$G^\mu = (8,1,0), \qquad W^\mu = (1,3,0), \qquad B^\mu = (1,1,0), \tag{2.3}$$

and, finally, the scalar sector consists of a single multiplet

$$H = \left(\begin{array}{c} H^+ \\ H^0 \end{array} \right) = (1, 2, +\tfrac{1}{2}), \tag{2.4}$$

usually called the Higgs doublet.

2.2 The Lagrangian

Now we can write down the individual pieces of the SM Lagrangian

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_Y - V \tag{2.5}$$

where $\mathcal{L}_{gauge}$ consists of the gauge-kinetic forms for the SM matter fermions, the Higgs doublet and the gauge fields, $\mathcal{L}_Y$ describes the Yukawa interactions of the fermions with the Higgs field and V is the scalar potential.

The derivative terms and the gauge interactions of the fermions read

$$\mathcal{L}_{gauge} \ni i\overline{L_j}\gamma_\mu D^\mu L_j + i\overline{Q_j}\gamma_\mu D^\mu Q_j + i\overline{e_{Rj}}\gamma_\mu D^\mu e_{Rj} + i\overline{u_{Rj}}\gamma_\mu D^\mu u_{Rj} + i\overline{d_{Rj}}\gamma_\mu D^\mu d_{Rj} \tag{2.6}$$

where it is summed over the generation index $j = 1, 2, 3$ and

$$D^\mu = \partial^\mu - ig_3 G_a^\mu T_a^c - ig_2 W_a^\mu T_a^L - ig_1 Y B^\mu .$$

Here g_1, g_2 and g_3 are $U(1)_Y$, $SU(2)_L$ and $SU(3)_c$ couplings, respectively, Y is the hypercharge of the given particle and T_a^c and T_a^L are the generators of $SU(3)_c$ and $SU(2)_L$ in the given representation:

$$T_a^c = \begin{cases} \frac{\lambda_a}{2} & \text{for } SU(3)_c \text{ triplets,} \\ 0 & \text{for } SU(3)_c \text{ singlets,} \end{cases} \qquad T_a^L = \begin{cases} \frac{\sigma_a}{2} & \text{for } SU(2)_L \text{ doublets,} \\ 0 & \text{for } SU(2)_L \text{ singlets,} \end{cases}$$

with σ_a and λ_a being Pauli (1.5) and Gell-Mann matrices, respectively.

Below (A.6) we explain that (2.6) would have the same form also if it is rewritten using the left-handed conjugated fields (2.2) instead of the right-handed fields. The situation is, however, more complicated for the Yukawa Lagrangian

$$-\mathcal{L}_Y = Y_{jk}^e (e^c)_{Lj}^T C H^\dagger L_k + Y_{jk}^d (d^c)_{Lj}^T C H^\dagger Q_k - Y_{jk}^u (u^c)_{Lj}^T C H^T i\sigma_2 Q_k + h.c. \tag{2.7}$$

$$= (Y^{e\dagger})_{jk} \overline{L_j} H e_{Rk} + (Y^{d\dagger})_{jk} \overline{Q_j} H d_{Rk} + (Y^{u\dagger})_{jk} \overline{Q_j} i\sigma_2 H^* u_{Rk} + h.c. \tag{2.8}$$

where Y^f for $f = e, d, u$ are general complex 3×3 matrices. In fact, the "*h.c.*" term at the first line corresponds to the terms explicitly written on the second line, which is also the reason why the conjugate transpose of the Yukawa couplings appear in (2.8). Although the notation (2.8) using the right-handed fields is more usual, we will stick to the form (2.7) since this one is used in the literature like [9] which is the key reference for our proton decay studies.

The scalar potential

$$V = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$$

is minimized by

$$\langle H^0 \rangle = v = \frac{\mu}{\sqrt{2\lambda}} \tag{2.9}$$

and without loss of generality v can be chosen real. After inserting this vacuum expectation value (VEV) into the gauge-kinetic form for the Higgs doublet

$$\mathcal{L}_{gauge} \ni (D_\mu H)^\dagger D^\mu H, \qquad D^\mu H = \left(\partial^\mu - i g_2 W_a^\mu \frac{\sigma_a}{2} - i g_1 \frac{1}{2} B^\mu \right) H \tag{2.10}$$

one finds that the charged weak gauge bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_{\mu 1} \mp W_{\mu 2} \right) \tag{2.11}$$

acquire mass

$$m_W^2 = \frac{1}{2} g_2 v^2.$$

The experimental value $m_W = 80.4$ GeV together with the weak coupling measurement then yield $v = 174$ GeV.[1] For the neutral massive gauge boson one gets

$$Z^\mu = \cos\theta_W W_3^\mu - \sin\theta_W B^\mu \tag{2.12}$$

where the so-called weak mixing angle θ_W satisfies

$$\sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}. \tag{2.13}$$

The mass of the Z-boson determined from (2.10) reads

$$m_Z = \frac{m_W}{\cos\theta_W}. \tag{2.14}$$

Finally, the linear combination of the neutral gauge bosons orthogonal to (2.12) corresponds to the SM photon and remains massless.

The $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group is, hence, broken to $SU(3)_c \times U(1)_Q$ where the electric charge Q of a given particle can be computed from the third component of the weak isospin T_3^L and the hypercharge Y as

$$Q = T_3^L + Y \tag{2.15}$$

The last piece of the Lagrangian (2.5) missing is the kinetic term for the gauge fields

$$\mathcal{L}_{gauge} \ni -\frac{1}{4} G_{a\mu\nu} G_a^{\mu\nu} - \frac{1}{4} W_{a\mu\nu} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{2.16}$$

[1] Let us note that this value is influenced by the definition (2.9), sometimes also the convention $\langle H^0 \rangle = \frac{v}{\sqrt{2}}$ is used, in that case $m_W^2 = \frac{1}{4} g_2 v^2$ and $v = 246$ GeV.

where

$$A^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{abc} A^\mu_a A^\mu_b$$

for $A_a = G_a, W_a, B$. Here g are the corresponding gauge couplings g_3, g_2, g_1, respectively, and f_{abc} are the structure constants of the given gauge group factor, i.e., the generators satisfy

$$[T_a, T_b] = i f_{abc} T_c \, .$$

In particular, $f_{abc} = 0$ for the Abelian $U(1)_Y$ corresponding to the gauge field B^μ.

2.3 Quark mixing

Due to the spontaneous symmetry breaking (2.9) the Yukawa interactions (2.7) generate also the Dirac mass terms for the charged leptons and quarks

$$\mathcal{L}_Y \ni v Y^e_{jk} (e^c)^T_{Lj} C e_{Lk} + v Y^d_{jk} (d^c)^T_{Lj} C d_{Lk} + v Y^u_{jk} (u^c)^T_{Lj} C u_{Lk} + h.c.$$

In order to obtain states with definite mass, the Yukawa couplings can be diagonalized by so-called biunitary transformation (A.7)

$$\begin{aligned} Y^e_{\text{diag}} &= E^T_C Y_e E, \\ Y^d_{\text{diag}} &= D^T_C Y^d D, \\ Y^u_{\text{diag}} &= U^T_C Y^u U \end{aligned} \tag{2.17}$$

which corresponds to the change of basis in the field generation space

$$e_{La} = (E^\dagger)_{aj} e_{Lj}, \qquad e_{Ra} = (E^\dagger_C)_{aj} e_{Rj}, \tag{2.18}$$

$$d_{La} = (D^\dagger)_{aj} d_{Lj}, \qquad d_{Ra} = (D^\dagger_C)_{aj} d_{Rj}, \tag{2.19}$$

$$u_{La} = (U^\dagger)_{aj} u_{Lj}, \qquad u_{Ra} = (U^\dagger_C)_{aj} u_{Rj}. \tag{2.20}$$

We use the same symbols for the fields corresponding to the "interaction" and "mass" basis, and these two will be distinguished by the choice of indices from the middle and the beginning of the alphabet, respectively.

Such a change of basis does not influence the interactions of the fermions with the Z-boson or the photon, however, it plays an important role for the charged currents mediated by the W boson. Indeed, rewriting the gauge interactions in (2.6) in terms of the physical fields $W^\pm$ (2.11) one obtains

$$\mathcal{L}_{gauge} \ni \frac{g_2}{\sqrt{2}} \left(\overline{u_{Lj}} \gamma^\mu W^+_\mu d_{Lj} + \overline{d_{Lj}} \gamma^\mu W^-_\mu u_{Lj} + \overline{\nu_{Lj}} \gamma^\mu W^+_\mu e_{Lj} + \overline{e_{Lj}} \gamma^\mu W^-_\mu \nu_{Lj} \right) . \tag{2.21}$$

The change of basis for the charged leptons (2.18) does not influence the form of these interactions since one can rotate the massless neutrino fields in an arbitrary way, hence, compensate for the rotation in the charged lepton generation space. On the other hand, using the physical quark fields denoted by the indices from the beginning of the alphabet in (2.19), (2.20) one obtains

$$\mathcal{L}_{gauge} \ni \frac{g_2}{\sqrt{2}}\overline{u_{La}}(U^\dagger)_{aj}\gamma^\mu W^+_\mu D_{jb}d_{Lb} + h.c. = \frac{g_2}{\sqrt{2}}(U^\dagger D)_{ab}\overline{u_{La}}\gamma^\mu W^+_\mu d_{Lb} + h.c. \quad (2.22)$$

The resulting charged-current interactions of the quarks are then in general *not* flavour diagonal. In case of two generations of quarks the 2×2 unitary matrix $U^\dagger D$ can be parametrized by a single angle (usually called the Cabibbo angle nowadays since it was N. Cabibbo [10] who first assumed the emergence of a mixing angle in the quark interactions), when the three unphysical phases are absorbed in the redefinition of the quark fields.

On the other hand, it was pointed out by Kobayashi and Maskawa [11] that in the case of three generations of quarks, the mixing matrix can not be made real in this way which, in turn, induces CP violation in the charged currents (2.22). Indeed, one can choose to make, e.g., the first row and the first column of the 3×3 matrix $U^\dagger D$ real by redefining the phases of the fields u_{La} and d_{Lb} (see, e.g., [12] for a detailed construction) which in fact corresponds to the ambiguity in U and D matrices (A.8). Such a redefinition, however, leads to absorbing only 5 phases whereas a general unitary 3×3 unitary matrix is parametrized by 6 phases (together with 3 angles).

In conclusion, the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix satisfies

$$U^\dagger D = K_1 V_{\text{CKM}} K_2 \quad (2.23)$$

where K_1 and K_2 are diagonal unitary matrices containing 3 and 2 phases, respectively, and can be parametrized in a "standard" way [13] as

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}. \end{aligned} \quad (2.24)$$

Here $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$ with $\theta_{12} \sim 13°$, $\theta_{13} \sim 0.2°$, $\theta_{23} \sim 2.4°$, and $\delta_{13} \sim 1.2$ rad [14]. Let us note that with the parametrization (2.24) it is not true that the first column and the first row of V_{CKM} is real as it was the case with the original parametrization of Kobayashi and Maskawa. The advantage of the standard parametrization of the CKM matrix is that the imaginary part of (2.24) is of the order of 10^{-3} which indeed corresponds to the magnitude of the CP violation in the quark sector determined by the Jarlskog invariant.

2.4 The triumph of the Standard Model

In this Section, we would like to stress that the SM is an extremely successful theory which poses strong constraints on possible physics beyond the SM.

SM well describes the physics at the energy scales stretching over many orders of magnitude: from the eV scale of the atomic physics up to the LHC energy scales in the 10 TeV ballpark no hints for new physics were observed. At the same time, the SM is tested with an extraordinary precision, e.g., the anomalous magnetic moment of the muon is measured with a relative uncertainty of about 10^{-9} which is compatible with the calculation where the QED contributions up to five loops and EW contributions up to two loops are included (see, e.g., the review [15]).

It is also remarkable how the predictions of the SM theory were often ahead of the experimental observations. For instance, in order to explain the CP violation in the weak interactions, Kobayashi and Maskawa suggested the third generation of matter fields in 1973 [11] when not even all the fermions from the second generation were observed experimentally. The need for the last missing piece of the second generation of fermions was even more pronounced[2]: besides the fact that a theory without the fourth quark could not explain the absence of the flavour-changing neutral currents (as pointed out by Glashow, Iliopoulos and Maiani [17]), complete families of fermions are needed in order to get a consistent theory with anomaly-free gauge symmetry.

Let us now discuss the problem of anomalies in the SM in more depth. It is interesting to turn around the requirement of the anomaly-free theory: if the SM fermion content is assumed, it can be shown (see, e.g., Section 22.4 of [18]) that the hypercharge of the five multiplets (2.1) can be determined when the cancellation of all the gauge anomalies (including the gravitational one) is assumed. The quantization of the electric charge of the SM chiral fermions can be explained in this way.

On the other hand, the global symmetries of the SM Lagrangian can be anomalous which also brings interesting implications. It was demonstrated by Adler, Bell and Jackiw [19, 20] that the observed decay rate for the $\pi^0 \to \gamma\gamma$ process can be explained by an anomalous chiral symmetry in the quark sector. Furthermore, the choice of the SM gauge group, together with the field content mentioned in Section 2.1, induces accidental global symmetries of the renormalizable Lagrangian which correspond to the conservation of the total lepton number L and baryon number B. Since these symmetries are anomalous, non-perturbative instanton processes can break B and L, however, such effects are very small in the present conditions and no lepton- or baryon-number-violating decays were observed (only upper bounds on the corresponding branching ratios are determined experimentally [14]). Interestingly, the global symmetry corresponding to the difference of the baryon and lepton numbers is anomaly-free in the SM.[3]

Also the particular choice of the SM scalar sector was supported by several obser-

[2]To such an extent that S. Glashow promised to eat his hat if the so-called charm quark was not found [16].

[3]Moreover, $U(1)_{B-L}$ can be gauged if the corresponding gauge bosons couple to extra fermions which are singlets with respect to the SM gauge group (see Section 22.4 of [18]), in particular, e.g., if a right-handed neutrino per fermion family is added. If, in addition, Dirac neutrinos are assumed, the hypercharge Y of the matter fields (2.1) is not uniquely determined by the requirement of anomaly-free theory since the hypercharge values $Y + \varepsilon(B - L)$ give non-anomalous solution for any ε.

vations even before the 2012 LHC discovery of a boson with mass 125 GeV [21–23], zero spin and positive parity [24], whose couplings to the other SM fields seem to be proportional to their masses [25].

First of all, the absence of the flavour-changing neutral currents already mentioned above is related to the scalar sector of the SM. Since the Yukawa couplings of the SM Higgs field (2.7) as well as the couplings of the neutral gauge bosons are diagonal in the generation space, the processes like $b \to s\gamma$ should be forbidden at the tree level and, hence, suppressed. We have seen in Section 2.3 that the quark interactions with the W boson are *not* flavour-diagonal, hence, the transitions like $b \to s\gamma$ can be generated at the loop level. The corresponding rare hadron decays (like $B \to K\mu^+\mu^-$, see, e.g., [26] for a review) were indeed observed, and the branching ratios obey the SM predictions posing strong constraints on the possible physics beyond the SM [27]. Contrarily, for massless neutrinos the lepton interactions with the W boson are flavour diagonal, hence, processes like $\mu \to e\gamma$ should be forbidden to all orders of perturbation theory. Indeed, the lepton-flavour-violating decays were not observed[4] and experiments constrain the corresponding branching ratios from above only [14].

Furthermore, the doublet nature of the only SM scalar field (2.4) implies the famous relation between the W and Z masses (2.14) which can be used as a consistency check of the SM spontaneous symmetry breaking pattern. If one defines

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W},$$

then the SM predicts $\rho = 1$ at the tree level. If, however, other scalar fields with hypercharges Y_r and VEVs v_r are added and if these fields are accommodated in the representation of $SU(2)_L$ with the highest weight T_r, then according to [12]

$$\rho = \frac{\sum_r |v_r|^2 \left[T_r\left(T_r+1\right) - Y_r^2\right]}{2\sum_r |v_r|^2 Y_r^2}. \tag{2.25}$$

This means that any number of scalar doublets with hypercharge $\pm\frac{1}{2}$ can be added to the SM preserving $\rho = 1$ at the tree level; however, the measurement of $\rho \sim 1$ constraints strongly other possible scalar representations.

In conclusion, SM is a mathematically consistent theory which is highly predictive, and a large number of its implications was confirmed experimentally. There are no discrepancies with the SM observed at colliders and in most of other terrestrial experiments. On the other hand, we will see in the next chapter that there are few "dark clouds" over the successful SM sky, in particular due to the cosmological observations and the phenomenon of neutrino oscillations.

[4]We will see in Sectionn 3.1 that neutrinos were proved to have tiny masses and the so-called neutrino oscillations violate the individual family lepton numbers. The neutrino mixing also implies that the processes like $\mu \to e\gamma$ are possible at the one-loop level, however, the decay rates are suppressed by powers of the m_ν/m_W ratio [28] and for $m_\nu \sim 1\,\text{eV}$ the predicted branching ratios are far from the reach of current experiments.

Part I

Beyond the Standard Model

Chapter 3

Problems of the Standard Model

Although the SM is an extremely successful theory, there are several experimental hints showing that the original Glashow-Weinberg-Salam model has to be extended. First, the non-vanishing mass of at least two neutrino species is an unquestionable fact nowadays as described in Section 3.1. Furthermore, the cosmological observations strongly indicate the existence of the so-called dark matter and the dominance of baryons over the antibaryons in our Universe. Neither of these phenomena can be explained within the SM as shown in Sections 3.2 and 3.3.

In Section 3.4 we describe the strong CP problem which is connected to the "unnatural" smallness of the SM parameter θ encoding the amount of the CP violation in the strong interactions. A note on the question of naturalness is in order here. As defined by t'Hooft [29], a physical parameter is allowed to be very small only if its replacement by zero would increase the symmetry of the system. The smallness of the parameter is then not spoiled by the quantum corrections at higher orders of the perturbation theory. For instance, the small electron Yukawa coupling is "natural", since its zero value would imply an additional chiral symmetry. The smallness of the parameter θ is, however, not natural in this sense.

Let us further note that often the so-called hierarchy problem is mentioned as one of the important problems of the SM, nevertheless, it will not be discussed here in detail. It is in fact another incarnation of the naturalness problem mentioned above: the Higgs mass at the EW scale is considered to be unnatural [29] since corrections to this mass of the order of the Planck scale (or other high scale present in the model) are claimed to be expected. The mechanisms, how to protect the Higgs mass by extra symmetry (like supersymmetry) were developed, or the possibility that in fact no fundamental scalars exist and the Higgs scalar is a composite particle was suggested. The new physics protecting the Higgs mass was, however, expected to be roughly at the TeV scale which is now disfavoured by the LHC searches.

3.1 Neutrino masses

In the original formulation of the SM given in Chapter 2 only left-handed neutrinos were present and these fields were assumed to be massless. Such assumption was, however, disproved recently as shown in Section 3.1.2. In order to describe the experiments which proved the neutrinos to be massive, the notion of neutrino oscillation is introduced in Section 3.1.1. Other experiments which intend to determine neutrino properties are then described in Sections 3.1.3 and 3.1.5. For the sake of this discussion the formalism concerning the neutrino mass terms and mixing is introduced in Section 3.1.4.

3.1.1 Neutrino oscillations

As in the case of quarks (see Section 2.3), the presence of the mass term for neutrinos implies the mixing in the lepton sector, i.e., the neutrino types ν_α, $\alpha = e, \mu, \tau$ produced in weak interactions together with the corresponding lepton do not coincide with their mass eigenstates[1]. For the sake of the present discussion, it is enough to assume that the left-handed neutrino states with a given flavour α are combined from the mass eigenstates ν_a as

$$|\nu_\alpha\rangle = V^*_{\alpha a} |\nu_a\rangle \, . \tag{3.1}$$

Here V is a unitary matrix[2] and we will distinguish between the flavour and mass eigenstates by the choice of indices from the Greek and Latin alphabet, respectively. Describing the evolution of the state (3.1) within the quantum mechanics as in [30,31], one gets

$$|\nu(t)\rangle = V^*_{\alpha a} e^{-iE_a t} |\nu_a\rangle$$

and the probability amplitude of detecting the neutrino at the time t as a flavour state $|\nu_\beta\rangle$ reads

$$A_{\nu_\alpha \to \nu_\beta}(t) = \langle \nu_\beta | \nu(t) \rangle = V_{\beta b} V^*_{\alpha a} e^{-iE_a t} \langle \nu_b | \nu_a \rangle = V_{\beta a} V^*_{\alpha a} e^{-iE_a t}. \tag{3.2}$$

Let us consider the simple case of two-flavour oscillations where one can assume

$$V = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

In this case

$$P_{\nu_1 \to \nu_2}(t) = |A_{\nu_1 \to \nu_2}(t)|^2 = \sin^2 2\theta \, \sin^2 \frac{(E_2 - E_1)t}{2} \approx \sin^2 2\theta \, \sin^2 \frac{\Delta m^2 t}{4E}$$

---

[1] The detailed explanation of the notation and of the emergence of the neutrino mixing matrix in the Lagrangian will be given in Section 3.1.4.

[2] We define (3.1) using V^* since then the identification of V with the so-called PMNS matrix will be straightforward – see Section 3.1.4.

with $\Delta m^2 = m_2^2 - m_1^2$. The last relation is based on

$$E_a = \sqrt{p_a^2 + m_a^2} \approx p_a + \frac{m_a^2}{2p_a} \approx p_a + \frac{m_a^2}{2E_a}$$

which holds for relativistic neutrinos and it was also assumed that the three-momentum of the mass eigenstates is equal: $p_a \equiv p \approx E$. If also the approximation $t \approx L$ is used, then one gets the dependence on the distance L travelled by neutrinos

$$P_{\nu_1 \to \nu_2}(L) \approx \sin^2 2\theta \, \sin^2 \left(\pi \frac{L}{L_{osc}} \right), \tag{3.3}$$

where

$$L_{osc} = \frac{4\pi E}{\Delta m^2} \approx 4\pi \times 197\,\mathrm{MeV\,fm}\,\frac{\mathrm{MeV}}{\mathrm{eV}^2} \left(\frac{E}{\mathrm{MeV}}\right) \left(\frac{\mathrm{eV}^2}{\Delta m^2}\right) \approx 2.48\,\mathrm{m} \left(\frac{E}{\mathrm{MeV}}\right) \left(\frac{\mathrm{eV}^2}{\Delta m^2}\right) \tag{3.4}$$

and (1.2) was used to reconstruct the metric units. In order to observe the sinusoidal dependence of the probability (3.3) on L, it is convenient if the experiment is set in such a way that $L \sim L_{osc}$. Moreover, the neutrinos are not produced and detected in one point, and if the path resolution is not good enough, the oscillations are averaged out. Similarly, good resolution in neutrino energy is needed to observe neutrino oscillations.

Interestingly, the measured neutrino oscillations can be approximately described by the two-flavour oscillations as will be shown below, hence, the explicit formula for three-flavour oscillations will not be given here. At the same time, the formula (3.3) corresponds to the oscillations in vacuum which will be enough for the order-of-magnitude estimates, although for precise determination of the oscillation parameters the matter effects have to be taken into consideration. The reader is referred, e.g., to the review [32] for a deeper account of these issues.

3.1.2 Oscillation experiments

Let us now briefly mention some of the oscillation experiments which led to determining the neutrino mass differences and mixing angles. We shall follow in this listing mainly the reference [32].

The first hint in favour of the existence of neutrino oscillations was the deficit in the flux of solar neutrinos observed by Davis et. al. [33] in the Homestake gold mine in the US. The spectrum of the electron neutrinos produced in the nuclear reactions in the Sun was predicted by the solar model [34], however, just around the third of the expected neutrino number was detected in the chlorine detector of Davis recording the $^{37}\mathrm{Cl}\,\nu_e \to {}^{37}\mathrm{Ar}\,e$ interactions. As we describe below, it was later confirmed that this was indeed due to $\nu_e \to \nu_\mu, \nu_\tau$ oscillations which can be effectively described by formula (3.3), unfortunately, solar neutrinos are produced in a large region, hence, averaging over L in (3.4)

$$P_{\nu_e \to \nu_\mu, \nu_\tau} \approx \frac{1}{2} \sin^2 2\theta_{sol}$$

is obtained. Consequently, it was hard to prove that the deficit in the solar neutrino flux was indeed caused by the neutrino oscillations.

The situation for so-called atmospheric neutrinos is much more favourable. These neutrinos are created in the upper part of the Earth's atmosphere where charged pions are produced due to the cosmic rays and decay later as

$$\pi^+ \to \mu^+\nu_\mu, \qquad \pi^- \to \mu^-\bar{\nu}_\mu.$$

Neither muons are stable and the decays

$$\mu^- \to e^-\bar{\nu}_e\nu_\mu, \qquad \mu^+ \to e^+\nu_e\bar{\nu}_\mu$$

imply the $\nu_\mu : \nu_e$ flux ratio to be approximately $2 : 1$. Moreover, since the cosmic rays are assumed to be isotropic, one expects the up-going and down-going flux of neutrinos to be approximately equal. The Super-Kamiokande (SK) experiment in the Kamioka mine in Japan is detecting these atmospheric neutrinos using the charged-current interaction $\nu_l N \to l N'$ with the nucleons in the water-Čerenkov detector and the type and direction of the final state leptons can be determined. Such measurements confirmed the expected behaviour for the electron neutrinos, however, the flux of muon neutrinos was strongly dependent on the direction. This dependence could be well fitted using the assumption of $\nu_\mu \to \nu_\tau$ oscillations with $\Delta m^2_{atm} \sim 10^{-3}\,\mathrm{eV}^2$ [35] and it was also the first strong evidence for the neutrino oscillations. Let us note that in a sense the experimentalists were lucky since the atmospheric neutrino setting $L \sim 10\,\mathrm{km} - 10\,000\,\mathrm{km}$ and $E \sim 1\,\mathrm{GeV}$ is well suited for observation of Δm^2_{atm} according to formula (3.4).[3]

The recent SK data together with data of other experiments like IceCube DeepCore located near the South Pole suggest[4] (see [36] and references therein)

$$\Delta m^2_{atm} \approx 2.5 \times 10^{-3}\,\mathrm{eV}^2, \qquad \sin^2\theta_{atm} \approx 0.5. \tag{3.5}$$

Another milestone in the history of oscillation experiments was the first observation of the solar neutrino *appearance* by the SNO experiment [37] (located in the Creighton Mine in Sudbury, Ontario, Canada) which, similarly as SK, is a water-Čerenkov detector, however, it uses the heavy water for the neutrino detection. The crucial improvement is brought by the deuteron interactions: comparing the rates of the processes $\nu_e\, d \to p\, p\, e$ and $\nu_{e,\mu,\tau}\, d \to \nu_{e,\mu,\tau}\, p\, n$, both the ν_e and $\nu_{\mu,\tau}$ flux could be measured by

[3] More precisely, $L_{osc} \sim 1000\,\mathrm{km}$ for atmospheric neutrinos with energy $\sim 1\,\mathrm{GeV}$, hence, in principle the first oscillation dip should be observed. In SK the energy resolution was not good enough, so rather the transition from $P_{\nu_\mu\to\nu_\mu} \approx 1$ for the down-going neutrinos to $P_{\nu_\mu\to\nu_\mu} \approx \frac{1}{2}$ for up-going neutrinos was observed. However, even the shape of this function gave a good enough evidence for neutrino oscillations.

[4] The precise values of the atmospheric mixing parameters depend on so-called neutrino mass hierarchy, however, we postpone introducing this notion below the formula (3.6) and give here the approximate values for Δm^2_{atm} and $\sin^2\theta_{atm}$ which hold for both neutrino hierarchies.

SNO, confirming that indeed, the deficit of ν_e observed by Davis can be explained by the flavour neutrino oscillations.

The director of the SNO experiment, Arthur B. McDonald, together with the leader of the SK experiment Takaaki Kajita, were awarded the 2015 Nobel Prize in Physics "for the discovery of neutrino oscillations, which shows that neutrinos have mass".

The solar neutrino experiments, however, do not have good enough sensitivity to Δm_{sol} and the determination of this quantity was based on the so-called reactor experiments. Here the $\bar{\nu}_e$ are produced in nuclear reactors usually few kilometres (or $\sim 100\,\text{km}$ in case of the KamLAND experiment in Japan) distant from a detector which uses the $\bar{\nu}_e p \to e^+ n$ interaction to detect the neutrinos and to determine their energy. Observing the neutrinos with energies $\sim 2 - 10\,\text{MeV}$, such experiments are sensitive to $\Delta m^2 \sim 10^{-5}\,\text{eV}^2$ (the formula (3.3) has the same form for antineutrinos as for neutrinos in the two-flavour case). The best fit using the data of the reactor experiment KamLAND and the solar neutrino experiments reads [36]

$$\Delta m^2_{sol} = 7.4 \times 10^{-5}\,\text{eV}^2, \qquad \sin^2\theta_{sol} = 0.3.$$

When studying the full problem of three-neutrino oscillations, the 3×3 matrix V present in (3.1) can be parametrized by three angles θ_{12}, θ_{23}, θ_{13} and one CP violating phase δ in a similar way as the CKM matrix (2.24). One finds out that

$$\theta_{atm} = \theta_{23}, \quad \Delta m^2_{atm} = |\Delta m^2_{23}|, \quad \theta_{sol} = \theta_{12}, \quad \Delta m^2_{sol} = \Delta m^2_{12} \tag{3.6}$$

where $\Delta m^2_{ij} = m^2_j - m^2_i$ and the two-flavour description of the oscillation experiments is allowed by the assumption $\theta_{13} \to 0$ and $|\Delta m^2_{23}| \gg \Delta m^2_{12}$. The quantities θ_{13}, δ and $\text{sgn}(\Delta m^2_{23})$ remain to be discussed.

The sign of Δm^2_{23} is still unknown, hence, two different scenarios are possible: either $m_1, m_2 < m_3$ which is usually called the normal hierarchy of neutrino masses or $m_3 < m_1, m_2$ corresponding to the so-called inverted hierarchy.

The non-zero value of θ_{13} was observed by the Daya Bay Reactor Neutrino Experiment [38] in China and other experiments like Double Chooz [39] in France which detect reactor $\bar{\nu}_e$ and are composed of a near detector needed in order to reduce systematic errors and a far detector in the distance of $L \sim 1\,\text{km}$ from the reactor corresponding to the first atmospheric oscillation dip. The precise best fit value [36] again depends on the neutrino mass hierarchy, for both of them approximately

$$\sin^2\theta_{13} \approx 0.02. \tag{3.7}$$

The last type of experiments we didn't mention, yet, are the so-called long-baseline experiments like T2K using the accelerator ν_μ from decays of charged pions and kaons which are produced by a proton beam hitting a fixed target. In T2K (Tokai to Kamioka) in Japan such ν_μ travel around $L \sim 300$ km to the SK detector and the observation of the appearance of ν_e in this neutrino beam was in fact the first hint of a possibly non-zero θ_{13} [40]. Future ambition of this experiment is the measurement of the CP violating phase δ, which is also a goal of another long-baseline experiment NOνA [41] in Fermilab, US. To achieve this, the measurements of the $\bar{\nu}_\mu$ beam will be compared with the ν_μ results and using this method the NOνA experiment can also have the potential to determine the neutrino mass hierarchy.

3.1.3 Direct measurements of neutrino masses

Unfortunately, the oscillation experiments can not determine all the neutrino properties, one of them is the absolute scale of the neutrino masses. In principle, the non-zero neutrino masses could be observed directly as described below; recent status of such measurements is summarized, e.g., in [42].

An exceptional occasion to determine the neutrino masses is a collapse of supernova: in 1987 a core-collapse supernova SN1987a in the Large Magellanic Cloud emitted a total energy of 3×10^{46} J, 99% of this energy was released by neutrinos which travelled the distance of about 165 000 lyr to the Earth and were detected by three independent neutrino detectors Kamiokande II in Japan, IMB in the US, and at Baksan in Russia. By comparison with the time when the particles at speed of light were detected one could conclude $m_\nu \lesssim 6\,\mathrm{eV}$ [43, 44]. However, since the expected rate for galactic core-collapse supernovae is 2-3 per century, this does not seem to be a practical way in which the neutrino masses can be determined.

More conventionally, neutrino masses could be directly determined by measuring the electron energy spectrum in the beta decay experiments. The maximum energy available for the electron in the $n \to e^- p^+ \bar{\nu}_e$ interaction depends on the effective mass of the electron (anti)neutrino

$$m_{\nu_e}^2 = \sum_{a=1}^{3} |V_{1a}|^2 m_a^2$$

where V was introduced in (3.1). Up to now only upper limits on this mass were determined by the Mainz Neutrino Mass Experiment [45] and the Troitsk experiment [46]: $m_{\nu_e} \lesssim 2$ eV. Currently, the beta decay experiment KATRIN is starting its operation [47] with expected sensitivity to the electron (anti)neutrino masses as low as 0.2 eV.

However, the strongest upper bound on the neutrino masses nowadays comes from cosmology. The measurement of the anisotropies in the cosmic microwave background by the Planck experiment suggests [48]

$$\sum_{a=1}^{3} m_a \leq 0.23\,\mathrm{eV} \tag{3.8}$$

(if the Λ-cold-dark-matter cosmological model is correct).

3.1.4 Interlude: Mixing of Dirac and Majorana neutrinos

Another open question about the neutrino properties is whether they are Dirac or Majorana particles. In order to explain how to look for the answer we introduce here the formalism concerning the neutrino masses and add also further details on neutrino mixing.

It was shown in Section 2.3 that although in the initial Lagrangian (2.6) the charged leptons need not correspond to the mass eigenstates, the rotation to the charged lepton mass basis does not change the shape of the Lagrangian, since it can be always absorbed into the rotation in the massless neutrinos. This is, however, not true when the neutrino masses do not vanish.

If neutrinos are Dirac particles, the situation is analogous to the case of quarks. The Yukawa Lagrangian (2.7) is supplemented by

$$-\mathcal{L}_Y \ni -Y^{\nu}_{\alpha\beta}(\nu^c)^T_{L\alpha} C H^T i\sigma_2 L_\beta + h.c. = Y^{\nu\dagger}_{\alpha\beta}\overline{L_\alpha} i\sigma_2 H^* \nu_{R\beta} + h.c. \tag{3.9}$$

which after inserting the VEV (2.9) implies

$$-\mathcal{L}_Y \ni v Y^{\nu}_{\alpha\beta}(\nu^c)^T_{L\alpha} C \nu_{L\beta} + h.c. \tag{3.10}$$

The general complex 3×3 matrix Y^ν can be again diagonalized by the biunitary transformation (A.7)

$$Y^{\nu}_{\rm diag} = N_C^T Y^\nu N \tag{3.11}$$

and the mass eigenstates correspond to

$$\nu_{La} = (N^\dagger)_{a\alpha}\nu_{L\alpha}, \qquad \nu_{Ra} = (N_C^\dagger)_{a\alpha}\nu_{R\alpha}. \tag{3.12}$$

Following the usual notation used in the description of neutrino oscillations (3.1) we distinguish the flavour and mass eigenstates by the choice of indices from the Greek and Latin alphabet, respectively. Let us stress that, historically, the neutrino types ν_α, $\alpha = e, \mu, \tau$ are denoted by the charged lepton which accompanies their production or detection induced by the weak charged currents (3.16), i.e., they correspond to the "flavour" eigenstates contrary to the case of quarks where the fields u, c, t etc. correspond to the mass eigenstates.

Let us note that in order to obtain $\sim$ eV neutrino masses from (3.10), $Y^\nu \sim 10^{-11}$ is needed which is usually considered as weird when compared to the size of other Yukawa couplings in the SM (although strictly speaking even such Yukawa couplings are natural in the t'Hooft sense).

The mass matrix diagonalization is slightly different if neutrinos are Majorana particles. Although the Majorana mass term for left-handed neutrinos (which are part of an $SU(2)_L$ doublet) can not be written in the Lagrangian in a renormalizable and at the same time $SU(2)_L$ invariant way, we will see in Chapter 4 that there are several extensions of the SM where effectively the Majorana mass term

$$\frac{1}{2}M^{\nu}_{\alpha\beta}\nu^T_{\alpha L} C \nu_{\beta L} + h.c. \tag{3.13}$$

is obtained. An important implication of the Majorana nature of the neutrinos is then the lepton number violation. Let us further stress that in the case of Majorana neutrinos, the right-handed neutrinos need not be introduced.

M^ν in (3.13) is a symmetric matrix, hence, according to (A.10) there exists a unitary matrix N such that

$$M^\nu_{\text{diag}} = N^T M^\nu N \tag{3.14}$$

where M^ν_{diag} is a real non-negative diagonal matrix and

$$\nu_{La} = (N^\dagger)_{a\alpha}\nu_{L\alpha}. \tag{3.15}$$

Formally, the change of basis (3.12) and (3.15) affects the charged-current Lagrangian (2.21) in the same way, hence, together with (2.18) one obtains both in Dirac and Majorana case[5]

$$\mathcal{L}_{gauge} \ni \frac{g_2}{\sqrt{2}}\overline{e_{La}}(E^\dagger)_{a\alpha}\gamma^\mu W^-_\mu N_{\alpha b}\nu_{Lb} + h.c. = \frac{g_2}{\sqrt{2}}(E^\dagger N)_{ab}\overline{e_{La}}\gamma^\mu W^-_\mu \nu_{Lb} + h.c. \tag{3.16}$$

In the Dirac case, one can perform identical analysis as for the CKM matrix in Section 2.3 finding that

$$E^\dagger N = K_3 V^D_{\text{PMNS}} K_4 \tag{3.17}$$

where K_3 and K_4 are diagonal unitary matrices containing 3 and 2 phases, respectively, and the unitary matrix V^D_{PMNS} can be parametrized by three angles and one phase in the same way as the CKM matrix (2.24). The subscript PMNS refers to the names of physicists who first considered the neutrino oscillations (see Section 3.1.1): Pontecorvo [49], Maki, Nakagawa, and Sakata [50].

On the other hand, in the case of the Majorana mass term (3.13), the phases of the physical neutrino fields can not be freely redefined when the masses are required to be real (this also corresponds to the fact that the diagonalization (3.14) uniquely determines the phases in N, see Corollary 3 in Appendix A.2). Consequently,

$$E^\dagger N = K_3 V^M_{\text{PMNS}} \tag{3.18}$$

where again K_3 is a diagonal unitary matrix containing 3 phases. V^M_{PMNS} then includes the so-called Majorana phases α_{21}, α_{31} and can be parametrized as [14]

$$V^M_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}. \tag{3.19}$$

The first factor is, similarly as in (2.24), a product of the matrices corresponding to the 2-3, 1-3 and 1-2 rotations, respectively, and can be used to parametrize V^D_{PMNS}. The

[5] Let us stress that the convention here is different than in the quark sector where the CKM matrix was defined by the Lagrangian term (2.22) containing W^+_μ.

CP violation in the neutrino sector is determined by the value of the phase δ (together with the Majorana phases).

Let us note that often the basis is chosen so that $E = \mathbb{1}$ in (3.17) or (3.18), hence, the charged lepton flavour eigenstates coincide with mass eigenstates, and the mass basis for left-handed neutrinos can be defined as

$$\nu_{La} = (V^{\dagger}_{\text{PMNS}})_{a\alpha}\nu_{L\alpha} \tag{3.20}$$

where the generic symbol V_{PMNS} is used for both V^{M}_{PMNS} and V^{D}_{PMNS}.

Let us now return to the key relation for explaining the phenomenon of neutrino oscillations (3.1). If we realize that the production of the *state* $|\nu_\alpha\rangle$ is due to the *h.c.* term in (3.16), then using the relation for the *field* ν_α (3.20), the formula (3.1) containing V^{*}_{PMNS} is obtained, hence, the matrix V can be identified with V_{PMNS}. The oscillation experiments are, however, not sensitive to the Majorana phases in (3.19) [51], hence, other experiments have to be devised in order to determine weather the neutrinos are Majorana or Dirac particles.

3.1.5 Dirac or Majorana?

We have now prepared all the tools to describe the strategy, how to determine if the neutrinos are Dirac or Majorana particles.

The lepton-number violating process of the neutrinoless double beta decay

$${}^{A}_{Z}\text{X} \rightarrow {}_{Z+2}^{\;\;\;\;A}\text{X}\; e^{-}e^{-} \tag{3.21}$$

depicted at the nucleon level in Figure 3.1 could prove the Majorana nature of the neutrinos and this interaction is searched for in experiments which are so numerous that we better choose to cite, e.g., the review [52] for reference.

The interaction similar to (3.21) where also two antineutrinos are emitted (corresponding to a Feynman diagram where, in contrast to Figure (3.1), the antineutrinos do not merge in the Majorana mass term but become outgoing particles) is called double beta decay and is preferred over the single beta decay by some nuclei such as ${}^{76}_{32}$Ge (${}^{76}_{33}$As is heavier than ${}^{76}_{32}$Ge, whereas in the decay to ${}^{76}_{34}$Se the energy of $\sim 2\,\text{MeV}$ is released). Observing the decays of such nuclei, it is searched for a peak in electron energy which would correspond to the situation when no neutrinos are emitted.

The difficulty of this task lies in the fact that the decay width for the process (3.21) is proportional to the neutrino mass, more precisely to the quantity $|\langle m_{ee}\rangle|^2$ where[6]

$$\langle m_{ee}\rangle \equiv \sum_{a=1}^{3}(V^{M}_{\text{PMNS}})^2_{1a}m_a \tag{3.22}$$

[6]Let us note that, e.g., in [32] it is claimed that $\langle m_{ee}\rangle$ corresponds to the element M^{ν}_{11} of the Majorana mass matrix (3.13), however, more precisely $\langle m_{ee}\rangle = (M^{\nu}_{11})^*$ in accordance with the formula (3.14).

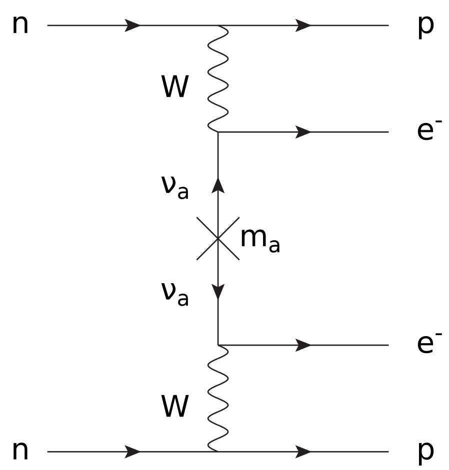

Figure 3.1: Feynman diagram describing the leading order contribution to the neutrinoless double beta decay (3.21).

as can be seen by observing the vertices in Figure 3.1. The quantity (3.22) is sensitive to the Majorana phases in (3.19) and constraining the value of (3.22) also allows to set the limits on the absolute scale of neutrino masses. Bearing in mind the nuclear matrix element uncertainties, the recent bound determined by the neutrinoless double beta decay experiments is around $|\langle m_{ee}\rangle| \lesssim 0.5$ eV [52].

3.2 Dark matter

Another phenomenon which can not be explained within the context of the SM is the problem of the so-called dark matter. Some of the observational evidence for dark matter will be listed here following mainly the reference [53] (and partly also the reviews [54–57]). One of the dark matter candidates, the axion, will be studied later in Section 5.1.

The discrepancy between the observed luminous matter distribution and the indirectly determined gravitational potential was first noticed in the galaxy clusters [58,59] where, on one hand, the total luminosity L can be measured and, on the other hand, the virial theorem can be used to determine the total mass M. This theorem relates the (internal) kinetic and potential energy of a system in a state of equilibrium as

$$2T + V = 0, \qquad T = \frac{1}{2}M\langle v^2\rangle, \qquad V = -\frac{1}{2}GM^2\left\langle \tfrac{1}{r}\right\rangle$$

where $\langle v\rangle$ is the mean (mass weighted) square velocity relative to the center of mass and $\langle \frac{1}{r}\rangle$ is the mean inverse separation. The virial theorem applies at least approximately to spherical clusters like the one in the Coma constellation and observation of several such clusters usually implies

$$\frac{M}{L} \sim (200 - 300)\, h\, \frac{M_\odot}{L_\odot}$$

where h is the Hubble constant in the units of $100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ (A.13) and $M_\odot$, $L_\odot$ is the mass and luminosity of the Sun. This result suggests that a large fraction of the matter is non-luminous.

Another hint for the presence of the extra non-luminous matter was given by the observation of the rotation curves of the type "Sc" spiral galaxies. Most of the luminous matter in galaxies is concentrated near the center of the galaxy, hence, the velocities of the stars in the outer parts were expected to follow the Kepler law, $v \propto r^{-\frac{1}{2}}$. On the other hand, the observation of 21 different galaxies led V. Rubin [60] to the conclusion that: "Most rotation curves are rising slowly even at the farthest measured point. Neither the high nor low luminosity Sc galaxies have falling rotation curves. Sc galaxies of all luminosities must have significant mass located beyond the optical image."

The two kinds of observations mentioned above lead to a rough estimate [53] $\Omega_M \sim 0.15$ for the fraction of the critical density of the Universe due to non-relativistic matter (A.16) (we will see that it is considerably improved by modern methods). It is known that most of this matter is non-luminous, i.e., not forming the ordinary stars or luminous gases. It still could be, however, formed by ordinary "baryonic" matter (nuclei and electrons) in non-luminous gases, black holes, neutron starts etc. The X-ray luminosity of clusters of galaxies partly shows that this is not the case (only the collisions of ordinary baryonic particles can produce the X-rays, see Section 1.9.B of [53]), however, we will mention here another kind of proof that the "dark matter" is not formed by baryons, based on the formation of the light elements in the early Universe.

Roughly 100 s after the Big bang the Universe consisted of free protons, neutrons and electrons in the thermal equilibrium with the photons (the neutrinos were already decoupled from the thermal bath) and the temperature was low enough that the formation of deuterium ^{2}D (consisting of one proton and one neutron) could start. Later, also heavier nuclei ^{3}He, ^{4}He, ^{6}Li, ^{7}Li were composed and the key input determining the amount of these elements produced is the ratio of the neutron and proton number density before the synthesis of deuterium started. For a detailed description of this so-called cosmological nucleosynthesis, careful consideration of the thermal history of the Universe is necessary (see, e.g., the Section 3.2 in [53]), however, the rule of thumb is that the higher the nucleon density was, the earlier the nucleosynthesis began, hence, the less time there was for neutrons to decay into protons. Consequently, the measurement of the element abundance in the Universe allows to determine

$$\eta \equiv \frac{n_B}{n_\gamma} \sim 6 \times 10^{-10} \tag{3.23}$$

where n_B and n_γ are the baryon and photon densities at the time of nucleosynthesis, respectively. The present value of the cosmic microwave background temperature (i.e. the present photon density) then allows to compute the nucleon density fraction $\Omega_B h^2 \sim 0.02$ [53] which is definitely inconsistent with the assumption that all the non-relativistic matter in the Universe is formed by the baryonic matter.

There is further evidence for the dark matter such as the effect of gravitational lensing, which allows even to visualize the dark matter in the galaxy clusters, however, let us proceed straight to the most precise quantitative results which can be nowadays determined by examining the anisotropies in the cosmic microwave background. The results of the Planck experiment [48] suggest

$$\Omega_M = 0.31, \qquad \Omega_M h^2 = 0.14, \qquad \Omega_B h^2 = 0.02, \quad \Rightarrow \quad \Omega_{DM} h^2 = 0.12. \tag{3.24}$$

Let us also note that the simulations of the structure formation in the Universe revealed that this dark matter had to be "cold", i.e., non-relativistic, when it decoupled from the thermal bath, which excludes neutrinos as a dominant component of the dark matter. Consequently, no SM particle can form the dark matter which is another motivation for the theories beyond the SM.

Different models for dark matter particles were proposed, one of the most popular options assumes certain supersymmetric partners of the neutral SM particles as the dark matter candidates. However, we refer the reader to reviews like [53–57] for general discussion of dark matter models and to Section 5.1 for a particular example of the dark matter candidate called axion.

3.3 Baryon-antibaryon asymmetry of the Universe

There is another contradiction with the expectations based on the SM which is connected to the cosmological observations and we will again follow the reference [53] in its discussion.

As explained in the previous section, the abundance of the light elements formed in the process of cosmological nucleosynthesis suggests that the ratio of baryon and photon number density obeyed (3.23) at the time of nucleosynthesis. On the other hand, if the amount of quarks and antiquarks before their annihilation (which took part before the nucleosynthesis) is assumed to be equal, then the thermal evolution would imply the leftover baryon to photon ratio to be much smaller. For this reason, initial asymmetry

$$\frac{n_B - n_{\bar{B}}}{n_\gamma} \sim \eta$$

has to be assumed, however, can not be explained within the SM.

In [61] A. Sakharov specified three conditions which are necessary for a theory to explain this asymmetry from the first principles:

1. Baryon number violation.

2. Violation of C and CP symmetries.

3. The Universe must at some time depart from a state of the thermal equilibrium.

The second condition is satisfied already in the SM (see Chapter 2): the Lagrangian of weak interactions violates both C and P symmetries, and the CP violation is induced by the non-zero phase δ_{13} in the CKM matrix (2.24).

On the other hand, the baryon number B is conserved perturbatively in the SM (the Lagrangian exhibits the $U(1)_B$ symmetry) as discussed in Section 2.4. However, even in the SM the baryon number is violated due to instanton processes. While at low temperatures these effects are highly suppressed, they could play an important role in the early Universe. Although there is a strong evidence that the net baryon number can not be produced in this way (see, e.g., Section 3.3 in [53] and references therein), the instanton processes can convert the baryon number density into lepton number density and vice versa. For this reason the explanation of the baryon-antibaryon asymmetry can be provided by the so-called leptogenesis [62] responsible for the net lepton number production. Also L is conserved in the SM, however, addition of, e.g., right-handed neutrinos with Majorana masses can provide a source of the lepton number violation necessary for the leptogenesis mechanism to work.[7]

Finally, the Universe is pulled from the state of the thermal equilibrium due to its expansion – this corresponds to the so-called "freeze-out" of the particle species from the thermal bath, which enables, e.g., the above mentioned mechanism of leptogenesis. Another explanation of the baryon-antibaryon asymmetry of the Universe is the so-called electroweak baryogenesis (see, e.g., [63] for a review) which assumes the departure from the thermal equilibrium due to the electroweak phase transition which is assumed to be first order. Also in this case, however, extra fileds have to be added to the SM for the phase transition to be strong enough.

An attempt to incorporate the mechanism of the electroweak baryogenesis in a particular model beyond the SM can be found in [1]. Moreover, the problem of the baryon-antibaryon asymmetry of the Universe can be considered as one of the motivations for building models incorporating baryon or lepton number violation in general.

3.4 Strong CP problem

In (2.6) the kinetic term for the gluon fields is present, however, in principle also a term of the form

$$\mathcal{L} \ni \theta \frac{g_3^2}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} G^{\mu\nu} G^{\rho\sigma} \tag{3.25}$$

[7] In fact, it can be shown (see Section 3.3 in [53]) that violation of $B-L$ is necessary in order to explain non-zero net baryon number of the Universe. $B-L$ is indeed violated in the heavy neutrino decays substantial for the leptogenesis mechanism, but, e.g., not in the processes involving the GUT scale leptoquarks in some of the unified theories (see Chapter 6).

could be included with θ being a dimensionless constant. As discussed, e.g., in Section 23.6 of [18], although this term is a total derivative irrelevant at the perturbative level, it can lead to CP violating phenomena in the low-energy non-perturbative regime of QCD. In particular, the electric dipole moment of the neutron

$$d_N \sim 10^{-16}|\theta| e\,\mathrm{cm}$$

is implied. Since, however, $d_N \lesssim 10^{-25} e\,\mathrm{cm}$ is constrained by experiments [14], the value of $|\theta| \lesssim 10^{-9}$ is required and the so-called strong CP problem refers to the question, how to explain this extreme smallness.

To be more precise, the parameter θ in (3.25) can be shifted due to a redefinition of the quark fields with flavour f and mass m_f:

$$q_f \to e^{i\alpha_f \gamma_5} q_f, \quad \Rightarrow \quad \theta \to \theta + 2\sum_f \alpha_f, \quad m_f \to e^{2i\alpha_f} m_f$$

and, hence, the physically observable quantity subject to the above experimental restrictions is in fact

$$\bar{\theta} = \theta - \arg \prod_f m_f. \tag{3.26}$$

This also suggests a simple solution of the strong CP problem: if any of the quark fields is massless, then the parameter θ can be always absorbed in the redefinition of the corresponding field. The current data, however, show that all the quarks are massive and the strong CP problem persists. One of its possible solutions was suggested by Peccei and Quinn [64, 65] and will be described in Section 5.1.

Chapter 4

Neutrino masses in simple SM extensions

In this chapter we recall several ways in which small neutrino masses can be explained, some of these mechanisms will be then employed in specific models studied in this thesis. In Section 4.1, the $d = 5$ operator providing the Majorana neutrino masses is introduced and its renormalizable realisations then lead to different SM extensions as described in Section 4.2. The options relevant for our further studies are detailed in Sections 4.3-4.5.

4.1 The Weinberg operator

It was argued in Section 3.1 that although in the original formulation of the SM neutrinos were assumed to be massless, there is a strong experimental evidence for non-zero neutrino masses at the scale of 10^{-1} eV: the lower bound on the (heaviest) neutrino mass is given by the atmospheric mass difference (3.5) determined by the oscillation experiments whereas the upper bound comes from cosmology (3.8). In this section we will show how the presence of a high energy scale where new physics emerges can give a clue to the extreme smallness of neutrino masses.

If we stick to the field content of the minimal Standard Model, only the Majorana mass term for neutrinos can be introduced. As pointed out in Appendix A.1, this would mean that neutrinos are neutral particles, which is true in the low energy broken phase of the SM – the neutrinos are electrically neutral and colorless. However, if the full $SU(3)_c \times SU(2)_L \times U(1)_Y$ theory is considered, the neutrinos are part of an $SU(2)_L$ doublet with hypercharge $-\frac{1}{2}$ and the simple Majorana mass term can not be introduced at the renormalizable level. Remarkably, the only 5-dimensional operator in expansion (1.1) that can be built from the SM fields (originally introduced by Weinberg [66]) gives rise to the neutrino Majorana mass term after spontaneous symmetry breaking:

$$\mathcal{L}^{d=5} = \frac{C^{\nu}_{jk}}{2M}(L^T_j i\sigma_2 H)C(H^T i\sigma_2 L_k) + h.c. \tag{4.1}$$

Here C^{ν} is a symmetric matrix in the generation space with j, k assigning the corresponding indices, the transpose of the Higgs and lepton doublets H, L and the matrix $i\sigma_2$ is due to the $SU(2)_L$ structure and for the lepton doublet L the transpose corresponds also to the Lorentz structure similarly as the C matrix. After the VEV insertion (2.9), the term (4.1) implies

$$M^{\nu} = C^{\nu}\frac{v^2}{M} \tag{4.2}$$

for the Majorana mass matrix (3.13), hence, if $C^{\nu} \sim \mathcal{O}(1)$ is assumed, the neutrino data suggest $M \sim 10^{13-14}$ GeV determining the scale where new physics has to emerge, since the effective description by (4.1) breaks down.

Let us emphasize that the operator (4.1) carries the lepton number $L = 2$ and similarly $B - L = -2$ where B is the baryon number, hence, breaks these accidental symmetries of the SM (see Section 2.4). This is why this effective operator can not be built in the SM even at the loop level, and a source of L violation has to be added in order to get a renormalizable realisation of this operator. Recall also that L (or B) and $B - L$ violation is well motivated by the problem of baryon-antibaryon asymmetry of the Universe mentioned in Section 3.3.

4.2 Renormalizable realizations of the Weinberg operator

Different ways in which the Weinberg operator (4.1) can be "opened" will be described here together with the basic properties of the corresponding models as summarized, e.g., in [67]. The scenarios which will play an important role for the core of this text will be detailed in the following sections.

4.2.1 Tree level realizations

At the tree level, there are two non-equivalent ways in which the 4 fields involved in the interaction (4.1) can be paired as depicted in Figure 4.1.

Fermionic mediators: type I and type III seesaw mechanism

First, one can construct a vertex out of H, L and a third particle with $Y = 0$ which must be a fermion in order to get a Lorentz-invariant structure. This option corresponds to the diagram on the l.h.s. of Figure 4.1. Since $2 \otimes 2 = 1 \oplus 3$ holds for $SU(2)$ representations, the mediator can be a singlet or a triplet with respect to the $SU(2)_L$

gauge group, and the $SU(2)_L$ singlet with $Y = 0$ can be readily identified with the right-handed neutrino ν_R. The extra interactions in the Lagrangian then read

$$y_R L^T i\sigma_2 H C \nu_R^c + h.c. \qquad \text{or} \qquad y_\Delta L^T i\sigma_2 \boldsymbol{\sigma} H \boldsymbol{\Delta}_F^c + h.c. \tag{4.3}$$

where y_R and y_Δ contain the coupling constants (the flavour-space dimensionality of these matrices depend on the details of the model), $\boldsymbol{\Delta}_F$ is an $SU(2)_L$ triplet and the boldface marks the vector nature of the given symbol. The first option in (4.3) corresponds to the neutrino Dirac mass term (3.9).[1] In order to conserve the fermion currents in the left diagram in Figure 4.1, Majorana mass terms

$$\frac{1}{2} m_R \nu_R^T C \nu_R + h.c. \qquad \text{or} \qquad \frac{1}{2} m_{\Delta_F} \boldsymbol{\Delta}_F^T C \boldsymbol{\Delta}_F + h.c. \tag{4.4}$$

have to be introduced. Here m_R and m_{Δ_F} are matrices with flavour-space dimensionality again depending on the details of the model. In any case, the larger $|m_R|$ or $|m_{\Delta_F}|$ is, the lighter the active neutrinos are as will be shown explicitly in Section 4.3 for the singlet case. For this reason, the mechanisms of this kind are called seesaw mechanisms, and the variant with a fermionic singlet ν_R originally studied by [68–70] is usually denoted as its type I. The case with the fermionic triplet mediator is usually called type III seesaw mechanism, and we refer the reader to the original study [71] for the details.

Let us note that if we assign the lepton number +1 to ν_R (or Δ_F), then the terms (4.3) conserve the lepton number, whereas (4.4) breaks it by 2 units. Alternatively, the lepton number of ν_R (or Δ_F) could vanish and (4.4) would conserve the lepton number, however, then (4.3) would break it. In short, there is no possible assignment of the lepton number allowing the L conservation within such models.

Let us further note that since only the neutrino mass differences are so far determined experimentally, it is still possible that one of the active neutrinos is massless, hence, only 2 copies of right-handed neutrinos or scalar triplets are required by the neutrino data.

Scalar mediator: type II seesaw mechanism

The other option how to open the operator (4.1) is to build a vertex out of two Higgs doublets H and two lepton doublets L as shown on the right diagram of Figure 4.1. In this case the mediator must be a scalar particle with hypercharge +1. Although one would naively expect again both singlet and triplet variant of $SU(2)_L$ representations, here the interaction of two left-handed neutrinos with $T_3^L = +\frac{1}{2}$ must be included, hence, only the option with the triplet field $\boldsymbol{\Delta}$ containing the $T_3^L = -1$ component is possible.

Also in this case the smallness of the neutrino masses is linked to the large mass of $\boldsymbol{\Delta}$; however, the detailed description of the key terms in the Lagrangian is postponed

[1] In order to get precisely the Dirac mass we also write ν_R^c in (4.3) instead of ν_R.

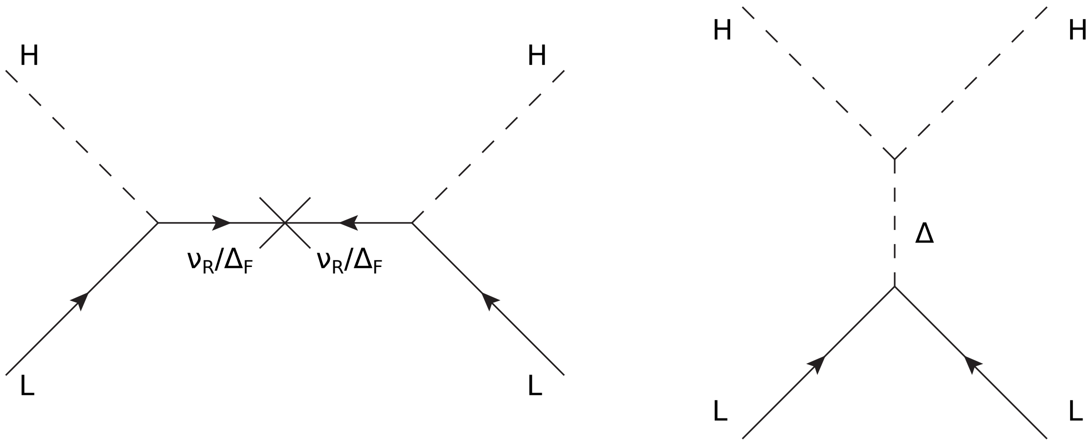

Figure 4.1: Different options how the Weinberg operator (4.1) can be built in a renormalizable way at the tree level. In the left diagram the mediator is a right-handed neutrino ν_R or a fermionic $SU(2)_L$ triplet Δ_F, which corresponds to the type I and type III seesaw mechanism, respectively; the former variant will be further studied in Section 4.3. On the right-hand side, the scalar $SU(2)_L$ triplet Δ induces the type II seesaw mechanism detailed in Section 4.4.

to Section 4.4. At this stage let us only note that models of this kind were first studied in combination with type I seesaw [72] usually motivated by an embedding of the SM to some bigger gauge group [73, 74] and are referred to as type II seesaw mechanism nowadays.

It is also already clear from the right diagram in Figure 4.1 that the interaction of $\boldsymbol{\Delta}$ either with the two lepton doublets or with the Higgs doublets does violate the lepton number (depending on the value of L assigned to $\boldsymbol{\Delta}$).

4.2.2 Loop diagrams

Finally, the Weinberg operator (4.1) can be build at the loop level, however, as we explain above, new fields providing lepton-number-violating interactions have to be added into the model. In this case the smallness of neutrino masses is partly due to the loop suppression and, hence, the extra fields need not to be as heavy as in the case of the seesaw mechanism above.

At the one loop level, a realization of the Weinberg operator was proposed by A. Zee [75] and requires adding a second Higgs doublet and an $SU(2)_L$ singlet scalar h^- with electric charge -1. It can be seen on the left diagram of Figure 4.2 that the interactions of the latter field are the source of the lepton number violation.

At two loops, A. Zee and K. S. Babu [76, 77] independently proposed a model with extra $SU(2)_L$ singlet scalars h^- and k^{--} with electric charges -1 and -2 allowing for

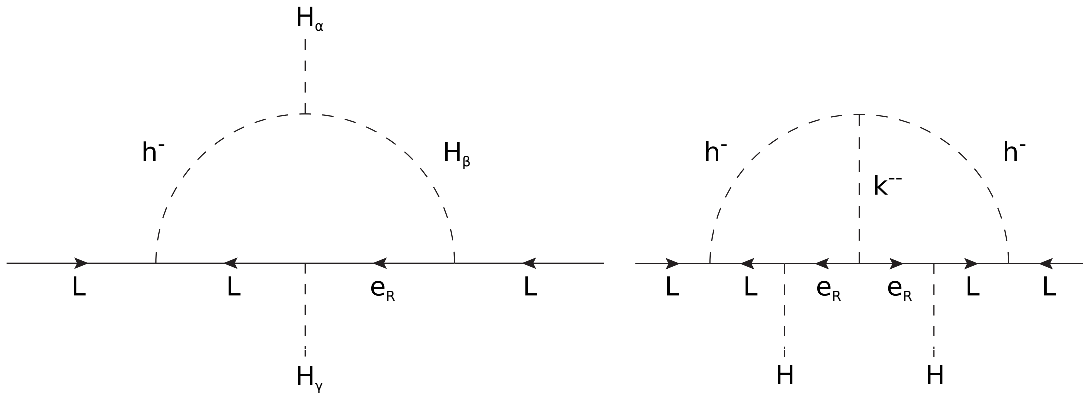

Figure 4.2: Sample of loop realization of the Weinberg operator (4.1). One-loop diagram on the left-hand side corresponds to the Zee mechanism where at least two different Higgs doublets (denoted by indices α, β, γ) are necessary. It will be shown in Section 4.5.1 that $\alpha \neq \beta$ is required by the vertex structure. The right diagram describes the Zee-Babu mechanism detailed in Section 4.5.2.

the right diagram in Figure 4.2.

These two models will be detailed in Section 4.5.

4.3 Type I seesaw mechanism

Let us now study the details of the model including the Majorana fermion ν_R which was introduced in the previous section.

As we explained above, only 2 right-handed neutrinos are required by the experimental data, on the other hand, the $B - L$ local symmetry is anomaly free and can be gauged if there is a right-handed neutrino for each family (see also Section 2.4). We will be interested in models with extended gauge symmetries in Part II and addition of three right-handed neutrinos will be considered, however, the general case when n species of the left-handed neutrinos are accompanied by n right-handed neutrinos will be assumed similarly as in the reference [30].

Although in the simple type I seesaw mechanism the Majorana mass term (A.4) for the left-handed neutrinos is absent, let us include it in order to obtain a formula applicable also to more complicated theories (such as those where type I and type II seesaw mechanisms are combined):

$$\mathcal{L}_{mass} \ni \frac{1}{2}\nu_L^T C M_L \nu_L + h.c.$$

with M_L being $n \times n$ symmetric matrix and ν_L denotes the n-dimensional vector of the left-handed neutrinos. Furthermore, the Dirac mass term is included in (4.3) after the VEV insertion (2.9)

$$\mathcal{L}_{mass} \ni \nu_L^T C M_D (\nu^c)_L + h.c. = \frac{1}{2}\left[\nu_L^T C M_D (\nu^c)_L + (\nu^c)_L^T C M_D^T \nu_L\right] + h.c.$$

where M_D is an $n \times n$ matrix and also ν^c has to be understood as an n-dimensional vector. Finally, the Majorana mass term for the right-handed neutrinos (4.4) can be rewritten using the properties of the particle-antiparticle conjugation given in Appendix A.1 as

$$\mathcal{L}_{mass} \ni \frac{1}{2}\nu_R^T C M_R^\dagger \nu_R + h.c. = \frac{1}{2}\left[\nu_R^T C M_R^\dagger \nu_R + (\nu_R)^{cT} C M_R (\nu_R)^c\right] = \frac{1}{2}(\nu^c)_L^T C M_R (\nu^c)_L + h.c.$$

with M_R being a symmetric $n \times n$ matrix.[2] In summary, the general neutrino mass term can be written in a compact form

$$\mathcal{L}_{mass} \ni \frac{1}{2} \begin{pmatrix} \nu_L^T & (\nu^c)_L^T \end{pmatrix} C \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ (\nu^c)_L \end{pmatrix} + h.c. \tag{4.5}$$

Let us now consider the regime when[3]

$$|M_L| \ll |M_D| \ll |M_R|. \tag{4.6}$$

The $2n \times 2n$ mass matrix in (4.5) can be then diagonalized using a unitary transformation which is given by

$$U = \begin{pmatrix} \mathbb{1} & \rho \\ -\rho^\dagger & \mathbb{1} \end{pmatrix} \tag{4.7}$$

up to $\mathcal{O}(\rho^2)$ corrections ($U^\dagger U = \mathbb{1} + \mathcal{O}(\rho^2)$, and ρ is considered to be a small perturbation). Indeed,

$$U^T \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} U = \begin{pmatrix} M_L - \rho^* M_D^T - M_D \rho^\dagger + \rho^* M_R \rho^\dagger & M_L \rho - \rho^* M_D^T \rho + M_D - \rho^* M_R \\ \rho^T M_L - \rho^T M_D \rho^\dagger + M_D^T - M_R \rho^\dagger & \rho^T M_L \rho + M_D^T \rho + \rho^T M_D + M_R \end{pmatrix}.$$

Applying (4.6), one can neglect the first two terms in the off-diagonal entries and the choice

$$\rho^* = M_D M_R^{-1} \tag{4.8}$$

gives (after neglecting also the first three terms in the lower right corner):

$$U^T \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} U \approx \begin{pmatrix} M^\nu & 0 \\ 0 & M_R \end{pmatrix},$$

with

$$M^\nu = M_L - M_D M_R^{-1} M_D^T. \tag{4.9}$$

[2]We apologize for slightly changing the notation with respect to (4.4), however, this notation leads to the assignment of neutrino mass matrix mostly used in literature.

[3]This setting can be motivated, e.g., in case of combined type I and type II seesaw where M_L is tiny as a result of the type II seesaw mechanism, M_D is assumed to lie at the EW scale for $\mathcal{O}(1)$ Yukawa couplings and M_R is determined by the high scale of the new physics.

Let us note that if the EW-scale M_D is assumed in the simplest type I seesaw case with $M_L = 0$, then $M_R \sim 10^{13}$ GeV is expected in order to get $M^\nu \sim$ eV which corresponds to the estimate below (4.2).

The $n \times n$ symmetric matrices M^ν and M_R can be further diagonalized in a usual way (A.10)

$$U_\nu^T M^\nu U_\nu = M^\nu_{\text{diag}}, \qquad U_R^T M_R U_R = M_R^{\text{diag}}$$

in order to obtain $2n$ massive Majorana particles. The form of the transformation U (4.7) and (4.8) then implies that the n light states are composed mostly of the original ν_L fields with an admixture of states $(\nu^c)_L$ being proportional to $|M_R|^{-1}|M_D|$, and similarly the n heavy states are mostly composed of $(\nu^c)_L$.

Let us stress that the $2n\times 2n$ neutrino mass matrix present in the general Dirac+Majorana mass term (4.5) is fully diagonalized by the $2n \times 2n$ unitary matrix

$$U_{D+M} = \begin{pmatrix} U_\nu & 0 \\ 0 & U_R \end{pmatrix} U. \tag{4.10}$$

The PMNS matrix then corresponds to the upper left $n \times n$ block of (4.10) which in general need not be unitary.

4.4 Type II seesaw mechanism

In order to find a formula for the neutrino masses induced by the right diagram in Figure 4.1, one has to specify the Lagrangian of the theory, namely, the Yukawa couplings of the scalar triplet $\mathbf{\Delta}$ and the scalar potential. It will be convenient to denote

$$\Delta = \frac{1}{\sqrt{2}} \mathbf{\Delta} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\Delta^+}{\sqrt{2}} & \Delta^{++} \\ \Delta^0 & -\frac{\Delta^+}{\sqrt{2}} \end{pmatrix}$$

where $\mathbf{\Delta} = (\Delta_1, \Delta_2, \Delta_3)$, and the states with definite T_3^L and electric charge read

$$\Delta^{++} = \frac{1}{\sqrt{2}}(\Delta_1 - i\Delta_2), \qquad \Delta^+ = \Delta_3, \qquad \Delta^0 = \frac{1}{\sqrt{2}}(\Delta_1 + i\Delta_2)$$

similarly as for the W bosons in the SM (2.11) which are also part of an $SU(2)_L$ triplet (with vanishing hypercharge in contrast to $Y = 1$ for Δ).

The Yukawa coupling to lepton doublets can be then written as

$$-\mathcal{L}_Y \ni \frac{1}{2} Y^\Delta_{jk} L_j^T C i\sigma_2 \Delta L_k + h.c. \tag{4.11}$$

which generates a (symmetric) Majorana mass matrix M^ν for left-handed neutrinos if the neutral component of Δ acquires a VEV:

$$\langle \Delta^0 \rangle = v_\Delta \quad \Rightarrow \quad M^\nu = v_\Delta Y^\Delta. \tag{4.12}$$

The reason for the smallness of the neutrino masses is now hidden in the minimization of the scalar potential as sketched below. Let us consider, e.g., the scalar potential used in the recent reappraisal of the type II seesaw mechanism [78]:

$$
\begin{aligned}
V = & - m_H^2 H^\dagger H + \frac{\lambda}{4}(H^\dagger H)^2 + M_\Delta^2 \mathrm{Tr}(\Delta^\dagger\Delta) + \mu(H^T i\sigma_2 \Delta^\dagger H + h.c.) + \lambda_1 (H^\dagger H) Tr(\Delta^\dagger\Delta) \\
& + \lambda_2 \left[Tr(\Delta^\dagger\Delta)\right]^2 + \lambda_3 Tr(\Delta^\dagger\Delta\Delta^\dagger\Delta) + \lambda_4 H^\dagger \Delta\Delta^\dagger H \qquad (4.13)
\end{aligned}
$$

where H is the usual Higgs doublet with $\langle H^0 \rangle = v$ and the term proportional to μ is the one responsible for the trilinear scalar interaction in the right diagram in Figure 4.1. One of the minimization conditions for the scalar potential above is then

$$
M_\Delta^2 = \frac{\mu v^2}{v_\Delta} - (\lambda_1 + \lambda_4) v^2 - 2(\lambda_2 + \lambda_3) v_\Delta^2 \qquad (4.14)
$$

and the masses of the charged scalars read

$$
m_{\Delta^{++}}^2 = \frac{\mu v^2}{v_\Delta} - 2\lambda_3 v_\Delta^2 - \lambda_4 v^2, \qquad m_{S^+}^2 = \frac{\mu v^2}{v_\Delta} + 2\mu v_\Delta - \frac{\lambda_4}{2}(v^2 + 2v_\Delta^2).
$$

Here

$$
S^+ = \frac{1}{\sqrt{v^2 + v_\Delta^2}} (v\Delta^+ - \sqrt{2} v_\Delta H^+) \qquad (4.15)
$$

and the orthogonal combination is the massless Goldstone boson.

Assuming that the dimensionful parameters M_Δ, μ correspond to some new physics scale, one can apply the limit $M_\Delta, \mu \gg v$ to (4.14) getting

$$
v_\Delta \approx \frac{\mu v^2}{M_\Delta^2} \sim \mathcal{O}(\frac{v^2}{M_\Delta}). \qquad (4.16)
$$

Consequently, $v_\Delta \ll v$. In this approximation, the mixing of the SM scalars with the components of Δ can be neglected as can be seen, e.g., in (4.15) and the masses of all three components become approximately equal:

$$
m_{\Delta^{++}}^2 \approx m_{\Delta^+}^2 \approx m_{\Delta^0}^2 \approx \frac{\mu v^2}{v_\Delta} \approx M_\Delta^2. \qquad (4.17)
$$

In conclusion, (4.12) together with (4.16) and (4.17) shows that, again, the suppression of neutrino masses is due to a large energy scale which corresponds to the mass of the scalar triplet. Similarly as in the type I seesaw case, if $Y^\Delta \sim \mathcal{O}(1)$, M_Δ is in the ballpark of some 10^{13-14} GeV.

4.5 Loop generation of neutrino masses

4.5.1 Zee model

Let us specify the Lagrangian supporting the left diagram in Figure 4.2 and, subsequently, evaluate the neutrino masses within this model originally introduced by A. Zee [75].

The trilinear coupling of the two Higgs doublets and the $SU(2)_L$ singlet scalar h^- arises from a term

$$V \ni \mu_{\alpha\beta} H_\alpha^T i\sigma_2 H_\beta h^- \tag{4.18}$$

where the nature of the $SU(2)_L$ structure containing the antisymmetric $i\sigma_2$ implies that $\mu_{\alpha\beta} = -\mu_{\beta\alpha}$. This is also the reason why the extra Higgs doublet had to be included in the theory. The most general Yukawa couplings for the leptons then read

$$\mathcal{L}_Y \ni f_{jk} L_j^T C i\sigma_2 L_k h^+ + Y^e_{\alpha ij} \overline{L}_i e_{Rj} H_\alpha + h.c. \tag{4.19}$$

where i, j are the generation indices and f is an antisymmetric matrix due to an extra antisymmetric $SU(2)_L$ contraction on top of the usual Lorenz structure present, e.g., in the Majorana mass term. In general

$$\langle H_1^0 \rangle = v_1, \qquad \langle H_2^0 \rangle = v_2$$

with the only constraint $v_1^2 + v_2^2 = v^2$ required in order to reproduce the correct weak boson masses. When switching to the mass basis for the scalar fields, for the most general case (4.19) the Yukawa interactions of the observed Higgs boson (corresponding to one of the mass eigenstates) may not be proportional to the fermion masses and even need not be diagonal in the lepton flavour space. This is a common feature of the two-Higgs-doublet models (see, e.g., the review [79]), however, as we explain in Section 2.4, the presence of the lepton-flavour-violating interactions is constrained experimentally. The simplest solution in this context is to introduce an extra global symmetry of the model as, e.g., in [80] which allows only one of the Higgs doublets to couple to the leptons. If, for instance,

$$Y^e_\alpha = 0 \qquad \text{for } \alpha = 2 \tag{4.20}$$

in (4.19) then the charged lepton masses m^e_i are determined by Y^e_1 only.

For such a setting it is simple to approximate the contribution of the left diagram in Figure 4.2 to the Majorana neutrino mass matrix M^ν_{ij}. Since $\beta = \gamma = 1$ in this diagram due to (4.20) the product of all the relevant couplings reads $f_{ij} \left(\frac{m^e_j}{v_1}\right)^2 \mu_{12}$ if we work in a basis where Y^e_1 is diagonal, hence, $(Y^e_1)_{jj}$ are proportional to the charged lepton masses m^e_j. If the contribution of the symmetric diagram is added, one concludes

$$M^\nu_{ij} \propto f_{ij} \mu_{12} \frac{m^{e2}_j - m^{e2}_i}{v_1^2}. \tag{4.21}$$

The full formula for the neutrino masses includes the factors coming from the loop integration and, of course, the suppression due to the potentially large masses of h^- and the second Higgs doublet. Regardless of the absolute size of M^ν_{ij}, one can conclude, however, that the neutrino mass matrix of the form (4.21) is excluded by the experimental data. Indeed, M^ν in (4.21) has all the diagonal elements equal to zero, in particular, it is traceless. As explained, e.g., in [81] such a matrix can not accommodate the oscillation data.

Of course, the situation becomes more complicated when the condition (4.20) is relaxed, and viable models were suggested [82, 83] featuring an interesting interplay between the neutrino data and the constraints implied by the absence of the lepton-flavour-violating interactions.

4.5.2 Zee-Babu model

Apart from the h^- scalar introduced above, also a doubly charged scalar k^{--} is included in this model in order to enable the right diagram in Figure 4.2. The Yukawa couplings of the leptons read

$$\mathcal{L}_Y \ni f_{jk} L_j^T C i\sigma_2 L_k h^+ + Y^e_{ij} \overline{L}_i e_{Rj} H + g_{ij} e^T_{Ri} C e_{Rj} k^{++} + h.c. \tag{4.22}$$

where i, j are the flavour indices, f is again antisymmetric whereas g is symmetric in the flavour space; moreover, the scalar potential has to be extended by a term

$$V \ni \mu (h^-)^2 k^{++} + h.c. \tag{4.23}$$

In the limit $m_h, m_k \gg m^e_a$ where m^e_a are the charged lepton masses, the full formula for the Majorana neutrino mass matrix M^ν reads [84]:

$$M^\nu_{ij} = 16 \mu I f_{ia} m^e_a g^*_{ab} m^e_b f_{jb}. \tag{4.24}$$

Here the loop integration gives

$$I = \frac{1}{(16\pi^2)^2} \frac{1}{M^2} \frac{\pi^2}{3} \tilde{I}(r), \qquad M \equiv \max(m_h, m_k), \quad r \equiv \frac{m_k^2}{m_h^2}$$

and the function $\tilde{I}(r)$ is close to 1 for a wide range of r:

$$\tilde{I}(r) = \begin{cases} 1 + \frac{3}{\pi^2}(\log^2 r - 1) & \text{for } r \gg 1, \\ 1 & \text{for } r \to 0. \end{cases}$$

The smallness of the neutrino masses (4.24) has several sources in this case. First, the couplings f and g can not be $\mathcal{O}(1)$ since the corresponding interactions induce processes like $\mu^- \to e^- \gamma$, $\mu^- \to e^- e^- e^+$ or violate the charged current universality (f mimics the charged current interactions, however, the couplings can differ for individual flavours contrary to the case of the weak interactions). As summarized in [84], the constraints on different entries of f, g vary, however, $|f|, |g| \lesssim 10^{-2} - 10^{-1}$ is a generic value if $m_h, m_k \sim 1$ TeV. Furthermore, the two-loop integration brings another suppression, hence, indeed, the new physics scale should be as low as $M, \mu \sim 1$ TeV to ensure realistic neutrino masses. This simple estimate is supported also by the numerical analysis of [84].

4.6 Probing the origin of neutrino masses at the LHC

In previous section we explained that extra scalars at the TeV scale can be expected in case of Zee-Babu model which opens the possibility to test the models for neutrino mass generation at colliders. Inspired by Chapter 9 of the review [85], also the collider signatures of the tree-level mechanisms for neutrino mass generation will be considered here showing that for particular parameter settings even these scenarios could be tested at the LHC.

Let us note that the models of neutrino mass generation can be revealed also due to lepton-flavour-violating interactions which are suppressed in the SM (see Section 2.4).

Type I seesaw

As estimated below (4.9), $M_R \sim 10^{13}\,\text{GeV}$ is expected if M_D is at the EW scale, i.e., if the Dirac neutrino Yukawa couplings (3.9) satisfy $Y^\nu \sim \mathcal{O}(1)$. On the other hand, in the setting with $M_R \sim 1\,\text{TeV}$ the correct neutrino masses can be achieved if $Y^\nu \sim 10^{-5}$ (it is interesting to compare this value with $Y^\nu \sim 10^{-11}$ in case of purely Dirac neutrinos). In such a case the kinematics allows the right-handed neutrinos to be, in principle, produced at the LHC, however, the production rate is suppressed by the $\nu_L - \nu_R$ mixing which, according to Section 4.3, is proportional to $|M_D|/|M_R| \sim 10^{-6}$. Consequently, the right-handed neutrinos in the type I seesaw model can not provide any distinguishing collider signatures by themselves.

Nevertheless, as we will show in Part II of this thesis, within the framework of theories with extended gauge symmetries the right-handed neutrino masses can be connected to the breaking scale of gauge symmetries like $U(1)_{B-L}$ or $SU(2)_R$ (the counterpart of $SU(2)_L$ in left-right symmetric models). Consequently, TeV-scale right-handed neutrinos can be accompanied by gauge bosons like Z' or W_R at the same scale. These gauge bosons can be then observable in the channels

$$\begin{aligned} q\bar{q}' &\to W_R^\pm \to e_a^\pm \nu_R, \\ q\bar{q} &\to Z' \to \nu_R \nu_R \end{aligned}$$

where e_a is a charged lepton of the a-th family. For the decay $\nu_R \to e_a^\pm W^\mp$ the probability to produce the positive- or negative-charge lepton is the same due to the Majorana nature of ν_R, hence, the production of W_R or Z' can be accompanied with the same-sign dileptons without missing transverse energy. Moreover, ν_R can be long-lived enough to provide displaced vertices in the detectors. In this way the type I seesaw could be revealed by the LHC, for quantitative discussion see [85].

Type II seesaw

Also the extra scalars which support the type II seesaw mechanism can have masses at the TeV scale for particular settings of the model parameters. The most spectacular collider signature is then provided by the doubly charged scalar Δ^{++} which decays into a pair of same-sign leptons if $v_\Delta \lesssim 10^{-4}\,\text{GeV}$. The branching ratios for the different combinations of the final state lepton flavours are then connected with the neutrino data due to (4.11) and strongly distinguish, e.g., the normal and inverted hierarchy of neutrino masses.

On the other hand $v_\Delta \lesssim 1\,\text{GeV}$ is constrained by the measurement of the ρ parameter (2.25) and for the range $10^{-4}\,\text{GeV} \lesssim v_\Delta \lesssim 1\,\text{GeV}$, Δ^{++} decays predominantly to a pair of W bosons providing also observable signal at the LHC. We again refer to [85] for quantitative results.

Type III seesaw

As will be discussed in Section 7.1.7, the type III seesaw mechanism is well motivated, e.g., in the models based on the $SU(5)$ gauge group, moreover, the gauge unification constraints predict the fermionic triplet Δ_F to be below the TeV scale in such models. The processes

$$q\bar{q}' \to W^{\pm *} \to \Delta_F^{\pm}\Delta_F^{0}$$
$$q\bar{q} \to Z^*/\gamma^* \to \Delta_F^{+}\Delta_F^{-}$$

are then available at the LHC and the decays of the Δ_F components lead again to multi-lepton states. However, the difference in the kinematics allows to distinguish these signals from the analogous type II seesaw signatures. As for the type I seesaw case, displaced vertices may occur due to small couplings of Δ_F to the SM fields.

Zee-Babu model

As we explained in Section 4.5.2, TeV scale scalars occur also in the setting with the neutrino masses generated at the two-loop level. The most spectacular collider signature is due to the decay of the doubly charged k^{++} scalar, where the branching ratios depend on the couplings g_{ij} (4.22) which, in turn, determine the neutrino mass matrix (4.24). The collider signatures are, hence, related to the neutrino properties in a straightforward way and, e.g., the normal and the inverted neutrino mass hierarchies could be distinguished or the CP violating phase δ could be predicted based on these branching ratios. A detailed numerical analysis which takes into account the current neutrino data and also the constraints on the Zee-Babu model parameters from the lepton-flavour-violating processes is available in [84].

Chapter 5

Scalar extensions of the Standard Model with neutrino-axion interconnection

In this chapter the bottom-up approach to building the models beyond SM is applied when the deficiencies of the SM mentioned in Chapter 3 are addressed by extending the SM as little as possible. The well-known axion solution to the strong CP problem which provides also a dark matter candidate is described in Section 5.1. This scheme is further extended in order to incorporate also the (small) neutrino masses as explained in Sections 5.2 and 5.3 which serve as an introduction to the full articles [1] and [2] attached to this thesis as Appendices B and C, respectively.

5.1 Axion phenomenology

In Section 3.4 we discussed the so-called strong CP problem connected to the extreme smallness of the coupling θ (3.25). We also explained that if one of the quarks was massless, it would be possible to set $\theta = 0$ using the freedom in the redefinition

$$q \to e^{i\alpha\gamma_5} q \tag{5.1}$$

of the massless quark which implies $\theta \to \theta + 2\alpha$. It was shown by R. Peccei and H. Quinn [64,65], that there is another option how to use the transformation (5.1) to zero out θ. If at least one of the quark masses is induced by the spontaneous symmetry breaking, i.e., by a VEV of a scalar field, then the shift (5.1) can correspond to an extra chiral $U(1)$ symmetry of the Lagrangian. This symmetry is nowadays usually called Peccei-Quinn symmetry and, e.g., a toy model of [64] with only one quark flavour and a singlet scalar φ including the Yukawa term

$$\mathcal{L}_Y \ni y\,\overline{q_L} q_R \varphi + h.c.$$

is invariant with respect to the $U(1)_{PQ}$ transformations $\varphi \to e^{-2i\alpha}\varphi$ and (5.1). It was shown in [64] that if we assign $\langle\varphi\rangle \equiv e^{i\beta}\lambda$, then the minimization of the full potential yields

$$\arg\left[y\, e^{i(\theta+\beta)}\right] = 0$$

which in turn implies the physical parameter $\bar{\theta}$ (3.26) equal to zero when the quark fields are rotated in such a way that real quark masses are obtained.

Also in the context of the full SM the parameter $\bar{\theta}$ can be dynamically set to zero if the $U(1)_{PQ}$ symmetry is introduced. The quark masses arise due to the non-zero VEV of the Higgs field anyway, however, in (2.7) H and H^* couple to the up- and down-type quarks, respectively. Since the up- and down-type quarks transform with respect to $U(1)_{PQ}$ in the same way, a second Higgs doublet has to be included in order to obtain Yukawa couplings invariant with respect to the PQ symmetry. One of the Higgs doublets, usually denoted as H_u then couples to the up quarks whereas H_d couples to the down quarks. Both H_u and H_d are charged with respect to $U(1)_{PQ}$, hence, this symmetry is spontaneously broken by their vacuum expectation values. Since $U(1)_{PQ}$ is already broken by the non-perturbative effects, the spontaneous symmetry breaking induces a pseudo-Goldstone boson with a small mass as pointed out by S. Weinberg and F. Wilczek [86, 87].

It can be shown (see, e.g., the reviews [88, 89]) that if the scalars ϕ_i with the PQ charges X_i and VEVs v_i exist in the theory, then the decay constant and the mass of this so-called axion are given by

$$f_a^2 = \sum_i X_i^2 v_i^2, \qquad m_a = m_\pi \frac{f_\pi}{f_a} \frac{\sqrt{z}}{1+z} \sum_i X_i^{(q)} \tag{5.2}$$

where m_π and f_π are the pion mass and decay constants, $z = \frac{m_u}{m_d}$ is the ratio of the up- and down-quark masses and $X_i^{(q)} \equiv X_i^R - X_i^L$ is the PQ charge of the quark q_i ($X_i^{R/L}$ corresponds to the PQ charge of the right/left-handed quark component). The couplings of the axion to the matter fields and photons are then proportional to m_a (or, equivalently, inversely proportional to the decay constant f_a). Unfortunately, with f_a at the EW scale, the axion mass is $m_a \sim \mathcal{O}(100\,\mathrm{keV})$ and its couplings are too strong, hence, the original Weinberg-Wilczek axion was quickly excluded even by the laboratory experiments [90].

If, however, a large PQ symmetry breaking scale is introduced, axions may become much lighter due to (5.2) and practically "invisible". Two options are mostly considered in the literature:

1. Either the ordinary quarks are assumed to be neural with respect to the $U(1)_{PQ}$ symmetry and only an extra heavy quark Q transforms as in (5.1). One ordinary Higgs doublet (neutral with respect to $U(1)_{PQ}$) is then sufficient, on the other hand, an extra discrete symmetry has to be introduced in order to forbid the hard mass term for Q and a weak singlet scalar σ has to be included. This scalar

interacts through the Yukawa coupling with the heavy quark Q only and its VEV can be made, in principle, arbitrarily large enabling $f_a \sim \langle\sigma\rangle$ to be phenomenologically viable. This scheme was considered by Kim, Shifman, Vainshtein, and Zakharov [91, 92] and the "KSVZ" axion has no tree level couplings to the SM leptons and the c, t, s, b quarks, however, the tree-level coupling to u and d quarks is induced due to the small mixing of the axion with π^0.

2. Similarly, the "DFSZ" axion occurs in a scheme first considered by Dine, Fischler, Srednicki and Zhitnitsky [93, 94]. Here, like in the Weinberg-Wilczek scenario, the two Higgs doublets H_u and H_d charged with respect to the $U(1)_{PQ}$ couple to the ordinary quarks, however, the PQ symmetry is at the same time broken by an extra singlet scalar σ which couples to H_u, H_d through the scalar potential. The VEV of σ is again assumed to be much larger than the EW scale, hence, $f_a \sim \langle\sigma\rangle$ holds as in the KSVZ case according to (5.2).

The above models are constrained by several astrophysical and cosmological observations. First, the weakness of the axion interactions with the ordinary matter enables these particles to carry out the energy from the stellar cores and, hence, to modify the evolution of the stars. The observed properties of the Sun and other stars at different stages of the stellar evolution then constrain g_{aee}, g_{aNN} and $C_{a\gamma\gamma}$, i.e., the couplings of axions to electrons, nucleons and photons, respectively. Since the electron-axion coupling is loop-suppressed in the case of the KSVZ axion, the translation of the bounds on g_{aee} to the limits on f_a differ for the KSVZ and DFSZ models (see, e.g., the reviews [88, 89] and references therein). On the other hand, the most stringent bound on f_a nowadays comes from the supernovae explosions limiting mainly the g_{aNN} coupling. This coupling is determined by the axion-π^0 mixing which is comparable in the KSVZ and DFSZ cases[1] and the observation of the SN1987a supernova (already mentioned in Section 3.1.3 as a source of information on neutrino properties) yields the common bound [89]

$$f_a \gtrsim 10^9\,\text{GeV} \tag{5.3}$$

corresponding to $m_a \lesssim 10^{-3}\,\text{eV}$.

Moreover, the lifetime of the axion with mass $m_a \lesssim 24\,\text{eV}$ is longer than the age of the Universe, hence, it can provide an explanation of the dark matter problem mentioned in Section 3.2. If the thermal production of axions is assumed (for $m_a \gtrsim 10^{-3}\,\text{eV}$, there is a period in the history of the early Universe when axions were in thermal equilibrium with the plasma), these particles would contribute to the hot dark matter with the critical density fraction [89]

$$\Omega_{a,\text{th}}\, h^2 \sim 10^{-2}\frac{m_a}{\text{eV}}.$$

[1] More precisely, there is also a contribution to g_{aNN} due to tree-level couplings of axion to the SM quarks which differ for the KSVZ and DFSZ axion, however, the two components of g_{aNN} are comparable and g_{aNN} is of the same order of magnitude for the KSVZ and DFSZ cases.

The hot dark matter option is, however, unfavoured by the structure formation mechanism as pointed out in Section 3.2; moreover, the thermal axions could form a significant part of the dark matter only if $m_a \sim 10\,\text{eV}$ which is not compatible with the bound (5.3).

On the other hand, there is also a non-thermal mechanism for cosmological axion production first described in [95–97]. It can be shown that the axion mass depends strongly on the temperature and $m_a \to 0$ in the early Universe when the temperature was much greater than the QCD scale. No particular value of $\bar{\theta}$ (3.26) was then chosen by the dynamics due to the shift symmetry of the axion field. Only when m_a became comparable to the Universe expansion rate H (A.12), $\bar{\theta}$ began to roll towards $\bar{\theta} = 0$ and oscillated around this value. These oscillations provided axion energy density which again depends on the (zero-temperature) axion mass and

$$\Omega_{a,\text{non-th}} \lesssim \mathcal{O}(1) \qquad \text{for} \qquad f_a \lesssim 10^{12}\,\text{GeV}. \tag{5.4}$$

Consequently, the values $f_a \gtrsim 10^{12}\,\text{GeV}$ are excluded, although several theoretical uncertainties enter this estimate (see, e.g., [89]).

The relic axions are nowadays searched for in the so-called microwave cavity haloscopes where the primordial axions should convert to microwave photons due to a strong magnetic field, and there are other experiments which can constrain the axion coupling to the photons. However, up to now the bounds (5.3) and (5.4) were not considerably improved and we refer the reader, e.g., to recent reviews [98, 99] for the future prospects of such experiments.

5.2 Massive neutrinos and invisible axion minimally connected

We have seen in Chapter 4 that the neutrino masses can be explained by an extension of the scalar sector of the SM only, namely, the weak triplet Δ induced the type II seesaw mechanism, and also the loop generation of the light neutrino masses was allowed by addition of extra scalars. In all these settings, however, there occurred an a priory unknown dimensionful parameter which could be determined only by the requirement of the realistic neutrino masses. Indeed, μ in the scalar potential of the type II seesaw (4.13) determined the neutrino masses through (4.12) and (4.16), similarly, μ_{12} from (4.18) in the Zee model occurs in (4.21) and μ from (4.23) sets the scale for neutrino masses within the Zee-Babu model by (4.24).

In [1], on the other hand, the corresponding trilinear couplings were replaced by quartic interactions with a singlet scalar field σ acquiring a VEV, and the neutrino sector was then related to the properties of σ. Moreover, the PQ symmetry was introduced and σ was identified with the scalar field responsible for the PQ symmetry breaking in the DFSZ scheme described in the previous section. This gave rise to realistic models incorporating small neutrino masses, solution to the strong CP problem and a dark matter candidate.

Let us note that other attempts to link the PQ symmetry and the massive neutrinos can be found in the literature (see [1] for the list of references) and, in particular, the simplest version of the Zee model extended by the axion was studied in [100, 101]. However, this scenario is excluded because of the wrong prediction on neutrino mixing as in the original Zee model (see Section 4.5.1), hence, in [1] we considered the variant of the Zee model proposed by Babu and Julio [83]. This model allows for realistic neutrino mixing and can be extended by the invisible axion, moreover, a second Higgs doublet needed to introduce the PQ symmetry is present in this model anyway. Also the type II seesaw and Zee-Babu models were considered in [1], and when these models were extended by a second Higgs doublet and a scalar singlet σ, scenarios with neutrino-axion interconnection were obtained.

The masses of the extra scalars (apart from σ) were assumed to lie close to the EW scale providing the possibility to test the proposed scenarios at colliders (see Section 4.6). Such setting requires extremely small values of some of the dimensionless scalar couplings, which is, however, natural in the t'Hooft sense (see the beginning of Chapter 3) since extra Poincaré symmetry corresponding to the decoupling of the σ field is introduced when the ultraweak scalar couplings are set exactly to zero.

Let us add that the presence of extra light scalars can also open the possibility of the electroweak baryogenesis [63] as shortly discussed in [1], however, a detailed study is needed to confirm if this solution to the issue of the baryon asymmetry of the Universe (see Section 3.3) is indeed applicable within the framework of the proposed models.

Technically, there were several tasks to be done in order to determine whether the models under consideration are viable or not and, in particular, following issues were addressed by the author of this thesis. First, the assignment of the PQ charges had to be figured out and it was revealed that, e.g., in case of the Zee-Babu model different options are possible for different choices of the terms in the scalar potential. Furthermore, the scalar spectrum had to be studied and since the scalar couplings were present also in the formulas for neutrino masses, the consistency with the neutrino data had to be checked. Finally, the predictions for the lepton-flavour-violating processes and the collider signatures of the extra scalars were studied confirming that the proposed models indeed comply with the current data.

5.3 Neutrino-axion-dilaton interconnection

The "axionized" type II seesaw model introduced above was further studied in [2] starting with the classically scale-invariant setting, i.e., without any dimensionful couplings present in the model. Similarly as in [102], the PQ scale was then dynamically generated through the Coleman-Weinberg mechanism [103] which also implied the EW symmetry breaking. The hierarchy between the PQ and the EW scales can be ensured by the smallness of certain scalar couplings, which can be again considered as natural in the t'Hooft sense since the zero values of these couplings implied the shift symmetry of the σ scalar. The spontaneous breaking of this shift symmetry by the VEV of σ also gives rise to a light pseudo-Goldstone boson called dilaton which is mainly formed by the real part of the field σ.

Due to the decreased number of free parameters in the scalar potential, this setting is even more constrained than the one of [1]. If the current measurements of the Higgs boson properties at the LHC are taken into account and also the boundedness of the scalar potential is required for the energies up to the Planck scale, only certain values of, e.g., $\tan\beta$ (the ratio of the VEVs of the two Higgs doublets) are allowed.

In order to determine such constraints, again the scalar spectrum had to be studied together with the implications for the neutrino masses. Furthermore, the renormalization group equations for the scalar couplings had to be specified and solved, which was the main contribution of the author of this thesis.

Part II

Unified Theories

Chapter 6

Do protons decay?

Since baryon number B is conserved in the SM Lagrangian (see Chapter 2), proton as the lightest baryon should be stable and, indeed, the proton decay is neither observed in nature nor in specially designed experiments (see Section 6.3).

On the other hand, there are several reasons why to expect the baryon number non-conservation. As we already mentioned in Section 2.4, even in the SM the baryon number is violated due to the instanton processes, yet the predicted proton decay rate is extremely small [104]. More importantly, the explanation of the baryon-antibaryon asymmetry of the Universe requires baryon number violation as explained in Section 3.3.

As we will see in the following section, in the expansion (1.1) the baryon-number-violating operators occur at the $d = 6$ level and we will inspect the implications of such operators for the proton decay. Let us anticipate that the unified theories which will be the subject of Chapters 7 and 8, may provide possible renormalizable realisation of the operators responsible for the proton decay.

6.1 $d = 6$ effective operators

In Section 4.1 we introduced the only independent $d = 5$ operator which can be constructed from SM fields (4.1), and we also explored some of the implications of the presence of this operator, in particular, the Majorana nature and the (tiny) mass of the neutrinos accompanied by the lepton number violation. Furthermore, we tried to find the renormalizable realizations of this operator in Section 4.2. Here, we would like to pursue a similar path for the $d = 6$ operators, although full consideration of all these operators is far beyond the scope of this text.

As derived in [105], there is in total $64 = 15 + 19 + 30$ independent $d = 6$ operators where the sum corresponds to the contributions containing 0, 2 and 4 fermion fields, respectively. Let us note that determining this number is not an easy task since seemingly different operators can be converted into each other by means of, e.g., Fierz transformations or equations of motion. The $d = 6$ operators introduce wide range of processes suppressed or absent in the SM, let us, e.g., mention that certain $d = 6$ operators containing 2 fermion fields induce the electric dipole moment of the electron [106]. Here, we will concentrate on the class of operators which violate the baryon number, hence, can contribute to the proton decay.

The 5 independent operators of this kind were first identified in [107] (correcting the earlier considerations [66] and [108]):

$$O^{(1)}_{abcd} = \varepsilon_{jkl}\varepsilon_{\alpha\beta}\left(d^T_{Raj}Cu_{Rbk}\right)\left(Q^T_{Lcl\alpha}CL_{Ld\beta}\right), \tag{6.1}$$

$$O^{(2)}_{abcd} = \varepsilon_{jkl}\varepsilon_{\alpha\beta}\left(Q^T_{Laj\alpha}CQ_{Lbk\beta}\right)\left(u^T_{Rcl}Ce_{Rd}\right), \tag{6.2}$$

$$O^{(3)}_{abcd} = \varepsilon_{jkl}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\left(Q^T_{Laj\alpha}CQ_{Lbk\beta}\right)\left(Q^T_{Lcl\gamma}CL_{Ld\delta}\right), \tag{6.3}$$

$$O^{(4)}_{abcd} = \varepsilon_{jkl}(\sigma_A\varepsilon)_{\alpha\beta}\cdot(\sigma_A\varepsilon)_{\gamma\delta}\left(Q^T_{Laj\alpha}CQ_{Lbk\beta}\right)\left(Q^T_{Lcl\gamma}CL_{Ld\delta}\right), \tag{6.4}$$

$$O^{(5)}_{abcd} = \varepsilon_{jkl}\left(d^T_{Raj}Cu_{Rbk}\right)\left(u^T_{Rcl}Ce_{Rd}\right) \tag{6.5}$$

where $\varepsilon_{\alpha\beta} \equiv (i\sigma_2)_{\alpha\beta}$ and ε_{jkl} are the totally antisymmetric tensors in two and three dimensions with $\alpha, \beta, \gamma, \delta$ and j, k, l denoting the $SU(2)_L$ and colour indices, respectively. Furthermore, a, b, c, d are the indices in the generation space of the SM fermions (2.1).

Interestingly, all the operators $O^{(1)-(5)}$ conserve the $B-L$ quantum number, which implies that the proton decay induced by these operators corresponds at the parton level to the interaction

$$q\,q \to \bar{q}\,\bar{l} \tag{6.6}$$

where the relevant combinations of the initial quarks are either $u\,d$ or $u\,u$ due to the proton quark content, the final antiquark can be $\bar{u}$, $\bar{d}$ or $\bar{s}$ for kinematical reasons and the final antilepton is either an antineutrino, e^+, or possibly μ^+ if kinematically allowed.

Let us now try to identify the fields which could be integrated out from a renormalizable theory in order to obtain these operators. Taking into account the quantum numbers of the fields in each bracket of (6.1)-(6.5), it is straightforward to identify the scalar fields which can mediate the interactions described by these effective operators. As stated in [66], the operators $O^{(1)}$, $O^{(2)}$, $O^{(3)}$, and $O^{(5)}$ can arise due to the exchange of a scalar field

$$T = (3, 1, -\tfrac{1}{3}) \tag{6.7}$$

and all these operators may induce the decay of the proton. Furthermore, the interaction $O^{(4)}$ can be mediated by a $(3, 3, -\frac{1}{3})$ scalar, and the linear combination [107]

$$\tilde{O}^{(5)}_{abcd} \equiv O^{(5)}_{cbad} - O^{(5)}_{cabd} = \varepsilon_{jkl}\left(u^T_{Raj}Cu_{Rbk}\right)\left(d^T_{Rcl}Ce_{Rd}\right)$$

(assigned as an independent operator $O^{(6)}$ in [66]) allows for the mediation by a $(3, 1, -\frac{4}{3})$ scalar. In $O^{(4)}$ and $\tilde{O}^{(5)}$, however, the contraction of the initial quark fields is antisymmetric in the generation indices, hence, these operators can not mediate any sizeable proton decay.

Furthermore, the operators $O^{(1)}$ and $O^{(2)}$ can be obtained also by integrating out a vector field as can be seen when the Fierz identity

$$(\overline{\psi_{1L}}\psi_{2R})(\overline{\psi_{3R}}\psi_{4L}) = \frac{1}{2}(\overline{\psi_{1L}}\gamma^{\mu}\psi_{4L})(\overline{\psi_{3R}}\psi_{2R})$$

is applied (recall that $\psi^T C = \overline{\psi^c}$ and that $(\psi_R)^c = (\psi^c)_L$ as derived in Appendix A.1).[1] Since we will be later interested in the vector-mediated proton decay, let us put these transformed operators explicitly as in [9][2]

$$O_I = k_1^2 \varepsilon_{jkl}\varepsilon_{\alpha\beta}\left(\overline{u^c_{Laj}}\gamma^{\mu}Q_{Lak\alpha}\right)\left(\overline{e^c_{Lb}}\gamma_{\mu}Q_{Lbl\beta}\right), \tag{6.8}$$

$$O_{II} = k_1^2 \varepsilon_{jkl}\varepsilon_{\alpha\beta}\left(\overline{u^c_{Laj}}\gamma^{\mu}Q_{Lak\alpha}\right)\left(\overline{d^c_{Lbl}}\gamma_{\mu}L_{Lb\beta}\right), \tag{6.9}$$

$$O_{III} = k_2^2 \varepsilon_{jkl}\varepsilon_{\alpha\beta}\left(\overline{d^c_{Laj}}\gamma^{\mu}Q_{Lak\alpha}\right)\left(\overline{u^c_{Lbl}}\gamma_{\mu}L_{Lb\beta}\right). \tag{6.10}$$

Here the notation (2.2) with left-handed conjugated fields instead of the right-handed fields was used, and the flavour structure where it is summed over the generation indices a, b in each bracket corresponds to the assumption that these operators arise from integrating out a heavy vector bosons. The operator O_I comes from the Fierz transformation of $O^{(2)}$ whereas O_{II} and O_{III} are different incarnations of $O^{(1)}$. Let us also note that, strictly speaking, the operators (6.8)-(6.10) do not formally fit in the expansion (1.1) since they already do include the dimension M^{-2} couplings k_1^2 and k_2^2, their meaning will become clear in a moment.

When the quantum numbers of the fields in brackets are again inspected, one finds that the vector bosons

$$X^{\mu} = (\bar{3}, 2, +\tfrac{5}{6}), \qquad X'^{\mu} = (\bar{3}, 2, -\tfrac{1}{6}) \tag{6.11}$$

could mediate the interactions O_{I-II}, and O_{III}, respectively. For this reason also the same coupling is used for operators O_I and O_{II} and if one assumes that in the renormalizable theory the vector bosons (6.11) correspond to a gauge group with a gauge coupling g_G, after integrating out such heavy vector bosons one gets

$$k_1 = \frac{g_G}{\sqrt{2}M_X}, \qquad k_2 = \frac{g_G}{\sqrt{2}M_{X'}}. \tag{6.12}$$

[1] Operators of the type $(\overline{\psi_{1R}}\psi_{2L})(\overline{\psi_{3R}}\psi_{4L})$ can not be translated into contractions containing γ-matrices when applying the Fierz transformations, hence, the interactions $O^{(3)-(6)}$ can not be mediated by vector currents.

[2] In this reference an extra operator O_{IV} including the right-handed neutrino ν_R occurs, however, we will not consider this operator here since in the models relevant for this text the seesaw mechanism is used and ν_R (or more precisely the mass eigenstate formed mostly by ν_R) is too heavy to be produced in the proton decay.

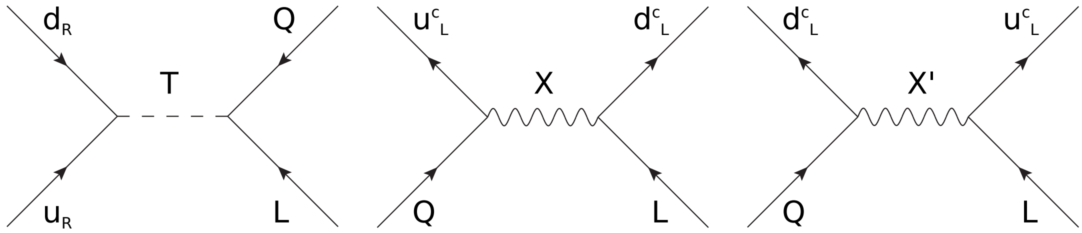

Figure 6.1: Three different renormalizable realization of the operator $O^{(1)}$ (6.1). The quark fields are intentionally assigned as right-handed fields in the first diagram or as left-handed conjugated fields in the other two diagrams since this corresponds also to the notation used for writing the operator (6.1) and its Fierz transformed versions (6.9) and (6.10), respectively. The quantum numbers of the mediators T (6.7), and X, X' (6.11) can be easily derived from the structure of the corresponding effective operators. The orientation of the Feynman diagrams corresponds to (6.6), i.e., to the way in which the proton decay can be mediated.

We will see in Section 6.3 that the experimental (non-)observation of the proton decay constraints the masses of X, X' to satisfy M_X, $M_{X'} \gtrsim 10^{16}$ GeV. Furthermore, in Chapter 7 these vector bosons will be recognized as an integral component of unification models.

In order to summarize the above results we show as an example the possible "openings" of the operator $O^{(1)}$ in Figure 6.1.

6.2 Partial proton decay rates

Since the proton decay rates will be computed for particular models considered in this text (see Chapter 8), explicit formulas for the partial rates induced by the $d = 6$ vector-mediated operators will be given here. Let us stress that these formulas can be derived using the effective description (6.8)-(6.10) only, the full high-energy theory enters here through the couplings (6.12) and the flavour structure of the operators. Furthermore, if higher precision is required then also the coefficients describing the running of the operator from the GUT scale down to the EW scale enter the proton decay formulas.

The interaction (6.6) translates into

$$p \ \rightarrow \ \text{meson} \ + \ \text{antilepton}, \tag{6.13}$$

at the hadron level and more than one operator among (6.8)-(6.10) can contribute to the process of this type with a given final state. Also the flavour structure of the operators (6.8)-(6.10) plays an important role: similarly as in case of the SM gauge interactions, these operators are written in the so-called interaction basis, where the contractions in each bracket are flavour diagonal. In this basis, however, the mass matrices are in general not diagonal and after their diagonalization (2.17) and (3.11) or (3.14) non-trivial flavour structure emerges.

Consequently, the partial proton decay rates read [9]:

$$\Gamma(p\to\pi^0 e^+_\beta) = \frac{m_p}{16\pi f_\pi^2} A_L^2 |\alpha|^2 (1+D+F)^2 \left\{ \left|c(e_\beta, d^C)\right|^2 + \left|c(e^C_\beta, d)\right|^2 \right\}, \tag{6.14}$$

$$\Gamma(p\to\eta e^+_\beta) = \frac{(m_p^2-m_\eta^2)^2}{48\pi m_p^3 f_\pi^2} A_L^2 |\alpha|^2 (1+D-3F)^2 \left\{ \left|c(e_\beta, d^C)\right|^2 + \left|c(e^C_\beta, d)\right|^2 \right\}, \tag{6.15}$$

$$\Gamma(p\to K^0 e^+_\beta) = \frac{(m_p^2-m_K^2)^2}{8\pi m_p^3 f_\pi^2} A_L^2 |\alpha|^2 \left[1+\frac{m_p}{m_B}(D-F)\right]^2 \times \left\{ \left|c(e_\beta, s^C)\right|^2 + \left|c(e^C_\beta, s)\right|^2 \right\}, \tag{6.16}$$

$$\Gamma(p\to\pi^+\overline{\nu}) = \frac{m_p}{8\pi f_\pi^2} A_L^2 |\alpha|^2 (1+D+F)^2 \sum_{l=1}^{3} \left|c(\nu_l, d, d^C)\right|^2, \tag{6.17}$$

$$\Gamma(p\to K^+\overline{\nu}) = \frac{(m_p^2-m_K^2)^2}{8\pi m_p^3 f_\pi^2} A_L^2 |\alpha|^2 \times \sum_{l=1}^{3} \left| \frac{2m_p}{3m_B} D\, c(\nu_l, d, s^C) + \left[1+\frac{m_p}{3m_B}(D+3F)\right] c(\nu_l, s, d^C) \right|^2, \tag{6.18}$$

where m_p, m_η and m_K denote the proton, η and kaon mass, respectively, m_B is an average baryon mass ($m_B \approx m_\Sigma \approx m_\Lambda$), f_π is the pion decay constant, $|\alpha|$, D and F are the parameters of the chiral Lagrangian, and A_L takes into account the renormalization from the EW scale to 1 GeV (see, e.g., [9] for the values of these parameters). Index β assigns the the charged lepton flavour and it is summed over neutrino flavour index l since the flavour of the neutrino in the final state is not determined in experiments (for similar reasons it is summed also over the charged lepton chiralities which gives rise to the incoherent sum in (6.14)-(6.16)). The flavour structure of the operators (6.8)-(6.10) when switching to the mass basis using the relations (2.17) and (3.11) or (3.14) determines the coefficients

$$c(e_\alpha, d^C_\beta) = k_1^2 (U_C^\dagger U)_{11} (D_C^\dagger E)_{\beta\alpha} + k_2^2 (D_C^\dagger U)_{\beta 1} (U_C^\dagger E)_{1\alpha}, \tag{6.19}$$

$$c(e^C_\alpha, d_\beta) = k_1^2 [(U_C^\dagger U)_{11} (E_C^\dagger D)_{\alpha\beta} + (U_C^\dagger D)_{1\beta} (E_C^\dagger U)_{\alpha 1}], \tag{6.20}$$

$$c(\nu_l, d_\alpha, d^C_\beta) = k_1^2 (U_C^\dagger D)_{1\alpha} (D_C^\dagger N)_{\beta l} + k_2^2 (D_C^\dagger D)_{\beta\alpha} (U_C^\dagger N)_{1l}, \tag{6.21}$$

where k_1 and k_2 are the effective couplings (6.12).

Let us stress that the unitary matrices U, D, E, N are partially determined by the low-energy constraints on the CKM (2.23) and PMNS (3.17), (3.18) matrices, however, U_C, D_C, E_C are completely unknown.

Let us also note that in case of more precise studies, one has to take into account also the renormalization of the operators (6.8)-(6.10) from the GUT scale to the EW scale as described in [4] attached to this thesis as Appendix E.

6.3 Experiments searching for the proton decay

In the beginning of this chapter we mentioned several reasons why violation of the baryon number can be expected. Consequently, experiments were searching for the baryon-number-violating processes since 1950's (for the full historical overview see, e.g., [9]), with negative results up to now. Only the lower limits on the proton lifetime are, hence, determined experimentally, and we will concentrate here on the measurements giving the strongest bounds nowadays. Moreover, some of the planned experiments will be mentioned.

When giving recent bounds on the partial proton decay widths, we should refer back to Section 3.1.2 where the Super-Kamiokande (SK) experiment was presented since, besides the neutrino oscillations, this water-Čerenkov detector also searches for the proton decay. The "golden" channel for this kind of detectors is $p \to \pi^0 e^+$ since, apart from the Čerenkov ring produced by the positron, also two photon rings can be observed since the neutral pion decays mostly via $\pi^0 \to \gamma\gamma$. The bound on the corresponding partial proton lifetime presented at the Moriond 2015 conference [109] (the results published in [110] give slightly lower bound) reads

$$\tau(p \to \pi^0 e^+) \equiv \frac{\tau_p}{BR(p \to \pi^0 e^+)} \geq 1.4 \times 10^{34}\,\mathrm{y}. \tag{6.22}$$

Also for the $p \to K^+\bar{\nu}$ channel the world's best limit is given by the SK experiment [111]

$$\tau(p \to K^+\bar{\nu}) \geq 5.9 \times 10^{33}\,\mathrm{y}. \tag{6.23}$$

The two bounds (6.22) and (6.23) are usually the most "dangerous" constraints for the unified models (for the full list of bounds on partial proton lifetimes see, e.g., [9], the updated values are given, e.g., in [110], [111]).

The successor of the SK experiment is planned to be the water-Čerenkov detector Hyper-Kamiokande (HK) to be placed in the neighbouring mine in Kamioka and designed to be about 20× larger than the SK, i.e., to consist of about 1 Mton of water. If no signal will be detected in 20 years of the data taking, the bounds

$$\tau(p \to \pi^0 e^+)_{\mathrm{HK}} \geq 2 \times 10^{35}\,\mathrm{y}, \qquad \tau(p \to K^+\bar{\nu})_{\mathrm{HK}} \geq 3 \times 10^{34}\,\mathrm{y} \tag{6.24}$$

are expected [112].

For the observation of the $p \to K^+\bar{\nu}$ channel the so-called Liquid Argon Time Projection Chamber (LAr TPC) technique is better suited and should be used for the planned experiment DUNE. Here a detector containing some 40 kton of liquid argon is planned to be built in a distance of about 1 300 km from Fermilab in order to serve also as a long-baseline neutrino experiment (see Section 3.1.2). The partial proton lifetime limit can be then improved up to

$$\tau(p \to K^+\bar{\nu})_{\mathrm{DUNE}} \geq 6 \times 10^{34}\,\mathrm{y} \tag{6.25}$$

after 20 years of data taking [113].

Let us note that the constraint $\tau_p \gtrsim 10^{34}$ y yields, due to the approximate behaviour of the amplitudes (6.14)-(6.18),

$$\Gamma \sim \frac{g_G^4}{M_{X^{(\prime)}}^4} m_p^5 \tag{6.26}$$

the value already mentioned in the end of the Section 6.1:

$$M_X, M_{X'} \gtrsim 10^{16}\,\mathrm{GeV} \tag{6.27}$$

(recall that $\tau_p \sim \Gamma^{-1}$ and the ordinary units are resurrected using the formula (1.3)).

Chapter 7

Properties of selected unified theories

In the previous chapter we inspected the effects of baryon number violation as one of the well-motivated steps beyond the SM. Here we consider the so-called Grand Unified Theories (GUTs) where the gauge group is a simple Lie group and show that the baryon number is indeed violated at the renormalizable level in such theories.

In the subsequent sections we describe the main features of the canonical GUTs based on the $SU(5)$ and $SO(10)$ gauge groups. Furthermore, since the article [3] deals with the $SU(5)\times U(1)$ gauge group, we introduce also this "flipped $SU(5)$" unification although it is a "partial", not "grand" unification. The particular unification models will be then discussed in the following chapter.

Let us note that we narrow our discussion to non-supersymmetric unified theories, since our main results concern models without supersymmetry. Besides, the TeV-scale superpartners (i.e., bosonic or fermionic counterparts of the SM fermions or bosons, respectively) were not observed by the LHC which constrains considerably the usual scenarios of the low-scale supersymmetry. Moreover, the superpartner mass spectrum, which is an important low-energy input for the supersymmetric unifications, is unknown, making, e.g., the accurate predictions of the partial proton decay widths within these models impossible.

7.1 $SU(5)$: a prototype GUT

The first published GUT model was based on the $SU(5)$ gauge group [114] and also due to its simplicity we believe (like the author of [67]) that it deserves to be referred to as a "prototype" GUT. The main features of GUTs will be explained on the example of $SU(5)$ similarly as in [67, 115], before that, the historical background for the first formulation of this model will be described.

In the early 1970's, the Yang-Mills theories [116] were slowly becoming a key concept of particle physics and it was explained also in [114] "how attractive it is for strong, weak and electromagnetic interactions to spring from a gauge theory based on the group $SU(3) \times SU(2) \times U(1)$".

However, the authors of [114] were disturbed by the fact that the electromagnetic and weak interactions were not truly unified – there were still two independent coupling constants. Moreover, the observed quantization of the electric charge could not be explained by the existing theory[1]. The solution of both these problems would be to describe electroweak interactions by a single simple gauge group (or a direct product of isomorphic simple factors). It was argued that if strong interactions were left aside and only the electromagnetic and weak interactions were unified, in any conceivable scheme, the generator of the electric charge did not admit fractional charges. Consequently, it was searched for an algebra that contains all $SU(3) \times SU(2) \times U(1)$ as a subalgebra and is simple or a direct product of isomorphic simple factors. Considering different possible algebras of rank at least 4, the authors "present a series of hypotheses and speculations leading inescapably to the conclusion that $SU(5)$ is the gauge group of the world."

Indeed, $SU(5)$ is the minimal solution of the problem above, and if fermions of one generation are grouped into the $\bar{5}$ and 10 representations, the theory is anomaly free. On the other hand, this theory gives several predictions that seemed to be correct back in 1974, but are in contradiction with the current measurements as will be summarized in Section 7.1.6. In order to get a phenomenologically viable model, several extensions of the minimal Georgi-Glashow model [114] have been proposed as described in Section 7.1.7, however, the problems of the $SU(5)$ unifications present a good motivation for more complicated models based on the $SO(10)$ gauge symmetry (see Section 7.2), and also for the flipped $SU(5)$ unification (see Section 7.3).

7.1.1 Group structure

$SU(5)$ is the Lie group of special (with determinant equal to 1) unitary 5×5 matrices, and the corresponding $su(5)$ Lie algebra consists of hermitian traceless 5×5 matrices. The generators T_a, $a = 1, \ldots, 24$ of the $su(5)$ algebra may be chosen in a way suggesting the embedding of its $su(3)_c \times su(2)_L \times u(1)_Y$ subalgebra: the first eight generators correspond to the $su(3)_c$ subalgebra, other three form the $su(2)_L$ subalgebra, and T_{24} corresponds to the SM hypercharge generator Y. Let us note that the normalization of the generators in the fundamental representation is conveniently fixed as

$$\mathrm{Tr}[T_a T_b] = \frac{1}{2}\delta_{ab}, \tag{7.1}$$

[1] We discussed in Section 2.4 that the quantization of the electric charge can be explained if anomaly-free SM gauge symmetry is required. On the other hand, we explained in the footnote at the end of the same section that if, e.g., Dirac neutrino masses are introduced and right-handed neutrinos are added to the theory, then the hypercharge assignment is not unique any more.

and we will see later that, in order to accommodate the SM particles with correct hypercharge in the $SU(5)$ multiplets, this normalization corresponds to $T_{24} = \sqrt{\frac{3}{5}}Y$. Explicitly, the generators of the $su(5)$ in the fundamental representation read

$$T_{1-8} = \begin{pmatrix} \frac{1}{2}\lambda_{1-8} & 0_{2\times 3} \\ 0_{3\times 2} & 0_{2\times 2} \end{pmatrix} \quad T_{9-11} = \begin{pmatrix} 0_{3\times 3} & 0_{2\times 3} \\ 0_{3\times 2} & \frac{1}{2}\sigma_{1-3} \end{pmatrix}$$

$$T_{24} = \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \sqrt{\frac{3}{5}} Y \tag{7.2}$$

where λ_j and σ_j are the Gell-Mann and Pauli matrices, respectively, and the dimensionality of the zero blocks is indicated by the subscript.

The remaining generators of the $su(5)$ algebra are the generalization of the σ_1 and σ_2 Pauli matrices:

$$T_{12} = \frac{1}{2} \begin{pmatrix} & & & 1 & 0 \\ & 0_{3\times 3} & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & 0_{2\times 2} \end{pmatrix} \qquad T_{13} = \frac{1}{2} \begin{pmatrix} & & & -i & 0 \\ & 0_{3\times 3} & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & 0_{2\times 2} \end{pmatrix}, \tag{7.3}$$

the other 10 generators have analogous shape with nonzero entries at other 5 positions in the upper right and the lower left submatrices. More precisely $(T_{14,15})_{ij} \neq 0$ for $(i, j) \in \{(1, 5), (5, 1)\}$, $(T_{16,17})_{ij} \neq 0$ for $(i, j) \in \{(2, 4), (4, 2)\}$, etc.

With this choice, the Cartan generators are the diagonal matrices T_3, T_8, T_{11}, and T_{24}. The weights of the fundamental 5-dimensional representation suggest that it decomposes under the SM subalgebra as

$$5 = (3, 1, -\tfrac{1}{3}) \oplus (1, 2, \tfrac{1}{2}) \tag{7.4}$$

where we indicate the transformation properties with respect to $SU(3)_c$ (determined by the T_3, T_8 eigenvalues), $SU(2)_L$ (corresponding to the T_{11} Cartan generator) and $Y = \sqrt{\frac{5}{3}}T_{24}$, respectively. Looking at these SM quantum numbers, the latter multiplet may accommodate the SM Higgs doublet.

Similarly, we find that the weights of the complex conjugate representation transform under the SM subalgebra as

$$\bar{5} = (\bar{3}, 1, \tfrac{1}{3}) \oplus (1, \bar{2}, -\tfrac{1}{2}). \tag{7.5}$$

Here the SM quantum numbers reveal that the anti-triplet of the right-handed down-type antiquarks and the doublet of the left-handed SM leptons (the representation $\bar{2}$ of SU(2) is equivalent to 2) may reside in the $\bar{5}_F$ fermionic representation.

The tensor product of two fundamental representations gives

$$5 \otimes 5 = 10_a \oplus 15_s \tag{7.6}$$

where the antisymmetric part decomposes under the SM subalgebra as

$$10 = (3, 2, \tfrac{1}{6}) \oplus (\bar{3}, 1, -\tfrac{2}{3}) \oplus (1, 1, 1). \tag{7.7}$$

Consequently, fermionic 10_F may accommodate the $SU(2)_L$ doublet of left-handed quarks, the color anti-triplet of right-handed up-type quarks, and the right-handed positron.

When looking for realistic models of fermion masses also the relations

$$\overline{10} \otimes \overline{10} = 5_s \oplus 45_a \oplus 50_s \tag{7.8}$$

$$10 \otimes \overline{5} = 5 \oplus 45 \tag{7.9}$$

become useful.

7.1.2 Gauge fields

If the $SU(5)$ gauge bosons corresponding to the 24 generators T_a are denoted as A_a^μ, then according to the definition (7.2) $A_{1-8}^\mu = G_{1-8}^\mu$ are the SM gluons, $A_{9-11}^\mu = W_{1-3}^\mu$ are the gauge fields corresponding to the $SU(2)_L$ and $A_{24}^\mu = B^\mu$ corresponds to the $U(1)_Y$. For the sake of definiteness the gauge fields A_{12-23}^μ corresponding to the generators (7.3) will be considered here in more detail.

Similarly as for $W_{1,2}^\mu$ in the SM, one finds that A_{12-23}^μ are not eigenvectors of the Cartan generators T_{11} and T_{24} which determine the weak isospin and the hypercharge quantum numbers, respectively. Indeed, since, e.g.,

$$[T_{24}, T_{12}] = -\frac{5}{6} i \sqrt{\frac{3}{5}} T_{13}, \qquad [T_{24}, T_{13}] = \frac{5}{6} i \sqrt{\frac{3}{5}} T_{12}$$

and similarly

$$[T_{11}, T_{12}] = -\frac{i}{2} T_{13}, \qquad [T_{11}, T_{13}] = \frac{i}{2} T_{12},$$

one finds that both T_{24} and T_{11} are diagonalized if we choose the basis containing the fields

$$(X_1^u)^\mu = \frac{1}{\sqrt{2}} (A_{12}^\mu + i A_{13}^\mu), \qquad (X_1^{u*})^\mu = \frac{1}{\sqrt{2}} (A_{12}^\mu - i A_{13}^\mu) \tag{7.10}$$

corresponding to the generators $\frac{1}{\sqrt{2}}(T_{12} - iT_{13})$, $\frac{1}{\sqrt{2}}(T_{12} + iT_{13})$, respectively. The weak isospin and the hypercharge of X_1^u read $+\frac{1}{2}$ and $+\frac{5}{6}$, respectively, hence, this gauge boson carries the electric charge $+\frac{4}{3}$. Similarly, one finds that

$$(X_1^d)^\mu = \frac{1}{\sqrt{2}} (A_{14}^\mu + i A_{15}^\mu) \tag{7.11}$$

is a field with the weak isospin $-\frac{1}{2}$ and the hypercharge $+\frac{5}{6}$, hence, the electric charge $+\frac{1}{3}$. If $X^u_{2,3}$ and $X^d_{2,3}$ are defined in analogous manner, one easily finds that X^u_{1-3} and X^d_{1-3} transform as an $SU(3)_c$ anti-triplet and each pair of X^u_i and X^d_i forms a weak doublet

$$X^\mu_i = \begin{pmatrix} X^u_i \\ X^d_i \end{pmatrix}^\mu = (\bar{3}_i, 2, +\tfrac{5}{6}). \tag{7.12}$$

The vector boson X^μ is, hence, exactly the one identified as the mediator of the effective operators $\mathcal{O}_I$, $\mathcal{O}_{II}$ (6.8), (6.9) and we will see in the following section that its interactions with the matter fields indeed induce the proton decay.

In order to summarize the definitions given above in one formula, one can rewrite

$$\sum_{a=1}^{24} A^\mu_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\sum_{a=1}^{8} G_a\lambda_a - \sqrt{\frac{2}{15}}B & \begin{matrix} X^{u*}_1 & X^{d*}_1 \\ X^{u*}_2 & X^{d*}_2 \\ X^{u*}_3 & X^{d*}_3 \end{matrix} \\ \begin{matrix} X^u_1 & X^u_2 & X^u_3 \\ X^d_1 & X^d_2 & X^d_3 \end{matrix} & \frac{1}{\sqrt{2}}\sum_{a=1}^{3} W_a\sigma_a + \sqrt{\frac{3}{10}}B \end{pmatrix}^\mu . \tag{7.13}$$

Let us also note that if we identify $g_5 T_{24} A^\mu_{24} = g_1 Y B^\mu$ in the covariant derivative where g_5 and g_1 are the unified and the SM hypercharge gauge couplings, respectively, according to the formula (7.2) we obtain $g_1 = \sqrt{\frac{3}{5}} g_5$ at the GUT scale. On the other hand, the $SU(3)$ and $SU(2)$ SM couplings satisfy $g_3 = g_2 = g_5$ at the GUT scale, hence, one obtains a firm prediction for the weak mixing angle (2.13)

$$\sin^2\theta_W(M_G) = \frac{g_1^2(M_G)}{g_1^2(M_G) + g_2^2(M_G)} = \frac{3}{8}. \tag{7.14}$$

7.1.3 Matter fields

As we describe in Section 7.1.1, the decomposition of the $SU(5)$ representations under the SM algebra suggests naturally how to accommodate the SM matter fields. Usually, the 5 and $\bar{5}$ representations are written as vectors with upper and lower indices, respectively, whereas the 10 is represented by an antisymmetric matrix with two upper indices. Moreover, for writing the Lagrangian in a compact form, it will be convenient to use the notation described in Appendix A.1 where only the left-handed fermionic fields are used. Using the lower indices to denote the three colors we get:

$$\bar{5}_F \equiv \psi = \begin{pmatrix} d^c_1 \\ d^c_2 \\ d^c_3 \\ e \\ -\nu \end{pmatrix}_L \qquad 10_F \equiv \chi = \begin{pmatrix} 0 & u^c_3 & -u^c_2 & u_1 & d_1 \\ -u^c_3 & 0 & u^c_1 & u_2 & d_2 \\ u^c_2 & -u^c_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix}_L . \tag{7.15}$$

Let us note that the relative minus sign between e and ν_L fields is due to the fact that e_L and ν_L transform as an $SU(2)_L$ *antidoublet* within the $\bar{5}$ representation of $SU(5)$ (7.5).

It follows from (2.6) and (A.6) that the gauge-kinetic form which reproduces the correct SM gauge interactions can be written as

$$\mathcal{L}_{gauge} \ni i\overline{\psi}\gamma_\mu D^\mu \psi + i\mathrm{Tr}[\overline{\chi}\gamma_\mu D^\mu \chi] \tag{7.16}$$

where

$$D^\mu \psi = (\partial^\mu - ig_5 \sum_a A_a^\mu T_a)\psi,$$
$$D^\mu \chi = \partial^\mu \chi - ig_5 \sum_a A_a^\mu (T_a\, \chi + \chi\, T_a^T).$$

and A_a^μ are the $SU(5)$ gauge bosons. Notice that the action of the $SU(5)$ generators on the matrix χ follows the fact that 10^{ij} is obtained from the tensor product of two fundamental representations (7.6).

Apart from the SM gauge interactions, the gauge-kinetic form (7.16) gives rise to extra interactions mediated by the gauge bosons X_j^u, X_j^d (7.10), (7.11):

$$\begin{aligned}\mathcal{L}_{gauge} \ni & \frac{g}{\sqrt{2}}\Big[(X_j^u)^\mu \left(\overline{d_j^c}\gamma_\mu e + \overline{e^c}\gamma_\mu d_j + \varepsilon_{jkl}\overline{u_l}\gamma_\mu u_k^c\right) \\ & +(X_j^d)^\mu \left(-\overline{d_j^c}\gamma_\mu \nu - \overline{e^c}\gamma_\mu u_j + \varepsilon_{jkl}\overline{d_l}\gamma_\mu u_k^c\right)\Big] + h.c. \\ = & \frac{g}{\sqrt{2}}\Big[\overline{d_j^c}\gamma_\mu X_j^{\mu T} i\sigma_2 L + \overline{e^c}\gamma_\mu X_j^{\mu T} i\sigma_2 Q_j + \varepsilon_{jkl}\overline{Q_l}^T X_j^\mu \gamma_\mu u_k^c\Big] + h.c.\end{aligned} \tag{7.17}$$

where we suppressed the subscript L since all the fields involved are left-handed, j, k, l are color indices, and X_j^μ is the weak doublet of gauge bosons defined in (7.12). On the third line, the $i\sigma_2$ matrix and the transpositions refer to the $SU(2)_L$ structure of the expressions. Let us emphasize that these interactions induce the $\mathcal{O}_I$ and $\mathcal{O}_{II}$ effective operators (6.8), (6.9) if X^μ is integrated out (e.g., the combination of the first and the third term on the third line lead to the interaction depicted on the second diagram of Figure 6.1 corresponding to the "opening" of the operator $\mathcal{O}_I$). Consequently, the extra gauge bosons have to be very heavy in order to avoid the quick proton decay as already stated by (6.27).

7.1.4 Scalar sector

In order to provide masses for the X^u and X^d gauge bosons, a scalar field with a VEV breaking the $SU(5)$ symmetry down to the SM gauge symmetry is needed. On the other hand, the rank of $SU(5)$ is equal to 4 which holds also for the SM gauge group, hence, the VEV should not reduce the rank. The following statement then explains the choice of the $SU(5)$ scalar sector.

Proposition 1. *Let G, G' be Lie groups and let the gauge symmetry G be spontaneously broken to G' by a VEV of the adjoint representation of G. Then the rank of G' is equal to the rank of G.*

Proof. The linear space of the adjoint representation can be formed by the generators of the Lie group G. Since these generators are Hermitian, the VEV of the adjoint representation can be always brought to the diagonal form by a unitary transformation U, hence, can be rewritten as $UDU^\dagger$ where D is a diagonal matrix.

Moreover, let the basis be chosen so that the Cartan generators H_i ($i = 1, 2 \ldots, n$, where n is the rank of G) are diagonal. Also in the set $\{UH_iU^\dagger\}_{i=1,2\ldots,n}$, all the matrices commute, hence, this corresponds to another choice of Cartan generators. Since the generators of G act on the adjoint representation via commutator, the relation

$$\left[UH_iU^\dagger, UDU^\dagger\right] = U\left[H_i, D\right]U^\dagger = 0, \qquad \forall i = 1, 2\ldots, n$$

proves the proposition.

□

Consequently, the 24-dimensional adjoint representation of the $SU(5)$ is usually included in the scalar sector of the $SU(5)$ unifications:

$$24_H \equiv \Sigma = \sum_{a=1}^{24} \Sigma_a T_a \qquad \langle\Sigma_{24}\rangle = V_G, \qquad \langle\Sigma_a\rangle = 0 \quad \forall a \neq 24.$$

As the action of the i-th generator of the adjoint representation on $\langle\Sigma\rangle$ reads

$$T_i(\langle\Sigma\rangle) = [T_i, \langle\Sigma\rangle] = V_G[T_i, T_{24}],$$

the SM subalgebra annihilates $\langle\Sigma\rangle$, whereas the generators T_{12-23} act on this state non-trivially, which justifies the desired pattern of the spontaneous symmetry breaking.

The properly normalized[2] gauge-kinetic form for the scalar Σ reads

$$\mathcal{L}_{gauge} \ni \mathrm{Tr}\left[(D_\mu\Sigma)^\dagger D^\mu\Sigma\right]$$

where

$$D^\mu\Sigma = \sum_{b=1}^{24} \partial^\mu\Sigma_b T_b - ig_5 \sum_{a,b=1}^{24} A_a^\mu \Sigma_b [T_a, T_b]$$

with A_a^μ being the (real) gauge fields corresponding to the $SU(5)$ gauge group. Using the defining relation for the structure constants $[T_a, T_b] = i\sum_c f_{abc}^{su(5)} T_c$ and the normalization (7.1), we obtain after spontaneous symmetry breaking

$$\mathcal{L}_{gauge} \ni \frac{1}{2} g_5^2 V_G^2 \sum_{a,c,d=1}^{24} f_{a\,24\,c}^{su(5)} f_{d\,24\,c}^{su(5)} A_a^\mu A_{d\mu}.$$

[2]For the individual real scalar fields Σ_a one has to obtain $\mathcal{L}_{gauge} \ni \frac{1}{2}\partial_\mu\Sigma_a\partial^\mu\Sigma_a$, which is ensured by the normalization (7.1) of the $SU(5)$ generators.

Consequently, after rewriting the real A^{μ}_{12-23} fields in terms of the complex $(X^u_i)^\mu$ and $(X^d_i)^\mu$ fields (7.10), (7.11), these acquire equal masses

$$M^2_{X^u} = M^2_{X^d} \equiv M_X = \frac{5}{12} g_5^2 V_G^2.$$

Recall that the experimental limit (6.27) applies to this mass which sets the $SU(5)$ symmetry breaking scale to $\sim 10^{16}$ GeV.

Moreover, we have to arrange for the SM symmetry breaking at the EW scale and as already explained in Section 7.1.1 the SM Higgs doublet can be accomodated in the fundamental representation of the $SU(5)$ (7.4):

$$5_H \equiv \Phi = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ H^+ \\ H^0 \end{pmatrix}.$$

The field $T = (3, 1, -\frac{1}{3})$ can be identified with the scalar (6.7) which could mediate the proton decay, and it, indeed, does induce this process due to its Yukawa couplings (see (7.18) below). For this reason, T is assumed to be very heavy compared to the 125 GeV SM Higgs boson which shares the same multiplet. The problem of separating these two masses in a natural way is sometimes called the doublet-triplet splitting problem.

We do not specify here the exact shape of the scalar potential since it is not necessary for a general discussion, nevertheless, this information will be supplied for the particular models studied in this thesis.

7.1.5 Yukawa interactions

At this stage it is possible to explain why the unified models usually predict certain relations between fermion masses. The Yukawa part of the minimal $SU(5)$ Lagrangian reads:

$$\mathcal{L}_Y = Y^\psi_{ab} (\psi_{ia})^T C \chi^{ij}_b \Phi^\dagger_j + Y^\chi_{ab} \varepsilon_{ijklm} (\chi^{ij}_a)^T C \chi^{kl}_b \Phi^m + h.c. \tag{7.18}$$

where ε is the totally-antisymmetric tensor in 5 dimensions, i, j, k, l, m and a, b are the $SU(5)$ and generation indices, respectively, and the transposition and the C matrix refer to the Lorentz structure of the expression (see Appendix A.1). After performing the $SU(5)$ contractions and inserting the VEV $\langle H^0 \rangle = v$, one finds

$$\mathcal{L}_Y \ni \left[Y^\psi_{ab} \left((d^c_a)^T C d_b + e^T_a C e^c_b \right) + 4 Y^\chi_{ab} \left(u^T_a C u^c_b + (u^c_a)^T C u_b \right) \right] v + h.c. + \dots$$

where we again suppress the subscript L for the left-handed fermions, as well as the color indices. Using the relation (A.2), it is easy to show that $(u^c_a)^T C u_b = u^T_b C u^c_a$, hence, only the symmetric part of Y^χ contributes to $\mathcal{L}_Y$. According to our definition of the Yukawa couplings in the SM (2.7) then

$$Y^u = (Y^u)^T = \frac{1}{8} Y^\chi \tag{7.19}$$

and this relation is important, e.g., for the computation of the proton decay rate. Indeed, $U = U_C$ can be chosen in relation (2.17) due to (A.10), hence, the flavour dependent coefficients (6.19)-(6.21) entering the partial decay rates simplify considerably [117].

Similarly, one also gets

$$Y^d = (Y^e)^T = Y^\psi. \tag{7.20}$$

Consequently, the masses of the charged leptons and the corresponding down-type quarks are predicted to be identical at the GUT scale.

7.1.6 Problems of the minimal $SU(5)$ unification

In the above description of the simplest Georgi-Glashow model we mentioned some of its predictions concerning the correlation among the SM parameters. Unfortunately, not all these relations comply with the current measurements as discussed in this section.

1. *Gauge coupling unification.* Except for the superheavy gauge bosons and the extra Higgs triplet, both with masses at the GUT scale, the Georgi-Glashow model does not contain any extra fields, hence, the running of the couplings corresponds to that in the SM. The predicted value of the weak mixing angle at the GUT scale (7.14) then suggests the value of about 0.20 at the electroweak scale. Although this was in accordance with the first rough measurements in the 1970's, current experimental data give the value of 0.23 with a good precision [14]. This issue is sometimes rephrased as the problem of the exact unification of the gauge couplings: although in the 1970's, the three SM gauge couplings seemed to coalesce in one point at a high energy scale (and actually it was one of the motivations for GUTs), this is no longer the case with the more accurate measurements of the gauge couplings at low energies. Let us mention as a curiosity that if TeV-scale superpartners are included in the running, this problem disappears. This was one of the motivations for supersymmetric unifications, however, we won't elaborate on them here for reasons mentioned at the beginning of this chapter.

2. *Massless neutrinos.* Another issue is that in the fermionic multiplets of the Georgi-Glashow model (7.15) only the left-handed neutrinos are present, hence, neutrinos turn out to be massless. Their masses may be introduced via extra $SU(5)$ singlets of right-handed neutrinos, however, in that case no predictions of their properties are possible. Moreover, similarly as in SM, if the neutrinos are assumed to be Dirac particles, then the Yukawa couplings present in the Dirac neutrino mass term would be very small compared to other Yukawa couplings, which calls for justification.

3. *Equal masses for down-type quarks and charged leptons.* Taking into account the running from the GUT scale, this equality does approximately hold for the third family, which in the late 1970's was considered as a successful and promising prediction of the $SU(5)$ unifications. However, later measurements of fermion masses from all three families together with the updated computation of their running from the EW scale to the GUT scale [118] reveal serious incompatibility with formulae like (7.20).

7.1.7 Realistic extensions of the Georgie-Glashow model

Let us briefly discuss several realistic models based on the $SU(5)$ gauge group in order to explain how the problems listed in the previous section could be overcome.

It was pointed out already in [119] that the third problem of the wrong relations for the fermion masses can be avoided by adding a 45-dimensional scalar representation (recall the relations (7.8) and (7.9) which suggest that 45_H can form similar Yukawa terms as 5_H). Alternatively, by introducing non-renormalizable operators the same goal may be achieved [120].

Furthermore, most of the mechanisms for the generation of small neutrino masses described in Chapter 4 are also applicable in the context of the $SU(5)$ unifications. The minimal option (considered, e.g., in [121]) is an addition of the right-handed neutrinos providing the type I seesaw mechanism described in Section 4.3. On the other hand, adding the 15-dimensional scalar representation (7.6) containing the $(1, 3, +1)$ multiplet called Δ in Section 4.4 provides the type-II-seesaw neutrino mass contribution. Such models were studied in the non-renormalizable case [122] as well as in the case of renormalizable models supplemented by the 45_H representation [123] . Similarly, the combination of type I and type III seesaw using the 24-dimensional fermionic representation, which contains both the weak singlet ν_R and the triplet Δ_F introduced in (4.3), was studied either in the non-renormalizable [124] or in the renormalizable [125] versions. Finally, even the Zee mechanism described in Section 4.5.1 can be applied in the framework of the $SU(5)$ unification [126] when 10_H containing the weak singlet scalar h^+ and the 45_H containing the second Higgs doublet are added.

In summary, addition of extra fields is necessary for any realistic extension of the Georgi-Glashow model, and this also helps to overcome the first problem mentioned in Section 7.1.6. If some of the additional fields acquire masses below the GUT scale, exact unification may be achieved for certain settings which considerably constrains the parameter space of this kind of models.

7.2 SO(10)

Given the problems of the simplest $SU(5)$ GUT listed in Section 7.1.6 and the general complexity of the realistic extensions of the Georgi-Glashow model mentioned in Section 7.1.7, it is natural to look for another simple gauge group which may serve the purposes of the gauge unification. The next-to-minimal GUTs are based on the $SO(10)$ gauge group which was first suggested in published form by Fritzsch and Minkowski [127] (although they admit that "after completion of this article [they] were informed that the $SO(10)$ model has also been considered by Georgi and Glashow"[3]). In [127], $SO(10)$ unification is presented only as one way among many others in which to group the 16 fermions of the SM (with the extra ν_R) in (one or more) multiplets of a simple group, however, it is concluded that only the $SO(10)$ and $SU(5)$ models are anomaly-free. The beauty of the models based on the $SO(10)$ gauge group lies exactly in the fact that the whole family of the fermions can be accommodated in the 16-dimensional spinor representation of the $SO(10)$ as will be shown in Section 7.2.1.

The rank of the $SO(10)$ group is equal to 5, hence, the symmetry breaking is usually more complicated (see Section 7.2.4). On the other hand, as we have shown in Section 4.1 the smallness of neutrino masses often calls for new physics at the energy scale of the order of $\sim 10^{13}$ GeV. The presence of the intermediate breaking scales is hence welcome. This holds also because the fields with masses below the unification scale can help with addressing the issue of the exact unification in a natural way.

Furthermore, we show in Section 7.2.5 that the Yukawa sector of the $SO(10)$ unifications can be rather constrained even if multiple scalar representations are included, hence, one can have a good grip on the flavour-dependent coefficients (6.19)-(6.21) present in the partial proton decay rates.

Due to the complexity of the $SO(10)$ unifications we do not attempt here to study all the variants in such a depth as those based on the $SU(5)$ gauge group, for more details on the group theory see, e.g., [128], for the other aspects [67] or [115]. The references studying the phenomenology of different $SO(10)$ models will be given in Section 7.2.4 and a particular model based on the $SO(10)$ gauge group will be detailed in Section 8.2.

7.2.1 Group structure

The special orthogonal group $SO(n)$ consists of the orthogonal matrices in n-dimensional vector space with determinant equal to $+1$, and the $so(n)$ algebra can be generated by $\frac{n(n-1)}{2}$ independent purely imaginary antisymmetric $n \times n$ matrices. For our purposes, however, it will not be necessary to specify the 45 generators of the $so(10)$ algebra, we will concentrate on the $SO(10)$ representations and their decomposition with respect to the $SO(10)$ subgroups following mainly the references [128] and [129].

There are two different subgroups of $SO(10)$ of rank 5, namely, the $SU(5) \times U(1)$ and $SO(6) \times SO(4) = SU(4) \times SU(2) \times SU(2)$. The decomposition with respect to the latter "Pati-Salam" subgroup will be handy since it can give an insight into the

[3] H. Georgi should have mentioned the $SO(10)$ model in a talk given at the APS meeting at William and Mary College in 1974.

quark-lepton unification and the left-right symmetry within the $SO(10)$ unification, whereas the former subgroup will be used to identify the SM multiplets within the $SO(10)$ representations, relying on the detailed study of $SU(5)$ above. Let us not that the $SU(5) \times U(1)$ subgroup can be embedded to $SO(10)$ in two non-equivalent ways [130], either the $SU(5)$ contains the SM hypercharge generator as in case of the Georgi-Glashow model, or the hypercharge can be a combination of the $SU(5)$ and $U(1)$ generators. The latter case corresponds to the so-called flipped $SU(5)$ subgroup, and the former embedding will be used for the decomposition of the $SO(10)$ representations.

The defining 10-dimensional representation of the $SO(10)$ is real (as any other defining representation of $SO(n)$ since $-T_a^* = T_a$ for purely imaginary generators), and can be decomposed with respect to the $SU(5)$ subgroup as

$$10 = 5 \oplus \bar{5}. \tag{7.21}$$

Consequently, two copies of the SM-Higgs-like scalar fields can be accommodated in this representation. The tensor product

$$10 \otimes 10 = 1_s \oplus 45_a \oplus 54_s \tag{7.22}$$

contains the adjoint representation and also the 54-dimensional representation which will play a role in the $SO(10)$ symmetry breaking; however, let us first consider two special representations of $SO(10)$ which can not be constructed by any tensor product, namely, the 16 and $\overline{16}$ spinor representations. For $SO(4n+2)$ the spinor representations denoted as D^{2n+1} and D^{2n} are complex and can be decomposed with respect to the $SU(2n+1)$ subgroup as [128]

$$D^{2n+1} = \sum_{j=0}^{n} [2j+1] \qquad D^{2n} = \sum_{j=0}^{n} [2j]$$

where $[k]$ is a representation corresponding to antisymmetric tensor product of k defining representations of $SU(2n+1)$. Recalling that $\overline{[k]} = [2n+1-k]$ holds for the $SU(2n+1)$ representations, in case of $n = 2$ then

$$\overline{16} \equiv D^5 = 5 + \overline{10} + 1, \qquad 16 \equiv D^4 = 1 + 10 + \bar{5}. \tag{7.23}$$

This means that both the matter multiplets (7.15) can be accommodated in the 16-dimensional representation together with a singlet field corresponding to ν_R, hence, the whole family of the SM fermions occupies a single multiplet in the case of $SO(10)$. The content of the tensor products

$$\overline{16} \otimes 16 = 1 \oplus 45 \oplus 210 \tag{7.24}$$

$$16 \otimes 16 = 10_s \oplus 120_a \oplus \overline{126}_s \tag{7.25}$$

will be useful for the construction of the Lagrangian, as well as the following decomposition with respect to the $SU(5)$ subgroup

$$120 = 5 \oplus \bar{5} \oplus 10 \oplus \overline{10} \oplus 45 \oplus \overline{45}, \tag{7.26}$$

$$126 = 1 \oplus \bar{5} \oplus 10 \oplus \overline{15} \oplus 45 \oplus \overline{50}. \tag{7.27}$$

In order to explore the gauge boson structure, it will be instructive to decompose the adjoint representation of the $SO(10)$ with respect to the $SU(4)_C \times SU(2)_L \times SU(2)_R$ subgroup:

$$45 = (15,1,1) \oplus (1,3,1) \oplus (1,1,3) \oplus (6,2,2). \tag{7.28}$$

Moreover, in order to identify the SM multiplets, the embedding $SU(3)_c \times U(1)_Y \subset SU(4)_C \times SU(2)_R$ has to be specified. One finds that

$$SU(3)_c \times U(1)_{B-L} \subset SU(4)_C \tag{7.29}$$

where $B-L$ denotes the difference of the baryon and lepton numbers (more precisely, the corresponding canonically normalized generator of $SU(4)_C$ is $\sqrt{\frac{3}{8}}(B-L)$). The SM hypercharge is then a combination of $B-L$ and the diagonal generator of $SU(2)_R$:

$$Y = T_3^R + \frac{1}{2}(B-L). \tag{7.30}$$

The decomposition of the $SU(4)_C$ multiplets with respect to the $SU(3)_c \times U(1)_{B-L}$ subgroup can be found, e.g., in [129]:

$$15 = (8,0) \oplus (3,+\tfrac{4}{3}) \oplus (\bar{3},-\tfrac{4}{3}) \oplus (1,0), \qquad 6 = (3,-\tfrac{2}{3}) \oplus (\bar{3},+\tfrac{2}{3}). \tag{7.31}$$

The decomposition with respect to the $SU(4)_C \times SU(2)_L \times SU(2)_R$ subgroup of the 54- and 210- dimensional representations present in the tensor products (7.22) and (7.24), respectively, will be useful when inspecting the symmetry breaking chains and can be found also in [129]:

$$54 = (1,1,1) \oplus (1,3,3) \oplus (20,1,1) \oplus (6,2,2), \tag{7.32}$$

$$210 = (1,1,1) \oplus (15,1,1) \oplus (6,2,2) \oplus (15,1,3) \oplus (15,3,1) \oplus (10,2,2) \oplus (\overline{10},2,2). \tag{7.33}$$

7.2.2 Gauge fields

In order to describe the $SO(10)$ gauge fields, let us decode the individual terms in the decomposition (7.28). Using the formula (7.31), one finds

$$(15,1,1) = (8,1,0) \oplus (3,1,+\tfrac{2}{3}) \oplus (\bar{3},1,-\tfrac{2}{3}) \oplus (1,1,0) \tag{7.34}$$

where the multiplets are denoted by the $SU(4)_C \times SU(2)_L \times SU(2)_R$ and SM quantum numbers on the left- and right-handed side of the equation, respectively. The embedding (7.29) suggests that the first term corresponds to the SM gluons and the last term accommodates the neutral gauge field corresponding to the $B-L$ symmetry. On top of that, the leptoquark gauge field with quantum numbers $(3,1,+\frac{2}{3})$ and its conjugate occur in unifications including the $SU(4)_C$ gauge group which were first studied by Pati and Salam [131]. These fields do not induce proton decay, however, they lead to other non-observed interactions such as $K_L^0 \to \mu^\pm e^\mp$ which constrains the mass of these vector bosons to be at least of the order of 10^6 GeV in the generic case [132].

The second and the third terms in (7.28) can be identified with the $SU(2)_L$ and $SU(2)_R$ gauge fields, respectively. It is worth noting here that introducing the left-right symmetry [133, 134] is an appealing way, in which to pair the right-handed SM matter fields into the $SU(2)_R$ doublets in the same fashion as the left-handed ones are paired into doublets of $SU(2)_L$. However, also the $SU(2)_R$ gauge bosons have to acquire large masses in order to comply with the experimental constraints. Nowadays, the collider searches or, e.g., the constraints from the neutral kaon mixing require the masses of $SU(2)_R$ gauge bosons to be at least few TeV [135].

Finally, (7.31) implies the decomposition

$$(6, 2, 2) = (3, 2, +\tfrac{1}{6}) \oplus (3, 2, -\tfrac{5}{6}) \oplus (\bar{3}, 2, +\tfrac{5}{6}) \oplus (\bar{3}, 2, -\tfrac{1}{6}), \tag{7.35}$$

where again the $SU(4)_C \times SU(2)_L \times SU(2)_R$ and SM quantum numbers are given on the left- and right-hand side, respectively. Hence, both the gauge boson multiplets mediating the proton decay (6.11) are present in $SO(10)$ unifications.

7.2.3 Matter fields

As already suggested below (7.23), the whole family of the SM fermions (including also the right-handed neutrino) can be accommodated in the spinor representation of the $SO(10)$ denoted by 16_F.

When rewriting the gauge interactions of 16_F using the SM matter fields, one finds that both $(\bar{3}, 2, +\frac{5}{6})$ and $(\bar{3}, 2, -\frac{1}{6})$ gauge bosons from (7.35) imply baryon number violation via terms of analogous form as in the $SU(5)$ unification (7.17) and the flipped $SU(5)$ unification (7.56), respectively. Moreover, other beyond-standard-model interactions are induced by $SU(2)_R$ and $SU(4)_C$ gauge bosons as already mentioned in the previous section. Since, however, the $SU(2)_R$ and $SU(4)_C$ breaking scales are typically rather large in the context of the $SO(10)$ unifications (see, e.g., [136]), i.e., well above the experimental limits ~ 1 TeV and ~ 1000 TeV mentioned in the previous section, corresponding interactions will not constrain the models discussed here, hence, their explicit form will not be needed.

7.2.4 Symmetry breaking patterns

In this section the spontaneous breaking of the $SO(10)$ gauge symmetry to the SM will be considered, whereas the breaking of the SM gauge group and the consequent fermion masses will be studied in the following section. There are multiple options for the $SO(10) \to$ SM symmetry breaking chains usually including one or more intermediate breaking scales. Here we will concentrate mainly on listing the possibilities for the first step of the symmetry breaking and the full chains will be given later for the selected models only (see Section 8.2).

Our approach will be to identify the $SO(10)$ representations which contain SM singlets, hence, can be used for the $SO(10)$ symmetry breaking. The quantum numbers of the multiplets containing these singlets with respect to the various $SO(10)$ subgroups will then suggest the possible symmetry breaking patterns. However, for an exhaustive mathematical discussion we refer the reader, e.g., to Chapter 24 of [128].

The smallest representation containing a SM singlet is apparently 16_H which decomposes under the $SU(5)$ subgroup as (7.23), hence, the only SM singlet contained in 16_H is also an $SU(5)$ singlet which suggests that 16_H triggers the intermediate $SU(5)$ symmetry stage. Similarly, the 126-dimensional representation (7.27) contains an $SU(5)$ singlet and no other SM singlets (recall that among the $SU(5)$ representations up to dimension 75 only the 1-, 24- and 75-dimensional ones contain a SM singlet). However, for non-supersymmetric scenarios the gauge unification constraints exclude the $SU(5)$ model as pointed out in Section 7.1.6, hence, the

$$SO(10) \xrightarrow{\langle 16_H \rangle \,/\, \langle 126_H \rangle} SU(5) \tag{7.36}$$

breaking is not phenomenologically viable. For this reason, we will concentrate on the symmetry breaking without the $SU(5)$ intermediate stage in the following.

The decomposition of the $SO(10)$ adjoint representation under the $SU(4)_C \times SU(2)_L \times SU(2)_R$ (7.28) suggests that also 45_H can trigger the $SO(10)$ symmetry breaking. It was found in Section 7.2.2 that the SM hypercharge gauge field resides in the $(15, 1, 1)$ submultiplet (7.34), the corresponding SM singlet is, hence, neutral with respect to the $SU(2)_R$ and also $U(1)_{B-L}$ due to (7.30). On the other hand, the SM singlet corresponding to the electrically neutral $SU(2)_R$ gauge field contained in the $(1, 1, 3)$ submultiplet is neutral with respect to $SU(4)_C$. This all suggests that both variants

$$SO(10) \xrightarrow{\langle 45_H \rangle} SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \tag{7.37}$$

$$SO(10) \xrightarrow{\langle 45_H \rangle} SU(4)_C \times SU(2)_L \times U(1)_R \tag{7.38}$$

are possible depending on the vacuum structure. Actually, the thorough study of the possible vacuum settings shows that also the $SU(5) \times U(1)$ intermediate stage may be triggered by $\langle 45_H \rangle$ (see Section 8.2), however, it can be shown that this option is not phenomenologically interesting.

Furthermore, the 54-dimensional representation is decomposed under the $SU(4)_C \times SU(2)_L \times SU(2)_R$ subgroup as (7.32) and it is easy to check that only the first term contains a SM singlet. This implies

$$SO(10) \xrightarrow{\langle 54_H \rangle} SU(4)_C \times SU(2)_L \times SU(2)_R. \tag{7.39}$$

Next, let us mention as a curiosity that the one-step breaking

$$SO(10) \xrightarrow{\langle 144_H \rangle} SM \tag{7.40}$$

is an option [137,138] where no other scalar representations are needed, however, extra matter fermions have to be added in order to accommodate realistic fermion masses.

We will stop our search for the SM singlets with the 210-dimensional representation (7.33). The $(15,1,1)$ term contains the SM singlet similarly as in the case of the 45_H and of course $(1,1,1)$ is a SM singlet, hence, the two variants

$$SO(10) \xrightarrow{\langle 210_H \rangle} SU(4)_C \times SU(2)_L \times SU(2)_R, \tag{7.41}$$

$$SO(10) \xrightarrow{\langle 210_H \rangle} SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \tag{7.42}$$

emerge. As in the case of 45_H also the $SU(5) \times U(1)$ symmetry stage can be reached by a particular vacuum structure.

Let us note that for the phenomenologically viable options (7.37)-(7.42) (apart from the one triggered by 144_H), the $U(1)_{B-L}$ subgroup is preserved and either 16_H or 126_H has to be added in order to break this symmetry. Broken $B-L$ then usually allows for right-handed neutrino Majorana masses as will be discussed in the next section, hence, specific value of $\langle 16_H \rangle$ or $\langle 126_H \rangle$ is necessary in order to provide realistic active neutrino masses through the type I seesaw mechanism (see Section 4.3). For a recent analysis of such phenomenology constraints combined with the requirement of the exact gauge coupling unification see, e.g., [136]. Also the options how to further break the symmetries (7.37)-(7.42) are listed in this article and references therein.

On the other hand, in [136] the so-called extended survival hypothesis [139] was assumed, i.e., only the scalars responsible for the spontaneous symmetry breaking at lower scales did not acquire unification scale masses. The dedicated study of the scalar potential may, however, show that this assumption is not fulfilled (if, for instance, a pseudo-Goldstone mode corresponding to an accidentally enhanced global symmetry appears) and the analysis of the phenomenology constraints becomes more complex. The case with 54_H was studied in great detail, e.g., in [140], in [141] the case with 210_H was scrutinized and [142] studied the phenomenology of both these variants. The option with 45_H was discarded, e.g., by the reference [143] where the 54_H case was favoured. The reasons for abandoning the 45_H option and its recent revival will be described in Section 8.2.

7.2.5 Yukawa couplings

Since all the SM fermions are accommodated in the 16-dimensional representation of $SO(10)$, the SM Higgs field has to be part of a representation present in the scalar product (7.25). The 10-dimensional scalar field can be in principle either real or complex, although the former option does not lead to a realistic fermion spectrum [144]. In the latter case, both 10_H and 10_H^* can couple independently to the SM fermions, however, the extra Yukawa coupling is usually avoided by introducing the Peccei-Quinn symmetry (see Section 5.1 for general discussion of this symmetry and [144] for its implementation in the $SO(10)$ unifications). The Yukawa Lagrangian then reads schematically

$$\mathcal{L}_Y = Y_{10} 16_F\, 16_F\, 10_H + Y_{120} 16_F\, 16_F\, 120_H + Y_{126} 16_F\, 16_F\, \overline{126}_H \tag{7.43}$$

where

$$Y_{10} = Y_{10}^T, \qquad Y_{120} = -Y_{120}^T, \qquad Y_{126} = Y_{126}^T \tag{7.44}$$

are matrices in the flavour space.

Let us note that the decomposition of the 10- and 126-dimensional representations with respect to the $SU(5)$ subgroup (7.21) and (7.27) suggest that these representations can accommodate two different copies of the SM Higgs doublets (recall that 5_H and 45_H was used for Yukawa couplings in the $SU(5)$ unifications), the corresponding VEVs will be denoted as $v^u_{10/126}$ and $v^d_{10/126}$. Similarly, the 120-dimensional representation (7.26) contains four Higgs-like fields, which can acquire different VEVs. Consequently, after the spontaneous symmetry breaking the Dirac mass terms for the up- and down-quarks, neutrinos and charged leptons contain the mass matrices [145]

$$M_u = v^u_{10} Y_{10} + v^u_{120} Y_{120} + v^u_{126} Y_{126}, \tag{7.45}$$

$$M_d = v^d_{10} Y_{10} + v^d_{120} Y_{120} + v^d_{126} Y_{126}, \tag{7.46}$$

$$M_D = v^u_{10} Y_{10} + v^D_{120} Y_{120} - 3 v^u_{126} Y_{126}, \tag{7.47}$$

$$M_e = v_{10} Y^d_{10} + v^e_{120} Y_{120} - 3 v^d_{126} Y_{126}, \tag{7.48}$$

respectively.

Clearly, at least two non-zero Yukawa matrices (7.44) will be needed: in case when all the mass matrices (7.45)-(7.48) are proportional to a single matrix, the quark mixing can not be accommodated ($U = D$ would hold for the rotation matrices in (2.17) which implies $V_{\mathrm{CKM}} = \mathbb{1}$ in (2.23)).

Furthermore, if no extra fine-tuning is at play, the Dirac neutrino mass matrix (7.47) would imply the neutrino masses of the similar magnitude as for the other SM fermions. On the other hand, if large Majorana masses of the right-handed neutrinos are introduced, this problem is naturally solved thanks to the type I seesaw mechanism described in Section 4.3.

As already anticipated in the previous section, the right-handed neutrino masses are usually connected to the $B - L$ breaking scale determined by the VEV of the 16_H or 126_H. Namely, in case of 16_H these Majorana masses can be generated at the two-loop level as will be shown in Section 8.1, however, as explained immediately afterwards, this is not phenomenologically viable. Another option is then to introduce non-renormalizable $d = 5$ operators if only 16_H is present.

On the other hand, the $SU(5)$ singlets of the right-handed neutrinos in 16_F (7.23) can couple to the $SU(5)$ singlet in 126_H (7.27) through the third term in (7.43), hence, the VEV direction of 126_H assigned as v^R_{126} can induce the right-handed neutrino masses at the tree level. This setting provides correlations among different fermion sectors and is usually preferred in recent studies.

The scalar 126_H also contains the weak triplet Δ with an induced VEV as in the type II seesaw mechanism described in Section 4.4 (recall that the 15-dimensional representation of the $SU(5)$ contained in (7.27) allowed the type II seesaw mechanism within the $SU(5)$ unifications as shown in Section 7.1.7). Consequently, the matrices entering the (generalized) type I seesaw formula (4.9) besides M_D (7.47) read

$$M_R = v_{126}^R Y_{126}, \tag{7.49}$$

$$M_L = v_{126}^L Y_{126}. \tag{7.50}$$

if $\langle\Delta\rangle = v_{126}^L$ is assigned.

In summary, the most economic choice of the Higgs sector which could accommodate realistic fermion masses corresponds to combination of 126_H and either 10_H or 120_H. Recent numerical analyses [145,146] show that such scenarios are highly constrained by the relations (7.45)-(7.49) (the dominance of the type I seesaw mechanism is assumed for the fits): for the $126_H + 10_H$ scalar content only the case of the normal neutrino mass hierarchy can be fitted and for the option $126_H + 120_H$ no consistent fits exist at all.

Let us note that if the scalar content $10_H + 126_H$ is chosen, then both the up- and down-quark mass matrices (7.45) and (7.46) are symmetric due to (7.44), which implies $U = U_C$ and $D = D_C$ in the formula (2.17) and firm predictions for the flavour structure of the proton partial decay rates (6.14)-(6.18) can be obtained.

7.3 Flipped $SU(5)$

Interestingly, the $SU(5) \times U(1)$ subgroup embedded in the $SO(10)$ in the "flipped" way (see Section 7.2.1) can be used as a gauge group by itself preserving the features of the full $SO(10)$ unification which enabled to avoid the problems of the standard $SU(5)$ model mentioned in Section 7.1.6. The unifications based on the $SU(5) \times U(1)_X$ gauge group were first studied in [130,147,148] and their basic structure will be described here omitting some of the details which are analogous to the case of the ordinary $SU(5)$. A particular flipped $SU(5)$ model will be later studied in Section 8.1.

7.3.1 Group structure

The key difference with respect to the ordinary $SU(5)$ unifications consists in the embedding of the SM hypercharge:

$$Y = \frac{1}{5}\left(X - \sqrt{\frac{5}{3}}T_{24}\right) \tag{7.51}$$

where T_{24} is again one of the $SU(5)$ generators (7.2) and the X charge corresponds to the $U(1)_X$ group factor.

This means that the $U(1)_Y$ is *not* a subgroup of $SU(5)$, hence, only the $SU(3)_c$ and $SU(2)_L$ couplings have to meet at one point, and the problem of the exact unification, mentioned as the first point in Section 7.1.6, disappears.

7.3.2 Gauge fields

Obviously, the hypercharge gauge field B^μ does not correspond to the T_{24} generator of $SU(5)$ as in (7.13), but there are two orthogonal combinations of A^μ_{24} and the $U(1)_X$ gauge field, one of them being B^μ and the other one a neutral gauge field with mass at the unification scale.

For the following discussion, however, only the mediator of the baryon number violation analogous to X^μ present in the ordinary $SU(5)$ will be important. In case of the flipped $SU(5)$, (7.12) is replaced by

$$X'^\mu = \left(\begin{array}{c} X'^u \\ X'^d \end{array} \right)^\mu = (\bar{3}, 2, -\tfrac{1}{6}) \tag{7.52}$$

due to the relation (7.51), i.e., the electric charges of X'^u and X'^d read $+\frac{1}{3}$ and $-\frac{2}{3}$, respectively. Let us note that this is exactly the gauge boson identified by the "opening" of the $d = 6$ operator $\mathcal{O}_{III}$ (6.10).

7.3.3 Matter fields

The different definition of the SM hypercharge (7.51) changes also the field content of the matter multiplets (7.15). Namely, if one defines the $SU(5) \times U(1)_X$ charges of these multiplets as[4]

$$10_F \equiv (10, +1), \qquad \bar{5}_F \equiv (\bar{5}, -3) \tag{7.53}$$

the last term in the decomposition (7.7) obtains, according to the formula (7.51), the quantum numbers of the right-handed neutrino which, subsequently, becomes an inevitable element of the theory solving partly the second problem mentioned in Section 7.1.6 (we will describe how to explain the smallness of neutrino masses within the flipped $SU(5)$ models in Section 7.3.5). On the other hand, in order to accommodate the right-handed electron, an extra $SU(5)$ singlet

$$1_F \equiv (1, +5) = (e^c)_L \tag{7.54}$$

has to be added. At the same time, the right-handed up-type quarks move to the $\bar{5}_F$ multiplet, whereas the right-handed down-type quarks become part of the 10_F

[4]This definition together with (7.54) is the only non-anomalous way, in which the $U(1)_X$ charges can be assigned and it also corresponds to the decomposition of the fermionic multiplet 16_F in $SO(10)$ under the maximal subgroup $SU(5) \times U(1)$.

multiplet. In summary, the alternative embedding of the SM hypercharge induces a flip

$$(e^c)_L \leftrightarrow (\nu^c)_L \qquad (d^c)_L \leftrightarrow (u^c)_L \tag{7.55}$$

in formula (7.15).

This flip influences, e.g., the gauge interactions of the fermions arising from a term analogous to (7.16). Recalling (7.52), the formula (7.17) is replaced by

$$\begin{aligned}\mathcal{L}_{gauge} \ni & \frac{g}{\sqrt{2}}\Big[(X_j'^u)^\mu \left(\overline{u_j^c}\gamma_\mu e + \overline{\nu^c}\gamma_\mu d_j + \varepsilon_{jkl}\overline{u_l}\gamma_\mu d_k^c\right) \\ & +(X_j'^d)^\mu\left(-\overline{u_j^c}\gamma_\mu \nu - \overline{\nu^c}\gamma_\mu u_j + \varepsilon_{jkl}\overline{d_l}\gamma_\mu d_k^c\right)\Big] + h.c. \\ = & \frac{g}{\sqrt{2}}\left[\overline{u_j^c}\gamma_\mu X_j'^{\mu T} i\sigma_2 L + \overline{\nu^c}\gamma_\mu X_j'^{\mu T} i\sigma_2 Q_j + \varepsilon_{jkl}\overline{Q_l}^T X_j'^\mu \gamma_\mu d_k^c\right] + h.c.\end{aligned} \tag{7.56}$$

in case of the flipped $SU(5)$. As in the $SU(5)$ case, the first and the third terms on the third line show that the operator $\mathcal{O}_{III}$ (6.10) can be obtained by integrating out the vector boson X'^μ (see the third diagram in Figure 6.1).

Also these interactions induce proton decay, hence, as in the ordinary $SU(5)$ case, X'^μ is assumed to be very heavy.

7.3.4 Scalar sector and Yukawa couplings

The SM Higgs field can be again accommodated in the fundamental representation of the $SU(5)$ if we define

$$5_H \equiv (5, -2).$$

This choice of the $U(1)_X$ charge also allows the Yukawa couplings of 5_H with the fermion fields (7.53) with an analogous structure as in the case of the ordinary $SU(5)$ (7.18), however, the flip (7.55) transforms the formulas (7.19), (7.20) into

$$Y^d = (Y^d)^T, \qquad Y^u = (Y^{\nu_D})^T. \tag{7.57}$$

The first formula suggests that $D = D_C$ in (2.17) which, again, allows to simplify the coefficients (6.19)-(6.21) and sometimes even to get partial proton decay rates independent of the flavour structure of the particular model [117].

The second equality in (7.57) relates the masses of the up-type quarks with the Dirac neutrino masses. This is in no contradiction with the observed masses if type I seesaw mechanism is employed and the right-handed neutrinos acquire large masses, hence, also the third problem mentioned in Section 7.1.6 can be addressed.

For completeness, let us add that the charged lepton Yukawa couplings are independent of the other parameters since they arise from the term

$$\mathcal{L}_Y \ni Y^1_{ab}\bar{5}_{Fai}1_{Fb}5^i_H$$

where i and a, b are the $SU(5)$ and generation indices, respectively, and the Lorentz structure is suppressed for simplicity.

Finally, the single-step $SU(5) \times U(1)_X \to SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry breaking can be triggered by

$$10_H \equiv (10, +1), \qquad \langle 10_H^{45} \rangle = V_G$$

where $ij = 45$ denotes the $SU(5)$ indices (recall the matrix notation (7.15)). Notice that the $SU(5) \times U(1)_X$ quantum numbers of 10_H coincide with those of 10_F (7.53), hence, the 10_H^{45} field corresponds to the right-handed neutrino entry, which shows that $\langle 10_H^{45} \rangle$, indeed, preserves the SM gauge group.

7.3.5 Neutrino mass generation

In the literature, mostly the supersymmetric flipped SU(5) models are studied because of the motivation from the string theory (see, e.g., [149] for a review). In that case the neutrino mass matrix is extended with fermionic partners of the singlet scalar fields, for further details, we refer the reader to studies like [150].

In the non-supersymmetric case, it was suggested, e.g., in [151] that the VEV of an extra scalar could provide the right-handed neutrino masses. Indeed, recalling the formula (7.8), one way in which the Majorana masses of the right-handed neutrinos can be generated is to introduce a scalar multiplet

$$50_H \equiv (50, -2), \qquad \langle 50_H \rangle = V_\nu$$

together with an extra term in the Yukawa Lagrangian

$$\mathcal{L}_Y \ni Y_{ab}^{50} 10_{Fa}^{ij} 10_{Fb}^{kl} 50_{Hijkl}.$$

In the last expression the Lorentz structure is again suppressed for simplicity, i, j, k, l and a, b are $SU(5)$ and generation indices, respectively. This term then provides the right-handed neutrino masses proportional to $Y^{50} V_\nu$. Let us note, however, that in such a scenario, the (symmetric) Y^{50} matrix is not correlated with other parameters of the model and, hence, the neutrino data do not considerably constrain the parameter space.

An alternative way in which the right-handed neutrino masses can be introduced within the framework of the flipped $SU(5)$ models will be presented in Section 8.1.

Chapter 8

Phenomenology of specific unified models

8.1 Witten's mechanism in the flipped SU(5) scenario

This section serves as an introduction to the full article [3] attached to this thesis as Appendix D. The motivation and inspiration for this work is described and several properties of the model under consideration are linked to the theory part of this thesis including the translation of the notation used in [3]. Finally, the main results are discussed.

It was explained in Section 7.3 that in the flipped SU(5) unifications the right-handed neutrinos are an integral part of the model, however, in order to introduce their Majorana masses (and, hence, to enable the type I seesaw mechanism described in Section 4.3) by means of the usual Higgs mechanism, a scalar 50-dimensional representation of $SU(5)$ has to be introduced. On the other hand, in [3] the right-handed neutrino masses were generated in an alternative way inspired by the Witten's idea [152].

In the original work [152] the neutrino masses were analysed in the context of a model based on the $SO(10)$ gauge symmetry spontaneously broken to $SU(5)$ by the VEV of a 16-dimensional scalar representation. On top of that, the 10-dimensional scalar representation contained the SM Higgs field responsible for the fermion masses. Witten observed that instead of introducing the 126-dimensional scalar which could provide the Majorana masses for the right-handed neutrinos at the tree level (see Section 7.2.5), the action of this representation can be mimicked by other fields which couple to the right-handed neutrinos (namely, the scalar 10 and the gauge 45 can be combined as $10 \otimes 45 \otimes 45 \ni 126$). The Majorna masses are then built at the loop level as shown in Figure 8.1.

Unfortunately, it was later shown that the symmetry breaking chain (7.36) is not consistent with the gauge coupling unification in most of the non-supersymmetric scenarios, hence, in realistic models the VEV of the 16-dimensional representation deter-

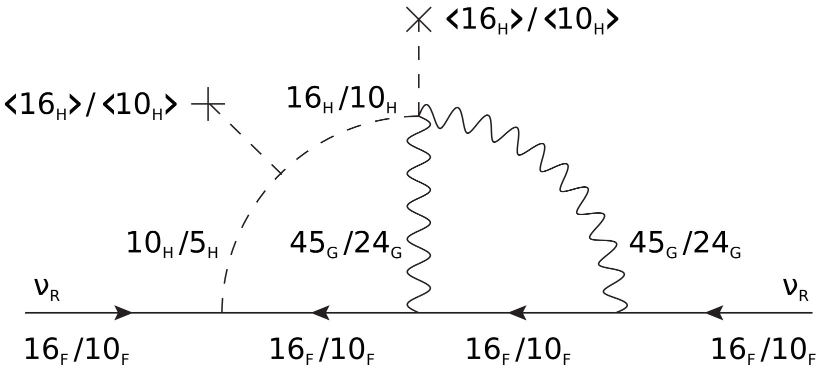

Figure 8.1: "Witten's loop" both in the original context of $SO(10)$ unification [152] and in the flipped $SU(5)$ model [3]. The first/second representation assigning the lines in this Feynman diagram correspond to these two implementations, respectively. Let us note that also the crossed diagrams contribute to the right-handed neutrino masses.

mining the size of the right-handed neutrino Majorana masses lies at an intermediate scale usually few orders of magnitude below the GUT scale. In this case the masses of the right-handed neutrinos are too low to produce small enough active neutrino masses in a generic case with $\mathcal{O}(1)$ Yukawa couplings.

On the other hand, if only the subgroup $SU(5) \times U(1)_X$ of the $SO(10)$ (studied in Section 7.3) is considered as the unified gauge group, the unification constraints are weakened, while very similar two-loop diagram can be built with the 5-, 10- and 24-dimensional representations of $SU(5)$ replacing the 10-, 16- and 45-dimensional representations of $SO(10)$, respectively (see Figure 8.1). In the flipped $SU(5)$ case $\langle 10_H \rangle$ at the GUT scale is consistent with the phenomenological constraints and realistic neutrino masses are obtained.

Combining the type I seesaw formula (4.9) (with $M_L = 0$) and the relation between the up-quark and Dirac neutrino Yukawa couplings in the flipped $SU(5)$ unifications (7.57), one finds that the light-neutrino mass matrix is related to the up-quark mass matrix M_u as

$$M^\nu = -M_u^T M_R^{-1} M_u.$$

Here the notation of the Section 4.3 is used, the notation employed in [3] corresponds to the translation $M^\nu \to m_{LL}$, $M_R \to M_\nu^M$. At the same time, the contribution of the loop diagram generating the mass matrix M_R can be evaluated using the parameters of the model. Consequently, M^ν can be in principle determined as well as the unitary matrix N (denoted as U_ν in the article [3]) used for the diagonalization of M^ν as in (3.14). This provides an extra information about the flavour structure of the model on top of the low energy constraints given by the fermion masses and the CKM and PMNS matrices (2.23), (3.18). Also the flavour-dependent coefficients (6.19)-(6.21) and, hence, the partial proton decay rates are then constrained as was shown both analytically and numerically (see Section III of Appendix D, the author of this thesis contributed to the analytical estimates and to the numerical results presented in the Figures 4-7). The predictions for the potentially observable decay channels $p \to \pi^0 e^+$ and $p \to \pi^0 \mu^+$ are then the main results of this article.

Let us note that after the publication of our results, we have found out that the basic structure of the Witten's loop in the flipped SU(5) context has already been considered in the context of string models [153].

8.2 Minimal $SO(10)$

As shown in Section 7.2.4, the $SO(10)$ gauge symmetry can be broken by the scalar 45-dimensional adjoint representation, and the rank of the gauge group has to be then further reduced. The viability of such schemes was shown only recently as described in Section 8.2.1 and we analysed one of the variants of this minimal $SO(10)$ model in the article [4] attached to this thesis as Appendix E. The methods used in this article together with the main results are summarized in Section 8.2.2.

8.2.1 $SO(10)$ broken by 45_H revived

It was found in Section 7.2.4 that the SM singlets reside in the following submultiplets of 45_H denoted by the $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ quantum numbers and acquiring the VEVs named in accordance with [4] as

$$\langle (1,1,1,0)_{45} \rangle \equiv \omega_{BL}, \qquad \langle (1,1,3,0)_{45} \rangle \equiv \omega_R. \tag{8.1}$$

The symmetry breaking options (7.37), (7.38) were found by simple inspection of the structure of the 45_H submultiplets, on the other hand, a deeper analysis reveals that for particular VEV settings the following subgroups can be obtained:

$$\begin{aligned}
\omega_R = 0, \omega_{BL} \neq 0: & \qquad SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}, && (8.2)\\
\omega_R \neq 0, \omega_{BL} = 0: & \qquad SU(4)_C \times SU(2)_L \times U(1)_R, && (8.3)\\
\omega_R \neq 0, \omega_{BL} \neq 0: & \qquad SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L}, && (8.4)\\
\omega_R = \omega_{BL} \neq 0: & \qquad \text{standard } SU(5) \times U(1), && (8.5)\\
\omega_R = -\omega_{BL} \neq 0: & \qquad \text{flipped } SU(5) \times U(1)_X. && (8.6)
\end{aligned}$$

This means that only the settings with $\omega_R \gg \omega_{BL}$ or $\omega_{BL} \gg \omega_R$ are phenomenologically viable as they do not contain the $SU(5)$ symmetry stage disfavoured by the unification constraints (see Section 7.2.4).

Regardless of the further symmetry breaking, it was observed in [143, 154], that only the settings with[1]

$$-2 < \frac{\omega_{BL}}{\omega_R} < -\frac{1}{2} \tag{8.7}$$

resembling the case (8.6) support the minimum of the tree-level scalar potential (see, e.g., the tree level formulas (7) and (8) in Appendix E). This observation led to abandoning the minimal schemes with only the 45_H scalar being responsible for the $SO(10)$ breaking.

Surprisingly, about 30 years later, it was found out [155] that the constraint (8.7) is released when quantum corrections are taken into account. In [4] an example of the one-loop formulas for the scalar masses is given showing that the loop corrections may easily ensure positive values of the scalar masses even if (8.7) is not satisfied (see the formulas (10), (11) in Appendix E). Moreover, the viability of the settings with $\omega_R \gg \omega_{BL}$ or $\omega_{BL} \gg \omega_R$ was ultimately demonstrated in the recent paper [156] where most of the important one-loop contributions to the effective potential were computed.

Consequently, the revived minimal $SO(10)$ settings with the Higgs sector consisting of $45_H \oplus 16_H$ or $45_H \oplus 126_H$ were further studied (in this discussion we neglect the SM symmetry breaking which can be ensured, e.g., by adding a 10_H as shown in Section 7.2.5). In particular, the issue of the neutrino masses within these models had to be solved, since assuming the extended survival hypothesis[2] the unification constraints suggest the $B-L$ breaking scale below 10^{12} GeV or 10^{10} GeV for the Higgs sector containing 16_H or 126_H, respectively [136].

As explained in Section 8.1, in case with $\langle 16_H \rangle$ well below the unification scale the right-handed neutrino masses induced by the Witten's mechanism are too low to provide realistic active neutrino masses in the generic case of $\mathcal{O}(1)$ Yukawa couplings. The same problem occurs if the $d = 5$ effective operator containing 16_H is introduced; moreover, in such a non-renormalizable scheme a number of new unconstrained parameters emerges.

On the other hand, using the 126_H representation, the right-handed neutrino masses can be introduced at the tree level and such a setting provides interesting correlations among the different fermion sectors as shown in Section 7.2.5. In [4] the VEV of the $SU(5)$ singlet field contained in 126_H (assigned as v_{126}^R in Section 7.2.5) is named

$$\langle (1, 1, 3, +2)_{126} \rangle \equiv \sigma \tag{8.8}$$

where the $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ quantum numbers are specified. Unfortunately, the value $|\sigma| \lesssim 10^{10}$ GeV setting the scale of the right-handed neutrino masses (7.49) again implies too heavy left-handed neutrinos if $\mathcal{O}(1)$ Yukawa couplings are assumed. However, it was shown in [157] that when the minimal survival hypothesis

[1]In the papers [143, 154] an extra Z_2 symmetry of the scalar potential was considered yielding the relation $-1 < \frac{\omega_{BL}}{\omega_R} < -\frac{2}{3}$, however, we refer here to the result obtained for the scalar potential considered in [4].

[2]Recall that this means that only the scalars responsible for the spontaneous symmetry breaking on lower scales do not acquire the unification scale masses as discussed in Section 7.2.4.

is abandoned and the full parameter space of the model is considered, settings with the $B-L$ breaking scale in the ballpark of $|\sigma| \sim 10^{13}$ GeV are reachable. Namely, the structure of the one-loop beta functions for the gauge coupling running revealed that this is achieved, for instance, if a scalar field with the SM quantum numbers $(6, 3, +\frac{1}{3})$ or $(8, 2, +\frac{1}{2})$ becomes light.

In order to obtain reliable quantitative predictions[3], however, a two-loop analysis of the unification constraints has to be performed. Let us note that computation at such a precision level has a good sense in the context of the $SO(10)$ broken by 45_H due to, e.g., the vanishing contribution of the leading Plack-scale correction described by the $d = 5$ effective operator of the form [159]

$$\frac{1}{M_{Pl}} \mathrm{Tr}\,(F_{\mu\nu} F^{\mu\nu} H_G)\,. \tag{8.9}$$

Here $F_{\mu\nu}$ is the unified gauge field strength tensor and H_G is a scalar field acquiring the VEV at the GUT scale. In our case $H_G = 45_H$ and (8.9) vanishes thanks to the antisymmetric nature of the 45-dimensional representation of $SO(10)$ (7.22). Otherwise, this type of operator induces the uncertainty in the matching of the gauge couplings with the size comparable or even greater than the error coming from neglecting the two-loop contributions to the beta functions making the efforts of going beyond the one-loop precision level pointless [158].

The two-loop analysis of the light octet case was performed in [160] and the one-loop results were shifted in such a way that the points in the parameter space which would survive also the Hyper-Kamiokande bound on the proton lifetime (6.24) suggest the mass of the $(8, 2, +\frac{1}{2})$ scalar to be at or below ~ 20 TeV, i.e., potentially within the reach of the LHC (see Figure 1 in Appendix E). The light sextet case was subsequently analysed in [4] as described in the following section.

8.2.2 Two-loop analysis of the minimal $SO(10)$ with a light sextet scalar

It was shown already in [155] that in the case of the light $(6, 3, +\frac{1}{3})$ scalar, the setting with $\omega_{BL} \gg |\omega_R|$ is preferred. Consequently, according to (8.2), the first symmetry breaking step is

$$SO(10) \xrightarrow{\mu_2} SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \equiv G'$$

with the matching scale conveniently defined as

$$\mu_2 = g\,\omega_{BL}. \tag{8.10}$$

[3]In particular, the errors in the proton lifetime estimates within approximately one order of magnitude are desirable since only then the next generation of experiments improving the limits (6.22) to (6.24) can distinguish among different models (see [158] for further details).

where g is a sample value of the unified coupling at the GUT scale. Let us note that small shifts in the matching scale have a minor effect, since the so-called threshold corrections will be taken into account as described below. The full symmetry breaking chain depends on the hierarchy of the VEVs $|\sigma|$ and $|\omega_R|$, namely

$$|\sigma| > |\omega_R| : \qquad G' \xrightarrow{\mu_1} \text{SM} \tag{8.11}$$

$$|\sigma| < |\omega_R| : \qquad G' \xrightarrow{\mu_1'} SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \xrightarrow{\mu_1} \text{SM} \tag{8.12}$$

where the matching scales are chosen as

$$\mu_1 = g|\sigma|, \qquad \mu_1' = g|\omega_R|. \tag{8.13}$$

Let us now summarize the necessary ingredients for the full two-loop analysis of the gauge coupling running and list the needed theoretical tools.

As anticipated, two-loop renormalization group equations (i.e., two-loop beta functions) have to be used for the gauge coupling running. This is rather straightforward in the case without multiple abelian group factors, since the relevant formulas for the two-loop beta functions are readily available [161]. If more than one $U(1)$ factor is contained in the gauge group, like in (8.12), the $U(1)$ mixing has to be taken into account [162–164] and instead of just two gauge couplings g_R, g_{BL}, the 2×2 matrix of couplings has to be considered. Since this topic is not so widely described in the literature, we include our notes on the gauge coupling running in the theories with multiple $U(1)$ gauge group factors in Appendix A.4. The details of the beta function computation for our model are given in Section IV.A.2 of Appendix E.

Furthermore, the two-loop running has to be accompanied with the one-loop matching [165, 166] at each symmetry breaking scale which takes into account the splitting of the heavy particle spectrum. For this reason the full scalar spectrum has to be computed (the formulas for the scalar masses in the minimal $SO(10)$ model are given in [157]) and again the case with multiple abelian group factors requires special attention as described in Appendix A.4. The matching conditions for the model under consideration are explicitly given in Section IV.A.3 of Appendix E.

Finally, the one-loop evolution of the effective four-fermion baryon-number-violating operators (6.8)-(6.10) has to be included [167,168] and the resulting coefficients entering the proton decay rates are computed in Section IV.B.1 of Appendix E.

All the above ingredients were incorporated by the author of this thesis into a numerical analysis where the parameter space of the minimal $SO(10)$ model with light scalar sextet was scanned and only the points with non-tachyonic spectrum and exact unification of the gauge couplings were accepted. Furthermore, long-enough proton lifetime (i.e., high-enough $SO(10)$ breaking scale ω_{BL}) and high-enough seesaw scale were required, getting again a picture shifted with respect to the result of the one-loop analysis in [157]. Namely, only few points survive the requirement $|\sigma| > 10^{12}$ GeV together with Hyper-Kamiokande bound on proton lifetime (6.24); if, moreover, $|\sigma| > 10^{13}$ GeV is enforced, *all* the surviving points can be excluded by the Hyper-Kamiokande experiment as depicted in Figure 5 of Appendix E.

Consequently, only the light-octet case may remain viable if the Hyper-Kamiokande does not observe the proton decay.

Chapter 9

Conclusions and outlook

In this thesis, different scenarios beyond the SM were studied, their compatibility with the current experimental data was tested and the possible ways in which some of these schemes can be distinguished by future experiments were suggested. In particular, we considered two unified models based on the $SO(10)$ and $SU(5) \times U(1)$ gauge groups (see Chapter 8), and also models linking the smallness of neutrino masses with the axion particle which provides a dark matter candidate together with a solution to the strong CP problem (see Chapter 5).

The latter class of models was analysed in [1] and [2] attached to this thesis as Appendices B and C. The scalar extensions of the SM in which the peculiar axion-neutrino interconnection was attainable in a simple manner were shown to feature several testable predictions. Namely, besides other features characteristic for the DFSZ model, in the current setup the axion entertains tiny couplings to the neutrinos; moreover, the extra scalars can be observed at colliders or mediate specific lepton-flavour-violating processes. In addition, a light pseudo-Goldstone boson called dilaton is present in the scenario considered in [2]. At the same time, all the proposed models comply with all the experimental constraints including, for instance, the Higgs boson properties.

On the other hand, the predictions of the unified models are usually related to the process of the proton decay. To this end, the main result of the article [3] attached to this thesis as Appendix D involved the correlations among the different partial proton decay rates. Such firm predictions were allowed by the constrained structure of the model where the right-handed neutrino masses were generated via a two-loop diagram within the context of the so-called flipped $SU(5)$ unification.

Furthermore, we resumed the previous studies of a model based on the $SO(10)$ gauge group broken by the adjoint representation. This minimal setting was only recently proved to be viable since the one-loop effective potential had to be considered in order to show that the scalar spectrum can be non-tachyonic along the potentially viable symmetry breaking chains. Subsequently, the study of the unification constraints determining the seesaw scale revealed that realistic masses of the active neutrinos can be attained in minimally fine-tuned schemes with either a color octet or a color sextet scalar accidentally light. A quantitative statement, however, requires a detailed two-loop analysis of the gauge-coupling running which was performed for the sextet case in [4] attached to this thesis as Appendix E. It was revealed that if the Hyper-Kamiokande experiment does not observe the proton decay, the parameter space will shrink in such a way that no points with the seesaw scale above 10^{13} GeV will be left. The model featuring a light color octet would be favoured in that case and it was shown in previous articles that this scalar could be within the reach of the LHC.

Let us note that the predictions of the proton decay rates made in the two articles on the unified models above could help to distinguish among different unified scenarios if the proton decay was observed by experiments like the Hyper-Kamiokande. Moreover, we would like to stress that the recent analyses [158] and [169] showed that the flipped $SU(5)$ unification and the models based on the $SO(10)$ gauge group broken by the adjoint representation are especially stable with respect to the effects of the Planck-suppressed operators which means that our predictions should not be significantly affected by the unknown physics at the Planck scale.

On the other hand, there are still several open questions concerning the models studied in this thesis. One of our aims is to incorporate also a solution to the problem of the baryon-antibaryon asymmetry of the Universe into the models with the neutrino-axion interconnection. At the same time, these models could be even more predictive if embedded into the framework of a theory with an extended gauge symmetry.

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Appendices

Appendix A

Miscellaneous technical details

A.1 Dirac, Majorana and Weyl spinors

The usual 4-component spinors will be mostly used in this text, however, in some cases the Weyl notation becomes useful; its basic features will be described here following mainly the reference [30].

Although the chirality operator γ_5 can be used as a dynamical variable only in the case of massless fields when it is proportional to the helicity operator [170], one can anyway define the chirality projector operators

$$P_L = \frac{1-\gamma_5}{2}, \qquad P_R = \frac{1+\gamma_5}{2}$$

and then act on a 4-component spinor ψ to get

$$\psi_L \equiv P_L\psi, \qquad \psi_R \equiv P_R\psi.$$

This decomposition is invariant with respect to any proper Lorenz transformation [170] and when the Weyl representation of the gamma matrices is used, then ψ_L and ψ_R can be represented by 2-component spinors.

Moreover, one can use the particle-antiparticle conjugation operator $\hat{C}$ in order to switch between left-handed and right-handed fermions. Indeed, if ψ is a solution of Dirac equation with positive energy, then

$$\hat{C}(\psi) \equiv \psi^c = C\overline{\psi}^T \tag{A.1}$$

is a solution with negative energy where C is a matrix satisfying

$$C^\dagger = C^T = C^{-1} = -C, \qquad C\gamma_\mu C^{-1} = -\gamma_\mu^T, \tag{A.2}$$

$C = i\gamma_2\gamma_0$ in the standard representation of gamma matrices. Using these relations, one easily finds

$$\hat{C}(P_L\psi) \equiv (\psi_L)^c = (\psi^c)_R \equiv P_R\hat{C}(\psi), \qquad \hat{C}(P_R\psi) \equiv (\psi_R)^c = (\psi^c)_L \equiv P_L\hat{C}(\psi),$$

which also means that instead of ψ_L and ψ_R one can use ψ_L and $(\psi^c)_L$ as the two independent components of ψ (each carrying two degrees of freedom).

Since the mass term in the Dirac Lagrangian reads

$$- \mathcal{L}_{mass} \ni m\overline{\psi}\psi = m\left(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L\right) \tag{A.3}$$

both the components ψ_L and ψ_R (or equivalently ψ_L and $(\psi^c)_L$) are needed to build a massive fermion. If these components are truly independent, then $\psi = \psi_L + \psi_R$ is called a Dirac spinor. On the other hand, for the so-called Majorana spinor

$$\psi = \psi_L + \eta(\psi_L)^c$$

where $\eta = e^{i\phi}$ is an arbitrary phase factor; hence, $\psi_R = \eta(\psi_L)^c$ and $\psi^c = \eta^*\psi$. The Majorana fermions are then equivalent to their antiparticles which also implies that they must be neutral fields.[1] The Majorana mass term for n fermion species (flavors) reads

$$- \mathcal{L}_{mass} \ni \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}\psi_{iL}^T C\psi_{jL} + h.c. \tag{A.4}$$

Let us note that since ψ_i and ψ_j are anti-commuting fermion operators, the relation (A.2) implies that the matrix M is symmetric.

If one needs to rewrite a Lagrangian in terms of ψ_L and $(\psi^c)_L$, the following relations can be also derived from (A.1) and (A.2):

$$\overline{\psi_R}\psi_L = \left((\psi^c)_L\right)^T C\psi_L \tag{A.5}$$

$$\overline{\psi_R}\gamma^\mu\psi_R = -\overline{(\psi^c)_L}\gamma^\mu(\psi^c)_L. \tag{A.6}$$

Let us note that the relation (A.6) implies that the terms $i\overline{\psi_R}\gamma^\mu D_\mu\psi_R$ and $i\overline{(\psi^c)_L}\gamma^\mu D_\mu(\psi^c)_L$ in Lagrangian are equivalent when the integration by parts is taken into account, provided that the two fields $(\psi^c)_L$ and ψ_R reside in complex conjugate representations whose generators are related by $T_a \to -T_a^* = -T_a^T$.

A.2 Matrix Diagonalization

To fix the notations, let us recall here the well-known result widely used in particle physics.

Proposition 2. *Let Y be a complex matrix. Then there exist unitary matrices U and U_C such that*

$$U_C^T Y U = Y_{\text{diag}} \tag{A.7}$$

[1] We will see in Section 4.2 that this does not mean that the field must occupy a singlet representation of a non-abelian gauge group, one can build a Majorana mass term for any real representation as well.

where Y_{diag} is a diagonal matrix with real non-negative entries. If these entries are non-zero and non-degenerate, then the matrices U and U_C are defined uniquely up to a simultaneous transformation

$$U \to UP, \qquad U_C \to U_C P^* \tag{A.8}$$

with P being a diagonal unitary matrix.

Proof. Since $Y^\dagger Y$ is a Hermitian matrix, it may be diagonalized by a unitary matrix U as $U^\dagger Y^\dagger Y U = Y_{\text{diag}}^2$ and the entries of Y_{diag}^2 are real non-negative. The columns of U are formed by the orthonormal eigenvectors of $Y^\dagger Y$ which are in case of non-degenerate eigenvalues defined up to a phase factor, hence the first relation in (A.8) is proved.

Let us define $Y_{\text{diag}} = \sqrt{Y_{\text{diag}}^2}$. If $(Y_{\text{diag}})_{ii} \neq 0 \ \forall i$, we put

$$U_C^* = Y U Y_{\text{diag}}^{-1}. \tag{A.9}$$

Since $U_C^\dagger U_C = Y_{\text{diag}}^{-1} U^T Y^T Y^* U^* Y_{\text{diag}}^{-1} = \mathbb{1}$, U_C is unitary and

$$U_C^T Y U = Y_{\text{diag}}^{-1} U^\dagger Y^\dagger Y U = Y_{\text{diag}}.$$

If one plugs $U \to UP$ into (A.9), the second relation in (A.8) is obtained realizing that the diagonal matrices Y_{diag}^{-1} and P commute.

In case $\left(Y_{\text{diag}}^2\right)_{ii} = 0$ for some i, we observe that (A.7) implies

$$U_C^T Y Y^\dagger U_C^* = Y_{\text{diag}}^2,$$

hence columns of U_C^* are constructed by the eigenvectors of $YY^\dagger$. However, in this way U_C is determined up to the transformations $U_C^* \to U_C^* \tilde{P}$ with $\tilde{P}$ being a diagonal unitary matrix in general independent of P, hence $\tilde{P}$ must be fixed by the requirements $(U_C^T Y U)_{ii} \in \mathbb{R}$ and $(U_C^T Y U)_{ii} \geq 0$.[2]

□

Corollary 3. *Let S be a symmetric complex matrix. Then there exists a unitary matrix V such that*

$$V^T S V = S_{\text{diag}} \tag{A.10}$$

where S_{diag} is a diagonal matrix with real non-negative entries. If these entries are non-zero and non-degenerate, then V is defined uniquely up to redefinition $V \to V\tilde{P}$ where $\tilde{P}$ is a diagonal matrix with entries equal to ± 1.

[2] Of course, this (more complicated) construction of U_C could be used also in the case of $(Y_{\text{diag}})_{ii} \neq 0$ $\forall i$. However, in case of degenerate non-zero eigenvalues, the ambiguity in U_C amounts to multiplication by a unitary matrix (instead of a mere phase) and $U_C^T Y U$ need not be diagonal any more.

Proof. According to Proposition 2, we can find U and U_C so that

$$S_{\text{diag}} = U_C^T S U = S_{\text{diag}}^T = U^T S^T U_C = U^T S U_C.$$

Consequently, $U^\dagger S^\dagger S U = U_C^\dagger S^\dagger S U_C = S_{\text{diag}}^2$ i.e. both U and U_C are composed of the eigenvectors of $S^\dagger S$ and, hence, they may differ only by the multiplication by a unitary diagonal matrix from the right: $U = U_C P$. If we choose $V \equiv U P^{-1/2} = U_C P^{1/2}$, then

$$V^T S V = P^{1/2} U_C^T S U P^{-1/2} = P^{1/2} S_{\text{diag}} P^{-1/2} = S_{\text{diag}}.$$

It is also clear that the redefinition $V \to V\tilde{P}$ with $\tilde{P}$ being general diagonal unitary matrix would not preserve the reality of the entries of S_{diag}, $\tilde{P}^2 = \mathbb{1}$ has to be satisfied, which proves the second part of the statement (and corresponds to the ambiguity in the definition of $P^{1/2}$ above).

□

A.3 Friedmann-Lemaître-Robertson-Walker metric

In order to elucidate the notation used in the cosmological considerations, the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) metric describing homogeneous and isotropic universe will be introduced here. The definition from [53] will be followed, however, with the opposite sign convention for the flat metric (1.4).

There are three different options for the 3-dimensional homogeneous isotropic space: the three-dimensional flat (Euclidean) space, a spherical surface in a four-dimensional Euclidean space, and a hyperspherical surface in four-dimensional pseudo-Euclidean space. For these three cases the FLRW metric can be then defined as

$$g_{ij} = -a(t)^2 \left(\delta_{ij} + K \frac{x^i x^j}{1 - K\boldsymbol{x}^2} \right), \qquad g_{i0} = 0, \qquad g_{00} = 1 \tag{A.11}$$

where

$$K = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperspherical} \\ 0 & \text{Euclidean} \end{cases}$$

and the boldface corresponds to the 3-vector nature of the symbol.

It is convenient to introduce the expansion rate

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \tag{A.12}$$

its value observed today at the Earth is usually called the Hubble constant:

$$H_0 \equiv H(t_0), \qquad h \equiv \frac{H_0}{100\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}}. \tag{A.13}$$

The value of H_0 enters many astronomical measurements, however, was not determined with a good precision[3], hence, many results of the cosmological observations are given as the multiples of (the powers of) h.

Furthermore, the Einstein field equations for the metric (A.11) and the proper energy density and pressure $\rho(t)$ and $p(t)$ imply

$$\dot{a}^2 + K = \frac{8\pi G\rho a^2}{3}, \tag{A.14}$$

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p) \tag{A.15}$$

where the second equation expresses just the energy conservation law whereas the first equation is the fundamental Friedmann equation governing the expansion of the Universe. The different options for the faith of the universe are analysed e.g. in Section 1.5 of [53], here we restrict ourselves to defining the critical density

$$\rho_{0,\mathrm{crit}} \equiv \frac{3H_0^2}{8\pi G},$$

and stating that according to the equation (A.14) the constant K equals +1, 0 or -1 if the present density ρ_0 is greater than, equal to or less than $\rho_{0,\mathrm{crit}}$. The different contributions to the present energy density are the vacuum energy, and the non-relativistic and relativistic matter which corresponds to the fractions

$$\Omega_\Lambda \equiv \frac{\rho_{V0}}{\rho_{0,\mathrm{crit}}}, \qquad \Omega_M \equiv \frac{\rho_{M0}}{\rho_{0,\mathrm{crit}}}, \qquad \Omega_R \equiv \frac{\rho_{R0}}{\rho_{0,\mathrm{crit}}}, \tag{A.16}$$

respectively. Moreover, if

$$\Omega_K \equiv -\frac{K}{a_0^2 H_0^2}$$

is defined, then

$$\Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K = 1.$$

A.4 Theories with multiple U(1) gauge group factors

Since the problem of the two-loop gauge coupling running in the theories with multiple $U(1)$ gauge group factors is not so often discussed in the literature, we include here the notes based on [162,164]. The formulas from these articles were applied in [4] attached to this thesis as Appendix E, although sometimes a more convenient form of these relations was used as shown in the following.

[3] Let us note that the evolution of $a(t)$ with time in the FLRW Universe can be determined by observing the light coming to us from distant stars. It can be shown that if the light with frequency ν_1 was emitted at time t_1, then the observed frequency ν_0 at time t_0 on Earth satisfies $\frac{\nu_0}{\nu_1} \equiv 1 + z = \frac{a(t_1)}{a(t_0)}$. Since the decrease in frequencies, i.e. the redshift, is observed, $a(t)$ is an increasing function of time suggesting that our universe is expanding. The fractional increase in wavelengths can be then expanded using the Hubble constant as $z = H_0(t_0 - t_1) + \ldots$.

A.4.1 Two-loop running

The two-loop renormalization group equations given in [162] will be rewritten here in a more suitable way which also suggests the form of the threshold corrections discussed in the next section.

Let us consider a theory with N abelian gauge bosons A_a^μ, n fermions ψ^k and m scalars ϕ^l and let us choose such a basis that the gauge interactions are diagonal in the space of the fermion and scalar species. The charge of the k-th fermion (or scalar) with respect to the a-th $U(1)$ factor can be then denoted by Q_k^a and

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^b F^{b\mu\nu} + i\bar{\psi}^k\gamma_\mu(\partial^\mu - ig_{ab}Q_k^a A_b^\mu)\psi^k + (\partial_\mu - ig_{ab}Q_l^a A_{b\mu})\phi^l(\partial^\mu - ig_{ab}Q_l^a A_b^\mu)\phi^{l*}.$$

Here the basis for the gauge fields was chosen so that the kinetic term is diagonal in the space of the individual $U(1)$ factors, hence the gauge couplings become $N \times N$ matrices. With this choice of basis there is still freedom in transformations

$$A_a^\mu \to O_{ab}A_b^\mu, \qquad g_{ab} \to g_{ac}(O^T)_{cb} \tag{A.17}$$

where O is an orthogonal matrix; thus, the N^2 entries of g_{ab} are unphysical and the matrix g may be always assumed to be triangular.

According to [162] the evolution equation for the matrix of the Abelian couplings g reads

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}t} = g_{ac}\beta_{cb}. \tag{A.18}$$

It is convenient to introduce the "reduced" coupling $G_{rb} := Q_r^a g_{ab}$, i.e. the coupling of the r-th fermion to the gauge boson A_b^μ (and similarly for scalars). Then [162]

$$\begin{aligned}
\beta_{ab} &= \beta_{ab}^{(1)} + \beta_{ab}^{(2)} \\
\beta_{ab}^{(1)} &= \frac{1}{8\pi}\left\{\frac{4}{3}\kappa_f G_{fa}G_{fb} + \frac{1}{3}\eta_s G_{sa}G_{sb}\right\} \\
\beta_{ab}^{(2)} &= \frac{1}{8\pi}\left\{-\frac{2\kappa_f}{(4\pi)^2}\mathrm{Tr}[G_{fa}G_{fb}YY^\dagger] + \frac{4}{(4\pi)^2}\left[\kappa_f\left(G_{fa}G_{fb}G_{fc}^2 + G_{fa}G_{fb}g_q^2C_2(F_q)\right)\right.\right. \\
&\quad \left.\left. +\eta_s\left(G_{sa}G_{sb}G_{sc}^2 + G_{sa}G_{sb}g_q^2C_2(S_q)\right)\right]\right\}
\end{aligned} \tag{A.19}$$

where $\kappa_f = 1, \frac{1}{2}$ for Dirac and Weyl fermions, respectively, and, similarly, $\eta_s = 1, \frac{1}{2}$ for complex and real scalars. It is summed over all the repeated indices, including, e.g., q that numbers the non-abelian couplings or c. Let us note that the terms in $\beta^{(1)}$ are the one-loop contributions whereas $\beta^{(2)}$ corresponds to two-loop corrections. It is also convenient to denote

$$\beta_{ab} = \frac{1}{8\pi}g_{ca}\gamma_{cd}g_{db}.$$

If we realize that

$$G_{fa}G_{fb} \equiv \sum_f G_{fa}G_{fb} = \sum_{f,c,d} Q_f^c g_{ca} Q_f^d g_{db} = g_{ca}(\Delta\gamma)_{cd}g_{db}$$

it is trivial to compute the coefficients γ_{cd}. In particular,

$$\gamma_{cd}^{(1)} = \frac{4}{3}\sum_f \kappa_f Q_f^c Q_f^d + \frac{1}{3}\sum_s \eta_s Q_s^c Q_s^d$$

and similarly for $\gamma_{cd}^{(2)}$. Here the term $G_{fa}G_{fb}G_{fe}^2$ might seem confusing, let us evaluate it as an example in the case of $N=2$:

$$\begin{aligned} G_{fa}G_{fb}G_{fe}^2 &\equiv \sum_f G_{fa}G_{fb}\sum_{e=1}^{2} G_{fe}^2 = \sum_f \sum_{c,d=1}^{2} Q_f^c g_{ca} Q_f^d g_{db} \sum_{e=1}^{2}(\sum_{h=1}^{2} g_{he}Q_f^h)^2 \\ &= \sum_{c,d=1}^{2} g_{ca}g_{db}\sum_f Q_f^c Q_f^d \left[(Q_f^1)^2(g_{11}^2+g_{12}^2) + (Q_f^2)^2(g_{21}^2+g_{22}^2) + 2Q_f^1 Q_f^2(g_{11}g_{21}+g_{12}g_{22})\right] \end{aligned}$$

where the last sum over fermions is exactly the contribution to γ_{cd}. If the compact notation with $\xi_r = 1$ for Dirac fermions and complex scalars and $\xi_r = \frac{1}{2}$ for Weyl fermions and real scalars is introduced, then in case of $N=2$

$$\begin{aligned} \gamma_{cd}^{(2)} = \frac{1}{(4\pi)^2}\Bigg\{ &-\sum_f 2\kappa_f Q_f^c Q_f^d \mathrm{Tr}[YY^\dagger] + \\ \sum_r 4\xi_r Q_r^c Q_r^d &\left[(Q_r^1)^2(g_{11}^2+g_{12}^2) + (Q_r^2)^2(g_{21}^2+g_{22}^2) + 2Q_r^1 Q_r^2(g_{11}g_{21}+g_{12}g_{22}) + \sum_q g_q^2 C_2(R_q)\right]\Bigg\} \end{aligned}$$

where the r runs over both scalars and fermions of the theory and q runs over the non-abelian gauge group factors.

It is important to note that due to the invariance (A.17), the same amount of information is contained in the combination $g_{ac}g_{bc} \equiv (gg^T)_{ab}$ as in the matrix g_{ab} on its own (and of course also the β-function is invariant under the transformation (A.17)). The evolution equation for this combination is

$$\begin{aligned} \frac{\mathrm{d}g_{ac}g_{bc}}{\mathrm{d}t} &= g_{ad}\beta_{dc}g_{bc} + g_{ac}g_{bf}\beta_{fc} = \frac{1}{8\pi}\left(g_{ad}g_{bc}g_{rd}g_{sc}\gamma_{rs} + g_{ac}g_{bf}g_{rf}g_{sc}\gamma_{rs}\right) \\ &= \frac{1}{4\pi}(gg^T)_{ar}\gamma_{rs}(gg^T)_{sb} \end{aligned} \tag{A.20}$$

where in the last equality we used that both gg^T and γ are symmetric matrices. For any invertible matrix A one obtains

$$\frac{\mathrm{d}A^{-1}}{\mathrm{d}t} = -A^{-1}\frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}$$

(which can be derived by rewriting $A^{-1} = A^{-1}AA^{-1}$ and applying the Leibnitz rule) hence

$$\frac{\mathrm{d}(gg^T)^{-1}}{\mathrm{d}t} = -\frac{1}{4\pi}\gamma. \tag{A.21}$$

We conclude that for solving equation (A.20) it is convenient to rewrite the coefficients γ in terms of the elements of the matrix gg^T instead of g

$$\gamma^{(2)}_{cd} = \frac{1}{(4\pi)^2}\left\{ -\sum_f 2\kappa_f Q^c_f Q^d_f \mathrm{Tr}[YY^\dagger] + \sum_r \left(4\xi_r Q^c_r Q^d_r \sum_{hj} Q^h_r (gg^T)_{hj} Q^j_r + \sum_q g^2_q C_2(R_q) \right) \right\}$$

and this approach was used for other beta functions in [164]. Moreover, as already anticipated, the equation (A.21) explains the form of the threshold corrections below.

Let us note that the equation (A.21) corresponds to equation (29) in Appendix E with the coefficients $a = \gamma^{(1)}$ and $b_i\alpha_i + c = 4\pi\gamma^{(2)}$.

A.4.2 Threshold corrections

Let us assume that the $U(1)^N$ symmetry considered above is broken to $U(1)^{\tilde{N}}$ symmetry ($\tilde{N} < N$) at a certain scale μ. Let the generators ($U(1)$ charges) of the two theories satisfy

$$\tilde{Q} = PQ \tag{A.22}$$

where P is an $\tilde{N} \times N$ matrix obeying $PP^T = \mathbb{1}$. Rewriting the relations (33) and (34) from [164] in the current notation one obtains

$$(\tilde{g}\tilde{g}^T)^{-1}(\mu) = P\left[(gg^T)^{-1}(\mu) - \Lambda(\mu)\right]P^T \tag{A.23}$$

where

$$\Lambda_{ab}(\mu) = \frac{1}{8\pi^2}\left(\frac{4}{3}\kappa_F Q^a_F Q^b_F \log\frac{M_F}{\mu} + \frac{1}{3}\eta_S Q^a_S Q^b_S \log\frac{M_S}{\mu}\right). \tag{A.24}$$

In the last equation it is summed over all the fermions F and scalars S that are integrated out at the scale μ. Let us note that thanks to the relation (A.22) the formulas (A.23) and (A.24) may be rewritten as

$$\begin{aligned}(\tilde{g}\tilde{g}^T)^{-1}(\mu) &= P\left[(gg^T)^{-1}(\mu)\right]P^T - \tilde{\Lambda}(\mu),\\ \tilde{\Lambda}_{ab}(\mu) &= \frac{1}{8\pi^2}\left(\frac{4}{3}\kappa_F \tilde{Q}^a_F \tilde{Q}^b_F \log\frac{M_F}{\mu} + \frac{1}{3}\eta_S \tilde{Q}^a_S \tilde{Q}^b_S \log\frac{M_S}{\mu}\right).\end{aligned}$$

These equations were used in the Section IV.A.3 of Appendix E, generalized for the case when the higher symmetry group contains also non-abelian factors.

Appendix B

Full article: Massive neutrinos and invisible axion minimally connected

See [1] for the full article.

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Appendix C

Full article: Neutrino-axion-dilaton interconnection

See [2] for the full article.

Appendix D

Full article: Witten's mechanism in the flipped SU(5) unification

See [3] for the full article.

Appendix E

Full article: Proton lifetime in the minimal SO(10) GUT and its implications for the LHC

See [4] for the full article.

Appendix F

Co-authorship statement

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Trieste, February 11, 2017

I am writing this letter at the request of Helena Kolešová about the inclusion of the two papers we co-authored as a part of her Ph.D. thesis.

Helena visited the particle theory group at SISSA from September 2014 to March 2015, supported by a Higher Education Erasmus fellowship. While profiting of some of the Ph.D. courses offered during her stay, Helena contributed to the completion of a paper and to the kickoff of a second one, in collaboration with Michal Malinský, Luca Di Luzio and a student of mine.

The project aimed at the construction of a minimal renormalizable extension of the standard electroweak model that may address in a structural way some of the shortcomings of the standard theory, both observational (neutrino masses and mixings, dark matter, barion asymmetry) and theoretical (strong CP, naturalness and stability of the electroweak scale). In the first paper (Phys.Rev. D91 (2015) no.5, 055014) we succeeded in achieving a scheme that by extending minimally the standard Higgs sector provides a strict connection between neutrino masses and the presence of an axion as a dark matter candidate. In the second paper (Phys.Rev. D93 (2016) no.1, 015009) we studied the implementation of spontaneously broken classical scale invariance. We showed that the model becomes sharply constrained, the extended scalar spectrum being controlled by two parameters, while the presence of a very light pseudo-dilaton opens the possibility of a consistent inflationary setup. Requiring strict naturalness further implies additional scalar states to be discovered at the LHC energy scales.

I must say that the expectations about Helena's potential and her contribution to the research activity were largely fulfilled. Her field theoretical and mathematical methods backgrounds allowed her to catch up with the reference literature and to become an active member of the collaboration in a very short time. A substantial part of the material involved in the two papers has been independently worked out/recovered by Helena herself.

In particular, in the first paper, Helena worked out in detail the Peccei-Quinn extension of the Babu-Zee model as well as the scalar spectrum in the type-II see-saw setup. In addition she provided a nice sum up of the phenomenological implications and constraints related to the presence of the exotic scalar states. In the second paper, Helena studied the Coleman-Weinberg potential in various toy models and computed very reliably the one-loop beta functions of the quartic scalar couplings.

In conclusion, I appreciated Helena's determination, reliability and depth and, last but not least, her passion for the field. I definitely expect Helena to pursue a successful academic career in high energy physics.

Yours sincerely

Stefano Bertolini