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Geometry of Membrane Sigma Models

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Abstrakt

Strunová teorie stále zůstává nadějným kandidátem na sjednocení teorie gravitace a kvantové teorie pole. Její důležitou součástí je relativistický popis pohybu vícerozměrných objektů nazývaných membrány (případně p-brány) v zakřiveném prostoročase. Ten je na úrovni klasické teorie pole popsán funkcionálem akce extremalizujícím objem variety vytnuté šířící se membránou. Tuto a související polní teorie souhrnně nazýváme membránové sigma modely.

Diferenciální geometrie představuje důležitý matematický nástroj při studiu strunové teorie. Ukazuje se, že strunová a membránová pozadí lze výhodně popsat pomocí objektů definovaných na direktním součtu tečného a kotečného fibrovaného prostoru k prostoročasové varietě. Studiem tohoto objektu se zabývá tzv. zobecněná geometrie. Její nedílnou součástí je teorie Leibnizových algebroidů, vektorových fibrovaných prostorů, jejichž moduly hladkých řezů jsou vybaveny Leibnizovou algebrou. Speciálními případy Leibnizových algebroidů jsou známější Lieovy a Courantovy algebroidy.

Tato práce je rozdělena do dvou hlavních částí. V první rekapitulujeme základy teorie Leibnizových algebroidů, zobecněné geometrie, rozšířené zobecněné geomerie a Nambu-Poissonových struktur. Cílem je poskytnout čtenáři ucelený základ matematické teorie použité v publikovaných pracech, text je kombinací známých i nových výsledků. Důraz je kladen především na pojem zobecněné metriky a příslušných ortogonálních transformací, jenž posloužily jako základ našeho výzkumu. Druhou hlavní částí je příloha tvořená čtyřmi články užívajícími zobecněnou geometrii na vybrané partie teorie strun a membrán. Práce jsou otištěny ve stejné podobě, v jaké byly publikovány.

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Abstract

String theory still remains one of the promising candidates for a unification of the theory of gravity and quantum field theory. One of its essential parts is relativistic description of moving multi-dimensional objects called membranes (or p-branes) in a curved spacetime. On the classical field theory level, they are described by an action functional extremalising the volume of a manifold swept by a propagating membrane. This and related field theories are collectively called membrane sigma models.

Differential geometry is an important mathematical tool in the study of string theory. It turns out that string and membrane backgrounds can be conveniently described using objects defined on a direct sum of tangent and cotangent bundles of the spacetime manifold. Mathematical field studying such object is called generalized geometry. Its integral part is the theory of Leibniz algebroids, vector bundles with a Leibniz algebra bracket on its module of smooth sections. Special cases of Leibniz algebroids are better known Lie and Courant algebroids.

This thesis is divided into two main parts. In the first one, we review the foundations of the theory of Leibniz algebroids, generalized geometry, extended generalized geometry, and Nambu-Poisson structures. The main aim is to provide the reader with a consistent introduction to the mathematics used in the published papers. The text is a combination both of well known results and new ones. We emphasize the notion of a generalized metric and of corresponding orthogonal transformations, which laid the groundwork of our research. The second main part consists of four attached papers using generalized geometry to treat selected topics in string and membrane theory. The articles are presented in the same form as they were published.

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Contents

List of used symbols

$\mathbb{R}$ .... field of real numbers
M .... smooth finite-dimensional manifold
$C^\infty(M)$ .... module of smooth real-valued functions on M
$\mathfrak{X}^p(M)$ .... module of smooth p-vector fields on M
$\Omega^p(M)$ .... module of smooth differential p-forms on M
$\Omega^\bullet(M)$ .... whole exterior algebra of M
$\mathcal{T}_p^q(M)$ .... module of tensors of type (p,q)
$\mathcal{T}(M)$ .... whole tensor algebra of M
$\langle\cdot,\cdot\rangle$ .... canonical pairing of two objects
$\Gamma(E)$ .... module of smooth global sections of vector bundle E over M
$\Gamma_U(E)$ .... module of smooth local sections of E defined on $U \subseteq M$
$\mathfrak{X}^p(E)$ .... module of global smooth sections of $\Lambda^p E$, similarly for $\Omega^p(E)$, $\mathcal{T}_p^q(E)$
$A := B$ .... expression B is used to define the expression A
$\mathrm{BDiag}(g,h)$ .... block diagonal matrix with g and h on the diagonal
$\mathcal{L}_X$ .... Lie derivative along a vector field $X \in \mathfrak{X}(M)$
i_X .... insertion operator (or inner product) of a vector field $X \in \mathfrak{X}(M)$
$[\cdot,\cdot]_S$ .... Schouten-Nijenhuis bracket of multivector fields
$\mathrm{Hom}(E,E')$ ..... smooth vector bundle morphisms from E to E' over the identity on the base
$\mathrm{End}(E)$ .... shorthand notation for $\mathrm{Hom}(E,E)$
$\mathrm{Aut}(E)$ .... subset of $\mathrm{End}(E)$ consisting of fiber-wise bijective maps

Chapter 1

Introduction

After more then forty years of history, string theory still stands as one of the most promising attempts to unify the gravitational and quantum physics. Originating as a quantum theory of one-dimensional strings moving in the space-time, it evolved throughout the years to include fermionic particles (superstring theory) and extended objects such as D-branes, membranes or p-branes. It also grew into a challenging and sophisticated theory. An effort of a single person to review the string theory is as difficult as to ask a crab living on a Normandy beach to describe the Atlantic Ocean. As a welcome side-effect, string theory fueled the development in various, old and new, areas of mathematics. In particular, attempting to be a theory of gravity, it pushed forward many areas of differential geometry.

Quite recently, one such mathematical theory rose to prominence as a useful tool in string theory. It was pioneered in early 2000s, as it appeared in three subsequent papers [43–45] of Nigel Hitchin and in the Ph.D. thesis [39] of his student Marco Gualtieri. In a nutshell, generalized geometry is a detailed study of the geometry of the generalized tangent bundle $TM \oplus T^*M$. Such a vector bundle has a surprisingly rich structure, in particular it possesses a canonical indefinite metric and a bracket operation. This allows one to describe various geometrical objects in a new intrinsic way. Note that Hitchin already recognized the possible applications of generalized geometry in string theory and commented on this several times in the above cited papers.

It turned out that certain symmetries of string theory can be naturally explained in terms of generalized geometry. For example, string T-duality can be viewed as an orthogonal transformation, see [22, 38, 86], following the work of P. Bouwknegt et al. [14, 16, 17]. Recently there has been found a way to describe D-branes as Dirac structures, isotropic and involutive subbundles of E, see [4–6]. For a nice overview of further interesting applications in physics and string theory see [64]. There are also ways to proceed in the opposite direction, constructing field theories out of generalized geometry mathematical objects. Besides the very well-known Poisson sigma models [54, 83] , there exist Courant sigma models [55, 82] using the AKSZ mechanism to construct actions from a given Courant algebroid, Dirac sigma models [68, 69] and Nambu sigma models [18, 59, 84]. Suitable modifications of generalized geometry can be useful in string related physics. For applications in M-theory, see the work of Hull [52] and Berman et al. in [8–10]. A nice modification of generalized geometry was used to describe supergravity effective actions in [24, 25]. Finally, note that generalized geometry is closely related to recently very popular modification of field theory, called double field theory. See the works of C.M. Hull, B. Zwiebach and O. Hohm [47, 50, 51], and especially the recent review paper [48].

By membrane sigma models we mean various field actions emanating from the bosonic part of Polyakov-like action for membrane as introduced by Howe-Tucker in [49] and for string case independently in [28] and [19], and named after Polyakov who used it to quantize strings in [79]. We focus in particular on its gauge-fixed version, which can be related to its dual version, non-topological Nambu sigma model as defined in [59]. A need to find a suitable mathematics underlying these field actions leads to a necessity to extend the tools of standard generalized geometry to more general vector bundles. Driven by this desire we refer to results of our work and consequently also of this thesis as to *geometry of membrane sigma models.*

1.1 Organization

This thesis is divided in two major parts.

The first half is an attempt to bring up a consistent introduction to the mathematics used in the papers. The intention is to give definitions and derive important properties in detail as well as enough examples to illustrate sometimes quite abstract theory. The main idea is to provide a self-contained text with a minimal necessity to refer to external literature, which inevitably leads to a rephrasing of old and well known results. We will stress new contributions where needed.

The second half consists of four published papers which use these mathematical tools to describe and develop several aspects of membrane sigma model theory. These papers are attached in the exactly same form as they were published in the journals. All of them were published as joint work with both my supervisors.

1.2 Overview of the theoretical introduction

Let us briefly summarize the content of the following chapters. This section is intended to be main navigation guide for the reader.

Chapter 2 brings up a quick review of elementary properties of Leibniz algebroids. We could take the liberty to disregard the chronological order in which this theory appeared in mathematical world. Generalizations usually develop from more elementary objects, but it is sometimes simpler to provide the original structures as special examples of the more general ones.

This is why we start with a general definition of Leibniz algebroids in Section 2.1. The basic and the most important example is the (higher) Dorfman bracket. Further, an induced Lie derivative on the tensor algebra of Leibniz algebroid is introduced.

A known subclass of Leibniz algebroids are those with a skew-symmetric bracket, denoted as Lie algebroids. They are described in Section 2.2. Most importantly, a Lie algebroid induces an analogue of the exterior differential on the module of its differential forms, allowing for a Cartan calculus on the exterior algebra. In fact, this exterior differential contains the same data as the original Lie algebroid, and can be used as its equivalent definition. The exterior algebra of multivectors on the Lie algebroid bundle can be equipped with an analogue of Schouten-Nijenhuis bracket, turning it into a Gerstenhaber algebra. This observation is essential for the definition of Lie bialgebroid.

A need of understanding Lie bialgebroids underpins the subject of Section 2.3, Leibniz algebroids with an invariant fiber-wise metric called Courant algebroids. They first appeared in the form of their most important example, the Dorfman bracket with an appropriate anchor and pairing. Finding a suitable set of axioms for this new type of algebroid proved necessary to define a double of Lie bialgebroid. We present this as one of the examples of Courant algebroids.

The second chapter is concluded by Section 2.4 dealing with linear connections on Leibniz algebroids, and related notions of torsion and curvature. For Lie algebroids, this is pretty straightforward and essentially it can be defined just by mimicking the ordinary theory of linear connections. For Courant and general Leibniz algebroids, this is more involved task. We present a new way how to approach this using a concept of local Leibniz algebroids. For Courant algebroids, one recovers the known definitions of torsion operator. To the best of our knowledge a working way how to define a curvature operator is presented, as well as a Ricci tensor and scalar. We explain this on the example which uses the Dorfman bracket.

Chapter 3 contains an introduction to the geometry of the generalized tangent bundle, usually called a generalized geometry. This thesis covers only a little portion of this wide branch of mathematics suitable for our needs. These are the reasons why we have completely omitted the original backbone of generalized geometry - generalized complex structures.

In Section 3.1 we study an orthogonal group of the direct sum of a vector space and its dual equipped with a natural pairing. Its Lie algebra can conveniently be described in terms of tensors. This also allows one to construct important special examples of orthogonal maps. We present a very useful simple ways of block matrix decompositions. Although almost trivial at first glance, this observation allows one to prove some quite complicated relations later. Since the natural pairing does not have a definite quadratic form, it allows for isotropic subspaces. This is a subject of Section 3.2. Examples of maximally isotropic subspaces are presented.

Everything can be without any issues generalized from vector spaces to vector bundles, replacing linear maps with vector bundle morphism, et cetera. One can consider a more general group of transformations preserving the fiber-wise pairing on generalized tangent bundle. We call this group the extended orthogonal group. Its corresponding Lie algebra can be shown to be a semi-direct product of the ordinary (fiber-wise) orthogonal Lie algebra and the Lie algebra of vector fields. This is what Section 3.3 is about.

The generalized tangent bundle has a canonical Courant algebroid structure, defined by the Dorfman bracket. It is natural to examine the Lie algebra of its derivations and the group of its automorphisms. This was one of the main reasons why generalized geometry and the Dorfman bracket came to prominence in string theory. We study these structures in Section 3.4 and Section 3.5. One can also find an explicit formula integrating the Dorfman bracket derivation to obtain a Dorfman bracket automorphism. To our belief this was not worked out in such detail before. It happens that some orthogonal maps do not preserve the bracket, and can be used to "twist" it, to define different (yet isomorphic) Courant algebroids. This defines a class of twisted Dorfman brackets, introduced in Section 3.6.

Dirac structures are maximally isotropic subbundles of the generalized tangent bundle which are involutive under Dorfman bracket. They constitute a way how to describe presymplectic and Poisson manifolds in terms of generalized geometry. They are described in Section 3.7.

For the purposes of applications of generalized geometry in string theory, the most important concept is the one of a generalized metric. It can be defined in several ways, where some of them make sense in a more general setting then others. In Section 3.8 we discuss all these possibilities and show when they are equivalent. The orthogonal group acts naturally on the space of generalized metrics. We examine the consequences and properties of this action in Section 3.9. In particular, we show that Seiberg-Witten open-closed relations can be interpreted in this way.

Having a fiber-wise metric, we may study its algebra of Killing sections. This is done in Section 3.10. We give an explicit way to integrate these infinitesimal symmetries to actual generalized metric isometries. A generalized metric is by definition positive definite. We can modify some of its definitions to include also the indefinite case. This up brings certain issues discussed in Section 3.11. Finally, to any generalized metric we may naturally assign a Courant algebroid metric compatible connection, called a generalized Bismut connection. Its various forms are shown in Section 3.12, and its scalar curvature is calculated.

Chapter 4 is to be considered as the most important of this thesis. Its main idea is to extend the concepts of generalized geometry to a higher generalized tangent bundle suitable for applications in membrane theory. This effort poses interesting problems to deal with. First note that there is no natural fiber-wise metric present. Instead, one can use a pairing with values in differential forms. However, its orthogonal group structure becomes quite complicated and depends on the rank of involved vector bundles. We examine this in Section 4.1. In particular, a set of examples of Dirac structures with respect to this pairing is very limited.

On the other hand, the Dorfman bracket generalizes into a bracket of very similar properties. We investigate its algebra of derivations and its group of automorphisms in detail in Section 4.2.

A next important step is to define an extended version of the generalized metric. For these reasons, we start with a description of a way how to use a manifold metric to induce a fiberwise metric on higher wedge powers of the tangent and cotangent bundle. There are several consequences following from the construction. In particular, one can calculate its signature out of the signature of the original metric. Moreover, for Killing equations and metric compatible connections, it is important to examine the way how Lie and covariant derivatives of the induced metric can be calculated using Lie and covariant derivatives of the original manifold metric. Finally, we derive a very important formula proving the relation of their determinants. All of this can be found in Section 4.3.

We proceed to the actual definition of a generalized metric in Section 4.4. Similarly to the ordinary generalized metric, one can express it either in terms of a metric and a differential form, or in terms of dual fields. One can show that in this extended case, one of the dual fields is not automatically a multivector, and the two dual fiberwise metrics are not induced from each other. Finally, we generalize the open-closed relations as a particular transformation of the generalized metric.

As we have already noted, there is no useful orthogonal group to encode the open-closed relations as an example of an orthogonal transformation. However, there is a natural pairing on the "doubled" vector bundle, produced from the original one by adding (in the sense of a direct sum) its dual bundle. In this "doubled formalism", as we call it, one can define a generalized metric in the usual sense. The membrane open-closed relations can be now easily encoded as an orthogonal transformation of a relevant generalized metric. The doubled formalism is a subject of Section 4.5.

Having now a larger vector bundle with working orthogonal structure, we would like to find a suitable integrability conditions for its subbundles. One such Leibniz algebroid is examined in Section 4.6. We show how closed differential forms and Nambu-Poisson structures can be realized as Dirac structures of this Leibniz algebroid. We conclude by showing how its bracket can be twisted by a certain orthogonal map, obtaining a twist similar to the one of Dorfman bracket.

One of the reasons to use the induced metric in the definition of generalized metric is the fact that Killing equations are easy to solve in this case, as we show in Section 4.7. It also allows us to find a simple example of a generalized metric compatible connection. We show how to interpret this connection in the doubled formalism and we calculate its scalar curvature in Section 4.8.

The final Chapter 5 of this thesis is devoted to a natural generalization of Poisson manifolds called Nambu-Poisson structures. In Section 5.1 we present and prove various equivalent ways how to express the fundamental identity for a Nambu-Poisson tensor. Interestingly, it can be shown that an algebraic part of the fundamental identity not only forces its decomposability, but also its complete skew-symmetricity of this tensor. Similarly to the ordinary Poisson case, Nambu-Poisson structures can be realized as certain involutive subbundles with respect to the Dorfman bracket. This interpretation allows one to easily define a twisted Nambu-Poisson structure. This and one quite interesting related observation are contained in Section 5.2.

Finally, in Section 5.3, we examine in detail the construction of the Nambu-Poisson structure induced diffeomorphism called a Seiberg-Witten map. It involves the flows of time-dependent vector fields which are for the sake of clarity recalled there.

1.3 Guide to attached papers

The second part of this thesis consists of four attached papers. All of them are available for download at arXiv.org archive, three of them can be openly accessed directly through the respective journals. Papers are presented here in their original form, exactly as they were published. We sometimes refer to equations in the theoretical introduction of this thesis to link the introduced mathematical theory with its applications in the papers.

The first attached paper is **p-Brane Actions and Higher Roytenberg Brackets** [61].

A main subject of this paper is a study of Nambu sigma model proposed in [59, 84]. We modify its action slightly to include a twist with a B-field. Using relations (4.49 - 4.51), its equivalence to a p-brane sigma model action is shown. We introduce a slightly modified higher analogue of Courant algebroid bracket (2.36) which we call in accordance with [42] a higher Roytenberg bracket. We use the results of [32] to show that the Poisson algebra of generalized charges of the Nambu sigma model closes and it can be described by the higher Roytenberg bracket. This generalizes the results [2, 42]. Next, we turned our attention to the topological Nambu sigma model. It can be viewed as a system with constrains. We have proved that a consistency of the constrains with time evolution forces the tensor Π defining the sigma model to be a Nambu-Poisson tensor. See Chapter 5 of this thesis for details. Using the Darboux coordinates of Theorem 5.1.2 we were able to explicitly solve the equations of motion. We have concluded the paper by showing that coefficients of the generalized Wess-Zumino term produced out of topological Nambu sigma model are exactly the structure functions of the higher Roytenberg bracket.

On the Generalized Geometry Origin of Noncommutative Gauge Theory [60].

Idea of this paper was to use the tools of generalized geometry to explain and simplify certain steps derived originally in [58], [62] and [63]. A main observation was that different factorizations of the generalized metric correspond to Seiberg-Witten open closed relations (3.119). For the first time we interpret this as an orthogonal transformation (3.17) of a generalized metric, see Section 3.9 of this thesis. Moreover, also adding a fluctuation F to the B-field background can be interpreted as an orthogonal transformation (3.16). These two transformation do not commute (this is in fact a direct consequence of Lemma 3.1.2). This immediately leads to the correct definition of non-commutative versions of the respective fields. We were able to use this formalism to quickly re-derive the identities essential for the equivalence of classical DBI action and its semi-classically noncommutative counterpart.

The third of the appended papers is **Extended generalized geometry and a DBI-type effective action for branes ending on branes** [56].

In this article, our intentions were to generalize the approach taken in the previous paper [60] to simplify the calculations in [59] leading to the proposal of a p-brane generalization of DBI effective action. To reach this goal, we were able to explain the p-brane open-closed relations (4.56 - 4.59) in terms of the factorization of the generalized metric (4.47). Higher generalized tangent bundle does not possess a canonical orthogonal group structure (it does for $p = 1$). This was the reason why we have approached this through the addition of its dual. See Section 4.5 of this thesis for details. Using the extended generalized geometry, we were able to show the equivalence of respective DBI actions very quickly.

Moreover, using the doubled formalism, we could define an analogue of a so called background independent gauge. Reason for this name comes from the famous paper [85], and it is related to the actual background independence of the corresponding non-commutative Yang-Mills action. For $p = 1$, this corresponds to the choice $\theta = B^{-1}$ in (3.119). We generalize this idea to $p \geq 1$ including also the case of a degenerate 2-form B for $p = 1$. A choice of suitable Nambu-Poisson tensor Π singles out particular directions on the p'-brane, which we call noncommutative directions. This allows us to derive a generalization of a double scaling limit, see [85], introducing an infinitesimal parameter ϵ into DBI action. Calculating an expansion of DBI in the first order of ϵ yields a possible generalization of a matrix model.

This lead us to the writing of the short paper **Nambu-Poisson Gauge Theory** [57].

The letter follows the ideas for $p = 1$ published in [76]. Nambu-Poisson theory gauge theory was originally invented in [46] as en effective theory on a M5-brane for a large longitudinal C-field in M-theory. Our idea was to start from scratch without any reference to M-theory and branes. We define covariant coordinates to be functions of space-time coordinates transforming in a prescribed way under gauge transformations parametrized by a $(p-1)$-form, using a prescribed $(p+1)$-ary Nambu-Poisson bracket. Following [62,63], we use the covariant coordinates to define a Nambu-Poisson gauge field and a corresponding Nambu-Poisson field strength. The second part of this paper is devoted to an explicit construction of covariant coordinates using the Seiberg-Witten map described in this thesis in Section 5.3. We propose a simple Yang-Mills action for this gauge theory and relate it to the DBI action expansion obtained in the above described paper [56].

1.4 Conventions

The main aim of this section is to introduce a notation used throughout this entire work. We always work with a finite-dimensional smooth second-countable Hausforff manifold, which we usually denote as M and its dimension as n. For implications of this definition, I would recommend an excellent book [71]. In particular, one can use the existence of a partition of unity and its implications. We consider all objects to be real, in particular all vector spaces, vector bundles, bilinear forms, etc.

Now, let us clarify our index notations. We reserve the small Latin letters (i, j, k etc.) to label the components corresponding to a set of local coordinates $(y^1, \dots, y^n)$ on M, or sometimes to some local frame field on M. We reserve small Greek letters (α, β, γ, etc.) to label the components with respect to some local basis of the module of smooth sections of a vector bundle E. We use the capital Latin letters (I, J, K etc.) to denote the *strictly ordered* multi-indices, that is $I = (i_1, \dots, i_p)$ for some $p \in \mathbb{N}$, where $1 \leq i_1 < \dots < i_p \leq n$. Particular value of p should be clear from the context. We always hold to the Einstein summation convention, where repeated indices (one upper, one lower) are assumed to be summed over their respective ranges. For example, $v^I w_I$ stands for the sum $\sum_{1 \leq i_1 < \dots < i_p \leq n} v^{(i_1 \dots i_p)} w_{(i_1 \dots i_p)}$.

If $(y^1, \dots, y^n)$ are local coordinates on M, by ∂_i we denote the corresponding partial derivatives and coordinate vector fields. By dy^I and ∂_I we denote the wedge products of coordinate 1-forms and vector fields:

$$dy^I = dy^{i_1} \wedge \dots \wedge dy^{i_p}, \ \partial_I = \partial_{i_1} \wedge \dots \partial_{i_p}. \tag{1.1}$$

By definition, dy^I and ∂_I form a local basis of $\Omega^p(M)$ and $\mathfrak{X}^p(M)$ respectively.

We will often use a generalized Kronecker symbol. We define it as follows

$$\delta^{j_1 \dots j_p}_{i_1 \dots i_p} = \begin{cases} +1 \text{ both } p\text{-indices are strictly ordered and one is an even permutation of the other,} \\ -1 \text{ both } p\text{-indices are strictly ordered and one is an odd permutaion of the other,} \\ \ 0 \text{ in all other cases.} \end{cases}$$

It is defined so that $(dy^J)_I = \delta^J_I$. A Levi-Civita symbol $\epsilon_{i_1 \dots i_p}$ can be then defined as $\epsilon_{i_1 \dots i_p} = \delta^{1 \dots p}_{i_1 \dots i_p}$.

Let E, E' be two vector bundles over M. By $\mathrm{Hom}(E, E')$ we mean a module of smooth vector bundle morphisms from E to E' over the identity map Id_M. Under our assumptions on M, $\mathrm{Hom}(E, E')$ coincides with the module of $C^\infty(M)$-linear maps from $\Gamma(E)$ to $\Gamma(E')$, and we will thus never distinguish between the vector bundle morphisms and the induced maps of sections. We define $\mathrm{End}(E) := \mathrm{Hom}(E, E)$, and $\mathrm{Aut}(E) := \{\mathcal{F} \in \mathrm{End}(E), \ \mathcal{F} \text{ is fiber-wise bijective}\}$.

Now, let $p \geq 0$ be a fixed integer, and $C \in \Omega^{p+1}(M)$ be a differential $(p+1)$-form on M. This induces a vector bundle morphism $C_\flat : \mathfrak{X}^p(M) \to \Omega^1(M)$ defined as

$$C_\flat(Q^J \partial_J) := Q^J C_{iJ} dy^i, \tag{1.2}$$

for all $Q = Q^J \partial_J \in \mathfrak{X}^p(M)$, and $C_{iJ} = C(\partial_i, \partial_{j_1} \dots, \partial_{j_p})$. It is straightforward to check that $C_\flat$ is a well-defined $C^\infty(M)$-linear map of sections (all indices are properly contracted). At each point $m \in M$, $C_\flat$ thus defines a linear map from $\Lambda^p T_m M$ to $T^*_m M$, with the corresponding matrix $(C_\flat|_m)_{i,J} \equiv [C|_m]_{iJ}$. Collecting those matrices, we get a matrix of functions $(C|_\flat)_{i,J} = C_{iJ}$. Here comes our convention. In the whole thesis, we will denote objects C, $C_\flat$ and $(C_\flat)_{i,J}$ with the single letter C, and the particular interpretation will always be clear from the context. By C^T we mean the transpose map from $\mathfrak{X}^1(M)$ to $\Omega^p(M)$. Note that $C^T(X) = i_X C$, for all $X \in \mathfrak{X}(M)$.

Similarly, for $\Pi \in \mathfrak{X}^{p+1}(M)$, we define the vector bundle morphism $\Pi^\sharp : \Omega^p(M) \to \mathfrak{X}(M)$ as

$$\Pi^\sharp(\xi_J dy^J) = \xi_J \Pi^{iJ} \partial_i, \tag{1.3}$$

for all $\xi \in \Omega^p(M)$, and $\Pi^{iJ} = \Pi(dy^i, dy^{j_1}, \dots, dy^{j_p})$. We again do not distinguish Π, $\Pi^\sharp$ and the matrix $(\Pi^\sharp)^{i,J} = \Pi^{iJ}$. The transpose map Π^T then maps from $\Omega^1(M)$ to $\mathfrak{X}^p(M)$.

These conventions may seem to be quite unusual when compared to the standard generalized geometry papers. They are however well suited for matrix multiplications, and proved to be the best choice to get rid of unnecessary $(-1)^p$ factors in all formulas.

Chapter 2

Leibniz algebroids and their special cases

In this chapter, we will introduce a framework useful to describe various algebraical and geometrical aspects of the objects living on vector bundles. Fields arising in string and membrane sigma models theory can be viewed as vector bundle morphisms of various powers of tangent and cotangent bundles, which justifies the efforts to understand the canonical structures coming with those vector bundles. We will proceed in a rather unhistorical direction, starting from the most recent definitions, and arriving to the oldest.

2.1 Leibniz algebroids

The basic idea goes as follows. Let $E \overset{\pi}{\to} M$ be a vector bundle. By definition, E is a collection of isomorphic vector spaces E_m at each point $m \in M$. Let us say that every E_m can be equipped with a Leibniz algebra bracket $[\cdot,\cdot]_m$, that is an $\mathbb{R}$-bilinear map from $E_m \times E_m$ to E_m, satisfying the Leibniz identity

$$[v,[v',v'']_m]_m = [[v,v']_m,v'']_m + [v',[v,v'']_m]_m, \tag{2.1}$$

for all $v, v', v'' \in E_m$. This bracket needs not to be skew-symmetric. Leibniz algebras were first introduced by Jean-Luis Loday in [73]. Now, suppose that this bracket changes smoothly from point to point. More precisely, if $e, e' \in \Gamma(E)$ are smooth sections, then formula

$$[e,e'](m) := [e(m),e'(m)]_m, \tag{2.2}$$

defines a smooth section $[e,e'] \in \Gamma(E)$. We have just constructed a simplest example of Leibniz algebroid[1]. Note that the bracket satisfies $[e, fe'] = f[e,e']$ for all $e, e' \in \Gamma(E)$ and $f \in C^\infty(M)$, and also the Leibniz identity

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']]. \tag{2.3}$$

This special case is too simple in a sense that the bracket depends only on the point-wise values of the incoming sections. In fact, all examples which we will encounter in this thesis are not of this type. Let us now give a formal definition of a general Leibniz algebroid.

[1]Consider that E has its typical fiber equipped with a Leibniz algebra bracket. If E can be locally trivialized by Leibniz algebra isomorphisms, one calls this example a *Leibniz algebroid bundle*.

Definition 2.1.1. Let $E \xrightarrow{\pi} M$ be a vector bundle, and $\rho \in \mathrm{Hom}(E, TM)$. Let $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ be an $\mathbb{R}$-bilinear map. We say that $(E, \rho, [\cdot,\cdot]_E)$ is a **Leibniz algebroid**, if

- For all $e, e' \in \Gamma(E)$, and $f \in C^\infty(M)$, there holds the *Leibniz rule*:
$$[e, fe']_E = f[e, e']_E + (\rho(e).f)e'. \tag{2.4}$$
- The bracket $[\cdot,\cdot]_E$ defines a Leibniz algebra on $\Gamma(E)$, it satisfies the Leibniz identity
$$[e, [e', e'']_E]_E = [[e, e']_E, e'']_E + [e', [e, e'']_E]_E, \tag{2.5}$$
for all $e, e', e'' \in \Gamma(E)$.

The map ρ is called the *anchor* of Leibniz algebroid, since it allows the sections of E to act on the module of smooth functions on M.

Let us make a few remarks to this definition.

- In the math literature, Leibniz algebroids are often called Loday algebroids. There is an extensive work on this topic by Y. Kosmann-Schwarzbach in [66] and especially by J. Grabowski an his collaborators [36]. However, we stick to the name Leibniz algebroid which was used in relation to Nambu-Poisson structures [41, 53], or in relation to generalized geometry, as in [7].
- In some literature, there is an another axiom present, namely that ρ is a bracket homomorphism:
$$\rho([e, e']_E) = [\rho(e), \rho(e')], \tag{2.6}$$
for all $e, e' \in \Gamma(E)$. However, it follows directly from the compatibility of Leibniz rule and Leibniz identity.
- Our motivating example thus corresponds to the case $\rho = 0$, so called *totally intransitive* Leibniz algebroid. Note that in the general case, there is no way how to consistently induce a Leibniz algebra bracket on the single fibers of E.
- There is an interesting subtlety in the presented definition of the Leibniz algebroid. The bracket $[\cdot,\cdot]_E$ has two inputs, but Leibniz rule controls only the right one. One can prove the following. Let $e' \in \Gamma(E)$ be a section such that $e'|_U = 0$ for some open subset $U \subseteq M$. Then $([e, e']_E)|_U = 0$. This can be shown by choosing $f = 1 - \chi$, where χ is a bump function supported inside U, and $\chi(m) = 1$ for chosen $m \in M$. Then $e' = (1-\chi)e'$ and consequently
$$[e, e']_E = [e, (1-\chi)e']_E = (1-\chi)[e, e']_E + (\rho(e).(1-\chi))e'.$$
Evaluating this at m, we get $[e, e']_E(m) = 0$. We can repeat this for any $m \in U$, and we get $([e, e']_E)|_U = 0$. This property allows one to restrict the second input of $[\cdot,\cdot]_E$ to $\Gamma_U(E)$. For a general Leibniz algebroid, there is however no way to do this in the first input.

Let us now examine some structures induced by Leibniz algebroid bracket on E. First note that it induces an analogue $\mathcal{L}^E$ of Lie derivative on the tensor algebra $\mathcal{T}(E)$. It is defined as follows. Assume $e, e' \in \Gamma(E)$, $\alpha \in \Gamma(E^*)$ and $f \in C^\infty(M)$.

1. On $\mathcal{T}_0^0(E) \cong C^\infty(M)$, we define it as $\mathcal{L}_e^E f = \rho(e).f$.
2. For $\mathcal{T}_0^1(E) \cong \Gamma(E)$, we set $\mathcal{L}_e^E e' = [e, e']_E$,
3. For $\mathcal{T}_1^0(E) \cong \Gamma(E^*)$, we define it by contraction
$$\langle \mathcal{L}_e^E \alpha, e' \rangle := \rho(e).\langle \alpha, e' \rangle - \langle \alpha, [e, e']_E \rangle. \tag{2.7}$$
It follows from the Leibniz rule that the right-hand side is $C^\infty(M)$-linear in e', and thus defines an element $\mathcal{L}_e \alpha \in \Gamma(E^*)$.
4. For general $\tau \in \mathcal{T}_q^p(E)$, $\mathcal{L}_e^E$ is defined as
$$\begin{aligned}[]
[\mathcal{L}_e^E \tau](e_1, \dots, e_q, \alpha_1, \dots, \alpha_p) = \rho(e).\tau(e_1, \dots, e_q, \alpha_1, \dots, \alpha_p) \\
- \tau(\mathcal{L}_e^E e_1, \dots, e_q, \alpha_1, \dots, \alpha_p) - \dots \\
\dots - \tau(e_1, \dots, e_q, \alpha_1, \dots, \mathcal{L}_e^E \alpha_p),
\end{aligned} \tag{2.8}$$
for all $e_1, \dots, e_q \in \Gamma(E)$ and $\alpha_1, \dots, \alpha_p \in \Gamma(E^*)$.

We can easily prove the following properties of the Lie derivative.

Lemma 2.1.2. *Lie derivative $\mathcal{L}_e^E$ satisfies the Leibniz rule:*
$$\mathcal{L}_e^E(\tau \otimes \sigma) = \mathcal{L}_e^E(\tau) \otimes \sigma + \tau \otimes \mathcal{L}_e^E(\sigma), \tag{2.9}$$
for all $\tau, \sigma \in \mathcal{T}(E)$. Moreover, it can be restricted to the exterior algebra $\Omega^\bullet(E)$, and forms its degree 0 derivation:
$$\mathcal{L}_e^E(\omega \wedge \omega') = \mathcal{L}_e^E(\omega) \wedge \omega' + \omega \wedge \mathcal{L}_e^E(\omega'), \tag{2.10}$$
for all $\omega, \omega' \in \Omega^\bullet(E)$. Next, the commutator of Lie derivatives is again a Lie derivative:
$$\mathcal{L}_e^E \mathcal{L}_{e'}^E - \mathcal{L}_{e'}^E \mathcal{L}_e^E = \mathcal{L}_{[e,e']_E}^E, \tag{2.11}$$
for all $e, e' \in \Gamma(E)$. Note that this also implies $\mathcal{L}_{[e,e]_E}^E = 0$, although in general $[e, e]_E \neq 0$. Finally, there holds also the identity
$$\mathcal{L}_e^E \circ i_{e'} - i_{e'} \circ \mathcal{L}_e^E = i_{[e,e']_E}, \tag{2.12}$$
where both sides are now considered as operators only on the submodule $\Omega^\bullet(E)$.

Proof. Leibniz rule (2.9) follows from the Leibniz rule (2.4) for the bracket $[\cdot, \cdot]_E$ and the definition formula (2.8). When $\tau = \omega \in \Omega^q(E)$, the expression on the right-hand side of (2.8) can be seen to be skew-symmetric in $(e_1, \dots, e_q)$, and the derivation property (2.10) then follows from the Leibniz rule (2.9). Finally, the left-hand side of (2.11) is a commutator and thus obeys (2.9). It thus suffices to prove this on tensors of type $(0,0)$, $(1,0)$ and $(0,1)$. For $f \in \mathcal{T}_0^0(E)$, the condition (2.11) is equivalent to (2.6), for $(1,0)$ it reduces to the Leibniz identity (2.5), and for $(0,1)$ it follows from the relation
$$\langle [\mathcal{L}_e^E, \mathcal{L}_{e'}^E]\alpha, e'' \rangle = [\rho(e), \rho(e')].\langle \alpha, e'' \rangle - \langle \alpha, [\mathcal{L}_e^E, \mathcal{L}_{e'}^E] e'' \rangle. \tag{2.13}$$
To finish the proof we have to show (2.12). Because both sides are derivations of degree -1 of the exterior algebra $\Omega^\bullet(E)$, we have to prove the result on forms of degree 0 and 1 only. The first case is trivial, the second gives
$$\rho(e).\langle \alpha, e' \rangle - \langle \mathcal{L}_e^E \alpha, e' \rangle = \langle \alpha, [e, e']_E \rangle,$$
which is precisely the definition of the Lie derivative on $\Omega^1(E)$. Note that Leibniz identity for $[\cdot, \cdot]_E$ was not used to prove (2.12). ■

We see that there is still one piece missing in the Cartan puzzle, namely the analogue of the differential: $d_E : \Omega^\bullet(E) \to \Omega^{\bullet+1}(E)$. There is however no way around this for general Leibniz algebroid. There are two reasons - usual Cartan's formula for differential not only fails to define a form of a higher degree, it does not give a tensorial object at all.

To conclude the subsection on Leibniz algebroids, we now bring up an example, which will play a significant role hereafter.

Example 2.1.3. Let $E = TM \oplus \Lambda^p T^*M$, where $p \geq 0$. We will denote the sections of E as formal sums $X + \xi$, where $X \in \mathfrak{X}(M)$ and $\xi \in \Omega^p(M)$. We define the anchor ρ simply as the projection onto TM: $\rho(X + \xi) = X$. The bracket, which we will denote as $[\cdot,\cdot]_D$ is defined as

$$[X + \xi, Y + \eta]_D := [X, Y] + \mathcal{L}_X \eta - i_y d\xi, \tag{2.14}$$

for all $X + \xi, Y + \eta \in \Gamma(E)$. It is a straightforward check to see that $(E, \rho, [\cdot,\cdot]_D)$ forms a Leibniz algebroid. The bracket $[\cdot,\cdot]_D$ is called the **Dorfman bracket**. For $p = 1$, it first appeared in [29], for $p > 1$ it appeared in [39, 45]. It proved to be a useful tool to describe Nambu-Poisson manifolds [41]. To illustrate the previous definitions, note that on $\Omega^1(E)$, the induced Lie derivative $\mathcal{L}^E$ has the form

$$\mathcal{L}^E_{X+\xi}(\alpha + Q) = \mathcal{L}_X \alpha + (d\xi)(Q) + \mathcal{L}_X Q, \tag{2.15}$$

for all $X + \xi \in \Gamma(E)$ and all $\alpha + Q \in \Omega^1(E)$.

2.2 Lie algebroids

Having the concept of Leibniz algebroid defined, it is easy to define a Lie algebroid. This structure is much older, it first appeared in [80]. Lie algebroids play the role of an "infinitesimal" object corresponding to Lie groupoids. While a Lie algebra is the tangent space at the group unit with the extra structure coming from the group multiplication, Lie algebroid is a vector bundle over a set of units of a Lie groupoid. However; not every Lie algebroid corresponds to a Lie groupoid, see [27]. For an extensive study of Lie groupoids, Lie algebroids and related subjects, see the book [74].

Definition 2.2.1. Let $(L, l, [\cdot,\cdot]_L)$ be a Leibniz algebroid. We say that $(L, l, [\cdot,\cdot]_L)$ is a **Lie algebroid**, if $[\cdot,\cdot]_L$ is skew-symmetric, and hence a Lie bracket. Leibniz identity (2.5) is now called the Jacobi identity (note that it can be reordered using the skew-symmetry of the bracket).

Example 2.2.2. There are several standard examples of Lie algebroids

1. Consider $L = TM$, $l = Id_{TM}$ and let $[\cdot,\cdot]_L = [\cdot,\cdot]$ be a vector field commutator.
2. A generalization of the example in the previous section, where each fiber of E_m is equipped with a Lie algebra bracket, with a smooth dependence on m. In particular, for $M = \{m\}$, we see that every Lie algebra is an example of Lie algebroid.
3. This is a classical example, which probably first appeared in [35]. According to [67], it was discovered independently by several authors during 1980s. Look there for a complete list of references. It is sometimes called Koszul or Magri bracket, or simply a cotangent Lie algebroid.

Let $\Pi \in \mathfrak{X}^2(M)$ be a bivector on M. Choose $L = T^*M$, and define the anchor ρ as $\rho(\alpha) = \Pi(\alpha)$ for all $\alpha \in \Omega^1(M)$. Finally, define the bracket $[\cdot,\cdot]_\Pi$ as

$$[\alpha,\beta]_\Pi = \mathcal{L}_{\Pi(\alpha)}\beta - i_{\Pi(\beta)}d\alpha. \tag{2.16}$$

This bracket is skew-symmetric when Π is, and satisfies the Jacobi identity if and only if Π is a Poisson bivector, that is $\{f,g\} = \Pi(df,dg)$ defines a Poisson bracket on M. We will investigate this in more detail in Section 5.

4. Consider a principal G-bundle $P \overset{\pi}{\to} M$. There is a $C^\infty(M)$-module $\Gamma_G(TP)$ of G-invariant vector fields on P, which turns out to be isomorphic to the module $\Gamma(TP/G)$ of sections of the quotient bundle TP/G (quotient with respect to the right translation of vector fields induced by the principal bundle group action). This isomorphism induces a bracket on $\Gamma(TP/G)$ using the vector field commutator on $\Gamma(TP)$. Finally, the tangent map $T(\pi) : TP \to TM$ descends to the quotient $\widehat{T}(\pi) : TP/G \to TM$, defining an anchor. For details of this construction, see the sections §3.1, §3.2 of [74]. The resulting Lie algebroid is called the *Atiyah-Lie algebroid*.

The newly imposed skew-symmetry of the bracket $[\cdot,\cdot]_L$ of a Lie algebroid allows for new structures on the exterior algebra $\Omega^\bullet(L)$ and multivector field algebra $\mathfrak{X}^\bullet(L)$. First, see that we finally have a differential (absent in general Leibniz algebroid case). We state this as a proposition.

Proposition 2.2.3. *Let $(L, l, [\cdot,\cdot]_L)$ be a Lie algebroid. We define an $\mathbb{R}$-linear map $d_L : \Omega^\bullet(L) \to \Omega^{\bullet+1}(L)$ as follows. Let $\omega \in \Omega^p(L)$, and $e_0,\dots,e_p \in \Gamma(L)$. Set*

$$\begin{aligned}(d_L\omega)(e_0,\dots,e_p) = &\sum_{i=0}^{p}(-1)^i l(e_i).\omega(e_1,\dots,\widehat{e_i},\dots,e_p)\\ &+\sum_{i<j}(-1)^{i+j}\omega([e_i,e_j]_L,e_0,\dots,\widehat{e_i},\dots,\widehat{e_j},\dots,e_p),\end{aligned} \tag{2.17}$$

where $\widehat{e_i}$ denotes an omitted term. Then the right-hand side of (2.17) is completely skew-symmetric in $(e_0,\dots,e_p)$, which proves that $d_L\omega \in \Omega^{p+1}(L)$. Moreover, d_L is a derivation of the exterior algebra of degree $+1$, that is

$$d_L(\omega\wedge\omega') = d_L\omega\wedge\omega' + (-1)^{|\omega|}\omega\wedge d_L\omega', \tag{2.18}$$

for all $\omega,\omega' \in \Omega^\bullet(L)$, such that $|\omega|$ is defined. Moreover, the map d_L squares to zero: $d_L^2 = 0$. Finally, let $\mathcal{L}^L$ be a Lie derivative defined by Leibniz algebroid structure on L. Then the Cartan magic formula holds:

$$\mathcal{L}^L_e\omega = d_L i_e\omega + i_e d_L\omega, \tag{2.19}$$

for all $e \in \Gamma(L)$ and $\omega \in \Omega^\bullet(L)$.

Proof. First, one can check the first assertion, which is quite straightforward. Next, one has to prove that the right-hand side is $C^\infty(M)$-linear in $e_0,\dots,e_p$ and thus d_L is a well-defined operator on tensors of L. This follows from the Leibniz rule (2.4). The most difficult step is to show (2.18), which is a quite tedious but straightforward proof by induction and we skip it here. To show $d_L^2 = 0$, one notes that $d_L^2 = \frac{1}{2}\{d_L,d_L\}$, where $\{\cdot,\cdot\}$ is the graded commutator, and thus d_L^2 is a derivation of $\Omega^\bullet(M)$ of degree 2. It thus suffices to check $d_L^2 = 0$ on degrees 0 and 1. One obtains

$$(d_L^2 f)(e,e') = \rho(e).\rho(e').f - \rho(e').\rho(e).f - \rho([e,e']_L).f,$$

which is precisely the homomorphism property (2.6). For $\alpha \in \Omega^1(E)$, we get

$$\begin{aligned}(d_L^2\alpha)(e,e',e'') = {}& ([\rho(e),\rho(e')] - \rho([e,e']_L).\alpha(e'') + cyclic\{e,e',e''\} \\ &+ \langle \alpha, [[e,e']_L, e'']_L + [e'',[e,e']_L]_L + [e',[e'',e]_L]_L \rangle.\end{aligned} \tag{2.20}$$

The first line again vanishes due to (2.6), and the second line due to Jacobi identity (2.5). The last assertion is an equality of two degree 0 derivations of $\Omega^\bullet(L)$, and it thus has to be verified for degree 0 and 1 forms, which is easy. ■

Interestingly, d_L is not only an induced structure, it contains all the information about the original Lie algebroid. More precisely, having any vector bundle L with a degree 1 derivation d_L of the exterior algebra $\Omega^\bullet(L)$ squaring to 0, we can define the anchor l as

$$l(e).f := \langle d_L f, e \rangle, \tag{2.21}$$

for all $f \in C^\infty(M)$ and $e \in \Gamma(L)$, and then a bracket $[\cdot,\cdot]_L$ by

$$\langle \alpha, [e,e']_L \rangle = l(e).\langle \alpha, e' \rangle - l(e').\langle \alpha, e \rangle - (d_L\alpha)(e,e'), \tag{2.22}$$

for all $e, e' \in \Gamma(L)$ and $\alpha \in \Omega^1(L)$. It then follows by simple calculations using just (2.18) that $(L, l, [\cdot,\cdot]_L)$ is a Lie algebroid.

The second object induced by a Lie algebroid is an analogue of Schouten-Nijenhuis bracket, we again define it using a proposition, this time without any proof. For a detailed discussion on this topic, see [67].

Proposition 2.2.4. *Let $(L, l, [\cdot,\cdot]_L)$ be a Lie algebroid. Then there is a unique bracket $[\cdot,\cdot]_L$ defined on the multivector field algebra $\mathfrak{X}^\bullet(L)$, which has the following properties:*

- *For $e \in \mathfrak{X}^1(L) \equiv \Gamma(L)$, and $f \in \mathfrak{X}^0(L) \equiv C^\infty(M)$, we have $[e,f]_L = \rho(e).f$.*
- *For $e, e' \in \mathfrak{X}^1(L) \equiv \Gamma(L)$, $[\cdot,\cdot]_L$ coincides with the Lie algebroid bracket.*
- *$(\mathfrak{X}^\bullet(L), [\cdot,\cdot]_L)$ forms a Gerstenhaber algebra, which amounts to the following:*
  1. *$[\cdot,\cdot]_L$ is a degree -1 map, that is $|[P,Q]_L| = |P| + |Q| - 1$ for $P, Q \in \mathfrak{X}^\bullet(L)$.*
  2. *For each $P \in \mathfrak{X}^\bullet(L)$, $[P,\cdot]$ is a derivation of the exterior algebra $\mathfrak{X}^\bullet(L)$ of degree $|P|-1$, that is there holds*
  $$[P, Q \wedge R]_L = [P,Q]_L \wedge R + (-1)^{(|P|-1)|Q|} Q \wedge [P,R]_L. \tag{2.23}$$
  3. *It is graded skew-symmetric, that is*
  $$[P,Q]_L = -(-1)^{(|P|-1)(|Q|-1)}[Q,P]_L. \tag{2.24}$$
  4. *It satisfies the graded Jacobi identity*
  $$[P,[Q,R]_L]_L = [[P,Q]_L,R]_L + (-1)^{(|P|-1)(|Q|-1)}[Q,[P,R]_L]_L. \tag{2.25}$$

  All properties are assumed to hold for all $P, Q, R \in \mathfrak{X}^\bullet(L)$ with a well-defined degree.

The bracket $[\cdot,\cdot]_L$ is called a Schouten-Nijenhuis bracket corresponding to the Lie algebroid $(L, l, [\cdot,\cdot]_L)$. For $L = TM$, it reduces to the original well-known Schouten-Nijenhuis bracket of multivector fields.

2.3 Courant algebroids

Let us now consider a second special case of Leibniz algebroids, the one most relevant for the generalized geometry. We will stick to the modern and more used definition, which views Courant algebroid as a Leibniz algebroid with an additional structure. Historically, though, it appeared to be a much more complicated object. It appeared in [72] as a double corresponding to a pair of compatible Lie algebroids (we will show this as an example) in an attempt to generalize the concept of Manin triple to the Lie algebroid setting. The modern definition accounts to the thesis [81] of Roytenberg, who proved that the original skew-symmetric bracket can be replaced by a Leibniz algebroid bracket together with a set of simpler axioms. We present this form here.

Definition 2.3.1. Let $E \xrightarrow{\pi} M$ be a vector bundle. By **fiber-wise metric** on E, we mean a symmetric $C^\infty(M)$-billinear non-degenerate form $\langle \cdot, \cdot \rangle_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$. It follows from the $C^\infty(M)$-bilinearity that for every $m \in M$ it defines a non-degenerate symmetric bilinear form (metric) on the fiber E_m.

Definition 2.3.2. Let $(E, \rho, [\cdot, \cdot]_E)$ be a Leibniz algebroid. Let $\langle \cdot, \cdot \rangle_E$ be a fiber-wise metric on E. We say that $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ forms a **Courant algebroid**, if

1. The form $\langle \cdot, \cdot \rangle_E$ is invariant with respect to the bracket:
$$\rho(e).\langle e', e'' \rangle_E = \langle [e, e']_E, e'' \rangle_E + \langle e', [e, e'']_E \rangle_E, \tag{2.26}$$
for all $e, e', e'' \in \Gamma(E)$. Equivalently, if $g_E \in \mathcal{T}_2^0(E)$ is a tensor corresponding to $\langle \cdot, \cdot \rangle_E$, we require $\mathcal{L}_e^E g_E = 0$ for all $e \in \Gamma(E)$.

2. For all $e, e' \in \Gamma(E)$, the symmetric part of the bracket is governed by ρ and $\langle \cdot, \cdot \rangle_E$ in the sense:
$$\langle [e, e]_E, e'' \rangle_E = \frac{1}{2} \rho(e'').\langle e, e \rangle_E, \tag{2.27}$$
for all $e, e' \in \Gamma(E)$. Equivalently, let $g_E : E \to E^*$ be the induced vector bundle isomorphism. Define an $\mathbb{R}$-linear map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ as $\mathcal{D} := g_E^{-1} \circ \rho^T \circ d$, where $\rho^T \in \mathrm{Hom}(T^*M, E^*)$ is the transpose of the anchor. We can then rewrite the axiom simply as
$$[e, e]_E = \frac{1}{2} \mathcal{D} \langle e, e \rangle_E, \tag{2.28}$$
and this can be polarized to
$$[e, e']_E = -[e', e]_E + \mathcal{D} \langle e, e' \rangle_E. \tag{2.29}$$
We see that $[\cdot, \cdot]_D$ is skew-symmetric up to the $\mathcal{D}$ of the function $\langle e, e' \rangle_E$.

We will call $\langle \cdot, \cdot \rangle_E$ or g_E the *Courant metric* on E.

Courant algebroid can be viewed as a generalization of the quadratic Lie algebra, since for $M = \{m\}$, it reduces to a Lie algebra equipped with a non-degenerate *ad*-invariant symmetric bilinear form. It was noted in [69] that the invariance of $\langle \cdot, \cdot \rangle_E$ cannot be achieved without the sacrifice of the skew-symmetry, i.e. there is no Lie algebroid with an invariant fiber-wise metric $\langle \cdot, \cdot \rangle_E$, unless it is totally intransitive, that is $\rho = 0$. Note that the control over the symmetric part of the bracket allows one to derive the Leibniz rule in its first input. We get
$$[fe, e']_E = f[e, e']_D - (\rho(e').f)e + \langle e, e' \rangle_E \mathcal{D} f. \tag{2.30}$$

Recall the remark under the Definition 2.1.1, where we have noted that general Leibniz bracket $[e, e']_E$ depends only on the values of section e' in an arbitrarily small neighborhood, but nothing can be said about e. This is not true for Courant algebroids, where we can use (2.30) to prove that $e|_U = 0$ implies $([e, e']_E)|_U = 0$. Altogether, we can restrict $[\cdot,\cdot]_E$ to the module of the local sections $\Gamma_U(E)$, which proves useful when working with a local basis for $\Gamma(E)$.

Remark 2.3.3. Interestingly, Kosmann-Schwarzbach has shown in [65] that the axiom of Leibniz rule (2.4) is superfluous in the definition of Courant algebroid, and can be derived from (2.26, 2.27).

Example 2.3.4. Let us now give a few examples of Courant algebroids.

1. Consider the Leibniz algebroid from Example 2.1.3 for $p = 1$. Then there is a canonical pairing $\langle\cdot,\cdot\rangle_E$ of vector fields and 1-forms on $\Gamma(E) = \mathfrak{X}(M) \oplus \Omega^1(M)$. Explicitly, the map $\mathcal{D}$ is then $\mathcal{D}(f) = 0 + df \in \Gamma(E)$. We have

$$[X + \xi, X + \xi]_D = d(i_X \xi) = \frac{1}{2}\mathcal{D}\langle X + \xi, X + \xi\rangle_E. \tag{2.31}$$

   The invariance of the pairing reduces to showing that

$$X.(\langle \eta, Z\rangle + \langle \zeta, Y\rangle) = \langle [X, Y], \zeta\rangle + \langle \mathcal{L}_X \eta - i_Y d\xi, Z\rangle + \langle \eta, [X, Z]\rangle + \langle Y, \mathcal{L}_X \zeta - i_Z d\xi\rangle.$$

   This is easy to show after one uses the definitions of $\mathcal{L}$ and the exterior differential d. Now, see that $[\cdot,\cdot]_D$ can be modified in the following way. Let $H \in \Omega^3(M)$ be a 3-form on M, and define a new bracket $[\cdot,\cdot]^H_D$ as

$$[X + \xi, Y + \eta]^H_D = [X + \xi, Y + \eta]_D - H(X, Y, \cdot), \tag{2.32}$$

   for all $X + \xi, Y + \eta \in \Gamma(E)$. Because H is completely skew-symmetric and $C^\infty(M)$-linear, both axioms (2.26, 2.27) remain valid also for $[\cdot,\cdot]^H_D$. Plugging into Leibniz identity (2.5) shows that it holds if and only if $dH = 0$, that is $H \in \Omega^3_{closed}(M)$. The bracket $[\cdot,\cdot]^H_D$ is called the **H-twisted Dorfman bracket**.

2. Let $(L, l, [\cdot,\cdot]_L)$ and $(L^*, l^*, [\cdot,\cdot]_{L^*})$ be a pair of Lie algebroids, where L^* is the dual vector bundle to L. One says that (L, L^*) forms a *Lie bialgebroid*, if d_L is a derivation of the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{L^*}$, that is[2]

$$d_L[\omega, \omega']_{L^*} = [d_L\omega, \omega']_{L^*} + (-1)^{|\omega|-1}[\omega, d_L\omega']_{L^*}, \tag{2.33}$$

   for all $\omega, \omega' \in \Omega^\bullet(L) \cong \mathfrak{X}^\bullet(L^*)$. Define $E = L \oplus L^*$. Denote the sections of E as ordered pairs (e, α), where $e \in \Gamma(L)$ and $\alpha \in \Gamma(L^*)$. The anchor ρ is defined as $\rho(e, \alpha) = l(e) + l^*(\alpha)$. The bracket $[\cdot,\cdot]_E$ is defined as

$$[(e, \alpha), (e', \alpha')]_E = \big([e, e']_L + \mathcal{L}^{L^*}_\alpha e' - i_\alpha(d_{L^*}e), [\alpha, \alpha']_{L^*} + \mathcal{L}^L_e \alpha' - i_{e'}(d_L\alpha)\big), \tag{2.34}$$

   for all $(e, \alpha), (e', \alpha') \in \Gamma(E)$. Finally, let $\langle\cdot,\cdot\rangle_E$ be a fiber-wise metric on E induced by the canonical pairing of L and L^*. Then $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ is a Courant algebroid, called the *double of the Lie bialgebroid* (L, L^*). The actual proof of this statement is straightforward but takes quite some time to go through. See [72] for details.

   Conversely, let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid, and L_1 and L_2 be two complementary Dirac structures, that is $E = L_1 \oplus L_2$. Then $L_2 \cong L_1^*$ and (L_1, L_2) can be equipped with a structure of Lie bialgebroid. (E, L_1, L_2) is called a *Manin triple*.

[2] This condition is in fact equivalent to the one with L and L^* interchanged [75].

3. Let us show an example of a double of Lie bialgebroid. Let $(L, l, [\cdot,\cdot]_L)$ be any Lie algebroid. Choose L^* to be a trivial Lie algebroid $(L^*, 0, 0)$. Since $[\cdot,\cdot]_{L^*} = 0$, the Lie bialgebroid condition (2.33) holds trivially. The resulting bracket on $E = L \oplus L^*$ is then

$$[(e,\alpha),(e',\alpha')]_E = ([e,e'], \mathcal{L}^L_e \alpha' - i_{e'} d_L \alpha), \tag{2.35}$$

for all $(e,\alpha),(e',\alpha') \in \Gamma(E)$. We will call it the **Dorfman bracket** corresponding to Lie algebroid $(L, l, [\cdot,\cdot]_L)$. For $L = (TM, Id_{TM}, [\cdot,\cdot])$, one obtains part 1. of this example.

4. Let $L = (TM, Id_{TM}, [\cdot,\cdot])$ be the canonical tangent bundle Lie algebroid, and set $L^* = (T^*M, \Pi, [\cdot,\cdot]_\Pi)$ to be the cotangent Lie algebroid from Example 2.2.2, 3. One can find in [75] that $d_{L^*} = -[\Pi, \cdot]_S$, where $[\cdot,\cdot]_S$ is the ordinary Schouten-Nijenhuis bracket on $\mathfrak{X}^\bullet(M)$. By definition of Schouten-Nijenhuis bracket, d_{L^*} is thus a derivation of $[\cdot,\cdot]_S$, which is exactly the Lie bialgebroid condition (2.33). The pair (L, L^*) therefore forms a Lie bialgebroid, called a *triangular Lie bialgebroid*. The resulting double bracket (2.34) on E can be after some effort written as

$$\begin{aligned}[X+\xi, Y+\eta]_E = [X+\Pi(\xi), Y+\Pi(\eta)] - \Pi\big(\mathcal{L}_{(X+\Pi(\xi))}\eta - i_{(Y+\Pi(\eta))} d\xi\big) \\ + \mathcal{L}_{(X+\Pi(\xi))}\eta - i_{(Y+\Pi(\eta))} d\xi,\end{aligned} \tag{2.36}$$

for all $X+\xi, Y+\eta \in \Gamma(E)$. Although complicated at first glance, it can be rewritten using the Dorfman bracket (2.14). Indeed, define a bundle map $e^\Pi : E \to E$ as $e^\Pi(X+\xi) = X + \Pi(\xi) + \xi$ for all $X+\xi \in \Gamma(E)$. We can then write

$$[X+\xi, Y+\eta]_E = e^{-\Pi}[e^\Pi(X+\xi), e^\Pi(Y+\eta)]_D. \tag{2.37}$$

Moreover, $\rho = pr_{TM} \circ e^\Pi$, and $\langle e^\Pi(X+\xi), e^\Pi(Y+\eta)\rangle_E = \langle X+\xi, Y+\eta\rangle_E$. This shows that bracket (2.36) is in fact just a "twist" of the Dorfman bracket. There is one important remark to be said. The bracket written in the form (2.36) in fact does not require Π to be a Poisson bivector in order to define a Courant algebroid. However; for general Π, the bracket (2.34) is not the same as (2.36).

If one uses $[\cdot,\cdot]^H_D$ in the formula (2.37) instead of $[\cdot,\cdot]_D$, one obtains a bracket which is in [42] called the *Roytenberg bracket*. We also use this name for its higher version.

There is a famous classification of Ševera of a particular class of Courant algebroids. Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be any Courant algebroid. Define a map $j : T^*M \to E$ as $j = g_E^{-1} \circ \rho^T$. One says that a Courant algebroid is *exact*, if there is a short exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\ j\ } E \xrightarrow{\ \rho\ } TM \longrightarrow 0. \tag{2.38}$$

In particular, ρ has to be surjective and j injective, and $\operatorname{Im} j = \ker \rho$. Note that inclusion $\operatorname{Im} j \subseteq \ker \rho$ holds for any Courant algebroid, because $\rho \circ \mathcal{D} = 0$. Ševera proved in [89] that up to an isomorphism, exact Courant algebroids are uniquely determined by a class $[H] \in H_3(M, \mathbb{R})$. In particular, there is an isotropic splitting $s : TM \to E$, $\langle s(X), s(Y)\rangle_E = 0$ for all $X, Y \in \mathfrak{X}(M)$, such that one can write

$$[s(X)+j(\xi), s(Y)+j(\eta)]_E = s([X,Y]) + j(\mathcal{L}_X\eta - i_Y d\xi - H(X,Y,\cdot)), \tag{2.39}$$

where $H \in \Omega^3_{closed}(M)$. For different splitting s' of the sequence, H changes to $H' = H + dB$ for some 2-form $B \in \Omega^2(M)$, but $[H'] = [H]$. Map $\mathbf{\Psi} : TM \oplus T^*M \to E$ defined as $\mathbf{\Psi}(X+\xi) =$

$s(X) + j(\xi)$ then defines a Courant algebroid isomorphism from $(TM \oplus T^*M, pr_{TM}, [\cdot,\cdot]_D^H)$ to $(E, \rho, [\cdot,\cdot]_E)$. Every exact Courant algebroid is thus isomorphic to a one equipped with an H-twisted Dorfman bracket.

Dorfman and H-twisted Dorfman brackets of Example 2.3.4, 1. are exact, whereas a Manin triple of Lie bialgebroid in general is not. This can be seen on example of the Dorfman bracket of a Lie algebroid, 2.3.4, 3. where any Lie algebroid L with a non-surjective anchor will give a non-exact Courant algebroid. On the other hand, the example 2.3.4, 4. is an example of exact Manin triple, in particular $[H] = [0]$ in this case.

To conclude, let us briefly note on the older, skew-symmetric version of Courant algebroid brackets. Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid. Define $[\cdot,\cdot]'_E$ to be its skew-symmetrization:

$$[e, e']'_E := \frac{1}{2}([e, e']_E - [e', e]_E) = [e, e']_E - \frac{1}{2}\mathcal{D}\langle e, e'\rangle_E. \tag{2.40}$$

Let us examine what happened to the Leibniz rule. Plugging into (2.40), we obtain

$$[e, fe']'_E = f[e, e']'_E + (\rho(e).f)e' - \frac{1}{2}\langle e, e'\rangle \mathcal{D}f. \tag{2.41}$$

Invariance of $\langle\cdot,\cdot\rangle_E$ with respect to the bracket $[\cdot,\cdot]'_E$ becomes

$$\rho(e).\langle e', e''\rangle_E = \langle [e, e']'_E + \frac{1}{2}\mathcal{D}\langle e, e'\rangle_E, e''\rangle_E + \langle e', [e, e'']'_E + \frac{1}{2}\mathcal{D}\langle e, e''\rangle_E\rangle_E, \tag{2.42}$$

for all $e, e', e'' \in \Gamma(E)$. Note that Leibniz rule for $[\cdot,\cdot]_E$ implies $\rho \circ \mathcal{D} = 0$. This also shows that $[\cdot,\cdot]'_E$ also satisfies the homomorphism property (2.6):

$$\rho([e, e']'_E) = [\rho(e), \rho(e')], \tag{2.43}$$

for all $e, e' \in \Gamma(E)$. The most complicated calculation is to see that Leibniz identity for $[\cdot,\cdot]'_E$ fails in the following sense. Define a map $T : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ as

$$T(e, e', e'') := \frac{1}{6}\langle [e, e']'_E, e''\rangle_E + cyclic\{e, e', e''\}. \tag{2.44}$$

Then there holds the following identity:

$$[[e, e']'_E, e'']'_E + [[e'', e]]'_E, e']'_E + [[e', e'']'_E, e]'_E = \mathcal{D}T(e, e', e''). \tag{2.45}$$

For the proof of this statement see [81]. Now let us just say that equations (2.41, 2.42, 2.43, 2.45) form a set of axioms of the original definition of Courant algebroid, as proposed in [72]. The skew-symmetric version of the bracket has its advantages, in particular in relation to strongly homotopy Lie algebras.

2.4 Algebroid connections, local Leibniz algebroids

For Lie algebroids there is a straightforward way to define linear connections [34]. For Courant algebroids, or even Leibniz algebroids, matters become more complicated, see [1]. Let us recall the basic definitions first.

Definition 2.4.1. Let E be a vector bundle. Map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is called a **linear connection on vector bundle** E, if

$$\nabla(fX, e) = f\nabla(X, e),\ \nabla(X, fe) = f\nabla(X, e) + (X.f)e, \tag{2.46}$$

for all $X \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$. We write $\nabla_X e := \nabla(X, e)$.

Remark 2.4.2. Equivalently, we can view ∇ as follows. Let $\mathcal{D}(E)$ be a vector bundle over M such that its space of sections $\Gamma(\mathcal{D}(E))$ has the form

$$\Gamma(\mathcal{D}(E)) = \{\mathcal{F} : \Gamma(E) \to \Gamma(E) \mid \mathcal{F}(fe) = f\mathcal{F}(e) + (X.f)e, \forall e \in \Gamma(E),\ \text{for } X \in \mathfrak{X}(M)\}. \tag{2.47}$$

Define $a : \Gamma(\mathcal{D}(E)) \to \mathfrak{X}(M)$ as $a(\mathcal{F}) = X$, and let $[\mathcal{F}, \mathcal{G}] = \mathcal{F} \circ \mathcal{G} - \mathcal{G} \circ \mathcal{F}$. Then $(\mathcal{D}(E), a, [\cdot, \cdot])$ is a Lie algebroid. We can then view linear connection ∇ as a vector bundle morphism $\nabla \in \mathrm{Hom}(TM, \mathcal{D}(E))$ defined as $\nabla(X) = \nabla_X$ fitting in the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "\nabla"] \arrow[dr, "1_{TM}"'] & \mathcal{D}(E) \arrow[d, "a"] \\ & TM \end{tikzcd} . \tag{2.48}$$

Note that both TM and $\mathcal{D}(E)$ are Lie algebroids. One can easily extend this definition to any Lie algebroid $(L, l, [\cdot, \cdot]_L)$. For more details concerning the construction of vector bundle $\mathcal{D}(E)$, see [74].

Every linear connection on E induces an analogue of the curvature operator. For $X, Y \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$ it is defined using the standard formula:

$$R(X, Y)e = \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e. \tag{2.49}$$

It is $C^\infty(M)$-linear in all inputs, hence $R \in \Omega^2(M) \otimes \mathcal{T}_1^1(E)$. In view of Remark (2.4.2), we may view R as a failure of ∇ to be a Lie algebroid morphism. If g_E is any fiber-wise metric on E, we can say that ∇ is metric compatible with g_E if

$$X.g_E(e, e') = g_E(\nabla_X e, e') + g_E(e, \nabla_X e'), \tag{2.50}$$

for all $X \in \mathfrak{X}(M)$ and $e, e' \in \Gamma(E)$. Obviously, there is no analogue of torsion for connections on a vector bundle. Now, let $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ be a Courant algebroid. One can define the Courant algebroid connection according to [1] as follows:

Definition 2.4.3. Let $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ be a Courant algebroid. A map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a **Courant algebroid connection**, if

$$\nabla(fe, e') = f\nabla(e, e'),\ \nabla(e, fe') = f\nabla(e, e') + (\rho(e).f)e', \tag{2.51}$$

for all $e, e' \in \Gamma(E)$, and ∇ is metric compatible with Courant metric $\langle \cdot, \cdot \rangle_E$ in the sense that

$$\rho(e).\langle e', e'' \rangle_E = \langle \nabla_e e', e'' \rangle_E + \langle e', \nabla_e e'' \rangle_E, \tag{2.52}$$

for all $e, e', e'' \in \Gamma(E)$. As usual, we will write $\nabla_e e' := \nabla(e, e')$.

As before, we can naively define a curvature operator R corresponding to ∇ as

$$R(e,e')e'' = \nabla_e \nabla_{e'} e'' - \nabla_{e'} \nabla_e e'' - \nabla_{[e,e']_E} e'', \tag{2.53}$$

for all $e, e', e'' \in \Gamma(E)$. This is $C^\infty(M)$-linear in e' and e'', but not in e. Instead, we get

$$R(fe,e')e'' = fR(e,e')e'' - \langle e,e'\rangle_E \nabla_{\mathcal{D}f} e''. \tag{2.54}$$

Let us remark that there is a class of connections which define a tensorial curvature operator. We say that ∇ is an induced Courant algebroid connection, if $\nabla_e = \nabla'_{\rho(e)}$ for some vector bundle connection ∇'. Because $\rho \circ D = 0$, the anomalous term in (2.54) disappears, and R is a well defined tensor on E. Second possibility is to restrict R to sections of some isotropic involutive subbundle $D \subseteq E$, where $\langle e,e'\rangle_E = 0$.

Unlike for vector bundle connections, there is a well defined analogue of the torsion operator. There are two independent, but essentially equivalent definitions. In [40], a torsion tensor $T \in \mathcal{T}_3(E)$ is given as

$$T(e,e',e'') = \langle \nabla_e e' - \nabla_{e'} e - [e,e']'_E, e''\rangle_E + \frac{1}{2}(\langle \nabla_{e''} e, e'\rangle_E - \langle \nabla_{e''} e', e\rangle_E, \tag{2.55}$$

for all $e, e', e'' \in \Gamma(E)$. By definition, it is skew-symmetric in (e, e'). In fact, the Courant metric compatibility condition (2.52) can be used to show that $T \in \Omega^3(E)$. In [1], a Courant algebroid torsion was defined as $C \in \Omega^3(E)$ in the form

$$C(e,e',e'') = \frac{1}{3}\langle [e,e']'_E, e''\rangle_E - \frac{1}{2}\langle \nabla_e e' - \nabla_{e'} e, e''\rangle_E + cyclic(e,e',e''). \tag{2.56}$$

This expression in manifestly completely skew-symmetric in all inputs, but at first glance it does not resemble the conventional definition of torsion operator. Interestingly, these two definitions coincide.

Lemma 2.4.4. *Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid, and ∇ a Courant algebroid connection. Then*

$$T(e,e',e'') = -C(e,e',e''). \tag{2.57}$$

Proof. This can be done by using (2.52) and Courant algebroid axioms (2.26, 2.28). Note that both T and C are well defined based on the Leibniz rule (2.1.1), but equivalent only due to the other Courant algebroid axioms. ■

For general Leibniz algebroids, there is no metric $\langle\cdot,\cdot\rangle_E$ present and definitions (2.55, 2.56) make no sense anymore. There is however a way to define a connection, a torsion and even a curvature operator for a special (and quite wide) class of Leibniz algebroids.

Definition 2.4.5. Let $(E, \rho, [\cdot,\cdot]_E)$ be a Leibniz algebroid. If there exists a $C^\infty(M)$-trilinear map $\mathbf{L} : \Gamma(E^*) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, such that

$$[fe,e']_E = f[e,e']_E - (\rho(e').f)e + \mathbf{L}(\mathsf{d}f, e, e'), \tag{2.58}$$

for all $e, e' \in \Gamma(E)$ and $f \in C^\infty(M)$, we call $(E, \rho, [\cdot,\cdot]_E, \mathbf{L})$ a **local Leibniz algebroid**. Here $\mathsf{d} : C^\infty(M) \to \Gamma(E^*)$ is an $\mathbb{R}$-linear map defined by

$$\langle \mathsf{d}f, e\rangle = \rho(e).f, \tag{2.59}$$

for all $e \in \Gamma(E)$ and $f \in C^\infty(M)$.

Note that $\mathbf{L}$ is not uniquely determined by equation (2.58), and has to be a part of the definition of a local Leibniz algebroid. Moreover, the compatibility of (2.58) with the homomorphism property (2.6) implies

$$\rho(\mathbf{L}(\mathsf{d}f, e, e')) = 0. \tag{2.60}$$

For a given Leibniz algebroid $(E, \rho, [\cdot,\cdot]_E)$ with a well-defined subbundle $\ker\rho$, one can always find $\mathbf{L}$ such that this property can be extended to $\rho(\mathbf{L}(\beta, e, e')) = 0$ for all $e, e' \in \Gamma(E)$ and $\beta \in \Gamma(E^*)$. To achieve this, choose some fiber-wise metric on E^* and define $\mathbf{L}(\beta, e, e') := 0$ for all $\beta \in \Gamma(\operatorname{Ann}\ker(\rho)^\perp)$.

Example 2.4.6. In fact, all examples in this thesis can be equipped with the structure of a local Leibniz algebroid.

- Let $(L, l, [\cdot,\cdot]_L)$ be a Lie algebroid. The choice of $\mathbf{L} = 0$ shows that $(L, l, [\cdot,\cdot]_L, \mathbf{L})$ is a local Leibniz algebroid.
- Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid. We see that from (2.30) that

  $$\mathbf{L}(\mathsf{d}f, e, e') = \langle e, e'\rangle_E g_E^{-1}(\mathsf{d}f). \tag{2.61}$$

  There is one canonical way to extend $\mathbf{L}$. Define

  $$\mathbf{L}(\beta, e, e') = \langle e, e'\rangle_E g_E^{-1}(\beta), \tag{2.62}$$

  for all $e, e' \in \Gamma(E)$ and $\beta \in \Gamma(E^*)$. However, note that this choice does not satisfy $\rho(\mathbf{L}(\beta, e, e')) = 0$. For E with well-defined subbundle $\ker\rho$, one can extend $\mathbf{L}$ trivially to some complement of $\operatorname{Ann}(\ker\rho)$. Note that different choices of this complement will lead to different extensions.
- Let $E = TM \oplus T^*M$ be equipped with the Dorfman bracket (2.14), and $\rho(X + \xi) = X$. The kernel of ρ is the subbundle $T^*M \subseteq E$. We have a short exact sequence

  $$0 \longrightarrow T^*M \xrightarrow{\ j\ } E \xrightarrow{\ \rho\ } TM \longrightarrow 0, \tag{2.63}$$

  where j is an inclusion. Choosing a complement of $\ker\rho$ corresponds to the choice of a splitting $s \in \operatorname{Hom}(TM, E)$ of this sequence. We can restrict ourselves to isotropic splittings, that is $\langle s(X), s(Y)\rangle_E = 0$ for all $X, Y \in \mathfrak{X}(M)$. The set of such splittings is in fact $\Omega^2(M)$, and for any $B \in \Omega^2(M)$ the complement to T^*M is exactly the subbundle

  $$G_B = \{X + B(X) \mid X \in TM\} \subseteq E. \tag{2.64}$$

  Note that $G_0 = TM$. This gives us also a splitting of E^*, in particular

  $$E^* = \operatorname{Ann}(\ker\rho) \oplus G_B. \tag{2.65}$$

  We can now define $\mathbf{L}$ to be trivial on G_B, let us write it with subscript B. We get

  $$\begin{aligned}\mathbf{L}_B(\alpha + V, e, e') &= \mathbf{L}_B(\alpha - B(V) + (V + B(V)), e, e')\\ &= \langle e, e'\rangle_E g_E^{-1}(\alpha - B(V)) = \langle e, e'\rangle_E(\alpha - B(V)).\end{aligned} \tag{2.66}$$

  We see how $\mathbf{L}$ can explicitly depend on the choice of the complement.

For an arbitrary Leibniz algebroid $(E, \rho, [\cdot,\cdot]_E)$ we can define a Leibniz algebroid connection ∇ in the same way as in Definition 2.4.3, except that we do not require the metric compatibility (2.52). Assume that this Leibniz algebroid is local. Then we can in fact define a torsion operator!

Proposition 2.4.7. *Let $(E, \rho, [\cdot,\cdot]_E, \mathbf{L})$ be a local Leibniz algebroid. Define an $\mathbb{R}$-bilinear map $T : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ as*

$$T(e, e') = \nabla_e e' - \nabla_{e'} e - [e, e']_E + \mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e'), \tag{2.67}$$

*for all $e, e' \in \Gamma(E)$. Here $(e_\lambda)_{\lambda=1}^k$ is an arbitrary local frame of E, and $(e^\lambda)_{\lambda=1}^k$ the corresponding dual one. Then T is $C^\infty(M)$-linear in e and e' and consequently $T \in \mathcal{T}_2^1(E)$. We call T a **torsion operator** corresponding to ∇.*

Proof. Direct calculation. ■

Let us emphasize that T is not in general skew-symmetric in (e, e'). This is not a problem since we can always take its skew-symmetric part. Moreover, its definition certainly depends on the choice of the map $\mathbf{L}$.

Interestingly, for induced connections $\nabla_e = \nabla'_{\rho(e)}$, this is not the case. To see this, choose the local frame $(e_\lambda)_{\lambda=1}^k$ adapted to the splitting $E = \ker(\rho) \oplus \ker(\rho)^\perp$ with respect to some fiber-wise metric g_E on E. Because ∇ is induced, only those e_λ in $\Gamma(\ker(\rho)^\perp)$ contribute to the sum in (2.67). But in this case $e^\lambda \in \mathrm{Ann}(\ker\rho)$, where the map $\mathbf{L}$ is determined uniquely[3]

For Courant algebroid $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ and $\mathbf{L}$ in the form (2.62), the torsion operator (2.67) can be simply related to the one defined by (2.55).

Proposition 2.4.8. *Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid, and $\mathbf{L}$ be a map defined by (2.62). Denote the torsion operator (2.55) as T_G, and let T be the torsion operator (2.67). Let ∇ be a Courant algebroid connection. Then*

$$T_G(e, e', e'') = \langle T(e, e'), e'' \rangle_E. \tag{2.68}$$

Proof. This can be verified by a direct calculation. Use the fact that

$$\langle \mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e'), e'' \rangle_E = \langle \nabla_{e''} e, e' \rangle_E. \tag{2.69}$$

This shows that for Courant algebroid connections, the symmetric part of the map $\mathbf{K}(e, e') = \mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e')$ does not depend on ∇ at all. Indeed, we have

$$\mathbf{K}(e, e') + \mathbf{K}(e', e) = \langle \nabla_{e''} e, e' \rangle_E + \langle e, \nabla_{e''} e' \rangle_E = \rho(e'').\langle e, e' \rangle_E.$$

This in fact proves that $T(e, e')$ is skew-symmetric in (e, e'), because

$$\langle e'', T(e, e') + T(e', e) \rangle_E = \langle e'', -[e, e']_E + [e', e]_E \rangle_E + \rho(e'').\langle e, e' \rangle_E = 0.$$

We have used the axiom (2.28) in the last step. ■

[3]Note that sections of the form $\mathrm{d}f$ for some $f \in C^\infty(M)$ locally generate $\Gamma(\mathrm{Ann}(\ker\rho))$.

For more general (local) Leibniz algebroids, there also exists a notion of a generalized torsion introduced for special examples in [24, 25]. They proceed as follows. Consider a local Leibniz algebroid $(E, \rho, [\cdot,\cdot]_E, \mathbf{L})$. Let $\mathcal{L}^E$ be the Lie derivative induced by $[\cdot,\cdot]_E$. In their paper this is called the *Dorfman derivative*. Consider a local holonomic frame $(e_\alpha)_{\alpha=1}^k$, that is $[e_\alpha, e_\beta]_E = 0$. We will use the shorthand notation $f_{,\alpha} := \rho(e_\alpha).f$. Let $e = v^\alpha e_\alpha$ and $e' = w^\beta e_\beta$. We have

$$\mathcal{L}^E_e e' = \{v^\alpha w^\beta{}_{,\alpha} - w^\alpha(v^\beta{}_{,\alpha} - v^\lambda{}_{,\mu} L^{\beta\mu}{}_{\lambda\alpha})\} e_\beta. \tag{2.70}$$

Their idea is to define a "covariantized" Dorfman derivative $\mathcal{L}^\nabla_e$ by replacing commas with semicolons:

$$\mathcal{L}^\nabla_e e' = \{v^\alpha w^\beta{}_{;\alpha} - w^\alpha(v^\beta{}_{;\alpha} - v^\lambda{}_{;\mu} L^{\beta\mu}{}_{\lambda\alpha})\} e_\beta. \tag{2.71}$$

Here $\nabla_{e_\alpha} e = v^\beta{}_{;\alpha} e_\beta$. This can be rewritten in terms of ∇ and $\mathbf{L}$ as

$$\mathcal{L}^\nabla_e e' = \nabla_e e' - \nabla_{e'} e + \mathbf{L}(e^\mu, \nabla_{e_\mu} e, e'). \tag{2.72}$$

Torsion operator T is in [24, 25] defined as difference of these two Lie derivatives:

$$T(e, e') = (\mathcal{L}^\nabla_e - \mathcal{L}^E_e) e'. \tag{2.73}$$

Comparing this with (2.67) we see from (2.72) that the two definitions coincide. Note the importance of the local frame holonomicity for a validity of this assertion.

We have shown that any local Leibniz algebroid allows one to define a tensorial torsion operator. We can use a very similar approach to get a well-defined curvature operator.

Proposition 2.4.9. *Let $(E, \rho, [\cdot,\cdot]_E, \mathbf{L})$ be a local Leibniz algebroid, such that $\rho(\mathbf{L}(\beta, e, e')) = 0$ for all $\beta \in \Gamma(E^*)$ and $e, e' \in \Gamma(E)$. Let ∇ be a Leibniz algebroid connection on E. Then the map R defined for all $e, e', e'' \in \Gamma(E)$ as*

$$R(e, e')e'' = \nabla_e \nabla_{e'} e'' - \nabla_{e'} \nabla_e e'' - \nabla_{[e,e']_E} e'' + \nabla_{\mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e')} e'', \tag{2.74}$$

is $C^\infty(M)$-linear in all inputs, and thus $R \in \mathcal{T}^1_3(E)$. We call R a generalized Riemann tensor.

Proof. The statement can be directly verified. One has to use (2.6) to show the $C^\infty(M)$-linearity in e''. The additional correction term containing $\mathbf{L}$ cancels the wrong term coming from the bracket term and its first input. The condition $\rho(\mathbf{L}(\beta, e, e')) = 0$ is necessary to keep the $C^\infty(M)$-linearity in e''. ■

First, note that because of the condition $\rho(\mathbf{L}(\beta, e, e')) = 0$, the additional term vanishes for induced connections, which in fact shows that in this case the usual curvature operator formula works and defines a tensorial R. Next, see that in general R is not skew-symmetric in (e, e'), which can be fixed by a skew-symmetrization if necessary.

Having a curvature operator R, we can define the corresponding Ricci tensor Ric as a contraction of R in two indices. Namely set

$$\mathrm{Ric}(e, e') = \langle e^\lambda, R(e_\lambda, e')e \rangle, \tag{2.75}$$

for all $e, e' \in \Gamma(E)$, where $(e_\lambda)_{\lambda=1}^k$ is some local frame of E, and $(e^\lambda)_{\lambda=1}^k$ the corresponding dual one of E^*. For Courant algebroid connections, R has some remarkable properties.

Proposition 2.4.10. *Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid with $\ker\rho \subseteq E$ being a well defined subbundle. Let $\mathbf{L}$ be defined trivially on some complement to $\mathrm{Ann}(\ker\rho)$. Let ∇ be a Courant algebroid connection. Then $R(e, e')$ is skew-symmetric in (e, e'), and*

$$\langle R(e,e')f, f'\rangle_E + \langle R(e,e')f', f\rangle_E = 0, \tag{2.76}$$

for all $e, e', f, f' \in \Gamma(E)$.

Proof. Let $E^* = \mathrm{Ann}(\ker\rho) \oplus V$ for some subbundle V, such that $\mathbf{L}|_V = 0$. Choose a local frame $e^\lambda = (g^j, f^k)$, where $(g^j)_{j=1}^m$ is a local frame of $\mathrm{Ann}(\ker\rho)$, and $(f^k)_{k=1}^{k-m}$ a local frame of V. Let $e_\lambda = (g_j, f_k)$ be a corresponding dual basis. Then $(f_k)_{k=1}^{k-m}$ generates $\ker\rho$, and $(g_i)_{i=1}^m$ its complement. We have

$$\begin{aligned}\langle \mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e), e'\rangle_E &= \langle \mathbf{L}(g^k, \nabla_{g_k} e, e), e'\rangle_E = \langle \langle \nabla_{g_k} e, e\rangle_E g_E^{-1}(g^k), e'\rangle_E \\ &= [\frac{1}{2}\rho(g_k).\langle e, e\rangle_E]\langle g^k, e'\rangle = \frac{1}{2}\rho(\langle g^k, e'\rangle g_k + \langle f^k, e'\rangle f_k).\langle e, e\rangle_E \\ &= \frac{1}{2}\rho(e').\langle e, e\rangle_E.\end{aligned}$$

This proves that for Courant algebroid connections, we have $\mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e) = [e, e]$, and the two non-trivial contributions in $R(e, e)$ cancel. Note that this shows that also $T(e, e) = 0$ for such an $\mathbf{L}$. The proof of (2.76) is analogous to the one for ordinary connections, using the metric compatibility (2.52). Note that in this process one has to use $\rho(\mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e)) = 0$. ■

Example 2.4.11. Consider $E = TM \oplus T^*M$ and the usual Dorfman bracket (2.14). Extend $\mathbf{L}$ to all $\gamma + Z \in \Gamma(E^*)$ as

$$\mathbf{L}(\gamma + Z, X + \xi, Y + \eta) = \langle X + \xi, Y + \eta\rangle_E (0 + \gamma). \tag{2.77}$$

This corresponds to the choice $B = 0$ in (2.66). Now consider a Courant algebroid connection ∇ on E. We have

$$\nabla_{X+\xi}(Y + \eta) = \nabla'_X(Y + \eta) + \nabla''_\xi(Y + \eta), \tag{2.78}$$

for all $X + \xi, Y + \eta \in \Gamma(E)$, for some vector bundle connection ∇' on E, and a map $\nabla'' : \Omega^1(M) \times \Gamma(E) \to \Gamma(E)$. Note that ∇'' must be $C^\infty(M)$-linear in the second input, and thus in fact $\nabla'' \in \mathfrak{X}^1(M) \otimes \mathcal{T}_1^1(E)$. We can view ∇'' as $C^\infty(M)$-linear map $\nabla'' : \Omega^1(M) \to \mathrm{End}(E)$. What are the implications of the Courant metric compatibility (2.52)? We get

$$X.\langle e, e'\rangle_E = \langle \nabla'_X e, e'\rangle_E + \langle e, \nabla'_X e'\rangle_E, \tag{2.79}$$

$$0 = \langle \nabla''_\xi e, e'\rangle_E + \langle e, \nabla''_\xi e'\rangle_E. \tag{2.80}$$

This implies that ∇' and ∇'' have to be of the block form

$$\nabla'_X = \begin{pmatrix} \nabla^M_X & \Pi_X \\ B_X & \nabla^M_X \end{pmatrix}, \ \nabla''_\xi = \begin{pmatrix} A_\xi & \theta_\xi \\ C_\xi & -A_\xi^T \end{pmatrix}, \tag{2.81}$$

where $\Pi_X, \theta_\xi \in \mathfrak{X}^2(M)$, $B_X, C_\xi \in \Omega^2(M)$, $A_\xi \in \mathrm{End}(TM)$, and ∇^M is an ordinary connection on M. ∇^M_X in the bottom-right corner of ∇'_X is the usual extension of ∇^M on 1-forms. All objects are assumed to be $C^\infty(M)$-linear in X and ξ. For the curvature tensor, we get

$$\begin{aligned} pr_1(R(X,Y)(Z+\zeta)) &= R^M(X,Y)Z + (\nabla^M_X\Pi)_Y(\zeta) - (\nabla^M_Y\Pi)_X(\zeta) \\ &+ \Pi_{T^M(X,Y)}(\zeta) + \Pi_X(B_Y(Z)) - \Pi_Y(B_X(Z)) \\ &- B_k(X,Y)(A^k(Z) + \theta^k(\zeta)), \end{aligned} \tag{2.82}$$

$$\begin{aligned} pr_2(R(X,Y)(Z+\zeta)) &= R^M(X,Y)\zeta + (\nabla^M_X B)_Y(Z) - (\nabla^M_Y B)_X(Z) \\ &+ B_{T^M(X,Y)}(Z) + B_X(\Pi_Y(\zeta)) - B_Y(\Pi_X(\zeta)) \\ &- B_k(X,Y)(C^k(Z) - (A^k)^T(\zeta)), \end{aligned} \tag{2.83}$$

$$\begin{aligned} pr_1(R(\xi,\eta)(Z+\zeta)) &= (A_\xi A_\eta + \theta_\xi C_\eta)(Z) + (A_\xi\theta_\eta - \theta_\xi A^T_\eta)(\zeta) - (\xi \leftrightarrow \eta) \\ &- \Pi_k(\xi,\eta)(A^k(Z) + \theta^k(\zeta)), \end{aligned} \tag{2.84}$$

$$\begin{aligned} pr_2(R(\xi,\eta)(Z+\zeta)) &= (C_\xi A_\eta - A^T_\xi C_\eta)(Z) + (C_\xi\theta_\eta + A^T_\xi A^T_\eta)(\zeta) - (\xi \leftrightarrow \eta) \\ &- \Pi_k(\xi,\eta)(C^k(Z) - (A^k)^T(\zeta)), \end{aligned} \tag{2.85}$$

$$\begin{aligned} pr_1(R(X,\eta)(Z+\zeta)) &= (\nabla^M_X A)_\eta(Z) + A_{\langle\eta, T^M(\cdot,X)\rangle}(Z) \\ &+ (\nabla^M_X\theta)_\eta(\zeta) + \theta_{\langle\eta,T^M(\cdot,X)\rangle}(\zeta) \\ &+ (\Pi_X C_\eta - \theta_\eta B_X)(Z) - (\Pi_X A^T_\eta + A_\eta\Pi_X)(\zeta), \end{aligned} \tag{2.86}$$

$$\begin{aligned} pr_2(R(X,\eta)(Z+\zeta)) &= (\nabla^M_X C)_\eta(Z) + C_{\langle\eta,T(\cdot,X)\rangle}(Z) \\ &- (\nabla^M_X A)^T_\eta(\zeta) - A^T_{\langle\eta,T(\cdot,X)\rangle}(\zeta) \end{aligned} \tag{2.87}$$

$$+ (B_X A_\eta + A^T_\eta B_X)(Z) + (B_X\theta_\eta - C_\eta\Pi_X)(\zeta). \tag{2.88}$$

By $(\nabla^M_X\Pi)_Y$ we mean the following. Bivector Π_Y depends $C^\infty(M)$-linearly on Y, and thus defines a tensor $\Pi \in \mathcal{T}^2_1(M)$. One can then calculate its covariant derivative $(\nabla^M_X\Pi) \in \mathcal{T}^2_1(M)$, and finally for each $Y \in \mathfrak{X}(M)$ the tensor $(\nabla_X\Pi)_Y \in \mathcal{T}^2_0(M)$. Similarly for the other objects. T^M denotes the torsion operator of the connection ∇^M.

Chapter 3

Excerpts from the standard generalized geometry

In this chapter, we will recall some basic facts about the standard generalized geometry, that is the geometry of the vector bundle $E = TM \oplus T^*M$. We will focus only on the topics relevant for the following chapters (including the papers). In the previous chapter, we have shown that E is equipped with the Courant algebroid bracket (2.14), and the natural pairing $\langle \cdot, \cdot \rangle_E$. Note that E is sometimes called the *generalized tangent bundle*. Pioneering works in generalized geometry are those of Hitchin [43,45] and especially the Ph.D. thesis of Gualtieri [39]. We will focus on a very detailed explicit analysis of the involved objects, which eventually will prove to be useful for physical applications. In the section describing the generalized metric, we have used the approach taken in [69].

3.1 Orthogonal group

First assume that V is an n-dimensional real vector space. The direct sum $W = V \oplus V^*$ is equipped with the canonical pairing $\langle \cdot, \cdot \rangle_W$, which defines a symmetric non-degenerate bilinear form on W. If $\mathcal{E} = (e_i)_{i=1}^n$ is any basis of V, there is a canonical basis $(e_1, \dots, e_n, e^1, \dots, e^n)$ of W, where $(e^i)_{i=1}^n$ is the basis of V^* dual to $\mathcal{E}$. In this basis, the pairing $\langle \cdot, \cdot \rangle_W$ has the matrix

$$g_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

where 1 denotes the $n \times n$ unit matrix. This proves that $\langle \cdot, \cdot \rangle_W$ has the signature (n, n). The group of operators on W preserving $\langle \cdot, \cdot \rangle_W$ is thus $O(n, n)$. Let $\mathcal{O} \in O(n, n)$. We will often use the formal block decomposition of linear maps on W, that is we will write

$$\mathcal{O} = \begin{pmatrix} O_1 & O_2 \\ O_3 & O_4 \end{pmatrix}, \tag{3.2}$$

where $O_1 \in \text{End}(V)$, $O_4 \in \text{End}(V^*)$ and $O_2 \in \text{Hom}(V^*, V)$, $O_3 \in \text{Hom}(V, V^*)$. The matrix g_W can be thus also viewed as a formal block decomposition of the isomorphism $g_W : V \oplus V^* \to V^* \oplus V$ induced by $\langle \cdot, \cdot \rangle_W$. The orthogonality condition

$$\langle \mathcal{O}(v + \alpha), \mathcal{O}(v' + \alpha') \rangle_W = \langle v + \alpha, v' + \alpha' \rangle \tag{3.3}$$

can be now rewritten in terms of O_i by expanding the 2×2 block matrix equation $\mathcal{O}^T g_W \mathcal{O} = g_W$. One obtains a set of three relations:

$$O_3^T O_1 + O_1^T O_3 = 0 \tag{3.4}$$
$$O_4^T O_2 + O_2^T O_4 = 0 \tag{3.5}$$
$$O_3^T O_2 + O_1^T O_4 = 1 \tag{3.6}$$

We can get more equations. First note that $\mathcal{O}^{-1} = g_W \mathcal{O} g_W$, which explicitly gives

$$\mathcal{O}^{-1} = \begin{pmatrix} O_4^T & O_2^T \\ O_3^T & O_1^T \end{pmatrix}. \tag{3.7}$$

The map $\mathcal{O}^{-1}$ is again orthogonal, there holds $\mathcal{O}^{-T} g_W \mathcal{O}^{-1} = g_W$. We have three more (of course not independent) equations:

$$O_4 O_3^T + O_3 O_4^T = 0, \tag{3.8}$$
$$O_2 O_1^T + O_1 O_2^T = 0, \tag{3.9}$$
$$O_2 O_3^T + O_1 O_4^T = 1. \tag{3.10}$$

We will now focus on the maps of the form $\mathcal{O} = \exp \mathcal{A}$, where $\mathcal{A} \in o(n,n)$. Lie algebra $o(n,n)$ is defined as the space of linear endomorphisms of W, which are skew-symmetric with respect to $\langle \cdot, \cdot \rangle_W$. Thus, every $\mathcal{A} \in o(n,n)$ thus has to satisfy the condition

$$\langle \mathcal{A}(v + \alpha), v' + \alpha' \rangle_W + \langle v + \alpha, \mathcal{A}(v' + \alpha') \rangle_W = 0, \tag{3.11}$$

for all $v + \alpha, v' + \alpha' \in W$. Let us write $\mathcal{A}$ in a formal block matrix form

$$\mathcal{A} = \begin{pmatrix} N & \Pi \\ B & N' \end{pmatrix}, \tag{3.12}$$

where $N \in \operatorname{End}(V)$, $N' \in \operatorname{End}(V^*)$, $\Pi \in \operatorname{Hom}(V^*, V)$ and $B \in \operatorname{Hom}(V, V^*)$. Plugging into (3.11) gives a block matrix equation $\mathcal{A}^T g_W + g_W \mathcal{A} = 0$, expansion of which yields a set of three equations

$$B + B^T = 0, \ \Pi + \Pi^T = 0, \ N' + N^T = 0. \tag{3.13}$$

This gives an easy way to interpret the conditions for the respective blocks. We see that map B has to be induced by a 2-form $B \in \Lambda^2 V^*$, Π by a bivector $\Pi \in \Lambda^2 V$, and $N' = -N^T$. The conclusion is that (as a vector space) the Lie algebra $o(n,n)$ can be decomposed as

$$o(n,n) \cong \operatorname{End}(V) \oplus \Lambda^2 V^* \oplus \Lambda^2 V, \tag{3.14}$$

where each $\mathcal{A} \in o(n,n)$ has a block form

$$\mathcal{A} = \begin{pmatrix} N & \Pi \\ B & -N^T \end{pmatrix} \tag{3.15}$$

for $N \in \operatorname{End}(V)$, $B \in \Lambda^2 V^*$ and $\Pi \in \Lambda^2 V$. Note that this simply reflects the fact that for any two finite-dimensional vector spaces V, W, one has $\Lambda^2(V \oplus W) \cong \bigoplus_{i=0}^2 \Lambda^i V \otimes \Lambda^{2-i} W$. We can now proceed with the examples of $O(n,n)$ transformations.

Example 3.1.1. Let us now show three main classes of $O(n,n)$ transformations.

1. **B-transform**: Choose $N = \Pi = 0$ in $\mathcal{A}$ of the form (3.15). Denote by e^B its exponential, that is $e^B = \exp \mathcal{A}$. Explicitly,
$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \tag{3.16}$$
As a map, it has the form $e^B(v+\alpha) = (v, \alpha + B(v))$, for all $v + \alpha \in W$. By construction, $e^B \in O(n,n)$. Below, it will play an important role in relation to the Dorfman bracket.

2. **Π-transform**: Now, choose $\mathcal{A}$ so that $N = B = 0$. Denote by e^Π its exponential, that is $e^\Pi = \exp \mathcal{A}$. Explicitly,
$$e^\Pi = \begin{pmatrix} 1 & \Pi \\ 0 & 1 \end{pmatrix}. \tag{3.17}$$
As a map, it has the form $e^\Pi(v+\alpha) = (v + \Pi(\alpha), \alpha)$, for all $v + \alpha \in W$. It will play an important role in the description of (Nambu-)Poisson structures.

3. **Group** $\mathrm{Aut}(V)$: Every invertible map $A \in \mathrm{Aut}(V)$ defines an $O(n,n)$ transformation $\mathcal{O}_A$ in the form
$$\mathcal{O}_A = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}. \tag{3.18}$$
As a map, it works as $\mathcal{O}_A(v+\alpha) = A(v) + A^{-T}(\alpha)$.

$O(n,n)$ as a Lie group has four connected components, and the above three examples generate its identity component. We will often make use of the following simple observation:

Lemma 3.1.2. *Let $\mathcal{M}$ be a block 2×2 matrix in the form*
$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.19}$$
The matrices A and D have to be square, but can be of different dimensions. Then

- *If A is invertible, there exists a unique decomposition*
$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix} \tag{3.20}$$
of $\mathcal{M}$ into a product of block lower unitriangular, block diagonal, and block upper unitriangular matrices.

- *If D is invertible, there exists a unique decomposition*
$$\mathcal{M} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} \tag{3.21}$$
of $\mathcal{M}$ into a product of block upper unitriangular, block diagonal, and block lower unitriangular matrices.

- *If $\mathcal{M}$ is invertible, and there exists some decomposition of $\mathcal{M}$ into a product of block lower unitriangular, block diagonal, and block upper unitriangular matrices, then A is invertible, and the decomposition is precisely of the form (3.20).*

- *If $\mathcal{M}$ is invertible, and there exists some decomposition of $\mathcal{M}$ onto a product of block upper unitriangular, block diagonal, and block lower unitriangular matrices, then D is invertible, and the decomposition is precisely of the form (3.21).*

Proof. If A is invertible, we can construct the right-hand side of (3.20) and verify by direct calculation that the product gives $\mathcal{M}$. Now assume that

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ U & 1 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix} \tag{3.22}$$

for some matrices U, V, S, T. Expanding the right-hand side gives

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} S & SV \\ US & T + USV \end{pmatrix}.$$

This shows that $S = A$, and since A is invertible, we get $U = CA^{-1}$, $V = A^{-1}B$ and $T = D - CA^{-1}B$, which are exactly the blocks in (3.20). This proves the uniqueness assertion.

The proof of the invertible D case is analogous.

Now assume that $\mathcal{M}$ is invertible and there exists some its decomposition of the form (3.22). We see that $\det \mathcal{M} = \det S \cdot \det T$, which forces both S and T to be invertible. But we have shown that S has to be A, and thus A is invertible, and we have shown that the blocks U, V, T are uniquely determined by $\mathcal{M}$. The proof for the other decomposition is analogous. ∎

Consider now a general orthogonal transformation $\mathcal{O}$, parametrized as in (3.2). If one assumes that either O_1 or O_4 is invertible, there always exists one of the decompositions in Lemma 3.1.2. Are the maps in the decomposition orthogonal? The answer is given by the following proposition.

Proposition 3.1.3. *Let $\mathcal{O} \in O(n,n)$ be parametrized as in (3.2).*

- *Let $O_1 \in \mathrm{End}(TM)$ be an invertible map. Then there exist $B \in \Lambda^2 V^*$ and $\Pi \in \Lambda^2 V$, such that*

  $$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} O_1 & 0 \\ 0 & O_1^{-T} \end{pmatrix} \begin{pmatrix} 1 & \Pi \\ 0 & 1 \end{pmatrix}. \tag{3.23}$$

  Moreover, any such B and Π are unique.

- *Let $O_4 \in \mathrm{End}(T^*M)$ be an invertible map. Then there exist $B' \in \Lambda^2 V^*$ and $\Pi' \in \Lambda^2 V$, such that*

  $$\mathcal{O} = \begin{pmatrix} 1 & \Pi' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O_4^{-T} & 0 \\ 0 & O_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B' & 1 \end{pmatrix}. \tag{3.24}$$

  Moreover, any such B' and Π' are unique.

Proof. Let O_1 be an invertible map. By Lemma 3.1.2, there exists a decomposition (3.20), and we get $B = O_3 O_1^{-1}$, $A = O_1$, and $\Pi = O_1^{-1} O_2$. We have to show that B is induced by a 2-form on V, that is $B + B^T = 0$. This reduces to $O_3 O_1^{-1} + O_1^{-T} O_3^T = 0$. Multiply this equation by O_1 from the right, and by O_1^T from the left. This gives $O_1^T O_3 + O_3^T O_1 = 0$. But this is exactly the equation (3.4). To show that $\Pi \in \Lambda^2 V$, we are required to prove that $O_1^{-1} O_2 + O_2^T O_1^{-T} = 0$. This reduces precisely to (3.9). At this point we know that

$$\mathcal{O} = e^B \begin{pmatrix} O_1 & 0 \\ 0 & O_4 - O_3 O_1^{-1} O_2 \end{pmatrix} e^{\Pi}.$$

Because e^B and e^Π are in $O(n,n)$, so has to be the middle block. This requires $O_4 - O_3 O_1^{-1} O_2 = O_1^{-T}$. The proof of the second part is analogous. ■

Note that there are elements of $O(n,n)$ which can be decomposed in both ways. However, not every orthogonal map, not even from the identity component of $O(n,n)$, can be written in this form. Consider for example $n = 2$ and $\mathcal{O} = e^B e^\Pi e^{-B} e^{2\Pi}$, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \Pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.25}$$

The resulting orthogonal map has the form

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.26}$$

It is a product of exponentials, hence it lies in the identity component of $O(n,n)$. On the other hand, it clearly cannot be decomposed in any way of Proposition 3.1.3.

3.2 Maximally isotropic subspaces

Having the pairing $\langle \cdot, \cdot \rangle_W$ with the signature (n,n), it is natural to study its isotropic subspaces. We say that subspace $P \subseteq W$ is isotropic, iff $\langle p, q \rangle = 0$ for all $p, q \in P$. In particular, we will be interested in *maximally isotropic* subspaces. Let us recall a well-known fact from the theory of quadratic forms. For the proof, see for example [70]. Note that maximally isotropic subspaces are sometimes called Lagrangian subspaces.

Lemma 3.2.1. *All maximally isotropic subspaces of* $(W, \langle \cdot, \cdot \rangle_W)$ *are* n*-dimensional.*

Because $\langle \cdot, \cdot \rangle_W$ is induced by the canonical pairing, there are two obvious maximally isotropic subspaces, namely V and V^*, viewed as subspaces of W. By definition, every orthogonal transformation $\mathcal{O} \in O(n,n)$ applied on V or V^* induces an isotropic subspace.

Example 3.2.2. Let us recall some standard examples of maximally isotropic subspaces of $(W, \langle \cdot, \cdot \rangle_W)$.

- Let $B \in \Lambda^2 V^*$, and define the subspace $G_B := e^B(V)$. Explicitly,

$$G_B = \{ v + B(v) \mid v \in V \} \subseteq V \oplus V^*. \tag{3.27}$$

  We can thus view G_B as a graph of the linear map $B \in \mathrm{Hom}(V, V^*)$. Conversely, let $B \in \mathrm{Hom}(V, V^*)$ be any linear map. One can always construct the subspace (3.27). It is always an n-dimensional subspace of W. One readily checks that G_B is isotropic if and only if $B \in \Lambda^2 V^*$.

- Let $\Pi \in \Lambda^2 V$, and define the subspace $G_\Pi := e^\Pi(V^*)$. Explicitly,

$$G_\Pi = \{ \alpha + \Pi(\alpha) \mid \alpha \in V^* \} \subseteq V \oplus V^*. \tag{3.28}$$

  We can thus view G_Π as a graph of the map $\Pi \in \mathrm{Hom}(V^*, V)$. Conversely, let $\Pi \in \mathrm{Hom}(V^*, V)$ be any linear map. Onc can always construct the subspace (3.28). It is always an n-dimensional subspace of W. One readily checks that G_Π is isotropic if and only if $\Pi \in \Lambda^2 V$.

- Let $\Delta \subseteq V$ be any subspace of V. Let $\mathrm{Ann}(\Delta) \subseteq V^*$ be the annihilator subspace of V^*, that is the vector space defined as
$$\mathrm{Ann}(\Delta) = \{\alpha \in V^* \mid \forall v \in \Delta,\ \alpha(v) = 0\}. \tag{3.29}$$
Then $\Delta \oplus \mathrm{Ann}(\Delta) \subseteq W$ forms a maximally isotropic subspace.

- Let $E \subseteq V$ be a vector subspace of V, and let $\theta \in \Lambda^2 E^*$. Define the vector space
$$L(E, \theta) = \{v + \alpha \in V \oplus V^* \mid v \in E, \text{ and } \alpha = \theta(v)\} \tag{3.30}$$
Then $L(E, \theta)$ is a maximally isotropic subspace. Moreover, every maximally isotropic subspace is of this form for some E and θ, see [39].

3.3 Vector bundle, extended group and Lie algebra

We can generalize everything from the previous two sections to the vector bundle $E = TM \oplus T^*M$. Subspaces will be replaced by subbundles, and linear maps are promoted to vector bundle morphisms over the identity map on M. For example, we define
$$O(n, n) = \{\mathcal{F} \in \mathrm{Aut}(E) \mid \langle \mathcal{F}(e), \mathcal{F}(e')\rangle_E = \langle e, e'\rangle_E \text{ for all } e \in \Gamma(E)\}. \tag{3.31}$$
Similarly for the orthogonal Lie algebra:
$$o(n, n) = \{\mathcal{F} \in \mathrm{End}(E) \mid \langle \mathcal{F}(e), e'\rangle_E + \langle e, \mathcal{F}(e')\rangle_E = 0 \text{ for all } e \in \Gamma(E)\}. \tag{3.32}$$
By a direct generalization of (3.14) we would arrive to
$$o(n, n) \cong \mathrm{End}(TM) \oplus \Omega^2(M) \oplus \mathfrak{X}^2(M). \tag{3.33}$$
We now have $O(n, n)$ transformations of the form of B-transforms, Π-transforms and $\mathrm{Aut}(TM)$ at our disposal, for $B \in \Omega^2(M)$ and $\Pi \in \mathfrak{X}^2(M)$. Finally, instead of maximally isotropic subspaces, we will talk about maximally isotropic subbundles of E. All examples from the previous subsection generalize naturally.

Of course, we can study also slightly more general objects. In particular, define extended automorphism group $\mathrm{EAut}(E)$ of E to be a group of fiber-wise bijective vector bundle morphisms over diffeomorphisms. Note that any $(\mathcal{F}, \varphi)$, where $\varphi \in \mathrm{Diff}(M)$, induces an automorphism $\mathcal{F}$ (denoted by the same letter) of $\Gamma(E)$. Indeed, let $e \in \Gamma(E)$. Define $\mathcal{F}(e) \in \Gamma(E)$ as $(\mathcal{F}(e))(\varphi(m)) = \mathcal{F}(e(m))$ for all $m \in M$.

Using this notation, we can define the extended orthogonal group $EO(n, n)$ as
$$EO(n, n) = \{(\mathcal{F}, \varphi) \in \mathrm{EAut}(E) \mid \langle \mathcal{F}(e), \mathcal{F}(e')\rangle_E \circ \varphi = \langle e, e'\rangle_E\}. \tag{3.34}$$
Its structure is in fact very simple, as the following lemma proves.

Lemma 3.3.1. *Let* $\mathrm{Diff}(M)$ *be the group of diffeomorphisms of* M*. Then*
$$EO(n, n) = O(n, n) \rtimes \mathrm{Diff}(M), \tag{3.35}$$
where $\mathrm{Diff}(M)$ *acts on* $O(n, n)$ *by conjugation:* $\varphi \blacktriangleright \mathcal{F}_0 := T(\varphi) \circ \mathcal{F}_0 \circ T(\varphi)^{-1}$*, for all* $\varphi \in \mathrm{Diff}(M)$*. Define the map* $(T(\varphi), \varphi) \in \mathrm{EAut}(E)$ *by putting* $T(\varphi)(X + \xi) = \varphi_*(X) + (\varphi^{-1})^*(\xi)$*, for all* $X + \xi \in \Gamma(E)$*.*

Proof. Let $(F, \varphi) \in EO(n,n)$. It is not difficult to show that $(T(\varphi), \varphi) \in EO(n,n)$. Define $\mathcal{F}_0 \in \mathrm{Aut}(A)$ as $\mathcal{F} = \mathcal{F}_0 \circ T(\varphi)$. Then, by definition, $\mathcal{F}_0 \in O(n,n)$. Moreover, $O(n,n)$ forms a normal subgroup of $EO(n,n)$. It only remains to determine the multiplication rule. Let $(\mathcal{G}, \psi) \in EO(n,n)$ and $\mathcal{G} = \mathcal{G}_0 \circ T(\psi)$. We get

$$\mathcal{F} \circ \mathcal{G} = \mathcal{F}_0 \circ [T(\varphi) \circ \mathcal{G}_0 \circ T(\varphi)^{-1}] \circ T(\varphi \circ \psi).$$

This proves the semi-direct structure assertion (3.35). ■

What is the Lie algebra $Eo(n,n)$ corresponding to the group $EO(n,n)$? First, recall the vector bundle $\mathcal{D}(E)$ defined in Remark 2.4.2. For any Lie algebroid $(L, l, [\cdot,\cdot]_L)$, any vector bundle morphism $R : L \rightarrow \mathcal{D}(E)$ preserving the brackets is called a *representation* of Lie algebroid L on the vector bundle E. See [74] for details.

We claim that $\Gamma(\mathcal{D}(E))$ is exactly the Lie algebra corresponding to $\mathrm{EAut}(E)$. To see this, assume that $(\mathcal{F}_t, \varphi_t)$ is a 1-parameter subgroup of automorphisms in $\mathrm{EAut}(E)$. In particular, φ_t is a 1-parameter subgroup of $\mathrm{Diff}(M)$, hence a flow of some vector field $X \in \mathfrak{X}(M)$. Define $\mathcal{F} : \Gamma(E) \rightarrow \Gamma(E)$ as $\mathcal{F}(e) := \frac{d}{dt}\big|_{t=0} \mathcal{F}_{-t}(e)$ for all $e \in \Gamma(E)$. Note that for $f \in C^\infty(M)$, we have $\mathcal{F}_{-t}(fe) = (f \circ \varphi_t)\mathcal{F}_{-t}(e)$. Differentiating this condition with respect to t at $t = 0$ gives

$$\mathcal{F}(fe) = f\mathcal{F}(e) + [\frac{d}{dt}\bigg|_{t=0} f(\varphi_t)]e = f\mathcal{F}(e) + (X.f)e. \tag{3.36}$$

This proves that $\mathcal{F} \in \Gamma(\mathcal{D}(E))$, and moreover $a(\mathcal{F}) = \frac{d}{dt}\big|_{t=0} \varphi_t$. We can also write the relation of $\mathcal{F}$ and $\mathcal{F}_t$ as $\exp{(t\mathcal{F})} := \mathcal{F}_{-t}$.

Let us now return to the Lie algebra $Eo(n,n)$. Let $(\mathcal{F}_t, \varphi_t)$ be a 1-parameter subgroup of $EO(n,n)$. The corresponding element of $Eo(n,n)$ will be $\mathcal{F} = \frac{d}{dt}\big|_{t=0} \mathcal{F}_{-t}$. Since $\mathcal{F}_t \in EO(n,n)$, we have

$$\langle \mathcal{F}_{-t}(e), \mathcal{F}_{-t}(e) \rangle_E = \langle e, e' \rangle_E \circ \varphi_t. \tag{3.37}$$

Differentiating this with respect to t at $t = 0$ leads us to the definition

$$Eo(n,n) := \{ \mathcal{F} \in \Gamma(\mathcal{D}(E)) \mid a(\mathcal{F}).\langle e, e' \rangle_E = \langle \mathcal{F}(e), e' \rangle_E + \langle e, \mathcal{F}(e') \rangle_E, \ \forall e, e' \in \Gamma(E) \}. \tag{3.38}$$

Recall that $(\mathcal{D}(E), a, [\cdot,\cdot])$ is the Lie algebroid defined in Remark 2.4.2. Lemma 3.3.1 suggests that $Eo(n,n)$ can be also written as a semi-direct product, this time of Lie algebras. Before proceeding to the lemma, note that there is a Lie algebroid representation $R : TM \rightarrow \mathcal{D}(E)$ of the Lie algebroid $(TM, Id_M, [\cdot,\cdot])$ which takes values in $Eo(n,n)$. Indeed, let $X \in \mathfrak{X}(M)$. Define $R(X) \in \Gamma(\mathcal{D}(E))$ as $R(X)(e) = [X + 0, e]_D$. It follows from (2.26) that $R(X) \in Eo(n,n)$. We can now state

Lemma 3.3.2. *Lie algebra $Eo(n,n)$ can be decomposed as*

$$Eo(n,n) = \mathfrak{X}(M) \ltimes o(n,n), \tag{3.39}$$

where $\mathfrak{X}(M)$ acts on $o(n,n)$ by Lie derivatives.

Proof. Let $\mathcal{F} \in Eo(n,n)$. Define $\mathcal{F}_0 \in o(n,n)$ as $\mathcal{F} = R(a(\mathcal{F})) + \mathcal{F}_0$. This proves the assertion on the level of vector spaces. First note that $[R(X), R(Y)] = R([X,Y])$ because R is a Lie algebroid representation. From (3.33) we see that every $\mathcal{F}_0 \in o(n,n)$ corresponds to a triplet $(N, B, \Pi) \in \mathrm{End}(TM) \oplus \Omega^2(M) \oplus \mathfrak{X}^2(M)$. Going through the construction of this correspondence above (3.14) it is straightforward to show that if $\mathcal{F}_0 \approx (N, B, \Pi)$, then $[R(X), \mathcal{F}_0] \approx (\mathcal{L}_X N, \mathcal{L}_X B, \mathcal{L}_X \Pi)$ where N is viewed as $(1,1)$-tensor on M. This proves the assertion (3.39) on the level of Lie algebras. ■

3.4 Derivations algebra of the Dorfman bracket

Let us now focus on the Dorfman bracket (2.14). It satisfies the Leibniz rule (2.4) in the right input, and Courant algebroid induced Leibniz rule (2.30) in the left input. We will now examine the Lie algebra $\mathrm{Der}(E)$ of its derivations defined as

$$\mathrm{Der}(E) = \{\mathcal{F} \in \Gamma(\mathcal{D}(E)) \mid \mathcal{F}([e,e']_D) = [\mathcal{F}(e),e']_D + [e,\mathcal{F}(e')]_D,\ \forall e,e' \in \Gamma(E)\}. \tag{3.40}$$

Let us emphasize that $\mathrm{Der}(E)$ is not a $C^\infty(M)$-module. Recall the map R defined just before Lemma 3.3.2. It follows from the Leibniz identity (2.5) that for every $X \in \mathfrak{X}(M)$, we have $R(X) \in \mathrm{Der}(E)$. Now observe that any $\mathcal{F} \in \mathrm{Der}(E)$ can be decomposed as

$$\mathcal{F} = R(a(\mathcal{F})) + \mathcal{F}_0. \tag{3.41}$$

Note that $\mathcal{F}_0$ is now $C^\infty(M)$-linear, or equivalently $\mathcal{F}_0 \in \mathrm{End}(E)$. Moreover, it is a difference of two derivations, hence itself a derivation. We can now focus on finding all $\mathcal{F}_0 \in \mathrm{Der}(E) \cap \mathrm{End}(E)$. First, there are now certain restrictions forced by the compatibility of the Leibniz rule (2.4) and the derivation property (3.40). Indeed, evaluating the derivation $\mathcal{F}_0$ on $[e, fe']$ in two ways, we obtain

$$\rho(\mathcal{F}_0(e)) = 0, \tag{3.42}$$

for all $e \in \Gamma(E)$. This shows that $\mathcal{F}_0$ must have a formal block form

$$\mathcal{F}_0 = \begin{pmatrix} 0 & 0 \\ F_{21} & F_{22} \end{pmatrix}, \tag{3.43}$$

where $F_{21} \in \mathrm{Hom}(TM, T^*M)$ and $F_{22} \in \mathrm{End}(T^*M)$. Next, there comes the compatibility with the left Leibniz rule (2.30). We obtain the condition

$$\langle e,e'\rangle_E \mathcal{F}_0(\mathcal{D}f) = \{\langle \mathcal{F}_0(e),e'\rangle_E + \langle e, \mathcal{F}_0(e')\rangle_E\}\mathcal{D}f, \tag{3.44}$$

which has to hold for all $e,e' \in \Gamma(E)$ and $f \in C^\infty(M)$. In particular, for $\langle e,e'\rangle_E = 0$ this implies $\langle \mathcal{F}_0(e),e'\rangle_E + \langle e,\mathcal{F}_0(e')\rangle_E = 0$. This immediately implies that $\langle F_{21}(X),Y\rangle + \langle X, F_{21}(Y)\rangle = 0$, and thus $F_{21}(X) = B(X)$ for $B \in \Omega^2(M)$. Choosing $e = X \in \mathfrak{X}(M)$ and $e' = \xi \in \Omega(M)$, we get

$$\langle \xi, X\rangle F_{22}(df) = \langle F_{22}(\xi), X\rangle df. \tag{3.45}$$

This has to hold for any (f, X, ξ), which is possible only if $F_{22}(\xi) = \lambda\xi$ for some $\lambda \in C^\infty(M)$. We see that $\mathcal{F}_0$ has to have the form

$$\mathcal{F}_0 = \begin{pmatrix} 0 & 0 \\ B & \lambda \cdot 1 \end{pmatrix}. \tag{3.46}$$

It remains to plug this into condition (3.40) to find the conditions on B and λ. We have

$$\mathcal{F}_0[X+\xi, Y+\eta]_D = B([X,Y]) + \lambda(\mathcal{L}_X\eta - i_Y d\xi), \tag{3.47}$$

$$[\mathcal{F}_0(X+\xi), Y+\eta]_D = -i_Y d(B(X) + \lambda\xi), \tag{3.48}$$

$$[X+\xi, \mathcal{F}_0(Y+\eta)]_D = \mathcal{L}_X(B(Y) + \lambda\eta). \tag{3.49}$$

Inserting this into (3.40) yields two independent equations

$$B([X,Y]) = \mathcal{L}_X(B(Y)) - i_Y d(B(X)), \tag{3.50}$$

$$\lambda \mathcal{L}_X \eta = \mathcal{L}_X(\lambda\eta). \tag{3.51}$$

Recall that $B(X) = -i_X B$. We can use the usual Cartan formulas to rewrite (3.50) as a condition $dB = 0$, that is $B \in \Omega^2_{closed}(M)$. Second equation forces λ to be locally constant, that is $\lambda \in \Omega^0_{closed}(M)$. We have just proved the following proposition.

Proposition 3.4.1. *Let* $\mathrm{Der}(E)$ *be the space of derivations of the Dorfman bracket* $[\cdot,\cdot]_D$*, that is (3.40) holds. Then as a vector space, it decomposes as*

$$\mathrm{Der}(E) \doteq \mathfrak{X}(M) \oplus \Omega^0_{closed}(M) \oplus \Omega^2_{closed}(M). \tag{3.52}$$

Every $\mathcal{F} \in \mathrm{Der}(E)$ *decomposes uniquely as* $\mathcal{F} = R(X) + \mathcal{F}_\lambda + \mathcal{F}_B$*, where* $R(X)(Y+\eta) = ([X,Y], \mathcal{L}_X\eta)$ *for all* $X, Y \in \mathfrak{X}(M)$ *and* $\eta \in \Omega^1(M)$*. Vector bundle endomorphisms* $\mathcal{F}_\lambda$ *and* $\mathcal{F}_B$ *are defined as*

$$\mathcal{F}_\lambda = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \cdot 1 \end{pmatrix}, \; \mathcal{F}_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \tag{3.53}$$

where $\lambda \in \Omega^0_{closed}(M)$ *and* $B \in \Omega^2_{closed}(M)$*. Nontrivial commutation relations are*

$$[R(X), R(Y)] = R([X,Y]), \tag{3.54}$$

$$[R(X), F_B] = F_{\mathcal{L}_X B}, \tag{3.55}$$

$$[\mathcal{F}_\lambda, \mathcal{F}_B] = \mathcal{F}_{\lambda B}. \tag{3.56}$$

On the Lie algebra level, we thus have

$$\mathrm{Der}(E) = \mathfrak{X}(M) \ltimes (\Omega^0_{closed}(M) \ltimes \Omega^2_{closed}(M)), \tag{3.57}$$

where $\Omega^0_{closed}(M)$*,* $\Omega^2_{closed}(M)$ *are viewed as Abelian Lie algebras,* $\Omega^0_{closed}(M)$ *acts on 2-forms in* $\Omega^2_{closed}(M)$ *by multiplication, and* $\mathfrak{X}(M)$ *acts on* $\Omega^0_{closed} \ltimes \Omega^2_{closed}(M)$ *by Lie derivatives. Finally, when we restrict to the subalgebra* $Eo(n,n)$*, we have*

$$\mathrm{Der}(E) \cap Eo(n,n) = \mathfrak{X}(M) \ltimes \Omega^2_{closed}(M). \tag{3.58}$$

Proof. We have proved the first part in the text above. The commutation relations can be directly calculated. ■

3.5 Automorphism group of the Dorfman bracket

Let us now examine the group of Dorfman bracket automorphisms. Its subgroup of orthogonal automorphisms is well-known for a long time and it is in fact one of the main reasons why the Dorfman bracket and generalized geometry play such an important role in string theory. We roughly follow the proof of Gualtieri in [39]. From the Courant algebroid perspective is makes sense to restrict to $EO(n,n)$, because in this case one obtains an automorphism of the whole Courant algebroid structure. However, for the sake of generalization to Leibniz algebroids where there is no pairing anymore, we will discuss the whole automorphism group. We define the Dorfman bracket automorphism group $\mathrm{Aut}_D(E)$ as

$$\mathrm{Aut}_D(E) := \{(\mathcal{F}, \varphi) \in \mathrm{EAut}(E) \mid [\mathcal{F}(e), \mathcal{F}(e')]_D = \mathcal{F}[e,e']_D, \; \forall e, e' \in \Gamma(E)\}. \tag{3.59}$$

This group decomposes similarly as $EO(n,n)$ in Lemma 3.3.1. Indeed, let $(\mathcal{F}, \varphi)$ in $\mathrm{Aut}_D(E)$. Then recall the map $(T(\varphi), \varphi)$, defined as $T(\varphi)(X+\xi) = \varphi_*(X) + (\varphi^{-1})^*(\xi)$ for all $X+\xi \in \Gamma(E)$. It follows from the usual properties of the Lie derivative and the exterior differential that $(T(\varphi), \varphi) \in \mathrm{Aut}_D(E)$. Define $\mathcal{F}_0 \in \mathrm{Aut}(E)$ as $\mathcal{F} = \mathcal{F}_0 \circ T(\varphi)$. It follows that $\mathcal{F}_0 \in \mathrm{Aut}_D(E)$, and we can thus focus on finding all vector bundle morphisms $\mathcal{F}_0$ over the identity preserving the bracket.

First note that it follows from the compatibility of (3.59) and Leibniz rule (2.4) that its projection using ρ satisfies $\rho(\mathcal{F}_0(e)) = \rho(e)$ for all $e \in \Gamma(E)$. This shows that $\mathcal{F}_0$ has to be of the block form

$$\mathcal{F}_0 = \begin{pmatrix} 1 & 0 \\ F_{21} & F_{22} \end{pmatrix}. \tag{3.60}$$

The Leibniz rule compatibility in the left input (2.30) gives the condition

$$\langle e, e' \rangle_E \mathcal{F}_0(\mathcal{D}f) = \langle \mathcal{F}_0(e), \mathcal{F}_0(e') \rangle_E \mathcal{D}f, \tag{3.61}$$

for all $e, e' \in \Gamma(E)$. Very similarly to the equation (3.44), this proves that $F_{21}(X) = B(X)$ for $B \in \Omega^2(M)$, and $F_{22}(\xi) = \lambda \cdot \xi$ for some $\lambda \in C^\infty(M)$. We require $\mathcal{F}_0$ to be fiber-wise bijective, and thus $\lambda(m) \neq 0$ for all $m \in M$. This restricts $\mathcal{F}_0$ to have the block form

$$\mathcal{F}_0 = \begin{pmatrix} 1 & 0 \\ B & \lambda \cdot 1 \end{pmatrix}. \tag{3.62}$$

Finally, we have to plug $\mathcal{F}_0$ into the condition (3.59). We have

$$\mathcal{F}_0[X+\xi, Y+\eta]_D = [X,Y] + B([X,Y]) + \lambda\{\mathcal{L}_X \eta - i_Y d\xi\}, \tag{3.63}$$

$$[\mathcal{F}_0(X+\xi), \mathcal{F}_0(Y+\eta)]_D = [X,Y] + \mathcal{L}_X(B(Y) + \lambda\eta) - i_Y d(B(X) + \lambda\xi). \tag{3.64}$$

Combining these two expressions gives the same conditions on B and λ as (3.40). We thus get $B \in \Omega^2_{closed}(M)$, and $\lambda \in \Omega^0_{closed}(M)$. Let $G(\Omega^0_{closed}(M))$ denote the Abelian group of everywhere non-zero closed 0-forms on M. Note that $G(\Omega^0_{closed}(M))$ is in fact isomorphic to a direct product of k copies of $(\mathbb{R} \setminus \{0\}, \cdot)$, where k is a number of connected components of M. We have just proved the following proposition:

Proposition 3.5.1. *Let* $\mathrm{Aut}_D(E)$ *be the group of automorphisms (3.59) of the Dorfman bracket (2.14). Then it has the following group structure:*

$$\mathrm{Aut}_D(E) = (\Omega^2_{closed}(M) \rtimes G(\Omega^0_{closed}(M))) \rtimes \mathrm{Diff}(M), \tag{3.65}$$

where $\Omega^2_{closed}(M)$ *is viewed as an Abelian group with respect to addition,* $G(\Omega^0_{closed}(M))$ *acts on* $\Omega^2_{closed}(M)$ *by multiplication, and* $\mathrm{Diff}(M)$ *acts on* $\Omega^2_{closed}(M) \rtimes G(\Omega^0_{closed}(M))$ *by inverse pullbacks. Every* $(\mathcal{F}, \varphi) \in \mathrm{Aut}_D(E)$ *can be uniquely decomposed as*

$$\mathcal{F} = e^B \circ \mathcal{S}_\lambda \circ T(\varphi), \tag{3.66}$$

where $e^B = \exp \mathcal{F}_B$*, and* $S_\lambda(X+\xi) = X + \lambda\xi$*, for unique* $B \in \Omega^2_{closed}(M)$ *and locally constant everywhere non-zero function* $\lambda \in G(\Omega^0_{closed}(M))$*. Finally, the subgroup of* $Aut_D(E)$ *consisting of (extended) orthogonal transformations is*

$$\mathrm{Aut}_D(E) \cap EO(n,n) = \Omega^2_{closed} \rtimes \mathrm{Diff}(M). \tag{3.67}$$

Proof. Only the multiplication rules remain to be proved. Let $\mathcal{G} = e^{B'} \circ S_{\lambda'} \circ T(\varphi')$. By the direct calculation, one obtains

$$\mathcal{F} \circ \mathcal{G} = e^{B + \lambda(\varphi^{-1})^* B'} \circ S_{\lambda(\varphi^{-1})^*\lambda'} \circ T(\varphi \circ \varphi'). \tag{3.68}$$

Symbolically, this yields the multiplication rule

$$(B, \lambda, \varphi) * (B', \lambda', \varphi') = (B + \lambda(\varphi^{-1})^* B', \lambda(\varphi^{-1})^*\lambda', \varphi \circ \varphi'), \tag{3.69}$$

which is exactly the double semi-direct product (3.65). ∎

Finally, let us show that every Dorfman bracket derivation $\mathcal{F} \in \mathrm{Der}(E)$ can explicitly be integrated to a 1-parameter subgroup $\exp(t\mathcal{F}) \subseteq \mathrm{Aut}_D(E)$ of the group of Dorfman bracket automorphisms. Note that $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and in general, t cannot be expanded to $\mathbb{R}$. $\exp(t\mathcal{F})$ is thus a 1-parameter subgroup with "certain conditions on parameters".

We have shown in Proposition 3.4.1 that every $\mathcal{F} \in \mathrm{Der}\, E$ can uniquely be written as $\mathcal{F} = R(X) + F_\lambda + \mathcal{F}_B$ for $X \in \mathfrak{X}(M)$, $\lambda \in \Omega^0_{closed}(M)$, and $B \in \Omega^2_{closed}(M)$. Let ϕ^X_t be the flow corresponding to X, for $t \in (-\epsilon, \epsilon)$. There lies the reason of t limitations: X may not be a complete vector field. We will now look for $\exp(t\mathcal{F})$ in the form

$$\exp(t\mathcal{F}) = e^{B(t)} \circ S_{\mu(t)} \circ T(\phi^X_{-t}), \tag{3.70}$$

where $B(t) \in \Omega^2_{closed}(M)$, and $\mu(t) \in G(\Omega^0_{closed}(M))$ for every $t \in (-\epsilon, \epsilon)$. We have

$$\exp(t\mathcal{F})(Y + \eta) = \phi^X_{-t*}(Y) + B(t)(\phi^X_{-t*}(Y)) + \mu(t)\phi^{X*}_t(\eta), \tag{3.71}$$

for all $Y + \eta \in \Gamma(E)$. Differentiating with respect to t at $t = 0$ gives the condition

$$\mathcal{F}(Y + \eta) = (1 + B(0))[X, Y] + [\frac{d}{dt}\Big|_{t=0} B(t)](Y) + [\frac{d}{dt}\Big|_{t=0} \mu(t)]\eta + \mu(0)\mathcal{L}_x \eta. \tag{3.72}$$

Comparing this with our parametrization of $\mathcal{F}$ gives the conditions on $B(t)$, and $\mu(t)$:

$$\frac{d}{dt}\Big|_{t=0} \mu(t) = \lambda,\ \mu(0) = 1,\ \frac{d}{dt}\Big|_{t=0} B(t) = B,\ B(0) = 0. \tag{3.73}$$

First two conditions give $\mu(t) = \exp t\lambda$. To find the solution for B, we will use the 1-parameter subgroup property of $\exp(t\mathcal{F})$. Note that we have $\phi^{X*}_t(\lambda) = \lambda$. This follows from the fact that flows cannot flow outside of the single connected component. Using the multiplication rule (3.68), we get

$$\exp(t\mathcal{F}) \circ \exp(s\mathcal{F}) = e^{B(t) + \exp t\lambda \cdot \phi^{X*}_t(B(s))} \circ S_{e^{(t+s)\lambda}} \circ T(\phi^X_{-(t+s)}). \tag{3.74}$$

Comparing this to $\exp((t + s)\mathcal{F})$ gives the condition

$$B(t + s) = B(t) + \exp t\lambda \cdot \phi^{X*}_t(B(s)). \tag{3.75}$$

Differentiate both sides with respect to s at $s = 0$. This yields

$$\dot{B}(t) = \exp t\lambda \cdot \phi^{X*}_t B, \tag{3.76}$$

and consequently

$$B(t) = \int_0^t \{\exp k\lambda \cdot \phi^{X*}_k B\} dk. \tag{3.77}$$

It is straightforward to check that such $B(t)$ indeed satisfies (3.75) and the two initial conditions (3.73). We have thus made our way to the following proposition:

Proposition 3.5.2. *Let $\mathcal{F} \in \mathrm{Der}(E)$ be a derivation of the Dorfman bracket. Then there is an $\epsilon > 0$ and a 1-parameter subgroup $\exp(t\mathcal{F}) \subseteq \mathrm{Aut}_D(E)$, where $t \in (-\epsilon, \epsilon)$, such that $\mathcal{F} = \frac{d}{dt}\big|_{t=0} \exp(t\mathcal{F})$. Explicitly, if $\mathcal{F} = R(X) + \mathcal{F}_\lambda + \mathcal{F}_B$ for $X \in \mathfrak{X}(M)$, $\lambda \in \Omega^0_{closed}(M)$, and $B \in \Omega^2_{closed}(M)$, we have*

$$\exp(t\mathcal{F}) = e^{B(t)} \circ S_{\mu(t)} \circ T(\phi^X_{-t}), \tag{3.78}$$

where $\mu(t) = \exp t\lambda$, ϕ^X_t is the flow of X, and

$$B(t) = \int_0^t \{\exp k\lambda \cdot \phi^{X*}_k B\} dk. \tag{3.79}$$

Proof. We have shown above that $\exp(t\mathcal{F})$ integrates $\mathcal{F}$. We just have to show that for each t, $\exp(t\mathcal{F}) \in \mathrm{Aut}_D(E)$. According to Proposition 3.5.1, this happens if and only if $B(t) \in \Omega^2_{closed}(M)$, and $\mu(t) \in G(\Omega^0_{closed}(M))$. But this clearly holds because d commutes with the integration and pullbacks. ■

3.6 Twisting of the Dorfman bracket

We have shown in Proposition 3.5.1 that $\mathcal{F} \in \mathrm{Aut}(E)$ preserving the bracket must be of the form (3.60) for $B \in \Omega^2_{closed}(M)$ and $\lambda \in \Omega^0_{closed}(M)$. Let us now focus only on $O(n,n)$ transformations, and thus set $\lambda \equiv 1$. In this case simply $\mathcal{F} = e^B$. What happens with the bracket for $dB \neq 0$? This is what we will examine in this section. Define a new bracket $[\cdot,\cdot]'_D$ as

$$[e, e']'_D = e^{-B}[e^B(e), e^B(e')]_D, \tag{3.80}$$

for all $e, e' \in \Gamma(E)$. Rewriting this bracket explicitly gives

$$\begin{aligned} [X+\xi, Y+\eta]'_D &= [X,Y] + \mathcal{L}_X(\eta + B(Y)) - i_Y d(\xi + B(X)) - B([X,Y]) \\ &= [X+\xi, Y+\eta]_D + \mathcal{L}_X(B(Y)) - i_Y d(B(X)) - B([X,Y]) \\ &= [X+\xi, Y+\eta]_D - dB(X,Y,\cdot). \end{aligned} \tag{3.81}$$

This proves that $[\cdot,\cdot]'_D$ is precisely the H-twisted Dorfman bracket (2.32), where $H = dB$. One can in fact show something more general: Twisted Dorfman brackets corresponding to the different representatives of the same cohomology class $[H] \in H_3(M,\mathbb{R})$ are related precisely by a B-transform.

Proposition 3.6.1. *Let $H \in \Omega^3_{closed}(M)$, and $B \in \Omega^2(M)$. Then*

$$[e^B(e), e^B(e')]^H_D = e^B([e,e']^{H+dB}_D). \tag{3.82}$$

Proof. Just repeat the calculation (3.81). ■

3.7 Dirac structures

In Section 3.2, we have introduced maximally isotropic subspaces and their examples. Generalizing this to the vector bundle $E = TM \oplus T^*M$, we have an additional structure at our disposal, namely the Dorfman bracket. It is a well known fact that subbundles of TM which are involutive with respect to the vector field commutator bracket are of an additional geometrical significance - they are tangent bundles to integral submanifolds of distributions. This justifies why it is interesting to study the subbundles of E involutive with respect to the Dorfman bracket $[\cdot,\cdot]_D$. In particular, one is interested in involutive subbundles, where the skew-symmetry "anomaly" (2.28) disappears. This is precisely the main idea leading to the definition of Dirac structures. Let us remark that study of Dirac structures was in fact the origin of all Courant algebroid brackets, see [26].

Definition 3.7.1. Let $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ be a Courant algebroid. A subbundle $L \subseteq E$ is called an **almost Dirac structure**, if for all $e, e' \in \Gamma(L)$, we have $\langle e, e'\rangle_E = 0$.

An almost Dirac structure L is called a **Dirac structure**, if L is involutive with respect to $[\cdot,\cdot]_E$, that is $[e,e']_E \in \Gamma(L)$ for all $e, e' \in \Gamma(L)$.

Note that $[\cdot,\cdot]_E|_{\Gamma(L)\times\Gamma(L)}$ together with $\rho|_{\Gamma(L)}$ forms a Lie algebroid structure on L.

We can identify several examples of almost Dirac structures of $E = TM \oplus T^*M$ in Example 3.2.2, if we just replace subspaces with subbundles, and elements of exterior powers of vector spaces by sections of corresponding vector bundles. We will now show under which conditions these become Dirac structures.

Example 3.7.2. Let $E = TM \oplus T^*M$ be equipped with the Dorfman bracket (2.14).

- Let G_B be the graph (3.27) of a 2-form $B \in \Omega^2(M)$. We can examine the involutivity. Let $X + B(X)$, $Y + B(Y) \in \Gamma(G_B)$. Then

$$[X + B(X), Y + B(Y)]_D = [X, Y] + \mathcal{L}_X(B(Y)) - i_Y d(B(X)).$$

  We see that $[X + B(X), Y + B(Y)]_D \in \Gamma(G_B)$ iff

$$\mathcal{L}_X(B(Y)) - i_Y d(B(X)) = B([X, Y]), \tag{3.83}$$

  for all $X, Y \in \mathfrak{X}(M)$. This is once more the condition (3.50), equivalent to $dB = 0$. We conclude that G_B is a Dirac structure iff $B \in \Omega^2_{closed}(M)$.

- Let G_Π be a graph (3.28) of a bivector $\Pi \in \mathfrak{X}^2(M)$. Involutivity condition implies the equation

$$[\Pi(\xi), \Pi(\eta)] = \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi), \tag{3.84}$$

  for all $\xi, \eta \in \Omega^1(M)$. We will show in Chapter 5 that this is equivalent to the Jacobi identity for $\{f, g\} := \Pi(df, dg)$. We conclude that G_Π is a Dirac structure if and only if $\Pi \in \mathfrak{X}^2(M)$ is a Poisson bivector.

- Let $\Delta \subseteq TM$ be a subbundle (that is in fact a smooth distribution on M). Let $L = \Delta \oplus \mathrm{Ann}(\Delta) \subseteq E$. Examining the involutivity condition shows that L is a Dirac structure, iff $[\Delta, \Delta] \subseteq \Delta$, that is Δ is an integrable distribution.

3.8 Generalized metric

We will now introduce a key concept for the applications of generalized geometry in string theory. In the context of generalized geometry, it appeared first in [39]. The name generalized metric was probably used for the first time by Hitchin in [43]. Generalized metric has several equivalent formulations, which we all present here.

Definition 3.8.1. Let E be a vector bundle with a fiber-wise metric $\langle \cdot, \cdot \rangle_E$. Let $\tau \in \mathrm{End}(E)$ be an involution of E, that is $\tau^2 = 1$. We say that τ is a **generalized metric**, if the formula

$$\mathbf{G}_\tau(e, e') := \langle e, \tau(e') \rangle_E, \tag{3.85}$$

for all $e, e' \in \Gamma(E)$, defines a positive definite fiber-wise metric $\mathbf{G}_\tau$ on E. When talking about generalized metric, we will not distinguish between τ and $\mathbf{G}_\tau$.

There are some remarks to be made about τ. It follows from the definition of a generalized metric, that τ must be symmetric with respect to $\langle \cdot, \cdot \rangle_E$, and consequently also orthogonal.

There is one nice property of involutive maps.

Lemma 3.8.2. *Let V be a finite-dimensional real vector space, and $A \in \mathrm{End}(V)$ satisfies $A^2 = 1$. Then A has eigenvalues ± 1 and it is diagonalizable, that is $V = V_+ \oplus V_-$, and $A(v_+ + v_-) = v_+ - v_-$.*

Proof. Let $p(z) = z^2 - 1$. Then $p(A) = 0$, and a minimal polynomial m_A of A thus must divide p. For $m_A(z) = z \pm 1$, this would imply $A = \pm 1$. In all other cases $m_A(z) = (z+1)(z-1)$. This shows that all roots of m_A have multiplicity 1, which is equivalent to A being diagonalizable. Moreover, its roots are precisely the eigenvalues of A. ∎

We can now use this result to reformulate the definition of generalized metric in case when $\langle \cdot, \cdot \rangle_E$ has a constant signature (the same at each fiber).

Proposition 3.8.3. *Let E be a vector bundle with a fiber-wise metric $\langle \cdot, \cdot \rangle_E$ of constant signature (p, q). Definition 3.8.1 of generalized metric τ is then equivalent to a definition of a positive subbundle $V_+ \subseteq E$ of maximal possible rank p.*

Proof. First, let $\tau \in \mathrm{End}(E)$ be a generalized metric. It induces an involution τ_m in each fiber E_m. By previous Lemma, there exist its ± 1 eigenspaces V_{m+} and V_{m-}, such that $E_m = V_{m+} \oplus V_{m-}$. By definition of generalized metric τ, V_{m+} and V_{m-} are positive definite and negative definite subspaces respectively. By our assumption, this implies $\dim V_{m+} = p$, and $\dim V_{m-} = q$. Now define

$$V_\pm := \ker(\tau \mp 1). \tag{3.86}$$

We have just proved that vector bundle morphisms $\tau \mp 1$ have both constant rank, and $V_\pm$ are thus well-defined subbundles of E. Moreover, $\mathrm{rank}\, V_+ = p$, and V_+ is a positive definite subbundle. Note that $E = V_+ \oplus V_-$, and $V_- = (V_+)^\perp$, where the orthogonal complement $\perp$ is taken with respect to $\langle \cdot, \cdot \rangle_E$.

Conversely, let $V_+ \subseteq E$ be a positive-definite subbundle or rank p.

First, having a vector space W with positive definite subspace $W_+ \subseteq W$ of dimension p with respect to signature (p, q) metric $\langle \cdot, \cdot \rangle_W$, define $W_- := (W_+)^\perp$. Clearly $W = W_+ \oplus W_-$. Is W_- a negative definite subspace? If there would be a non-zero strictly positive vector $w \in W_-$, we could define $W'_+ = W \oplus \mathbb{R}\{w\}$, which would be a positive definite subspace of W of dimension $p + 1$. This cannot happen. If $w \in W_-$ would be a non-zero isotropic vector, we can take any nonzero $v \in W_+$, and define $w' = v + w$. Then $\langle w', w' \rangle_W = \langle v, v \rangle_W > 0$, and $W'_+ = W_+ \oplus \mathbb{R}\{w'\}$ would be a positive definite subspace of W of dimension $p + 1$, which is again impossible. We conclude that necessarily $\langle w, w \rangle_W < 0$.

To a positive definite subbundle V_+ of rank p, we can define $V_- = (V_+)^\perp$, where $\perp$ is taken with respect to $\langle \cdot, \cdot \rangle_E$. This is a well-defined rank q subbundle, which is by the previous discussion negative definite, and $V = V_+ \oplus V_-$. We can now define $\tau \in \mathrm{End}(E)$ as $\tau(e_+ + e_-) := e_+ - e_-$, for $e_\pm \in V_\pm$. One checks easily that τ satisfies all properties required by Definition 3.8.1. ∎

To get back to $E = TM \oplus T^*M$, we now bring an interpretation of the generalized metric most useful for actual calculations.

Proposition 3.8.4. *Let E be a vector bundle with a fiber-wise metric $\langle \cdot, \cdot \rangle_E$ of signature (n, n), and let $E = L \oplus L^*$, where L and L^* are isotropic subbundles with respect to $\langle \cdot, \cdot \rangle_E$. Note that L^* can be identified with the vector bundle dual to L.*

Generalized metric τ on E is then equivalent to a unique pair (g, B), where $g \in \Gamma(S^2 L^*)$ is a positive definite fiber-wise metric on L, and $B \in \Omega^2(L)$ is a 2-form on L.

Proof. Let τ be a generalized metric. We meet the requirements of Proposition 3.8.3, and thus $E = V_+ \oplus V_-$, where $\operatorname{rank} V_\pm = n$, and V_+ and V_- form the positive and negative definite subbundles with respect to $\langle \cdot, \cdot \rangle_E$. Now, because $V_+ \cap L = V_+ \cap L^* = \{0\}$, we see that V_+ must be a graph of some vector bundle morphism $A \in \operatorname{Hom}(L, L^*)$. A can uniquely be decomposed as $A = g + B$, where $g \in \Gamma(S^2 L^*)$, and $B \in \Omega^2(L)$. Every section $e \in \Gamma(V_+)$ can be thus written as $e = X + (g+B)(X)$, where $X \in \Gamma(L)$. We obtain

$$\langle e, e \rangle_E = \langle X + (g+B)(X), X + (g+B)(X) \rangle_E = 2\langle g(X), X \rangle_E = 2g(X, X). \tag{3.87}$$

Note that the canonical pairing between L and L^* is provided by $\langle \cdot, \cdot \rangle_E$. Because V_+ is the positive definite subbundle, we see that $g(X, X) > 0$ for all nonzero $X \in \Gamma(L)$, proving the positivity of the metric g. See that A is always a vector bundle isomorphism. Also note that V_- must be for the same reasons the graph of some vector bundle morphism $\widetilde{A} \in \operatorname{Hom}(L, L)$. From $\langle V_+, V_- \rangle_E = 0$ it follows that $\widetilde{A} = -g + B = -A^T$.

Conversely, let (g, B) be a pair, where $g \in \Gamma(S^2 L^*)$ is a positive definite metric and $B \in \Omega^2(L)$. We can define V_+ to be the graph of $A = g + B$. Repeating the calculation (3.87) shows that V_+ is a positive definite subbundle of rank n. This by Proposition 3.8.3 defines a generalized metric on the vector bundle E. ■

In the rest of this section, we will assume $E = TM \oplus T^*M$, and $L = TM$, $L^* = T^*M$. These satisfy the requirements of the previous proposition. However, keep in mind that everything works also for general L and L^*.

We will now rewrite the map τ and the corresponding fiber-wise metric $\mathbf{G}_\tau$ in terms of g and B. First, note that we can explicitly construct the projectors $P_\pm : E \to V_\pm$. Define two isomorphisms $\mathbf{\Psi}_\pm : TM \to V_\pm$ as

$$\mathbf{\Psi}_\pm(X) = X + (\pm g + B)(X), \tag{3.88}$$

for all $X \in \mathfrak{X}(M)$. Next, note that we can rewrite $X + \xi \in \Gamma(E)$ as

$$\begin{aligned} X + \xi &= \frac{1}{2}(X + (g+B)(X)) + \frac{1}{2}(X + (-g+B)(X)) \\ &+ \frac{1}{2}(g^{-1}(\xi) + (g+B)(g^{-1}(\xi))) - \frac{1}{2}(g^{-1}(\xi) + (-g+B)(g^{-1}(\xi))) \\ &- \frac{1}{2}(g^{-1}B(X) + (g+B)(g^{-1}B(X))) + \frac{1}{2}(g^{-1}B(X) + (-g+B)(g^{-1}B(X))) \\ &= \frac{1}{2}\mathbf{\Psi}_+\big(X + g^{-1}(\xi) - g^{-1}B(X)\big) + \frac{1}{2}\mathbf{\Psi}_-\big(X - g^{-1}(\xi) + g^{-1}B(X)\big). \end{aligned} \tag{3.89}$$

We thus obtain

$$P_\pm(X + \xi) = \frac{1}{2}\mathbf{\Psi}_\pm\big(X \pm g^{-1}(\xi) \mp g^{-1}B(X)\big). \tag{3.90}$$

We have defined $V_\pm$ as ± 1 eigenbundles of τ. Hence

$$\begin{aligned} \tau(X + \xi) &= \frac{1}{2}\mathbf{\Psi}_+(X + g^{-1}(\xi) - g^{-1}B(X)) - \frac{1}{2}\mathbf{\Psi}_-(X - g^{-1}(\xi) + g^{-1}B(X)) \\ &= g^{-1}(\xi) - g^{-1}B(X) + (g - Bg^{-1}B)(X) + Bg^{-1}(\xi). \end{aligned} \tag{3.91}$$

This proves that τ has a formal block form

$$\tau = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}. \tag{3.92}$$

Fiber-wise metric $\mathbf{G}_\tau$ in the block form is obtained from τ by multiplying it by matrix g_E, which is the same as (3.1). Thus

$$\mathbf{G}_\tau = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}. \tag{3.93}$$

Now recall Lemma 3.1.2. We see that the top-left and bottom-right blocks are invertible, and thus both decompositions of $\mathbf{G}_\tau$ exist. We find

$$\mathbf{G}_\tau = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{3.94}$$

This proves that $\mathbf{G}_\tau = (e^{-B})^T \mathcal{G}_E e^{-B}$, where $\mathcal{G}_E$ is the block diagonal metric

$$\mathcal{G}_E = \mathrm{BDiag}(g, g^{-1}). \tag{3.95}$$

This observation gives us two interesting facts. First, note that the blocks in the decomposition are unique, which re-proves the uniqueness assertions of the preceding proposition. Next, this helps us to prove that not every positive definite fiber-wise metric on E is a generalized metric. We see that $\det(\mathbf{G}_\tau) = 1$. Define $\mathbf{G} := \lambda \mathbf{G}_\tau$, where $\lambda \neq 1$ is a positive real constant. $\mathbf{G}$ is clearly a positive definite fiber-wise metric on E, but $\det(\mathbf{G}) = \lambda^{2n} \neq 1$. We can now give the last equivalent definition of the generalized metric.

Proposition 3.8.5. *Let E be a vector bundle, and $\langle \cdot, \cdot \rangle_E$ be a fiber-wise metric on E. We say that fiber-wise metric $\mathbf{G}$ is a generalized metric, if $\mathbf{G}$ is positive definite and $\mathbf{G} \in \mathrm{Hom}(E, E^*)$ defines an orthogonal map. We use the fact that the dual vector bundle E^* is naturally equipped with an induced fiber-wise metric $\langle \cdot, \cdot \rangle_{E^*} = g_E^{-1}$.*

We claim that this definition coincides with Definition 3.8.1.

Proof. Denote by $g_E \in \mathrm{Hom}(E, E^*)$ the vector bundle isomorphism induced by a fiber-wise metric $\langle \cdot, \cdot \rangle_E$. Let $\tau \in \mathrm{End}(E)$ be a generalized metric according to Definition 3.8.1. The properties of τ can be now written as

$$g_E \tau = \tau^T g_E, \; \tau^T g_E \tau = g_E, \; \tau^2 = 1. \tag{3.96}$$

The metric $\mathbf{G}_\tau$ and τ are related simply as $\mathbf{G}_\tau = g_E \tau$. We have to show that $\langle e, e' \rangle_E = \langle \mathbf{G}_\tau(e), \mathbf{G}_\tau(e') \rangle_{E^*}$. This can be rewritten as the condition

$$\mathbf{G}_\tau g_E^{-1} \mathbf{G}_\tau = g_E. \tag{3.97}$$

Plugging in for $\mathbf{G}_\tau$ translates into $\tau^T g_E \tau = g_E$. Conversely, let $\mathbf{G}$ be a generalized metric according to the definition in 3.8.5. This implies that $\mathbf{G}$ satisfies (3.97). Define $\tau := g_E^{-1} \circ \mathbf{G}$. We have

$$g_E \tau - \tau^T g_E = g_E(g_E \mathbf{G}) - (\mathbf{G} g_E^{-1}) g_E = \mathbf{G} - \mathbf{G} = 0.$$

This proves that τ is symmetric with respect to $\langle \cdot, \cdot \rangle_E$. Then

$$\tau^T g_E \tau = (\mathbf{G} g_E^{-1}) g_E (g_E^{-1} \mathbf{G}) = \mathbf{G} g_E^{-1} \mathbf{G} = g_E,$$

where we have used (3.97) in the last step. This proves that τ is orthogonal with respect to $\langle \cdot, \cdot \rangle_E$. The property $\tau^2 = 1$ follows automatically (any map which is both symmetric and orthogonal is an involution). ■

To conclude this section, note that there is no actual reason to choose the map $A \in \mathrm{Hom}(TM, T^*M)$ in order to describe V_+ in the proof of Proposition 3.8.4. One can as well describe V_+ using the map $A^{-1} \in \mathrm{Hom}(T^*M, TM)$, which decomposes as $A^{-1} = G^{-1} + \Pi$, where G is positive definite metric on M, and $\Pi \in \mathfrak{X}^2(M)$. The two descriptions are related as

$$(g + B)^{-1} = G^{-1} + \Pi. \tag{3.98}$$

One can find G and Π explicitly in terms of (g, B) as

$$G = g - Bg^{-1}B, \tag{3.99}$$
$$\Pi = -g^{-1}B(g - Bg^{-1}B)^{-1}. \tag{3.100}$$

To obtain this use the decomposition (3.21) for $\mathbf{G}_\tau$ in the form (3.93), and then note that $\mathbf{G}_\tau$ can be also decomposed as

$$\mathbf{G}_\tau = \begin{pmatrix} 1 & 0 \\ \Pi & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\Pi \\ 0 & 1 \end{pmatrix}. \tag{3.101}$$

A comparison of the blocks gives exactly the relations (3.99, 3.100).

3.9 Orthogonal transformations of the generalized metric

There is a natural action of the orthogonal group on the space of generalized metrics. We will analyze this action mainly in terms of the corresponding fields (g, B). We consider $E = TM \oplus T^*M$. Let τ be a generalized metric, and $\mathcal{O} \in O(n, n)$. Define $\tau' \in \mathrm{End}(E)$ as

$$\tau' := \mathcal{O}^{-1}\tau\mathcal{O}. \tag{3.102}$$

Then $\mathbf{G}_{\tau'} = g_E\tau' = O^T g_E \tau \mathcal{O} = \mathcal{O}^T \mathbf{G}_\tau \mathcal{O}$. Clearly $\tau'^2 = 1$. This means that τ' is also a generalized metric. Corresponding eigenbundles are related as

$$V_\pm^{\tau'} = \mathcal{O}^{-1} V_\pm^\tau. \tag{3.103}$$

We have also proved that there is always a unique pair (g, B) corresponding to τ. Let (g', B') be a pair corresponding to τ'. How are (g', B') and (g, B) related? Let $A = g + B$, and $A' = g' + B'$. By definition, $V_+^\tau = G_A$, and $V_+^{\tau'} = G_{A'}$, where G_A and $G_{A'}$ are the graphs of the respective vector bundle morphisms. We will use the notation introduced in Section 3.1. Let $X \in \mathfrak{X}(M)$. We have

$$\mathcal{O}^{-1}(X + A(X)) = (O_4^T + O_2^T A)(X) + (O_3^T + O_1^T A)(X).$$

Define $Y = (O_4^T + O_2^T A)(X)$. Then

$$\mathcal{O}^{-1}(X + A(X)) = Y + (O_3^T + O_1^T A)(O_4^T + O_2^T A)^{-1}(Y).$$

If the inverse of $O_4^T + O_2^T A$ exists, we get the following formula for A':

$$A' = (O_3^T + O_1^T A)(O_4^T + O_2^T A)^{-1}. \tag{3.104}$$

Recall the isomorphisms $\mathbf{\Psi}_\pm \in \mathrm{Hom}(TM, V_\pm)$ defined by (3.88), and let $\widetilde{\mathbf{\Psi}}_\pm \in \mathrm{Hom}(T^*M, V_\pm)$ be similarly induced isomorphisms: $\widetilde{\mathbf{\Psi}}_\pm(\xi) = \xi + (\pm G^{-1} + \Pi)(\xi)$, for all $\xi \in \Omega^1(M)$. Define new vector bundle morphisms $\mathbf{\Phi}_\pm$, $\mathbf{\Upsilon}_\pm$ by the following commutative diagram (in fact there are two independent diagrams, one for $+$, one for $-$):

$$\begin{tikzcd}
T^*M \arrow[r, "\mathbf{\Upsilon}_\pm"] \arrow[d, "\widetilde{\mathbf{\Psi}}^\tau_\pm"'] & T^*M \arrow[d, "\widetilde{\mathbf{\Psi}}^{\tau'}_\pm"] \\
V^\tau_\pm \arrow[r, "\mathcal{O}^{-1}"] & V^{\tau'}_\pm \\
TM \arrow[u, "\mathbf{\Psi}^\tau_\pm"] \arrow[r, "\mathbf{\Phi}_\pm"] \arrow[uu, bend left=60, "A"] & TM \arrow[u, "\mathbf{\Psi}^{\tau'}_\pm"'] \arrow[uu, bend right=60, "A'"']
\end{tikzcd} \tag{3.105}$$

All involved maps are vector bundle isomorphisms, and so have to be $\mathbf{\Phi}_\pm$, $\mathbf{\Upsilon}_\pm$. We can find explicit formulas:

$$\mathbf{\Phi}_+ = O_4^T + O_2^T A,\ \mathbf{\Phi}_- = O_4^T - O_2^T A^T, \tag{3.106}$$

$$\mathbf{\Upsilon}_+ = O_1^T + O_3^T A^{-1},\ \mathbf{\Upsilon}_- = O_1^T - O_3^T A^{-T}. \tag{3.107}$$

This proves that the inverse in (3.104) exists. These maps in fact transform between the involved Riemannian metrics.

Proposition 3.9.1. *There hold the following conjugation relations:*

$$g'^{-1} = \mathbf{\Phi}_\pm g^{-1} \mathbf{\Phi}_\pm^T, \tag{3.108}$$

$$g' - B' g'^{-1} B' = \mathbf{\Upsilon}_\pm (g - B g^{-1} B) \mathbf{\Upsilon}_\pm^T. \tag{3.109}$$

Proof. We will prove only one of the four equations, because the other ones follow in the same way. First note that $\langle \mathbf{\Psi}^\tau_+(X), \mathbf{\Psi}^\tau_+(Y)\rangle_E = 2g(X,Y)$, and similarly for τ'. Hence

$$2g(X,Y) = \langle \mathbf{\Psi}^\tau_+(X), \mathbf{\Psi}^\tau_+(Y)\rangle_E = \langle \mathcal{O}\mathbf{\Psi}^{\tau'}_+(\mathbf{\Phi}_+(X)), \mathcal{O}\mathbf{\Psi}^{\tau'}_+(\mathbf{\Phi}_+(Y))\rangle_E = 2g'(\mathbf{\Phi}_+(X), \mathbf{\Phi}_+(Y)).$$

Thus $g = \mathbf{\Phi}_+^T g' \mathbf{\Phi}_+$, which is precisely (3.108) with the $+$ sign. ■

Now, note that formula (3.104) can be rewritten as

$$A' = \mathbf{\Upsilon}_+ A \mathbf{\Phi}_+^{-1} = \mathbf{\Phi}_-^{-T} A \mathbf{\Upsilon}_-^T. \tag{3.110}$$

The latter expression can be found as the analogue of (3.104) derived using the subbundle V_-. Together with the previous proposition, we can find transformation rules for B'.

$$B' = ((\mathbf{\Upsilon}_+ - \mathbf{\Phi}_+^{-T})g + \mathbf{\Upsilon}_+ B)\mathbf{\Phi}_+^{-1} = (-(\mathbf{\Upsilon}_- - \mathbf{\Phi}_-^{-T})g + \mathbf{\Upsilon}_- B)\mathbf{\Phi}_-^{-1}. \tag{3.111}$$

Using the orthogonal group, we can describe the set of all generalized metrics in a more intrinsic way. To do so, we first need to prove two following lemmas.

Lemma 3.9.2. *The action of $O(n,n)$ on the set of generalized metrics is transitive.*

Proof. Let $\mathbf{G}$ and $\mathbf{G}'$ be two generalized metrics on E. Then $\mathbf{G} = [e^{-B}]^T \mathcal{G}_E e^{-B}$, and $\mathbf{G}' = [e^{-B'}]^T \mathcal{G}'_E e^{-B'}$. Because e^{-B} and $e^{-B'}$ are $O(n,n)$ transformations, it suffices to show that there exists $\mathcal{O} \in O(n,n)$, such that $\mathcal{G}'_E = \mathcal{O}^T \mathcal{G}_E \mathcal{O}$. For any two Riemannian metrics g and g' on M, there is a vector bundle isomorphism $N \in \mathrm{Aut}(TM)$, such that $g' = N^T g N$. Define $\mathcal{O} := \mathcal{O}_N$, where $\mathcal{O}_N$ is a block diagonal map

$$\mathcal{O}_N = \begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix}. \tag{3.112}$$

Obviously $\mathcal{O}_N \in O(n,n)$ and $\mathcal{G}'_E = \mathcal{O}_N^T \mathcal{G}_E \mathcal{O}_N$. This finishes the proof. ■

On the other hand, the action of $O(n,n)$ is not free, as the next lemma shows.

Lemma 3.9.3. *Let* $\mathbf{G}$ *be a generalized metric. Let* $O(n,n)_{\mathbf{G}} \subseteq O(n,n)$ *be its stabilizer subgroup. Then*

$$O(n,n)_{\mathbf{G}} \cong O(n) \times O(n). \tag{3.113}$$

Proof. Any morphism $\mathcal{O}$ stabilizing $\mathbf{G}$ must preserve the subbundles V_+ and V_-, it is thus block diagonal with respect to decomposition $E = V_+ \oplus V_-$. Moreover, $\langle \cdot, \cdot \rangle_E$ has the form

$$\langle e_+ + e_-, e'_+ + e'_- \rangle_E = \langle e_+, e'_+ \rangle_+ - \langle e_-, e'_- \rangle_-, \tag{3.114}$$

for all $e_\pm, e'_\pm \in V_\pm$, and $\langle \cdot, \cdot \rangle_\pm$ are positive definite fiber-wise metrics on $V_\pm$. This proves that $\mathcal{O} \in O(n,n)$ iff both its diagonal blocks are in $O(n)$. We conclude that $O(n,n)_{\mathbf{G}} = O(n) \times O(n)$. ■

These two observations are sufficient to describe the set of all generalized metrics on E.

Proposition 3.9.4. *The set of all generalized metrics is the coset space* $O(n,n)/(O(n) \times O(n))$.

Proof. There always exists at least one generalized metric, for example $\mathcal{G}_E$ for some Riemannian metric g. On every manifold M, there exists some g (it is constructed using the partition of unity). The space of all generalized metrics is its orbit by Lemma 3.9.2. But every orbit is isomorphic to $O(n,n)/O(n,n)_{\mathbf{G}}$, and thus by Lemma 3.9.3 to $O(n,n)/(O(n) \times O(n))$. ■

Example 3.9.5. Let us conclude this section with a few examples of the $O(n,n)$ actions on the generalized metric $\mathbf{G}$ described by a pair of fields (g, B).

- Let $Z \in \Omega^2(M)$ be a 2-form. Set $\mathcal{O} = e^{-Z}$. In particular, we have

  $$O_1 = 1,\ O_2 = 0,\ O_3 = -Z,\ O_4 = 1. \tag{3.115}$$

  Hence $\mathbf{\Phi}_\pm = 1$, $\mathbf{\Upsilon}_\pm = 1 + Z(\pm g + B)^{-1}$. We get $g' = g$, and

  $$B' = (\mp(\mathbf{\Upsilon}_\pm - 1)g + \mathbf{\Upsilon}_\pm B) = B + Z. \tag{3.116}$$

  Of course, we could have seen this directly from $\mathbf{G}' = (e^{-Z})^T[(e^{-B})^T \mathcal{G}_E e^{-B}]e^{-Z}$. In particular, if B and B' are related by a gauge transformation, $B' = B + da$ for $a \in \Omega^1(M)$, we can interpret the gauge transformation as the $O(n,n)$ transformation of the generalized metric $(g, B) \mapsto (g, B + da)$.

- Let $\theta \in \mathfrak{X}^2(M)$ be a 2-vector. Set $\mathcal{O} = e^\theta$. In particular, we have

  $$O_1 = 1,\ O_2 = \theta,\ O_3 = 0,\ O_4 = 1. \tag{3.117}$$

  This implies $\mathbf{\Phi}_\pm = 1 + \theta(\pm g + B)$, $\mathbf{\Upsilon}_\pm = 1$. The resulting relations are more clear in terms of dual fields (G, Π) and (G', Π') defined by (3.98). We get

  $$G' = G,\ \Pi' = \Pi - \theta. \tag{3.118}$$

  One can thus write the relations as

  $$\frac{1}{g + B} = \frac{1}{g' + B'} + \theta. \tag{3.119}$$

  These are precisely the open-closed relations of Seiberg-Witten as they appeared in [85].

- Let $N \in \mathrm{Aut}(E)$, and let $\mathcal{O} = \mathcal{O}_N$. Then
$$O_1 = N,\ O_2 = 0,\ O_3 = 0,\ O_4 = N^{-T}, \tag{3.120}$$
and consequently $\mathbf{\Phi}_{\pm} = N^{-1}$, and $\mathbf{\Upsilon}_+ = N^T$. We get $g' = N^T g N$, and $B' = N^T B N$. This proves that a change of frame in TM and its consequences for g and B can be incorporated as a special case of $O(n,n)$ transformation.

- Consider $M = \mathbb{R}^{d+1}$ for $d > 0$, and coordinates $(x^\mu, x^\bullet)$, $\mu \in \{1, \dots, d\}$. Define the vector bundle morphism $T \in \mathrm{Aut}(E)$ as
$$T(\partial_\mu) = \partial_\mu,\ T(dx^\mu) = dx^\mu,\ T(\partial_\bullet) = dx^\bullet,\ T(dx^\bullet) = \partial_\bullet. \tag{3.121}$$
It is easy to see that $T \in O(n,n)$, where now $n = d+1$. We can write the matrix of T in block form as
$$T = \begin{pmatrix} 1_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1_d & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.122}$$
where 1_d is a $d \times d$ identity matrix. We can write the matrix of metric g, and matrix of $B \in \Omega^2(M)$ in block forms as
$$g = \begin{pmatrix} \hat{g} & g_\bullet \\ g_\bullet^T & g_{\bullet\bullet} \end{pmatrix},\ B = \begin{pmatrix} \hat{B} & B_\bullet \\ -B_\bullet^T & 0 \end{pmatrix}, \tag{3.123}$$
where $\hat{g}$ is a $d \times d$ matrix $(\hat{g})_{\mu\nu} = g_{\mu\nu}$, and similarly with other components. We thus have
$$O_1 = \begin{pmatrix} 1_d & 0 \\ 0 & 0 \end{pmatrix},\ O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ O_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ O_4 = \begin{pmatrix} 1_d & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.124}$$
$$\mathbf{\Phi}_{\pm} = \begin{pmatrix} 1_d & 0 \\ \pm g_\bullet^T - B_\bullet^T & \pm g_{\bullet\bullet} \end{pmatrix} = \begin{pmatrix} 1_d & 0 \\ \pm g_\bullet^T - B_\bullet^T & 1 \end{pmatrix} \begin{pmatrix} 1_d & 0 \\ 0 & \pm g_{\bullet\bullet} \end{pmatrix}. \tag{3.125}$$
These maps are invertible, because $g_{\bullet\bullet} > 0$. We get
$$\mathbf{\Phi}_{\pm}^{-1} = \begin{pmatrix} 1_d & 0 \\ 0 & \pm\frac{1}{g_{\bullet\bullet}} \end{pmatrix} \begin{pmatrix} 1_d & 0 \\ \mp g_\bullet^T + B_\bullet^T & 1 \end{pmatrix} = \begin{pmatrix} 1_d & 0 \\ \pm\frac{1}{g_{\bullet\bullet}}(\mp g_\bullet^T + B_\bullet^T) & \pm\frac{1}{g_{\bullet\bullet}} \end{pmatrix} \tag{3.126}$$
The simplest way to (g', B') is now through (3.104), because we have already calculated $\mathbf{\Phi}_+^{-1} \equiv (O_4^T + O_2^T A)^{-1}$. We have
$$O_3^T + O_1^T A = \begin{pmatrix} \hat{g} + \hat{B} & g_\bullet + B_\bullet \\ 0 & 1 \end{pmatrix}. \tag{3.127}$$
Plugging into (3.104) now gives
$$A' = \begin{pmatrix} \hat{g} + \hat{B} + \frac{1}{g_{\bullet\bullet}}(g_\bullet + B_\bullet)(-g_\bullet^T + B_\bullet^T) & \frac{1}{g_{\bullet\bullet}}(g_\bullet + B_\bullet) \\ \frac{1}{g_{\bullet\bullet}}(-g_\bullet^T + B_\bullet^T) & \frac{1}{g_{\bullet\bullet}} \end{pmatrix} \tag{3.128}$$
Reading off the symmetric and skew-symmetric part, we obtain
$$\hat{g}' = \hat{g} + \frac{1}{g_{\bullet\bullet}}(B_\bullet B_\bullet^T - g_\bullet g_\bullet^T),\ g'_\bullet = \frac{1}{g_{\bullet\bullet}} B_\bullet,\ g'_{\bullet\bullet} = \frac{1}{g_{\bullet\bullet}}, \tag{3.129}$$
$$\hat{B}' = \hat{B} + \frac{1}{g_{\bullet\bullet}}(g_\bullet B_\bullet^T - B_\bullet g_\bullet^T),\ B'_\bullet = \frac{1}{g_{\bullet\bullet}} g_\bullet. \tag{3.130}$$

But these are exactly the well-known Buscher rules [20] emerging from string theory's T-duality. The generalized geometry thus allows to describe T-duality as an orthogonal transformation of the generalized metric.

3.10 Killing sections and corresponding isometries

Let $\mathbf{G}$ be a given generalized metric on E. Up to now, we have discussed only the isometries formed from $O(n,n)$ maps, concluding that $O(n) \times O(n)$ is the subgroup preserving $\mathbf{G}$. We can generalize the notion of isometry as follows.

Definition 3.10.1. We define the group $\mathrm{EIsom}(\mathbf{G})$ of extended isometries as

$$\mathrm{EIsom}(\mathbf{G}) = \{(\mathcal{F}, \varphi) \in \mathrm{EAut}(E) \mid \mathbf{G}(\mathcal{F}(e), \mathcal{F}(e')) \circ \varphi = \mathbf{G}(e, e')\}. \tag{3.131}$$

We have shown that $\mathrm{EIsom}(\mathbf{G}) \cap O(n,n) \cong O(n) \times O(n)$. We do not intend to find the whole group $\mathrm{EIsom}(\mathbf{G})$. Instead, we will find an important class of examples - solutions to the Killing equation. To find it, let us assume that $(\mathcal{F}_t, \varphi_t) \subseteq \mathrm{EIsom}(\mathbf{G})$ is a one-parameter subgroup and define $\mathcal{F} \in \Gamma(\mathcal{D}(E))$ as $\mathcal{F} = \frac{d}{dt}\big|_{t=0} \mathcal{F}_{-t}$. By differentiating (3.131) with respect to t at $t = 0$, we obtain

$$a(\mathcal{F}).\mathbf{G}(e, e') = \mathbf{G}(\mathcal{F}(e), e') + \mathbf{G}(e, \mathcal{F}(e')), \tag{3.132}$$

for all $e, e' \in \Gamma(E)$. This is still a way too complicated equation to solve, and we therefore restrict to $\mathcal{F}$ in the form $\mathcal{F}(e') = [e, e']_D$ for fixed $e \in \Gamma(E)$, and all $e' \in \Gamma(E)$. Note that $\mathcal{F} \in \mathrm{Der}(E)$, and $a(\mathcal{F}) = \rho(e)$. Requiring $\mathcal{F}$ to satisfy (3.132) leads to the definition of Killing equation.

Definition 3.10.2. Let $\mathbf{G}$ be a generalized metric. We say that $e \in \Gamma(E)$ is a **Killing section** of $\mathbf{G}$, if it satisfies the **Killing equation**

$$\rho(e).\mathbf{G}(e', e'') = \mathbf{G}([e, e']_D, e'') + \mathbf{G}(e', [e, e'']_D), \tag{3.133}$$

for all $e', e'' \in \Gamma(E)$.

Now assume that $\mathbf{G} \approx (g, B)$. We will examine the condition (3.133) in terms of the fields g and B. Recall that $\mathbf{G} = (e^{-B})^T \mathcal{G}_E e^{-B}$, where $\mathcal{G}_E = \mathrm{BDiag}(g, g^{-1})$. Moreover, we can use the fact that $e^{-B}[e, e']_D = [e^{-B}(e), e^{-B}(e')]_D^{dB}$, following from (3.82). Finally, note that $\rho(e^{-B}(e)) = \rho(e)$. These observations allow us to rewrite (3.133) as

$$\rho(e^{-B}(e)).\mathcal{G}_E(f', f'') = \mathcal{G}_E([e^{-B}(e), f']_D^{dB}, f'') + \mathcal{G}_E(f', [e^{-B}(e), f'']_D^{dB}), \tag{3.134}$$

for all $f', f'' \in \Gamma(E)$. This proves that e is a Killing section of $\mathbf{G}$, iff $f := e^{-B}(e)$ is a Killing section of $\mathcal{G}_E$. We will thus find all Killing sections of simpler generalized metric $\mathcal{G}_E$, but using dB-twisted Dorfman bracket instead. Write $f = X' + \xi'$ for $X' \in \mathfrak{X}(M)$ and $\xi' \in \Omega^1(M)$. Plugging into (3.134) for f, and writing $f' = Y + \eta$, $f'' = Z + \zeta$, we obtain

$$\begin{aligned} X'.\{g(Y, Z) + g^{-1}(\eta, \zeta)\} = {} & g([X', Y], Z) + g(Y, [X', Z]) \\ & + g^{-1}(\mathcal{L}_{X'}\eta - i_Y d\xi', \zeta) + g^{-1}(\eta, \mathcal{L}_{X'}\zeta - i_Z d\xi') \\ & - g^{-1}(dB(X', Y, \cdot), \zeta) - g^{-1}(\eta, dB(X', Z, \cdot)). \end{aligned} \tag{3.135}$$

This gives three equations

$$X'.g(Y,Z) = g([X',Y],Z) + g(Y,[X',Z]), \tag{3.136}$$

$$X'.g^{-1}(\eta,\zeta) = g^{-1}(\mathcal{L}_{X'}\eta,\zeta) + g^{-1}(\eta,\mathcal{L}_{X'}\zeta), \tag{3.137}$$

$$0 = g^{-1}(i_Y d\xi',\zeta) + g^{-1}(dB(X',Y,\cdot),\zeta) \tag{3.138}$$

all valid for all $Y, Z \in \mathfrak{X}(M)$ and $\eta,\zeta \in \Omega^1(M)$. They are equivalent to

$$\mathcal{L}_{X'}g = 0,\ d\xi' = -i_{X'}dB \tag{3.139}$$

Now let $X' + \xi' = e^{-B}(X+\xi) = X + \xi - B(X)$. We arrive to the following proposition:

Proposition 3.10.3. *Let* $\mathbf{G}$ *be a generalized metric corresponding to fields* (g,B)*. A section* $e = X + \xi$ *is a Killing section of* $\mathbf{G}$*, iff*

$$\mathcal{L}_X g = 0,\ d\xi = -\mathcal{L}_X B. \tag{3.140}$$

Since $\mathcal{F} = [X+\xi,\cdot] \in \mathrm{Der}(E)$, we can use the result of Proposition (3.5.2) to find the corresponding automorphism of Dorfman bracket. By construction, we expect $\exp(t\mathcal{F})$ to be an element of $\mathrm{EIsom}(\mathbf{G})$. Also note that $\mathcal{F} \in Eo(n,n)$, and thus $\exp(t\mathcal{F}) \in EO(n,n)$. Note that $\mathcal{F} = R(X) + \mathcal{F}_{d\xi}$. Using the Killing equation, we evaluate the integral defining the 2-form $B(t)$:

$$B(t) = \int_0^t \{\phi_k^{X*}(d\xi)\}dk = -\int_0^t \{\phi_k^{X*}(\mathcal{L}_X B)\}dk = B - \phi_t^{X*}B.$$

The corresponding 1-parameter subgroup of $\mathrm{Aut}_D(E)$ then has the form

$$\exp(t\mathcal{F}) = e^{B(t)} \circ T(\phi_{-t}^X). \tag{3.141}$$

Finally, we are able to prove the following statement:

Proposition 3.10.4. *Let* $\mathcal{F} = [e,\cdot]_D$*, where* $e \in \Gamma(E)$ *is a Killing section of* $\mathbf{G}$*. Then* $\exp(t\mathcal{F}) \in \mathrm{Aut}_D(E)$ *is an extended isometry of* $\mathbf{G}$*,* $\exp(t\mathcal{F}) \in \mathrm{EIsom}(\mathbf{G})$*.*

Proof. This is a direct calculation. Note that for $\mathbf{G} = (e^{-B})^T \mathcal{G}_E e^{-B}$, we get

$$\mathbf{G}(\exp(t\mathcal{F})(e), \exp(t\mathcal{F})(e')) = \mathcal{G}_E\big(e^{\phi_t^{X*}B}(T(\phi_{-t}^X)(e)), e^{\phi_t^{X*}B}(T(\phi_{-t}^X)(e'))\big).$$

Now note that $e^{\phi_t^{X*}B} \circ T(\phi_{-t}^X) = T(\phi_{-t}^X) \circ e^B$. Condition (3.131) then becomes

$$\mathcal{G}_E(T(\phi_{-t}^X)(f), T(\phi_{-t}^X)(f')) = \mathcal{G}_E(f,f') \circ \phi_t^X. \tag{3.142}$$

Rewriting this using the definition of $\mathcal{G}_E$, we obtain the condition $\phi_{-t}^{X*}g = g$. Here we use the second of the conditions (3.140) and the fact that Killing vector field X generates a flow preserving the metric g. ■

3.11 Indefinite case

The concept of generalized metric has proved to be a useful tool to encode a Riemannian metric g and a 2-form B into a single object $\mathbf{G}$, or its equivalents τ and V_+. For applications of this tool in physics, we should discuss also the case when g is an indefinite metric. We clearly have to abandon the interpretation using the definite subbundles $V_{\pm}$. The obvious candidate for indefinite generalized metric is $\mathbf{G}$ in the block form (3.93), since all expressions make sense. For given metric g and 2-form B, we *define* generalized metric $\mathbf{G}$ to be a fiber-wise metric

$$\mathbf{G} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}. \tag{3.143}$$

A first question comes with the invertibility of the map $g - Bg^{-1}B$. To answer it, we prove the following lemma:

Lemma 3.11.1. *Let V be a finite-dimensional vector space, $g \in S^2V^*$ be a non-degenerate bilinear form on V, and $B \in \Lambda^2V^*$. Let $A \in \mathrm{Hom}(V, V^*)$ be a linear map defined as $A = g+B$.*

Then the map A is invertible if and only if the bilinear form $g - Bg^{-1}B$ is non-degenerate.

Proof. First, assume that A is invertible. Then so is $A^T = g - B$. Next, note that we can write $g - Bg^{-1}B = (g + B)g^{-1}(g - B)$. This proves that $g - Bg^{-1}B$ is non-degenerate. In fact, we can take the determinant of this formula to get

$$[\det(g + B)]^2 = \det(g)\det(g - Bg^{-1}B). \tag{3.144}$$

This proves the converse statement. ■

For a positive definite g, the map $g + B$ is always invertible. However, for indefinite g, this is no more true. Consider for example

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.145}$$

The map $A = g + B$ is then certainly singular.

Then comes a question of the signature of $G = g - Bg^{-1}B$. But this in fact follows from the observation in the proof above, because $G = A^T g^{-1} A$, and the signature of G thus must be same as the one of g. If the signature of g is (p, q), we can take the square root of (3.144) to obtain

$$\det(g + B) = \pm[(-1)^q \det(g)]^{\frac{1}{2}}[(-1)^q \det(g - Bg^{-1}B)]^{\frac{1}{2}}. \tag{3.146}$$

Note the $\pm$ sign in the formula. For $g > 0$, determinant on the left-hand side is always positive. Indeed, define $f(t) = \det(g + tB)$. This is a continuous nonzero function of t, and $f(0) > 0$. This proves that $f(1) > 0$.

For indefinite g, the signs for $\det(g)$ and $\det(g + B)$ can be different. Consider for example

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}. \tag{3.147}$$

Then $\det(g + B) = -1 + \lambda^2$, and $\det(g) = -1$. We can thus choose $\lambda > 1$ and the two signs differ (the reason is of course the singularity at $\lambda = 1$).

We can still consider an orthogonal transformation of generalized metric $\mathbf{G}$. Consider $\mathcal{O} \in O(n, n)$ and simply define $\mathbf{G}' := \mathcal{O}^T \mathbf{G} \mathcal{O}$. Is there always metric g' and 2-form B' such that

$$\mathbf{G}' = \begin{pmatrix} 1 & B' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g' & 0 \\ 0 & g'^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B' & 1 \end{pmatrix} \tag{3.148}$$

holds true? A partial answer is given by the following proposition. We use the notation of sections 3.1 and 3.9.

Proposition 3.11.2. *Let $\mathbf{G}$ be a generalized metric in a sense of (3.143), and define $\mathbf{G}' = \mathcal{O}^T \mathbf{G} \mathcal{O}$ for $\mathcal{O} \in O(n,n)$.*

There exist metric g' and 2-form B' such that $\mathbf{G}'$ has the form (3.143), if and only if the map $\mathbf{\Phi}_+ \equiv O_4^T + O_2^T(g+B)$ is invertible.

Proof. First note that bottom-right corner of $\mathbf{G}'$ defines a fiber-wise bilinear form h' on T^*M given by formula

$$h' = O_4^T g^{-1} O_4 + O_2^T B g^{-1} O_4 - O_4^T g^{-1} B O_2 + O_4^T (g - B g^{-1} B) O_4. \tag{3.149}$$

Next see that h' can be written as

$$h' = \mathbf{\Phi}_+ g^{-1} \mathbf{\Phi}_+^T. \tag{3.150}$$

This can be verified directly, using the orthogonality property $O_4^T O_2 + O_2^T O_4 = 0$ for $\mathcal{O}$. Taking the determinant of this relation gives

$$\det(h') = [\det(\mathbf{\Phi}_+)]^2 / \det(g). \tag{3.151}$$

This proves that h' is invertible if and only if $\mathbf{\Phi}_+$ is.

Now assume that $\mathbf{G}'$ has the form (3.148) for metric g' and $B' \in \Omega^2(M)$. In this case $h' = g'^{-1}$, and (3.151) proves that $\mathbf{\Phi}_+$ is invertible.

Conversely, let $\mathbf{\Phi}_+$ be an invertible map. Formula (3.151) proves that h' is invertible (and thus a fiber-wise metric on T^*M). Define $g' := h'^{-1}$. Now recall Proposition 3.8.5. Even in indefinite case, $\mathbf{G} \in \mathrm{Hom}(E, E^*)$ is an orthogonal map, and so is $\mathbf{G}' = \mathcal{O}^T \mathbf{G} \mathcal{O}$. Moreover, this map has invertible bottom-right corner h'. We can now use an analogue of Proposition 3.1.3 to show that $\mathbf{G}'$ can be decomposed as (3.148), proving that $B' \in \Omega^2(M)$. ∎

Note that in indefinite case, $\mathbf{\Phi}_+$ is not always invertible, not even in the case when $g+B$ is. Consider $\mathcal{O} = e^{-\Theta}$ of Example 3.9.5 for $n = 2$. Define

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \; \Theta = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}. \tag{3.152}$$

Then $\det(g+B) = b^2 - 1$, and for $b \neq \pm 1$, $g+B$ is invertible. The map $\mathbf{\Phi}_+ = 1 - \Theta(g+B)$ has the explicit form

$$\mathbf{\Phi}_+ = \begin{pmatrix} 1+tb & t \\ t & 1+tb \end{pmatrix}. \tag{3.153}$$

Now $\det(\mathbf{\Phi}_+) = (1+tb)^2 - t^2$. Equation $\det(\mathbf{\Phi}_+) = 0$ now has two roots: $t = -1/(b \pm 1)$. For every b, we can thus find Θ making $\mathbf{\Phi}_+$ a singular matrix, and consequently $\mathbf{G}'$ is indecomposable in the sense of (3.148).

We see that everything in principle carries out to the indefinite case, but in every step one has to make assumptions concerning the invertibility of involved maps. In order to avoid unnecessary discussions here and there, we thus usually stick to the positive definite case.

3.12 Generalized Bismut connection

Let $\mathbf{G}$ be a generalized metric on the Courant algebroid $E = TM \oplus T^*M$. We may look for Courant algebroid connections compatible with $\mathbf{G}$. Recall that by Definition 2.4.3, ∇ is a Courant algebroid connection if it preserves the Courant metric $\langle \cdot, \cdot \rangle_E$. There exists a simple example of such connection. It first appeared in [33] and was studied from the perspective of Courant algebroids in [40]. Before we introduce its definition, recall that $\mathbf{G}$ induces an operator $C : V_\pm \to V_\mp$ defined as

$$C(\mathbf{\Psi}_\pm(X)) = \mathbf{\Psi}_\mp(X), \tag{3.154}$$

for all $X \in \mathfrak{X}(M)$. By direct calculation using the projectors (3.90), one arrives to an explicit formula

$$C(X + \xi) = X - \xi + 2B(X), \tag{3.155}$$

for all $X + \xi \in \Gamma(E)$. Note that by definition $C^2 = 1$, and C is an anti-orthogonal map with respect to $\langle \cdot, \cdot \rangle_E$. For any $e \in \Gamma(E)$, denote $e_\pm \equiv P_\pm(e)$ to simplify the notation.

Definition 3.12.1. The **generalized Bismut connection** ∇ is for $e, e' \in \Gamma(E)$ defined as

$$\nabla_e e' = ([e_+, e'_-]_D)_- + ([e_-, e'_+]_D)_+ + ([C(e_+), e'_+]_D)_+ + ([C(e_-), e'_-]_D)_-. \tag{3.156}$$

We have used a more abstract definition as it appeared in [40]. It is easy to see that $\nabla_{fe} e' = f \nabla_e e'$. This follows from the fact that $\langle V_+, V_- \rangle_E = 0$, and $(e_+)_- = (e_-)_+ = 0$ for all $e \in \Gamma(E)$. To prove the second property note that $\rho \circ C = \rho$, and we get

$$\begin{aligned}\nabla_e(fe') &= f\nabla_e e' + (\rho(e_+).f)e'_- + (\rho(e_-).f)e'_+ + (\rho(e_+).f)e'_+ + (\rho(e_-).f)e'_- \\ &= f\nabla_e e' + (\rho(e).f)e'.\end{aligned}$$

We will prove the metric compatibility with $\langle \cdot, \cdot \rangle_E$ later. It is easier to determine the action of the connection on the sections of the special form. Let $e = \mathbf{\Psi}_+(X)$, and $e' = \mathbf{\Psi}_-(Y)$. Only one of the four terms contributes, namely the first one. Next note that $\mathbf{\Psi}_\pm = e^B \mathbf{\Psi}^0_\pm$, where $\mathbf{\Psi}^0_\pm(X) = X \mp g(X)$. Hence

$$[\mathbf{\Psi}_+(X), \mathbf{\Psi}_-(Y)]_D = e^B [\mathbf{\Psi}^0_+(X), \mathbf{\Psi}^0_-(Y)]^H_D.$$

Here $H = dB$. This simplifies the calculation, because $P_-(e^B(Y + \eta)) = \frac{1}{2}\mathbf{\Psi}_-(Y - g^{-1}(\eta))$. Then

$$[\mathbf{\Psi}^0_+(X), \mathbf{\Psi}^0_-(Y)]^H_D = [X, Y] - \mathcal{L}_X(g(Y)) - i_Y d(g(X)) - H(X, Y, \cdot).$$

Combining all the observations gives

$$\nabla_e e' = \mathbf{\Psi}_-\big[\frac{1}{2}\{[X, Y] + g^{-1}(\mathcal{L}_X(g(Y)) + i_Y d(g(X)))\} + \frac{1}{2} g^{-1} H(X, Y, \cdot))\big]. \tag{3.157}$$

The terms not containing B form a well known object. Indeed, we have

$$\nabla^{LC}_X Y = \frac{1}{2}\{[X, Y] + g^{-1}(\mathcal{L}_X(g(Y)) + i_y d(g(X)))\}, \tag{3.158}$$

where ∇^{LC} is the Levi-Civita connection on M corresponding to the metric g. We can thus write

$$\nabla_e e' = \mathbf{\Psi}_-(\nabla^{LC}_X Y + \frac{1}{2} g^{-1} H(X, Y, \cdot)). \tag{3.159}$$

Repeating the same procedure for all the possible combination of $\pm$ signs, we would prove the following lemma.

Lemma 3.12.2. *Let ∇ be the generalized Bismut connection defined by (3.156), and $H = dB$. Then*

$$\nabla_{\mathbf{\Psi}_{\pm}(X)}(\mathbf{\Psi}_{+}(Y)) = \mathbf{\Psi}_{+}(\nabla^{+}_{X}Y), \tag{3.160}$$

$$\nabla_{\mathbf{\Psi}_{\pm}(X)}(\mathbf{\Psi}_{-}(Y)) = \mathbf{\Psi}_{-}(\nabla^{-}_{X}Y), \tag{3.161}$$

for all $X, Y \in \mathfrak{X}(M)$. $\nabla^{\pm}$ is a couple of connections on M defined as

$$\nabla^{\pm}_{X}Y = \nabla^{LC}_{X}Y \mp \frac{1}{2}g^{-1}H(X,Y,\cdot). \tag{3.162}$$

Connections $\nabla^{\pm}$ are metric compatible with g, and equations (3.160 - 3.162) can be considered as an equivalent definition of the generalized Bismut connection.

This lemma has one immediate consequence. We can now prove that ∇ is in fact induced by an ordinary vector bundle connection. To show this, let $e = 0+\alpha$ for $\alpha \in \Omega^{1}(M)$. It suffices to show that $\nabla_{e} = 0$. But in this case $e = \mathbf{\Psi}_{+}(\frac{1}{2}g^{-1}(\alpha)) - \mathbf{\Psi}_{-}(\frac{1}{2}g^{-1}(\alpha))$, and the statement follows from (3.160, 3.161).

Our aim now is to find an expression for the generalized Bismut connection acting on a section in a general form $e = Y + \eta$. To do so, introduce an auxiliary connection $\widehat{\nabla}_{e} = e^{-B}\nabla_{e^{B}(e)}e^{B}$. We have

$$e^{B}(Y+\eta) = \frac{1}{2}\mathbf{\Psi}_{+}(Y + g^{-1}(\eta)) + \frac{1}{2}\mathbf{\Psi}_{-}(Y - g^{-1}(\eta)).$$

In particular $e^{B}(X) = \frac{1}{2}\mathbf{\Psi}_{+}(X) + \frac{1}{2}\mathbf{\Psi}_{-}(X)$. This shows that

$$\begin{aligned} e^{B}\widehat{\nabla}_{X}(Y+\eta) &= \frac{1}{2}\nabla_{\mathbf{\Psi}_{+}(X)}\mathbf{\Psi}_{+}(Y+g^{-1}(\eta)) + \frac{1}{2}\nabla_{\mathbf{\Psi}_{+}(X)}\mathbf{\Psi}_{-}(Y-g^{-1}(\eta)) \\ &= \frac{1}{2}\mathbf{\Psi}_{+}(\nabla^{+}_{X}(Y+g^{-1}(\eta))) + \frac{1}{2}\mathbf{\Psi}_{-}(\nabla^{-}_{X}(Y-g^{-1}(\eta))) \\ &= \frac{1}{2}e^{B}\mathbf{\Psi}^{0}_{+}(\nabla^{+}_{X}(Y+g^{-1}(\eta))) + \frac{1}{2}e^{B}\mathbf{\Psi}^{0}_{-}(\nabla^{-}_{X}(Y-g^{-1}(\eta))) \end{aligned}$$

Hence

$$\widehat{\nabla}_{X}(Y+\eta) = \frac{1}{2}\mathbf{\Psi}^{0}_{+}(\nabla^{+}_{X}(Y+g^{-1}(\eta))) + \frac{1}{2}\mathbf{\Psi}^{0}_{-}(\nabla^{-}_{X}(Y-g^{-1}(\eta))). \tag{3.163}$$

It is now simple to plug in from (3.162) for $\nabla^{\pm}$. This results in the formal block form of $\widehat{\nabla}$:

$$\widehat{\nabla}_{X} = \begin{pmatrix} \nabla^{LC}_{X} & -\frac{1}{2}g^{-1}H(X,g^{-1}(\star),\cdot) \\ -\frac{1}{2}H(X,\star,\cdot) & \nabla^{LC}_{X} \end{pmatrix}. \tag{3.164}$$

The $\star$ symbol indicates where the input enters. In other words, we have

$$\widehat{\nabla}_{X}(Y+\eta) = (\nabla^{LC}_{X}Y - \frac{1}{2}g^{-1}H(X,g^{-1}(\eta),\cdot)) + (\nabla^{LC}_{X}\eta - \frac{1}{2}H(X,Y,\cdot)). \tag{3.165}$$

Now it is easy to write the original $\mathbf{G}$-compatible connection ∇. One obtains

$$\nabla_{X} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} \nabla^{LC}_{X} & -\frac{1}{2}g^{-1}H(X,g^{-1}(\star),\cdot) \\ -\frac{1}{2}H(X,\star,\cdot) & \nabla^{LC}_{X} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{3.166}$$

This is the form of generalized Bismut connection as it appeared in [33].

Proposition 3.12.3. *Generalized Bismut connection ∇ is compatible with the pairing $\langle\cdot,\cdot\rangle_E$, and with the generalized metric $\mathbf{G}$.*

Proof. Directly from the definition of $\widehat{\nabla}$, it is easy to see that ∇ is compatible with $\langle\cdot,\cdot\rangle_E$ and $\mathbf{G}$, if and only if $\widehat{\nabla}$ is compatible with $\langle\cdot,\cdot\rangle_E$ and $\mathcal{G}_E$. The latter property can be easily checked explicitly using the block form (3.164) of $\widehat{\nabla}$. ∎

Now recall the definition of the torsion operator (2.67). We will use $\mathbf{L}$ suitable for Courant algebroids, namely

$$\mathbf{L}(\beta, e, e') = \langle e, e'\rangle_E g_E^{-1}(\beta). \tag{3.167}$$

We will make advantage of the simpler connection $\widehat{\nabla}$ to calculate T. Indeed, we have

$$T(e^B(e), e^B(e')) = e^B\big(\widehat{\nabla}_e e' - \widehat{\nabla}_{e'} e - [e,e']^H_D + \mathbf{L}(e^\lambda, \widehat{\nabla}_{e_\lambda} e, e')\big).$$

Thus $T(e^B(e), e^B(e')) = e^B\widehat{T}(e,e')$, where $\widehat{T}$ is the torsion of connection $\widehat{\nabla}$ with H-twisted Dorfman bracket. Let us now calculate $\widehat{T}$ explicitly. One gets

$$\begin{aligned}\widehat{T}(X+\xi, Y+\eta) = &-\frac{1}{2}g^{-1}H(X, g^{-1}(\eta), \cdot) + \frac{1}{2}g^{-1}H(Y, g^{-1}(\xi), \cdot)\\ &-\frac{1}{2}H(X,Y,\cdot) - \frac{1}{2}H(g^{-1}(\xi), g^{-1}(\eta), \cdot).\end{aligned} \tag{3.168}$$

This proves that $\widehat{\nabla}$ and consequently ∇ is torsion-free if and only if $H = 0$. Torsion T can be now calculated in a straightforward manner using e^B and $\widehat{T}$.

According to the remark under (2.74), we may define the curvature operator of ∇ using the usual formula:

$$R(e,e')e'' = \nabla_e\nabla_{e'}e'' - \nabla_{e'}\nabla_e e'' - \nabla_{[e,e']_D}e'', \tag{3.169}$$

for all $e, e', e'' \in \Gamma(E)$. Using the relation of ∇ to $\widehat{\nabla}$, we get the expression

$$R(e^B(e), e^B(e'))e^B(e'') = e^B\big(\widehat{\nabla}_e\widehat{\nabla}_{e'}e'' - \widehat{\nabla}_{e'}\widehat{\nabla}_e e'' - \widehat{\nabla}_{[e,e']^H_D}e''\big). \tag{3.170}$$

Hence $R(e^B(e), e^B(e'))e^B(e'') = e^B(\widehat{R}(e,e')e'')$, where $\widehat{R}$ is the curvature operator of $\widehat{\nabla}$ using the H-twisted Dorfman bracket $[\cdot,\cdot]^H_D$. It is not difficult to calculate $\widehat{R}$ explicitly. We get

$$\begin{aligned}\widehat{R}_1(X,Y)(Z+\zeta) = {} & R^{LC}(X,Y)Z - \frac{1}{2}g^{-1}((\nabla^{LC}_X H)(Y, g^{-1}(\zeta), \cdot) - (\nabla^{LC}_Y H)(X, g^{-1}(\zeta), \cdot))\\ &+\frac{1}{4}g^{-1}H(X, g^{-1}H(Y,Z,\cdot),\cdot) - \frac{1}{4}g^{-1}H(Y, g^{-1}H(X,Z,\cdot),\cdot)\end{aligned} \tag{3.171}$$

$$\begin{aligned}\widehat{R}_2(X,Y)(Z+\zeta) = {} & R^{LC}(X,Y)\zeta - \frac{1}{2}(\nabla^{LC}_X H)(Y,Z,\cdot) + \frac{1}{2}(\nabla^{LC}_Y H)(X,Z,\cdot)\\ &+\frac{1}{4}H(X, g^{-1}H(Y, g^{-1}(\zeta),\cdot),\cdot) - \frac{1}{4}H(Y, g^{-1}H(X, g^{-1}(\zeta),\cdot),\cdot).\end{aligned} \tag{3.172}$$

We have used $\widehat{R}_1$ and $\widehat{R}_2$ to denote the TM and T^*M components of $\widehat{R}$ respectively. One can now calculate the corresponding Ricci tensor $\widehat{\mathrm{Ric}}$, defined as $\widehat{\mathrm{Ric}}(e,e') = \langle e^\lambda, \widehat{R}(e_\lambda, e')e\rangle$. Note that it has only two non-trivial components (with respect to the block decomposition). One obtains

$$\widehat{\mathrm{Ric}}(X,Y) = \mathrm{Ric}^{LC}(X,Y) - \frac{1}{4}H(Y, g^{-1}H(\partial_k, X, \cdot), g^{-1}(dy^k)), \tag{3.173}$$

$$\widehat{\mathrm{Ric}}(\xi, Y) = -\frac{1}{2}(\nabla^{LC}_{\partial_k}H)(Y, g^{-1}(\xi), g^{-1}(dy^k)). \tag{3.174}$$

Finally, we may use the generalized metric $\mathcal{G}_E$ to calculate the trace of $\widehat{\mathrm{Ric}}$ and obtain the corresponding scalar curvature $\widehat{\mathcal{R}}$. One gets

$$\widehat{\mathcal{R}} = \widehat{\mathrm{Ric}}(\mathcal{G}_E^{-1}(e^\lambda), e_\lambda) = \mathcal{R}(g) - \frac{1}{4} H_{ijk} H^{ijk}. \tag{3.175}$$

To conclude this section, note that we can use this result to calculate the scalar curvature $\mathcal{R}$ of the generalized Bismut connection.

Proposition 3.12.4. *Let ∇ be the generalized Bismut connection corresponding to the generalized metric $\mathbf{G}$. Let* Ric *be its Ricci tensor, and let $\mathcal{R}$ be the scalar function on M defined as*

$$\mathcal{R} = \mathrm{Ric}(\mathbf{G}^{-1}(e^\lambda), e_\lambda), \tag{3.176}$$

where $(e_\lambda)_{\lambda=1}^{2n}$ is some local frame on E. Then $\mathcal{R} = \widehat{\mathcal{R}}$, that is

$$\mathcal{R} = \mathcal{R}(g) - \frac{1}{4} H_{ijk} H^{ijk}. \tag{3.177}$$

Proof. The result follows from the definition of the connection $\widehat{\nabla}$, the relation (3.170), and the fact that $\mathbf{G} = (e^{-B})^T \mathcal{G}_E e^{-B}$. ■

We see that the scalar curvature of the generalized Bismut connection does not depend on B, but only on a cohomology class $[H]$ of its exterior derivative $H = dB$.

Chapter 4

Extended generalized geometry

The main aim of this chapter is to generalize the objects of the standard generalized geometry in order to work also on the vector bundle $E = TM \oplus \Lambda^p T^*M$. The main issue is that the most straightforward generalization of the orthogonal group $O(n,n)$ suitable for E does not seem to be useful for a description of the generalized metric. This lead us to the idea of embedding the generalized geometry of E into the larger vector bundle $E \oplus E^*$, already equipped with the canonical $O(d,d)$ structure.

4.1 Pairing, Orthogonal group

Let $E = TM \oplus \Lambda^p T^*M$. We have $\Gamma(E) = \mathfrak{X}(M) \oplus \Omega^p(M)$. Define a non-degenerate $C^\infty(M)$-bilinear symmetric form $\langle \cdot, \cdot \rangle_E : \Gamma(E) \to \Gamma(E) \to \Omega^{p-1}(M)$ as

$$\langle X + \xi, Y + \eta \rangle_E = i_X \eta + i_Y \xi, \tag{4.1}$$

for all $X + \xi, Y + \eta \in \Gamma(E)$. Although it is not an ordinary $C^\infty(M)$-valued pairing, one can still define its orthogonal group $O(E)$ as usual, that is

$$O(E) = \{\mathcal{F} \in \mathrm{Aut}(E) \mid \langle \mathcal{F}(e), \mathcal{F}(e') \rangle_E = \langle e, e' \rangle_E, \forall e, e' \in \Gamma(E)\}. \tag{4.2}$$

We will now examine its Lie algebra $o(E)$, defined as

$$o(E) = \{\mathcal{F} \in \mathrm{End}(E) \mid \langle \mathcal{F}(e), e' \rangle_E + \langle e, \mathcal{F}(e') \rangle_E = 0, \forall e, e' \in \Gamma(E)\}. \tag{4.3}$$

In fact, the structure of this algebra greatly depends on p and the dimension n of the manifold M. Write $\mathcal{F}$ in the formal block form as

$$\mathcal{F} = \begin{pmatrix} A & \Pi \\ -C^T & A' \end{pmatrix}. \tag{4.4}$$

Plugging $\mathcal{F}$ into the condition (4.3) gives the following set of equations:

$$i_Y C^T(X) + i_X C^T(Y) = 0, \tag{4.5}$$

$$i_Y A'(\xi) + i_{A(Y)} \xi = 0, \tag{4.6}$$

$$i_{\Pi(\xi)} \eta + i_{\Pi(\eta)} \xi = 0. \tag{4.7}$$

One can now discuss the consequences of these equations. This is a straightforward but a little bit technical linear algebra. We present only the results in the form of a proposition.

Proposition 4.1.1. *Let $\mathcal{F} \in \mathrm{End}(E)$ have a formal block form (4.4). Then, depending on p, we have the following conditions for $\mathcal{F} \in o(E)$.*

1. *$p = 0$: All fields are arbitrary, that is*

$$o(E) = \mathrm{End}(TM) \oplus \mathfrak{X}(M) \oplus \Omega^1(M) \oplus C^\infty(M). \tag{4.8}$$

2. *$p = 1$: In this case $o(E) = o(n,n)$, and thus $A' = -A^T$, $\Pi \in \mathfrak{X}^2(M)$, $C \in \Omega^2(M)$, and*

$$o(E) = \mathrm{End}(TM) \oplus \mathfrak{X}^2(M) \oplus \Omega^2(M). \tag{4.9}$$

3. *$1 < p < n-1$: In this case $A = \lambda \cdot 1$, $A' = -\lambda \cdot 1$, where $\lambda \in C^\infty(M)$, $\Pi = 0$, and $C \in \Omega^{p+1}(M)$. Hence,*

$$o(E) = \mathfrak{X}^{p+1}(M) \oplus C^\infty(M). \tag{4.10}$$

4. *$p = n-1$: $A = \lambda \cdot 1$, $A' = -\lambda \cdot 1$, for $\lambda \in C^\infty(M)$. $\Pi \in \mathfrak{X}^n(M)$, and $C \in \Omega^n(M)$. Thus,*

$$o(E) = \mathfrak{X}^n(M) \oplus \Omega^n(M) \oplus C^\infty(M). \tag{4.11}$$

5. *$p = n$: In this case $A = \lambda \cdot 1$, $A' = -\lambda \cdot 1$, for any $\lambda \in C^\infty(M)$, $C = \Pi = 0$, and therefore*

$$o(E) = C^\infty(M). \tag{4.12}$$

We see that possible choices for $o(E)$ are very different for different values of p. In particular note that for $1 < p < n-1$, there is no $(p+1)$-vector Π defining a skew-symmetric transformation, and thus no e^Π defining an orthogonal transformation. This proves that for general p, Nambu-Poisson manifolds cannot be realized as Dirac structures. This was proved by Zambon in [91], and it has in fact lead to the theory of Nambu-Dirac manifolds examined by Hagiwara in [41].

Example 4.1.2. For a general p, there are thus fewer generic examples of orthogonal transformations, let us recall them here

- C-transform: Let $C \in \Omega^{p+1}(M)$. It defines a map $C \in \mathrm{Hom}(\Lambda^p TM, T^*M)$, and we will define $e^C \in \mathrm{Aut}(E)$ by its formal block form

$$e^C = \begin{pmatrix} 1 & 0 \\ -C^T & 1 \end{pmatrix}. \tag{4.13}$$

  Note that for $p = 1$, we have $C = -C^T$. It follows from Proposition 4.1.1 that $e^C \in O(E)$.

- Let $\lambda \in C^\infty(M)$ be everywhere non-zero smooth function. Define the map $\mathcal{O}_\lambda$ as

$$\mathcal{O}_\lambda(X + \xi) = \lambda X + \frac{1}{\lambda}\xi, \tag{4.14}$$

  for all $X + \xi \in \Gamma(E)$. Obviously $O_\lambda \in O(E)$.

4.2 Higher Dorfman bracket and its symmetries

Let us now examine the Dorfman bracket from Example 2.1.3. Recall that it is defined as

$$[X + \xi, Y + \eta]_D = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi, \tag{4.15}$$

for all $X + \xi$, $Y + \eta \in \Gamma(E)$. To distinguish it from its $p = 1$ version, we sometimes refer to it as the *higher* Dorfman bracket. We have shown that for $\rho = pr_{TM}$, the triplet $(E, \rho, [\cdot, \cdot]_E)$ forms a Leibniz algebroid. Due to its structure similar to $p = 1$ Dorfman bracket, it also has some properties similar to Courant algebroid axioms:

Lemma 4.2.1. *Let $[\cdot, \cdot]_D$ be the Dorfman bracket (4.15), and $\langle \cdot, \cdot \rangle_E$ be the pairing (4.1). Then*

$$[X + \xi, X + \xi]_D = \frac{1}{2} d\langle X + \xi, X + \xi \rangle_E, \tag{4.16}$$

for all $X + \xi \in \Gamma(E)$, and the pairing $\langle \cdot, \cdot \rangle_E$ is invariant with respect to $[\cdot, \cdot]_D$ in the sense that

$$\mathcal{L}_{\rho(X+\xi)} \langle Y + \eta, Z + \zeta \rangle_E = \langle [X + \xi, Y + \eta]_D, Z + \zeta \rangle_E + \langle Y + \eta, [X + \xi, Z + \zeta]_D \rangle_E, \tag{4.17}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, and $\xi, \eta, \zeta \in \Omega^p(M)$.

Proof. Direct calculation and definitions. ■

We can directly generalize the derivations algebra and the automorphism group of the Dorfman bracket. The derivation of the results is completely analogous to the $p = 1$ case provided in Section 3.4 and Section 3.5. We thus omit proofs of following propositions.

Proposition 4.2.2. *Define the Lie algebra $\mathrm{Der}\, E$ of derivations of the Dorfman bracket (4.15) as in (3.40). Then as a vector space, it decomposes as*

$$\mathrm{Der}(E) \doteq \mathfrak{X}(M) \oplus \Omega^0_{closed}(M) \oplus \Omega^{p+1}_{closed}(M). \tag{4.18}$$

Every $\mathcal{F} \in \mathrm{Der}(E)$ decomposes uniquely as $\mathcal{F} = R(X) + \mathcal{F}_\lambda + \mathcal{F}_C$, where $R(X)(Y + \eta) = ([X, Y], \mathcal{L}_X \eta)$, for all $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Omega^p(M)$. Vector bundle endomorphisms $\mathcal{F}_\lambda$ and $\mathcal{F}_C$ are defined as

$$\mathcal{F}_\lambda = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \cdot 1 \end{pmatrix}, \; \mathcal{F}_C = \begin{pmatrix} 0 & 0 \\ -C^T & 0 \end{pmatrix}, \tag{4.19}$$

where $\lambda \in \Omega^0_{closed}(M)$ and $C \in \Omega^{p+1}_{closed}(M)$. Nontrivial commutation relations are

$$[R(X), R(Y)] = R([X, Y]), \tag{4.20}$$

$$[R(X), F_C] = F_{\mathcal{L}_X B}, \tag{4.21}$$

$$[\mathcal{F}_\lambda, \mathcal{F}_B] = \mathcal{F}_{\lambda C}. \tag{4.22}$$

On the Lie algebra level, we thus have

$$\mathrm{Der}(E) = \mathfrak{X}(M) \ltimes (\Omega^0_{closed}(M) \ltimes \Omega^{p+1}_{closed}(M)), \tag{4.23}$$

where $\Omega^0_{closed}(M)$, $\Omega^{p+1}_{closed}(M)$ are viewed as Abelian Lie algebras, $\Omega^0_{closed}(M)$ acts on forms in $\Omega^{p+1}_{closed}(M)$ by multiplication, and $\mathfrak{X}(M)$ acts on $\Omega^0_{closed} \ltimes \Omega^{p+1}_{closed}(M)$ by Lie derivatives.

Proposition 4.2.3. *Let* $\mathrm{Aut}_D(E)$ *be the group of automorphisms (3.59) of the Dorfman bracket (4.15). Then it has the following group structure:*

$$\mathrm{Aut}_D(E) = (\Omega^{p+1}_{closed}(M) \rtimes G(\Omega^0_{closed}(M))) \rtimes \mathrm{Diff}(M), \tag{4.24}$$

where $\Omega^{p+1}_{closed}(M)$ *is viewed as an Abelian group with respect to addition,* $G(\Omega^0_{closed}(M))$ *acts on* $\Omega^{p+1}_{closed}(M)$ *by multiplication, and* $\mathrm{Diff}(M)$ *acts on* $\Omega^{p+1}_{closed}(M) \rtimes G(\Omega^0_{closed}(M))$ *by inverse pullbacks. Every* $(\mathcal{F}, \varphi) \in \mathrm{Aut}_D(E)$ *can uniquely be decomposed as*

$$\mathcal{F} = e^C \circ \mathcal{S}_\lambda \circ T(\varphi), \tag{4.25}$$

where $e^C = \exp \mathcal{F}_C$*, and* $S_\lambda(X + \xi) = X + \lambda\xi$ *for the unique* $C \in \Omega^{p+1}_{closed}(M)$ *and* $\lambda \in G(\Omega^0_{closed}(M))$*. By* $G(\Omega^0_{closed}(M))$ *we mean the multiplicative group of everywhere non-zero closed* 0*-forms (locally constant functions).*

Similarly to the $p = 1$ Dorfman bracket, we expect something interesting to happen when we twist it with a non-trivial C-transformation. Let $C \in \Omega^{p+1}(M)$, and in general $dC \neq 0$. Define a new bracket $[\cdot,\cdot]'_D$ as

$$[e, e']'_D = e^{-C}[e^C(e), e^C(e')]_D, \tag{4.26}$$

for all $e, e' \in \Gamma(E)$. It turns out, due calculations similar to Section 3.6, that $[\cdot,\cdot]'_D = [\cdot,\cdot]^{dC}_D$, where the H-twisted higher Dorfman bracket $[\cdot,\cdot]^H_D$ is for given $H \in \Omega^{p+2}_{closed}(M)$ defined as

$$[X + \xi, Y + \eta]^H_D = [X + \xi, Y + \eta]_D - H(X, Y, \cdot), \tag{4.27}$$

for all $X + \xi$, $Y + \eta \in \Gamma(E)$. There holds also a complete analogue of Proposition 3.6.1, where all objects can straightforwardly be replaced by their $p > 1$ generalizations.

4.3 Induced metric

Before proceeding to an analogue of the generalized metric suitable for $E = TM \oplus \Lambda^p T^*M$, let us examine in detail the following construction. Let $g \in \Gamma(S^2T^*M)$ be an arbitrary metric on M. Our intention is to define an induced fiber-wise metric $\widetilde{g}$ on $\Lambda^p TM$. First, let us define a type $(0, 2p)$ tensor $\widetilde{g}$ on M as

$$\widetilde{g}(V_1, \dots, V_p, W_1, \dots, W_p) = \sum_{\sigma \in S_p} (-1)^{|\sigma|} g(V_{\sigma(1)}, W_1) \times \dots \times g(V_{\sigma(p)}, W_p), \tag{4.28}$$

for all $V_1, \dots, V_p, W_1, \dots, W_p \in \mathfrak{X}(M)$. First note that $\widetilde{g}$ is skew-symmetric in first and last p inputs. Moreover, one can interchange $(V_1, \dots, V_p)$ and $(W_1, \dots, W_p)$:

$$\widetilde{g}(V_1, \dots, V_p, W_1, \dots, W_p) = \widetilde{g}(W_1, \dots, W_p, V_1, \dots, V_p). \tag{4.29}$$

This proves that $\widetilde{g}$ defines a fiber-wise symmetric bilinear form on $\Lambda^p T^*M$. In local coordinates, it has the form

$$\widetilde{g}_{i_1 \dots i_p j_1 \dots j_p} = \delta^{k_1 \dots k_p}_{i_1 \dots i_p} g_{k_1 j_1} \dots g_{k_p j_p}. \tag{4.30}$$

To prove that $\widetilde{g}$ is a fiber-wise metric, define a type $(2p, 0)$ tensor $\widetilde{g}^{-1}$ on M as

$$\widetilde{g}^{-1}(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p) = \sum_{\sigma \in S_p} g^{-1}(\alpha_{\sigma(1)}, \beta_1) \times \dots \times g^{-1}(\alpha_{\sigma(p)}, \beta_p), \tag{4.31}$$

for all $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p \in \Omega^1(M)$. Now note that we can view $\widetilde{g}$ as a vector bundle morphism $\widetilde{g}$ from $\Lambda^p TM$ to $\Lambda^p T^*M$, and $\widetilde{g}^{-1}$ as a vector bundle morphism from $\Lambda^p T^*M$ to $\Lambda^p TM$. It is straightforward to check that $\widetilde{g}^{-1} \circ \widetilde{g} = 1$. This proves that $\widetilde{g}$ is non-degenerate and thus a fiber-wise metric on $\Lambda^p TM$.

There is now one interesting question to pose. What is the signature of $\widetilde{g}$ for given signature (r,s) of g? The answer is given by the following lemma

Lemma 4.3.1. *Let g be a metric of signature (r,s), and let $\widetilde{g}$ be the fiber-wise metric on $\Lambda^p TM$ defined by (4.28). Then $\widetilde{g}$ has the signature $(\binom{n}{p} - N(r,s,p), N(r,s,p))$, where the number $N(r,s,p)$ is given by a formula*

$$N(r,s,p) := \sum_{k=1}^{\lceil p/2 \rceil} \binom{s}{2k-1} \binom{r}{p-2k+1}. \tag{4.32}$$

Proof. Choose an orthonormal frame $(E_i)_{i=1}^n = (e_1, \dots, e_r, f_1, \dots, f_s)$ for g:

$$g(e_i, e_j) = \delta_{ij},\ g(f_i, f_j) = -\delta_{ij},\ g(e_i, f_j) = 0. \tag{4.33}$$

We can calculate $\widetilde{g}$ in the basis $E_I = E_{i_1} \wedge \dots \wedge E_{i_p}$. One gets $\widetilde{g}(E_I, E_J) = \pm \delta_I^J$, proving that E_I form an orthonormal basis for $\widetilde{g}$. It thus remains to track the $\pm$ sign. For given odd $j \in \{1, \dots, p\}$ there is $N_j(r,s,p) := \binom{s}{j}\binom{r}{p-j}$ different strictly ordered p-indices I, such that exactly j indices in I correspond to negative norm orthonormal basis vectors. These are precisely the p-indices I where $\widetilde{g}(E_I, E_I) = -1$. Resulting $N(r,s,p)$ is just a sum of $N_j(r,s,p)$ over all odd j:

$$N(r,s,p) = \sum_{j \text{ odd},\ 1 \leq j \leq p} \binom{s}{j} \binom{r}{p-j}.$$

This is exactly the formula (4.32). ■

We can now calculate some relevant examples. For a positive definite g, we have $(r,s) = (n,0)$, and thus $N(n,0,p) = 0$. This means that also $\widetilde{g}$ is positive definite. For Lorentzian g, we have two possibilities: $(r,s) = (1,d)$ or $(r,s) = (d,1)$. Note that for p even, one gets $N(r,s,p) = N(s,r,p)$.

- $(r,s) = (d,1)$: We get

$$N(d,1,p) = \sum_{k=1}^{\lceil p/2 \rceil} \binom{1}{2k-1} \binom{d}{p-2k+1} = \binom{d}{p-1}. \tag{4.34}$$

- $(r,s) = (1,d)$: For even p, we get $N(d,1,p) = N(1,d,p)$. For odd p, we obtain

$$N(1,d,p) = \sum_{k=1}^{\lceil p/2 \rceil} \binom{d}{2k-1} \binom{1}{p-2k+1} = \binom{d}{p}. \tag{4.35}$$

By construction of $\widetilde{g}$, it is clear that geometrical properties of $\widetilde{g}$ will follow from those of g. As an example, we can calculate its Lie derivative.

Lemma 4.3.2. *Let $\widetilde{g}$ be the fiber-wise metric (4.28). Then we have*

$$(\mathcal{L}_X \widetilde{g})(P,Q) = X.\widetilde{g}(P,Q) - \widetilde{g}(\mathcal{L}_X P, Q) - \widetilde{g}(P, \mathcal{L}_X Q), \tag{4.36}$$

for all $P, Q \in \mathfrak{X}^p(M)$, where on the left-hand side $\widetilde{g}$ is viewed as a tensor $\widetilde{g} \in \mathcal{T}^0_{2p}(M)$. Moreover, this Lie derivative can be calculated as

$$\begin{aligned}&(\mathcal{L}_X \widetilde{g})(V_1, \dots, V_p, W_1, \dots, W_p) = \\ &\sum_{\sigma \in S_p} (-1)^{|\sigma|} \sum_{k=1}^{p} g(V_{\sigma(1)}, W_1) \times \dots \times (\mathcal{L}_X g)(V_{\sigma(k)}, W_k) \times \dots \times g(V_{\sigma(p)}, W_p).\end{aligned} \tag{4.37}$$

In particular, $\mathcal{L}_X g = 0$ implies $\mathcal{L}_X \widetilde{g} = 0$.

Proof. Equation (4.36) follows from the definition of Lie derivative and the fact that it commutes with contractions. Equation (4.37) follows from the fact that

$$(\phi_t^{X*} \widetilde{g})(V_1, \dots, V_p, W_1, \dots, W_p) = \sum_{\sigma \in S_p} (-1)^{|\sigma|} (\phi_t^{X*} g)(V_{\sigma_1}, W_1) \times \dots \times (\phi_t^{X*} g)(V_{\sigma(p)}, W_p). \tag{4.38}$$

Now differentiate this at $t = 0$ to obtain (4.37). ■

Remark 4.3.3. The converse statement is not true. Consider $M = \mathbb{R}^2$, and the Minkowski metric

$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.39}$$

Its algebra of Killing vectors is spanned by generators of two translations and a Lorentz boost. For $p = 2$, the metric $\widetilde{g}$ is given by single component $\widetilde{g}_{(12)(12)} = -1$. If $X = X^1 \partial_1 + X^2 \partial_2$, the Killing equation for $\widetilde{g}$ gives

$$X^1{}_{,1} + X^2{}_{,2} = 0. \tag{4.40}$$

To be in Killing algebra of g, X^1 and X^2 have to have the form

$$X^1 = cx^2 + a, \ X^2 = cx^1 + b,$$

for $a, b, c \in \mathbb{R}$. Note that such X^1, X^2 indeed solve (4.40). On the other hand, (4.40) has many more solutions, for example $X^1 = f(x^2)$, $X^2 = 0$ for an arbitrary smooth function f. This shows that the Killing algebra of $\widetilde{g}$ can be strictly larger than the one of g. Also, note that Killing algebra for $\widetilde{g}$ does not have to be finite-dimensional.

Now, see that in chosen coordinates, one can view $\widetilde{g}_{IJ}$ as a square $\binom{n}{p} \times \binom{n}{p}$ matrix, and g_{ij} as a square $n \times n$ matrix. Are determinants of these matrices related?

Lemma 4.3.4. *Let A be a square $n \times n$ matrix, denote its components as $A^i{}_j$. Define an $\binom{n}{p} \times \binom{n}{p}$ matrix B labeled by strictly ordered p-indices I and J as*

$$B^I{}_J = \delta^I_{k_1 \dots k_p} A^{k_1}{}_{j_1} \dots A^{k_p}{}_{j_p}. \tag{4.41}$$

Then A is invertible if and only if B is invertible. Moreover, there holds a determinant formula:

$$\det(B) = [\det(A)]^{\binom{n-1}{p-1}}. \tag{4.42}$$

Proof. Let A be invertible. Define B^{-1} as

$$(B^{-1})^I{}_J = \delta^I_{k_1 \dots k_p} (A^{-1})^{k_1}{}_{j_1} \dots (A^{-1})^{k_p}{}_{j_p}. \tag{4.43}$$

It is straightforward to check that $B^I{}_J (B^{-1})^J{}_K = \delta^I_K$. The opposite statement follows from the formula (4.42). Its proof is more complicated and can be found in the Appendix A. ■

To conclude this section, we shall examine how a covariant derivative acts on $\widetilde{g}$. This will be important for the generalized Bismut connection in one of the following sections.

Lemma 4.3.5. *Let ∇ be any connection on M. Let $\widetilde{g}$ be the metric (4.28) viewed as $(0, 2p)$-tensor. Then we can write its covariant derivative as*

$$(\nabla_X \widetilde{g})(P, Q) = X.\widetilde{g}(P, Q) - \widetilde{g}(\nabla_X P, Q) - \widetilde{g}(P, \nabla_X Q), \tag{4.44}$$

for all $P, Q \in \mathfrak{X}^p(M)$. Moreover, one has

$$\begin{aligned}(\nabla_X \widetilde{g})(V_1, \dots, V_p, W_1, \dots, W_p) &= \\ \sum_{\sigma \in S_p} (-1)^{|\sigma|} \sum_{k=1}^{p} g(V_{\sigma(1)}, W_1) \times \cdots \times &(\nabla_X g)(V_{\sigma(k)}, W_k) \times \cdots \times g(V_{\sigma(p)}, W_p).\end{aligned} \tag{4.45}$$

In particular, if $\nabla_X g = 0$, then $\nabla_X \widetilde{g} = 0$.

Proof. Equation (4.44) follows from the definition of covariant derivative and the fact that it commutes with tensor contractions. Next, let $m \in M$, and γ be the integral curve of $X \in \mathfrak{X}(M)$ starting at m. Let τ^γ_t be the parallel transport from $m \equiv \gamma(0)$ to $\gamma(t)$ induced by connection ∇. Then

$$(\tau^\gamma_{-t} \widetilde{g})(V_1, \dots, V_p, W_1, \dots, W_p) = \sum_{\sigma \in S_p} (-1)^{|\sigma|} \sum_{k=1}^{p} (\tau^\gamma_{-t} g)(V_{\sigma(1)}, W_1) \times \cdots \times (\tau^\gamma_{-t} g)(V_{\sigma(p)}, W_p).$$

Differentiation of this equation with respect to t at $t = 0$ gives the assertion of the lemma. ■

4.4 Generalized metric

We would like to define a positive definite fiber-wise metric $\mathbf{G}$ on E of similar properties as the generalized metric on $TM \oplus T^*M$ defined in Section 3.8. There is no canonical fiber-wise metric on E, except for the $\Omega^{p-1}(M)$-valued pairing $\langle \cdot, \cdot \rangle_E$. The definition suitable for the generalization to E is the formal block form (3.93), in particular its form $\mathbf{G} = (e^{-C})^T \mathcal{G}_E e^{-C}$. For any metric g, define

$$\mathcal{G}_E = \begin{pmatrix} g & 0 \\ 0 & \widetilde{g}^{-1} \end{pmatrix}, \tag{4.46}$$

where $\widetilde{g}$ is the induced metric on $\Lambda^p TM$ defined by (4.28). Let $C \in \Omega^{p+1}(M)$, and let e^{-C} be the $O(E)$ map defined by (4.13). Then set

$$\mathbf{G} := \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & \widetilde{g}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C^T & 1 \end{pmatrix} = \begin{pmatrix} g + C\widetilde{g}^{-1}C^T & C\widetilde{g}^{-1} \\ \widetilde{g}^{-1}C^T & \widetilde{g}^{-1} \end{pmatrix}. \tag{4.47}$$

If g is positive definite, then so is $\mathbf{G}$. For g of a general signature (r,s), one can determine the signature of $\mathbf{G}$ using Lemma 4.3.1. The reason why we first consider only $\widetilde{g}$ induced by g and $C \in \Omega^{p+1}(M)$ follows from the physics - the inverse of $\mathbf{G}$ naturally appears in the Hamiltonian density of gauge-fixed Polyakov-like action for a p-brane. It appeared in exactly this form in the paper of Duff and Lu [30]. See also related concepts in [52] suitable for various M-geometries.

There is still one characterization, which could possibly survive the generalization, because the dual bundle E^* can be equipped with a $\mathfrak{X}^{p-1}(M)$-valued pairing $\langle \cdot, \cdot \rangle_{E^*}$, defined similarly to (4.1). We can then ask if $\mathbf{G}$ viewed as $\mathbf{G} \in \mathrm{Hom}(E, E^*)$ defines an orthogonal map with respect to $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_{E^*}$. But this is not true for $p > 1$. This can be most easily seen from the fact that $\mathcal{G}_E$ itself is not orthogonal for $p > 1$, and thus even in the $C = 0$ case, $\mathbf{G}$ is not orthogonal.

Note that in order to introduce the orthogonal transformations, we have to allow for more general fields in generalized metric. In particular, for $p > 1$ we would not assume that $\widetilde{g}$ is induced from g via (4.28). Moreover, the vector bundle morphism $C \in \mathrm{Hom}(\Lambda^p TM, T^*M)$ does not have to be induced by a $(p+1)$-form C. Note that every positive definite fiber-wise metric $\mathbf{G}$ on E can then be decomposed as (4.47). It will turn out that a generalized metric $\mathbf{G}'$ related to $\mathbf{G}$ by open-closed relations will not have $\widetilde{G}$ and G related by (4.28).

For positive definite g, the symmetric bilinear form $g + C\widetilde{g}^{-1}C^T$ is invertible, and thus defines a metric on M. Using the decomposition Lemma (3.1.2), we see that there exists a unique decomposition

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ -\Pi_N^T & 1 \end{pmatrix} \begin{pmatrix} G_N & 0 \\ 0 & \widetilde{G}_N^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\Pi_N \\ 0 & 1 \end{pmatrix}, \tag{4.48}$$

where the fields G_N, $\widetilde{G}_N$ and Π_N have the form

$$G_N = g + C\widetilde{g}^{-1}C^T, \tag{4.49}$$

$$\widetilde{G}_N = \widetilde{g} + C^T g^{-1} C, \tag{4.50}$$

$$\Pi_N = -(g + C\widetilde{g}^{-1}C^T)^{-1} C \widetilde{g}^{-1} = -g^{-1} C (\widetilde{g} + C^T g^{-1} C)^{-1}. \tag{4.51}$$

There is a historical reason behind the N subscript. Fields $(G_N, \widetilde{G}_N, \Pi_N)$ correspond to the Nambu sigma model dual to the membrane sigma model described by fields $(g, \widetilde{g}, C)$. Note that in general, $\widetilde{G}_N$ is not induced from G_N via (4.28), and $\Pi_N \in \mathrm{Hom}(\Lambda^p T^*M, TM)$ is not induced by $\Pi_N \in \mathfrak{X}^{p+1}(M)$.

Example 4.4.1. Let us show an example proving the preceding assertion. Consider $M = \mathbb{R}^3$, and let g be the Euclidean metric on $\mathbb{R}^3$. Consider $p = 2$. Let $(\partial_{(12)}, \partial_{(13)}, \partial_{(23)})$ be a local basis of $\mathfrak{X}^2(M)$. The induced metric $\widetilde{g}$ has the unit matrix in this basis. Any $C \in \mathrm{Hom}(\Lambda^2 TM, T^*M)$ induced by a $(p+1)$-form C has the matrix

$$C = \begin{pmatrix} 0 & 0 & c \\ 0 & -c & 0 \\ c & 0 & 0 \end{pmatrix},$$

where $c := C_{123}$. We have

$$G = \begin{pmatrix} 1+c^2 & 0 & 0 \\ 0 & 1+c^2 & 0 \\ 0 & 0 & 1+c^2 \end{pmatrix}, \; \widetilde{G} = \begin{pmatrix} 1+c^2 & 0 & 0 \\ 0 & 1+c^2 & 0 \\ 0 & 0 & 1+c^2 \end{pmatrix}.$$

This shows that $\widetilde{G}$ is not of the form (4.28) because such $\widetilde{G}$ must be quadratic in the elements of G, for example $\widetilde{G}_{(12)(12)} = \delta^{kl}_{12} G_{k1} G_{l2} = G_{12}^2 = (1+c^2)^2 \neq 1 + c^2$. The vector bundle morphism Π_N has the matrix

$$\Pi_N = \begin{pmatrix} 0 & 0 & -(1+c^2)^{-1}c \\ 0 & (1+c^2)^{-1}c & 0 \\ -(1+c^2)^{-1}c & 0 & 0 \end{pmatrix},$$

which shows that Π_N in this case *is induced* by a 3-vector $\Pi_N \in \mathfrak{X}^3(M)$.

Example 4.4.2. Finding a case when Π_N is not induced by $(p+1)$-vector is also not difficult, but one has to go to higher dimensions. Consider $n = 5$, $p = 2$, $M = \mathbb{R}^5$ and g the Euclidean metric. There are 10 strictly ordered 2-indices, let us order them lexicographically. Define a 3-form C as $C = dx^1 \wedge dx^2 \wedge dx^3 + dx^3 \wedge dx^4 \wedge dx^5$. We get $G_N = 2g + dx^3 \otimes dx^3$, and thus

$$G_N^{-1} = \frac{1}{2} g^{-1} - \frac{1}{6} \partial_3 \otimes \partial_3.$$

Then $(\Pi_N)^{iJ} = -G_N^{ik} C_{kJ}$, because $\widetilde{g}_{IJ} = \delta_I^J$. We can now simply calculate Π_N explicitly. One obtains

$$(\Pi_N)^{1(23)} = -\frac{1}{2},\ (\Pi_N)^{2(13)} = \frac{1}{2},\ (\Pi_N)^{3(12)} = -\frac{1}{3}. \tag{4.52}$$

This proves that Π_N is not induced by a 3-vector.

To conclude this section, we can briefly discuss the $p \geq 1$ analogue of the open-closed relations (3.119). Let $\Pi \in \mathrm{Hom}(\Lambda^p T^*M, TM)$ be any vector bundle morphism. Define new generalized metric $\mathbf{G}'$ as

$$\mathbf{G}' = (e^{\Pi})^T \mathbf{G} e^{\Pi}. \tag{4.53}$$

Recall that $e^{\Pi} \in \mathrm{Aut}(E)$ is defined as

$$e^{\Pi} = \begin{pmatrix} 1 & \Pi \\ 0 & 1 \end{pmatrix}. \tag{4.54}$$

We immediately see that this has a solution by rewriting (4.48) as $\mathbf{G}^{-1} = e^{\Pi_N} \mathcal{G}_N^{-1} (e^{\Pi_N})^T$, where $\mathcal{G}_N = \mathrm{BDiag}(G, \widetilde{G}^{-1})$. This proves that $\mathbf{G}'^{-1} = e^{(\Pi_N - \Pi)} \mathcal{G}_N^{-1} (e^{(\Pi_N - \Pi)})^T$. We thus have

$$G'_N = G_N,\ \widetilde{G}'_N = \widetilde{G}_N,\ \Pi'_N = \Pi_N - \Pi. \tag{4.55}$$

We see that everything works as in the case of ordinary open-closed relations. We can also use (4.55) to write down the explicit relations between $(g, \widetilde{g}, C)$ and new fields, usually denoted as $(G, \widetilde{G}, \Phi)$. We get

$$g + C\widetilde{g}^{-1}C^T = G + \Phi \widetilde{G}^{-1} \Phi^T, \tag{4.56}$$

$$\widetilde{g} + C^T g^{-1} C = \widetilde{G} + \Phi^T g^{-1} \Phi, \tag{4.57}$$

$$C\widetilde{g}^{-1} = \Phi \widetilde{G}^{-1} - (G + \Phi \widetilde{G}^{-1} \Phi^T)\Pi, \tag{4.58}$$

$$g^{-1} C = G^{-1} \Phi - \Pi(\widetilde{G} + \Phi^T G^{-1} \Phi). \tag{4.59}$$

Similarly to the above examples, $\widetilde{G}$ is in general not in the form (4.28), and Φ is not necessarily induced by a $(p+1)$-form $\Phi \in \Omega^{p+1}(M)$.

4.5 Doubled formalism

This section will provide a more rigid framework for the generalized metric $\mathbf{G}$ defined by (4.47). The main clue leading to this approach was the fact that open-closed relations (4.56 - 4.59) can be rewritten in the formal block matrix form

$$\begin{pmatrix} g & C \\ -C^T & \widetilde{g} \end{pmatrix}^{-1} = \begin{pmatrix} G & \Phi \\ -\Phi^T & \widetilde{G} \end{pmatrix}^{-1} + \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix}. \tag{4.60}$$

Let W be a vector bundle $W = TM \oplus \Lambda^p TM$. Define the following vector bundle morphisms:

$$\mathcal{G} = \begin{pmatrix} g & 0 \\ 0 & \widetilde{g} \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}, \ \mathcal{H} = \begin{pmatrix} G & 0 \\ 0 & \widetilde{G} \end{pmatrix}, \ \Xi = \begin{pmatrix} 0 & \Phi \\ -\Phi^T & 0 \end{pmatrix}, \ \Theta = \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix}. \tag{4.61}$$

We can now rewrite (4.60) in the form resembling the original open-closed relations (3.119):

$$(\mathcal{G} + \mathcal{B})^{-1} = (\mathcal{H} + \Xi)^{-1} + \Theta. \tag{4.62}$$

See that $\mathcal{G}$ and $\mathcal{H}$ are positive definite fiber-wise metrics on W, $\mathcal{B}, \Xi \in \Omega^2(W)$, and $\Theta \in \mathfrak{X}^2(W)$. This suggests that we should focus on the generalized geometry of W. In particular, to consider the vector bundle $V = W \oplus W^*$. This vector bundle is equipped with a natural pairing $\langle \cdot, \cdot \rangle_V$, and thus also with a natural orthogonal group $O(d,d)$, where $d = n + \binom{n}{p}$. This configuration allows one to define a generalized metric on V using the formalism of Section 3.8. By the generalized metric we mean all forms equivalent to Definition 3.8.1. Let us see how this allows one to describe the generalized metric of Section 4.4. Note that we do not assume that $C \in \Omega^{p+1}(M)$, and $\widetilde{g}$ is in general ont of the form (4.28).

Definition 4.5.1. Let $\mathbf{G}_V$ be a generalized metric on $V = W \oplus W^*$, where $W = TM \oplus \Lambda^p TM$. We can view $\mathbf{G}_V$ as an element of $\mathrm{Hom}(V, V^*)$. Note that E and E^* are subbundles of both V and V^*.

We say that $\mathbf{G}_V$ is a **relevant generalized metric**, if $\mathbf{G}_V(E) \subseteq E^*$.

Let us now show that the restriction of a relevant generalized metric $\mathbf{G}_V$ to the subbundle E is exactly the generalized metric $\mathbf{G}$ defined by (4.47). First, note that every $\mathbf{G}_V$ is uniquely determined by a positive definite metric $\mathcal{G}$ on W, $\mathcal{B} \in \Omega^2(W)$, and has the formal block form

$$\mathbf{G}_V = \begin{pmatrix} \mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B} & \mathcal{B}\mathcal{G}^{-1} \\ -\mathcal{G}^{-1}\mathcal{B} & \mathcal{G}^{-1} \end{pmatrix}. \tag{4.63}$$

It is straightforward to show that $\mathbf{G}_V$ is a relevant generalized metric, if and only if $\mathcal{G}$ and $\mathcal{B}$ have the form

$$\mathcal{G} = \begin{pmatrix} g & 0 \\ 0 & \widetilde{g} \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}, \tag{4.64}$$

where g is a Riemannian metric on M, $\widetilde{g}$ is a positive definite fiber-wise metric on $\Lambda^p TM$, and $C \in \mathrm{Hom}(\Lambda^p TM, T^*M)$. We have shown that $\mathbf{G}_V$ is uniquely determined by a map $\mathcal{A} = \mathcal{G} + \mathcal{B}$. This map now reads

$$\mathcal{A} = \begin{pmatrix} g & C \\ -C^T & \widetilde{g} \end{pmatrix}. \tag{4.65}$$

To see how $\mathbf{G}_V$ and $\mathbf{G}$ fit together, note that V can also be written as $V = E \oplus E^*$. Moreover, E and E^* are complementary maximally isotropic subbundles of V, and $\langle \cdot, \cdot \rangle_V$ coincides with the canonical pairing of E and E^*. Because $\mathbf{G}_V$ is a relevant generalized metric, we see that the involution $\mathcal{T}_V$ corresponding to $\mathbf{G}_V$ satisfies $\mathcal{T}_V(E) \subseteq E^*$. We can write $\mathcal{T}_V$ as a block matrix with respect to the splitting $V = E \oplus E^*$ as

$$\mathcal{T}_V = \begin{pmatrix} 0 & \mathbf{H} \\ \mathbf{G} & \mathbf{N} \end{pmatrix}. \tag{4.66}$$

Now $\mathcal{T}_V$ has to be symmetric with respect to $\langle \cdot, \cdot \rangle_V$, and $\mathcal{T}_V^2 = 1$. These two properties give $\mathbf{N} = 0$, and $\mathbf{H} = \mathbf{G}^{-1}$. An examination of $\mathbf{G}$ shows that it is exactly the generalized metric (4.47). Moreover, the corresponding eigenbundles $V_\pm$ have the form

$$V_\pm = \{e \pm \mathbf{G}(e) \mid e \in E\}. \tag{4.67}$$

In the isotropic splitting $V = E \oplus E^*$, a relevant generalized metric $\mathbf{G}_V$ is thus described by a pair $(\mathbf{G}, \mathbf{0})$, where $\mathbf{G}$ is a positive definite fiber-wise metric in E, and $\mathbf{0} \in \Omega^2(E)$. Again, let us emphasize that this description does not single out $\mathbf{G}$ where $\widetilde{g}$ is an induced metric (4.28), and $C \in \Omega^{p+1}(M)$.

We can consider $O(d,d)$ transformations an their action on the generalized metric $\mathbf{G}_V$, similarly to Section 3.9. Clearly, not for any $\mathcal{O}_V \in O(d,d)$, the new metric $\mathbf{G}'_V := \mathcal{O}_V^T \mathbf{G}_V \mathcal{O}_V$ is again a relevant one.

Example 4.5.2. Let us show some examples of $O(d,d)$-transformations. We will follow the structure of Example 3.9.5. We only have to discuss the conditions under which the new generalized metric $\mathbf{G}'_V$ becomes relevant. We assume that $\mathbf{G}_V$ is of the form (4.63), where $\mathcal{G}$ and $\mathcal{B}$ are parametrized by $(g, \widetilde{g}, C)$ as in (4.64).

- Let $\mathcal{Z} \in \Omega^2(V)$, and choose $\mathcal{O}_V = e^{-\mathcal{Z}}$. The new generalized metric $\mathbf{G}'_V = \mathcal{O}_V^T \mathbf{G}_V \mathcal{O}_V$ is described by a pair $(\mathcal{G}', \mathcal{B}')$, and we get

  $$\mathcal{G}' = \mathcal{G},\ \mathcal{B}' = \mathcal{B} + \mathcal{Z}. \tag{4.68}$$

  Clearly $(\mathcal{G}', \mathcal{B}')$ describes a relevant metric, if and only if $\mathcal{B}'$ is again block off-diagonal. This happens if and only if $\mathcal{Z}$ is block off-diagonal:

  $$\mathcal{Z} = \begin{pmatrix} 0 & Z \\ -Z^T & 0 \end{pmatrix}, \tag{4.69}$$

  where $Z \in \operatorname{Hom}(\Lambda^p TM, T^*M)$. The fiber-wise metric $\mathbf{G}'$ corresponding to $\mathbf{G}'_V$ is then described by a triplet $(g, \widetilde{g}, C + Z)$.

- Let $\Theta \in \mathfrak{X}^2(V)$. Define $\mathcal{O}_V = e^{\Theta}$. The new generalized metric $\mathbf{G}'_V = \mathcal{O}_V^T \mathbf{G}_V \mathcal{O}_V$ is described by a pair $(\mathcal{H}, \Xi)$, and we get the relation

  $$(\mathcal{G} + \mathcal{B})^{-1} = (\mathcal{H} + \Xi)^{-1} + \Theta. \tag{4.70}$$

  To see which Θ give a relevant $\mathbf{G}'_V$ is similar to the previous example - one just has to switch to the dual fields describing $\mathbf{G}_V$. In particular, if $\mathcal{A} = \mathcal{G} + \mathcal{B}$, let $\mathcal{A}^{-1} = \mathcal{G}_N^{-1} + \Theta_N$ for a positive definite fiber-wise metric $\mathcal{G}_N$ on V, and $\Theta_N \in \mathfrak{X}^2(V)$. One can show that for a relevant $\mathbf{G}_V$ they have the form

  $$\mathcal{G}_N = \begin{pmatrix} G_N & 0 \\ 0 & \widetilde{G}_N \end{pmatrix},\ \Theta_N = \begin{pmatrix} 0 & \Pi_N \\ -\Pi_N^T & 0 \end{pmatrix}, \tag{4.71}$$

where $(G_N, \widetilde{G}_N, \Pi_N)$ are the fields (4.49 - 4.51). Similarly to the $p=1$ case, we have $\Theta'_N = \Theta_N - \Theta$. This proves that $\mathbf{G}'_V$ is a relevant generalized metric, if and only if Θ is block off-diagonal:

$$\Theta = \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix}, \tag{4.72}$$

where $\Pi \in \mathrm{Hom}(\Lambda^p T^* M, TM)$. To conclude, if $\mathbf{G}'$ corresponding to $\mathbf{G}'_V$ is described by a triplet $(G, \widetilde{G}, \Phi)$, we get exactly the equation (4.60). This is in turn equivalent to the set of equations (4.56 - 4.59).

- Let $\mathcal{N} \in \mathrm{End}\, V$, and choose $\mathcal{O}_V = \mathcal{O}_{\mathcal{N}}$, where $\mathcal{O}_{\mathcal{N}}$ is defined as in (3.120). The new generalized metric $\mathbf{G}'_V = \mathcal{O}_V^T \mathbf{G}_V \mathcal{O}_V$ is described by a pair $(\mathcal{G}', \mathcal{B}')$, where

$$\mathcal{G}' = \mathcal{N}^T \mathcal{G} \mathcal{N},\ \mathcal{B}' = \mathcal{N}^T \mathcal{B} \mathcal{N}. \tag{4.73}$$

Criteria for $\mathcal{N}$ to give a relevant generalized metric $\mathbf{G}'_V$ are in this case more intricate. Indeed, let $\mathcal{N}$ have the block form

$$\mathcal{N} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}. \tag{4.74}$$

Then $\mathbf{G}'_V$ defines a relevant generalized metric, if and only if

$$N_1^T g N_2 + N_3^T g N_4 = 0, \tag{4.75}$$
$$N_2^T \widetilde{g} N_1 + N_4^T \widetilde{g} N_3 = 0, \tag{4.76}$$
$$N_2^T C N_4 - N_4^T C^T N_2 = 0, \tag{4.77}$$
$$N_1^T C N_3 - N_3^T C^T N_1 = 0. \tag{4.78}$$

Let $(g', \widetilde{g}', C')$ be the fields corresponding to the fiber-wise metric $\mathbf{G}'$. There are two simple solutions to this set of equations. First consider $N_2 = N_3 = 0$. This gives

$$g' = N_1^T g N_1,\ \widetilde{g}' = N_4^T \widetilde{g} N_4,\ C' = N_1^T C N_4. \tag{4.79}$$

The second example is $N_1 = N_4 = 0$. In this case, one obtains

$$g' = N_3^T \widetilde{g} N_3,\ \widetilde{g}' = N_2^T g N_2,\ C' = -N_3^T C^T N_2. \tag{4.80}$$

This example shows that criteria to give a relevant generalized metric are not universal - they can depend on the original generalized metric $\mathbf{G}_V$.

4.6 Leibniz algebroid for doubled formalism

We have just shown that the vector bundle $V = W \oplus W^*$ proves to be a natural way to describe the generalized geometry on the vector bundle E. To complete this discussion, we have to introduce a suitable bracket structure on V. In particular, we would like to define a bracket $[\cdot, \cdot]_V$, which will restrict to the higher Dorfman bracket on (4.15) on E. To do so, we will use the following lemma:

Lemma 4.6.1. *Let $(E, \rho, [\cdot,\cdot]_E)$ be a Leibniz algebroid. Then there is always a Leibniz algebroid structure on $V = E \oplus E^*$, restricting to $[\cdot,\cdot]_E$ on E. In particular, define*

$$[e + \alpha, e' + \alpha']_V = [e, e']_E + \mathcal{L}^E_e \alpha', \tag{4.81}$$

for all $e, e' \in \Gamma(E)$, and $\alpha, \alpha' \in \Gamma(E^*)$. The anchor $\rho_V \in \mathrm{Hom}(V, TM)$ is defined as $\rho_V = \rho \circ pr_1$. Then $(V, \rho_V, [\cdot,\cdot]_V)$ is a Leibniz algebroid. By $\mathcal{L}^E$ we mean the induced Lie derivative (2.7).

Proof. The Leibniz rule for $[\cdot,\cdot]_V$ follows from (2.10). For the Leibniz identity, we have

$$[e + \alpha, [e' + \alpha', e'' + \alpha'']_V]_V = [e, [e', e'']_E]_E + \mathcal{L}^E_e \mathcal{L}^E_{e'} e'', \tag{4.82}$$

$$[[e + \alpha, e' + \alpha']_V, e'' + \alpha'']_V = [[e, e']_E, e'']_E + \mathcal{L}^E_{[e,e']_E} e'', \tag{4.83}$$

$$[e' + \alpha', [e + \alpha, e'' + \alpha'']_V]_V = [e', [e, e'']_E]_E + \mathcal{L}^E_{e'} \mathcal{L}^E_e e''. \tag{4.84}$$

One now sees that the Leibniz identity for $[\cdot,\cdot]_V$ follows from the one of $[\cdot,\cdot]_E$ and its property (2.11). The bracket $[\cdot,\cdot]_V$ clearly restricts to $[\cdot,\cdot]_E$ on E. ∎

Now consider $E = TM \oplus \Lambda^p T^*M$ with the higher Dorfman bracket (4.15). We have already calculated $\mathcal{L}^E$ for this bracket, see (2.15). We will denote the sections of V as (X, P, α, ξ) for $X \in \mathfrak{X}(M)$, $P \in \mathfrak{X}^p(M)$, $\alpha \in \Omega^1(M)$, and $\xi \in \Omega^p(M)$. Then,

$$[(X, P, \alpha, \xi), (Y, Q, \beta, \eta)]_V = \big([X, Y], \mathcal{L}_X Q, \mathcal{L}_X \beta + (d\xi)(Q), \mathcal{L}_X \eta - i_Y d\xi\big), \tag{4.85}$$

for all $(X, P, \alpha, \xi), (Y, Q, \beta, \eta) \in \Gamma(V)$. We claim that this is the bracket most suitable to describe the generalized geometry on $E \subseteq V$. The bracket (4.85) was briefly mentioned by Hagiwara in [41] to describe Nambu-Dirac structures.

Remark 4.6.2. Note that with respect to the splitting $V = W \oplus W^*$, this bracket strongly resembles the usual Dorfman bracket. Indeed, see that there is a Leibniz algebroid bracket on W defined as

$$[(X, P), (Y, Q)]_W = ([X, Y], \mathcal{L}_X Q), \tag{4.86}$$

for all $(X, P), (Y, Q) \in \Gamma(W)$. Then $\mathcal{L}^W_{(X,P)}(\beta, \eta) = (\mathcal{L}_X \beta, \mathcal{L}_X \eta)$, and we have

$$[(X, P, \alpha, \xi), (Y, Q, \beta, \eta)]_V = \big([(X, P), (Y, Q)]_W, \mathcal{L}^W_{(X,P)}(\beta, \eta) - i_{(Y,Q)} d_W(\alpha, \xi)\big), \tag{4.87}$$

where $i_{(Y,Q)} d_W(\alpha, \xi)$ is a formal operation *defined* as $i_{(Y,Q)} d_W(\alpha, \xi) := (-(d\xi)(Q), i_Y d\xi)$. There is no actual definition of the differential d_W.

We have introduced the bracket (4.85) in order to be able to define Dirac structures of the vector bundle V. Since we have the fiber-wise metric $\langle \cdot, \cdot \rangle_V$, we can study involutive maximally isotropic subbundles of V. One can roughly follow Section 3.7. We will make use of the following simple technical Lemma

Lemma 4.6.3. *Let $T \in \mathcal{T}^p_q(M)$ be a tensor field on M, completely skew-symmetric in all upper indices, and in all lower indices (equivalently $T \in \mathrm{Hom}(\Lambda^p TM, \Lambda^q TM)$), such that $\mathcal{L}_X T = 0$ for all $X \in \mathfrak{X}(M)$. Then*

1. *For $p = q$, $T = \lambda \cdot 1$, where $\lambda \in \Omega^0_{closed}(M)$.*
2. *For $p \neq q$, $T = 0$.*

Proof. See Appendix A. ■

Example 4.6.4. Let us bring up some important examples of Dirac structures of V.

- By definition (Lemma 4.6.1), E and E^* define Dirac structures of V.
- Subbundles W and W^* define Dirac structures of V.
- Let $\mathcal{B} \in \Omega^2(W)$ be an arbitrary 2-form on W. We know that $e^{\mathcal{B}} \in O(d,d)$. This implies that $e^{\mathcal{B}}(W)$ is a maximally isotropic subbundle of V. Let $\mathcal{B}$ be of the block form

$$\mathcal{B} = \begin{pmatrix} B & C \\ -C^T & \widetilde{B} \end{pmatrix}, \tag{4.88}$$

where $B \in \Omega^2(M)$, $\widetilde{B} \in \Omega^2(\Lambda^p TM)$, and $C \in \mathrm{Hom}(\Lambda^p TM, T^*M)$. The subbundle $e^B(W)$ is a graph of $\mathcal{B} \in \mathrm{Hom}(W, W^*)$, that is

$$G_{\mathcal{B}} = \{(w, \mathcal{B}(w)) \mid w \in W\} \subseteq V. \tag{4.89}$$

The involutivity condition $[G_{\mathcal{B}}, G_{\mathcal{B}}]_V \subseteq G_{\mathcal{B}}$ gives two conditions

$$\mathcal{L}_X(B(Y) + C(Q)) + d(-C^T(X) + \widetilde{B}(P))(Q) = B([X,Y]) + C(\mathcal{L}_X Q), \tag{4.90}$$

$$\mathcal{L}_X(-C^T(Y) + \widetilde{B}(Q)) - i_Y d(-C^T(X) + \widetilde{B}(P)) = -C^T([X,Y]) + \widetilde{B}(\mathcal{L}_X Q)). \tag{4.91}$$

These two equations have to hold for all $X, Y \in \mathfrak{X}(M)$ and for all $P, Q \in \mathfrak{X}^p(M)$. First note that there is only one term containing the pair (P,Q) in the first condition. This implies $d(\widetilde{B}(P)) = 0$ for all $P \in \mathfrak{X}^p(M)$. Hence $df \wedge \widetilde{B}(P) = 0$ for all $P \in \mathfrak{X}^p(M)$. For $p < n$, this gives $\widetilde{B} = 0$. For $p = n$, there is no non-trivial $\widetilde{B}$, because $\Lambda^p TM$ has rank 1.

The part of (4.90) containing the pair (X,Y) gives $\mathcal{L}_X(B(Y)) = B([X,Y])$. This is equivalent to $\mathcal{L}_X B = 0$. Using Lemma 4.6.3, we get $B = 0$. There remain only two non-trivial equations. First, consider the part of (4.91) containing the pair (X,Y). One obtains

$$\mathcal{L}_X(C^T(Y)) - i_Y d(C^T(X)) = C^T([X,Y]). \tag{4.92}$$

This condition is $C^\infty(M)$-linear in Y, but in general not in X. This forces the following condition to hold for every $f \in C^\infty(M)$:

$$df \wedge [i_X C^T(Y) + i_Y C^T(X)] = 0. \tag{4.93}$$

We see that necessarily $i_X C^T(Y) + i_Y C^T(X) = 0$. This proves that C has to be induced by $C \in \Omega^{p+1}(M)$, $C^T(X) = i_X C$. Using this in (4.92) gives $i_Y i_X dC = 0$, that is $C \in \Omega^{p+1}_{closed}(M)$. There still remains one condition for the terms containing the pair (X,Q) in (4.90). One can show that it already follows from $dC = 0$. We have just proved that $G_{\mathcal{B}}$ is a Dirac structure of V, if and only if $\mathcal{B}$ is of the form

$$\mathcal{B} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}, \; C \in \Omega^{p+1}_{closed}(M). \tag{4.94}$$

Now note that we have three more isotropic subbundles which we can produce using $e^{\mathcal{B}}$. In particular $e^{\mathcal{B}}(W^*)$, $e^{\mathcal{B}}(E)$, and $e^{\mathcal{B}}(E^*)$. None of these other choices give something interesting.

- Let $\Theta \in \mathfrak{X}^2(W)$. Then $e^{\Theta}(W^*)$ is a maximally isotropic subbundle of V. Let Θ have the block form
$$\Theta = \begin{pmatrix} \pi & \Pi \\ -\Pi^T & \widetilde{\pi} \end{pmatrix}. \tag{4.95}$$
The subbundle $e^{\Theta}(W^*)$ is in fact the graph of the map Θ:
$$G_{\Theta} = \{(\Theta(\mu), \mu) \mid \mu \in \Gamma(W^*)\} \subseteq W \oplus W^*. \tag{4.96}$$
Let us examine the consequences of the involutivity condition $[G_{\Theta}, G_{\Theta}]_V \subseteq G_{\Theta}$. We get the set of two equations
$$\begin{aligned} [\pi(\alpha) + \Pi(\xi), \pi(\beta) + \Pi(\eta)] &= \pi\big(\mathcal{L}_{\pi(\alpha)+\Pi(\xi)}\beta + (d\xi)(-\Pi^T(\beta) + \widetilde{\pi}(\eta))\big) \\ &\quad + \Pi\big(\mathcal{L}_{\pi(\alpha)+\Pi(\xi)}\eta - i_{\pi(\beta)+\Pi(\eta)}d\xi\big), \end{aligned} \tag{4.97}$$
$$\begin{aligned} \mathcal{L}_{\pi(\alpha)+\Pi(\xi)}(-\Pi^T(\beta) + \widetilde{\pi}(\eta)) &= -\Pi^T\big(\mathcal{L}_{\pi(\alpha)+\Pi(\xi)}\beta + (d\xi)(-\Pi^T(\beta) + \widetilde{\pi}(\eta))\big) \\ &\quad + \widetilde{\pi}\big(\mathcal{L}_{\pi(\alpha)+\Pi(\xi)}\eta - i_{\pi(\beta)+\Pi(\eta)}d\xi\big). \end{aligned} \tag{4.98}$$
Since these two equations have to hold for all $\alpha, \beta \in \Omega^1(M)$ and $\xi, \eta \in \Omega^p(M)$, we can find these equivalent to the set of more simpler equations:
$$[\pi(\alpha), \pi(\beta)] = \pi(\mathcal{L}_{\pi(\alpha)}\beta), \tag{4.99}$$
$$[\pi(\alpha), \Pi(\eta)] = \Pi(\mathcal{L}_{\pi(\alpha)}\eta), \tag{4.100}$$
$$[\Pi(\xi), \pi(\beta)] = \pi\big(\mathcal{L}_{\Pi(\xi)}\beta - (d\xi)(\Pi^T(\beta))\big) - \Pi(i_{\pi(\beta)}d\xi), \tag{4.101}$$
$$[\Pi(\xi), \Pi(\eta)] = \pi\big((d\xi)(\widetilde{\pi}(\eta))\big) + \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi), \tag{4.102}$$
$$\mathcal{L}_{\pi(\alpha)}(\Pi^T(\beta)) = \Pi^T(\mathcal{L}_{\pi(\alpha)}\beta), \tag{4.103}$$
$$\mathcal{L}_{\pi(\alpha)}(\widetilde{\pi}(\eta)) = \widetilde{\pi}(\mathcal{L}_{\pi(\alpha)}\eta), \tag{4.104}$$
$$\mathcal{L}_{\Pi(\xi)}(\Pi^T(\beta)) = \Pi^T\big(\mathcal{L}_{\Pi(\xi)}\beta - (d\xi)(\Pi^T(\beta))\big) + \widetilde{\pi}(i_{\pi(\beta)}d\xi). \tag{4.105}$$
Let us focus on the very first equation. It can be rewritten as $\mathcal{L}_{\pi(\alpha)}\pi = 0$. It is not linear in α, which forces $\pi(\alpha) \wedge \pi(\beta) = 0$, for all $\alpha, \beta \in \Omega^1(M)$. This means that vector fields $\pi(\alpha)$ and $\pi(\beta)$ are linearly dependent for all α, β. This would mean that the map π has rank at most 1 everywhere. But π is skew-symmetric, and thus $\pi = 0$. This radically simplifies the rest of the equations. In particular, we are left with the two of them:
$$[\Pi(\xi), \Pi(\eta)] = \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi), \tag{4.106}$$
$$\mathcal{L}_{\Pi(\xi)}(\Pi^T(\beta)) = \Pi^T\big(\mathcal{L}_{\Pi(\xi)}\beta - (d\xi)(\Pi^T(\beta))\big). \tag{4.107}$$
First note that the first equation can be rewritten as
$$(\mathcal{L}_{\Pi(\xi)}\Pi)(\eta) = -\Pi(i_{\Pi(\eta)}d\xi), \tag{4.108}$$
where Π on the left-hand side is viewed as a type $(0, p+1)$ tensor. We will show in the next chapter that this condition forces Π to be induced by a $(p+1)$-vector $\Pi \in \mathfrak{X}^{p+1}(M)$, which is moreover a Nambu-Poisson tensor. Compare with (5.13) of Lemma 5.1.4 and use Lemma 5.1.6 to prove the skew-symmetry of Π.

The second condition equation can then easily be seen to be the transpose to the first one. Note that there is no condition on $\widetilde{\pi}$ whatsoever. We conclude that $e^{\Theta}(W^*) = G_{\Theta}$ is a Dirac structure, if and only if
$$\Theta = \begin{pmatrix} 0 & \Theta \\ -\Theta^T & \widetilde{\pi} \end{pmatrix}, \ \Theta \in \mathfrak{X}^{p+1}(M), \ \widetilde{\pi} \in \mathfrak{X}^2(\Lambda^p TM), \tag{4.109}$$
and Θ is a Nambu-Poisson tensor. The bivector $\widetilde{\pi}$ on $\Lambda^p TM$ can be completely arbitrary.

- Let $L \subseteq E$ be an arbitrary subbundle of E. We can define a subbundle Δ of V as

$$\Delta = L \oplus L^{\perp}, \tag{4.110}$$

where $L^{\perp} \subseteq E^*$ is the annihilator subbundle of L. By definition, Δ is a maximally isotropic subbundle of V. Then Δ is a Dirac structure of V iff L is involutive under higher Dorfman bracket (4.15).

To conclude this section, let us consider the twisting of the bracket $[\cdot,\cdot]_V$. We follow the idea of (4.26) and below. Let $\mathcal{B} \in \Omega^2(W)$ be of the same block form as in (4.88). We define a new bracket $[\cdot,\cdot]'_V$ as

$$[v, v']'_V = e^{-\mathcal{B}}[e^{\mathcal{B}}(v), e^{\mathcal{B}}(v')]_V, \tag{4.111}$$

for all $v, v' \in \Gamma(V)$. We expect to obtain the bracket in the form

$$[v, v']'_V = [v, v']_V - d\mathcal{B}(v, v'), \tag{4.112}$$

where $d\mathcal{B} : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ is to be determined now. The calculation shows that for $v = (X, P, \alpha, \xi)$, $v' = (Y, Q, \beta, \eta)$, we have $pr_W d\mathcal{B}(v, v') = 0$, and

$$\begin{aligned} pr_{T^*M}(d\mathcal{B}(v, v')) &= -\mathcal{L}_X(C(Q) + B(Y)) + d(C^T(X) - \widetilde{B}(P))(Q) \\ &\quad + B[X, Y] + C(\mathcal{L}_X Q), \end{aligned} \tag{4.113}$$

$$\begin{aligned} pr_{\Lambda^p T^*M}(d\mathcal{B}(v, v')) &= \mathcal{L}_X(C^T(Y) - \widetilde{B}(Q)) - i_Y d(C^T(X) - \widetilde{B}(P)) \\ &\quad - C^T([X, Y]) + \widetilde{B}(\mathcal{L}_X Q). \end{aligned} \tag{4.114}$$

Now, note that $d\mathcal{B}(v, v')$ is $C^\infty(M)$-linear in v', but for a general $\mathcal{B}$ it is not $C^\infty(M)$-linear in v. Let us now *require* this property. This implies that for any $f \in C^\infty(M)$ there must hold

$$df \wedge i_X B(Y) + (Y.f)B(X) = 0, \tag{4.115}$$

$$(df \wedge \widetilde{B}(P))(Q) = 0. \tag{4.116}$$

The first condition can be rewritten as $i_Y(df \wedge B(X)) = 0$. It then follows that these two conditions imply $B = \widetilde{B} = 0$. Next, from (4.114) we obtain the condition

$$df \wedge (i_X C^T(Y) + i_Y C^T(X)) = 0. \tag{4.117}$$

This proves that C has to be induced by $C \in \Omega^{p+1}(M)$. Let us see how $d\mathcal{B}$ looks now. We obtain

$$d\mathcal{B}(v, v') = \big(0, 0, -(i_X dC)(Q), i_Y i_X dC\big). \tag{4.118}$$

One can conclude that the twist by $e^{\mathcal{B}}$ defines a $C^\infty(M)$-bilinear map $d\mathcal{B}$, if and only if $B = \widetilde{B} = 0$, and $C \in \mathrm{Hom}(\Lambda^p TM, T^*M)$ is induced by a $(p+1)$-form $C \in \Omega^{p+1}(M)$.

4.7 Killing sections of generalized metric

One can naturally generalize Section 3.10 to the $p \geq 1$ setting. In particular, we can define Killing sections as in (3.133). Also the derivation of the explicit form of Killing equations is very similar. Assume that $\widetilde{g}$ is of the form (4.28), and $C \in \Omega^{p+1}(M)$. One can see that $e \in \Gamma(E)$ is a Killing section of $\mathbf{G} = (e^{-C})^T \mathcal{G}_E e^{-C}$, if and only if there holds

$$\rho(e^{-C}(e)).\mathcal{G}_E(f', f'') = \mathcal{G}_E([e^{-C}(e), f']_D^{dC}, f'') + \mathcal{G}_E(f', [e^{-C}(e), f'']_D^{dC}), \tag{4.119}$$

for all $f', f'' \in \Gamma(E)$. Let $f := e^{-C}(e)$. We can thus study the Killing equation for the section f and the metric $\mathcal{G}_E$, but now using the twisted bracket $[\cdot,\cdot]_D^{dC}$. Let $f = X' + \xi'$ for $X' \in \mathfrak{X}(M)$ and $\xi' \in \Omega^p(M)$, and write $f' = Y + \eta$, $f'' = Z + \zeta$. One gets

$$\begin{aligned} X'.\{g(Y,Z) + \widetilde{g}^{-1}(\eta,\zeta)\} = {} & g([X',Y],Z) + g(Y,[X',Z]) \\ & + \widetilde{g}^{-1}\big(\mathcal{L}_{X'}\eta - i_Y d\xi' - dC(X',Y,\cdot),\zeta\big) \\ & + \widetilde{g}^{-1}\big(\eta, \mathcal{L}_{X'}\zeta - i_Z d\xi' - dC(X',Z,\cdot)\big). \end{aligned} \tag{4.120}$$

This yields a set of four separate equations

$$X'.g(Y,Z) = g([X',Y],Z) + g(Y,[X',Z]), \tag{4.121}$$

$$X'.\widetilde{g}^{-1}(\eta,\zeta) = \widetilde{g}^{-1}(\mathcal{L}_{X'}\eta,\zeta) + \widetilde{g}^{-1}(\eta,\mathcal{L}_{X'}\zeta), \tag{4.122}$$

$$0 = \widetilde{g}^{-1}(-i_Y d\xi' - dC(X',Y,\cdot),\zeta), \tag{4.123}$$

$$0 = \widetilde{g}^{-1}(\eta, -i_Z d\xi' - dC(X',Z,\cdot)). \tag{4.124}$$

The first equation is an ordinary Killing equation for the vector field X', that is $\mathcal{L}_{X'}g = 0$. The second equation yields $\mathcal{L}_{X'}\widetilde{g}^{-1} = 0$. This in turn yields $\mathcal{L}_{X'}\widetilde{g} = 0$. For $\widetilde{g}$ of the form (4.28), Lemma 4.3.2 shows that this already follows from $\mathcal{L}_{X'}g = 0$. Last two equations force

$$d\xi' = -i_{X'}dC. \tag{4.125}$$

Now let $e = X + \xi$. Then $X' = X$, and $\xi' = \xi + i_X C$. Plugging into the above conditions gives:

Proposition 4.7.1. *Let $e = X + \xi$. Then e satisfies a generalized Killing equation (3.133) for $\mathbf{G} = (e^{-C})^T \mathcal{G}_E e^{-C}$, if and only if the following conditions hold:*

$$\mathcal{L}_X g = 0,\ d\xi = -\mathcal{L}_X C. \tag{4.126}$$

Moreover, one can also restate Proposition 3.10.4 for the $p > 1$ case. It has the completely same form and there is no reason to repeat it here explicitly.

4.8 Generalized Bismut connection II

We will now generalize the notion of generalized Bismut connection defined in Section 3.12. Let $\mathbf{G}$ be a generalized metric (4.47) on $E = TM \oplus \Lambda^p T^*M$. We assume that $\widetilde{g}$ is of the form (4.28). Let $H = dC$. We are looking for an example of the vector bundle connection on E compatible with $\mathbf{G}$. The most straightforward way is to use the form (3.166) and replace g with $\widetilde{g}$ where necessary to make it work. We define the connection ∇ as

$$\nabla_X = \begin{pmatrix} 1 & 0 \\ -C^T & 1 \end{pmatrix} \begin{pmatrix} \nabla_X^{LC} & \frac{1}{2} g^{-1} H(X,\cdot,\widetilde{g}^{-1}(\star)) \\ -\frac{1}{2} H(X,\star,\cdot) & \nabla_X^{LC} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C^T & 1 \end{pmatrix}. \tag{4.127}$$

By ∇^{LC} we mean the Levi-Civita connection of g acting on vector fields and on p-forms. Using Lemma 4.3.5 it is easy to check that ∇_X is metric compatible with $\mathbf{G}$.

We can now examine this connection from a more conceptual viewpoint, using the doubled formalism on $V = W \oplus W^*$. Let $\mathbf{G}_V$ be the generalized metric on $V = W \oplus W^*$ corresponding to $\mathbf{G}$, and let $\mathbf{\Psi}_\pm : W \to V_\pm$ be the two isomorphisms induced by $\mathbf{G}_V$. Let $\mathcal{G}$ and $\mathcal{B}$ be the fields corresponding to the generalized metric $\mathbf{G}_V$. Let ∇^* be the dual connection induced by ∇ on the vector bundle E^*. Explicitly

$$\nabla^*_X = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \nabla^{LC}_X & \frac{1}{2} H(X, \cdot, \star) \\ -\frac{1}{2} \widetilde{g}^{-1} H(X, g^{-1}(\star), \cdot) & \nabla^{LC}_X \end{pmatrix} \begin{pmatrix} 1 & -C \\ 0 & 1 \end{pmatrix}. \tag{4.128}$$

Define a new connection ∇^V on $V = E \oplus E^*$ as a block diagonal combination of ∇ and ∇^*:

$$\nabla^V_X (e + \alpha) = \nabla_X e + \nabla^*_X \alpha, \tag{4.129}$$

for all $e \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$. By construction, ∇^V is compatible with the generalized metric $\mathbf{G}_V$. Moreover, it can be written in a way resembling the original generalized Bismut connection. Define a bilinear map $\mathcal{H} : \Gamma(W) \times \Gamma(W) \to \Gamma(W^*)$ using the twisting map $d\mathcal{B}$ defined by (4.118). Let $w, w' \in \Gamma(W)$. We can view them as elements of $\Gamma(V)$. Define

$$\mathcal{H}(w, w') = pr_{W^*} d\mathcal{B}(w, w'). \tag{4.130}$$

With respect to the splitting $V = W \oplus W^*$, the connection ∇^V can be written in the block form

$$\nabla^V_X = \begin{pmatrix} 1 & 0 \\ \mathcal{B} & 1 \end{pmatrix} \begin{pmatrix} \nabla^{LC}_X & -\frac{1}{2} \mathcal{G}^{-1} \mathcal{H}(X, \mathcal{G}^{-1}(\star)) \\ -\frac{1}{2} \mathcal{H}(X, \star) & \nabla^{LC}_X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{B} & 1 \end{pmatrix}, \tag{4.131}$$

where ∇^{LC} acts diagonally on $W = TM \oplus \Lambda^p TM$ and on $W^* = T^*M \oplus \Lambda^p T^*M$. Note that this connection is also compatible with the natural pairing $\langle \cdot, \cdot \rangle_V$ on V, that is

$$X.\langle v, v' \rangle_V = \langle \nabla^V_X v, v'' \rangle_V + \langle v, \nabla^V_X v'' \rangle_V, \tag{4.132}$$

for all $v, v' \in \Gamma(V)$. By definition, we obtain ∇ by restriction of ∇^V onto the subbundle $E \subseteq V$. In accordance with Lemma 3.12.2, one can write ∇^V using the isomorphisms $\mathbf{\Psi}_\pm$ and a pair of connections $\nabla^\pm$ on the vector bundle W. One gets

$$\nabla_{\mathbf{\Psi}_\pm(w)} (\mathbf{\Psi}_+(w')) = \mathbf{\Psi}_+(\nabla^+_w w'), \tag{4.133}$$
$$\nabla_{\mathbf{\Psi}_\pm(w')} (\mathbf{\Psi}_-(w')) = \mathbf{\Psi}_-(\nabla^-_w w'), \tag{4.134}$$

for all $w, w' \in \Gamma(W)$, and $\nabla^\pm$ is a pair of connections on W defined as

$$\nabla^\pm_w w' = \nabla^{LC}_{pr_1 w} w' \mp \frac{1}{2} \mathcal{G}^{-1} \mathcal{H}(w, w'). \tag{4.135}$$

Returning to the original connection ∇, we may calculate its torsion and curvature operators. Define a simpler connection $\widehat{\nabla}$ using the formula $\nabla_X = e^C \widehat{\nabla}_X e^{-C}$. This is a connection compatible with $\mathcal{G}_E$, having the form of the middle block as in (4.127). By definition, it will have the same scalar curvature as ∇. Let $X, Y \in \mathfrak{X}(M)$ and $Z + \zeta \in \Gamma(E)$. We get

$$\begin{aligned}
\widehat{R}_1(X,Y)(Z+\zeta) = {} & R^{LC}(X,Y)Z \\
& + \frac{1}{2}g^{-1}(\nabla^{LC}_X H)(Y,\cdot,\widetilde{g}^{-1}(\zeta)) - \frac{1}{2}g^{-1}(\nabla^{LC}_Y H)(X,\cdot,\widetilde{g}^{-1}(\zeta)) \\
& - \frac{1}{4}g^{-1}H(X,\cdot,\widetilde{g}^{-1}H(Y,Z,\cdot)) + \frac{1}{4}g^{-1}H(Y,\cdot,\widetilde{g}^{-1}H(X,Z,\cdot)),
\end{aligned} \tag{4.136}$$

$$\begin{aligned}
\widehat{R}_2(X,Y)(Z+\zeta) = {} & R^{LC}(X,Y)\zeta \\
& - \frac{1}{2}(\nabla^{LC}_X H)(Y,Z,\cdot) + \frac{1}{2}(\nabla^{LC}_Y H)(X,Z,\cdot) \\
& + \frac{1}{4}H(X,g^{-1}H(Y,\cdot,\widetilde{g}^{-1}(\zeta)),\cdot) - \frac{1}{4}H(Y,g^{-1}H(X,\cdot,\widetilde{g}^{-1}(\zeta)),\cdot)
\end{aligned} \tag{4.137}$$

The Ricci tensor $\widehat{\mathrm{Ric}}$ has only two non-trivial components. Namely, we get

$$\widehat{\mathrm{Ric}}(X,Y) = \mathrm{Ric}^{LC}(X,Y) + \frac{1}{4}H(Y,g^{-1}(dy^k),\widetilde{g}^{-1}H(\partial_k,X,\cdot)), \tag{4.138}$$

$$\widehat{\mathrm{Ric}}(\xi,Y) = \frac{1}{2}(\nabla^{LC}_{\partial_k} H)(Y,g^{-1}(dy^k),\widetilde{g}^{-1}(\xi)). \tag{4.139}$$

Finally, define a scalar curvature $\widehat{\mathcal{R}} = \widehat{\mathrm{Ric}}(\mathcal{G}_E^{-1}(e^\lambda), e_\lambda)$. One obtains

$$\widehat{\mathcal{R}} = \mathcal{R}(g) - \frac{1}{4}H_{ijK}H^{ijK}. \tag{4.140}$$

The indices of H are raised by metric g, and $\mathcal{R}(g)$ is the scalar curvature of the Levi-Civita connection of g. Let $\mathcal{R}$ be the scalar curvature of the original connection ∇ defined using the generalized metric $\mathbf{G} = (e^{-C})^T \mathcal{G}_E e^{-C}$, that is $\mathcal{R} = \mathrm{Ric}(\mathbf{G}^{-1}(e^\lambda), e_\lambda)$. By construction, the two connections have the same scalar curvature, that is we get $\mathcal{R} = \widehat{\mathcal{R}}$.

Chapter 5

Nambu-Poisson structures

In this chapter, we will discuss in detail the definitions and structures induced by a Nambu bracket on a manifold. This $(p+1)$-ary bracket was for $p=2$ introduced in 1972 by Y. Nambu in [78] as an attempt to generalize the classical Hamiltonian mechanics. Nambu defines a trinary bracket $\{f,g,h\}$ for a triplet of functions of three variables (x,y,z) as

$$\{f,g,h\} = \frac{\partial(f,g,h)}{\partial(x,y,z)}. \tag{5.1}$$

He notes that such a bracket has some remarkable properties, in particular it is completely skew-symmetric and it satisfies the Leibniz rule.

$$\{ff',g,h\} = f\{f',g,h\} + \{f,g,h\}f'. \tag{5.2}$$

Interestingly, he does not attempt to generalize the third usual property of Poisson bracket, the Jacobi identity. An axiomatic definition of Nambu brackets was introduced more than twenty years later in [87], where the term Nambu-Poisson manifold appears for the first time. The author already suspects and emphasizes throughout his paper that Nambu-Poisson structures are much more rigid objects than Poisson structures. This was finally proved by several different people in 1996, for example in [3]. For the complete list of references and many more interesting remarks, see the survey [88] of I. Vaisman.

5.1 Nambu-Poisson manifolds

Definition 5.1.1. Let $p \geq 1$ be a fixed integer, and let $\{\cdot,\dots,\cdot\} : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)$ be an $\mathbb{R}$-$(p+1)$-linear map. We say that $\{\cdot,\dots,\cdot\}$ is a **Nambu bracket**, if the following properties hold:

1. The map $\{\cdot,\dots,\cdot\}$ is completely skew-symmetric.

2. It satisfies the Leibniz rule:

$$\{f_1,\dots,f_{p+1} \cdot g_{p+1}\} = \{f_1,\dots,f_{p+1}\}g_{p+1} + f_{p+1}\{f_1,\dots,g_{p+1}\}. \tag{5.3}$$

3. It satisfies the **fundamental identity**:

$$\begin{aligned}\{f_1,\ldots,f_p,\{g_1,\ldots,g_{p+1}\}\} = \{\{f_1,\ldots,f_p,g_1\},\ldots,g_{p+1}\} \\ + \cdots + \{g_1,\ldots,\{f_1,\ldots,f_p,g_{p+1}\}\}.\end{aligned} \tag{5.4}$$

Both the Leibniz rule and the fundamental identity are assumed to hold for all involved smooth functions.

We see that for $p = 1$, the definition reduces to the ordinary Poisson bracket on M. Next, note that for any $f \in C^\infty(M)$, the bracket $\{g_1,\ldots,g_p\}' := \{f, g_1,\ldots,g_p\}$ defines again a Nambu-Poisson bracket. Both axioms can be read as follows. To any p-tuple $(f_1,\ldots,f_p)$ of smooth functions, we may assign an operator

$$X_{(f_1,\ldots,f_p)} := \{f_1,\ldots,f_p,\cdot\}. \tag{5.5}$$

Leibniz rule proves that $X_{(f_1,\ldots,f_p)}$ is a vector field on M, $X_{(f_1,\ldots,f_p)} \in \mathfrak{X}(M)$. The fundamental identity then requires $X_{(f_1,\ldots,f_p)}$ to be a derivation of the bracket $\{\cdot,\ldots,\cdot\}$.

It follows from the Leibniz rule (5.3) that $\{\cdot,\ldots,\cdot\}$ in fact depends only on differentials of incoming functions. It thus makes sense to define a **Nambu-Poisson tensor** Π by

$$\Pi(df_1,\ldots,df_{p+1}) := \{f_1,\ldots,f_{p+1}\}, \tag{5.6}$$

for all $f_1,\ldots,f_{p+1} \in C^\infty(M)$. The complete skew-symmetry of the bracket $\{\cdot,\ldots,\cdot\}$ is clearly equivalent to Π being a $(p+1)$-vector, $\Pi \in \mathfrak{X}^{p+1}(M)$. The fundamental identity can be then rewritten simply as

$$\mathcal{L}_{X_{(f_1,\ldots,f_p)}}\Pi = 0. \tag{5.7}$$

Now, let us recall the fundamental theorem for the theory of Nambu-Poisson manifolds. The proof can be found for example in [3, 31, 88], and we thus omit it here.

Theorem 5.1.2. *Let $p \geq 2$, and $\Pi \in \mathfrak{X}^{p+1}(M)$. Then Π is a Nambu-Poisson tensor, if and only if for every $x \in M$, such that $\Pi(x) \neq 0$, there is a neighborhood $U \ni x$, and a set of local coordinates $(x^1,\ldots,x^n)$ on U, such that locally*

$$\Pi = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{p+1}}. \tag{5.8}$$

The components of Π in this coordinates are thus $\Pi^{i_1\ldots i_{p+1}} = \epsilon^{i_1\ldots i_{p+1}}$, and the corresponding bracket $\{\cdot,\ldots,\cdot\}$ has the local form

$$\{f_1,\ldots,f_{p+1}\} = \frac{\partial(f_1,\ldots,f_{p+1})}{\partial(x^1,\ldots,x^{p+1})} \tag{5.9}$$

We will call $(x^1,\ldots,x^n)$ the ***Darboux coordinates*** *for Π.*

Note that the only if part of this theorem is not true for $p = 1$. The simple counter-example is the canonical Poisson structure Π on $\mathbb{R}^{2n}$, $n > 1$:

$$\Pi = \sum_{j=1}^{n} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial p_j}. \tag{5.10}$$

In these coordinates, the component matrix Π^{ij} is invertible. On the other hand, the matrix in coordinates (5.8) has always the rank 2, hence it cannot be invertible. This proves that Π defined by (5.10) is not decomposable. For Poisson manifolds, there holds a more subtle statement, called Darboux-Weinstein theorem [90]. We will now use Theorem 5.1.2 to reformulate the fundamental identity in several ways more useful for our purposes.

Lemma 5.1.3. *Let $\Pi \in \mathfrak{X}^{p+1}(M)$, and $p > 1$. Then Π satisfies the fundamental identity (5.7), if and only if there holds*

$$\mathcal{L}_{\Pi(\xi)}\Pi = -\langle \Pi, d\xi \rangle \Pi, \tag{5.11}$$

for all $\xi \in \Omega^p(M)$.

Proof. If part is simple, for any p-tuple $(f_1, \dots, f_p)$ choose $\xi = df_1 \wedge \dots \wedge df_p$. Then $\Pi(\xi) = (-1)^p X_{(f_1,\dots,f_p)}$, and since $d\xi = 0$, we get $\mathcal{L}_{X_{(f_1,\dots,f_p)}}\Pi = 0$.

Conversely, assume that Π is a Nambu-Poisson tensor. It suffices to prove (5.11) for ξ in the form $\xi = g \cdot df_1 \wedge \dots \wedge df_p$, where $g, f_1, \dots, f_p \in C^\infty(M)$. At points $x \in M$ where $\Pi(x) = 0$ the statement holds trivially. We can thus assume that we can work locally with Π in the form (5.8). We have

$$\mathcal{L}_{\Pi(\xi)}\Pi = (-1)^p \mathcal{L}_{gX_{(f_1,\dots,f_p)}}\Pi = (-1)^{p+1} X_{(f_1,\dots,f_p)} \wedge i_{dg}\Pi.$$

We have used the fundamental identity (5.7) to get rid of one term. We can write

$$i_{dg}\Pi = \sum_{r=1}^{p} (-1)^{r+1} \frac{\partial g}{\partial x^r} \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^{r-1}} \wedge \frac{\partial}{\partial x^{r+1}} \wedge \dots \wedge \frac{\partial}{\partial x^p},$$

and consequently, we obtain

$$\begin{aligned}
(-1)^{p+1} X_{(f_1,\dots,f_p)} \wedge i_{dg}\Pi &= (-1)^{p+1} \sum_{r=1}^{p} \frac{\partial}{\partial x^1} \wedge \dots \wedge X^r_{(f_1,\dots,f_p)} \frac{\partial g}{\partial x^r} \frac{\partial}{\partial x^r} \wedge \dots \wedge \frac{\partial}{\partial x^p} \\
&= (-1)^{p+1} (\sum_{r=1}^{p} X^r_{(f_1,\dots,f_p)} \frac{\partial g}{\partial x^r})\Pi = (-1)^{p+1}(X_{(f_1,\dots,f_p)}.g)\Pi \\
&= (-1)^{p+1} \langle \Pi, df_1 \wedge \dots \wedge df_p \wedge dg \rangle \Pi = -\langle \Pi, d\xi \rangle \Pi.
\end{aligned}$$

This proves the assertion of the Lemma. ∎

Note that (5.11) does not hold for $p = 1$. To include this case, one must modify it as

$$\mathcal{L}_{\Pi(\xi)}\Pi = -(\langle \Pi, d\xi \rangle \Pi - \frac{1}{p+1} i_{d\xi}(\Pi \wedge \Pi)). \tag{5.12}$$

For $p > 1$, the decomposability implies $\Pi \wedge \Pi = 0$, and the proof is then just an easy alteration. For $p = 1$, this is not necessarily true, as illustrates the example (5.10). The proof of (5.12) is in this case left for an interested reader. We will now derive a more conceptual reformulation of the fundamental identity, which will give an immediate geometrical description of Nambu-Poisson structures.

Lemma 5.1.4. *Let $p \geq 1$, and $\Pi \in \mathfrak{X}^{p+1}(M)$. Then Π is a Nambu-Poisson tensor iff*

$$[\mathcal{L}_{\Pi(\xi)}\Pi](\eta) = -\Pi(i_{\Pi(\eta)} d\xi), \tag{5.13}$$

for all $\xi, \eta \in \Omega^p(M)$.

Proof. For $p = 1$, this follows from

$$(\mathcal{L}_{\Pi(\xi)}\Pi)(\eta) + \Pi(i_{\Pi(\eta)} d\xi) = \frac{1}{2} i_\eta i_\xi [\Pi, \Pi]_S, \tag{5.14}$$

for all $\xi, \eta \in \Omega^1(M)$. This can be verified directly in coordinates, or using several explicit forms of the Schouten-Nijenhuis bracket $[\cdot,\cdot]_S$. See for example [67].

We will focus on the $p > 1$ case here. Assume that Π is a Nambu-Poisson tensor. At points where $\Pi(x) = 0$, the equation (5.13) holds trivially. We can thus again assume that Π is of the form (5.8). Note that (5.13) is $C^\infty(M)$-linear in η. Moreover, Π and $\Pi(\xi)$ only have components with indices ranging only in $\{1, \dots, p+1\}$. We thus have to check (5.13) only for η in the form $\eta = dx^{[r]} := dx^1 \wedge \ldots \wedge \widehat{dx^r} \wedge \ldots \wedge dx^{p+1}$, where $r \in \{1, \dots, p+1\}$ and $\widehat{dx^r}$ denotes the omitted term. Choose one such η. Both sides of (5.13) are vector fields. If we examine the k-th component of the both sides for $k \neq r$, the left hand side vanishes. The right-hand side gives $(-1)^{r+1}\epsilon^{kJ}(d\xi)_{rJ}$. The only non-trivial contribution to the sum can come from $J = [k] := (1, \dots, \hat{k}, \dots, p+1)$, but then $(d\xi)_{r[k]} = 0$ because $r \in [k]$. Thus also the right-hand side vanishes. For $k = r$, the left-hand side gives

$$(\mathcal{L}_{\Pi(\xi)}\Pi)(dx^{[r]})^r = (\mathcal{L}_{\Pi(\xi)}\Pi)^{r[r]} = (-1)^{r+1}(\mathcal{L}_{\Pi(\xi)}\Pi)^{1\dots p+1} = (-1)^r \sum_{q=1}^{p+1} \xi_{J,q}\epsilon^{qJ} = (-1)^r (d\xi)_{1\dots p}.$$

The right-hand side can be rewritten as

$$-[\Pi(i_{\Pi(dx^{[r]})}d\xi)]^r = (-1)^r \Pi^{rJ}(d\xi)_{rJ} = (-1)^r (d\xi)_{1\dots p}.$$

A comparison of both sides gives the result. Conversely, if (5.13) holds, we can plug in $\xi = df_1 \wedge \ldots \wedge df_p$ to obtain the fundamental identity (5.7). ■

We can immediately use this lemma to prove some important observations. First note that the identity (5.13) has two parts, differential and algebraical. To see this, rewrite it in some local coordinates $(y^1, \dots, y^n)$. We write $\xi = \xi_I dy^I$, $\eta = dy^J$, and take the k-th component of the result. We get

$$\begin{aligned}[(\mathcal{L}_{\Pi(\xi)}\Pi)(\eta)]^k = (\mathcal{L}_{\Pi(\xi)}\Pi)^{kJ} &= \xi_I \big(\Pi^{nI}\Pi^{kJ}{}_{,n} - \Pi^{kI}{}_{,n}\Pi^{nJ} - \sum_{q=1}^{p} \Pi^{j_q I}{}_{,n}\Pi^{kj_1\dots n\dots j_p}\big) \\ &\quad - \xi_{I,n}\big(\Pi^{kI}\Pi^{nJ} + \sum_{q=1}^{p} \Pi^{j_q I}\Pi^{kj_1\dots n\dots j_p}\big).\end{aligned}$$

The right-hand side gives

$$-\Pi(i_{\Pi(\eta)}d\xi)^k = -\Pi^{kL}(d\xi)_{mL}\Pi^{mJ} = -\xi_{I,n}\delta^{nI}_{mL}\Pi^{kL}\Pi^{mJ}.$$

The terms proportional to ξ_I give the *differential part* of the fundamental identity:

$$\Pi^{nI}\Pi^{kJ}{}_{,n} - \Pi^{kI}{}_{,n}\Pi^{nJ} - \sum_{q=1}^{p} \Pi^{j_q I}{}_{,n}\Pi^{kj_1\dots n\dots j_p} = 0. \tag{5.15}$$

The terms proportional to $\xi_{I,n}$ give the quadratic equation for Π, the *algebraical part* of the fundamental identity:

$$\Pi^{kI}\Pi^{nJ} + \sum_{q=1}^{p} \Pi^{j_q I}\Pi^{kj_1\dots n\dots j_p} = \delta^{nI}_{mL}\Pi^{kL}\Pi^{mJ}. \tag{5.16}$$

This one is trivially satisfied for $p = 1$. For $p > 1$ it is in fact this part which forces Π to be decomposable. We can use this to immediately prove the following:

Lemma 5.1.5. *Let $p > 1$. Let Π be a Nambu-Poisson tensor, and $f \in C^\infty(M)$. Then $\Pi' := f\Pi$ is also a Nambu-Poisson tensor. In particular, every $\Pi \in \mathfrak{X}^n(M), n = \dim M$, is a Nambu-Poisson tensor.*

Proof. Multiplication of Π by f does not change the algebraic part (5.16). Because Π is Nambu-Poisson, we can choose Darboux coordinates where $\Pi^{iJ} = \epsilon^{iJ}$. It suffices to prove the differential identity, choose $\xi = dx^I$. We thus have to show that

$$\mathcal{L}_{(f\Pi)(dx^I)}(f\Pi) = 0. \tag{5.17}$$

We have

$$\begin{aligned}\mathcal{L}_{(f\Pi)(dx^I)}(f\Pi) &= ((f\Pi)(dx^I).f)\Pi + f\mathcal{L}_{(f\Pi)(dy^I)}\Pi \\ &= f(\Pi(dx^I).f)\Pi + f^2\mathcal{L}_{\Pi(dy^I)}\Pi - f(\Pi(dy^I) \wedge i_{df}\Pi) \\ &= f\big((\Pi(dx^I).f)\Pi - \Pi(dy^I) \wedge i_{df}\Pi\big).\end{aligned}$$

We have used the fundamental identity for Π in the last step. It thus suffices to show that

$$(\Pi(dx^I).f)\Pi = \Pi(dy^I) \wedge i_{df}\Pi. \tag{5.18}$$

This is equation which has a single non-trivial component, that is $(1, \dots, p)$. We get

$$\epsilon^{nI}\partial_n f = \epsilon^{kI}\partial_n f \epsilon^{nJ}\epsilon_{kJ}.$$

The only non-trivial contribution is possible for $I = [r]$ for $r \in \{1, \dots, p+1\}$. We then get

$$(-1)^{r+1}\partial_r f = (-1)^{r+1}\partial_n f \epsilon^{nJ}\epsilon_{rJ} = (-1)^{r+1}\partial_r f.$$

This finishes the proof of the first part. The second part follows easily. Every $\Pi \in \mathfrak{X}^n(M)$ can be locally written as $\Pi = f\partial_1 \wedge \dots \wedge \partial_n$. The n-vector $\widetilde{\Pi} = \partial_1 \wedge \dots \wedge \partial_n$ is (at least locally well defined) Nambu-Poisson tensor, and we can use the preceding proof to show that $\Pi = f\widetilde{\Pi}$ satisfies the fundamental identity. ■

There is one very interesting observation, noted in [37]. If one assumes that Leibniz rule (5.3) holds in every input (instead of in just one as we did), and that fundamental identity (5.4) holds, then the complete skew-symmetry of the bracket already follows. We reformulate this in the language of the corresponding tensors and vector bundle morphisms:

Lemma 5.1.6. *Let $\Pi \in \mathrm{Hom}(\Lambda^p T^*M, TM)$ be an arbitrary vector bundle morphism satisfying (5.16), where now $\Pi^{iJ} := \langle dy^i, \Pi(dy^J)\rangle$. Then Π is induced by a $(p+1)$-vector on M, that is $\Pi(\xi) = \Pi(\cdot, \xi)$ for $\Pi \in \mathfrak{X}^{p+1}(M)$.*

Proof. We will show that the algebraical identity (5.16) implies that $\Pi^{iJ} = 0$, whenever $i \in J$, in arbitrary coordinates. In particular, for any non-zero $\alpha \in T_x^*M$, we can choose the local coordinates in M, such that $dy^1|_x = \alpha$. This will prove that $\langle \alpha, \Pi(\alpha \wedge dy^{i_2} \wedge \dots \wedge dy^{i_p}\rangle = 0$, which implies the complete skew-symmetry of Π. Let us thus prove this assertion, let $i \in J$. Choose $k = n = i$ and $J = I$ in (5.16). We obtain

$$(\Pi^{iJ})^2 + \sum_{q=1}^{p} \Pi^{j_q J}\Pi^{ij_1 \dots i \dots j_p} = \delta^{iJ}_{mL}\Pi^{iL}\Pi^{mI}.$$

By assumption, we have $i = j_r$ for some $r \in \{1, \dots, p\}$. Thus only the $q = r$ term on the left-hand side contributes to the sum. Moreover, the right-hand side vanishes identically, because δ^{iJ}_{mL} is completely skew-symmetric in top indices. We thus get

$$2(\Pi^{iJ})^2 = 0,$$

which proves the assertion. ■

5.2 Leibniz algebroids picture

Recall now Example 2.2.2 and the Koszul bracket induced on cotangent bundle by a Poisson tensor. Can one construct such a bracket also for a Nambu-Poisson structure?

Proposition 5.2.1. *Let $\Pi \in \mathrm{Hom}(\Lambda^p T^*M, TM)$ be a vector bundle morphism. Let $L = \Lambda^p T^*M$, and define the $\mathbb{R}$-bilinear bracket $[\cdot,\cdot]_\Pi : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ as*

$$[\xi, \eta]_\Pi := \mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi, \tag{5.19}$$

for all $\xi, \eta \in \Omega^p(M)$. Then $(L, \Pi, [\cdot,\cdot]_\Pi)$ is a Leibniz algebroid, if and only if Π is a Nambu-Poisson tensor.

Proof. The Leibniz rule (2.4) holds for any Π. We will show that the Leibniz identity (2.5) for $[\cdot,\cdot]_\Pi$ is equivalent to the fundamental identity (5.7) for Π. In particular, we will use its version (5.13). For $\xi, \eta \in \Omega^p(M)$, define the vector field $V(\xi, \eta)$ as

$$V(\xi, \eta) := [\Pi(\xi), \Pi(\eta)] - \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi). \tag{5.20}$$

We claim that $V(\xi, \eta) = 0$ if and only if Π is a Nambu-Poisson tensor. This follows from the fact that $\mathcal{L}$ commutes with contractions, and thus

$$\begin{aligned} [\Pi(\xi), \Pi(\eta)] &= \mathcal{L}_{\Pi(\xi)}(\Pi(\eta)) = (\mathcal{L}_{\Pi(\xi)}\Pi)(\eta) + \Pi(\mathcal{L}_{\Pi(\xi)}\eta) \\ &= \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi) + \{(\mathcal{L}_{\Pi(\xi)}\Pi)(\eta) + \Pi(i_{\Pi(\eta)}d\xi)\}. \end{aligned}$$

We see that Π satisfies (5.13) if and only if $V(\xi, \eta) = 0$. Combining this with Lemma 5.1.6 proves that $V(\xi, \eta) = 0$ if and only if Π is a Nambu-Poisson tensor. Moreover, note that we have

$$V(\xi, \eta) = [\Pi(\xi), \Pi(\eta)] - \Pi([\xi, \eta]_\Pi). \tag{5.21}$$

Let us now examine the Leibniz identity for $[\cdot,\cdot]_\Pi$:

$$[\xi, [\eta, \zeta]_\Pi]_\Pi = \mathcal{L}_{\Pi(\xi)}(\mathcal{L}_{\Pi(\eta)}\zeta - i_{\Pi(\zeta)}d\eta) - i_{\Pi(\mathcal{L}_{\Pi(\eta)}\zeta - i_{\Pi(\zeta)}d\eta)}d\xi, \tag{5.22}$$

$$[[\xi, \eta]_\Pi, \zeta]_\Pi = \mathcal{L}_{\Pi(\mathcal{L}_{\Pi(\xi)} - i_{\Pi(\eta)}d\xi)}\zeta - i_{\Pi(\zeta)}d(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi), \tag{5.23}$$

$$[\eta, [\xi, \zeta]_\Pi]_\Pi = \mathcal{L}_{\Pi(\eta)}(\mathcal{L}_{\Pi(\xi)}\zeta - i_{\Pi(\zeta)}d\xi) - i_{\Pi(\mathcal{L}_{\Pi(\xi)}\zeta - i_{\Pi(\zeta)}d\xi)}d\eta. \tag{5.24}$$

Arranging this into the Leibniz identity for $[\cdot,\cdot]_\Pi$ and using the Cartan formulas to rewrite several terms, one arrives to the condition

$$\mathcal{L}_{V(\xi,\eta)}\zeta - i_{V(\xi,\zeta)}d\eta + i_{V(\eta,\zeta)}d\xi = 0. \tag{5.25}$$

We are now ready to finish the proof. First, when Π is a Nambu-Poisson tensor, then $V(\xi, \eta) = 0$, hence (5.25) holds. This proves that $(L, \Pi, [\cdot,\cdot]_\Pi)$ is a Leibniz algebroid. Conversely, if $(L, \Pi, [\cdot,\cdot]_\Pi)$ is a Leibniz algebroid, we know that the anchor Π is a bracket homomorphism (2.6). Glancing at (5.21), we see that this is equivalent to $V(\xi, \eta) = 0$. Hence Π is a Nambu-Poisson tensor. ■

There is a natural way how to explain the origin of the bracket (5.19). Consider now arbitrary $\Pi \in \mathrm{Hom}(\Lambda^p T^*M, TM)$. We view its graph G_Π as a subbunle of E:

$$G_\Pi = \{\Pi(\xi) + \xi \mid \xi \in \Lambda^p T^*M\}. \tag{5.26}$$

We can now study its involutivity. Note that G_Π is not an isotropic subbundle with respect to the pairing (4.1). Instead, it forms an example of an almost Nambu-Dirac structure, defined and studied in detail in [41].

Proposition 5.2.2. *The subbundle G_Π is involutive under the higher Dorfman bracket (4.15), if and only if Π is a Nambu-Poisson tensor.*

Proof. Let $\Pi(\xi) + \xi$ and $\Pi(\eta) + \eta$ be sections of G_Π. Then

$$[\Pi(\xi) + \xi, \Pi(\eta) + \eta]_D = [\Pi(\xi), \Pi(\eta)] + \mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)} d\xi. \tag{5.27}$$

The right-hand side is again a section of G_Π iff

$$[\Pi(\xi), \Pi(\eta)] = \Pi\big(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)} d\xi\big). \tag{5.28}$$

Using the properties of Lie derivative, this is equivalent to

$$(\mathcal{L}_{\Pi(\xi)}\Pi)(\eta) = -\Pi(i_{\Pi(\eta)} d\xi). \tag{5.29}$$

This is exactly the fundamental identity for Π written in the form (5.13). ∎

We can use this to clarify the structure of the bracket (5.19). Indeed, define a vector bundle isomorphism $\mathbf{\Psi} \in \mathrm{Hom}(\Lambda^p T^*M, G_\Pi)$ as $\mathbf{\Psi}(\xi) = \Pi(\xi) + \xi$. Assume that Π is a Nambu-Poisson tensor. The relation is then

$$\mathbf{\Psi}([\xi, \eta]_\Pi) = [\mathbf{\Psi}(\xi), \mathbf{\Psi}(\eta)]_D. \tag{5.30}$$

The anchor for $[\cdot, \cdot]_\Pi$ is then in fact a composition $\rho \circ \Psi$. Indeed, we have

$$(\rho \circ \Psi)(\xi) = \Pi(\xi). \tag{5.31}$$

This gives an alternative proof of the "if part" of Proposition 5.2.1.

To conclude this section, see that Proposition 5.2.2 allows one to easily define the twisted version of Nambu-Poisson structure.

Definition 5.2.3. Let $H \in \Omega^{p+2}(M)$ be a closed $(p+2)$-form, and let $[\cdot, \cdot]_D^H$ be a twisted Dorfman bracket (4.27). Let $\Pi \in \mathrm{Hom}(\Lambda^p T^*M, TM)$, and let $G_\Pi \subseteq E$ be its graph (5.26). We say that Π is an ***H*-twisted Nambu-Poisson tensor** if G_Π defines a subbundle involutive under $[\cdot, \cdot]_D^H$.

This definition by itself does not point at all to the statement of the following proposition, which may seem somewhat surprising. It was first observed and proved in [18].

Proposition 5.2.4. *For $p = 1$, Definition 5.2.3 gives a usual H-twisted Poisson manifold, where $\Pi \in \mathfrak{X}^2(M)$ satisfies the condition*

$$\frac{1}{2}[\Pi, \Pi]_S(\xi, \eta, \zeta) = H(\Pi(\xi), \Pi(\eta), \Pi(\zeta)), \tag{5.32}$$

for all $\xi, \eta, \zeta \in \Omega^1(M)$. Recall that $[\cdot, \cdot]_S$ is the Schouten-Nijenhuis bracket of multivector fields. For $p > 1$, Definition 5.2.3 gives no new structure at all.

Proof. For $p = 1$, the statement follows from the fact that

$$\frac{1}{2}[\Pi, \Pi]_S(\xi, \eta, \cdot) = [\Pi(\xi), \Pi(\eta)] - \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi). \tag{5.33}$$

The involutivity of G_Π under $[\cdot, \cdot]^H_\Pi$ gives the condition

$$[\Pi(\xi), \Pi(\eta)] = \Pi(\mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}d\xi - H(\Pi(\xi), \Pi(\eta), \cdot)) \tag{5.34}$$

A combination of these two relations gives the assertion of the proposition. For $p > 1$, we can rewrite the involutivity of G_Π under $[\cdot, \cdot]^H_D$ as

$$(\mathcal{L}_{\Pi(\xi)}\Pi)(\eta) = -\Pi(i_{\Pi(\eta)}d\xi + H(\Pi(\xi), \Pi(\eta), \cdot)). \tag{5.35}$$

Now recall that the algebraic part (5.16) of the fundamental identity comes from the failure of (5.13) to be $C^\infty(M)$-linear in ξ. But the addition of H does not change this part! Hence Π satisfies (5.16). It was shown in [3] that this in fact proves that there is a local frame $(e_\lambda)^n_{\lambda=1}$, such that $\Pi = e_1 \wedge \ldots \wedge e_{p+1}$. Thus the only components of H contributing to (5.35) are those corresponding to the first $p + 1$ vectors of the frame. But H is a $(p + 2)$-form, and those components vanish due to skew-symmetry. Hence H in no way contributes to (5.35) and Π satisfies the untwisted fundamental identity (5.13). ■

5.3 Seiberg-Witten map

We will now show that given a p-form A, one can use a Nambu-Poisson tensor Π to define a diffeomorphism of M. It is a direct generalization of the Seiberg-Witten map [85]. In the presented form, it was introduced for $p = 1$ in [63] as a dual analogue of Moser's lemma in symplectic geometry [77]. Its generalization to $p > 1$ were presented in [59] and [23]. Let us first recall a few facts about time-dependent vector fields and their flows.

Definition 5.3.1. Let $I \subseteq \mathbb{R}$ be an open interval, and let $V_\bullet : I \to \mathfrak{X}(M)$ be a map. A value of this map at given $t \in I$ is a vector field denoted as V_t. We say that $V_\bullet$ is a **time-dependent vector field**, if in every local coordinate set $(y^1, \ldots, y^n)$ the components of V_t depend smoothly on t. General time-dependent tensor fields are defined analogously.

Remark 5.3.2. Equivalently, we may view $V_\bullet$ as a vector field on the extended manifold $M \times I$ in the form

$$V_\bullet(x, t) = V_t(x) + \partial_t. \tag{5.36}$$

This interpretation is however not useful for higher tensor fields.

For a time-dependent vector field, a notion of integral curve still makes sense, except that one has to specify its *starting time*. For $s \in I$, $\gamma_s : J \to M$ is an integral curve starting at $m \in M$ at the time s, if $\dot{\gamma_s}(t) = V_t(\gamma_s(t))$ for all $t \in J$, and $\gamma_s(s) = m$. Here $J \subseteq I$ is an open interval. The vector field local flow theorem generalizes as follows:

Proposition 5.3.3. *Let $V_\bullet$ be a time-dependent vector field defined on an open interval $I \subseteq \mathbb{R}$. Then there exists an open subset $\mathcal{E} \subseteq I \times I \times M$ and a smooth map $\psi : \mathcal{E} \to M$ called a* ***time-dependent local flow of*** $V_\bullet$*, such that*

1. *For each $s \in I$ and $m \in M$, the set $\mathcal{E}_{s,m} = \{t \in I \mid (t, s, m) \in \mathcal{E}\}$ is an open subinterval of I containing s. The map $\psi_{s,m} := \psi(\cdot, s, m) : \mathcal{E}_{s,m} \to M$ is a maximal integral curve of $V_\bullet$ starting at m at the time s.*

2. *For any* $t \in \mathcal{E}_{s,m}$*, and* $q = \psi_{s,m}(t)$*, there holds* $\mathcal{E}_{t,q} = \mathcal{E}_{s,m}$*, and* $\psi_{t,q} = \psi_{s,p}$*.*

3. *For any* $(t,s) \in I \times I$*, the set* $M_{t,s} = \{m \in M \mid (t,s,m) \in \mathcal{E}\}$ *is an open subset of* M*. The map* $\psi_{t,s} := \psi(t,s,\cdot) : M_{t,s} \to M_{s,t}$ *is a diffeomorphism, and* $\psi_{t,s}^{-1} = \psi_{s,t}$*.*

4. *Let* $m \in M_{t,s}$*, ant* $\psi_{t,s}(m) \in M_{v,t}$*. Then* $p \in M_{v,s}$ *and*

$$\psi_{v,t} \circ \psi_{t,s} = \psi_{v,s}. \tag{5.37}$$

Proof. The proof is in fact a careful application of the ordinary local flow theorem for a vector field $V_\bullet$ mentioned in Remark 5.3.2. For details see [71]. ■

Having a local flow, there is a well defined generalization of Lie derivative. Let $T_\bullet$ be a time-dependent tensor field, and $\psi_{t,s}$ be a local flow of a time-dependent vector field $V_\bullet$. Define a new time-dependent tensor field as

$$(\mathcal{L}^{\tau}_{V_\bullet} T_\bullet)_s = \tfrac{d}{dt}\big|_{t=s} \psi^*_{t,s}(T_t). \tag{5.38}$$

A direct calculation similar to the one for ordinary tensor fields shows that

$$(\mathcal{L}^{\tau}_{V_\bullet} T_\bullet)_s = \partial_s T_s + \mathcal{L}_{V_s} T_s. \tag{5.39}$$

We have used the superscript τ to distinguish the generalization from the ordinary Lie derivative standing on the right-hand side (which is assumed to be given by usual algebraic formula). Lie derivative is a tool useful to describe the invariance of tensor fields with respect to flows. Let us show that for time-dependent tensor fields, $\mathcal{L}^\tau$ plays the same role.

Lemma 5.3.4. *Let* $V_\bullet$ *be a time-dependent vector field. Let* $T_\bullet$ *be a time-dependent tensor field satisfying* $\mathcal{L}^\tau_{V_\bullet} T_\bullet = 0$*. Then for any* $(t,s) \in I \times I$*, one has* $\psi^*_{t,s}(T_t) = T_s$ *on* $M_{t,s}$*.*

Proof. First let us show that the assumption in fact implies that $\frac{d}{dt}\psi^*_{t,s}(T_t) = 0$, for all $t \in I$. We will now use the composition rule (5.37). Indeed, one has

$$\frac{d}{dt}\psi^*_{t,s}(T_t) = \frac{d}{da}\bigg|_{a=0} \psi^*_{t+a,s}(T_{t+a}) = \psi^*_{t,s}\, \frac{d}{da}\bigg|_{a=0} \psi^*_{t+a,t}(T_{t+a}) = \psi^*_{t,s}(\mathcal{L}^\tau_{V_\bullet}(T_\bullet)_t) = 0. \tag{5.40}$$

This proves that $\psi^*_{t,s}(T_t) = T'_s$ for some $T'_\bullet$ and all $t \in I$. Setting $t = s$ shows that $T'_\bullet = T_\bullet$. ■

We now have all ingredients prepared to introduce the Seiberg-Witten map. Let $\Pi \in \mathfrak{X}^{p+1}(M)$ be a Nambu-Poisson tensor, and let $A \in \Omega^p(M)$. Denote $F = dA$. We can use F to define a new Nambu-Poisson tensor Π' as follows. Let $e^F \in \mathrm{Aut}(E)$ be the map (4.13) induced by F. Let $G_\Pi \subseteq E$ be a graph of Π which is by definition involutive under the Dorfman bracket. We have shown in Proposition 4.2.3 that e^F is an automorphism of the Dorfman bracket. This proves that the subbundle $e^F(G_\Pi)$ is also involutive under the Dorfman bracket. If there is $\Pi' \in \mathrm{Hom}(\Lambda^p T^*M, TM)$ such that $G_{\Pi'} = e^F(G_\Pi)$, we know that Π' is again a Nambu-Poisson tensor. Let $\Pi(\xi) + \xi \in \Gamma(G_\Pi)$. We have

$$e^F(\Pi(\xi) + \xi) = \Pi(\xi) + (1 - F^T\Pi)(\xi). \tag{5.41}$$

If Π' exists, there must hold $\Pi(\xi) = \Pi'(1 - F^T\Pi)(\xi)$. Let us assume that $1 - F^T\Pi$ is an invertible map. Hence

$$\Pi' = \Pi(1 - F^T\Pi)^{-1}. \tag{5.42}$$

Now define a time-dependent tensor field $\Pi_\bullet$ as

$$\Pi_t = \Pi(1 - tF^T\Pi)^{-1}. \tag{5.43}$$

Assume that it is well defined for $I = (-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$. Clearly $\Pi_0 = \Pi$, $\Pi_1 = \Pi'$. Using the same argument as above, Π_t is a Nambu-Poisson tensor for every $t \in I$. In particular, it satisfies the fundamental identity (5.13). Plug $\xi = A$ into this condition. It gives $\mathcal{L}_{\Pi_t(A)}\Pi_t = -\Pi_t F^T \Pi_t$. Next, examine the time derivative of Π_t. One obtains

$$\partial_t \Pi_t = \Pi(1 - tF^T\Pi)^{-1} F^T \Pi(1 - F^T\Pi)^{-1} = \Pi_t F^T \Pi_t. \tag{5.44}$$

If we define a time-dependent vector field $A^\#_\bullet$ as $A^\#_t = \Pi_t(A)$, we have just proved that $\mathcal{L}^\tau_{A^\#_\bullet}\Pi_\bullet = 0$. Using Lemma 5.3.4 we see that $\psi^*_{t,s}\Pi_t = \Pi_s$. In particular, define a diffeomorphism $\rho_A := \psi_{1,0}$. The map $\rho_A \in \mathrm{Diff}(M)$ is called the **Seiberg-Witten map**. By construction, $\rho^*_A(\Pi') = \Pi$. Note that it is essential that Π_t is a Nambu-Poisson tensor for every $t \in I$.

To conclude this section, note that for $p > 1$, the form $F \in \Omega^{p+1}(M)$ used to define a time-dependent tensor field $\Pi_\bullet$ does not have to be closed to define a set of Nambu-Poisson tensors. This is true because in fact for any $F \in \Omega^{p+1}(M)$ there holds

$$\Pi_t = (1 - \frac{t}{p+1}\langle \Pi, F \rangle)^{-1}\Pi. \tag{5.45}$$

This can be proved easily in Darboux coordinates (5.8) for Π. This shows that Π_t is just a scalar multiple of Π and the assertion follows from Lemma 5.1.5.

Chapter 6

Conclusions and outlooks

We have introduced an extension of the generalized geometry suitable for a description of membrane sigma models. Our intention was to follow the outline of the standard generalized geometry, in particular in the case of generalized metric. This was not possible until we have introduced a doubled formalism. To our delight, we were able to use it to significantly simplify the calculations required in particular to relate commutative and semi-classically non-commutative p-DBI actions. Moreover, it proved useful to discover the membrane analogue of background-independent gauge and the double scaling limit. Of course, this formalism is mathematically interesting in its own right. We should focus on the future prospects of the ideas presented in this thesis. We will now point out the sections which require further investigation.

Let us start with the mathematical side of things. We have defined connections on local Leibniz algebroids in Section 2.4. This direction is definitely worth of pursuing. For example, one can study a class of Courant algebroid connections which are compatible with the generalized metric and their torsion operator (2.55) vanishes. A set of such connections is larger then in the case of the ordinary Riemannian geometry. However, it turns out that they such connections have a quite nice form allowing for the calculation of their scalar curvature. Interestingly, this scalar function is exactly the one multiplying the integral density in string effective actions. One should relate this viewpoint to the generalized geometry treatment of string effective actions in [11, 12]. It would be also necessary to generalize such Courant algebroid connections to the Leibniz algebroid setting, in particular using the doubled formalism of Sections 4.5 and 4.6.

Killing sections of Section 3.10 have an interesting role in string theory, since one can construct generalized charges using such sections. Those charges are conserved in time evolution if and only if the respective sections satisfy the generalized Killing equations. Moreover, such sections are closely related to T-duality, see [38]. There has to exist some link between these observations. It would be also important to find a geometrical explanation to membrane duality rotations od Duff and Lu in [30]. Understanding T-duality analogues for membranes could possibly give an equivalent derivation od p-DBI actions.

There are several directions where to proceed with the physics presented in the attached papers. A Nambu sigma model proposed using AKSZ construction in [18] is a little bit different from the one defined in [59]. It would be interesting to analyze this disparity. In particular, the latter version used also in our paper [61] is not invariant with respect to worldvolume reparametrizations. Is there a way to define an invariant and possibly more general Nambu sigma model action?

An important feature of topological Poisson sigma models is the existence of the gauge transformations, see for example [13]. It is in fact a direct consequence of the fact that topological Poisson sigma models are a theory with constraints, and constraints themselves are integrals of motion. Since the topological Nambu sigma model is also a theory with constraints, there should be a similar process leading to its gauge symmetries. Nambu-Poisson structures can be possibly an interesting object on its own. Usual Poisson structures and Poisson sigma models turned out to be a crucial element in the integration of Lie algebroids. Is there a similar use for Nambu-Poisson structures and Nambu sigma models?

By construction, standard generalized geometry (and its extended variant presented here) is not suitable to describe supersymmetric theories due to its lack of Grassmanian variables. There are its extensions used in supergravity [24, 25] and M-theory [52]. It would be interesting to modify generalized geometry to work in a world of graded geometry, in particular supermanifolds in the sense of [21]. The proposed p-DBI action in [59] is obviously only the bosonic part of a (yet unknown) full supersymmetric action. An understanding of generalized (super)geometry could give us answers necessary to derive it.

The guiding principle of "doubling" and related construction of generalized metric can be easily generalized to more general vector bundles. There is an intriguing relation between Leibniz algebroids and Lie algebra representations, see [7]. This reference provides a huge class of interesting Leibniz algebroid examples, which can treated similarly as we did within our extended generalized geometry. This can be useful to understand better the spherical T-duality, [15].

The author hopes that this thesis and his research proved once more the importance of understanding the geometry underlying the theoretical physics. Not only it brings new ways how to understand and verify known things, but it could also provide the missing tools to push the knowledge of our world further.

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Appendices

Appendix A

Proofs of technical Lemmas

- **Lemma 4.3.4**

Proof. Let us prove the formula (4.42). Both sides can be viewed as smooth functions of matrix elements ${A^i}_j$. We will restrict to the open subset $GL(n,\mathbb{R}) = \mathbb{R}^{n,n} \setminus \det^{-1}(\{0\})$. It is dense in $\mathbb{R}^{n,n}$, and the general result will follow by the continuity of both functions. Recall that there holds a formula

$$\frac{\partial \det(A)}{\partial {A^i}_j} = \det(A){(A^{-1})^j}_i, \tag{A.1}$$

Hence we get

$$\frac{\partial}{\partial {A^i}_j}[\det A]^{\binom{n-1}{p-1}} = \binom{n-1}{p-1}[\det A]^{\binom{n-1}{p-1}}{(A^{-1})^j}_i.$$

This proves that for $F = [\det A]^{\binom{n-1}{p-1}}$, we have

$$\frac{\partial}{\partial {A^i}_j}\ln|F| = \binom{n-1}{p-1}{(A^{-1})^j}_i. \tag{A.2}$$

We will now show that the same equation holds for B, that is

$$\frac{\partial}{\partial {A^i}_j}\ln|\det B| = \binom{n-1}{p-1}{(A^{-1})^j}_i. \tag{A.3}$$

Let us calculate this explicitly. We get

$$\begin{aligned}\frac{\partial}{\partial {A^i}_j}\det(B) &= \det(B){(B^{-1})^J}_I\frac{\partial {B^I}_J}{\partial {A^i}_j} = \det(B){(B^{-1})^J}_I\frac{\partial}{\partial {A^i}_j}[\delta^I_{k_1\dots k_p}{A^{k_1}}_{j_1}\dots {A^{k_p}}_{j_p}] \\ &= \det(B){(B^{-1})^J}_I\sum_{r=1}^{p}\delta^I_{k_1\dots k_p}{A^{k_1}}_{j_1}\dots\delta^{k_r}_i\delta^{j_r}_j\dots {A^{k_p}}_{j_p}.\end{aligned}$$

One can now insert a unit matrix to get

$$\begin{aligned}\frac{\partial}{\partial A^i{}_j}\det(B) &= \det(B)(B^{-1})^J{}_I\sum_{r=1}^{p}\delta^I_{k_1\dots k_p}A^{k_1}{}_{j_1}\dots A^{k_r}{}_m\dots A^{k_p}{}_{j_p}\delta^{j_r}_j(A^{-1})^m{}_i\\ &= \det(B)(B^{-1})^J{}_I\sum_{r=1}^{p}B^I{}_{j_1\dots m\dots j_p}\delta^{j_r}_j(A^{-1})^m{}_i =\\ &= \det(B)\sum_{J,j\in J}(A^{-1})^j{}_i = \det(B)\binom{n-1}{p-1}(A^{-1})^j{}_i.\end{aligned}$$

This proves the equation (A.3). We thus have

$$\det(B) = K[\det(A)]^{\binom{n-1}{p-1}}. \tag{A.4}$$

for some K which is locally constant on $GL(n,\mathbb{R})$. To finish the proof, we have to prove that $K = 1$ on both components of $GL(n,\mathbb{R})$. For the group unit component, we can choose $A = 1$. In this case $B^I{}_J = \delta^I_J$, and thus $\det(B) = 1$. For the second component, choose A to be Minkowskian metric of signature $(n-1,1)$. From the proof of Lemma 4.3.1, we see that B is diagonal metric with ± 1 on the diagonal, where the number of negative ones is $N(n-1,1,p) = \binom{n-1}{p-1}$. We have

$$\det(A) = -1,\ \det(B) = (-1)^{\binom{n-1}{p-1}}.$$

We see that again $K = 1$. This finishes the proof for $A \in GL(n,\mathbb{R})$, and the assertion of the lemma follows by the continuity. ■

- **Lemma 4.6.3**

Proof. Let us work in fixed local coordinate system $(y^1,\dots,y^n)$ on M. Let $I = (i_1 < \dots < i_p)$ be a strictly ordered p-index, and $J = (j_1 < \dots < j_q)$ a strictly ordered q-index. By assumption, we have

$$0 = (\mathcal{L}_X T)^I_J = X^m T^I_{J,m} + \sum_{r=1}^{q} X^m{}_{,j_q} T^I_{j_1\dots m\dots j_q} - \sum_{l=1}^{p} X^{i_l}{}_{,m} T_J^{i_1\dots m\dots i_p}, \tag{A.5}$$

for all $X \in \mathfrak{X}(M)$. In particular, it must hold also for fX, where $f \in C^\infty(M)$. This gives a necessary condition

$$\sum_{r=1}^{q} f_{,j_r} X^m T^I_{j_1\dots m\dots j_q} = \sum_{l=1}^{p} X^{i_l} f_{,m} T_J^{i_1\dots m\dots i_p}. \tag{A.6}$$

First assume that $I \neq J$. In particular, this includes the case $p \neq q$. With no loss of generality, we may assume that there is $i_a \in I$, such that $i_a \neq J$. Choose $X = \partial_{i_a}$, and $f = y^{i_a}$. This gives

$$\sum_{r=1}^{q} \delta^{i_a}_{j_r} T^I_{j_1\dots i_a\dots j_q} = T^I_J. \tag{A.7}$$

But because $i_a \neq J$, the Kronecker symbol in the left-hand side sum is always zero. Hence $T^I_J = 0$. We can now assume that $p = q$. Moreover, we have already proved that

$T_I^J = \lambda_I \delta_I^J$, where $\lambda_I \in C^\infty(M)$. We want to show that $\lambda_I = \lambda_J$ for all (I, J). First consider (I, J), such that both p-indices differ *in exactly one index*. There is thus $i_a \in I$, and $j_b \in J$, such that $I \setminus \{i_a\} = J \setminus \{j_b\}$. Choose $X = \partial_{i_a}$ and $f = y^{j_b}$ in (A.6). This gives $\lambda_I = \lambda_J$.

Now let I and J be general p-indices. There is always a chain $[K_0, \ldots, K_m]$ of p-indices, where $I = K_0$, $J = K_m$, and (K_i, K_{i+1}) differ in exactly one index. This proves $\lambda_I = \lambda_J$. Hence $T_J^I = \lambda \cdot \delta_J^I$ for $\lambda \in C^\infty(M)$. As a map T thus has a form $T = \lambda \cdot 1$. By definition of Lie derivative, we have

$$\mathcal{L}_X(T(Q)) = (\mathcal{L}_X T)(Q) + T(\mathcal{L}_X Q), \tag{A.8}$$

for all $Q \in \mathfrak{X}^p(M)$. Using the assumption and the above form of T, we get

$$\mathcal{L}_X(\lambda Q) = \lambda \mathcal{L}_X Q, \tag{A.9}$$

for all $Q \in \mathfrak{X}(M)$. This is possible, if and only if $\lambda \in \Omega^0_{closed}(M)$. ■

Appendix B

Paper 1: p-Brane Actions and Higher Roytenberg Brackets

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p-brane actions and higher Roytenberg brackets

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Abstract: Motivated by the quest to understand the analog of non-geometric flux compactification in the context of M-theory, we study higher dimensional analogs of generalized Poisson sigma models and corresponding dual string and p-brane models. We find that higher generalizations of the algebraic structures due to Dorfman, Roytenberg and Courant play an important role and establish their relation to Nambu-Poisson structures.

Keywords: Flux compactifications, p-branes, Sigma Models

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1 Introduction

To relate ten-dimensional superstring theory to particle physics and cosmology in four-dimensional spacetime, it is necessary to compactify the superfluous dimensions. Introducing fluxes in this context helps to overcome problems of more standard Calabi-Yau compactifications, but at the same time the underlying geometric structures become more general: the notion of a compactifying manifold needs to be relaxed, allowing patching not only by diffeomorphisms but also by more general string symmetry transformations. The resulting non-geometric flux compactifications can appear in the T-duals of geometric flux compactifications [1, 2]. An example are toroidal compactifications with R-fluxes, where non-associative structures arise [3], whose quantization is related to twisted Poisson sigma models [4]. Poisson sigma models [5, 6] are also at the heart of Kontsevich's approach to deformation quantization [7]. For a recent review with a comprehensive list of references in the more general context of AKSZ topological field theory, we refer to [8]. See also [9] for an interesting conception of membrane symmetries.

From a mathematical point of view, it is known that Poisson sigma models are intimately connected to a lot of interesting differential geometry. The fields of Poisson sigma models can be interpreted as Lie algebroid morphisms [10] and can be further generalized in terms of generalized (complex) geometry [11, 12]. It was observed by Alekseev and Strobl in [13], that the current algebra of sigma models naturally involves the structures of generalized geometry [14, 15], such as Dorfman bracket and Dirac structures. This was further developed by Ekstrand and Zabzine in [16] and Bonelli and Zabzine in [17]. Recently, D-branes have been identified with Dirac structures [18]. In [19], Halmagyi observed that in the Hamiltonian of the Polyakov model, characterized by a 2-form B and a bivector Π, appears a more general form of world sheet currents and found their algebra to close under a more general bracket, which he calls a Roytenberg bracket. Finally, in [20], Halmagyi shows that the same bracket appears if one lifts the first order action to a three-manifold using Stokes theorem.

The known string theories as well as supergravity are naturally embedded in eleven-dimensional M-theory, whose building blocks are membranes and five-branes. This motivates the study of higher dimensional analogs of the structures that we have described above. In this article, we would like to go beyond the Courant sigma-model, which is already a higher version of the Poisson sigma-model on an open three-dimensional membrane, but still features a bi-vector field. Generalizing this (twisted) Poisson bi-vector to a $(p+1)$-vector field we face the question how to generalize the Jacobi identity that governs the $p=1$ case. One possibility is to impose the condition of a vanishing Schouten bracket, but that will be non-trivial only for even p. Another possibility is to impose the so-called fundamental identity of a Nambu-Poisson structure [21]. Evidence for the latter choice comes from the study of actions for multiple membranes in M-theory [22], see [23] for a recent review and many references. Local symmetries in M-theory and their relation to generalized geometry were discussed in [24–26]. For $p=1$, the consistency of the equations of motion of the topological sigma model action implies the Jacobi identity. For $p>1$, the Nambu-Poisson fundamental identity has an algebraic as well as a differential part and it is thus not clear how it could be related to a consistency condition for differential equations of motion. In this article we solve this problem and study the relevant higher algebraic and geometric structures. A suitable higher generalization of Poisson sigma models has recently been proposed by two of us [27]: this Nambu-sigma model features a $(p+1)$-dimensional world volume and corresponding higher-order tensor fields on a target manifold. The topological version of the model can also be obtained by an AKSZ construction [28].

This paper is organized as follows: In section 2 we review the relevant models and compute the Hamiltonian. In section 3 we use a $(p+1)$-vector Π to twist a higher Dorfman bracket and obtain a new Courant bracket like structure, which we call a higher Roytenberg bracket. In section 4 we discuss the charge algebra of the model and its relation to the higher Roytenberg bracket. In section 5 we verify the consistency of the topological part of the p-brane action. We find that Π should satisfy the fundamental identity of a Nambu-Poisson structure (differential as well as algebraic part). In section 6 we derive the equations of motion of the topological model and find an explicit non-trivial solution. In section 7 we lift the topological part of the action to a $(p+2)$-dimensional world volume and derive generalized Wess-Zumino terms that involve the structure functions of the higher Roytenberg bracket. In the appendices we summarize relevant facts about the higher Roytenberg bracket and Nambu Poisson structures.

2 Nambu sigma model and p-brane action

In this section we review the Nambu sigma model following [27, 29], compute the corresponding Hamiltonian and remark on the dual p-brane action.

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Let us consider a $(p+1)$-dimensional world volume Σ with a set of local coordinates $(\sigma^0, \dots, \sigma^p)$. We assume that σ^μ are Cartesian coordinates for a Lorentzian metric h with signature $(-,+,\dots,+)$ on Σ. Furthermore, we consider an n-dimensional target manifold M, equipped with a $(p+1)$-vector Π and a $(p+1)$-form B. We also choose some local coordinates $(y^1, \dots, y^n)$ on M. Lower case Latin characters will always correspond to these coordinates. We will use upper case Latin characters to denote strictly ordered multi-indices (mostly p-indices), that is $I = (i_1, \dots, i_p)$, where $i_1 < \dots < i_p$. We will assume that M is equipped with a metric tensor field G with local components G_{ij}, and a fiber-wise metric $\widetilde{G}$ on the vector bundle $\Lambda^p TM$ with components $\widetilde{G}_{IJ}$ in a local section basis $\partial_I \equiv \frac{\partial}{\partial y^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{i_p}}$. Metric matrices with upper indices denote as usual the corresponding inverses. For the components of the smooth map $X : \Sigma \to M$ we will use the following notation: $X^i = y^i(X)$, $dX^I = dX^{i_1} \wedge \ldots \wedge dX^{i_p}$, and $\widetilde{\partial X}^I = (dX^I)_{1\ldots p}$ where the latter denotes the $1 \ldots p$ component of the world volume form dX^I.

The "Nambu-sigma model" action, as introduced in [27, 29], is

$$S[\eta, \widetilde{\eta}, X] := \int d^{p+1}\sigma \Big[-\frac{1}{2}(G^{-1})^{ij}\eta_i\eta_j + \frac{1}{2}(\widetilde{G}^{-1})^{IJ}\widetilde{\eta}_I\widetilde{\eta}_J + \eta_i\partial_0 X^i + \widetilde{\eta}_I\widetilde{\partial X}^I - \Pi^{iJ}\eta_i\widetilde{\eta}_J - B_{iJ}\partial_0 X^i \widetilde{\partial X}^J \Big], \quad (2.1)$$

where $\eta_i, \widetilde{\eta}_J$ are auxiliary fields, which transform under change of local coordinates on M according to their index structure.

The canonical momenta corresponding to the fields X^i are

$$P_i = \eta_i - B_{iJ}\widetilde{\partial X}^J. \quad (2.2)$$

Starting with the canonical Hamiltonian $H_{can}[X, P, \widetilde{\eta}] = \int d^p\sigma P_i \partial_0 X^i - \mathcal{L}(X, P, \widetilde{\eta})$ and substituting the Euler-Lagrange equation for $\widetilde{\eta}_J$, we obtain the Hamiltonian[1]

$$H[X, P] = \frac{1}{2}\int d^p\sigma \left[(G^{-1})^{ij}K_iK_j + \widetilde{G}_{IJ}\widetilde{K}^I\widetilde{K}^J\right], \quad (2.3)$$

where

$$K_i := \eta_i = P_i + B_{iK}\widetilde{\partial X}^K, \quad (2.4)$$

$$\widetilde{K}^I := -\widetilde{G}^{IJ}\widetilde{\eta}_J = \widetilde{\partial X}^I - \Pi^{mI}K_m. \quad (2.5)$$

Here and in the rest of the paper, the integration over $d^p\sigma$ means the integration over the space-like coordinates $(\sigma^1, \dots, \sigma^p)$ of Σ. The Hamiltonian can be conveniently written in matrix notation: the components of the $(p+1)$-vector Π^{iJ} form an $n \times \binom{n}{p}$ rectangular matrix Π with row index i and column index J; similarly for B. Likewise, G and $\widetilde{G}$ are

[1] Note that $\partial_0 X^i$ cannot be directly expressed in terms of P_i but it still drops out of H_{can} in the computation, as it should. The construction is robust in the sense that first using the equations of motion for $\widetilde{\eta}$ and η and then constructing the Hamiltonian yields the same result.

$n \times n$ and $\binom{n}{p} \times \binom{n}{p}$ matrices corresponding to the metrics G and $\widetilde{G}$, respectively. Next, we define $(n + \binom{n}{p})$-row column vectors

$$\mathcal{K} = \begin{pmatrix} K_i \\ \widetilde{K}^I \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} P_i \\ \widetilde{\partial X}^I \end{pmatrix}.$$

Note that these vectors have the same index structure as coordinate expressions of sections of $T^*M \oplus \Lambda^p TM$. The defining equations (2.4) and (2.5) can then be rewritten as $\mathcal{K} = \mathbf{A}\mathcal{V}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -\Pi^T & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & B \\ -\Pi^T & \mathbf{1} - \Pi^T B \end{pmatrix}. \tag{2.6}$$

Note that $\mathbf{A}$ can always be inverted, i.e. we can uniquely express the fields P and $\widetilde{\partial X}$ using K and $\widetilde{K}$. We can view $\mathbf{A}$ as the matrix of a linear endomorphism of $T^*M \oplus \Lambda^p TM$. Finally, we can define the matrix $\mathbf{G} = \begin{pmatrix} G^{-1} & 0 \\ 0 & \widetilde{G} \end{pmatrix}$ and view it as the matrix of the fiberwise metric on $T^*M \oplus \Lambda^p TM$. Then, we can rewrite the Hamiltonian (2.3) as

$$H[X, P] = \int d^p\sigma [\mathcal{V}^T(\mathbf{A}^T \mathbf{G} \mathbf{A})\mathcal{V}]. \tag{2.7}$$

Let us note that the matrix $\mathbf{A}^T\mathbf{G}\mathbf{A}$ has a natural interpretation as a (twisted) higher $(p > 1)$ analog of the generalized metric of $p = 1$ generalized geometry.

If we start again with the action (2.1), and integrate out the fields $\widetilde{\eta}_J$ using their equations of motion, we get the action

$$S[X, \eta] = \int d^{p+1}\sigma \Big[-\frac{1}{2}\eta^T G^{-1}\eta - \frac{1}{2}\widetilde{K}^T \widetilde{G}\widetilde{K} + \partial_0 X^T(\eta - B\widetilde{\partial X}) \Big], \tag{2.8}$$

where η, $\widetilde{\eta}$, K, $\widetilde{K}$, $\partial_0 X$ and $\widetilde{\partial X}$ are column vectors defined in the obvious way. We next use the Euler-Lagrange equations to eliminate η in (2.8) and get

$$S[X] = \int d^{p+1}\sigma \Big[\frac{1}{2}\partial_0 X^T g\, \partial_0 X - \frac{1}{2}\widetilde{\partial X}^T \widetilde{g}\, \widetilde{\partial X} - \partial_0 X^T (B + C)\widetilde{\partial X} \Big], \tag{2.9}$$

where

$$g = (G^{-1} + \Pi \widetilde{G} \Pi^T)^{-1}, \tag{2.10}$$

$$\widetilde{g} = (\widetilde{G}^{-1} + \Pi^T G \Pi)^{-1}, \tag{2.11}$$

and

$$C = -g\Pi\widetilde{G} = -G\Pi\widetilde{g}. \tag{2.12}$$

The action (2.9) is just the Polyakov-style Howe-Tucker membrane action introduced by Deser-Zumino [30], Brink-Di Vecchia-Howe [31] and Howe-Tucker [32] with properly fixed gauge (coordinates on Σ), see [27]. For $p = 1$ case, see [33].

The background fields $G, \widetilde{G}, \Pi$ can also be expressed in terms of $g, \widetilde{g}, C$:

$$G = g + C\widetilde{g}^{-1}C^T, \tag{2.13}$$

$$\widetilde{G} = \widetilde{g} + C^T g^{-1} C, \tag{2.14}$$

and

$$\Pi = -g^{-1}C\widetilde{G}^{-1} = -G^{-1}C\widetilde{g}^{-1}\,. \tag{2.15}$$

The relations between $G, \widetilde{G}, \Pi$ and $g, \widetilde{g}, C$ are higher p-brane version [27] of the well-known open-closed string relations, cf. also [9]. We can write these relations in terms of the higher generalized metric $\mathbf{A}^T\mathbf{G}\mathbf{A}$ as

$$\mathbf{A}^T\mathbf{G}\mathbf{A} = \mathbf{a}^T\mathbf{g}\mathbf{a}\,, \text{ where } \mathbf{g} = \begin{pmatrix} g^{-1} & 0 \\ 0 & \tilde{g} \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} 1 & B+C \\ 0 & 1 \end{pmatrix}.$$

The Hamiltonian corresponding to the action (2.9) features the inverse of the matrix $\mathbf{a}^T\mathbf{g}\mathbf{a}$.

Instead of the B-field, it is sometimes more convenient to introduce a $(p+1)$-form Φ and write $\tilde{\mathbf{A}} = \left(\begin{smallmatrix} 1 & \Phi \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ -\Pi^T & 1 \end{smallmatrix}\right)$. Redefining $\tilde{\mathbf{a}} = \left(\begin{smallmatrix} 1 & C \\ 0 & 1 \end{smallmatrix}\right)$ and equating $\tilde{\mathbf{A}}^T\mathbf{G}\tilde{\mathbf{A}} = \tilde{\mathbf{a}}^T\mathbf{g}\tilde{\mathbf{a}}$ provides an alternative derivation of the general open-closed p-brane relations of [27]. This new approach should also be useful in the context of effective actions for multiple branes ending on branes.

3 Higher Roytenberg bracket

In this section we will recall some of the algebraic structures needed in the following. The name "Roytenberg bracket" was introduced by Halmagyi [19], since the bracket was originally introduced by Roytenberg in [34]. We present a higher analog of this bracket here, which is essentially a higher Dorfman bracket twisted by a $(p+1)$-vector Π as well as by a $(p+1)$-form H. For further reading on higher Dorfman bracket see e.g. [35] or [36].

Let $E = TM \oplus \Lambda^p T^*M$. We define a non-degenerate and $C^\infty(M)$-bilinear pairing $\langle\cdot,\cdot\rangle : \Gamma(E)\times\Gamma(E) \to \Omega^{p-1}(M)$ as

$$\langle V+\xi, W+\eta\rangle = i_V(\eta) + i_W(\xi), \tag{3.1}$$

for vector fields $V, W \in \mathfrak{X}(M)$ and p-forms $\xi, \eta \in \Omega^p(M)$. We define the anchor map $\rho : E \to TM$ as the projection onto the first direct summand of E, and denote by the same character also the induced map of sections $\rho(V+\xi) = V$. The Dorfman bracket is the $\mathbb{R}$-bilinear bracket on sections $[\cdot,\cdot]_D : \Gamma(E)\times\Gamma(E)\to\Gamma(E)$, defined as

$$[V+\xi, W+\eta]_D = [V,W] + \mathcal{L}_V(\eta) - i_W(d\xi), \tag{3.2}$$

for all $V, W \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^p(M)$. This bracket is a particular example of a Leibniz algebroid bracket, see [35]. If we define $\mathcal{D} : \Omega^{p-1}(M) \to \Gamma(E)$ as $\mathcal{D} = j \circ d$, where $j : \Omega^p(M) \hookrightarrow \Gamma(E)$ is the inclusion, we have the following properties of Dorfman bracket:

1. Derivation property:

$$[e_1, [e_2, e_3]_D]_D = [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D\,, \tag{3.3}$$

for all $e_1, e_2, e_3 \in \Gamma(E)$.

$$[e_1, fe_2]_D = f[e_1, e_2]_D + (\rho(e_1).f)e_2\,, \tag{3.4}$$

for all $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

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2. $\langle\cdot,\cdot\rangle$ is E-invariant in the following sense:

$$\mathcal{L}_{\rho(e_1)}(\langle e_2, e_3\rangle) = \langle [e_1, e_2]_D, e_3\rangle + \langle e_2, [e_1, e_3]_D\rangle\,, \tag{3.5}$$

for all $e_1, e_2, e_3 \in \Gamma(E)$.

3. Dorfman bracket is skew-symmetric up to "coboundary", that is

$$[e, e]_D = \frac{1}{2}\mathcal{D}\langle e, e\rangle\,, \tag{3.6}$$

for all $e \in \Gamma(E)$.

This bracket can be easily modified in two ways:

Firstly, given a $(p+2)$-form $H \in \Omega^{p+2}(M)$, we can define H-twisted higher Dorfman bracket on E as

$$[V + \xi, W + \eta]_D^{(H)} = [V, W] + \mathcal{L}_V(\eta) - i_W(d\xi) + i_W i_V H. \tag{3.7}$$

The form H has to be closed, in order to keep the property (3.3). All the other properties of higher Dorfman bracket are also valid for the H-twisted case.

Secondly, assume that we have an arbitrary $C^\infty(M)$-linear map of sections $\Pi^\# : \Omega^p(M) \to \mathfrak{X}(M)$, for example the map induced by a $(p+1)$-vector Π on M:

$$\Pi^\#(\xi) = (-1)^p i_\xi \Pi = \xi_K \Pi^{iK}\partial_i, \tag{3.8}$$

for all $\xi \in \Omega^p(M)$. Define new anchor map $\rho : E \to TM$ as

$$\rho(V + \xi) = V - \Pi^\#(\xi)\,, \tag{3.9}$$

and the "twisted" inclusion of $\Omega^p(M)$ into $\Gamma(E)$ as

$$j(\xi) = \xi + \Pi^\#(\xi)\,. \tag{3.10}$$

Denote as pr_2 the projection onto the second summand of E. Using this notation, one can define new non-degenerate pairing $\langle\cdot,\cdot\rangle_R$:

$$\langle e_1, e_2\rangle_R = i_{\rho(e_1)}(pr_2(e_2)) + i_{\rho(e_2)}(pr_2(e_1))\,, \tag{3.11}$$

for all $e_1, e_2 \in \Gamma(E)$. Finally, we define the following bracket on $\Gamma(E)$:

$$[e_1, e_2]_R = [\rho(e_1), \rho(e_2)] + j\big(\mathcal{L}_{\rho(e_1)}(pr_2(e_2)) - i_{\rho(e_2)}(d(pr_2(e_1))) + i_{\rho(e_2)} i_{\rho(e_1)} H\big)\,, \tag{3.12}$$

for all $e_1, e_2 \in \Gamma(E)$. We refer to $[\cdot,\cdot]_R$ as higher Roytenberg bracket. This bracket together with the anchor map (3.9) defines again a Leibniz algebroid, i.e., it satisfies (3.3) and (3.4). More interestingly, it also satisfies (3.5) and (3.6) with respect to the pairing (3.11). All of the properties are straightforward to check; see also [28]. In appendix A we present the coordinate form of the higher Roytenberg bracket. For $p = 1$ we get exactly the structure functions of [20].

4 Charge algebra

In this section we study the algebra of the currents that appear in the Hamiltonian associated to the Nambu-sigma model. We find that the corresponding charge algebra is governed by the higher Roytenberg bracket that we have discussed in the previous section.

Let us return to the Hamiltonian (2.3). The canonical equal-time Poisson brackets are

$$\{X^i(\sigma), P_j(\sigma')\} = \delta^i_j \delta(\sigma - \sigma')\,,$$

where σ, σ' are the space-like p-tuples of world volume coordinates. We consider the generalized charges

$$Q_f(V+\xi) = \int d^p\sigma\, f(\sigma)[V^i K_i + \xi_J \widetilde{K}^J]\,, \tag{4.1}$$

corresponding to the currents K^i and $\widetilde{K}_J$ that appear explicitly in the Hamiltonian. Here $V + \xi \in \Gamma(E)$ and $f \in C^\infty(\Sigma)$ is a test function. The appearance of Courant algebroid structures in the current algebra was first observed by Alekseev and Strobl in [13] for the Poisson-sigma model, i.e. the special case $p = 1$. More general observations from the supergeometry point of view were done by Guttenberg in [37]. Here we will calculate the charge algebra for $p \geq 1$, following the approach of Ekstrand and Zabzine, who integrated the currents to generalized charges. In fact, we shall consider more general charges, involving background fields Π and B. This can be done in a straightforward manner; however it is easier to use the results of [16]: With $\widetilde{Q}_f(V + \xi)$ defined as

$$\widetilde{Q}_f(V+\xi) = \int d^p\sigma\, f(\sigma)\big[V^i P_i + \xi_J \widetilde{\partial X}^J\big]\,, \tag{4.2}$$

the Poisson bracket is

$$\begin{aligned}\{\widetilde{Q}_f(V+\xi), \widetilde{Q}_g(W+\eta)\} &= \\ &- \widetilde{Q}_{fg}([V+\xi, W+\eta]_D) - \int d^p\sigma\, g(\sigma)(df \wedge X^*(\langle V+\xi, W+\eta\rangle))_{1\ldots p}\,,\end{aligned} \tag{4.3}$$

where $[\cdot,\cdot]_D$ is the higher Dorfman bracket (3.2) and $\langle\cdot,\cdot\rangle$ is the pairing (3.1). We can use this result to find the Poisson brackets for the charges Q as defined in (4.1). The key is the following relation between charges Q and $\tilde{Q}$:

$$Q_f(V+\xi) = \widetilde{Q}_f\big(V - \Pi^\#(\xi) + \xi + i_{V-\Pi^\#(\xi)}(B)\big)\,. \tag{4.4}$$

The resulting Poisson bracket of the charges is

$$\begin{aligned}\{Q_f(V+\xi), Q_g(W+\eta)\} &= \\ &- Q_{fg}([V+\xi, W+\eta]_R) - \int d^p\sigma\, g(\sigma)(df \wedge X^*(\langle V+\xi, W+\eta\rangle_R))_{1\ldots p}\,,\end{aligned} \tag{4.5}$$

where $[\cdot,\cdot]_R$ is the higher Roytenberg bracket (3.12) and $\langle\cdot,\cdot\rangle_R$ is the pairing (3.11). The calculation is straightforward but quite lengthy and we omit it here.

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Let us note that choosing constant test functions $f = g = 1$, one finds that the charge algebra (4.5) closes and it is described by the higher Roytenberg bracket. For the special case $p = 1$, this was already observed by Halmagyi [19].

Using this result, we can determine conditions for the conservation of such charges. To avoid the anomalous term in (4.5), we shall consider only the charges

$$Q(V+\xi) := Q_1(V+\xi)\,, \tag{4.6}$$

for a constant test function $f = 1$. We are interested to obtain conditions on $V+\xi \in \Gamma(E)$, which would guarantee that

$$\{Q(V+\xi), H\} = 0\,, \tag{4.7}$$

where H is the Hamiltonian (2.3). The left hand side of this condition can be conveniently rewritten using the Leibniz rule for Poisson bracket:

$$\begin{aligned}\{Q(V+\xi),H\} =& \\ &\frac{1}{2}\{Q(V+\xi), Q_{K_i}((G^{-1})^{ij}\partial_j)\} + \frac{1}{2}\{Q(V+\xi), Q_{(G^{-1})^{ij}\partial_j}(\partial_i)\} \\ &+ \frac{1}{2}\{Q(V+\xi), Q_{\widetilde{K}^I}(\widetilde{G}_{IJ}dy^J)\} + \{Q(V+\xi), Q_{\widetilde{G}_{IJ}\widetilde{K}^J}(dy^I)\}\,.\end{aligned} \tag{4.8}$$

Now we can use (4.5) to carry out the straightforward but tedious calculation that leads to the following result. Let $\mathcal{L}_W$ be the Lie derivative with respect to the vector field $W = V - \Pi^{\#}(\xi)$. The following set of conditions ensure that the charge $Q(V+\xi)$ is conserved:

$$\mathcal{L}_W(G)_{ij} = G_{\rm in}\Pi^{nL}\big(W^m dB_{mjL} - (d\xi)_{jL}\big) + (i \leftrightarrow j)\,, \tag{4.9}$$

$$\mathcal{L}_W(\widetilde{G})_{IJ} = \widetilde{G}_{IL}\Pi^{nL}\big(W^m dB_{mnJ} - (d\xi)_{nJ}\big) + (I \leftrightarrow J)\,, \tag{4.10}$$

$$\mathcal{L}_W(\Pi)^{kI} = \big(\Pi^{kJ}\Pi^{nI} - (\widetilde{G}^{-1})^{IJ}(G^{-1})^{kn}\big)\big(W^m dB_{mnJ} - (d\xi)_{nJ}\big)\,. \tag{4.11}$$

(Here $\widetilde{G}$ is viewed as a $2p$-times covariant tensor field on M.) Let us observe that there exists a particular simplification of these conditions: choosing

$$d\xi = i_W(dB)\,, \tag{4.12}$$

all terms on the right-hand side vanish and we get a new set of conditions

$$\mathcal{L}_W(G) = \mathcal{L}_W(\widetilde{G}) = \mathcal{L}_W(\Pi) = 0\,. \tag{4.13}$$

The special choice (4.12) can be rewritten as

$$\mathcal{L}_W(B) = d(\xi - i_W(B))\,. \tag{4.14}$$

The particular solution (4.13) to the more general conditions (4.9)–(4.11) implies that the image of $V+\xi$ under the anchor map (3.9) preserves the background fields $G, \widetilde{G}, \Pi$ and preserves the $(p+1)$-form field B up to an exact term.

The conditions (4.9)–(4.11) have an interesting geometrical meaning. Let $(\cdot,\cdot)$ be the fiberwise metric on $TM \oplus \Lambda^p T^* M$ given by $\mathbf{G}^{-1}$, the inverse of matrix $\mathbf{G}$ appearing in the Hamiltonian (2.7):

$$(V+\xi, W+\eta) := \begin{pmatrix} V \\ \xi \end{pmatrix}^T \begin{pmatrix} G & 0 \\ 0 & \widetilde{G}^{-1} \end{pmatrix} \begin{pmatrix} W \\ \eta \end{pmatrix} . \tag{4.15}$$

Let $e = V + \xi \in \Gamma(TM \oplus \Lambda^p T^* M)$. The conditions (4.9)–(4.11) are equivalent to the equation

$$\rho(e).(e_1, e_2) = ([e, e_1]_R, e_2) + (e_1, [e, e_2]_R) \,, \tag{4.16}$$

for all $e_1, e_2 \in \Gamma(TM \oplus \Lambda^p T^* M)$. In the other words, the charge $Q(V+\xi)$ is conserved, if $e = V + \xi$ is a "Killing section" of the fiberwise metric $(\cdot,\cdot)$ (4.15) with respect to the higher Roytenberg bracket.

5 Topological model, consistency of constraints

In this section we examine the topological sigma model, which is obtain from (2.1) by setting $G^{-1} = \widetilde{G}^{-1} = 0$. We will show that algebra of constraints closes on shell and that the constraints are compatible with time evolution. The consistency of the constraints is ensured by the vanishing of certain structure functions of the higher Roytenberg bracket, which in turn is related to the fundamental identity of a Nambu-Poisson structure.

The action has the form

$$S[\eta, \widetilde{\eta}, X] := \int d^{p+1}\sigma \big[\eta_i \partial_0 X^i + \widetilde{\eta}_I \widetilde{\partial X}^I - \Pi^{iJ} \eta_i \widetilde{\eta}_J - B_{iJ} \partial_0 X^i \widetilde{\partial X}^J \big] \,. \tag{5.1}$$

The canonical Hamiltonian of this model can be written as

$$H[X, \widetilde{\eta}, P] = - \int d^p\sigma \big[\widetilde{\eta}_I \big(\widetilde{\partial X}^I - \Pi^{kI} (P_k + B_{kJ} \widetilde{\partial X}^J) \big) \big] \,, \tag{5.2}$$

with canonical momenta P_k as given in (2.2). Using the notation of (2.4) and (2.5), we have

$$H[X, \widetilde{\eta}, P] = - \int d^p\sigma \widetilde{\eta}_I \widetilde{K}^I \,. \tag{5.3}$$

Looking at Lagrange-Euler equation for $\widetilde{\eta}_I$, we obtain

$$\widetilde{K}^I = 0 \,, \tag{5.4}$$

which should be viewed as a set of constraints, with $\widetilde{\eta}_I$ being the corresponding Lagrange multipliers. $\widetilde{K}^I$ as well as H can be expressed in terms of the charges (4.1) for special choices of test functions:

$$\widetilde{K}^I(\sigma) = Q_{\delta(\sigma - \cdot)}(dy^I) \,, \tag{5.5}$$

$$H = -Q_{\widetilde{\eta}_I}(dy^I) \,. \tag{5.6}$$

The constraint algebra and time evolution of constraints can therefore be expressed in terms of the Roytenberg bracket by equation (4.5). In terms of the structure functions of the Roytenberg bracket (cf. appendix) we obtain the following current algebra

$$\begin{aligned}\{\widetilde{K}^I(\sigma),\widetilde{K}^J(\sigma')\} = -\delta(\sigma-\sigma')(R^{IJk}K_k + S^{IJ}_K\widetilde{K}^K)(\sigma') \\ -\big(d(\delta(\sigma-\cdot))\wedge X^*(\langle dy^I,dy^J\rangle_R)\big)_{1\dots p}(\sigma')\,.\end{aligned} \tag{5.7}$$

It is hence natural to ask for R to vanish. This leads precisely to the condition (B.6) for $\xi = dy^J$, $\eta = dy^I$ and $H = -dB$. Imposing the condition $R = 0$ is thus equivalent to the assumption that Π fulfills the differential part of the fundamental identity for a $(-dB)$-twisted Nambu-Poisson tensor[2] and we shall henceforth assume that this is the case. Even then, there still seems to be a problem with the anomalous last term in (5.7), since in general the expression $\langle dy^J, dy^I\rangle_R$ doesn't vanish. To see it let us note that vanishing of $\langle dy^J, dy^I\rangle_R$ is equivalent to

$$i_{\Pi^\#(dy^J)}(dy^I) + i_{\Pi^\#(dy^I)}(dy^J) = 0\,. \tag{5.8}$$

The anomalous terms can be dealt with using secondary constraints and consistency of these constraints turns out to be ensured by the algebraic part of the fundamental identity for a Nambu-Poisson tensor. Indeed, geometrically (5.8) implies that the graph of Π, $G_\Pi = \{\xi + \Pi^\#(\xi)|\ \xi\in\Omega^p(M)\}$, is isotropic with respect to the canonical pairing $\langle\cdot,\cdot\rangle$ (3.1) on $\Gamma(TM\oplus\Lambda^pT^*M)$. However, as was noticed by Zambon in [36], such (nontrivial) Π exists only for $p=1$ and $p = \mathrm{dim}M-1$. For $1<p<\mathrm{dim}M-1$, we are forced to add the following set of constraints to the system:

$$\chi_q^{IJ} \equiv (X^*\langle dy^I,dy^J\rangle_R)_{1\dots\hat{q}\dots p} = 0\,. \tag{5.9}$$

The new constraints (5.9) do not contain any P_m's and thus they Poisson commute with each other, i.e.

$$\{\chi_q^{IJ}(\sigma),\chi_r^{KL}(\sigma')\} = 0. \tag{5.10}$$

The Poisson brackets between the new constraints χ_q^{IJ} and the constraints $\widetilde{K}^M(\sigma')$ are

$$\begin{aligned}\{\chi_q^{IJ}(\sigma),\widetilde{K}^M(\sigma')\} &= \big(S^{MI}{}_K\chi_q^{KJ} + S^{MJ}{}_K\chi_q^{IK}\big)(\sigma)\delta(\sigma-\sigma') \\ &\quad + \sum_{\substack{r=1\\ r\neq q}}^{p}\mathrm{sgn}(r,q)\big(X^*(i_{\Pi^\#(dy^M)}\langle dy^I,dy^J\rangle_R)\big)_{1\dots\hat{r}\dots\hat{q}\dots p}(\sigma)\frac{\partial\delta(\sigma'-\cdot)}{\partial\sigma^r}(\sigma)\,,\end{aligned}$$

where $\mathrm{sgn}(r,q)$ is just a sign, irrelevant for the discussion. The first term clearly vanishes for $\chi_q^{IJ} = 0$. The second term, in fact, also weakly vanishes (i.e. it vanishes when the constraints equations are used; this is denoted by "$\approx$"). To see this, it is sufficient to show that

$$\big(X^*(i_{\Pi^\#(dy^M)}\langle dy^I,dy^J\rangle_R)\big)_{1\dots\hat{r}\dots\hat{q}\dots p} \approx 0\,, \tag{5.11}$$

[2] Note that for $p>1$ the twisting of Nambu-Poisson structures is redundant since it just leads again to an ordinary Nambu-Poisson structure.

Evaluating the left hand side expression at a $p \in \Sigma$ with $\Pi(X(p)) = 0$, clearly gives zero. If $\Pi(X(p)) \neq 0$, the validity of (5.11) can be shown to be a consequence of the following observation made in [38]

$$\langle dy^I, dy^J \rangle_R |_{\Lambda^{p-1}\rho(G_\Pi)} = 0\,, \tag{5.12}$$

where ρ denotes the projection onto the first summand of the graph G_Π. The reasoning itself is not very illuminating and we skip the details here.[3]

Since the Hamiltonian is of the form (5.3), the constraints $\widetilde{K}^I$ and χ_q^{IJ} are consistent with the dynamics, i.e., they weakly Poisson commute with the Hamiltonian. This follows immediately from the above discussion of the constraints algebra.

To conclude this section, we shall investigate the conservation of charges (4.6) with respect to the dynamics governed by the Hamiltonian (5.3). This is again simple using (5.6) and (4.5). For the charge $Q(V+\xi)$ to be conserved, one gets the condition

$$\mathcal{L}_{\Pi^\#(\widetilde{\eta})}(V) = \Pi^\#(i_{\Pi^\#(\widetilde{\eta})} i_V dB), \tag{5.13}$$

where we have introduced the section $\widetilde{\eta} := \eta_J dy^J$ of the pullback bundle $X^*(\bigwedge^p T^*M)$. Note that the charge $Q(0+\xi)$ is conserved for arbitrary $\xi \in \Omega^p(M)$.

Given the results of this section, we will shall henceforth assume that Π is a Nambu-Poisson tensor (which may be twisted in the case $p=1$).

6 Equations of motion, solution

In this section we will derive the equations of motion of the topological action (5.1) using the Hamiltonian formalism and previous results. Using the natural coordinates associated with every Nambu-Poisson structure (for $p>1$), we will find an explicit solution of these equations. The calculations involve the higher Roytenberg bracket via the charge algebra. They are again quite long, but straightforward and we will mostly just state the results.

Straight from the definition, one can calculate the equations of motion for the X^m fields. Indeed, the calculation of

$$\dot{X}^m(\sigma) = \{X^m(\sigma), H\}$$

is just an easy application of the Leibniz rule for the Poisson bracket. Of course, among the equations of motion we will find also the constrains $\widetilde{K}^I = 0$. The most difficult part comes with the calculation of

$$\dot{P}_i(\sigma) = \{P_i(\sigma), H\}\,.$$

[3] Alternatively, one can introduce new constraints $\chi_{rq}^{MIJ} := \left(X^*(i_{\Pi^\#(dy^M)}\langle dy^I, dy^J\rangle_R)\right)_{1\ldots\hat{r}\ldots\hat{q}\ldots p} = 0$. These will obviously Poisson commute with each other and with all χ_q^{IJ}'s. Hence, we just have to check their Poisson brackets with the $\widetilde{K}^I$'s. Doing this, new anomalous terms proportional to $X^*(i_{\Pi^\#(dy^N)} i_{\Pi^\#(dy^M)}\langle dy^I, dy^J\rangle_R)$ will appear. We can treat these again as new constraints and repeat the procedure until we arrive at anomalous terms containing $(p-1)$-contractions with $i_{\Pi^\#(sth)}$. By (5.12) this is identically equal to zero. Note that all these auxiliary constraints follow already from the first ones, i.e., from $\chi_q^{IJ} = 0$ by the above discussion.

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This can be done again using (4.5). First, note that

$$P_i = K_i - B_{iL}\big(\Pi^{mL}K_m + \widetilde{K}^L\big)\,. \tag{6.1}$$

Hence

$$P_i(\sigma) = Q_{\delta(\sigma-\cdot)}\big(\partial_i - \Pi^{\#}(i_{\partial_i}B) - i_{\partial_i}B\big)\,.$$

Now, using (5.6) and (4.5), one gets the following result: The fields $X^m(\sigma), P_m(\sigma)$ and $\widetilde{\eta}_J(\sigma)$ of the sigma model defined by action (5.1) evolve in accordance with the following set of equations:

$$\begin{aligned}
\widetilde{\partial X}^I &= \Pi^{mI}(P_m + B_{mK}\widetilde{\partial X}^K)\,, && (6.2)\\
\dot{X}^m &= \Pi^{mJ}\widetilde{\eta}_J\,, && (6.3)\\
\dot{P}_m &= -\Pi^{kJ}{}_{,m}P_k - (d\widetilde{\eta}_{mN}\wedge dX^N)_{1\dots p} + \Pi^{kJ}B_{kmL}(d\widetilde{\eta}_J\wedge dX^L)_{1\dots p} && (6.4)\\
&\quad -\widetilde{\eta}_J\left(\Pi^{kJ}B_{mL,k} + \Pi^{kJ}{}_{,m}B_{kL} + \sum_{n=1}^{p}\Pi^{kJ}{}_{,l_n}B_{ml_1\dots k\dots l_p} - \Pi^{kJ}(dB)_{kmL}\right)\widetilde{\partial X}^L\,.
\end{aligned}$$

In particular, for $B = 0$, we get the equations of motion for the untwisted sigma model:

$$\begin{aligned}
\widetilde{\partial X}^I &= \Pi^{mI}P_m\,, && (6.5)\\
\dot{X}^m &= \Pi^{mJ}\widetilde{\eta}_J\,, && (6.6)\\
\dot{P}_m &= -\widetilde{\eta}_J\Pi^{kJ}{}_{,m}P_k - (d\widetilde{\eta}_{mN}\wedge dX^N)_{1\dots p}\,. && (6.7)
\end{aligned}$$

Now we will show that there always exists a non-trivial solution of the field equations (6.2)–(6.3). We will use the natural local coordinates that are associated with every Nambu-Poisson tensor, namely $(x^1,\dots,x^n)$, such that

$$\Pi = \frac{\partial}{\partial x^1}\wedge\dots\wedge\frac{\partial}{\partial x^{p+1}}\,, \tag{6.8}$$

which exist around every point $x\in M$, where $\Pi(x)\neq 0$ (see e.g. [39]). In these coordinates the components of Π can be expressed in terms of the Levi-Civita symbol:

$$\Pi^{i_1\dots i_{p+1}} = \epsilon^{i_1\dots i_{p+1}}. \tag{6.9}$$

This choice of local coordinates simplifies the equations of motion considerably. We define a p-index $[r] = (1,\dots,\hat{r},\dots p{+}1)$ and $(p{-}1)$-index $[p,q] = (1,\dots,\hat{p},\dots,\hat{q},\dots,p{+}1)$. (The hats denote omitted indices.)

The constraints (5.9) are in these coordinates equivalent to

$$\frac{\partial X^m}{\partial\sigma^k} = 0,$$

for $m > p+1$ and $k\in\{1,\dots,p\}$. This is not straightforward to see, one has to use the consequences of (5.12). Furthermore, the equations (6.3) impose

$$\dot{X}^m = 0,$$

for $m > p+1$ and we thus get $X^m = C^m$ for $m > p+1$, where $C^m\in\mathbb{R}$ are arbitrary real constants. One can then easily deduce the following solution of (6.2)–(6.4):

(i) For $m \leq p+1$,

$$X^m = f^m,$$

where $f^m \in C^\infty(\Sigma)$ are arbitrary smooth functions on Σ;

(ii) for $m > p+1$,

$$X^m = C^m,$$

where $C^m \in \mathbb{R}$ are arbitrary real constants;

(iii) for $r \leq p+1$,

$$\widetilde{\eta}_{[r]} = (-1)^{r+1}\dot{X}^r,$$

and if $I \neq [r]$,

$$\widetilde{\eta}_I = E_I,$$

where E_I are arbitrary constants in space-like variables on Σ.

(iv) for $r \leq p+1$,

$$P_r = (-1)^{r+1}(1 - B_{1\dots p+1})\widetilde{\partial X}^{[r]};$$

(v) for $m > p+1$,

$$P_m = \int d\sigma^0 \left[\sum_{k,r=1}^{p+1} \left[(dB_{km[r]} - B_{m[r],k})\dot{X}^k \widetilde{\partial X}^{[r]}\right] + \sum_{\substack{r,q=1 \\ r\neq q}}^{p+1} \sum_{k=1}^{p+1} B_{km[r,q]}(d\dot{X}^k \wedge dX^{[r,q]})_{1\dots p} \right].$$

Although straightforward, it is actually a lengthy computation to verify that this solution to equations (6.2) and (6.3) indeed also solves the equation (6.4).

There is a nice geometrical interpretation of the solutions for X: Π defines a $(p+1)$-dimensional foliation in M, and $(x^1, \dots, x^n)$ are coordinates adapted to this foliation. Hence the fields X are constant in the directions transversal to this foliation.

7 Generalized Wess-Zumino terms

In this section, we encounter yet another way how the higher Roytenberg bracket appears in the context of the Nambu sigma model: Lifting the topological terms of the model to $(p+2)$ dimensions, the structure functions appear as coefficients in the resulting generalized Wess-Zumino terms. This resembles the $p = 1$ case, where the generalized Wess-Zumino terms are topological if and only if the associated Roytenberg relations are satisfied.

We shall use the Lagrangian formalism this time and follow essentially the classic approach of Wess, Zumino, and Witten [40, 41], adapted to the twisted Poisson sigma model by Halmagyi in [20].

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Define the p-forms A_i and 1-forms $\widetilde{A}_J$ as

$$
\begin{aligned}
A_i &= \eta_i d\sigma^1 \wedge \ldots \wedge d\sigma^p, \\
\widetilde{A}_J &= \widetilde{\eta}_J d\sigma^0.
\end{aligned}
$$

Choosing the orientation on Σ as $o(\sigma^0, \sigma^1, \ldots, \sigma^p) = +1$ and introducing an auxiliary Minkowski world volume metric, the action (2.1) can be rewritten as

$$
\begin{aligned}
S[X, A, \widetilde{A}] = \int_\Sigma -\frac{1}{2}(G^{-1})^{ij} A_i \wedge *A_j - \frac{1}{2}(\widetilde{G}^{-1})^{IJ} \widetilde{A}_I \wedge *\widetilde{A}_J \\
+ dX^i \wedge A_i + \widetilde{A}_J \wedge dX^J - \Pi^{iJ} \widetilde{A}_J \wedge A_i - X^*(B)\,,
\end{aligned} \tag{7.1}
$$

Topological part of this action has the form

$$
\begin{aligned}
S_{\text{top}}[X, A, \widetilde{A}] = \int_\Sigma dX^i \wedge A_i + \widetilde{A}_J \wedge dX^J - \Pi^{iJ} \widetilde{A}_J \wedge A_i \\
+ \frac{1}{2}\widetilde{A}_I \wedge \widetilde{A}_J \wedge M^{IJ} - X^*(B)\,,
\end{aligned} \tag{7.2}
$$

where we have added a new term $\frac{1}{2}\widetilde{A}_I \wedge \widetilde{A}_J \wedge M^{IJ}$, which is zero on Σ[4] and where

$$
\begin{aligned}
M^{IJ} &= \frac{1}{2} X^* \big(i_{\Pi^\#(dy^I)}(dy^J) - i_{\Pi^\#(dy^J)}(dy^I) \big) \\
&= \frac{1}{2} \sum_{r=1}^{p} (-1)^{r-1} \Pi^{j_r I} (dX^{j_1} \wedge \ldots \wedge \widehat{dX^{j_r}} \wedge \ldots \wedge dX^{j_p}) - (I \leftrightarrow J)\,.
\end{aligned} \tag{7.3}
$$

(The hat denotes a factor that is omitted.)

Let us suppose that $\Sigma = \partial N$, where N is a smooth $(p+2)$-dimensional manifold. Using Stoke's theorem, we can lift the action to N:

$$
S_{\text{top}}[X, A, \widetilde{A}] = \int_N d(\mathcal{L})_{\text{top}}\,. \tag{7.4}
$$

$$
\begin{aligned}
d(\mathcal{L}_{\text{top}}) &= -(dX^i - \Pi^{iJ} \widetilde{A}_J) \wedge dA_i + d\widetilde{A}_J \wedge (dX^J - \Pi^{iJ} A_i - M^{JK} \widetilde{A}_K) \\
&\quad - \Pi^{iJ}{}_{,k} dX^k \wedge \widetilde{A}_J \wedge A_i + \frac{1}{2} \widetilde{A}_I \wedge \widetilde{A}_J \wedge dM^{IJ} \\
&\quad - \frac{p!}{(p+2)!} dB_{klJ} \wedge dX^k \wedge dX^l \wedge dX^J\,.
\end{aligned}
$$

We define new fields ψ^i and $\widetilde{\psi}^J$ as

$$
\psi^i = dX^i - \Pi^{iJ} \widetilde{A}_J\,, \tag{7.5}
$$

$$
\widetilde{\psi}^J = dX^J - \Pi^{iJ} A_i - M^{JK} \wedge \widetilde{A}_K\,. \tag{7.6}
$$

[4] This term is zero on $\Sigma = \partial N$; however, we assume an arbitrary extension of $\widetilde{A}$ on N, hence it is in general non-zero on N.

We observe that

$$
\begin{aligned}
dM^{IJ} &= \frac{1}{2}\sum_{r=1}^{p} \Pi^{j_r I}{}_{,k} dX^{j_1\ldots k\ldots j_p} - (I \leftrightarrow J) \\
&= \frac{1}{2}\sum_{r=1}^{p} \Pi^{j_r I}{}_{,k} \big(\widetilde{\psi}^{j_1\ldots k\ldots j_r} + \Pi^{i j_1\ldots k\ldots j_p} A_i + M^{j_1\ldots k\ldots j_p, K} \wedge \widetilde{A}_K\big) - (I \leftrightarrow J)\,.
\end{aligned}
\tag{7.7}
$$

Putting the above expression for dM^{IJ} and redefinition of the fields into $d(\mathcal{L}_{\text{top}})$, one finds that

$$
\begin{aligned}
d(\mathcal{L}_{\text{top}}) = & -\psi^i \wedge dA_i + d\widetilde{A}_J \wedge \widetilde{\psi}^J + Q'^{Ji}{}_k \psi^k \wedge \widetilde{A}_J \wedge A_i + \frac{1}{2} F'_{kl}{}^i \psi^k \wedge A_i \wedge \psi^l \\
& - \frac{1}{2} H'_{klJ}\ \psi^k \wedge \psi^l \wedge \widetilde{\psi}^J + D'_{kM}{}^J \widetilde{A}_M \wedge \widetilde{\psi}^J \wedge \psi^k \\
& - \frac{1}{2} S'^{LM}{}_J \widetilde{A}_L \wedge \widetilde{A}_M \wedge \widetilde{\psi}^J - \frac{1}{2} R'^{LJi} \widetilde{A}_L \wedge \widetilde{A}_J \wedge A_i \\
& - \left(\frac{1}{2} H'_{klL} \psi^k \wedge \psi^l + D'_{lL}{}^I \widetilde{A}_I \wedge \psi^l + \frac{1}{2} S'^{IJ}{}_L \widetilde{A}_I \wedge \widetilde{A}_J\right) \wedge \widetilde{A}_N \wedge M^{NL}\,,
\end{aligned}
\tag{7.8}
$$

where Q', F', H', D', S', R' are structure functions of skew-symmetric version of the higher Roytenberg bracket (see appendix A) corresponding to a re-scaled 3-form flux

$$
H_{jlK} = \frac{1}{(p+1)(p+2)} (dB)_{jlK}\,.
$$

8 Conclusion

In this article, we have studied higher dimensional analogs of generalized Poisson sigma models and the corresponding dual string and p-brane models. In this context, we have found that higher algebraic structures related to a generalization of the Roytenberg bracket play an important role and that Nambu-Poisson structures are the appropriate $p > 1$ generalization of the Poisson structures that are relevant for the $p = 1$ case.

Let us summarize the main results: By a Legendre transformation, we have obtained the Hamiltonian corresponding to the Nambu sigma model that had been introduced in [27] and identified as a dual to the gauge-fixed Polyakov-style Howe-Tucker p-brane action. The resulting quadratic form can be viewed as higher-dimensional analog of a generalized metric (see e.g. [42]). Starting with the definition of a twisted higher Dorfman bracket (see [35]) and using a $(p+1)$-vector Π, we have further twisted this structure and have obtained a new Courant bracket like structure, which we call a higher Roytenberg bracket. Its $p = 1$ version was originally introduced by Roytenberg in [34]. We define a higher analog in coordinate-free intrinsic form, such that its properties, which resemble that of higher Dorfman brackets, can be easily verified. The algebraic structures related to this new bracket play a fundamental role throughout this article. Next, we have defined generalized charges for the model, with a complicated structure that is parameterized by sections of the vector bundle $TM \oplus \Lambda^p T^*M$. We have found that we can use previous results of Ekstrand and Zabzine [16] to calculate the world sheet algebra of the charges. It turns out that the Poisson bracket of the charges closes under the higher Roytenberg bracket up to an anomaly. This anomaly vanishes if one restricts to some isotropic subbundle $TM \oplus \Lambda^p T^*M$ with respect to a twisted pairing $\langle \cdot, \cdot \rangle_R$. One can further find the parameterizing sections of the charges, such that they are conserved under time evolution. We have been let to a set of partial differential equations that generalize the ones found by Halmagyi in [19]. The equations have an interesting geometrical interpretation: they constitute Killing equations with respect to a certain fiber-wise metric. The topological part of the p-brane action turns out to be a system with constraints, as expected. We have analyzed the consistency of these constraints under time evolution and with the constraint algebra itself. The constraints can be written in the terms of the generalized charges that we have introduced in this article and the calculation of their Poisson bracket can be carried out using the higher Roytenberg bracket. Consistency under time evolution forces certain structure functions of the higher Roytenberg bracket to vanish, which is equivalent to the differential part of the fundamental identity satisfied by a Nambu-Poisson tensor. However, an anomalous term remains in the Poisson bracket, which can be dealt with using secondary constraints for the model. We have shown that these secondary constraints are compatible with time evolution, provided that the algebraic part of the fundamental identity of a Nambu-Poisson structure also holds. It is thus natural to consider the background $(p+1)$-vector field Π to be a Nambu-Poisson structure. We have derive explicit expressions for the equations of motion of the topological model, using once more results for the charge algebra. This has been possible, since the canonical momenta P_m can be rewritten in the terms of generalized charges. Using special coordinates, whose existence is guaranteed locally for any Nambu-Poisson structure, we have been able to simplify the equations of motion and find an explicit non-trivial solution. This is similar to the use of Darboux-Weinstein coordinates in the case of Poisson sigma models. Finally, we have present the analog of the calculation of Halmagyi in [20]: we have lifted the topological part of the action to a $(p+2)$-dimensional world volume N, such that $\Sigma = \partial N$, using Stoke's theorem. After some redefinitions of the fields, the resulting Lagrangian density (generalized Wess-Zumino terms) incorporates the fields coupled to new background fields, which are the structure functions of the skew-symmetric version of the higher Roytenberg bracket that we have introduce in this paper. The generalized Wess-Zumino terms are topological if and only if the higher Roytenberg relations are satisfied (see appendix A).

Studying the consistency of the topological model, one is let to a set of constraints that are usually understood as constraints on the embedding fields X and eventually imply conditions on the multi-vector Π, but that can also be interpreted as constraints on the auxiliary fields η and $\widetilde{\eta}$. This was already observed by Halmagyi in the case $p = 1$ in [19]. Halmagyi does not further comment on the implication of this observation, but we can in fact now understand this in the present context: the constraints on the auxiliary fields effectively reduce the available dimensionality of target space for the other fields of the model. The multi-vector Π is of maximal rank in this subspace. It therefore factorizes and is thus forced to be of Nambu-Poisson type. This is true for $p > 1$ and confirms the results that we have obtained in this article using more sophisticated methods. The observation and the conclusion is, however, also valid in the well-studied $p = 1$ case: a factorized bi-vector (i.e. $\Pi = V_1 \wedge V_2$ with suitable vector fields V_1 and V_2) will indeed ensure the consistency of the equations of motion, but this is just a special example of a more general Poisson bi-vector satisfying the Jacobi identity, which also ensures consistency.

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A Higher Roytenberg bracket, structure functions

Here we summarize the local form of the higher Roytenberg bracket (3.12) twisted by a $(p+2)$-form flux

$$H = \frac{1}{(p+1)(p+2)} H_{klJ} dX^k \wedge dX^l \wedge dX^J ,$$

where $dX^J \equiv dX^{j_1} \wedge \ldots \wedge dX^{j_p}$ and $J = (j_1, \ldots, j_p)$ denotes an ordered multi-index with $j_1 < \ldots < j_p$.

Let $(y^1, \ldots, y^n)$ be a set of local coordinates on M. Denote $\partial_k = \frac{\partial}{\partial y^k}$ and $dy^K = dy^{k_1} \wedge \ldots \wedge dy^{k_p}$. Then, one has

$$[\partial_k, \partial_l]_R = F_{kl}{}^m \partial_m + H_{klL} dy^L , \tag{A.1}$$

$$[\partial_k, dy^J]_R = Q_k^{mJ} \partial_m + D^J_{k\,L} dy^L , \tag{A.2}$$

$$[dy^I, dy^J]_R = R^{IJm} \partial_m + S^{IJ}{}_L dy^L . \tag{A.3}$$

The structure functions have the following form (Roytenberg relations):

$$F_{kl}{}^m = H_{klJ} \Pi^{mJ} , \tag{A.4}$$

$$Q_k^{mJ} = -\Pi^{mJ}{}_{,k} + H_{lkL} \Pi^{lJ} \Pi^{mL} , \tag{A.5}$$

$$D^J_{k\,L} = H_{lkL} \Pi^{lJ} , \tag{A.6}$$

$$\begin{aligned} R^{IJm} &= \Pi^{nI} \Pi^{mJ}{}_{,n} - \Pi^{nJ} \Pi^{mI}{}_{,n} \\ &\quad - \sum_{r=1}^{p} \Pi^{j_r I}{}_{,k} \Pi^{m j_1 \ldots k \ldots j_p} + \Pi^{kI} \Pi^{lJ} \Pi^{mL} H_{klL} , \end{aligned} \tag{A.7}$$

$$S^{IJ}{}_L = - \sum_{r=1}^{p} \Pi^{j_r I}{}_{,k} \delta_L^{j_1 \ldots k \ldots j_p} + \Pi^{kI} \Pi^{lJ} H_{klL} . \tag{A.8}$$

We denote by a prime the structure functions of the skew-symmetrized version of the higher Roytenberg bracket. For example $S'^{IJ}{}_L = \frac{1}{2} \left(S^{IJ}{}_L - S^{JI}{}_L \right)$.

B Nambu-Poisson structures

Here we recall some fundamental properties of Nambu-Poisson structures [21] as needed in this paper. For details see, e.g., [38] or [35].

For any $(p+1)$-vector field A on M we define the induced map $A^\# : \Omega^p(M) \to \mathfrak{X}(M)$ as $A^\#(\xi) = (-1)^p i_\xi A = \xi_K A^{iK} \partial_i$.

Let Π be a $(p+1)$-vector field on M. We call Π a Nambu-Poisson structure, if

$$\mathcal{L}_{\Pi^\#(df_1 \wedge \ldots \wedge df_p)}(\Pi) = 0\,, \tag{B.1}$$

for all $f_1, \ldots, f_p \in C^\infty(M)$.

Lemma 1. *For arbitrary $p \geq 1$ the condition (B.1) can be stated in the following equivalent ways:*

1. *The graph $G_\Pi = \{\Pi^\#(\xi) + \xi \mid \xi \in \Omega^p(M)\}$ is closed under the higher Dorfman bracket (3.2);*

2. *for any $\xi, \eta \in \Omega^p(M)$ it holds that*

$$(\mathcal{L}_{\Pi^\#(\xi)}(\Pi))^\#(\eta) = -\Pi^\#(i_{\Pi^\#(\eta)}(d\xi))\,; \tag{B.2}$$

3. *let $[\cdot,\cdot]_\pi : \Omega^p(M) \times \Omega^p(M) \to \Omega^p(M)$ be defined as*

$$[\xi, \eta]_\pi := \mathcal{L}_{\Pi^\#(\xi)}(\eta) - i_{\Pi^\#(\eta)}(d\xi)\,, \tag{B.3}$$

for all $\xi, \eta \in \Omega^p(M)$. Then it holds that

$$[\Pi^\#(\xi), \Pi^\#(\eta)] = \Pi^\#([\xi, \eta]_\pi)\,, \tag{B.4}$$

for all $\xi, \eta \in \Omega^p(M)$;

4. *for any $\xi \in \Omega^p(M)$ it holds that*

$$\mathcal{L}_{\Pi^\#(\xi)}(\Pi) = -\left(i_{d\xi}(\Pi)\Pi - \frac{1}{p+1} i_{d\xi}(\Pi \wedge \Pi) \right). \tag{B.5}$$

There seems to be a natural way to define a twisted Nambu-Poisson structure: Let Π be a $(p+1)$-vector on M. Let $H \in \Omega^{p+2}(M)$, such that $dH = 0$. We call Π an H-twisted Nambu-Poisson structure, if the graph G_Π of Π is closed under H-twisted higher Dorfman bracket (3.7). Equivalently, a H-twisted Nambu-Poisson structure can be defined using the condition

$$(\mathcal{L}_{\Pi^\#(\xi)}(\Pi))^\#(\eta) = -\Pi^\#(i_{\Pi^\#(\eta)}(d\xi - i_{\Pi^\#(\xi)} H))\,, \tag{B.6}$$

for all $\xi, \eta \in \Omega^p(M)$. This definition is correct, however, for $p > 1$ there occurs an interesting thing: The fundamental identity (B.1) splits into two parts – one part is a differential identity similar to the Jacobi identity for the Poisson bivector, the other part of the identity

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is purely algebraic. Interestingly for $p > 1$, the fundamental identity ensures the existence of coordinates $(x^1, \ldots, x^n)$ around every point x where $\Pi(x) \neq 0$, such that

$$\Pi = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{p+1}} \,. \tag{B.7}$$

For details, see e.g. [39]. Conversely, every decomposable $(p+1)$-vector whose support is an integrable distribution is Nambu-Poisson. The algebraic part of (B.1) comes from the fact that (B.2) is not $C^\infty(M)$-linear in ξ. If we now consider (B.6), we see that if we add a part that is $C^\infty(M)$-linear in ξ, the algebraic part of identity will stay untouched. This means that a Π satisfying (B.6) is in fact still an ordinary Nambu-Poisson tensor, satisfying (B.2). The concept of an H-twisted Nambu-Poisson tensor is therefore redundant for $p > 1$, as has already been noticed in [28].

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Appendix C

Paper 2: On the Generalized Geometry Origin of Noncommutative Gauge Theory

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On the generalized geometry origin of noncommutative gauge theory

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Abstract: We discuss noncommutative gauge theory from the generalized geometry point of view. We argue that the equivalence between the commutative and semiclassically noncommutative DBI actions is naturally encoded in the generalized geometry of D-branes.

Keywords: D-branes, Non-Commutative Geometry, Differential and Algebraic Geometry, Sigma Models

Dedicated to Bruno Zumino on the occasion of his 90th birthday

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Contents

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1 Introduction

Generalized geometry [1, 2] recently appeared to be a powerful mathematical tool for the description of various aspects of string and field theories. Here we mention only few instances of its relevance that are more or less directly related to the present paper. Topological and non-topological Poisson sigma models are known to be intimately related to a lot of interesting differential, in particular generalized, geometry. For instance, the topological Poisson sigma models are of interest for the integration of Poisson manifolds (and Lie algebroids) [3] and are at the heart of deformation quantization [4]. Field equations of (topological) Poisson sigma models can be interpreted as Lie algebroid morphisms [5] and as such can further be generalized in terms of generalized (complex) geometry [6, 7]. Poisson sigma models can be twisted by a 3-form H-field [8] and also generalized to Dirac sigma models [7], where the graph defined by the corresponding (possibly twisted) Poisson structure is replaced by a more general Dirac structure. In turn, at least in some instances, D-branes can be related to Dirac structures [9, 10], or coisotropic submanifolds [11]. In [12], it has been observed that the current algebra of sigma models naturally involves structures of generalized geometry, such as the Dorfman bracket and Dirac structures. This was further developed in [13] and [14]. In [15], it was observed that in the first order (non-topological) Poisson sigma model characterized by a 2-form B and a bivector θ, a more general form of world-sheet currents appears. Their algebra has been shown to close under a more general bracket, the so called Roytenberg bracket [16]. In [17], it has been shown that the structure constants of the Roytenberg bracket appear if one lifts the topological part of first order Poisson sigma characterized by a 2-form B and a bivector θ to a three-dimensional WZW term. It this respect, generalized geometry is relevant for discussions of non-geometric backgrounds.

Noncommutativity of open strings, more precisely of their endpoints, in the presence of a B-field was recognized in [18, 19] and [20]. A thorough discussion of noncommutativity in string theory followed in the famous article of Seiberg and Witten [21], where, among other things, also the equivalence of commutative and noncommutative gauge theories was discussed via a field redefinition known under the name Seiberg-Witten map. In particular, it was argued that the higher derivative terms in the noncommutative version of the Dirac-Born-Infeld (DBI) action can be viewed as corrections to the usual DBI action, the effective D-brane action. For reviews on noncommutativity in string theory we refer, e.g., to [22, 23]. Let us also note that the (semiclassical) noncommutativity of D-branes can be seen as the (semiclassical) noncommutativity of the string endpoints in the open topological Poisson sigma model [3], which fits naturally to their role in both the integration as well as deformation quantization of Poisson structures.

The purpose of the present paper is to unravel the generalized geometry origin of noncommutative gauge theory. We will mainly focus on the equivalence between the commutative and semiclassically noncommutative DBI actions (and closely related issues) and argue that the necessity of such an equivalence can be seen and naturally interpreted within generalized geometry. In the discussion, non-topological Poisson sigma models play a role. Roughly speaking, we intend to convince the reader that the equivalence of commutative and semiclassically noncommutative DBI actions is encoded in two different ways of expressing a generalized metric on a D-brane.

Before going into a more detailed description of the individual sections, let us note that almost everything in this paper is presented in a form suitable for a direct generalization to Nambu-Poisson structures and M-theory membranes, cf. [24, 25]. We will discuss this in detail in a forthcoming paper.

The paper is organized as follows.

In the second section, we review basic definitions of generalized geometry. We emphasize the behavior of a generalized metric under orthogonal transformations of $TM \oplus T^*M$. This allows us to recover the formulas relating, via a bivector θ, the closed background fields g, B and the open string backgrounds G and Φ. It comes as a relation between two generalized metrics, which are connected by the action of a certain orthogonal transformation induced by the bivector θ. Finally, we recall the definition of the Dorfman bracket, Dirac structures and their relation to D-branes. In the latter we follow the proposal of [10], where D-branes correspond to leaves of foliations defined by Dirac structures.

In the third section, we observe that adding the gauge field F on D-brane volume corresponds to an action of an orthogonal transformation on the natural generalized metric on the D-brane, the pullback of the generalized space-time metric defined by the closed backgrounds g and B. The natural question is whether the so obtained generalized metric can again be rewritten in the open string variables (with some gauge field F' and a possibly modified bivector θ'). The positive answer is given by two different factorizations of an orthogonal transformations defined by a bivector and a 2-form, in our case θ and F. As a consequence, we find a generalization of open-closed relations of Seiberg and Witten, which includes the field strengths F and F', the latter one closely related to the nocommutative gauge field strength. This equality, crucial for our discussion of DBI actions, also hints towards the appearance of the semiclassical Seiberg-Witten map, once one recalls its interpretation as the local coordinate change between the two (Poisson) bivectors θ and θ'.

In the fourth section, we use the above mentioned relation between open and closed variables (including gauge fields) to show that non-topological Poisson-sigma model, its Hamiltonian and the corresponding Polyakov action are manifestly invariant under the open-closed field redefinitions as they geometrically correspond to the same generalized metric.

In the fifth section, we briefly recall the interpretation of the semiclassical Seiberg-Witten map as a local diffeomorphism on the D-brane world volume relating the noncommutativity parameters (Poisson bivectors) θ and θ'. This interpretation is the most relevant one for our discussion in the final section. When considering D-branes which are symplectic leaves of θ, the Seiberg-Witten map is naturally interpreted in terms of the corresponding Dirac structure.

In the final section, we discuss the equivalence of commutative and semiclassically noncommutative DBI action of a D-brane. We show that this equivalence is a direct consequence of the (gauge field dependent) open-closed relations combined with a Seiberg-Witten map. The discussion here is not completely new. However, what we believe is new and interesting is the clear generalized geometry origin of its main ingredients as developed in previous sections. Everything works very naturally for a D-brane which is a symplectic leaf of the Poisson structure, describing the noncommutativity.

We believe that analogous results hold also for more general D-branes, i.e. those which are related to more general Dirac structures than the ones defined by graphs of Poisson tensors. For such D-branes, Dirac sigma models of [7] should replace the Poisson sigma models.

2 Generalized geometry

2.1 Fiberwise metric, generalized metric

In this section we recall some basic facts regarding generalized geometry, see, e.g., [2, 26]. Although most of the involved objects can be defined in a more general framework, we focus on a particular choice of vector bundle. Namely, let M be a smooth manifold and $E = TM \oplus T^*M$. A fiberwise metric $(\cdot,\cdot)$ on E is a $C^\infty(M)$-bilinear map $(\cdot,\cdot) : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$, such that for each $p \in M$, $(\cdot,\cdot)_p : E_p \times E_p \to \mathbb{R}$ is a symmetric non-degenerate bilinear form. There exists a canonical fiberwise metric $\langle\cdot,\cdot\rangle$ on E, defined as

$$\langle V+\xi, W+\eta\rangle = i_V(\eta) + i_W(\xi), \tag{2.1}$$

for every $(V+\xi),(W+\eta) \in \Gamma(E)$. This fiberwise metric has signature (n,n), where n is a dimension of M. Hence, we denote by $O(n,n)$ the set of vector bundle automorphisms preserving this fiberwise metric. That is

$$O(n,n) = \{O \in \Gamma(Aut(E)) \mid (\forall e_1, e_2 \in \Gamma(E))\ (\langle Oe_1, Oe_2\rangle = \langle e_1, e_2\rangle)\}. \tag{2.2}$$

There are three important examples of $O(n,n)$ transformations, which we will use in the sequel. Let $B \in \Omega^2(M)$ be a 2-form on M. In this paper we will always denote the induced vector bundle morphism from TM to T^*M by the same letter, i.e., we define

$$B(V) = -i_V B = B(\cdot, V), \tag{2.3}$$

for all $V \in \mathfrak{X}(M)$. Correspondingly, the map e^B is given as

$$e^B(V + \xi) = V + \xi + B(V). \tag{2.4}$$

In the block matrix form

$$e^B \begin{pmatrix} V \\ \xi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} V \\ \xi \end{pmatrix}, \tag{2.5}$$

for all $(V + \xi) \in \Gamma(E)$. Similarly, let $\theta \in \Lambda^2\mathfrak{X}(M)$ be a bivector. The induced vector bundle morphism is again denoted by the same letter, that is

$$\theta(\xi) := -i_\xi \theta = \theta(\cdot, \xi), \tag{2.6}$$

for all $\xi \in \Omega^1(M)$. Correspondingly, we have e^θ

$$e^\theta(V + \xi) = V + \xi + \theta(\xi). \tag{2.7}$$

In the block matrix form

$$e^\theta \begin{pmatrix} V \\ \xi \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V \\ \xi \end{pmatrix}, \tag{2.8}$$

for all $(V + \xi) \in \Gamma(E)$. Finally, let $N : TM \to TM$ be any invertible smooth vector bundle morphism over identity. We define the map O_N as

$$O_N(V + \xi) := N(V) + N^{-T}(\xi), \tag{2.9}$$

where $N^{-T} : T^*M \to T^*M$ denotes the map transpose to N^{-1}. In the block matrix form

$$O_N \begin{pmatrix} V \\ \xi \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix} \begin{pmatrix} V \\ \xi \end{pmatrix}. \tag{2.10}$$

Any $O(n,n)$ transformation with the invertible upper-left block can be uniquely decomposed as a product of the form

$$e^{-B} O_N e^{-\theta}. \tag{2.11}$$

More explicitly, for $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ in $O(n,n)$, i.e., $A_{21}^T A_{11} + A_{11}^T A_{21} = 0$, $A_{12}^T A_{22} + A_{22}^T A_{12} = 0$ and $A_{21}^T A_{12} + A_{11}^T A_{22} = 1$, we find $N = A_{11}$, $\theta = -A_{11}^{-1} A_{12}$ and $B = -A_{21} A_{11}^{-1}$.

Let now $\tau : \Gamma(E) \to \Gamma(E)$ be a $C^\infty(M)$-linear map of sections, such that $\tau^2 = 1$. For $e_1, e_2 \in \Gamma(E)$, we put

$$(e_1, e_2)_\tau := \langle \tau(e_1), e_2 \rangle. \tag{2.12}$$

If such $(.,.)_\tau$ defines a positive definite fiberwise metric, we refer to it as a generalized metric on E. From now on, we will always assume that this is the case. Since $(\cdot,\cdot)_\tau$ is symmetric, τ is a symmetric map, that is,

$$\langle \tau(e_1), e_2\rangle = \langle e_1, \tau(e_2)\rangle, \tag{2.13}$$

for all $e_1, e_2 \in \Gamma(E)$. Also, because $\tau^2 = 1$, it is orthogonal and thus $\tau \in O(n,n)$. Moreover, from $\tau^2 = 1$, we get two eigenbundles V_+ and V_-, corresponding to $+1$ and -1 eigenvalues of τ, respectively. Using the fact that $(\cdot,\cdot)_\tau$ is positive definite, we get that $\langle\cdot,\cdot\rangle$ is positive definite on $\Gamma(V_+)$ and negative definite on $\Gamma(V_-)$. Finally, we can observe that $V_+^\perp = V_-$ with respect to $\langle\cdot,\cdot\rangle$ and vice versa, and using the knowledge of the signature of $\langle\cdot,\cdot\rangle$, we get the direct sum decomposition

$$E = V_+ \oplus V_-. \tag{2.14}$$

Conversely, for any subbundle V of E of rank n, on which $\langle\cdot,\cdot\rangle$ is positive definite, set $\tau|_V := +1$ and $\tau|_{V^\perp} = -1$ to get a generalized metric on E.

From positive definiteness on V_+, we have $V_+ \cap TM = 0$ and $V_+ \cap T^*M = 0$, and the same for V_-. This means that V_+ and V_- can be viewed as graphs of invertible smooth vector bundle morphisms:

$$V_+ = \{V + A(V) \mid V \in TM\} \equiv \{A^{-1}(\xi) + \xi \mid \xi \in T^*M\}, \tag{2.15}$$
$$V_- = \{V + A'(V) \mid V \in TM\} \equiv \{A'^{-1}(\xi) + \xi \mid \xi \in T^*M\}, \tag{2.16}$$

where $A, A' : TM \to T^*M$, respectively. We can view A as covariant 2-tensor field on M, and write uniquely $A = g + B$, where g is a symmetric part of A and B a skew-symmetric part of A. From the positive definiteness of V_+ we get that g is a Riemannian metric on M, whereas B can be an arbitrary 2-form on M. Using the orthogonality of V_+ and V_-, we see that $A' = -g + B$. From this equivalent formulation, i.e. using g and B, we can uniquely reconstruct τ. This will give

$$\tau(V+\xi) = (g - Bg^{-1}B)(V) - g^{-1}B(V) + Bg^{-1}(\xi) + g^{-1}(\xi), \tag{2.17}$$

for all $(V + \xi) \in \Gamma(E)$. In the block matrix form,

$$\tau\begin{pmatrix} V \\ \xi \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}\begin{pmatrix} V \\ \xi \end{pmatrix}. \tag{2.18}$$

The corresponding fiberwise metric $(\cdot,\cdot)_\tau$ can then be written in the block matrix form

$$(V+\xi, W+\eta)_\tau = \begin{pmatrix} V \\ \xi \end{pmatrix}^T \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}\begin{pmatrix} W \\ \eta \end{pmatrix}. \tag{2.19}$$

The important observation is that the block matrix in formula (2.19) can be written as a product of simpler matrices. Namely,

$$\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{2.20}$$

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Note the important fact that the 2-form B *does not* have to be closed, and this will remain true throughout the whole paper. Nevertheless, we assume that B is globally defined, i.e. $H = dB$ globally.[1] We thus consider only the models with trivial H-flux. The case of the non-trivial H-flux will be discussed elsewhere.

There exists a natural action of the group $O(n,n)$ on the space of generalized metrics. For each $O \in O(n,n)$ and given τ define $\tau' = O^{-1}\tau O$. Clearly $\tau'^2 = 1$ and

$$\langle \tau'(e_1), e_2 \rangle = \langle \tau(O(e_1)), O(e_2) \rangle = (O(e_1), O(e_2))_\tau.$$

Hence $(\cdot,\cdot)_{\tau'}$ is again a generalized metric. We may use the notation $(\cdot,\cdot)_{\tau'} = O(\cdot,\cdot)_\tau$.

2.2 Factorizations of generalized metric, open-closed relations

Let us start with a (different) generalized metric $\mathbf{H}$, described by a Riemannian metric G and a 2-form Φ. Hence

$$\mathbf{H} = \begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi & 0 \end{pmatrix}. \tag{2.21}$$

Let θ be a 2-vector field on M. The action of the $O(n,n)$ map $e^{-\theta}$ on the generalized metric $\mathbf{H}$ gives us a new generalized metric $\mathbf{G}$, which has the form

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix} \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix}. \tag{2.22}$$

By the previous discussion, there exists a unique Riemannian metric g and a 2-form B, such that

$$\mathbf{G} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{2.23}$$

Comparing the two expressions (2.22) and (2.23) of $\mathbf{G}$, we get the matrix equations

$$g - Bg^{-1}B = G - \Phi G^{-1}\Phi, \tag{2.24}$$

$$Bg^{-1} = \Phi G^{-1} - (G - \Phi G^{-1}\Phi)\theta, \tag{2.25}$$

which can be uniquely solved for G and Φ. Since $e^{-\theta}$ is invertible, we can proceed the other way around as well. We also know how the corresponding endomorphism $\tau_{\mathbf{H}}$ is changed by $e^{-\theta}$. Namely, we have

$$\tau_{\mathbf{G}} = e^{\theta} \tau_{\mathbf{H}} e^{-\theta}. \tag{2.26}$$

From that, we can easily find the relation between $+1$ eigenbundles:

$$V_+^{\mathbf{G}} = e^{\theta} V_+^{\mathbf{H}}. \tag{2.27}$$

Since

$$V_+^{\mathbf{G}} = \{\xi + (g+B)^{-1}(\xi) \mid \xi \in T^*M\},$$

[1] More precisely, we assume that the corresponding integral cohomology class $[H]$ is trivial.

and

$$V_+^{\mathbf{H}} = \{\xi + (G+\Phi)^{-1}(\xi) \mid \xi \in T^*M\},$$

we get using the above formula that

$$(g+B)^{-1} = \theta + (G+\Phi)^{-1}. \tag{2.28}$$

Formulae (2.24) and (2.25) are the symmetric and antisymmetric parts of (2.28). If θ is Poisson, (2.28) is the Seiberg-Witten formula[2] relating closed and open string backgrounds in the presence of a noncommutative structure represented by θ. In particular, for given g, B and θ, we can find a unique G and Φ, and conversely, for given G, Φ and θ, there exists a unique pair g and B.

For $\Phi = 0$ the open-closed relations can be given a slightly more geometric interpretation [10]. Consider the inverse $\mathbf{G}^{-1}$ of the generalized metric $\mathbf{G}$. If we exchange the tangent and cotangent bundles TM and T^*M, respectively, $\mathbf{G}^{-1}$ has the same properties as $\mathbf{G}$. Obviously, $\mathbf{G}^{-1}$ and $\mathbf{G}$ have identical graphs as well as ± 1-eigenbundles. The open-closed relations, for $\Phi = 0$, is a simple consequence of that.

2.3 Dorfman bracket, Dirac structures, D-branes

Here we briefly recall some relevant facts concerning the Dorfman bracket and Dirac structures, see, e.g., [2, 26, 28]. Our vector bundle $E = TM \oplus T^*M$ can be equipped with a structure of a Courant algebroid. The corresponding Courant bracket is the antisymmetrization of the Dorfman bracket:

$$[V+\xi, W+\eta]_D = [V,W] + \mathcal{L}_V(\eta) - i_W(d\xi), \tag{2.29}$$

for all $(V+\xi) \in \Gamma(E)$. The corresponding pairing is the canonical fiberwise metric (2.1).

A Dirac structure is a (smooth) subbundle L of E, which is maximally isotropic with respect to $\langle\cdot,\cdot\rangle$ and involutive under the Dorfman bracket (2.29).

Let θ be a rank-2 contravariant tensor field on M. As before, define a vector bundle morphism $\theta : T^*M \to TM$ by $\theta(\xi) = \theta(\cdot,\xi)$. Define a subbundle G_θ of E as its graph, that is

$$G_\theta = \{\xi + \theta(\xi) \mid \xi \in T^*M\}. \tag{2.30}$$

It is known that G_θ is a Dirac structure with respect to the Dorfman bracket, if and only if θ is a Poisson bivector. Similarly, let B be any rank-2 covariant tensor field on M. Define $B(V) = B(V,\cdot)$ and its graph G_B as

$$G_B = \{V + B(V) \mid V \in TM\}. \tag{2.31}$$

Again, one can show that G_B is a Dirac structure, if and only if B is a closed 2-form on M.

Furthermore, for any closed $B \in \Omega^2(M)$, one has

$$e^B[V+\xi, W+\eta]_D = [e^B(V+\xi), e^B(W+\eta)]_D, \tag{2.32}$$

[2]For an earlier appearance of this type of formulae in the context of duality rotations see [27].

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and

$$\langle e^B(V+\xi), e^B(W+\eta)\rangle = \langle V+\xi, W+\eta\rangle, \tag{2.33}$$

for all $(V+\xi),(W+\eta) \in \Gamma(E)$. In the other words, e^B is an automorphism of the corresponding Courant algebroid. Note that (2.32) is no longer true for e^θ, where $\theta \in \Lambda^2\mathfrak{X}(M)$, but (2.33) holds.

Generally, a Dirac structure L provides a singular foliation of M by presympletic leaves, which is generated by its image $\rho(L)$ of the Dirac structure under the anchor map. We refer to [10] for arguments in favor of the identification "D-branes $\sim$ leaves of foliations defined by Dirac structures". In the case we will consider later, L will be given as a graph of a Poisson tensor θ and the corresponding foliation of M will be the foliation generated by Hamiltonian vector fields, i.e., by symplectic leaves of θ. Hence, in this case we will identify the symplectic leaves and D-branes.

3 Gauge field as an orthogonal transformation of the generalized metric

Let us start with a given Riemannian metric g and 2-form B. Further, let F be a 2-form (at this point an arbitrary one[3]). The gauge transformation defines new 2-form $B' = B + F$. To the pair (g, B) corresponds the generalized metric $\mathbf{G}$, see (2.23). The generalized metric $\mathbf{G}'$ corresponding to the pair $(g, B+F)$ has the following block matrix form:

$$\mathbf{G}' = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}, \tag{3.1}$$

that is, $\mathbf{G}'$ is related to $\mathbf{G}$ by the $O(n,n)$ transform e^{-F}. As shown before, we can always get $\mathbf{G}$ by action of $O(n,n)$ transformation $e^{-\theta}$ on the generalized metric $\mathbf{H}$, where $\mathbf{H}$ is described by fields G and Φ, see (2.21).

One may ask, if there is a bivector θ' on M, such that we get $\mathbf{G}'$ by the action of $e^{-\theta'}$ on the generalized metric $\mathbf{H}'$, which is described by the same G as $\mathbf{H}$, but by gauged 2-form $\Phi' = \Phi + F'$ for some gauge field F'. This can be achieved under some assumptions, however, only up to a certain additional $O(n,n)$ action. In particular, there exists a vector bundle morphism $N : TM \to TM$, such that

$$\mathbf{G}' = \begin{pmatrix} 1 & 0 \\ \theta' & 1 \end{pmatrix} \begin{pmatrix} N^T & 0 \\ 0 & N^{-1} \end{pmatrix} \mathbf{H}' \begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix} \begin{pmatrix} 1 & -\theta' \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

where

$$\mathbf{H}' = \begin{pmatrix} 1 & \Phi' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi' & 1 \end{pmatrix}.$$

Indeed, examine the block matrix decomposition:

$$\mathbf{G}' = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix} \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}.$$

[3] Later, when discussing DBI action, F will be closed and defined only on a submanifold of M supporting a D-brane. In which case, all expression involving F will make sense only when considered on the D-brane.

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It suffices to consider the three rightmost matrices in the above expression. Since we want to modify Φ to $\Phi + F'$, we may proceed by inserting $1 = e^{-F'}e^{F'}$:

$$\begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix}\begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(\Phi + F') & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ F' & 1 \end{pmatrix}\begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}.$$

Now it is enough to note that the product of the last three matrices, can be uniquely decomposed into a product of a diagonal and an upper triangular block matrix — of course, only if we assume that $(1 + \theta F)$ is invertible. For this, use the decomposition of $e^{-\theta}e^{-F} \in O(n, n)$ according to (2.11) as

$$e^{-\theta}e^{-F} = e^{-F'}O_N e^{-\theta'}, \tag{3.3}$$

with $F' \in \Omega^2(M), \theta' \in \Lambda^2\mathfrak{X}(M)$ and $N \in \Gamma(\mathrm{Aut}(TM))$. What we find are the following expression for θ', F' and N:

$$\theta' = (1 + \theta F)^{-1}\theta = \theta(1 + F\theta)^{-1}, \tag{3.4}$$

$$F' = F(1 + \theta F)^{-1} = (1 + F\theta)^{-1}F, \tag{3.5}$$

$$N = 1 + \theta F. \tag{3.6}$$

Comparing (3.1) and (3.2), we get the equalities

$$g - (B + F)g^{-1}(B + F) = N^T(G - (\Phi + F')G^{-1}(\Phi + F'))N \tag{3.7}$$

and

$$(B + F)g^{-1} = N^T(\Phi + F')G^{-1}N^{-T} - N^T(G - (\Phi + F')G^{-1}(\Phi + F'))N\theta'. \tag{3.8}$$

Taking the determinant of (3.7), we find that

$$\det(g - (B + F)g^{-1}(B + F)) = \det(N)^2 \cdot \det(G - (\Phi + F')G^{-1}(\Phi + F')). \tag{3.9}$$

This equality will play the central role when later discussing the DBI action.

Furthermore, following the same type of arguments leading to (2.28) we see that the equations (3.7) and (3.8) can equivalently be written as

$$(g + B + F)^{-1} = \theta' + (N^T(G + \Phi + F')N)^{-1}. \tag{3.10}$$

Finally, let us examine the objects F' and θ' using the tools described in subsection 2.3. We will concentrate on the case important for the discussion of the DBI action and non-commutative gauge theory. Therefore, in the rest of this section, we assume that θ is Poisson and F is closed. θ' is a bivector on M. For the graphs of θ and θ' we have

$$e^F G_\theta = G_{\theta'}. \tag{3.11}$$

Since e^F is an automorphism of Dorfman bracket, $G_{\theta'}$ has to be again a Dirac structure of E. Hence, θ' is a Poisson bivector. Similarly, one can see that

$$e^\theta G_F = G_{F'}. \tag{3.12}$$

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This is no more an automorphism of Dorfman bracket but it preserves the (maximal) isotropy property of G_F. Hence $G_{F'}$ is an isotropic subbundle of E and F' is therefore a 2-form on M. Let us also note, that F' doesn't need to be closed. The last remark: In case that $(1+\theta F)$ is not invertible, $e^F G_\theta$ still makes perfect sense as a Dirac structure. Similarly, $e^\theta G_F$ will still define an almost Dirac structure.

4 Non-topological Poisson-sigma model and Polyakov action

In this section we review the non-topological Poisson-sigma model from the generalized geometry point of view developed in the previous sections.

Let us consider a 2-dimensional world-sheet Σ with a set of local coordinates (σ^0, σ^1). We assume that σ^μ are Cartesian coordinates for a Lorentzian metric h with signature $(-,+)$ on Σ. Furthermore, we consider an n-dimensional target manifold M, equipped with a metric G, 2-vector θ and a 2-form Φ. We can assume Σ with a non-empty boundary $\partial\Sigma$. On M assume an abelian gauge field A coupling to the boundary (and extending to Σ, the field strength being $F = dA$). We also choose some local coordinates $(y^1, \dots, y^n)$ on M. Lower case Latin characters will always correspond to these coordinates. For the components of the smooth map $X : \Sigma \to M$ we will use the following notation: $X^i = y^i(X)$. In this section it will be convenient to introduce the following notation: We put $\bar{G} := N^T G N$, $\bar{\Phi} := N^T \Phi N$ and $\bar{F}' := N^T F' N$ and introduce auxiliary fields η_i and $\widetilde{\eta}_j$, which transform under change of local coordinates on M according to their index structure. We combine them in a $2n$-dimensional column vector $\Psi^T := (\eta, \tilde{\eta})$. We also introduce another $2n$-dimensional column vector $V^T := (\partial_0 X, \partial_1 X)$. Finally, we define a $2n \times 2n$ matrix[4]

$$\bar{\mathbf{G}} = \begin{pmatrix} -\bar{G} & -\bar{\Phi} - \bar{F}' \\ \bar{\Phi} + \bar{F}' & \bar{G} \end{pmatrix}^{-1} + \begin{pmatrix} 0 & \theta' \\ -\theta' & 0 \end{pmatrix}. \tag{4.1}$$

Our (non-topological) Poisson-sigma model action is

$$S[\eta, \widetilde{\eta}, X] := \int d^2\sigma \frac{1}{2} \Psi^T \bar{\mathbf{G}} \Psi + \Psi^T V. \tag{4.2}$$

Using relations (3.7), (3.8), the action (4.2) can equivalently be written as

$$S[\eta, \widetilde{\eta}, X] := \int d^2\sigma \frac{1}{2} \Psi^T \tilde{\mathbf{G}} \Psi + \Psi^T V, \tag{4.3}$$

where

$$\tilde{\mathbf{G}} = \begin{pmatrix} -g & -B - F \\ B + F & g \end{pmatrix}^{-1} \tag{4.4}$$

with g, B and F being related to G, Φ and F' by (3.7), (3.8) and (3.5). Integrating out the auxiliary fields η and $\tilde{\eta}$ we obtain the Polyakov action expressed equivalently either in open or closed variables

$$S[X] := -\frac{1}{2} \int d^2\sigma V^T \bar{\mathbf{G}}^{-1} V = -\frac{1}{2} \int d^2\sigma V^T \tilde{\mathbf{G}}^{-1} V. \tag{4.5}$$

[4]Here, we neither need to assume that θ is Poisson nor that F is closed.

Actually, all this can be seen rather straightforwardly. For this, note that relations (3.7), (3.8) can alternatively be expressed as the equality of matrices $\bar{\mathbf{G}} = \tilde{\mathbf{G}}$. The relations in the form (3.10) and their transposes are obtained from the nonzero off-diagonal blocks after the similarity transformation with the block matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ is applied to the equality $\bar{\mathbf{G}} = \tilde{\mathbf{G}}$.

The generalized metric $\mathbf{G}'$ can explicitly be seen either in the Hamiltonian corresponding to the Polyakov action (4.5) or in the Hamiltonian corresponding to the action (4.2) after the equations of motions for one half of the auxiliary fields, the $\tilde{\eta}$s, are used. As can straightforwardly be checked, these Hamiltonians are identical. To write down the result we introduce a new $2n$-dimensional column vector $\Upsilon^T := (\partial_1 X, \eta)$. The auxiliary fields η_i become the canonical momenta and the Hamiltonian is

$$H[X,\eta] = \frac{1}{2}\int d\sigma^1 \Upsilon^T \mathbf{G}' \Upsilon\,, \tag{4.6}$$

where $\mathbf{G}'$ the matrix given by the two equivalent decompositions (3.1) and (3.2). Hence, we have the same Hamiltonian using either the closed or the open variables. Let us note that, for $F = 0$, the relation between the action (4.2) and the action (4.5) with G' expressed as in (3.1) can be found in [29]. The Hamiltonian (4.6) with G' given by (3.2) can be found, again for $F = 0$, in [17]. Polyakov actions like the first one in (4.5) appeared (with $F = 0$) in [30] in the context of Poisson-Lie T-duality.

5 Seiberg-Witten map

For an approach to the non-abelian case, using cohomological methods akin to the ones of Zumino's famous decent equations [31], see [32, 33]. Here we follow the approach of [34–36], where it was shown that the Seiberg-Witten field redefinition from the commutative to the non-commutative setting has its origin in a change of coordinates given by a map $\rho : M \to M$, such that $\rho^*(\theta') = \theta$.[5] This map can be derived using a generalization of Moser's lemma: Consider the family of Poisson bivectors

$$\theta_t = \theta(1 + tF\theta)^{-1} \tag{5.1}$$

parameterized by $t \in [0,1]$. Of course, we have to presume that the formula is well-defined. To see that these θ_t are indeed Poisson for all t, simply observe that $G_{\theta_t} = e^{tF} G_\theta$ holds for the respective graphs.[6] Partial differentiation of (5.1) with respect to t leads to the differential equation

$$\partial_t \theta_t = -\theta_t F \theta_t.$$

For $F = dA$, this can be rewritten as

$$\partial_t \theta_t = -\mathcal{L}_{\theta_t(A)} \theta_t,$$

[5] As said before, here we assume only topologically trivial $[H]$-flux. The interested reader may find some relevant discussion concerning nontrivial H and the related non-commutative gerbe in [37].

[6] Let us note again that $e^{tF} G_\theta$ is a bona-fide Dirac structure even for non-invertible $(1 + tF\theta)$.

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with a vector field $\theta_t(A) := \theta_t(\cdot, A)$, with initial condition $\theta_0 = \theta$. This differential equation can be integrated to a flow ϕ_t, such that $\phi_t^*(\theta_t) = \theta$. Thus $\rho = \phi_1$. Obviously, ρ explicitly depends on the choice of gauge potential A, hence we shall use the notation ρ_A. To avoid possible confusion, we will for a moment notationally distinguish between the tensor itself and its components in coordinates. Therefore we introduce the matrix $(\boldsymbol{\theta})^{ij} := \theta^{ij}$. Also, denote $J^i{}_k = \frac{\partial \rho^i}{\partial x^k}$. We have

$$\rho_A^*(\theta'^{kl}) = J^k{}_i J^l{}_j \theta^{ij}.$$

We thus get that

$$\det \rho_A^*(\boldsymbol{\theta'}) = J^2 \det \boldsymbol{\theta}. \tag{5.2}$$

Let us assume for a moment that $\boldsymbol{\theta}$ is invertible. From (3.4) we see that so is $\rho_A^* \boldsymbol{\theta'}$. We immediately have that

$$J^{-2} = \det\left(\boldsymbol{\theta}(\rho_A^* \boldsymbol{\theta'})^{-1}\right). \tag{5.3}$$

For degenerate θ and hence also θ' the formula (5.3) still makes sense and we can argue as follows: Since the map ρ_A is infinitesimally generated by the vector field $\theta_t(A)$, and the kernels of all θ_t's are the same, we see that ρ_A only changes coordinates on the symplectic leaves (of θ). We can thus restrict ourselves to the non-degenerate case in order to carry out the computation of the Jacobian.

In the next section, we will discuss the case when the Poisson structure θ (i.e., the corresponding Dirac structure) will be used, following the suggestion of [10], to define the D-branes as its symplectic leaves. The above argument shows that we can safely restrict our discussion without the loss of generality to any of the respective D-branes. In such a case, the (Seiberg-Witten) map ρ_A is a diffeomorphism of the D-brane world-volume D. The Poisson structures θ_t have in fact the same symplectic foliations for all t. Actually, all Poisson structures θ_t, including in particular θ and θ', are Morita equivalent [38].

Finally, on the level of Dirac structures, the Seiberg Witten map is the map of graphs $\rho^* : G'_\theta \mapsto G_\theta$. More explicitly,

$$\{\theta'(\eta) + \eta, \eta \in T^*M\} \mapsto \{\theta(\eta) + \eta, \eta \in T^*M\} = \{N\theta' N^{-T}(\eta) + N^{-T}(\eta), \eta \in T^*M\}.$$

Hence, the Seiberg-Witten map can be seen as the map induced by the $O(n,n)$ transformation O_N entering the decomposition (2.11), if one considers D-branes which are symplectic leaves.

6 Noncommutative gauge theory and DBI action

In the previous sections we have described all ingredients needed for our discussion of noncommutativity of D-branes as a consequence of their generalized geometry. Namely, we have seen that the relations (2.24), (2.25), (3.7) and the (semiclassical) Seiberg-Witten have their root in generalized geometry. Actually, it is know for quite some time [36] that the equivalence of the commutative and (semiclassically) noncommutative DBI actions follows once one has established (2.24), (2.25), (3.7) and has understood the (semiclassical) Seiberg-Witten map as a (local) D-brane diffeomorphism. Nevertheless, according to our best knowledge, the direct relation to generalized geometry is new. Moreover, the discussion generalizes to the case of M-theory branes [24, 25] and will be elaborated in detail in a forthcoming paper. Here we will include the derivation of the equivalence of the commutative and (semiclassically) noncommutative DBI actions for the sake of completeness and the reader's convenience. For related work based on dualities, see [39].

Assume that we have a D-brane D of dimension d, i.e, a submanifold of target space-time M equipped with a line bundle with a connection A and corresponding field strength F. Also, consider the restrictions (pullbacks) of the background fields (open and closed ones) to D. While describing the Seiberg-Witten map in the previous section, we have seen that it is quite natural to assume that there is a relation between the D-brane and the Poisson tensor θ.[7] Namely, assume that our D-brane is of a particular kind, i.e., one which comes as symplectic leaf of the Poisson structure θ.[8] As argued before, under this assumption, the Seiberg-Witten map is a D-brane diffeomorphism.

Before we turn to the discussion of the DBI action and its commutative and noncommutative description, we discuss the relation between the effective closed and open string coupling constants g_s and G_s, respectively [21]. These are related as

$$G_s = g_s \left(\frac{\det(G+\Phi)}{\det(g+B)} \right)^{1/2}.$$

We can use the formula for the determinant of a sum of a symmetric matrix S and an antisymmetric matrix A, $|S+A| = |S|^{1/2}|S - AS^{-1}A|^{1/2}$, and the relation (2.24) to rewrite this as

$$G_s = g_s \left(\frac{\det G}{\det g} \right)^{1/4}. \tag{6.1}$$

A most intriguing relation is obtained from (6.1) and the relation (3.7), again using the above mentioned formula for the determinant of a sum of a symmetric and an antisymmetric matrix:

$$\frac{1}{g_s} \det{}^{1/2}(g+B+F) = \frac{1}{G_s} \det{}^{1/2}(1+\theta F) \det{}^{1/2}(G+\Phi+F'). \tag{6.2}$$

Integrating over the D-brane world-volume

$$\int d^d x \frac{1}{g_s} \det{}^{1/2}(g+B+F) = \int d^d x \frac{1}{G_s} \det{}^{1/2}(1+\theta F) \det{}^{1/2}(G+\Phi+F'), \tag{6.3}$$

recalling (5.3), and performing the change of coordinates according to the Seiberg-Witten map, we finally obtain a relation between the commutative and semiclassically noncommutative DBI actions

$$S^{c}_{\mathrm{DBI}} := \int d^d x \frac{1}{g_s} \det{}^{1/2}(g+B+F) = \int d^d x \frac{1}{\hat{G}_s} \det{}^{1/2}\left(\frac{\hat{\theta}}{\theta}\right) \det{}^{1/2}(\hat{G}+\hat{\Phi}+\hat{F}') =: S^{\mathrm{nc}}_{\mathrm{DBI}}. \tag{6.4}$$

[7] Recall, in accordance with our above discussion of the open-closed relations, here we start from a given closed background (g, B), pick a θ and determine uniquely the open variables (G, Φ).

[8] It is straight-forward to modify everything to the case where the D-brane is a submanifold, such that the restriction of θ to it defines a regular Poisson structure, i.e. a Poisson structure having constant rank.

The hat "^" has the following meaning: On matrix elements of θ it is defined as $\hat{\theta}^{ij} := \rho_A^*(\theta^{ij})$, and similarly for the other objects. As a result of this definition, $\hat{F}'$ is the semiclassically noncommutative field strength, which under the gauge transformation $\delta A = d\lambda$ transforms semiclassically noncommutatively, i.e.,

$$\begin{aligned}\delta \hat{F}'_{ij} &= \{\hat{F}'_{ij}, \tilde{\lambda}\},\\ \tilde{\lambda} &= \sum \frac{(\theta_t(A) + \partial_t)^n(\lambda)}{(n+1)!}|_{t=0}.\end{aligned}$$

Here, the curly bracket is the Poisson bracket corresponding to the Poisson tensor θ and $\tilde{\lambda}$ is the (semiclassical) noncommutative gauge parameter.

Let us note: The commutative DBI action S^c_{DBI} on the l.h.s. in (6.4) is the effective D-brane action obtained from the Polyakov action (4.5). Expressed directly in terms of the matrix $\tilde{\mathbf{G}}$, the action S^c_{DBI} is the integral of

$$\det{}^{1/4}\tilde{\mathbf{G}} \tag{6.5}$$

up to the inverse of the closed coupling constant g_s. Hence, an alternative — but completely equivalent — way of obtaining the relation between the commutative and semiclassically noncommutative DBI actions (6.4) is to start from the matrix equality $\tilde{\mathbf{G}} = \bar{\mathbf{G}}$. This makes the relation to the Polyakov action more transparent. We leave the details to the reader.

Finally, the Hamiltonian (4.6) can equivalently be expressed using either the "commutative" (3.1) or "noncommutative" (3.2) decompositions of the generalized metric $\mathbf{G}'$. This is maybe the most direct hint from generalized geometry about the necessity of a relation like (6.4).

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Appendix D

Paper 3: Extended generalized geometry and a DBI-type effective action for branes ending on branes

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Extended generalized geometry and a DBI-type effective action for branes ending on branes

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Abstract: Starting from the Nambu-Goto bosonic membrane action, we develop a geometric description suitable for p-brane backgrounds. With tools of generalized geometry we derive the pertinent generalization of the string open-closed relations to the p-brane case. Nambu-Poisson structures are used in this context to generalize the concept of semi-classical noncommutativity of D-branes governed by a Poisson tensor. We find a natural description of the correspondence of recently proposed commutative and noncommutative versions of an effective action for p-branes ending on a p'-brane. We calculate the power series expansion of the action in background independent gauge. Leading terms in the double scaling limit are given by a generalization of a (semi-classical) matrix model.

Keywords: p-branes, Differential and Algebraic Geometry, M-Theory

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Contents

1 Introduction

Among the most intriguing features of fundamental theories of extended objects are novel types of symmetries and concomitant generalized notions of geometry. Particularly interesting examples of these symmetries are T-duality in closed string theory and the equivalence of commutative/noncommutative descriptions in open string theory. These symmetries have their natural settings in generalized geometry and noncommutative geometry. Low energy effective theories link the fundamental theories to potentially observable phenomena in (target) spacetime. Interestingly, the spacetime remnants of the stringy symmetries can

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fix these effective theories essentially uniquely without the need of actual string computations: "string theory with no strings attached."

The main objective of this paper is to study this interplay of symmetry and geometry in the case of higher dimensional extended objects (branes). More precisely, we intended to extend, clarify and further develop the construction outlined in [1] that tackles the quest to find an all-order effective action for a system of multiple p-branes ending on a p'-brane. The result for the case of open strings ending on a single D-brane is well known: the Dirac-Born-Infeld action provides an effective description to all orders in α' [2–4]. The way that this effective action has originally been derived from first principles in string theory is rather indirect: the effective action is determined by requiring that its equations of motion double as consistency conditions for an anomaly free world sheet quantization of the fundamental string. A more direct target space approach can be based on T-duality arguments. Moreover, there is are equivalent commutative and non-commutative descriptions [5], where the equivalency condition fixes the action essentially uniquely [6, 7]. This "commutative-noncommutative duality" has been used also to study the non-abelian DBI action [6, 8]. In the context of the M2/M5 brane system a generalization has been proposed in [9].

In this paper, we focus only on the bosonic part of the action. The main idea of [1], inspired by [9], was to introduce open-closed membrane relations, and a Nambu-Poisson map which can be used to relate ordinary higher gauge theory to a new Nambu gauge theory [10–13]. See also the work of P.-M. Ho et al. [14–17] and K. Furuuchi et al. [18, 19] on relation of M2/M5 to Nambu-Poisson structures. It turns out that the requirement of "commutative-noncommutative duality" determines the bosonic part of the effective action essentially uniquely. Interesting open problems are to determine, in the case of a M5-brane, the form of the full supersymmetric action and to check consistency with κ-symmetry and (nonlinear) selfduality.

Nambu-Poisson structures were first considered by Y. Nambu already in 1973 [20], and generalized and axiomatized more then 20 years later by L. Takhtajan [21]. The axioms of Nambu-Poisson structures, although they seem to be a direct generalization of Poisson structures, are in fact very restrictive. This was already conjectured in the pioneering paper [21] and proved three years later in [22, 23]. For a modern treatment of Nambu-Poisson structures see [24–26].

Matrix-model like actions using Nambu-Poisson structures are a current focus of research (see e.g. [27–30]) motivated by the works of [31–35] and others. See also [36, 37] for further reference. Among the early approaches, the one closest to ours is the one of [38, 39], which uses κ-symmetry as a guiding principle and features a non-linear self-duality condition. It avoids the use of an auxiliary chiral scalar [40] with its covariance problems following a suggestion of [41]. For these and alternative formulations, e.g., those of [42], based on superspace embedding and κ-symmetry, we refer to the reviews [43, 44].

Generalized geometry was introduced by N. Hitchin in [45–47]. It was further elaborated in [48]. Although Hitchin certainly recognized the possible importance for string backgrounds, and commented on it in [45], this direction is not pursued there. Recently, a focus of applications of generalized geometry, is superstring theory and supergravity. Here we mention closely related work [49, 50]. The role of generalized geometries in M-theory was previously examined by C.M. Hull in [51]. A further focus is the construction of the field theories based on objects of generalized geometry. This is mainly pursued in [52, 53] and in [54], see also [55]. Generalized geometry (mostly Courant algebroid brackets) was also used in relation to worldsheet algebras and non-geometric backgrounds. See, for example, [56–58] and [59, 60]. One should also mention the use of generalized geometry in the description of T-duality, see [61], or the lecture notes [62]. An outline of the relation of T-duality with generalized geometry can be found in [63]. Finally, there is an interesting interpretation of D-branes in string theory as Dirac structures of generalized geometry in [64, 65]. Finally, in [66], we have used generalized geometry to describe the relation between string theory and non-commutative geometry.

This paper is organized as follows: in section 3, we review basic facts concerning classical membrane actions. In particular, we recall how gauge fixing can be used to find a convenient form of the action. We show that the corresponding Hamiltonian density is a fiberwise metric on a certain vector bundle. We present background field redefinitions, generalizing the well-known open-closed relations of Seiberg and Witten.

In section 4, we describe the sigma model dual to the membrane action. It is a straightforward generalization of the non-topological Poisson sigma model of the $p = 1$ case.

Section 5 sets up the geometrical framework for the field redefinitions of the previous sections. An extension of generalized geometry is used to describe open-closed relations as an orthogonal transformation of the generalized metric on the vector bundle $TM \oplus \Lambda^p TM \oplus T^*M \oplus \Lambda^p T^*M$. Compared to the $p = 1$ string case, we find the need for a second "doubling" of the geometry. The split in TM and $\Lambda^p TM$ has its origin in gauge fixing of the auxiliary metric on the $p+1$-dimensional brane world volume and the two parts are related to the temporal and spatial worldvolume directions. To the best of our knowledge, this particular structure $W \oplus W^*$ with $W = TM \oplus \Lambda^p TM$ has not been considered in the context of M-theory before.

In section 6, we introduce the $(p+1)$-form gauge field F as a fluctuation of the original membrane background. We show that this can be viewed as an orthogonal transformation of the generalized metric describing the membrane backgrounds. On the other hand, the original background can equivalently be described in terms of open variables and this description can be extended to include fluctuations. Algebraic manipulations are used to identify the pertinent background fields. The construction requires the introduction of a target manifold diffeomorphism, which generalizes the (semi-classical) Seiberg-Witten map from the string to the $p > 1$ brane case.

This map is explicitly constructed in section 7 using a generalization of Moser's lemma. The key ingredient is the fact that Π, which appears in the open-closed relations, can be chosen to be a Nambu-Poisson tensor. Attention is paid to a correct mathematical formulation of the analogue of a symplectic volume form for Nambu-Poisson structures.

Based on the results of the previous sections, we prove in section 8 the equivalence of a commutative and semiclassically noncommutative DBI action. We present various forms of the same action using determinant identities of block matrices. Finally, we compare our action to existing proposals for the M5-brane action.

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In section 9, we show that the Nambu-Poisson structure Π can be chosen to be the pseudoinverse of the $(p+1)$-form background field C. In analogy with the $p = 1$ case, we call this choice "background independent gauge". However, for $p > 1$ we have to consider both algebraic and geometric properties of C in order to obtain a well defined Nambu-Poisson tensor Π. The generalized geometry formalism developed in section 5 is used to derive the results in a way that looks formally identical to the much easier $p = 1$ case. (This is a nice example of the power of generalized geometry.)

In section 10, we introduce a convenient splitting of the tangent bundle and rewrite all membrane backgrounds in coordinates adapted to this splitting using a block matrix formalism. We introduce an appropriate generalization of the double scaling limit of [5] to cut off the series expansion of the effective action.

In the final section 11 of the paper, we use background independent gauge, double scaling limit, and coordinates adapted to the non-commutative directions to expand the DBI action up to first order in the scaling parameter. It turns out that this double scaling limit cuts off the infinite series in a physically meaningful way. We identify a possible candidate for the generalization of a matrix model. For a discussion of the underlying Nambu-Poisson gauge theory we refer to [11].

2 Conventions

Thorough the paper, $p > 0$ is a fixed positive integer. Furthermore, we assume that we are given a $(p+1)$-dimensional compact orientable worldvolume Σ with local coordinates $(\sigma^0, \dots, \sigma^p)$. We may interpret σ^0 as a time parameter. Integration over all coordinates is indicated by $\int d^{p+1}\sigma$, whereas the integration over space coordinates $(\sigma^1, \dots, \sigma^p)$ is indicted as $\int d^p\sigma$. Indices corresponding to the worldvolume coordinates are denoted by Greek characters $\alpha, \beta, \dots$, etc. As usual, $\partial_\alpha \equiv \frac{\partial}{\partial \sigma^\alpha}$. We assume that the n-dimensional target manifold M is equipped with a set of local coordinates $(y^1, \dots, y^n)$. We denote the corresponding indices by lower case Latin characters $i, j, k, \dots$, etc. Upper case Latin characters $I, J, K, \dots$, etc. will denote strictly ordered p-tuples of indices corresponding to (y) coordinates, e.g., $I = (i_1, \dots, i_p)$ with $1 \leq i_1 < \dots < i_p \leq n$. We use the shorthand notation $\partial_J \equiv \frac{\partial}{\partial y^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_p}}$ and $dy^J = dy^{j_1} \wedge \dots \wedge dy^{j_p}$. The degree q-parts of the exterior algebras of vector fields $\mathfrak{X}(M)$ and forms $\Omega(M)$ are denoted by $\mathfrak{X}^q(M)$ and $\Omega^q(M)$, respectively.

Where-ever a metric g on M is introduced, we assume that it is positive definite, i.e., (M, g) is a Riemannian manifold. With this choice we will find a natural interpretation of membrane backgrounds in terms of generalized geometry. For any metric tensor g_{ij}, we denote, as usually, by g^{ij} the components of the inverse contravariant tensor.

We use the following convention to handle $(p+1)$-tensors on M. Let $B \in \Omega^{p+1}(M)$ be a $(p+1)$-form on M. We define the corresponding vector bundle map $B_\flat : \Lambda^p TM \to T^*M$ as $B_\flat(Q) = B_{iJ} Q^J dy^i$, where $Q = Q^J \partial_J$. We do not distinguish between vector bundle morphisms and the induced $C^\infty(M)$-linear maps of smooth sections. We will usually use the letter B also for the $\binom{n}{p} \times n$ matrix of $B_\flat$ in the local basis ∂_J of $\mathfrak{X}^p(M)$ and dy^i of $\Omega^1(M)$, that is $(B)_{i,J} = \langle \partial_i, B_\flat(\partial_J) \rangle$. Similarly, let $\Pi \in \mathfrak{X}^{p+1}(M)$; the induced map $\Pi^\sharp : \Lambda^p T^*M \to TM$ is defined as $\Pi^\sharp(\xi) = \Pi^{iJ} \xi_J \partial_i$ for $\xi = \xi_J dy^J$. We use the letter Π also for the $\binom{n}{p} \times n$ matrix of $\Pi^\sharp$, that is $(\Pi)^{i,J} = \langle dy^i, \Pi^\sharp(dy^J)\rangle$. Clearly, with these conventions $(B)_{i,J} = B_{iJ}$ and $(\Pi)^{i,J} = \Pi^{iJ}$.

Let $X : \Sigma \to M$ be a smooth map. We use the notation $X^i = y^i \circ X$, and correspondingly $dX^i = d(X^i) = X^*(dy^i)$. Similarly, $dX^J = X^*(dy^J)$. We reserve the symbol $\widetilde{\partial X}^J$ for spatial components of the p-form dX^J, that is, $\widetilde{\partial X}^J = (dX^J)_{1\ldots p}$. We define the generalized Kronecker delta $\delta^{j_1\ldots j_p}_{i_1\ldots i_p}$ to be $+1$ whenever the top p-index constitutes an even permutation of the bottom one, -1 if for the odd permutation, and 0 otherwise. In other words, $\delta^{j_1\ldots j_p}_{i_1\ldots i_p} = p! \cdot \delta^{[j_1}_{[i_1} \ldots \delta^{j_p]}_{i_p]}$. We use the convention $\epsilon_{i_1\ldots i_p} \equiv \epsilon^{i_1\ldots i_p} \equiv \delta^{1\ldots p}_{i_1\ldots i_p} \equiv \delta^{i_1\ldots i_p}_{1\ldots p}$. Thus, in this notation we have $\widetilde{\partial X}^I = \partial_{l_1} X^{i_1} \cdots \partial_{l_p} X^{i_p} \epsilon^{l_1\ldots l_p}$.

3 Membrane actions

The most straightforward generalization of the relativistic string action to higher dimensional world volumes is the Nambu-Goto p-brane action, simply measuring the volume of the p-brane:

$$S_{NG}[X] = T_p \int d^{p+1}\sigma \sqrt{\det\left(\partial_\alpha X^i \partial_\beta X^j g_{ij}\right)}, \tag{3.1}$$

where g_{ij} are components of the positive definite target space metric g, and $X : \Sigma \to M$ is the n-tuple of scalar fields describing the p-brane. In a similar manner as for the string action, one can introduce an auxiliary Riemannian metric h on Σ and find the classically equivalent Polyakov action of the p-brane:

$$S_P[X,h] = \frac{T'_p}{2} \int d^{p+1}\sigma \sqrt{h}\Big(h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j g_{ij} - (p-1)\lambda\Big), \tag{3.2}$$

where $\lambda > 0$ can be chosen arbitrarily (but fixed), and $T'_p = \lambda^{\frac{p-1}{2}} T_p$. Using the equations of motion for $h^{\alpha\beta}$'s:

$$\frac{1}{2} h_{\alpha\beta}\big(h^{\gamma\delta} g_{\gamma\delta} - (p-1)\lambda\big) = g_{\alpha\beta}, \tag{3.3}$$

where $g_{\alpha\beta} = [X^*(g)]_{\alpha\beta} \equiv \partial_\alpha X^i \partial_\beta X^j g_{ij}$, in S_P, one gets back to (3.1). In the rest of the paper, we will choose $T_p \equiv 1$. Using reparametrization invariance, one can always (at least locally) choose coordinates $(\sigma^0, \ldots, \sigma^p)$ such that $h_{00} = \lambda^{p-1} \det h_{ab}$, $h_{0a} = 0$, where h_{ab} denotes the space-like components of the metric. In this gauge, the first term in action (3.2) splits into two parts, one of them containing only the spatial derivatives of X^i and the spatial components of the metric h. Using now the equations of motion for h_{ab}, one gets the gauge fixed Polyakov action[1]

$$S_P^{gf}[X] = \frac{1}{2} \int d^{p+1}\sigma \big\{\partial_0 X^i \partial_0 X^j g_{ij} + \det\left(\partial_a X^i \partial_b X^j g_{ij}\right)\big\}. \tag{3.4}$$

[1] The gauge constraints on h_{a0}, h_{0b} and h_{00} imply an energy-momentum tensor with vanishing components $T_{a0} = T_{0a}$ and T_{00}. These constraints must be considered along with the equations of motion of the action (3.4), to ensure equivalence with the actions (3.1) and (3.2). As discussed in [67], the subgroup of the diffeomorphism symmetries that remains after gauge fixing is a symmetry of the gauge-fixed p-brane action (3.4) and also transforms the pertinent components of the energy-momentum tensor into one another (even if they are not set equal to zero). The constraints can thus be consistently imposed at the level of states.

The second term can be rewritten in a more convenient form once we define

$$\widetilde{g}_{IJ} = \sum_{\pi\in\Sigma_p} sgn(\pi) g_{i_{\pi(1)}j_1} \ldots g_{i_{\pi(p)}j_p} \equiv \delta_I^{k_1\ldots k_p} g_{k_1 j_1} \ldots g_{k_p j_p}. \tag{3.5}$$

Using this notation, one can write

$$S_P^{gf}[X] = \frac{1}{2} \int d^{p+1}\sigma \{ \partial_0 X^i \partial_0 X^j g_{ij} + \widetilde{\partial X}^I \widetilde{\partial X}^J \widetilde{g}_{IJ} \}. \tag{3.6}$$

From now on, assume that g is a positive definite metric on M. Note that from the symmetry of g it follows that $\widetilde{g}_{IJ} = \widetilde{g}_{JI}$. We can view $\widetilde{g}$ as a fibrewise bilinear form on the vector bundle $\Lambda^p TM$. Moreover, at any $m \in M$, one can define the basis (E_I) of $\Lambda^p T_m M$ as $E_I = e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $(e_1, \ldots, e_n)$ is the orthonormal basis for the quadratic form $g(m)$ at $m \in M$. In this basis one has $\widetilde{g}(m)(E_I, E_J) = \delta_{I,J}$, which shows that $\widetilde{g}$ is a positive definite fibrewise metric on $\Lambda^p TM$.

For any $C \in \Omega^{p+1}(M)$, we can add the following coupling term to the action:

$$S_C[X] = -i \int_\Sigma X^*(C) = -i \int d^{p+1}\sigma \partial_0 X^i \widetilde{\partial X}^J C_{iJ}. \tag{3.7}$$

The resulting gauge fixed Polyakov action $S_P^{\text{tot}}[X] = S_P^{gf}[X] + S_C[X]$ has the form

$$S_P^{\text{tot}}[X] = \frac{1}{2} \int d^{p+1}\sigma \{ \partial_0 X^i \partial_0 X^j g_{ij} + \widetilde{\partial X}^I \widetilde{\partial X}^J \widetilde{g}_{IJ} - 2i \partial_0 X^i \widetilde{\partial X}^J C_{iJ} \}. \tag{3.8}$$

This can be written in the compact matrix form by defining an $(n + \binom{n}{p})$-row vector

$$\Psi = \begin{pmatrix} i\partial_0 X^i \\ \widetilde{\partial X}^J \end{pmatrix}.$$

The action then has the block matrix form

$$S_P^{\text{tot}}[X] = \frac{1}{2} \int d^{p+1}\sigma \left\{ \Psi^\dagger \begin{pmatrix} g & C \\ -C^T & \widetilde{g} \end{pmatrix} \Psi \right\}. \tag{3.9}$$

From now on, unless explicitly mentioned, we may assume that $\widetilde{g}$ is not necessarily of the form (3.5), i.e., $\widetilde{g}$ can be any positive definite fibrewise metric on $\Lambda^p TM$. Any further discussions will, of course, be valid also for the special case (3.5). Since g is non-degenerate, we can pass from the Lagrangian to the Hamiltonian formalism and vice versa. The corresponding Hamiltonian has the form

$$H_P^{\text{tot}}[X,P] = -\frac{1}{2} \int d^p\sigma \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}^T \begin{pmatrix} g^{-1} & -g^{-1}C \\ -C^T g^{-1} & \widetilde{g} + C^T g^{-1} C \end{pmatrix} \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}. \tag{3.10}$$

The expression $\widetilde{g} + C^T g^{-1} C$ in the Hamiltonian and a similar expression $g + C\widetilde{g}^{-1} C^T$ play the role of "open membrane metrics" and first appeared in the work of Duff and Lu [68] already in 1990. Hamilton densities for membranes have also been discussed around that time, see e.g. [67].[2] The block matrix in the Hamiltonian can be viewed as positive definite fibrewise metric $\mathbf{G}$ on $T^*M \oplus \Lambda^p TM$ defined on sections as

$$\mathbf{G}(\alpha + \mathbf{Q}, \beta + \mathbf{R}) = \begin{pmatrix} \alpha \\ \mathbf{Q} \end{pmatrix}^T \begin{pmatrix} g^{-1} & -g^{-1}C \\ -C^T g^{-1} & \widetilde{g} + C^T g^{-1} C \end{pmatrix} \begin{pmatrix} \beta \\ \mathbf{R} \end{pmatrix}, \tag{3.11}$$

for all $\alpha, \beta \in \Omega^1(M)$ and $\mathbf{Q}, \mathbf{R} \in \mathfrak{X}^p(M)$. For $p = 1$ and $\widetilde{g} = g$, one gets exactly the inverse of the generalized metric corresponding to a Riemannian metric g and a 2-form C. Note that, analogously to the $p = 1$ case, $\mathbf{G}$ can be written as a product of block lower triangular, diagonal and upper triangular matrices:

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ -C^T & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & \widetilde{g} \end{pmatrix} \begin{pmatrix} 1 & -C \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

Before we proceed with our discussion of the corresponding Nambu sigma models, let us introduce another parametrization of the background fields g and C. In analogy with the $p = 1$ case, we shall refer to g and C as to the closed background fields. Let $\mathbf{A}$ denote the matrix in the action (3.9), that is,

$$\mathbf{A} = \begin{pmatrix} g & C \\ -C^T & \widetilde{g} \end{pmatrix}. \tag{3.13}$$

This matrix is always invertible, explicitly:

$$\mathbf{A}^{-1} = \begin{pmatrix} (g + C\widetilde{g}^{-1}C^T)^{-1} & -(g + C\widetilde{g}^{-1}C^T)^{-1}C\widetilde{g}^{-1} \\ \widetilde{g}^{-1}C^T(g + C\widetilde{g}^{-1}C^T)^{-1} & (\widetilde{g} + C^T g^{-1} C)^{-1} \end{pmatrix}. \tag{3.14}$$

Further, let us assume an arbitrary but fixed $(p+1)$-vector $\Pi \in \mathfrak{X}^{p+1}(M)$ and consider a matrix $\mathbf{B}$ of the form

$$\begin{aligned} \mathbf{B} &= \begin{pmatrix} G & \Phi \\ -\Phi^T & \widetilde{G} \end{pmatrix}^{-1} + \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix} \\ &= \begin{pmatrix} (G + \Phi\widetilde{G}^{-1}\Phi^T)^{-1} & -(G + \Phi\widetilde{G}\Phi^T)^{-1}\Phi\widetilde{G}^{-1} + \Pi \\ \widetilde{G}^{-1}\Phi^T(G + \Phi\widetilde{G}^{-1}\Phi^T)^{-1} - \Pi^T & (\widetilde{G} + \Phi^T G^{-1}\Phi)^{-1} \end{pmatrix} \end{aligned} \tag{3.15}$$

such that the equality $\mathbf{A}^{-1} = \mathbf{B}$, i.e.,

$$\begin{pmatrix} g & C \\ -C^T & \widetilde{g} \end{pmatrix}^{-1} = \begin{pmatrix} G & \Phi \\ -\Phi^T & \widetilde{G} \end{pmatrix}^{-1} + \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix} \tag{3.16}$$

holds. This generalization was introduced and used in [1]. Again, in analogy with the case $p = 1$, we will refer to G and Φ as to the open backgrounds. More explicitly, we have the

[2] We believe that the Hamiltonian (3.10) has been known, in this or a similar form, to experts for a long time but we were not able to trace it in even older literature, cf. [69] for the string case. More recently, the Hamiltonian as well as the open membrane metrics appeared, e.g., in [70]. We thank D. Berman for bringing this paper to our attention.

following set of open-closed relations:

$$g + C\widetilde{g}^{-1}C^T = G + \Phi\widetilde{G}^{-1}\Phi^T, \tag{3.17}$$

$$\widetilde{g} + C^T g^{-1} C = \widetilde{G} + \Phi^T G^{-1}\Phi, \tag{3.18}$$

$$g^{-1}C = G^{-1}\Phi - \Pi(\widetilde{G} + \Phi^T G^{-1}\Phi), \tag{3.19}$$

$$\Phi\widetilde{G}^{-1} = C\widetilde{g}^{-1} + (g + C\widetilde{g}^{-1}C^T)\Pi. \tag{3.20}$$

For fixed Π, given $(g, \widetilde{g}, C)$ there exist unique $(G, \widetilde{G}, \Phi)$ such that the above relations are fulfilled, and vice versa. The explicit expressions are most directly seen from the equality $\mathbf{A} = \mathbf{B}^{-1}$, again using the formula for the inverse of the block matrix $\mathbf{B}$. In particular,

$$g^{-1} = (1 - \Phi\Pi^T)^T G^{-1}(1 - \Phi\Pi^T) + \Pi\widetilde{G}\Pi^T, \tag{3.21}$$

$$\widetilde{g}^{-1} = (1 - \Phi^T\Pi)^T\widetilde{G}^{-1}(1 - \Phi^T\Pi) + \Pi^T G\Pi, \tag{3.22}$$

and the explicit expression for C can be found straightforwardly. Obviously, the inverse relations are obtained simply by interchanging $g \leftrightarrow G$, $\widetilde{g} \leftrightarrow \widetilde{G}$, $C \leftrightarrow \Phi$, and $\Pi \leftrightarrow -\Pi$. Using these relations, we can write the action (3.9) equivalently in terms of the open backgrounds G, Φ and the (so far auxiliary) $(p+1)$-vector Π.

In terms of the corresponding Hamiltonian (3.10), the above open-closed relations give just another factorization of the matrix $\mathbf{G}$. This time we have

$$\mathbf{G} = \begin{pmatrix} 1 & \Pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi^T & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & \widetilde{G} \end{pmatrix} \begin{pmatrix} 1 & -\Phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Pi^T & 1 \end{pmatrix}. \tag{3.23}$$

In the sequel it will be convenient to distinguish the respective expressions of above introduced matrices $\mathbf{A}$ and $\mathbf{G}$ in the closed and open variables. For the former we we shall use $\mathbf{A_c}$ and $\mathbf{G_c}$ and for the latter we introduce $\mathbf{A}_o$ and $\mathbf{G}_o$, respectively. Hence the open-closed relations can be expressed either way: $\mathbf{A} \equiv \mathbf{A}_c = \mathbf{A}_o \equiv \mathbf{B}^{-1}$ or $\mathbf{G}_c = \mathbf{G}_o$. Note, that the latter form is just equivalent to the statement about the decomposability of a 2x2 block matrix with the invertible upper left block as a product of lower triangular, diagonal, and upper triangular block matrices, the triangular ones having unit matrices on the diagonal. Note that for $p = 1$ and $\widetilde{g} = g$, the open-closed relations (see [5]) are usually written simply as

$$\frac{1}{g + C} = \frac{1}{G + \Phi} + \Pi. \tag{3.24}$$

To conclude this section, note that taking the determinant of the matrix $\mathbf{A}_c$, we may prove the useful identity:

$$\det\,(\widetilde{g} + C^T g^{-1} C) = \frac{\det\widetilde{g}}{\det g}\det\,(g + C\widetilde{g}^{-1}C^T). \tag{3.25}$$

To show this, just note that $\mathbf{A}_c$ can be decomposed in two different ways, either

$$\mathbf{A}_c = \begin{pmatrix} 1 & 0 \\ -C^T g^{-1} & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & (\widetilde{g} + C^T g^{-1} C) \end{pmatrix} \begin{pmatrix} 1 & g^{-1}C \\ 0 & 1 \end{pmatrix},$$

or as

$$\mathbf{A}_c = \begin{pmatrix} 1 & C\widetilde{g}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (g + C\widetilde{g}^{-1}C^T) & 0 \\ 0 & \widetilde{g} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\widetilde{g}^{-1}C^T & 1 \end{pmatrix}.$$

Taking the determinant of both expressions and comparing them yields (3.25).

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4 Nambu sigma model

In analogy with the $p = 1$ case, we may ask whether there is a Nambu sigma model classically equivalent to the action (3.9). To see this, introduce new auxiliary fields η_i and $\widetilde{\eta}_J$, which transform according to their index structure under a change of coordinates on M. Define an $(n + \binom{n}{p})$-row vector $\Upsilon = \begin{pmatrix} i\eta_i \\ \widetilde{\eta}_J \end{pmatrix}$. The corresponding (non-topological) Nambu sigma model then has the form:

$$S_{NSM}[X, \eta, \widetilde{\eta}] = -\int d^{p+1}\sigma \left\{ \frac{1}{2} \Upsilon^\dagger \mathbf{A}^{-1} \Upsilon + \Upsilon^\dagger \Psi \right\}, \tag{4.1}$$

where $\mathbf{A}$ can be either of $\mathbf{A_o}$ and $\mathbf{A_c}$, supposing that the open-closed relations $\mathbf{A_o} = \mathbf{A_c}$ hold. Using the equations of motion for Υ, one gets back the Polyakov action (3.9). For the detailed treatment of Nambu sigma models see [71].

Yet another parametrization of $\mathbf{A}^{-1}$ — using new background fields $G_N, \widetilde{G}_N, \Pi_N$, which we refer to as Nambu background fields[3] — can be introduced

$$\mathbf{A}^{-1} = \begin{pmatrix} G_N^{-1} & \Pi_N \\ -\Pi_N^T & \widetilde{G}_N^{-1} \end{pmatrix}. \tag{4.2}$$

We will denote as $\mathbf{A}_N$ the matrix $\mathbf{A}$ expressed with help of Nambu background fields $G_N, \widetilde{G}_N, \Pi_N$. Using (3.14), one gets the correspondence between closed and Nambu sigma background fields:

$$G_N = g + C\widetilde{g}^{-1}C^T, \tag{4.3}$$

$$\widetilde{G}_N = \widetilde{g} + C^T g^{-1} C, \tag{4.4}$$

$$\Pi_N = -(g + C\widetilde{g}^{-1}C^T)^{-1} C\widetilde{g}^{-1} = -g^{-1}C(\widetilde{g} + C^T g^{-1} C)^{-1}. \tag{4.5}$$

Clearly, G_N is a Riemannian metric on M and $\widetilde{G}_N$ is a fibrewise positive definite metric on $\Lambda^p TM$. It is important to note that in general, for $p > 1$, $\Pi_N : \Lambda^p T^*M \to TM$ is not necessarily induced by a $(p+1)$-vector on M. This also means that it is not in general a Nambu-Poisson tensor. However; for $p = 1$, it is easy to show that Π_N is a bivector.

Also note that even if $\widetilde{g}$ is a skew-symmetrized tensor product of g's (3.5), $\widetilde{G}_N$ is not in general the skew-symmetrized tensor product of G_N's.

The converse relations are:

$$g = (G_N^{-1} + \Pi_N \widetilde{G}_N \Pi_N^T)^{-1}, \tag{4.6}$$

$$\widetilde{g} = (\widetilde{G}_N^{-1} + \Pi_N^T G_N \Pi_N)^{-1}, \tag{4.7}$$

$$C = -(G_N^{-1} + \Pi_N \widetilde{G}_N \Pi_N^T)^{-1} \Pi_N \widetilde{G}_N = -G_N \Pi_N (\widetilde{G}_N^{-1} + \Pi_N^T G_N \Pi_N)^{-1}. \tag{4.8}$$

Again, it is instructive to pass to the corresponding Hamiltonians. First, find the canonical Hamiltonian to (4.1), that is

$$H^c_{NSM}[X, P, \widetilde{\eta}] = \int d^p\sigma P_i \partial_0 X^i - \mathcal{L}[X, P, \widetilde{\eta}].$$

[3] Here, instead of fixing Π and finding open variables in terms of closed ones, we fix Φ to be zero and find, again using the open-closed relations, unique $G_N, \widetilde{G}_N, \Pi_N$ as functions of $\mathfrak{g}, \widetilde{g}$ and C, or vice versa.

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Second, use the equations of motion to get rid of $\widetilde{\eta}$. In analogy with the $p = 1$ case, one expects that resulting Hamiltonian H_{NSM} coincides with (3.10), that is

$$H_{NSM}[X,P] = H_P^{\text{tot}}[X,P].$$

Indeed, we get

$$H_{NSM}[X,P] = -\frac{1}{2}\int d^p\sigma \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}^T \begin{pmatrix} G_N^{-1} + \Pi_N \widetilde{G}_N \Pi_N^T & \Pi_N \widetilde{G}_N \\ \widetilde{G}_N \Pi_N^T & \widetilde{G}_N \end{pmatrix} \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}. \tag{4.9}$$

If one plugs (4.3)–(4.4) to (4.9), one obtains exactly the Hamiltonian (3.10). The matrix $\mathbf{G}$ can be thus written as

$$\mathbf{G} = \begin{pmatrix} 1 & \Pi_N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G_N^{-1} & 0 \\ 0 & \widetilde{G}_N \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Pi_N^T & 1 \end{pmatrix} \tag{4.10}$$

when using the Nambu background fields, in which case we shall introduce the notation $\mathbf{G}_N$ for it. This shows that to any $g, \widetilde{g}, C$ one can uniquely find $G_N, \widetilde{G}_N, \Pi_N$ and vice versa, since they both come from the respective unique decompositions of the matrix $\mathbf{G}$.

Note that for $p = 1$ and $\widetilde{g} = g$, relations (4.3)–(4.5) are usually written simply as

$$\frac{1}{g+C} = \frac{1}{G_N} + \Pi_N. \tag{4.11}$$

We will refer to the Poisson sigma model, when expressed — using Π — in open variables $(G, \widetilde{G}, \Phi)$ as to augmented Poisson sigma model.

5 Geometry of the open-closed brane relations

For $p = 1$, the open-closed relations (3.24) can naturally be explained using the language of generalized geometry. We have developed this point of view in [66]. One expects that similar observations apply also for $p > 1$ case. In the previous section we have already mentioned the possibility to define the generalized metric on the vector bundle $TM \oplus \Lambda^p T^* M$ by the inverse of the matrix (3.12). Here we discuss an another approach to a generalization of the generalized geometry starting from equation (3.16). Denote $W = TM \oplus \Lambda^p TM$.

The main goal of this section is to show that we can without any additional labor adapt the whole formalism of [66] to the vector bundle $W \oplus W^*$.

Define the maps $\mathcal{G}$, $\mathcal{B} : W \to W^*$ using block matrices as

$$\mathcal{G}\begin{pmatrix} V \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & \widetilde{g} \end{pmatrix} \begin{pmatrix} V \\ \mathbf{P} \end{pmatrix}, \quad \mathcal{B}\begin{pmatrix} V \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix} \begin{pmatrix} V \\ \mathbf{P} \end{pmatrix}, \tag{5.1}$$

for all $V + \mathbf{P} \in \Gamma(W)$. Next, define the map $\Theta : W^* \to W$ as

$$\Theta \begin{pmatrix} \alpha \\ \Sigma \end{pmatrix} = \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \Sigma \end{pmatrix}, \tag{5.2}$$

for all $\alpha + \Sigma \in \Gamma(W^*)$. Then define $\mathcal{H}, \Xi : W \to W^*$ as in (5.1) using the fields $G, \widetilde{G}, \Phi$ instead of $g, \widetilde{g}, C$. The open-closed relations (3.16) can be then written as simply as

$$\frac{1}{\mathcal{G}+\mathcal{B}} = \frac{1}{\mathcal{H}+\Xi} + \Theta. \tag{5.3}$$

We see that they have exactly the same form as (3.24) for $p = 1$. The purpose of this section is to obtain these relations from the geometry of the vector bundle $W \oplus W^*$.

We define an inner product $\langle \cdot, \cdot \rangle : \Gamma(W \oplus W^*) \times \Gamma(W \oplus W^*) \to C^\infty(M)$ on $W \oplus W^*$ to be the natural pairing between W and W^*, that is:

$$\langle V + \mathbf{P} + \alpha + \Sigma, W + \mathbf{Q} + \beta + \Psi \rangle = \beta(V) + \alpha(W) + \Psi(\mathbf{P}) + \Sigma(\mathbf{Q}),$$

for all $V, W \in \mathfrak{X}(M)$, $\alpha, \beta \in \Omega^1(M)$, $\mathbf{P}, \mathbf{Q} \in \mathfrak{X}^p(M)$, and $\Sigma, \Psi \in \Omega^p(M)$. Note that this pairing has the signature $(n + \binom{n}{p}, n + \binom{n}{p})$.

Now, let $\mathcal{T} : W \oplus W^* \to W \oplus W^*$ be a vector bundle endomorphism squaring to identity, that is, $\mathcal{T}^2 = 1$. We say that $\mathcal{T}$ is a generalized metric on $W \oplus W^*$, if the fibrewise bilinear form

$$(E_1, E_2)_{\mathcal{T}} \equiv \langle E_1, \mathcal{T}(E_2) \rangle,$$

defined for all $E_1, E_2 \in \Gamma(W \oplus W^*)$, is a positive definite fibrewise metric on $W \oplus W^*$. It follows from definition that $\mathcal{T}$ is orthogonal and symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$. Moreover, it defines two eigenbundles $V_\pm \subset W \oplus W^*$, corresponding to eigenvalues ± 1 of $\mathcal{T}$. It follows immediately from the properties of $\mathcal{T}$, that they are both of rank $n + \binom{n}{p}$, orthogonal to each other, and thus

$$W \oplus W^* = V_+ \oplus V_-.$$

Moreover, V_+ and V_- form the positive definite and negative definite subbundles of $\langle \cdot, \cdot \rangle$, respectively. From the positive definiteness of V_+ it follows that V_+ has zero intersection both with W and W^*, and is thus a graph of a unique vector bundle isomorphism $\mathcal{A} : W \to W^*$. The map $\mathcal{A}$ can be written as a sum of a symmetric and a skew-symmetric part with respect to $\langle \cdot, \cdot \rangle$: $\mathcal{A} = \mathcal{G} + \mathcal{B}$. From the positive definiteness of V_+, it follows that $\mathcal{G}$ is a positive definite fibrewise metric on W. From the orthogonality of V_+ and V_- we finally obtain that:

$$V_\pm = \{(V + \mathbf{P}) + (\pm \mathcal{G} + \mathcal{B})(V + \mathbf{P}) \; | V + \mathbf{P} \in W\}.$$

The map $\mathcal{T}$, or equivalently the fibrewise metric $(\cdot, \cdot)_{\mathcal{T}}$ can be reconstructed using the data $\mathcal{G}$ and $\mathcal{B}$ to get

$$(V + \mathbf{P} + \alpha + \Sigma, W + \mathbf{Q} + \beta + \Psi)_{\mathcal{T}} = \begin{pmatrix} V + \mathbf{P} \\ \alpha + \Sigma \end{pmatrix}^T \begin{pmatrix} \mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B} & \mathcal{B}\mathcal{G}^{-1} \\ -\mathcal{G}^{-1}\mathcal{B} & \mathcal{G}^{-1} \end{pmatrix} \begin{pmatrix} W + \mathbf{Q} \\ \beta + \Psi \end{pmatrix}.$$

Note that the above block matrix can be decomposed as a product

$$\begin{pmatrix} \mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B} & \mathcal{B}\mathcal{G}^{-1} \\ -\mathcal{G}^{-1}\mathcal{B} & \mathcal{G}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{B} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{G} & 0 \\ 0 & \mathcal{G}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathcal{B} & 1 \end{pmatrix}.$$

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The maps $\mathcal{G}, \mathcal{B}$ can be parametrized as

$$\mathcal{G}\begin{pmatrix} V \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} g & D \\ D^T & \widetilde{g} \end{pmatrix}\begin{pmatrix} V \\ \mathbf{Q} \end{pmatrix},$$
$$\mathcal{B}\begin{pmatrix} V \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} B & C \\ -C^T & \widetilde{B} \end{pmatrix}\begin{pmatrix} V \\ \mathbf{Q} \end{pmatrix},$$

where g is a symmetric covariant 2-tensor on M, $C, D : \Lambda^p TM \to T^*M$ are vector bundle morphisms, $B \in \Omega^2(M)$, and $\widetilde{g}$ and $\widetilde{B}$ are symmetric and skew-symmetric fibrewise bilinear forms on $\Lambda^p TM$, respectively. The fields $g, \widetilde{g}, D$ are not arbitrary, since $\mathcal{G}$ has to be a positive definite fibrewise metric on W. One immediately gets that $g, \widetilde{g}$ have to be positive definite. The conditions imposed on D can be seen from the equalities

$$\begin{aligned}\begin{pmatrix} g & D \\ D^T & \widetilde{g} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ D^T g^{-1} & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & \widetilde{g} - D^T g^{-1} D \end{pmatrix}\begin{pmatrix} 1 & g^{-1}D \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & D\widetilde{g}^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g - D\widetilde{g}^{-1}D^T & 0 \\ 0 & \widetilde{g} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \widetilde{g}^{-1}D^T & 1 \end{pmatrix}.\end{aligned}$$

We see that there are two equivalent conditions on D: the fibrewise bilinear form $\widetilde{g} - D^T g^{-1} D$, or 2-tensor $g - D\widetilde{g}^{-1}D^T$ have to be positive definite. Inspecting the action (3.9), we see that only the case when $B = \widetilde{B} = D = 0$ is relevant for our purpose.

Now, let us turn our attention to the explanation of the open-closed relations. For this, consider the vector bundle automorphism $\mathcal{O} : W \oplus W^* \to W \oplus W^*$, orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$, that is,

$$\langle \mathcal{O}(E_1), \mathcal{O}(E_2) \rangle = \langle E_1, E_2 \rangle,$$

for all $E_1, E_2 \in \Gamma(W \oplus W^*)$. Given a generalized metric $\mathcal{T}$, we can define a new map $\mathcal{T}' = \mathcal{O}^{-1}\mathcal{T}\mathcal{O}$. It can be easily checked that $\mathcal{T}'$ is again a generalized metric. Obviously, the respective eigenbundles V_+ are related using $\mathcal{O}$, namely:

$$V_+^{\mathcal{T}'} = \mathcal{O}^{-1}(V_+^{\mathcal{T}}). \tag{5.4}$$

We have also proved that every generalized metric $\mathcal{T}$ corresponds to two unique fields $\mathcal{G}$ and $\mathcal{B}$. This means that to given $\mathcal{G}$ and $\mathcal{B}$, and an orthogonal vector bundle isomorphism $\mathcal{O}$, there exists a unique pair $\mathcal{H}$, Ξ corresponding to $\mathcal{T}' = \mathcal{O}^{-1}\mathcal{T}\mathcal{O}$. We will show that open-closed relations are a special case of this correspondence. Also, note that $(\cdot, \cdot)_{\mathcal{T}}$ and $(\cdot, \cdot)_{\mathcal{T}'}$ are related as

$$(\cdot, \cdot)_{\mathcal{T}'} = (\mathcal{O}(\cdot), \mathcal{O}(\cdot))_{\mathcal{T}}. \tag{5.5}$$

Now, consider an arbitrary skew-symmetric morphism $\Theta : W^* \to W$, that is

$$\langle \alpha + \Sigma, \Theta(\beta + \Psi) \rangle = -\langle \Theta(\alpha + \Sigma), \beta + \Psi \rangle,$$

for all $\alpha, \beta \in \Omega^1(M)$, and $\Sigma, \Psi \in \Omega^p(M)$. It can easily be seen that the vector bundle isomorphism $e^{\Theta} : W \oplus W^* \to W \oplus W^*$, defined as

$$e^{\Theta}\begin{pmatrix} V + \mathbf{Q} \\ \alpha + \Sigma \end{pmatrix} = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V + \mathbf{Q} \\ \alpha + \Sigma \end{pmatrix},$$

for all $V + \mathbf{Q} + \alpha + \Sigma \in \Gamma(W \oplus W^*)$, is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$. Its inverse is simply $e^{-\Theta}$. Let $\mathcal{T}$ be the generalized metric corresponding to $\mathcal{G} + \mathcal{B}$. Note that $V_+^{\mathcal{T}}$ can be expressed as

$$V_+^{\mathcal{T}} = \{(\mathcal{G} + \mathcal{B})^{-1}(\alpha + \Sigma) + (\alpha + \Sigma) \mid (\alpha + \Sigma) \in W^*\}.$$

Using the relation (5.4), we obtain that

$$V_+^{\mathcal{T}'} = e^{-\Theta} V_+^{\mathcal{T}} = \{\big((\mathcal{G} + \mathcal{B})^{-1} - \Theta\big)(\alpha + \Sigma) + (\alpha + \Sigma) \mid (\alpha + \Sigma) \in W^*\}.$$

We see that the vector bundle morphism $\mathcal{H} + \Xi$ corresponding to $\mathcal{T}'$ satisfies

$$(\mathcal{H} + \Xi)^{-1} = (\mathcal{G} + \mathcal{B})^{-1} - \Theta.$$

But this is precisely the relation (5.3). We also know how to handle this relation on the level of the positive definite fibrewise metrics $(\cdot, \cdot)_{\mathcal{T}}$ and $(\cdot, \cdot)_{\mathcal{T}'}$. From (5.5) we get the relation

$$\begin{pmatrix} \mathcal{H} - \Xi \mathcal{H}^{-1} \Xi & \mathcal{B} \mathcal{H}^{-1} \\ -\mathcal{H}^{-1} \Xi & \mathcal{H}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\Theta & 1 \end{pmatrix} \begin{pmatrix} \mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B} & \mathcal{B}\mathcal{G}^{-1} \\ -\mathcal{G}^{-1}\mathcal{B} & \mathcal{G}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}.$$

Using the decomposition of the matrices, we can write this also as

$$\begin{pmatrix} 1 & \Xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{H}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Xi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\Theta & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{G} & 0 \\ 0 & \mathcal{G}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}.$$

Comparing both expressions, we get the explicit form of open-closed relations:

$$\mathcal{H} - \Xi \mathcal{H}^{-1} \Xi = \mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B}, \tag{5.6}$$
$$\Xi \mathcal{H}^{-1} = (\mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B})\Theta + \mathcal{B}\mathcal{G}^{-1}, \tag{5.7}$$
$$\mathcal{H}^{-1} = (1 + \Theta\mathcal{B})\mathcal{G}^{-1}(1 - \mathcal{B}\Theta) - \Theta\mathcal{G}\Theta. \tag{5.8}$$

We have proved that for given $\mathcal{G}, \mathcal{B}$ and any Θ, $\mathcal{H}$ and Ξ can be found uniquely. Inverse relations can be obtained by interchanging $\mathcal{G} \leftrightarrow \mathcal{H}$, $\mathcal{B} \leftrightarrow \Xi$ and $\Theta \leftrightarrow -\Theta$. Note that, actually, the last equation follows from the first two. Now let us turn our attention to the case of $\mathcal{G} + \mathcal{B}$ in the form (5.1). One has

$$\mathcal{G} - \mathcal{B}\mathcal{G}^{-1}\mathcal{B} = \begin{pmatrix} g + C\widetilde{g}^{-1}C^T & 0 \\ 0 & \widetilde{g} + C^T g^{-1} C \end{pmatrix},$$
$$\mathcal{B}\mathcal{G}^{-1} = \begin{pmatrix} 0 & C\widetilde{g}^{-1} \\ -C^T g^{-1} & 0 \end{pmatrix}, \qquad \mathcal{G}^{-1} = \begin{pmatrix} g^{-1} & 0 \\ 0 & \widetilde{g}^{-1} \end{pmatrix}.$$

Parametrize Θ as

$$\Theta = \begin{pmatrix} \pi & \Pi \\ -\Pi^T & \widetilde{\pi} \end{pmatrix},$$

where $\pi \in \mathfrak{X}^2(M)$, $\Pi : \Lambda^p T^*M \to TM$, and $\widetilde{\pi}$ is skew-symmetric fibrewise bilinear form on $\Lambda^p T^*M$. Right-hand side of (5.7) is then

$$\begin{pmatrix} g + C\widetilde{g}^{-1}C^T & 0 \\ 0 & \widetilde{g} + C^T g^{-1} C \end{pmatrix} \begin{pmatrix} \pi & \Pi \\ -\Pi^T & \widetilde{\pi} \end{pmatrix} + \begin{pmatrix} 0 & C\widetilde{g}^{-1} \\ -C^T g^{-1} & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} (g + C\widetilde{g}^{-1}C^T)\pi & (g + C\widetilde{g}^{-1}C^T)\Pi + C\widetilde{g}^{-1} \\ -(\widetilde{g} + C^T g^{-1} C)\Pi^T - C^T g^{-1} & (\widetilde{g} + C^T g^{-1} C)\widetilde{\pi} \end{pmatrix}.$$

We see that to obtain a generalized metric where $\mathcal{H}$ is block diagonal, and Ξ is block off-diagonal, we have to choose $\pi = \widetilde{\pi} = 0$. This means that we choose Θ to be of the form (5.2). Defining

$$\mathcal{H} = \begin{pmatrix} G & 0 \\ 0 & \widetilde{G} \end{pmatrix}, \; \Xi = \begin{pmatrix} 0 & \Phi \\ -\Phi^T & 0 \end{pmatrix},$$

it is now straightforward to see that the set of equations (5.6)–(5.8) gives exactly the open-closed relations (3.17)–(3.20). The relations between the open membrane variables and Nambu fields $G_N, \widetilde{G}_N, \Pi_N$ can be explained in a similar fashion. Indeed, note that the map $\mathcal{G} + \mathcal{B}$ is invertible, and its inverse, the vector bundle morphism from W^* to W, can be split into symmetric and skew-symmetric part:

$$(\mathcal{G} + \mathcal{B})^{-1} = \mathcal{H}_N^{-1} + \Theta_N, \tag{5.9}$$

where $\mathcal{H}_N$ is a fibrewise positive definite metric on W, and Θ_N is a skew-symmetric fibrewise bilinear form on W^*. Parametrizing them as

$$\mathcal{H}_N = \begin{pmatrix} G_N & 0 \\ 0 & \widetilde{G}_N \end{pmatrix}, \quad \Theta_N = \begin{pmatrix} 0 & \Pi_N \\ -\Pi_N^T & 0 \end{pmatrix},$$

and expanding (5.9), we obtain exactly the set of equations (4.3)–(4.5).

6 Gauge field F as transformation of the fibrewise metric

In this section, we would like to develop the equalities required in the discussion of DBI actions. In the previous sections we have shown how the closed and open membrane actions are related using the generalized geometry point of view. One expects that it is also true for their versions taking into account the fluctuations. The following paragraphs show that it is true "up to an isomorphism", fluctuated backgrounds cannot be related simply by open-closed relations in the form (3.17)–(3.20).

We also show that corresponding open backgrounds are essentially uniquely fixed, there is no ambiguity at all. For $p = 1$, we have already used this observation in [66].

The idea is the following: suppose that we would like to add a fluctuation F to the $(p+1)$-form C. At this point we consider F to be defined globally on the entire manifold M, although everything works also in the case when F is defined only on a some submanifold of M.[4]

[4]Later, this submanifold will correspond to a p'-brane, $p' \geq p$, where p-branes can end.

Going from C to $C+F$ corresponds to replacing $\mathbf{G}_c$ in the Hamiltonian (3.10) with $\mathbf{G}_c^F$, defined as

$$\mathbf{G}_c^F = \begin{pmatrix} 1 & 0 \\ -F^T & 1 \end{pmatrix} \mathbf{G}_c \begin{pmatrix} 1 & -F \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -(C+F)^T & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & \widetilde{g} \end{pmatrix} \begin{pmatrix} 1 & -(C+F) \\ 0 & 1 \end{pmatrix}. \tag{6.1}$$

The matrix $\begin{pmatrix} 1 & -F \\ 0 & 1 \end{pmatrix}$ corresponds to an endomorphism of $T^*M \oplus \Lambda^p TM$, which we denote as e^{-F}. Note that unlike in the $p=1$ case, e^{-F} is not orthogonal with respect to the canonical pairing (valued in $\mathfrak{X}^{p-1}(M)$) on $T^*M \oplus \Lambda^p TM$, defined as:

$$\langle \alpha + \mathbf{Q}, \beta + \mathbf{R} \rangle = i_\alpha \mathbf{R} + i_\beta \mathbf{Q},$$

for all $\alpha, \beta \in \Omega^1(M)$ and $\mathbf{Q}, \mathbf{R} \in \mathfrak{X}^p(M)$. It can be shown that any orthogonal F has to be identically 0. On the other hand, its transpose map, $(e^{-F})^T \equiv e_{-F}$, which is an endomorphism of $TM \oplus \Lambda^p T^*M$, is orthogonal with respect to the canonical pairing (valued in $\Omega^{p-1}(M)$) on $TM \oplus \Lambda^p T^*M$ iff F is a $(p+1)$-form in M. This pairing is defined as

$$\langle V + \Sigma, W + \Xi \rangle = i_V \Sigma + i_W \Xi,$$

for all $V, W \in \mathfrak{X}(M)$ and $\Sigma, \Xi \in \Omega^p(M)$. In this notation, the transformation (6.1) can be written as

$$\mathbf{G}_c^F = e_{-F} \mathbf{G}_c e^{-F} \equiv (e^{-F})^T \mathbf{G}_c e^{-F}. \tag{6.2}$$

We know that $\mathbf{G}$ can be rewritten as $\mathbf{G}_o$ in the open variables $(G, \widetilde{G}, \Phi)$, corresponding to augmented Nambu sigma model. If we define the automorphism e^Π of $T^*M \oplus \Lambda^p TM$ as

$$e^\Pi \begin{pmatrix} \alpha \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Pi^T & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mathbf{Q} \end{pmatrix},$$

we can express $\mathbf{G}_o$ as

$$\mathbf{G}_o = e_\Pi \begin{pmatrix} 1 & 0 \\ -\Phi^T & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & \widetilde{G} \end{pmatrix} \begin{pmatrix} 1 & -\Phi \\ 0 & 1 \end{pmatrix} e^\Pi, \tag{6.3}$$

where $e_\Pi = (e^\Pi)^T$. Dually to the previous discussion, e^Π is an orthogonal transformation of $T^*M \oplus \Lambda^p TM$; although e_Π, for non-zero Π, is never orthogonal on $TM \oplus \Lambda^p T^*M$.

Now, it is natural to ask whether to the gauged closed variables $(g, \widetilde{g}, C+F)$ there correspond some open variables and hence an augmented Nambu sigma model, described by some Π' and $(G, \widetilde{G}, \Phi + F')$, where F' describes a fluctuation of the background Φ. More precisely, we ask whether one can write $\mathbf{G}_o^F$ in the form

$$\mathbf{G}_o^F \stackrel{?}{=} e_{\Pi'} \begin{pmatrix} 1 & 0 \\ -(\Phi + F')^T & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & \widetilde{G} \end{pmatrix} \begin{pmatrix} 1 & -(\Phi + F') \\ 0 & 1 \end{pmatrix} e^{\Pi'}. \tag{6.4}$$

Translated into the language of the corresponding automorphisms of $T^*M \oplus \Lambda^p TM$, this boils down to the question

$$e^\Pi e^{-F} \stackrel{?}{=} e^{-F'} e^{\Pi'}, \tag{6.5}$$

for some Π' and F'. In general, this is not possible. Explicitly the equation (6.5) reads

$$\begin{pmatrix} 1 & -F \\ \Pi^T & 1-\Pi^T F \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1-F'\Pi' & -F' \\ \Pi'^T & 1 \end{pmatrix}.$$

This implies $\Pi^T F = 0$, which, of course, in general is not satisfied. The decomposition on the right-hand side therefore has to contain a block-diagonal term. Note that $e^{-F'}$ is upper triangular, whereas $e^{\Pi'}$ is lower triangular. For a matrix to have a decomposition into a product of a block upper triangular, diagonal and lower triangular matrix, it has to have an invertible bottom right block, that is $1-\Pi^T F$. Hence, we assume that $1-\Pi^T F$ is an invertible $\binom{n}{p} \times \binom{n}{p}$ matrix. We are now looking for a solution of the equation

$$e^{\Pi} e^{-F} = e^{-F'} \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} e^{\Pi'}, \tag{6.6}$$

where $M : T^*M \to T^*M$ and $N : \Lambda^p TM \to \Lambda^p TM$ are (necessarily) invertible vector bundle morphisms.

We can decompose $e^{\Pi} e^{-F}$ as

$$\begin{pmatrix} 1 & -F(1-\Pi^T F)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+F(1-\Pi^T F)^{-1}\Pi^T & 0 \\ 0 & 1-\Pi^T F \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (1-\Pi^T F)^{-1}\Pi^T & 1 \end{pmatrix}. \tag{6.7}$$

From this we see that $F' = F(1-\Pi^T F)^{-1}$, $\Pi' = \Pi(1-F^T\Pi)^{-1}$ and $N = 1-\Pi^T F$. To find an alternative description of F', Π' and M, examine the inverse of the equation (6.6):

$$e^{F} e^{-\Pi} = e^{-\Pi'} \begin{pmatrix} M^{-1} & 0 \\ 0 & N^{-1} \end{pmatrix} e^{F'}. \tag{6.8}$$

The left hand side of this equation is

$$e^{F} e^{-\Pi} = \begin{pmatrix} 1-F\Pi^T & F \\ -\Pi^T & 1 \end{pmatrix},$$

which shows that $1-\Pi^T F$ is invertible iff $1-F\Pi^T$ is invertible. The decomposition of $e^F e^{-\Pi}$ reads

$$\begin{pmatrix} 1 & 0 \\ -\Pi^T(1-F\Pi^T)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1-F\Pi^T & 0 \\ 0 & 1+\Pi^T(1-F\Pi^T)^{-1}F \end{pmatrix} \begin{pmatrix} 1 & (1-F\Pi^T)^{-1}F \\ 0 & 1 \end{pmatrix}. \tag{6.9}$$

We thus get that $F' = (1-F\Pi^T)^{-1}F$, $\Pi' = (1-\Pi F^T)^{-1}\Pi$ and $M = (1-F\Pi^T)^{-1}$.

We can conclude that the fields F', Π', and vector bundle morphisms M, N in the decomposition (6.6) have one of the following equivalent forms:

$$F' = F(1-\Pi^T F)^{-1} = (1-F\Pi^T)^{-1}F, \tag{6.10}$$

$$\Pi' = \Pi(1-F^T\Pi)^{-1} = (1-\Pi F^T)^{-1}\Pi, \tag{6.11}$$

$$M = 1 + F(1 - \Pi^T F)^{-1}\Pi^T = 1 + F'\Pi^T = (1 - F\Pi^T)^{-1}, \tag{6.12}$$
$$N = 1 - \Pi^T F = \big(1 + \Pi^T(1 - F\Pi^T)^{-1}F\big)^{-1} = (1 + \Pi'^T F)^{-1}. \tag{6.13}$$

Thus, we have found a factorization of $\mathbf{G}_o^F$ in the form

$$\mathbf{G}_o^F = e_{\Pi'} \begin{pmatrix} M^T & 0 \\ 0 & N^T \end{pmatrix} e_{-(\Phi+F')} \begin{pmatrix} G^{-1} & 0 \\ 0 & \widetilde{G} \end{pmatrix} e^{-(\Phi+F')} \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} e^{\Pi'}. \tag{6.14}$$

Comparing this to $\mathbf{G}_c^F$, in particular comparing the respective bottom right blocks, we get the important identity

$$\widetilde{g} + (C+F)^T g^{-1}(C+F) = N^T\big(\widetilde{G} + (\Phi+F')^T G^{-1}(\Phi+F')\big)N. \tag{6.15}$$

Similarly, comparing the top left blocks of the inverses, one gets

$$g + (C+F)\widetilde{g}^{-1}(C+F)^T = M^{-1}\big(G + (\Phi+F')\widetilde{G}^{-1}(\Phi+F')\big)^T M^{-T}. \tag{6.16}$$

Equivalently, one can gauge the matrix $\mathbf{A}_c$, i.e., set

$$\mathbf{A}_c^F = \begin{pmatrix} g & (C+F) \\ -(C+F)^T & \widetilde{g} \end{pmatrix}. \tag{6.17}$$

To express this matrix in open variables we introduce the following notation: $\bar{G}^{-1} := M^T G^{-1} M$, $\bar{\widetilde{G}} = N^T \widetilde{G} N$, $\bar{\Phi} := M^{-1}\Phi N$ and $\bar{F}' := M^{-1}F'N$. If we now put

$$\mathbf{A}_o^F = \begin{pmatrix} \bar{G} & (\bar{\Phi} + \bar{F}') \\ -(\bar{\Phi} + \bar{F}')^T & \bar{\widetilde{G}} \end{pmatrix}^{-1} + \begin{pmatrix} 0 & \Pi' \\ -\Pi'^T & 0 \end{pmatrix}, \tag{6.18}$$

the (gauged) open-closed relations are equivalent to $\mathbf{A}_c^F = \mathbf{A}_o^F$. As in the previous sections, using the matrices $\mathbf{A}_c^F$, $\mathbf{A}_o^F$, $\mathbf{G}_c^F$ and $\mathbf{G}_o^F$, one can write down the corresponding Polyakov or (augmented) Nambu sigma models, i.e.,

$$S_P^{tot,F}[X] = \frac{1}{2}\int d^{p+1}\sigma\{\Psi^\dagger \mathbf{A}_c^F \Psi\} = \frac{1}{2}\int d^{p+1}\sigma\{\Psi^\dagger \mathbf{A}_o^F \Psi\}, \tag{6.19}$$
$$S_{NSM}^F[X,\eta,\eta'] = -\int d^{p+1}\sigma\{\Upsilon^\dagger \mathbf{A}_c^{F\,-1}\Upsilon + \Upsilon^\dagger\Psi\} = -\int d^{p+1}\sigma\{\Upsilon^\dagger \mathbf{A}_c^{F\,-1}\Upsilon + \Upsilon^\dagger\Psi\}, \tag{6.20}$$
$$\begin{aligned} H_P^{tot,F}[X,P] = H_{NSM}^F[X,P] &= -\frac{1}{2}\int d^p\sigma \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}^T \mathbf{G_c^F} \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix} \\ &= -\frac{1}{2}\int d^p\sigma \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}^T \mathbf{G_o^F} \begin{pmatrix} iP \\ \widetilde{\partial X} \end{pmatrix}. \end{aligned} \tag{6.21}$$

7 Seiberg-Witten map

In the previous section, we have developed the correspondence between closed and open fields, including their respective fluctuations. However, they are not related simply by open-closed relations. Instead, the discussion brings new vector bundle isomorphisms M and N, defined by (6.12), (6.13), respectively, into the picture. The determinant of the left-hand side of (6.16) seems to be a likely candidate to appear in the "commutative" membrane DBI action, whereas the determinant on the right-hand side of (6.16) seems to contain as a factor a likely candidate to appear in its "noncommutative" counterpart.

This observation suggests that we should look for a change of coordinates on the manifold M, the Jacobian of which could cancel the $\det^2(N)$ factor coming under the determinant from the right-hand side of (6.16). The resulting diffeomorphism will be called a Seiberg-Witten map in analogy to the string $p = 1$ case. We use a direct generalization of the semi-classical construction used first in [7]. The most intriguing part will be to define carefully a substitute for a determinant of a Nambu-Poisson $(p+1)$-vector.

In the following, let Π be a Nambu-Poisson $(p+1)$-vector (see appendix A) on M. We can examine the F-gauged tensor $\Pi' = (1 - \Pi F^T)^{-1}\Pi$.[5] We will now show that for $p > 1$ this tensor is always a Nambu-Poisson $(p+1)$-vector, whereas for $p = 1$ it is a Poisson bivector if F is closed.

First, for $p > 1$, one can see that

$$\Pi' = \left(1 - \frac{1}{p+1}\langle \Pi, F\rangle\right)^{-1} \Pi, \tag{7.1}$$

where $\langle \Pi, F\rangle = \Pi^{iJ}F_{iJ} \equiv Tr(\Pi F^T)$. For this, one has to prove that

$$\Pi = (1 - \Pi F^T)\left(1 - \frac{1}{p+1}\langle \Pi, F\rangle\right)^{-1} \Pi. \tag{7.2}$$

This can easily be checked in coordinates $(x^1, \ldots, x^n)$ in which (A.7) holds, and hence, for Π with components $\Pi^{iJ} = \epsilon^{iJ}$. Now, using (7.1) and lemma A.2, we see that Π' is again a Nambu-Poisson tensor.

To include the $p = 1$ case: for $p \geq 1$, and F closed, we can use the fact that $G_{\Pi'} = e_{-F}G_{\Pi}$, where G_{Π} and $G_{\Pi'}$ are graphs of the maps $\Pi^\sharp$ and $\Pi'^\sharp$, respectively (see lemma A.1). This is easily verified using (6.11). It can be seen that the Dorfman bracket (A.1) satisfies $[e_{-F}(V+\xi), e_{-F}(W+\eta)]_D = e_{-F}[V+\xi, W+\eta]_D$, whenever F is closed. But this implies that $G_{\Pi'}$ is closed under the Dorfman bracket, which is according to A.1 equivalent to the Nambu-Poisson fundamental identity. On the other hand, note that for $p > 1$, F' is not necessarily a $(p+1)$-form.

Next, see that the scalar function in front of Π in (7.1) is related to the determinant of the vector bundle isomorphism $1 - \Pi F^T$. For $p > 1$, any Nambu-Poisson tensor and any $(p+1)$-form F, its holds

$$\det\left(1 - \Pi F^T\right) = \left(1 - \frac{1}{p+1}\langle \Pi, F\rangle\right)^{p+1}. \tag{7.3}$$

To prove this identity, note that both sides are scalar functions. We may therefore use any local coordinates on M. Again, use those in which (*A.7*) holds. The rest of the proof is straightforward.

[5] We assume that $1 - \Pi F^T$ is invertible. In a more formal approach we also could treat Π' as a formal power series in Π.

Further on, assume that F is closed, that is at least locally $F = dA$ for a p-form A. Define a 1-parametric family of tensors $\Pi'_t := (1 - t\Pi F^T)^{-1}\Pi$, cf. Footnote 5. This is obviously chosen so that $\Pi'_0 = \Pi$ and $\Pi'_1 = \Pi'$. Differentiation of Π'_t with respect to t gives:

$$\partial_t \Pi'_t = \Pi'_t F^T \Pi'_t. \tag{7.4}$$

This equation can be rewritten as

$$\partial_t \Pi'_t = -\mathcal{L}_{A_t^\sharp} \Pi'_t, \tag{7.5}$$

where the time-dependent vector field $A_t^\sharp$ is defined as $A_t^\sharp = \Pi'^\sharp_t(A)$. To see this, note that Π'_t is, using similar arguments as above, a Nambu-Poisson tensor. Then recall the property (A.3), and choose $\xi = A$ and $\eta = dy^J$. Contracting the resulting vector field equality with dy^i gives exactly $\mathcal{L}_{A_t^\sharp}\Pi'_t = -\Pi'_t F^T \Pi'_t$. Equation (7.5) states precisely that the flow ϕ_t corresponding to $A_t^\sharp$, together with condition $\Pi'_0 = \Pi$, maps Π_t to Π, that is,

$$\phi_t^*(\Pi'_t) = \Pi. \tag{7.6}$$

We have thus found the map $\rho_A \equiv \phi_1$, which gives $\rho_A^*(\Pi') = \Pi$. This is the $p \geq 1$ analogue of the well known semiclassical Seiberg-Witten map. Obviously, it preserves the singular foliation defined by Π. We emphasize the dependence of this map on the p-form A by an explicit addition of the subscript A.

Denote $J^i{}_k = \frac{\partial \widehat{X}^i}{\partial x^k}$, with $\widehat{X}^i := \rho_A^*(x^i)$ being *covariant* coordinates. We have

$$\rho_A^*(\Pi'^{j_1,\dots j_{p+1}}) = J^{j_1}_{i_1} \dots J^{j_{p+1}}_{i_{p+1}} \Pi^{i_1\dots i_{p+1}}. \tag{7.7}$$

Further, denote by $|J|$ the determinant of J^i_k in some (arbitrarily) chosen local coordinates $(x^1,\dots,x^n)$ on M. One can choose, for instance, the special coordinates $(\widetilde{x}^i,\dots\widetilde{x}^n)$ on M in which (A.7) holds. We will use the notation $|\widetilde{J}|$ for the determinant of the matrix $\widetilde{J}^i_k = \frac{\partial \widetilde{x}^i(\rho_A(x))}{\partial \widetilde{x}^k}$. From now, for any function φ (e.g., a matrix component, determinant, etc.), the symbol $\widehat{\varphi}$ will always denote the function defined as $\widehat{\varphi}(x) \equiv \rho_A^*(\varphi)(x) = \varphi(\rho_A(x))$. Recall now the definition (A.8) of the density $|\Pi(x)|$.[6] By definition of $|J|$, we then have

$$|J| = |\widetilde{J}| \frac{|\widehat{\Pi}(x)|^{\frac{1}{p+1}}}{|\Pi(x)|^{\frac{1}{p+1}}} \tag{7.8}$$

The Jacobian $|\widetilde{J}|$ can easily be calculated using (7.1) and (7.7). Indeed, the equation (7.7) can be, in $(\widetilde{x})$ coordinates, rewritten as

$$\left(1 - \frac{1}{p+1}\langle \widehat{\Pi}, \widehat{F}\rangle\right)^{-1} \epsilon^{j_1\dots j_{p+1}} = \epsilon^{j_1\dots j_{p+1}} \widetilde{J}^1_{i_1} \dots \widetilde{J}^{p+1}_{i_{p+1}} \epsilon^{i_1\dots i_{p+1}}.$$

To justify this, note that Seiberg-Witten map acts nontrivially only in the directions of the first $(p+1)$-coordinates. The Jacobi matrix $\widetilde{J}$ of ρ_A in $(\widetilde{x})$ coordinates is thus a block upper

[6] For $p = 1$, one can (around every regular point of the characteristic distribution) define $|\Pi(x)|$ to be the Jacobian of the transformation to the Darboux-Weinstein coordinates. This gives a good definition even if Π is degenerate.

triangular with identity matrix in the bottom right block. Moreover, the determinant of $\widetilde{J}$ is then equal to the determinant of the top left block. We can divide both sides with $\epsilon^{j_1 \dots j_{p+1}}$. We thus remain with the equation

$$\left(1 - \frac{1}{p+1}\langle \widehat{\Pi}, \widehat{F} \rangle\right)^{-1} = \widetilde{J}^1_{i_1} \dots \widetilde{J}^{p+1}_{i_{p+1}} \epsilon^{i_1 \dots i_{p+1}} = |\widetilde{J}|.$$

Putting this back into (7.8), we obtain the useful relation

$$|J|^{p+1} = \left(1 - \frac{1}{p+1}\langle \widehat{\Pi}, \widehat{F} \rangle\right)^{-(p+1)} \frac{|\widehat{\Pi}(x)|}{|\Pi(x)|}, \tag{7.9}$$

or using (7.3)[7]

$$|J|^{p+1} = \det\left(1 - \widehat{\Pi}\widehat{F}^T\right)^{-1} \frac{|\widehat{\Pi}(x)|}{|\Pi(x)|}. \tag{7.10}$$

Note that this expression does not depend on the choice of the Darboux coordinates in which the densities $|\Pi(x)|$ are calculated. We discuss this subtlety in the appendix A under (A.9). We see that $|\Pi(x)|$ itself transforms as in (A.10). Fortunately, the determinant of the block M in (A.9) does not depend on the coordinates $(\widetilde{x}^1, \dots \widetilde{x}^{p+1})$. Since these are the only coordinates changed by the Seiberg-Witten map, we get $(\det M)(x) = (\det M)(\rho_A(x))$. In other words, these determinants cancel out in the fraction $|\widehat{\Pi}(x)|/|\Pi(x)|$, as expected.

The following observation is in order: the Nambu-Poisson tensor Π_t does not depend on the choice of the gauge p-potential A. As already mentioned, the Nambu-Poisson map ρ_A does: an infinitesimal gauge transformation $\delta A = d\lambda$ — with a $(p-1)$-form gauge transformation parameter λ — induces a change in the flow, which is generated by the vector field $X_{[\lambda,A]} = \Pi^{iJ} d\Lambda_J \partial_i$, where

$$\Lambda = \sum_{k=0}^{\infty} \frac{(\mathcal{L}_{A_t^\sharp} + \partial_t)^k(\lambda)}{(k+1)!}\Big|_{t=0}, \tag{7.11}$$

is the semiclassically noncommutative $(p-1)$-form gauge parameter. This is the p-brane analog of the exact Seiberg-Witten map for the gauge transformation parameter. It is straightforwardly obtained by application of the BCH formula to $\rho^*_{A+d\lambda}(\rho^*_A)^{-1}$. Finally, in analogy with the $p=1$ case, we define the (components of the) semiclassically noncommutative field strength to be

$$\widehat{F}'_{i_1,\dots,i_{p+1}} = \rho^*_A F'_{i_1,\dots,i_{p+1}}, \tag{7.12}$$

i.e., the components of F' evaluated in the covariant coordinates. Infinitesimally, components of $\hat{F}$ transform as

$$\delta\widehat{F}' = \Pi^{iJ} d\Lambda_J \partial_i \hat{F}', \tag{7.13}$$

which justifies the adjectives "semiclassically noncommutative".

[7]For $p=1$, one can derive this relation by calculating $|\widetilde{J}|$ in Darboux-Weinstein coordinates directly from (7.7) and the definition of Π', and then use (7.8).

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8 Nambu gauge theory; equivalence of commutative and semiclassically noncommutative DBI action

Here we consider a system of multiple open M2 branes ending on an M5 brane. We would like to describe this system by an effective action that is exact, for slowly varying fields, to all orders in the coupling constant. Since we focus only on the bosonic part of this action, we do not need to restrict ourselves to the values $p = 2$ and $p' = 5$ and our construction is valid for arbitrary values of p and p' such that $p \leq p'$. Our goal is thus the construction of an effective action for a p'-brane with open p-branes ending on it while being submerged in a C_{p+1}-background. The construction is based on two guiding principles: firstly, this effective action should have dual descriptions similar to the commutative and non-commutative ones of the D-brane and open strings[8] and secondly, it should feature expressions that also appear in the p-brane action (6.19).

Denote the p'-brane submanifold as N. We shall now clarify the geometry underlying the following discussion. Originally, $g, \widetilde{g}, C$ were assumed to be the closed membrane backgrounds in the ambient background manifold M. Hereafter, we denote by the same characters their pullbacks to the p'-brane N. This makes sense since all of them are covariant tensor fields on M. Little subtlety comes with the Nambu-Poisson tensor Π. We have basically two options. First, we would like to restrict some Nambu-Poisson tensor in M to the p'-brane. This in fact requires N to be a Nambu-Poisson submanifold of M. The latter option is to *choose* the Nambu-Poisson tensor Π on N after we restrict the other backgrounds to N. The open membrane variables $G, \widetilde{G}, \Phi$, calculated using the membrane open-closed relations (3.17)–(3.20), are assumed to be calculated entirely on N, using the pullbacks of closed variables. Finally, the field F is assumed to be a $(p+1)$-form defined and having components only in N. All the discussion related to Seiberg-Witten map in the previous section is assumed to take place on the submanifold N.

The open-closed membrane relations (6.16) immediately imply

$$\det[g + (C+F)\tilde{g}^{-1}(C+F)^T] = \det{}^2[1 - F\Pi^T] \cdot \det[G + (\Phi + F')\tilde{G}^{-1}(\Phi + F')^T], \quad (8.1)$$

where $F' = (I - F\Pi^T)^{-1}F$. Obviously, in order get a sensible action we have to form an integral density, which can be integrated over the world volume of the larger p'-brane. And, in order to obtain a noncommutative action from the right hand side of (8.1), we have to apply the Seiberg-Witten map ρ_A^* to it. It would be tempting to take the square root of the identity (8.1) to construct the action. But, recall (7.10) and notice the factor $\det{}^{-(p+1)}[1 - F\Pi^T]$ appearing in it upon the application of the Seiberg-Witten map. Hence, not the square root but the $2(p+1)$-th root of (8.1) is the most natural choice to enter the effective action that we look for. As we already said, the Lagrangian density must be an integral density, and therefore we need to multiply that piece of the action by a proper power of the determinant of the pullback of the target space metric. These considerations

[8] Actually, our exposition so far closely followed our previous work [71], where the role of generalized geometry was emphasized.

fix the action essentially uniquely and we postulate

$$S_{p\text{-DBI}} = -\int d^{p'+1}x \,\frac{1}{g_m}\, \det{}^{\frac{p}{2(p+1)}}(g)\cdot \det{}^{\frac{1}{2(p+1)}}\left[g+(C+F)\tilde{g}^{-1}(C+F)^T\right], \tag{8.2}$$

where g_m is a "closed membrane" coupling constant. The integration is over the p'-brane and the fields g, $\tilde{g}$, and C in this expression are the pull-backs of the corresponding background target space fields to this p'-brane. Asking for

$$\begin{aligned}\frac{1}{g_m}&\det{}^{\frac{p}{2(p+1)}} g\cdot \det{}^{\frac{1}{2(p+1)}}\left[g+(C+F)\tilde{g}^{-1}(C+F)^T\right]\\ &=\frac{1}{G_m}\det{}^{\frac{p}{2(p+1)}}(G)\det{}^{\frac{1}{(p+1)}}\left[1-\Pi F^T\right]\cdot\det{}^{\frac{1}{2(p+1)}}\left[G+(\Phi+F')\tilde{G}^{-1}(\Phi+F')^T\right],\end{aligned} \tag{8.3}$$

it follows from (8.1) that the closed and open coupling constants g_m and G_m must be related as

$$G_m = g_m\,(\det G/\det g)^{\frac{p}{2(p+1)}}\,. \tag{8.4}$$

As desired, the action (8.2) is exactly equal to its "noncommutative" dual

$$S_{p\text{-NCDBI}} = -\int d^{p'+1}x\,\frac{1}{\widehat{G}_m}\,\frac{|\widehat{\Pi}|^{\frac{1}{p+1}}}{|\Pi|^{\frac{1}{p+1}}}\,\det{}^{\frac{p}{2(p+1)}}\widehat{G}\cdot\det{}^{\frac{1}{2(p+1)}}\left[\widehat{G}+(\widehat{\Phi}+\widehat{F}')\widehat{\tilde{G}}^{-1}(\widehat{\Phi}+\widehat{F}')^T\right], \tag{8.5}$$

where as before $\widehat{\ }$ denotes objects evaluated at covariant coordinates[9] and $\widehat{F}'$ is the Nambu (NC) field strength (7.12). This follows from integrating of (8.3) followed by the change of integration variables on its right hand side according to the Seiberg-Witten map.

The factor involving the quotient of $|\widehat{\Pi}|$ and $|\Pi|$ vanishes for constant $|\Pi|$, but it is essential for the gauge invariance of (8.5) in all other cases.

Let us give two alternative, but equivalent, expressions for the action (8.2), which might turn out to be useful when looking for supersymmetric generalizations. The first one is obvious:

$$S_{p\text{-DBI}} = -\int d^{p'+1}x\,\frac{1}{g_m}\,\det{}^{\frac{1}{2}}(g)\cdot\det{}^{\frac{1}{2(p+1)}}\left[1+g^{-1}(C+F)\tilde{g}^{-1}(C+F)^T\right]. \tag{8.6}$$

A very similar expression can be found using (3.25)

$$S_{p\text{-DBI}} = -\int d^{p'+1}x\,\frac{1}{g_m}\,\det{}^{\frac{1}{2}}(g)\cdot\det{}^{\frac{1}{2(p+1)}}\left[1+\tilde{g}^{-1}(C+F)^T g^{-1}(C+F)\right]. \tag{8.7}$$

For the second one, let us note that $\det\tilde{g} = \det^{\binom{p'}{p-1}} g$, in the case of factorizable $\tilde{g}$. Hence, in this case:

$$S_{p\text{-DBI}} = -\int d^{p'+1}x\,\frac{1}{g_m}\,\det{}^{\frac{p-\binom{p'}{p-1}}{2(p+1)}} g\cdot\det{}^{\frac{1}{2(p+1)}}\begin{pmatrix} g & (C+F)\\ -(C+F)^T & \tilde{g}\end{pmatrix}. \tag{8.8}$$

[9] Let us emphasize that this is not a coordinate transformation of a tensor. We just evaluate the component functions in different coordinates.

Let us note that in the case of a D-brane, i.e., $p = 1$, we get indeed the DBI D-brane action. In the other extreme case, $p = p'$, we get[10]

$$S_M = -\int d^{p+1}x \frac{1}{g_m} \det{}^{\frac{1}{2(p+1)}} \begin{pmatrix} g & (C+F) \\ -(C+F)^T & \tilde{g} \end{pmatrix} . \tag{8.9}$$

Now we can compare our action, e.g, to the DBI part of the M5-brane action in equation (2.9) of [38, 39]. Their action is, up to conventions,

$$S' = -\int d^6x \sqrt{\det g} \sqrt{1 + \frac{1}{3}\mathrm{tr}k - \frac{1}{6}\mathrm{tr}k^2 + \frac{1}{18}(\mathrm{tr}\, k)^2} , \tag{8.10}$$

where $k^i_j = (dA+C)^{ikl}(dA+C)_{jkl}$ is the modified field strength. (See also [72], for an early proposal with a similar index structure.) The form of the polynomial in k in the action has been determined by lengthy computation based on κ-symmetry and the requirement of non-linear self-duality, the self-duality relations being consistently decoupled from the background. More precisely, in [38, 39], it is shown that consistency of the non-linear self-duality is restrictive enough that demanding κ-symmetry gives its explicit form, which can be obtained without a priori specifying the form of the polynomial in the action. At the same time the projector specifying the κ-symmetry and the form of the polynomial are determined.

To our surprise, we found that this action S' can be interpreted as a low-energy (second order in k) approximation of our p-DBI action (8.2). Indeed,for $p = 2$ and $p' = 5$ we have $d^{p'+1}x = d^6x$, $\frac{1}{2(p+1)} = \frac{1}{6}$ and

$$det^{\frac{1}{6}}(1 + k) = \sqrt{1 + \frac{1}{3}\mathrm{tr}k - \frac{1}{6}\mathrm{tr}k^2 + \frac{1}{18}(\mathrm{tr}\, k)^2 + \dots} .$$

The fact that two very different approaches (one based on non-linear self-duality and κ-symmetry, the other on commutative/non-commutative duality) give rise to the same action in the low energy limit is very encouraging and seems to indicate that our proposal can indeed be extended to a full supersymmetric action.

Finally, let us mention that noncommutative structures in the context of the M5 brane have previously been discussed, for example, in [73] and [74]. However, the type of non-commutativity discussed in these earlier papers is the well-known deformation of the commutative point-wise multiplication along a (constant) Poisson tensor that already appeared in the $p = 1$ string theory case. This is very different from the notion of noncommutativity that we argue to be pertinent for $p > 1$ and in particular for the $p = 2$ case relevant for the M5 brane: for $p > 1$, we do not deform the commutative product — our "noncommutativity" has rather to be understood in the Nambu-Poisson sense as explained in detail above, cf. the remark at the end of the previous section.

9 Background independent gauge

For $p = 1$, assuming that the pullback of the background 2-form C to the p'-brane N is non-degenerate and closed (that is symplectic), one can choose the bivector Π to be

[10] The notation S_M will be justified later.

the inverse of C (that is a Poisson bivector corresponding to the symplectic structure C). Solving the open-closed relations then gives

$$G = -Cg^{-1}C, \; \Phi = -C. \tag{9.1}$$

This is known as the background independent gauge [5]. Our aim is to generalize this construction for $p \geq 1$, even giving milder assumptions on C for $p = 1$.

Let us start on the level of linear algebra first. Assume that V is a finite-dimensional vector space. Let g be an inner product on V, and $C \in \Lambda^2 V^*$ a 2-form. Let $P : V \to V$ denote a projector orthogonal with respect to g, such that

$$\ker(C) = \ker(P),$$

where C is viewed as a map $C : V \to V^*$. Then there exists a unique bivector $\Pi \in \Lambda^2 V$, satisfying

$$\Pi C = P\,, \; P\Pi = \Pi. \tag{9.2}$$

The reader can find the proof of this statement in proposition B.1 of appendix B.

Recall that open-closed relations for $p = 1$ have the form

$$\frac{1}{g + C} = \frac{1}{G + \Phi} + \Pi. \tag{9.3}$$

This equality can be rewritten as

$$G + \Phi = (1 - (g + C)\Pi)^{-1}(g + C). \tag{9.4}$$

Using (9.2), one gets

$$G + \Phi = P'^T gP' - Cg^{-1}C - C,$$

where $P' = 1 - P$. From this we can read of the symmetric and skew-symmetric part to get

$$G = P'^T gP' - Cg^{-1}C\,, \; \Phi = -C. \tag{9.5}$$

We can view this as a generalization of (9.1), not assuming a non-degenerate C. See that G is again a positive definite metric, and $G + \Phi$ is thus invertible. Note that we are now on the level of a single vector space V, not discussing any global properties of Π yet.

We would like to generalize this procedure to $p \geq 1$ case. Our goal is to find a suitable choice for Π, such that $\Phi = -C$. Assume that $C : \Lambda^p V \to V^*$ is a linear map, g is an inner product on V, and $\widetilde{g}$ is an inner product on $\Lambda^p V$. The key is to keep in mind the open-closed relations (5.3). We see that by defining

$$\mathcal{G} = \begin{pmatrix} g & 0 \\ 0 & \widetilde{g} \end{pmatrix}, \; \mathcal{B} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix},$$

we get an inner product $\mathcal{G}$ on $W \equiv V \oplus \Lambda^p V$, and a bilinear skew-symmetric form $\mathcal{B} \in \Lambda^2 W^*$.

The situation is thus analogous to the previous one, if we replace V by W, the metric g by $\mathcal{G}$, and the 2-form C by $\mathcal{B}$. If we define $\mathcal{P}$ to be an orthogonal projector with respect to $\mathcal{G}$ with $\ker(\mathcal{P}) = \ker(\mathcal{B})$, we may again apply proposition B.1 to see that there exists a unique $\Theta \in \Lambda^2 W$, such that

$$\Theta\mathcal{B} = \mathcal{P}\,, \mathcal{P}\Theta = \Theta. \tag{9.6}$$

Now we can solve the open-closed relations (5.3) for this choice of Θ, using the same calculation as we did in order to obtain (9.5). One gets

$$\mathcal{H} = \mathcal{P}'^T\mathcal{G}\mathcal{P}' - \mathcal{B}\mathcal{G}^{-1}\mathcal{B}\,,\ \Xi = -\mathcal{B}, \tag{9.7}$$

where $\mathcal{P}' = 1 - \mathcal{P}$. Exploring what $\mathcal{B}$ and Ξ are, leads to $\Phi = -C$, as intended. However, we do not know whether $\mathcal{H}$ and Θ obtained by this procedure are of the suitable form, that is whether $\mathcal{H}$ is block-diagonal and Θ block-off-diagonal. This can be easily proved by examining the projector $\mathcal{P}$. Clearly, one has

$$\ker\mathcal{B} = \ker C^T \oplus \ker C \subseteq V \oplus \Lambda^p V.$$

Therefore we have that $\mathrm{Im}(\mathcal{P}) = \ker\mathcal{B}^{\perp} = (\ker C^T)^{\perp(g)} \oplus (\ker C)^{\perp(\widetilde{g})}$. This proves that in a block form, we have

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & \widetilde{P} \end{pmatrix},$$

where $P : V \to V$ is an orthogonal projector with respect to g, and $\widetilde{P} : \Lambda^p V \to \Lambda^p V$ is an orthogonal projector with respect to $\widetilde{g}$. This and the relation (9.7) imply that $\mathcal{H}$ is block-diagonal. The second equality in (9.6) then proves that Θ is block-off-diagonal, that is

$$\Theta = \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix},$$

where $\Pi : \Lambda^p V^* \to V$. We can now simply extract all the relations from (9.6). The equality $\Theta\mathcal{B} = \mathcal{P}$ gives

$$\begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix}\begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & \widetilde{P} \end{pmatrix},$$

which translates into

$$\Pi C^T = -P\,,\ \Pi^T C = -\widetilde{P}. \tag{9.8}$$

Rewriting the equation $\mathcal{B}\mathcal{P} = \mathcal{B}$, we get

$$\begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & \widetilde{P} \end{pmatrix} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix},$$

which translates into

$$C\widetilde{P} = C\,,\ C^T P = C^T. \tag{9.9}$$

Also see that $\ker(\widetilde{P}) = \ker(C)$, and $\ker(P) = \ker(C^T)$. The equality $\mathcal{P}\Theta = \Theta$ gives

$$\begin{pmatrix} P & 0 \\ 0 & \widetilde{P} \end{pmatrix}\begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix},$$

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and thus

$$P\Pi = \Pi\,, \;\; \widetilde{P}\Pi^T = \Pi^T. \tag{9.10}$$

Finally, we may examine (9.7) to find

$$G = P'^T g P' + C\widetilde{g}^{-1}C^T\,, \;\; \widetilde{G} = \widetilde{P}'^T\widetilde{g}\widetilde{P}' + C^T g^{-1} C\,, \;\; \Phi = -C. \tag{9.11}$$

We have thus shown that, corresponding to the orthogonal projectors P and $\widetilde{P}$ and the linear map $C : \Lambda^p V \to V^*$, there exists a unique linear map $\Pi : \Lambda^p V^* \to V$, such that (9.8) and (9.10) hold. Plugging this Π into open-closed relations (5.3) gives (9.11).

To use this for our purposes, we have to impose conditions on C to ensure that Π is a Nambu-Poisson tensor.

For $p > 1$, first observe that the linear map $\Pi : \Lambda^p V^* \to V$ induced (at a chosen point on M) by a Nambu-Poisson tensor has rank either 0 or $p+1$. Since Π always has the same rank as C, we get the first assumption on the linear map C.

There will always arise problems with the smoothness of Π at points $x \in N$, where $C(x) = 0$. If this set has measure zero, we can change the area of integration in DBI action from N to an open submanifold N', where $C(x) \neq 0$. If not, we cannot go to the background-independent gauge. Let us hereafter assume that $C(x) \neq 0$ for all $x \in N$, and therefore that $\mathrm{rank}(C) = p+1$.

Now assume that the linear map C is induced by a $(p+1)$-form $C \in \Lambda^{p+1}V^*$. Note that in this case, we always have the estimate $\mathrm{rank}(C) \geq p+1$.

Let $D \subseteq V$ denote the non-degenerate subspace of C^T orthogonal (with respect to g) to its kernel, that is $D = \ker(C^T)^\perp$. Assumption on the rank of C thus means that $\dim(D) = p+1$. From the skew-symmetry of C, we have that $C \in \Lambda^{p+1}D^*$. It is thus a top-level form on D. Choose now an orthonormal basis $(e_1, \ldots, e_{p+1})$ of D. We see that

$$C = \lambda \cdot e^1 \wedge \ldots \wedge e^{p+1}, \tag{9.12}$$

where $\lambda \neq 0$. Now, choosing an arbitrary complementary basis $(f_1, \ldots, f_{p'-p})$ of $\ker(C^T) \equiv D^\perp$, one can find counterexamples to the assumption that, for a general $\widetilde{g}$, the map Π is a $(p+1)$-vector (although it has a correct rank). We thus have to add the second assumption: $\widetilde{g}$ has to be of the special skew-symmetrized tensor product form (3.5).

In this case we find that $\Lambda^p D$ is spanned by orthonormal basis of the form $e_1 \wedge \ldots \wedge \hat{e}_r \wedge \ldots \wedge e_{p+1}$. This allows us to write Π explicitly as

$$\Pi = -\frac{1}{\lambda} \cdot e_1 \wedge \ldots \wedge e_{p+1}. \tag{9.13}$$

It is easy to show that such a Π indeed satisfies (9.8) and (9.10), and since such a Π is unique, this is the one. We can thus conclude that for $\mathrm{rank}(C) = p+1$, and $\widetilde{g}$ in the form (3.5), Π is a $(p+1)$-vector, more precisely $\Pi \in \Lambda^{p+1}D$.

We now turn our attention to global properties. If we assume that $C(x) \neq 0$ on the p'-brane, we can define the subspace D at every point, defining a smooth subbundle (it is an orthogonal complement to the kernel of constant rank vector bundle morphism C^T). Around any point, we can choose a local orthonormal frame $(e_1, \ldots, e_{p+1})$, forming a local basis for the sections of D. The expression (9.13) proves that Π is a smooth $(p+1)$-vector on the p'-brane, since $\frac{1}{\lambda}$ is a smooth function.

Finally, we have to decide under which conditions Π forms a Nambu-Poisson tensor. In the view of lemma A.3, we see that the sufficient and necessary condition is that the subbundle D defines an integrable distribution in N. This distribution has to be regular, and thus, this condition is equivalent to the involutivity of D under vector field commutator: $[D, D] \subseteq D$.

One can find a simple equivalent criterion for C to define an integrable distribution D. In order to do so, assume now that $(e_1, \ldots, e_{p+1}, f_1, \ldots f_{p'-p})$ is a positively oriented orthonormal local frame for N, such that $(e_1, \ldots, e_{p+1})$ is a local orthonormal frame for D. The metric volume form Ω_g is then by definition

$$\Omega_g = e^1 \wedge \ldots \wedge e^{p+1} \wedge f^1 \wedge \ldots \wedge f^{p'-p}.$$

Having a volume form, one can form the Hodge dual of C. Using (9.12) we get

$$*C = \lambda \cdot f^1 \wedge \ldots \wedge f^{p'-p}.$$

We see that $D = \ker(*C)^T$, $(*C)^T : TN \to \Lambda^{p'-p-1}T^*N$. But forms with integrable kernel distribution have their own name, they are called integrable forms, see appendix B for the definition and basic properties. We can conclude that Π is a Nambu-Poisson $(p+1)$-vector if and only if $*C$ is an integrable everywhere non-vanishing $(p'-p)$-form on N. Note that the Hodge star is defined with respect to the induced metric on N.

There exists a nice sufficient integrability condition: if C is a $(p+1)$-form of rank $p+1$, such that $\delta C = 0$, then $*C$ is integrable. By δ we denote the codifferential defined using the Hodge duality. Note that $\delta C = 0$ are the non-homogeneous charge free Maxwell equations for the field strength C. Also, note that in the whole discussion, we do not need the integrability of the distribution $D^\perp$. Since C is already a non-vanishing $(p+1)$-form of rank $p+1$, the sufficient condition for integrability of $D^\perp$ is $dC = 0$. Interestingly, both D and $D^\perp$ are integrable regular distributions if C is a $(p+1)$-form of rank $p+1$, satisfying the Maxwell equations $dC = 0$, $\delta C = 0$.

For $p = 1$, the discussion is very similar, except that the rank of C can be any nonzero even integer not exceeding n. This adds another condition on dC. In particular, the necessary and sufficient condition on C to define a Poisson tensor Π is the integrability of the regular smooth distribution D, and a condition $dC|_{\Gamma(D)} = 0$.

10 Non-commutative directions, double scaling limit

By the construction of the preceding section, we have the decompositions

$$TM = D \oplus D^\perp, \; \Lambda^p TM = \widetilde{D} \oplus \widetilde{D}^\perp,$$

where $\widetilde{D} = \Lambda^p D$. We say that tangent vectors contained in D point in "non-commutative" directions. Because D is integrable, around each point there are coordinates such that D is spanned by coordinate tangent vectors corresponding to first $p+1$ of these coordinates.

These local coordinates are accordingly called "non-commutative" coordinates. This terminology comes from the fact that for $p = 1$, we have $\{x^i, x^j\} = \Pi^{ij}$. The right-hand side is non-vanishing when both x^i and x^j correspond to D. This gives non-vanishing quantum-mechanical commutator of these coordinates.

We can thus write all involved quantities in the block matrix form corresponding to this decomposition. From the orthogonality of respective subspaces, the matrices of g and $\widetilde{g}$ will be block diagonal:

$$g = \begin{pmatrix} g_\bullet & 0 \\ 0 & g_\circ \end{pmatrix}, \; \widetilde{g} = \begin{pmatrix} \widetilde{\mathfrak{g}}_\bullet & 0 \\ 0 & \widetilde{g}_\circ \end{pmatrix},$$

where $g_\bullet$ is a positive definite fibrewise metric on D, $\mathfrak{g}_\circ$ is a positive definite fibrewise metric on $D^\perp$ and $\widetilde{g}_\bullet$ and $\widetilde{g}_\circ$ are positive definite fibrewise metrics on $\widetilde{D}$ and $\widetilde{D}^\perp$, respectively. In the same fashion we obtain

$$C = \begin{pmatrix} C_\bullet & 0 \\ 0 & 0 \end{pmatrix}, \; \Pi = \begin{pmatrix} \Pi_\bullet & 0 \\ 0 & 0 \end{pmatrix}, \; F = \begin{pmatrix} F_\bullet & F_{\mathbf{I}} \\ F_{\mathbf{II}} & F_\circ \end{pmatrix}.$$

Examine how the F-gauged tensor Π' looks like in this block form. We have

$$1 - F^T \Pi = \begin{pmatrix} 1 - F_\bullet^T \Pi_\bullet & 0 \\ -F_{\mathbf{I}}^T \Pi_\bullet & 1 \end{pmatrix}.$$

Hence

$$\Pi' \equiv \Pi (1 - F^T \Pi)^{-1} = \begin{pmatrix} \Pi_\bullet (1 - F_\bullet^T \Pi_\bullet)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Denote $\Pi'_\bullet = \Pi_\bullet (1 - F_\bullet^T \Pi_\bullet)^{-1}$. We also have $\Pi'_\bullet = (1 - \Pi_\bullet F_\bullet^T)^{-1} \Pi_\bullet$. Also, note that in this formalism P and $\widetilde{P}$ are simply given as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; \widetilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the defining equations of Π can be written as

$$\Pi_\bullet C_\bullet^T = -1 \,, \; \Pi_\bullet^T C_\bullet = -1. \tag{10.1}$$

Having this in hand, recall that for $p = 1$, the background independent gauge could be obtained in a completely different way. It was obtained by Seiberg and Witten in [5] as a following limit of the relation (4.11). Reintroducing the Regge slope α' into description, the relation between closed variables g, C and Nambu fields G_N, Π_N is explicitly

$$G_N = g - (2\pi\alpha')^2 C g^{-1} C^T, \; \frac{1}{2\pi\alpha'} \Pi_N = -(2\pi\alpha') g^{-1} C \big(g - (2\pi\alpha')^2 C g^{-1} C \big)^{-1}.$$

Now one would like to do the zero slope limit $\alpha' \to 0$ in a way such that G_N and Π_N remain finite. This clearly requires the simultaneous scaling of the metric g. Scaling the g as a whole will not work, since the resulting G_N will not be a metric. The correct answer is given by scaling the non-commutative part $g_\bullet$ and commutative part $g_\circ$ of the metric g differently. The resulting maps G_N and Π_N also split accordingly as

$$G_{N\bullet} = g_\bullet - (2\pi\alpha')^2 C_\bullet g_\bullet^{-1} C_\bullet^T,\ G_{N\circ} = g_\circ,$$
$$\frac{1}{2\pi\alpha'}\Pi_{N\bullet} = -(2\pi\alpha')g_\bullet^{-1}C_\bullet(g_\bullet - (2\pi\alpha')^2 C_\bullet g_\bullet^{-1} C_\bullet)^{-1}.$$

Now, scaling $g_\bullet \propto \epsilon$, $g_\circ \propto 1$, $\alpha' \propto \epsilon^{\frac{1}{2}}$ as $\epsilon \mapsto 0$ gives in this limit

$$G_{N\bullet} = -C_\bullet g_\bullet^{-1} C_\bullet^T, \qquad G_{N\circ} = g_\circ,$$
$$\Pi_{N\bullet} = C_\bullet^{-1}.$$

Replacing Π_N by Π and G_N by G is exactly the background independent gauge. This double scaling limit was then used to determine which terms should be kept in the expansion of the DBI action. We would like to find an analogue of this in our $p > 1$ case.[11] We immediately see that first naive answer would be wrong. One of the relations is

$$G_{N\bullet} = g_\bullet + C_\bullet \widetilde{g}_\bullet^{-1} C_\bullet^T.$$

Note that $\widetilde{g}_\bullet$ is again a skew-symmetrized p-fold tensor product of $g_\bullet$. This suggests that if $g_\bullet \propto \epsilon$, then $\widetilde{g}_\bullet \propto \epsilon^p$. This would imply that $C_\bullet \propto \epsilon^{\frac{p}{2}}$ in order to keep $G_{N\bullet}$ finite (we have included ϵ into C). But the second relation is

$$\widetilde{G}_{N\bullet} = \widetilde{g}_\bullet + C_\bullet^T g_\bullet^{-1} C.$$

This shows that $\widetilde{G}_N \to 0$ as $\epsilon \to 0$. This is clearly not very plausible. However, this can still be fixed by using the remaining gauge fixing freedom of the Polyakov action (3.2) by scaling also the ratio between g and $\widetilde{g}$. The biggest issue comes with the fact that $\widetilde{g}_\circ$ is not a tensor product of $g_\circ$'s only. In fact, every component $(\widetilde{g}_\circ)_{IJ}$ contains as many $g_\bullet$'s as the number of "commutative" indices in I (or J) is. This means that every component of $\widetilde{g}_\circ$ should scale differently. We must thus abandon the idea of scaling just g, we have to scale $\widetilde{g}$ independently! The correct answer is given by the geometry of the vector bundle $W = TM \oplus \Lambda^p TM$ again. We immediately see that scaling $\mathcal{G}_\bullet \propto \epsilon$, $\mathcal{G}_\circ \propto 1$ and $\mathcal{B} \propto \epsilon^{\frac{1}{2}}$ gives in limit $\epsilon \to 0$ the background independent gauge. This corresponds to

$$g_\bullet \propto \epsilon,\ \widetilde{g}_\bullet \propto \epsilon,\ g_\circ \propto 1,\ \widetilde{g}_\circ \propto 1,\ C_\bullet \propto \epsilon^{\frac{1}{2}}. \tag{10.2}$$

Let us note that in the case of an M5 brane a scaling treating directions differently was described in [75] and [76]. It would be interesting to compare the scaling in these papers with the one introduced here.

11 Matrix model

Now we will apply the previous generalization of the background independent gauge. We will use the double scaling limit to cut off the power series expansion of the DBI action. It

[11] See [9] for a previous discussion of the double scaling limit in the context of the M2/M5 system that came to different conclusions regarding the appropriate powers of ϵ.

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turns out that we find an action describing a natural $p > 1$ (semi-classical) analogue of a matrix model with higher brackets and an interacting with the gauge field F. It will be of order $2(p+1)$ in the matrix variables $\widehat{X}^a$, and at most quadratic in F. The term of order $2(p+1)$ in $\widehat{X}^a$'s and constant in F gives a possible $p > 1$ analogue of the semiclassical pure matrix model.

Assume that C satisfies all the conditions required for Π to be a Nambu-Poisson tensor on N. From (8.6), we have that Lagrangian of the commutative p-DBI action has the form

$$\mathcal{L}_{p-\mathrm{DBI}} = -\frac{1}{g_m} \det{}^{\frac{1}{2}}(g) \cdot \det{}^{\frac{1}{2(p+1)}} [1 + g^{-1}(C+F)\widetilde{g}^{-1}(C+F)^T].$$

Note that the second determinant is the determinant of the vector bundle endomorphism $X : TM \to TM$, where $X = 1 + g^{-1}(C+F)\widetilde{g}^{-1}(C+F)^T$. In the block form $X : D \oplus D^\perp \to D \oplus D^\perp$, we have

$$X = \begin{pmatrix} 1+g_\bullet^{-1}(C_\bullet+F_\bullet)\widetilde{g}_\bullet^{-1}(C_\bullet+F_\bullet)^T+g_\bullet^{-1}F_{\mathbf{I}}\widetilde{g}_\circ^{-1}F_{\mathbf{I}}^T & g_\bullet^{-1}(C_\bullet+F_\bullet)\widetilde{g}_\bullet^{-1}F_{\mathbf{II}}^T + g_\bullet^{-1}F_{\mathbf{I}}\widetilde{g}_\circ^{-1}F_\circ^T \\ g_\circ^{-1}F_{\mathbf{II}}\widetilde{g}_\bullet^{-1}(C_\bullet+F_\bullet)^T + g_\circ^{-1}F_\circ\widetilde{g}_\circ^{-1}F_{\mathbf{I}}^T & 1 + g_\circ^{-1}F_{\mathbf{II}}\widetilde{g}_\bullet^{-1}F_{\mathbf{II}}^T + g_\circ^{-1}F_\circ\widetilde{g}_\circ^{-1}F_\circ^T \end{pmatrix}.$$

Here we have used the following notations for the blocks of F

$$F = \begin{pmatrix} F_\bullet & F_{\mathbf{I}} \\ F_{\mathbf{II}} & F_\circ \end{pmatrix}.$$

This can be decomposed as a product

$$X = \begin{pmatrix} g_\bullet^{-1}(C_\bullet+F_\bullet) & 0 \\ 0 & 1 \end{pmatrix} Y \begin{pmatrix} \widetilde{g}_\bullet^{-1}(C_\bullet+F_\bullet)^T & 0 \\ 0 & 1 \end{pmatrix},$$

where the vector bundle endomorphism $Y : \widetilde{D} \oplus D^\perp \to \widetilde{D} \oplus D^\perp$ is

$$Y = \begin{pmatrix} 1 + \Pi_\bullet'^T(g_\bullet + F_{\mathbf{I}}\widetilde{g}_\circ^{-1}F_{\mathbf{I}}^T)\Pi_\bullet'\widetilde{g}_\bullet & \widetilde{g}_\bullet^{-1}(F_{\mathbf{II}}^T - \widetilde{g}_\bullet\Pi_\bullet'^T F_{\mathbf{I}}`\widetilde{g}_\circ^{-1}F_\circ^T) \\ g_\circ^{-1}(F_{\mathbf{II}} - F_\circ\widetilde{g}_\circ^{-1}F_{\mathbf{I}}^T\Pi_\bullet'\widetilde{g}_\bullet) & 1 + g_\circ^{-1}F_{\mathbf{II}}\widetilde{g}_\bullet^{-1}F_{\mathbf{II}}^T + g_\circ^{-1}F_\circ\widetilde{g}_\circ^{-1}F_\circ^T \end{pmatrix}.$$

Writing Y in block form as

$$Y = \begin{pmatrix} Y_\bullet & Y_{\mathbf{I}} \\ Y_{\mathbf{II}} & Y_\circ \end{pmatrix},$$

note that $Y_\bullet$ is an invertible matrix. This is true because it is a top left block of the matrix Y coming from positive definite matrix $g + (C+F)\widetilde{g}^{-1}(C+F)$ by multiplying it by invertible block-diagonal matrices. Hence, we can write

$$\det(Y) = \det(Y_\bullet)\det(Y_\circ - Y_{\mathbf{I}}Y_\bullet^{-1}Y_{\mathbf{II}}). \tag{11.1}$$

The second matrix has the form

$$\begin{aligned} Y_\circ - Y_{\mathbf{I}}Y_\bullet^{-1}Y_{\mathbf{II}} = 1 &+ g_\circ^{-1}F_{\mathbf{II}}(1 - Y_\bullet^{-1})\widetilde{g}_\bullet^{-1}F_{\mathbf{II}}^T + g_\circ^{-1}F_\circ\widetilde{g}_0^{-1}F_\circ^T + g_\circ^{-1}F_{\mathbf{II}}Y_\bullet^{-1}\Pi_\bullet'^T F_{\mathbf{I}}\widetilde{g}_\circ^{-1}F_\circ^T \\ &+ g_\circ^{-1}F_\circ\widetilde{g}_\circ^{-1}F_{\mathbf{I}}^T\Pi_\bullet'\widetilde{g}_\bullet Y_\bullet^{-1}\widetilde{g}_\bullet^{-1}F_{\mathbf{II}}^T - g_\circ^{-1}F_\circ\widetilde{g}_\circ^{-1}F_{\mathbf{I}}^T\Pi_\bullet'\widetilde{g}_\bullet Y_\bullet^{-1}\Pi_\bullet'^T F_{\mathbf{I}}\widetilde{g}_\circ^{-1}F_\circ^T. \end{aligned}$$

At this point, we will employ the double scaling limit introduced above. Namely, in the $\det^{\frac{1}{2(p+1)}}(Y)$, we wish to keep only the terms scaling at most as ϵ^1. Note that $(Y_\bullet - 1) \propto \epsilon$. Also, $Y_\bullet^{-1} = 1 - (Y_\bullet - 1) + o(\epsilon^2)$. Using this, we can write

$$Y_\circ - Y_{\mathbf{I}} Y_\bullet^{-1} Y_{\mathbf{II}} = 1 + g_\circ^{-1} \Big(F_{\mathbf{II}} \Pi'^T_\bullet g_\bullet \Pi'_\bullet F^T_{\mathbf{II}} + (F_{\mathbf{II}} \Pi'^T_\bullet F_I + F_\circ) \widetilde{g}_\circ^{-1} (F_{\mathbf{II}} \Pi'^T_\bullet F_I + F_\circ)^T \Big) + o(\epsilon^2).$$

The whole term in parentheses after g_0^{-1} is of order ϵ^1. Therefore, we have

$$\begin{aligned} \det{}^{\frac{1}{2(p+1)}}(Y_\circ - Y_{\mathbf{I}} Y_\bullet^{-1} Y_{\mathbf{II}}) = 1 &+ \frac{1}{2(p+1)} \operatorname{tr}(g_\circ^{-1} F_{\mathbf{II}} \Pi'^T_\bullet g_\bullet \Pi'_\bullet F^T_{\mathbf{II}}) \\ &+ \frac{1}{2(p+1)} \operatorname{tr}\Big(g_\circ^{-1} (F_{\mathbf{II}} \Pi'^T_\bullet F_I + F_\circ) \widetilde{g}_\circ^{-1} (F_{\mathbf{II}} \Pi'^T_\bullet F_I + F_\circ)^T \Big) + o(\epsilon^2). \end{aligned}$$

For the first factor in (11.1), we have

$$\det{}^{\frac{1}{2(p+1)}}(Y_\bullet) = 1 + \frac{1}{2(p+1)} \operatorname{tr}\left(\Pi'^T_\bullet (g_\bullet + F_{\mathbf{I}} \widetilde{g}_\circ^{-1} F^T_{\mathbf{I}}) \Pi'_\bullet \widetilde{g}_\bullet \right) + o(\epsilon^2).$$

Putting all together, we obtain

$$\begin{aligned} \det{}^{\frac{1}{2(p+1)}}(Y) = 1 &+ \frac{1}{2(p+1)} \operatorname{tr}\left(\Pi'^T_\bullet (g_\bullet + F_{\mathbf{I}} \widetilde{g}_\circ^{-1} F^T_{\mathbf{I}}) \Pi'_\bullet \widetilde{g}_\bullet \right) + \frac{1}{2(p+1)} \operatorname{tr}(g_\circ^{-1} F_{\mathbf{II}} \Pi'^T_\bullet g_\bullet \Pi'_\bullet F^T_{\mathbf{II}}) \\ &+ \frac{1}{2(p+1)} \operatorname{tr}\Big(g_\circ^{-1} (F_{\mathbf{II}} \Pi'^T_\bullet F_I + F_\circ) \widetilde{g}_\circ^{-1} (F_{\mathbf{II}} \Pi'^T_\bullet F_I + F_\circ)^T \Big) + o(\epsilon^2). \end{aligned} \tag{11.2}$$

Now, comparing the definitions of scalar densities corresponding to Π and Π', it is clear that

$$\det(C_\bullet + F_\bullet) = \pm \det(1 - \Pi F^T) \cdot |\Pi(x)|^{-(p+1)}.$$

Here we assume that one chooses the basis of $\Lambda^p D$ induced by the basis of D. The sign $\pm$ depends on the ordering of that basis. Next, see that $\det(\widetilde{g}_\bullet) = \det{}^{\binom{p}{p-1}}(g_\bullet) = \det{}^p(g_\bullet)$. This shows that

$$S_{p\text{-DBI}} = \mp \int d^{p'+1}x \frac{1}{g_m} \frac{\det^{\frac{1}{p+1}}(1 - \Pi F^T)}{|\Pi(x)|^{\frac{1}{p+1}} \det^{\frac{1}{2}}(g_\bullet)} \det{}^{\frac{1}{2}}(g) \det{}^{\frac{1}{2(p+1)}}(Y).$$

Changing the coordinates according to Seiberg-Witten map, we get the noncommutative DBI action in the form:

$$S_{p\text{-NCDBI}} = \mp \int d^{p'+1}x \frac{1}{\widehat{g}_m} \frac{\det^{\frac{1}{2}}(\widehat{g})}{|\Pi(x)|^{\frac{1}{p+1}} \det^{\frac{1}{2}}(\widehat{g}_\bullet)} \det{}^{\frac{1}{2(p+1)}}(\widehat{Y}).$$

In the last part of the discussion assume that the distribution $D^\perp$ is also integrable, so we can use the set of local coordinates $(x^1, \dots, x^{p+1}, x^{p+2}, \dots, x^{p'+1})$ on N, such that $(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^{p+1}})$ span D, and $(\frac{\partial}{\partial x^{p+2}}, \dots, \frac{\partial}{\partial x^{p'+1}})$ span $D^\perp$. All quantities with indices in $D^\perp$ are now assumed to be in this coordinate basis. Under this assumptions, the integral density in the action can be written as

$$\det{}^{\frac{1}{2}}(g) = \det{}^{\frac{1}{2}}(g_\bullet) \cdot \det{}^{\frac{1}{2}}(g_\circ).$$

Finally, to distinguish the noncommutative and commutative coordinates, we reserve the letters (a, b, c) for labeling the coordinates $(x^1, \ldots, x^{p+1})$, (i, j, k) for labeling the coordinates $(x^{p+2}, \ldots, x^{p'+1})$, (A, B, C) for p-indices containing only noncommutative indices (thus p-indices labeling $\widetilde{D}$) and (I, J, K) for p-indices containing at least one commutative index (thus p-indices labeling $\widetilde{D}^{\perp}$). Also, note that from the definition of ρ_A, we have

$$\widehat{\Pi}'^{aB} = \{\widehat{X}^a, \widehat{X}^{b_1}, \ldots, \widehat{X}^{b_p}\},$$

where $\{\cdot, \ldots, \cdot\}$ is the Nambu-Poisson bracket corresponding to Π, $\widehat{X}^a = \rho_A^*(x^a)$, and $B = (b_1, \ldots, b_p)$. To simplify the expressions, we shall also use the shorthand notation $\{\cdot, \widehat{X}^A\} \equiv \{\cdot, \widehat{X}^{a_1}, \ldots, \widehat{X}^{a_p}\}$. Finally, we also introduce usual index raising/lowering conventions, for example, $\widehat{F}^k{}_A = \sum_{n=1}^{p'+1} \widehat{g}^{kn} \widehat{F}_{nA} = \widehat{g}^{kl} \widehat{F}_{lA}$, or $\widehat{F}_k{}^A = \widehat{\widetilde{g}}^{AB} \widehat{F}_{kB}$ for multiindices. Note that since both g and $\widetilde{g}$ are block diagonal, no confusion concerning range of summation appears. Implementing this notation, we can write

$$\begin{aligned} S_{p\text{-NCDBI}} = \mp \int d^{p'+1}x \frac{1}{\widehat{g}_m} \frac{\det^{\frac{1}{2}}(\widehat{g}_\circ)}{|\Pi(x)|^{\frac{1}{p+1}}} \bigg(& 1 + \frac{1}{2(p+1)} \{\widehat{X}^a, \widehat{X}^A\}\{\widehat{X}_a, \widehat{X}_A\} \\ & + \frac{1}{2(p+1)} \{\widehat{X}^a, \widehat{X}^A\} \widehat{F}_a{}^I \widehat{F}_{bI} \{\widehat{X}^b, \widehat{X}_A\} + \frac{1}{2(p+1)} \{\widehat{X}^a, \widehat{X}^A\} \widehat{F}_{kA} \widehat{F}^k{}_B \{\widehat{X}_a, \widehat{X}^B\} \\ & + \frac{1}{2(p+1)} (\widehat{F}_{kA} \{\widehat{X}^a, \widehat{X}^A\} \widehat{F}_{aJ} + \widehat{F}_{kJ})(\widehat{F}^k{}_B \{\widehat{X}^b, \widehat{X}^B\} \widehat{F}_b{}^J + \widehat{F}^{kJ}) \bigg) + \cdots . \end{aligned}$$

Note that the first non-cosmological term $\{\widehat{X}^a, \widehat{X}^A\}\{\widehat{X}_a, \widehat{X}_A\}$ can be rewritten as

$$\{\widehat{X}^a, \widehat{X}^A\}\{\widehat{X}_a, \widehat{X}_A\} = \frac{1}{p!} \widehat{g}_{a_1 b_1} \cdots \widehat{g}_{a_{p+1} b_{p+1}} \{\widehat{X}^{a_1}, \ldots, \widehat{X}^{a_{p+1}}\}\{\widehat{X}^{b_1}, \ldots, \widehat{X}^{b_{p+1}}\}, \tag{11.3}$$

where summation now goes over all (not strictly ordered) $(p+1)$-indices $(a_1, \ldots, a_{p+1})$ and $(b_1, \ldots, b_{p+1})$. Here, we have used the fact that $\widetilde{g}_\bullet$ is a skew-symmetrized p-fold tensor product of $g_\bullet$. We can even drop the restriction of the summations to noncommutative directions, since the Nambu-Poisson bracket takes care of this automatically. This term corresponds to a $p > 1$ generalization of the matrix model. Note that using the double scaling limit for the expansion of (11.2) leads to a series in positive integer powers of ϵ, automatically truncating higher-order powers in F. This gives an independent justification of the independent scaling of $\widetilde{g}_\bullet$ and $\widetilde{g}_\circ$ in (10.2).

12 Conclusions and discussion

In this paper we have extended, clarified and further developed the construction outlined in [1]. We discussed in detail the bosonic part of an all-order effective action for a system of multiple p-branes ending on a p'-brane. The leading principle was to have an action allowing, similarly to the DBI action, for two mutually equivalent descriptions: a commutative and a "noncommutative" one. As explained in the main body of the paper, the noncommutativity means a semicalssical one, in which the Poisson tensor is replaced by a

Nambu-Poisson one.[12] It turned out that this requirement determines the bosonic part of the effective action essentially uniquely.

In our derivation of the action, generalized geometry played an essential role. All key ingredients, have their origin in the generalized geometry. It already has been appreciated in the literature that the presence of a (p+1)-form leads to a generalized tangent space $TM \oplus \Lambda^p T^*M$. Although, this observation perfectly applies also in our situation, we found it very useful to double it, i.e., to consider the the extended/doubled generalized tangent space $W \oplus W^*$, with $W = TM \oplus \Lambda^p TM$.

Let us comment on this more: in the string case, p=1, the sum of the background fields $g+B$ plays a prominent role. It enters naturally the Polyakov action, the DBI action, Buscher's rules, etc. In generalized geometry, one way define a generalized metric, is to give a subbundle of the generalized tangent bundle $TM \oplus T^*M$ of maximal rank, on which the natural (+) pairing on generalized tangent bundle is positive definite. Such a subbundle can be characterized as a graph of the map from $TM \to T^*M$ defined by the sum $g+B$. Therefore, it is quite natural to look for a formalism which would allow for a natural "sum" of a metric and a higher rank (p+1)-form. What this sum should be is indicated by the Polyakov type membrane action in its matrix form (3.9). From here it is just a small step to recognize the doubled generalized tangent bundle as a right framework for a meaningful interpretation of the "sum" of the metric and a higher rank $(p+1)$-form. This observation is further supported by the form of the open closed relations in the doubled form (3.16) and the matrix form of the Nambu sigma model (4.1). Finally, the corresponding Hamiltonian (3.10), cf. also (4.9), tells us what the relation to the generalized metric on $TM \oplus \Lambda^p T^*M$ is. Hence, at the end, we do not really use the full doubled generalized tangent bundle, we use it only for a nice embedding of the generalized tangent bundle $TM \oplus \Lambda^p T^*M$.[13]

Nevertheless, we found the doubled generalized geometry quite intriguing. Extending on the above comments: since on the doubled generalized tangent bundle there is a natural function-valued non-degenerated pairing $\langle .,. \rangle$, we can mimic the standard constructions with $TM \oplus T^*M$. For instance, one can speak of the orthogonal group, define the generalized metric using an involutive endomorphism $\mathcal{T}$ on $W \oplus W^*$, such that $\langle \mathcal{T}, . \rangle$ defines a fibre-wise metric on the doubled generalized tangent bundle, etc.

However, we are still facing a problem; we lack a canonical Courant algebroid structure. The reason lies basically in very limited choices for the anchor map $\rho : W \oplus W^* \to TM$, which leave us only with a projection onto the tangent bundle TM. The map ρ is therefore "too simple" to control the symmetric part of any bracket. However, we can still consider Leibniz algebroid structures on $W \oplus W^*$. There are several possibilities to do this. To choose the one suitable for p-brane backgrounds, one can consider the action of the map $e^{\mathcal{B}}$: $W \oplus W^* \to W \oplus W^*$, where $\mathcal{B}$ is a general section of $\Lambda^2 W$, viewed as a map from W to W^*, and extended to $\mathrm{End}(W \oplus W^*)$ by zeros. The map $e^{\mathcal{B}}$ is thus an analogue of the usual B-field

[12]Let us notice, that in our approach to noncommutativity of fivebrane, the ordinary point-wise product remains undeformed.

[13]The doubled generalized geometry formalism can also be introduced for the p=1 string case and allows an elegant formulation of the theory. For any p, the appearance of TM and $\Lambda^p TM$ (and similarly of T^*M and $\Lambda^p T^*M$) is related to the split into one temporal and p spatial world-sheet directions.

transform of generalized geometry $TM \oplus T^*M$. It turns our that there is a Leibniz algebroid, such that the condition for $e^{\mathcal{B}}$ to be an isomorphism of the bracket forces $\mathcal{B}$ to take the block off-diagonal form (5.1), with $C \in \Omega^{p+1}_{\text{closed}}(M)$. This bracket coincides with the one defined by Hagiwara in [24] to study Nambu-Dirac manifolds. Moreover, Nambu-Poisson manifolds appear naturally as its Nambu-Dirac structures. Interestingly, its full group of orthogonal automorphisms can be calculated, giving (for $p > 1$) a semi-direct product $\text{Diff}(M) \ltimes (\Omega^{p+1}_{\text{closed}} \rtimes G)$, where G is the group of locally constant non-zero functions on M. Notably, this coincides with the group of all automorphisms of higher Dorfman bracket, see e.g. [25].

Relating our approach, based on the generalized geometry on the vector bundle $W \oplus W^*$, with the usual generalized geometries in M-theory and supergravity [49–51, 70], we notice the following. A choice of a generalized geometry is subject to the field content one wants to describe. In principle, one can double each of of them and use the advantages of having a natural function-valued pairing as we did for our case of interest in this paper. However, the field content coming with such a doubled generalized geometry is much bigger then we started with and we have to reduce it accordingly.

Finally, let us again notice the striking similarity with the result of [38, 39] — based on a very different approach — and discussed after equation (8.10). We find worth to pursue a deeper understanding of this similarity in the future.

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A Nambu-Poisson structures

Here we recall some fundamental properties of Nambu-Poisson structures [21] as needed in this paper. For details see, e.g., [24] or [25].

For any $(p+1)$-vector field A on M we define the induced map $A^\sharp : \Omega^p(M) \to \mathfrak{X}(M)$ as $A^\sharp(\xi) = (-1)^p i_\xi A = \xi_K A^{iK} \partial_i$.

Also, for an alternative formulation of the fundamental identity, we need to recall the Dorfman bracket, i.e., the $\mathbb{R}$-bilinear bracket on the sections of $TM \oplus \Lambda^p T^*M$, defined as

$$[V + \xi, W + \eta]_D = [V, W] + \mathcal{L}_V \eta - i_W d\xi, \tag{A.1}$$

for all $V, W \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^p(M)$.

Let Π be a $(p+1)$-vector field on M. We call Π a Nambu-Poisson structure if

$$\mathcal{L}_{\Pi^\sharp(df_1\wedge\ldots\wedge df_p)}(\Pi) = 0\,, \tag{A.2}$$

for all $f_1, \ldots, f_p \in C^\infty(M)$.

Lemma A.1. *For an arbitrary $p \geq 1$, the condition (A.2) can be stated in the following equivalent ways:*

1. *The graph $G_\Pi = \{\Pi^\sharp(\xi) + \xi \mid \xi \in \Omega^p(M)\}$ is closed under the Dorfman bracket (A.1);*
2. *for any $\xi, \eta \in \Omega^p(M)$ it holds that*
$$(\mathcal{L}_{\Pi^\sharp(\xi)}(\Pi))^\sharp(\eta) = -\Pi^\sharp(i_{\Pi^\sharp(\eta)}(d\xi))\,; \tag{A.3}$$
3. *let $[\cdot,\cdot]_\pi : \Omega^p(M) \times \Omega^p(M) \to \Omega^p(M)$ be defined as*
$$[\xi,\eta]_\pi := \mathcal{L}_{\Pi^\sharp(\xi)}(\eta) - i_{\Pi^\sharp(\eta)}(d\xi)\,, \tag{A.4}$$
for all $\xi, \eta \in \Omega^p(M)$. Then it holds that
$$[\Pi^\sharp(\xi), \Pi^\sharp(\eta)] = \Pi^\sharp([\xi,\eta]_\pi)\,, \tag{A.5}$$
for all $\xi, \eta \in \Omega^p(M)$;
4. *for any $\xi \in \Omega^p(M)$ it holds that*
$$\mathcal{L}_{\Pi^\sharp(\xi)}(\Pi) = -\Big(i_{d\xi}(\Pi)\Pi - \frac{1}{p+1} i_{d\xi}(\Pi\wedge\Pi) \Big)\,. \tag{A.6}$$

For $p > 1$, around any point $x \in M$, where $\Pi(x) \neq 0$, there exist local coordinates $(x^1, \ldots, x^n)$, such that

$$\Pi(x) = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{p+1}}. \tag{A.7}$$

In this coordinates $\Pi^{iJ} = \delta^{iJ}_{1\ldots p+1} = \epsilon^{iJ}$.

For $p > 1$, a Nambu-Poisson tensor can be multiplied by any smooth function, and one gets again a Nambu-Poisson tensor:

Lemma A.2. *Let Π be a Nambu-Poisson tensor, and $p > 1$. Let $f \in C^\infty(M)$ be a smooth function on M. Then $f\Pi$ is again a Nambu-Poisson tensor. For $p = 1$ this is not true in general.*

This lemma has a simple useful consequence

Lemma A.3. *Let $n = p+1$. Then any $\Pi \in \Gamma(\Lambda^{p+1}TM)$ is a Nambu-Poisson tensor.*

There is an interesting little technical detail. One of the equivalent reformulations of fundamental identity was the closedness of the graph G_Π under the Dorfman bracket. But see that the both, the definition of G_Π and the involutivity condition have a good meaning also for *any* vector bundle morphism $\Pi^\sharp : \Lambda^p T^*M \to TM$. We may ask whether there exists $\Pi^\sharp$, which is not induced by $(p+1)$-vector on M. The answer is given by the following lemma:

Lemma A.4. *Let $\Pi^\sharp : \Lambda^p T^* M \to TM$ be a vector bundle morphism, such that its graph*

$$G_\Pi = \{\Pi^\sharp(\xi) + \xi \mid \xi \in \Omega^p(M)\},$$

is closed under higher Dorfman bracket (A.1). Let Π be a contravariant $(p+1)$-tensor defined by

$$\Pi(\alpha, \xi) = \langle \alpha, \Pi^\sharp(\xi) \rangle,$$

for all $\alpha \in \Omega^1(M)$ and $\xi \in \Omega^p(M)$. Then Π is a $(p+1)$-vector, and hence a Nambu-Poisson tensor.

Proof. The closedness of G_Π under the Dorfman bracket can immediately be rewritten as (A.3), where Π is now not necessarily a $(p+1)$-vector. This relation is tensorial in η, so choose $\eta = dy^J$, and look at the i-th component of the identity. The left-hand side is

$$\begin{aligned}(\mathcal{L}_{\Pi^\sharp(\xi)}\Pi)^{iJ} = \xi_K \Big(\Pi^{mK}\Pi^{iJ}{}_{,m} - \Pi^{iK}{}_{,m}\Pi^{mJ} - \sum_{r=1}^{p} \Pi^{j_r K}{}_{,m} \Pi^{ij_1 \ldots m \ldots j_p} \Big) \\ - \xi_{K,m} \Big(\Pi^{iK}\Pi^{mJ} + \sum_{r=1}^{p} \Pi^{j_r K} \Pi^{ij_1 \ldots m \ldots j_p} \Big).\end{aligned}$$

The right-hand side of (A.3) is

$$-\Pi^\sharp(i_{\Pi^\sharp(dy^J)} d\xi)^i = \Pi^{iM}\Pi^{lJ}(d\xi)_{lM} = -\xi_{K,m}\big(\Pi^{iM}\Pi^{lJ}\delta^{mK}_{lM}\big).$$

The terms proportional to ξ_K form the differential part of the identity, whereas the terms proportional to $\xi_{K,m}$ form the algebraic part:

$$\Pi^{iK}\Pi^{mJ} + \sum_{r=1}^{p} \Pi^{j_r K}\Pi^{ij_1\ldots m \ldots j_p} = \Pi^{iM}\Pi^{lJ}\delta^{mK}_{lM}.$$

We will use this algebraic identity to show that $\Pi^{kM} = 0$, whenever $k \in M$. This will prove that Π is a $(p+1)$-vector. To do this, choose $m = i = k$, and $K = J = M$ in the above identity. Assume that $m_q = k$, where $M = (m_1 \ldots m_p)$. Then, the only non-trivial term in the sum is the one for $r = q$. Right-hand side vanishes due to skew-symmetry of the symbol δ. Hence, we obtain

$$2(\Pi^{kM})^2 = 0.$$

This proves that $\Pi^{kM} = 0$, and Π is thus a $(p+1)$-vector. $\square$

A.1 Scalar density

Interestingly, the coordinates $(x^1, \ldots x^n)$, in which Π has the form (A.7), allow us to define a well-behaved scalar density $|\Pi(x)|$ of weight $-(p+1)$. Let $(y^1, \ldots, y^n)$ be arbitrary local coordinates. Define the function $|\Pi(x)|$ as

$$|\Pi(x)| = \det\left(\frac{\partial y^i}{\partial x^j}\right)^{p+1}, \tag{A.8}$$

that is, the Jacobian of the coordinate transformation $y^i = y^i(x^k)$. This is indeed a scalar density (with respect to a change $y \mapsto \widetilde{y}$) of weight $-(p+1)$, as can easily be seen using the chain rule.

For $p = 1$, let Π^{ij} be the matrix of Π in (y) coordinates. We can ask, whether $|\Pi(x)| = \det \Pi^{ij}$ whenever Π is decomposable. The answer is clearly negative for $n > 2$, where $\det \Pi^{ij} = 0$. The case $p = 1$, $n = 2$ is a special case contained in the next question. Let $p \geq 1$ and $n = p+1$. Let Π^{iJ} be the matrix of the vector bundle map $\Pi^\sharp$. For $n = p+1$, this is a square $n \times n$ matrix. We can thus ask whether $|\Pi(x)| = \det \Pi^{iJ}$. It is of course modulo the sign, depending on the ordering of the basis of $\Omega^p(M)$. Now, see that

$$\Pi(x) = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{p+1}} = |\Pi(x)|^{\frac{1}{n}} \frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^{p+1}}.$$

This means that $|\Pi(x)|^{\frac{1}{n}} = \Pi^{1\ldots n}(x)$. The determinant of Π^{iJ} is up to sign the n-th power of $\Pi^{1\ldots n}$, and thus $\det \Pi^{iJ} = \pm|\Pi(x)|$.

Further, we have to be careful with the dependence of $|\Pi(x)|$ on the choice of the special local coordinates $(x^1, \ldots, x^n)$. Let $(x'^1, \ldots, x'^n)$ is another set of such coordinates, that is

$$\Pi(x) = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^{p+1}} = \frac{\partial}{\partial x'^1} \wedge \cdots \wedge \frac{\partial}{\partial x'^{p+1}}. \tag{A.9}$$

Denote by J the Jacobi matrix of the transformation $\widetilde{x}^i = \widetilde{x}^i(x^k)$. We can split it as

$$J = \begin{pmatrix} J_\epsilon & K \\ L & M \end{pmatrix},$$

where the top-left block J_ϵ is a $(p+1) \times (p+1)$ submatrix corresponding to the first $p+1$ of both sets of coordinates. The condition in (A.9) forces $\det(J_\epsilon) = 1$ and $L = 0$. We thus get the important observation that

$$\det J = \det M,$$

and moreover $\det M = \det M(x^{j>p+1})$. This implies that $|\Pi(x)|$ transforms, with respect to the change the special coordinates (x), as

$$|\Pi(x)| = \det(M)^{p+1} |\Pi(x)|', \tag{A.10}$$

where $|\Pi(x)|'$ is calculated with respect to (x') coordinates on M.

B Background independent gauge

B.1 Pseudoinverse of a 2-form

Proposition B.1. *Let V be a finite-dimensional vector space. Let g be an inner product on V, and $C \in \Lambda^2 V^*$ a 2-form on V. Let $P : V \to V$ an orthogonal projector, such that* $\ker(P) = \ker(C)$. *Then there exists a unique 2-vector Π, such that*

$$\Pi C = P, \quad P\Pi = \Pi.$$

Proof. Let $\mathbf{C}$, $\mathbf{g}$ and $\mathbf{P}$ be the matrices of C, g, P, respectively, in an arbitrary fixed basis of V. First construct the map $\widetilde{C} \equiv g^{-1}C : V \to V$. This map is skew-symmetric with respect to g. Indeed, we have

$$\mathbf{g}^{-1}(\mathbf{g}^{-1}\mathbf{C})^T\mathbf{g} = -\mathbf{g}^{-1}\mathbf{C}.$$

Denote $\widetilde{\mathbf{C}} = \mathbf{g}^{-1}\mathbf{C}$. Let $\mathbf{A}$ be the matrix diagonalizing $\mathbf{g}$, that is $\mathbf{A}^T\mathbf{g}\mathbf{A} = \mathbf{1}$. Finally, define the matrix $\widetilde{\mathbf{C}}' = \mathbf{A}^{-1}\widetilde{\mathbf{C}}\mathbf{A}$. This matrix is skew-symmetric (in the ordinary sense). Standard linear algebra says that there exists a standard block-diagonal form of the matrix $\widetilde{\mathbf{C}}'$. In more detail, one can find an orthogonal matrix $\mathbf{O}$ and a matrix $\mathbf{\Sigma}$, such that $\widetilde{\mathbf{C}}' = \mathbf{O}\mathbf{\Sigma}\mathbf{O}^T$, where $\mathbf{\Sigma}$ has the form

$$\mathbf{\Sigma} = \operatorname{diag}\left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}, 0, \dots, 0 \right).$$

where $k = \frac{1}{2}\operatorname{rank}(\widetilde{\mathbf{C}}')$, and $\lambda_1, \dots, \lambda_k > 0$. Note that the matrix $\mathbf{O}$ is not unique, and the matrix $\mathbf{\Sigma}$ is unique up to the reordering of the 2×2 blocks.

This shows that we can write $\mathbf{C} = \mathbf{g}\mathbf{A}\mathbf{O}\mathbf{\Sigma}\mathbf{O}^T\mathbf{A}^{-1}$. Define $\mathbf{\Delta}_{2k} = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)$, where the number of 1's is $2k$. The (unique) matrix $\mathbf{P}$ can be now written as $\mathbf{P} = \mathbf{A}\mathbf{O}\mathbf{\Delta}_{2k}\mathbf{O}^T\mathbf{A}^{-1}$. Let $\mathbf{\Pi}$ be the matrix of a bivector we are looking for. The equation $\Pi C = P$ translates into

$$\mathbf{\Pi}\mathbf{g}\mathbf{A}\mathbf{O}\mathbf{\Sigma}\mathbf{O}^T\mathbf{A}^{-1} = \mathbf{A}\mathbf{O}\mathbf{\Delta}_{2k}\mathbf{O}^T\mathbf{A}^{-1}.$$

We thus get that $(\mathbf{O}^T\mathbf{A}^{-1}\mathbf{\Pi}\mathbf{g}\mathbf{A}\mathbf{O})\mathbf{\Sigma} = \mathbf{\Delta}_{2k}$. This means that

$$\mathbf{\Pi} = \mathbf{A}\mathbf{O}\mathbf{\Sigma}^+\mathbf{O}^T\mathbf{A}^{-1}\mathbf{g}^{-1},$$

where $\mathbf{\Sigma}^+\mathbf{\Sigma} = \mathbf{\Delta}_{2k}$. Now it is easy to see that Π is a bivector, if and only if $\mathbf{\Sigma}^+$ is, and that $P\Pi = \Pi$ holds if and only if $\mathbf{\Delta_{2k}}\mathbf{\Sigma}^+ = \mathbf{\Sigma}^+$. This fixes $\mathbf{\Sigma}^+$ and thus $\mathbf{\Pi}$ uniquely. It coincides with the Moore-Penrose pseudoinverse of the matrix $\mathbf{\Sigma}$, and it is given, in the block form, as

$$\mathbf{\Sigma}^+ = \begin{pmatrix} \mathbf{\Sigma_0}^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{\Sigma_0}$ is the invertible top left $2k \times 2k$ block of $\mathbf{\Sigma}$. □

B.2 Integrable forms

Let M be a smooth manifold, and let C be a $(p+1)$-form on M. The form C is called an integrable form if it holds

$$C(\mathbf{P}) \wedge C = 0, \tag{B.1}$$

$$C(\mathbf{P}) \wedge dC = 0, \tag{B.2}$$

for all $\mathbf{P} \in \mathfrak{X}^p(M)$, where on the left-hand side $C(\mathbf{P})$ denotes the value of the induced vector bundle morphism $C : \Lambda^p TM \to T^*M$ when evaluated on $(\mathbf{P})$. The condition (B.1) is in fact a very restrictive one. Also, it is very similar to the algebraic part of Nambu-Poisson fundamental identity:

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Lemma B.2. *Let C be a $(p+1)$-form. Then C satisfies (B.1) if and only if it is decomposable around every point $x \in M$, such that $C(x) \neq 0$. That means that there exists a $(p+1)$-tuple $(\alpha_1, \ldots, \alpha_{p+1})$ of linearly independent 1-forms , such that locally*

$$C = \alpha_1 \wedge \ldots \wedge \alpha_{p+1}.$$

Proof. Let us proceed by induction on p. The $p = 0$ case is a trivial statement, any 1 form is decomposable. Now choose $p > 0$. Assume that statement holds for all p-forms, and let C be a $(p+1)$-form satisfying (B.1). We have to show that it is decomposable.

Let $x \in M$, such that $C(x) \neq 0$. First, see that for any $V \in \mathfrak{X}(M)$, such that $(i_V(C))(x) \neq 0$, the p-form $i_V(C)$ satisfies (B.1), and thus, by induction hypothesis, is decomposable. Let us take any $\mathbf{Q} \in \mathfrak{X}^{p-1}(M)$. We have to show that

$$(i_V C)(\mathbf{Q}) \wedge (i_V C) = 0.$$

But this can be rewritten as

$$i_V \big(C(V \wedge \mathbf{Q}) \wedge C \big) = 0,$$

which follows from the assumptions on C, taking $\mathbf{P} = V \wedge \mathbf{Q}$. Second, take the original condition (B.1) and apply i_V to both sides with an arbitrary $V \in \mathfrak{X}(M)$. One gets

$$i_V(C(\mathbf{P})) \cdot C - C(\mathbf{P}) \wedge i_V(C) = 0.$$

But $i_V(C(\mathbf{P}))$ is a scalar function, and since C is a nonzero $(p+1)$-form at x, there have to exist $V \in \mathfrak{X}(M)$ and $\mathbf{P} \in \mathfrak{X}^p(M)$, such that $\lambda \equiv i_V(C(\mathbf{P})) \neq 0$, at least at some neighborhood of x. Thus, locally we can write

$$C = \frac{1}{\lambda} C(\mathbf{P}) \wedge i_V(C).$$

Since $\lambda(x) \neq 0$, also $(i_V(C))(x) \neq 0$. We can now apply the induction hypothesis to this p-form to get p linearly independent 1-forms $(\alpha_1, \ldots, \alpha_p)$, such that

$$i_V C = \alpha_1 \wedge \ldots \wedge \alpha_p.$$

This finishes the proof, because taking $\alpha_{p+1} = \frac{(-1)^p}{\lambda} C(\mathbf{P})$ leads to the desired decomposition. □

Let us now clarify where integrable forms got their name from:

Definition B.3. Let C is a $(p+1)$-form. Denote by M' the open submanifold of M, where $C \neq 0$. The kernel distribution K of C is a distribution on M', defined at every $x \in M'$ as

$$K_x = \{ V \in T_x M \mid i_V(C(x)) = 0 \}.$$

Note that this distribution is not necessarily a smooth one.

We can now relate integrability of distributions to the integrability od forms.

Lemma B.4. *Let C be a $(p+1)$-form. Then C integrable if and only if K is an integrable $(n-(p+1))$-dimensional regular smooth distribution on M'.*

Proof. First assume that C is an integrable $(p+1)$-form. Then by the previous lemma, around every point of $x \in M'$, there exists a $(p+1)$-tuple of linearly independent 1-forms, such that locally

$$C = \alpha_1 \wedge \ldots \wedge \alpha_{p+1}. \tag{B.3}$$

The subspace K_x can be determined easily as

$$K_x = \{V \in T_x M \mid i_V(\alpha_i(x)) = 0, \forall i \in \{1, \ldots, p+1\}\}.$$

This is a set of k linearly independent linear equations for the components of V. The dimension of K_x is thus $n-(p+1)$. To see that this is a smooth regular distribution, note that K is the kernel of a smooth vector bundle morphism of a constant rank, and hence a subbundle of TM'. Hence, a smooth distribution in M'.

To see that it is also integrable, plug the expression (B.3) into the second defining equation (B.2). It turns out that it is equivalent to

$$d\alpha_j \wedge \alpha_1 \wedge \ldots \wedge \alpha_{p+1} = 0, \tag{B.4}$$

for all $j \in \{1, \ldots, p+1\}$. Now take any $V \in \Gamma(K)$, and plug it into (B.4). It gives $i_V(d\alpha_j) = 0$ for all $j \in \{1, \ldots, p+1\}$. But this is, using the Cartan formula for $d\alpha_j$, equivalent to involutivity of the subbundle K under the commutator of vector fields, which is in turn, using the Frobenius integrability theorem, equivalent to the integrability of K.

Conversely, assume that K is integrable $((n-(p+1))$-dimensional regular smooth distribution. At every $x \in M'$, there is a neighborhood $U_x \ni x$, and a set of local coordinates $(x^1, \ldots, x^{(n-(p+1))}, y^1, \ldots, y^{p+1})$, such that sections of the subbundle K are on U_x spanned by $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{(n-(p+1))}})$. Then C has to be annihilated by all vectors of K, so it has to have the local form

$$C = \lambda \cdot dy^1 \wedge \ldots \wedge dy^{p+1}. \tag{B.5}$$

We see that this C clearly satisfies (B.1). Since we are on M', we have $\lambda \neq 0$. We set $\alpha_1 = \lambda dy^1$, and $\alpha_i = dy^i$ for $i = 2, \ldots, p+1$. The second condition for integrable $(p+1)$-forms translates as (B.4). Obviously, this holds for the above defined α_j's.

At $x \in M \setminus M'$ the integrability conditions (B.1), (B.2) hold trivially and we can conclude that C is an integrable $(p+1)$-form. □

Remark B.5. One can extend the distribution K to the whole manifold M. For each $x \in M \setminus M'$, define $K_x = \{0\}$. By this extension one gets a smooth singular distribution on M. However, even for integrable $(p+1)$-forms, K is not integrable in general. For details see [77].

Let us conclude this section by relating the concepts of integrable $(p+1)$-forms to Nambu-Poisson structures. This is given by the following lemma.

Lemma B.6. *Let M be an orientable smooth manifold. Let Ω be the corresponding volume form. Let C be a $(p+1)$-form on M. Define a $(p+1)$-vector Π by equation*

$$i_\Pi \Omega = C.$$

Then Π is a Nambu-Poisson $(n-(p+1))$-vector if and only if C is an integrable $(p+1)$-form.

Proof. Clearly, $\Pi(x) = 0$ if and only if $C(x) = 0$. Let Π be a Nambu-Poisson tensor. By previous comment, at singular points of Π, C vanishes. The conditions on integrability are, at these points, satisfied trivially. Assume that $\Pi(x) \neq 0$. Then there exist local coordinates $(x^1, \ldots, x^n)$ around x, such that

$$\Pi = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-(p+1)}}.$$

In these coordinates, the volume form Ω is

$$\Omega = \omega \cdot dx^1 \wedge \ldots \wedge dx^n,$$

where $\omega \neq 0$. We thus see that C has the explicit form

$$C = \omega \cdot dx^{n-(p+1)+1} \wedge \ldots \wedge dx^n.$$

It is easy to check that it satisfies both integrability conditions (B.1), (B.2).

The converse statement follows basically from the proof of the previous lemma. There, we have shown that C can be, for an integrable $(p+1)$-form, written (around any point where $C(x) \neq 0$) in the local form (B.5). Writing the volume form in these local coordinates as

$$\Omega = g \cdot dx^1 \wedge \ldots dx^{(n-(p+1))} \wedge dy^1 \wedge \ldots dy^{p+1},$$

one finds the local expression for Π as

$$\Pi = \frac{\lambda}{g} \cdot \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-(p+1)}}.$$

Note that this is a top-level multivector field on the submanifold N'. In the view of lemma A.3, one would expect that this is enough. Inspection of the fundamental identity shows that all partial derivatives are contracted with the components of Π, so in the fundamental identity there are no partial derivatives in transversal directions. We can now apply (the proof of) lemma A.3 to conclude that Π is a Nambu-Poisson tensor on M. □

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Appendix E

Paper 4:
Nambu-Poisson Gauge Theory

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Nambu–Poisson gauge theory

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ABSTRACT

We generalize noncommutative gauge theory using Nambu–Poisson structures to obtain a new type of gauge theory with higher brackets and gauge fields. The approach is based on covariant coordinates and higher versions of the Seiberg–Witten map. We construct a covariant Nambu–Poisson gauge theory action, give its first order expansion in the Nambu–Poisson tensor and relate it to a Nambu–Poisson matrix model.

1. Introduction

In this letter, we introduce a higher analogue of noncommutative (abelian) pure gauge theory. What we consider here is a deformation, in the presence of a background $(p+1)$-rank Nambu–Poisson tensor, of an abelian gauge theory with a p-form gauge potential, i.e., a $(p-1)$-gerbe connection. Our approach, for $p > 1$, is similar to that of [1] which deals with the more familiar case of $p = 1$. A Nambu–Poisson gauge theory was pioneered by P.-M. Ho et al. in [2] as the effective theory of M5-brane for a large longitudinal C-field background in M-theory. Related work can be found in their papers [3–5].

We formulate the theory independently of string/M-theory. Nevertheless, the motivation comes from M-theory branes; more explicitly from an effective DBI-type theory proposed for the description of multiple M2-branes ending on an M5-brane, where the Nambu–Poisson 3-tensor enters as a pseudoinverse of the 3-form field C [6,7]. We develop the theory at a semiclassical level, briefly commenting on the issue of quantization at the end.

The paper is organized as follows: After discussing conventions in Section 2, we introduce in Section 3 covariant coordinates, which transform nontrivially with respect to gauge transformations parametrized by a $(p-1)$-form, the gauge transformation being described in terms of a $(p+1)$-bracket arising from a background Nambu–Poisson $(p+1)$-tensor. Based on these covariant coordinates, we introduce Nambu–Poisson gauge fields in Section 4. In Section 5, we construct Nambu–Poisson gauge fields explicitly, using a suitable generalization [6–8] of the Seiberg–Witten map [9], starting from an ordinary $(p-1)$-form gauge potential. We give explicit expressions for all components of the Nambu–Poisson field strength. In Section 6, we give the corresponding (semiclassically) "noncommutative" action and its first order expansion in the Nambu–Poisson tensor. Up to this order the result is unambiguous, because quantum corrections from any type of quantization of the Nambu–Poisson structure will only affect higher orders. We conclude the letter by relating the action to (the semiclassical version of) a Nambu–Poisson matrix model.

We only briefly comment on deformation quantization of Nambu–Poisson structures in this letter. A satisfactory description of Nambu–Poisson noncommutative gauge theory beyond the semiclassical level will require a suitable analogue of Kontsevich's formality, solving in particular the deformation quantization problem for an arbitrary Nambu–Poisson structure.

2. Conventions

We assume that n-dimensional space-time M is equipped with a rank $p+1$ Nambu–Poisson structure Π, with $1 < p < n$.[1] The corresponding Nambu–Poisson bracket is denoted by $\{\cdot, \ldots, \cdot\}$. In

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[1] The discussion could be extended to include also the well known case $p = 1$, but for clarity and brevity we concentrate here on $p > 1$ and refer to [7] for $p = 1$.

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order to keep notation close to the familiar $p=1$ case, we write $\{f,\lambda\} := \Pi(df, d\lambda) = \frac{1}{p!}\Pi^{ij_1\dots j_p}\partial_i f(d\lambda)_{j_1\dots j_p}$ for a $(p-1)$-form λ and a function f. In the special case, where $d\lambda$ factorizes as a product $d\lambda = d\lambda_1 \wedge \dots \wedge d\lambda_p$, we have $\{f,\lambda\} \equiv \{f,\lambda_1,\dots,\lambda_p\}$. We consider a set of local coordinates $(x^1,\dots,x^n)$ on M and denote the corresponding indices by lower case Latin characters i, j, k, etc. Upper case Latin characters I, J, K, etc. denote strictly ordered p-tuples of indices, i.e. $J=(j_1,\dots,j_p)$ with $1 \leqslant j_1 < \dots < j_p \leqslant n$. With this notation, $\Pi(df,d\lambda) = \Pi^{iJ}\partial_i f(d\lambda)_J$. Often, we will omit indices altogether, implicitly implying matrix multiplication of the underlying rectangular matrices as in $(\Pi F^T)^i_j = \Pi^{iK}F_{Kj}$. We use Roman characters a, B, etc. for indices and multi-indices taking values only in the "noncommutative" directions $1,\dots,p+1$.

3. Covariant coordinates

Before we introduce in the next section the Nambu–Poisson gauge potential[2] $\hat{A}$ and field strength $\hat{F}$, let us define "covariant coordinates"[3] as functions $\hat{x}^i(x)$, $i=1,\dots,n$ of the space–time coordinates $\{x^i\}_{i=1}^n$, which transform under gauge transformations parametrized by a $(p-1)$-form Λ as

$$\delta_\Lambda \hat{x}^i = \{\hat{x}^i, \Lambda\}, \tag{1}$$

where the bracket is a $p+1$ Nambu–Poisson bracket (cf. Section 2 for notation). We assume our fixed (but arbitrarily chosen) coordinates x^i to be invariant under gauge transformations. We also assume that they can be expanded around any point $x \in M$, at least in the sense of formal power series, as $\hat{x}^i = x^i + \cdots$. Hence, at least formally, we can always solve for x^i as functions of covariant coordinates $\hat{x}^i$, i.e. $x^i = \hat{x}^i + \cdots$. We denote by ρ the (formal) diffeomorphism on M corresponding to this change of local variables on M and write $\hat{x}^i = \rho^*(x^i)$ for the respective local coordinate functions. The change of coordinates defined by ρ^* is also called "covariantizing map". The diffeomorphism ρ can be used to define a new Nambu–Poisson structure Π' with bracket $\{\cdot,\dots,\cdot\}'$:

$$\begin{aligned}\rho^*\big(\{x^{i_1},\dots,x^{i_{p+1}}\}'\big) &:= \{\rho^* x^{i_1},\dots,\rho^* x^{i_{p+1}}\} \\ &\equiv \{\hat{x}^{i_1},\dots,\hat{x}^{i_{p+1}}\}.\end{aligned} \tag{2}$$

4. Nambu–Poisson gauge fields

Here and in the subsequent sections, we follow closely the semiclassical parts of [10,11], where the $p=1$ case is described. Using covariant coordinates $\hat{x}^i$, we define the Nambu–Poisson ("noncommutative") gauge potential with components labeled by upper indices $i=1,\dots,n$ as[4]

$$\hat{A}^i = \hat{x}^i - x^i = \rho^*(x^i) - x^i. \tag{3}$$

Its gauge transformation follows from (1)

$$\delta_\Lambda \hat{A}^i = \{\hat{A}^i, \Lambda\} + \{x^i, \Lambda\}. \tag{4}$$

Next, we introduce the contravariant tensor F' with components $F'^{i_1\dots i_{p+1}}$ as the difference of the Nambu–Poisson structures Π', see Eq. (2), and Π:

$$F'^{i_1\dots i_{p+1}} = \Pi'^{i_1\dots i_{p+1}} - \Pi^{i_1\dots i_{p+1}}. \tag{5}$$

[2] This is the higher analog of the $p=1$ noncommutative gauge potential.

[3] Covariant with respect to the gauge transformation (4). For $p=1$ they correspond to background independent operators of [9]; they are actually dynamical fields.

[4] See [12–14] for an alternative approach related to area-preserving diffeomorphisms.

Covariantizing the individual components of this tensor using the diffeomorphism ρ, we obtain the Nambu–Poisson ("noncommutative") field strength $\hat{F}'$ with components

$$\hat{F}'^{i_1\dots i_{p+1}} := \rho^*\big(F'^{i_1\dots i_{p+1}}\big). \tag{6}$$

Using (5) and a hat to denote the application of ρ^*,

$$\begin{aligned}\hat{F}'^{i_1\dots i_{p+1}} &= \hat{\Pi}'^{i_1\dots i_{p+1}} - \hat{\Pi}^{i_1\dots i_{p+1}} \\ &= \rho^*\big(\Pi'^{i_1\dots i_{p+1}}\big) - \rho^*\big(\Pi^{i_1\dots i_{p+1}}\big).\end{aligned} \tag{7}$$

Rewriting this with the help of (2) as

$$\hat{F}'^{i_1\dots i_{p+1}} = \{\hat{x}^{i_1},\dots,\hat{x}^{i_{p+1}}\} - \{x^{i_1},\dots,x^{i_{p+1}}\}(\hat{x}), \tag{8}$$

the gauge transformation of $\hat{F}'$ can be easily determined:

$$\delta_\Lambda \hat{F}'^{i_1\dots i_{p+1}} = \{\hat{F}'^i, \Lambda\}. \tag{9}$$

From now on we will assume without loss of generality that the local coordinates x^i are adapted to the Nambu–Poisson structure Π, i.e., $\{x^i\}$ are local coordinates around some point M, where Π is non-zero, such that[5]

$$\Pi = \partial_1 \wedge \dots \wedge \partial_{p+1}. \tag{10}$$

With this choice of coordinates, we find

$$\hat{F}'^{i_1\dots i_{p+1}} = \{\hat{x}^{i_1},\dots,\hat{x}^{i_{p+1}}\} - \{x^{i_1},\dots,x^{i_{p+1}}\}, \tag{11}$$

where the second bracket is in fact either zero or equal to the $p+1$ epsilon symbol in the noncommutative directions $1,\dots,p+1$. Roman indices $a_1,\dots,a_{p+1}$ shall henceforth denote these directions. Furthermore, we will focus on the case where for the covariantizing map ρ^* acts trivially (i.e. $\hat{x}^i = x^i$) on coordinates x^i with indices in the commutative directions $p+2,\dots,n$. It follows from (1) that only the covariant coordinates in the noncommutative directions transform non-trivially under gauge transformations and that the gauge fields $\hat{A}^i$ are trivial for $i=p+2,\dots,n$. Also, all the field strengths, except those indexed solely by noncommutative indices $i=1,\dots,p+1$, will automatically be zero. With these conventions, we can use the $p+1$ epsilon tensor to lower the index on $\hat{A}^a$ and introduce another kind of gauge potential uniquely determined by complete antisymmetrization of its non-zero components $\hat{A}_B$ labeled by strictly ordered p-tuples of indices, with individual indices taking values in the labels of the noncommutative directions

$$\hat{A}_B := \epsilon_{aB}\hat{A}^a. \tag{12}$$

The components $\hat{A}_B$ transform in a more familiar looking manner (but recall that we are still dealing with a $p+1$ Nambu–Poisson bracket and a $(p-1)$-form gauge parameter Λ):

$$\delta_\Lambda \hat{A}_B = (d\Lambda)_B + \{\hat{A}_B, \Lambda\}. \tag{13}$$

Similarly, we define the corresponding field strength with components $\hat{F}'_{aB}$ by

$$\hat{F}'_{aB} = \epsilon_{aC}\big(\hat{\Pi}'^{bC} - \Pi^{bC}\big)\epsilon_{bB}. \tag{14}$$

The components $\hat{F}'_{aB}$ transform as expected

$$\delta_\Lambda \hat{F}'_{aB} = \{\hat{F}'_{aB}, \Lambda\}. \tag{15}$$

A straightforward check reveals that $\hat{F}'_{aB}$ can be consistently extended to be antisymmetric in all of its indices. Finally, $\hat{F}'_{aB}$ can be

[5] Here we ignore, for simplicity, points where Π could possibly be zero and focus on globally non-degenerate Nambu–Poisson structures.

expressed in terms of the gauge potential $\hat{A}_B$. For this, we need a $(p+1-q)$-ary Nambu bracket defined as[6]

$$\{\cdot,\ldots,\cdot\}^{i_1\ldots i_q} := \{x^{i_1},\ldots,x^{i_q},\cdot,\ldots,\cdot\}.$$

Now, using (3), (11), (12) and (14) we obtain

$$\hat{F}'_{1\ldots p+1} = (d\hat{A})_{1\ldots p+1} + \sum_{r=0}^{p-1}\sum_{\sigma\in S(r,n-r)} (-1)^{\sum_{k=r+1}^{p+1}(\sigma(k)-1)} \times \operatorname{sgn}(\sigma)\{\hat{A}_{[\sigma(r+1)]},\ldots,\hat{A}_{[\sigma(p+1)]}\}^{\sigma(1)\ldots\sigma(r)}, \tag{16}$$

where $\sigma\in S(r,n-r)$ is an $(r,n-r)$ shuffle, and $[a]$ is the multi-index $1\cdots(a-1)(a+1)\cdots(p+1)$. This formula is a generalization to $p>1$ of the well-known $p=1$ formula for the (noncommutative) field strength that involves the 2-bracket ("commutator") of gauge fields.

In the next section we will use a higher analog of the Seiberg–Witten map in order to construct explicit expressions for the covariant coordinates and noncommutative gauge fields. This will allow us to also supplement the remaining components of the Nambu–Poisson gauge field strength (14), i.e., the ones with at least one index in a commutative direction.

5. Nambu–Poisson gauge fields via Seiberg–Witten map

We start with a brief summary of the relevant facts concerning the Seiberg–Witten map as it applies in the present context. We refer the reader to a detailed exposition in [7]. All order solution to the Seiberg–Witten map related to Nambu–Poisson M5-brane theory can be found in [8].

Let us consider a p-form gauge potential a on M with corresponding field strength $F=da$. Infinitesimally, under a gauge transformation given by a $(p-1)$-form λ,

$$\delta_\lambda a = d\lambda, \qquad \delta_\lambda F = 0. \tag{17}$$

Using the $(p+1)$-form F we construct from a given Nambu–Poisson tensor Π the F-gauged tensor which we denote for now by Π_F,[7]

$$\Pi_F := \left(1-\Pi F^T\right)^{-1}\Pi = \Pi\left(1-F^T\Pi\right)^{-1}. \tag{18}$$

These expressions are to be interpreted as matrix equations for the corresponding maps sending p-forms to 1-forms, cf. Section 2. The superscript T stands for the transposed map. For $p>1$, the $(p+1)$-tensor Π_F is always a Nambu–Poisson one,[8] furthermore, we also have due to factorizability of Π,

$$\Pi_F = \left(1-\frac{1}{p+1}\langle\Pi,F\rangle\right)^{-1}\Pi, \tag{19}$$

where $\langle\Pi,F\rangle = \Pi^{iJ}F_{iJ} \equiv \operatorname{Tr}(\Pi F^T)$.

Now we define a 1-parametric family of Nambu–Poisson tensors $\Pi_t := (1-t\Pi F^T)^{-1}\Pi$, cf. Footnote 7, interpolating between Π and Π_F. Differentiation of Π_t with respect to t gives:

$$\partial_t\Pi_t = \Pi_t F^T\Pi_t. \tag{20}$$

This equation can be rewritten as

$$\partial_t\Pi_t = -\mathcal{L}_{A_t^\sharp}\Pi_t, \tag{21}$$

where the time-dependent vector field $A_t^\sharp$ is defined as $A_t^\sharp = \Pi_t^\sharp(a) = \Pi_t^{iJ}a_J\partial_i$ and $\mathcal{L}_{A_t^\sharp}$ is the corresponding Lie derivative. Eq. (21) implies that the flow ϕ_t corresponding to $A_t^\sharp$, together with the initial condition $\Pi_0=\Pi$, maps Π_t to Π, that is,

$$\phi_t^*(\Pi_t) = \Pi. \tag{22}$$

We have thus found the map $\rho_a := \phi_1$, such that $\rho_a^*(\Pi') = \Pi$. This is the higher form gauge field ($p>1$) analogue of the well known semiclassical Seiberg–Witten map. We emphasize the dependence of this map on the p-form a by an explicit addition of the subscript a. The following observation is important: The Nambu–Poisson tensor Π_t is gauge invariant (because it depends on the p-potential a only via the gauge invariant $p+1$ form field strength $f=da$), but the Nambu–Poisson map ρ_a is not: The infinitesimal gauge transformation $\delta_\lambda a = d\lambda$, with a $(p-1)$-form gauge transformation parameter λ, induces a change in the flow, which is generated by the vector field $X_{[\lambda,a]} = \Pi^{iJ}d\Lambda_J\partial_i$, where the $(p-1)$-form Λ, explicitly given by

$$\Lambda = \sum_{k=0}^{\infty}\frac{(\mathcal{L}_{A_t^\sharp}+\partial_t)^k(\lambda)}{(k+1)!}\bigg|_{t=0}, \tag{23}$$

is the semiclassically noncommutative $(p-1)$-form gauge parameter. This leads to the following rule for the gauge transformation of coordinates $\hat{x}_a^i := \rho_a^*(x^i)$, cf. (1):

$$\delta_\lambda\hat{x}_a^i = \{\hat{x}_a^i,\Lambda\}. \tag{24}$$

Hence, the generalized Seiberg–Witten map provides us with an explicit construction, based on ordinary higher gauge fields, of the covariant coordinates $\hat{x}^i$ that we introduced in Section 3. As a consequence, we can identify $\hat{x}^i \equiv \hat{x}_a^i$ and $\Pi' \equiv \Pi_F$. Moreover, $\hat{x}^i = \hat{x}_a^i = x^i$, for the "commutative" directions $i=p+2,\ldots,n$. All discussion of the previous Sections 3 and 4 applies directly.

Having the ordinary p-form gauge field a at our disposal we can now define the full Nambu–Poisson field strength $\hat{F}'$ with all components (in noncommutative as well as in commutative directions), such that its components in the noncommutative directions $x^1,\ldots,x^{p+1}$ coincide with those of $\hat{F}'_{aB}$ (14).

For this let

$$F' := F\left(1-\Pi^T F\right)^{-1} = \left(1-F\Pi^T\right)^{-1}F \tag{25}$$

and define

$$\hat{F}'_{iJ} := \rho_A^* F'_{iJ}, \tag{26}$$

i.e., the components of F' evaluated in the covariant coordinates. It is a rather straightforward check to see that for all indices $i_1,\ldots,i_{p+1}$ taking values only in the set $\{1,\ldots,p+1\}$ we get exactly the $\hat{F}'_{aB}$ of (14).

Now we turn our attention to the remaining components of $\hat{F}'$ (including commutative directions). Starting from (25) and (26), we can with the help of Footnote 7 and the explicit expression for Π in coordinates (10) use a construction very similar to the one leading to (16). We find that the resulting expressions involve a covariant scalar function that depends on $\hat{A}$ (and hence via the generalized Seiberg–Witten map also on the ordinary p-form gauge potential a):

$$f[\hat{A}] := 1 + \sum_{r=0}^{p}\sum_{\sigma\in S(r,n-r)}(-1)^{\sum_{k=r+1}^{p+1}(\sigma(k)-1)} \times \operatorname{sgn}(\sigma)\{\hat{A}_{[\sigma(r+1)]},\ldots,\hat{A}_{[\sigma(p+1)]}\}^{\sigma(1)\ldots\sigma(r)}.$$

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[6] With some abuse of notation we allow also for the case $p=q$, i.e., the "1-ary" bracket, which will become useful later.

[7] We assume that $1-\Pi F^T$ is invertible. In a more formal approach we also could treat Π_F as a formal power series in Π.

[8] Even for a non-closed F.

Firstly, let us consider $\hat{F}'_{aK}$ with the index a taking on values in $\{1,\dots,p+1\}$, and K containing at least one index in one of the commutative directions $p+2,\dots,n$. We find

$$\hat{F}'_{aK} = f[\hat{A}]\hat{F}_{aK}, \tag{27}$$

where $\hat{F}_{aK} = \rho^* F_{aK}$ is the component F_{aK} of the ordinary (commutative) field strength evaluated at the covariant coordinates $\hat{x}^i$. Secondly, for the components of $\hat{F}'$ with index k taking value in $\{p+2,\dots,n\}$, and A containing only the indices lying in the set $\{1,\dots,p+1\}$,

$$\hat{F}'_{kA} = f[\hat{A}]\hat{F}_{kA}. \tag{28}$$

Finally, for the components $\hat{F}'_{kL}$, where k takes value in the set $\{p+2,\dots,n\}$ and L contains at least one index of the same set, we have

$$\hat{F}'_{kL} = \hat{F}_{kL} + f[\hat{A}]\sum_{a=1}^{p+1}(-1)^{a+1}\hat{F}_{k[a]}\hat{F}_{aL}. \tag{29}$$

Under (ordinary) infinitesimal gauge transformations δ_λ, all components of $\hat{F}'$ transform as

$$\delta_\lambda \hat{F}' = \{\hat{F}', \Lambda\}, \tag{30}$$

justifying calling it "Nambu–Poisson" or "(semiclassically) noncommutative" field strength.

Note that unlike for the noncommutative components, the full tensor $\hat{F}'$ cannot be extended to be a totally antisymmetric one.

6. Action

For simplicity, we assume Euclidean space–time signature.[9] The action

$$\frac{1}{g}\int_M d^n x \hat{F}'_{iJ}\hat{F}'^{iJ} \tag{31}$$

is by construction invariant under ordinary commutative as well as under Nambu–Poisson (semiclassically noncommutative) gauge transformations. This can easily be verified using partial integration. The coupling constant g is dimensionless in $n = 2(p+1)$ spacetime dimensions, i.e. for example for $p = 1$, $n = 4$ (NC Maxwell) and for $p = 2$, $n = 6$ (M2–M5 system). In the following we will set $g = 1$.

We expand $\hat{F}'$ in a power series in Π

$$\hat{F}'_{iJ} = F_{iJ} + A_L\Pi^{kL}F_{iJ,k} + F_{iL}\Pi^{kL}F_{kJ} + o(\Pi^2). \tag{32}$$

The corresponding expansion of the action (31) is

$$\int_M d^n x\hat{F}'_{iJ}\hat{F}'^{iJ} = \int_M d^n x\bigg\{F_{iJ}F^{iJ} - \frac{1}{p+1}F_{iJ}F^{iJ}F_{kL}\Pi^{kL} + 2F^{iJ}F_{iL}\Pi^{kL}F_{kJ}\bigg\} + o(\Pi^2). \tag{33}$$

A quantization of the underlying Nambu–Poisson structure will not add quantum corrections to the action at the given order of expansion.

[9] Another simple possibility would be consider the Minkowskian space–time, with Π extending in the spatial directions only. In case of a general metric g we would have to use the inverse metric matrix elements evaluated in the covariant coordinates to rise the indices of $\hat{F}'$ and the density defined by the metric also evaluates in the covariant coordinates.

Shifting the components $\hat{F}'_{1\dots p+1}$ of the Nambu–Poisson field strength by the constants $\epsilon_{1\dots p+1}$, will not affect the gauge invariance of the action (31). Using (11) and (14) we see that the action (31) with shifted $\hat{F}'$ takes the form of a semiclassical version of a Nambu–Poisson matrix model:

$$\begin{aligned} S_M &= \int d^n x\{\hat{x}^a, \hat{x}^A\}\{\hat{x}_a, \hat{x}_A\} \\ &= \int d^n x\frac{1}{p!}\{\hat{x}^{a_1},\dots,\hat{x}^{a_{p+1}}\}\{\hat{x}_{a_1},\dots,\hat{x}_{a_{p+1}}\}, \end{aligned} \tag{34}$$

where the summation in the second expression runs over all (not strictly ordered) $(p+1)$-indices $(a_1,\dots,a_{p+1})$ and $(b_1,\dots,b_{p+1})$, with all of them in the noncommutative direction. We could actually drop the a priori restriction of the summation to noncommutative directions, since the Nambu–Poisson bracket automatically takes care of this. For a more detailed discussion of the (semiclassical) matrix model we refer to [7].

Given an appropriate quantization $[\cdot,\dots,\cdot]$ of the Nambu–Poisson bracket and trace of the quantized Nambu–Poisson structure, the Nambu–Poisson matrix model takes the form

$$\tilde{S}_M = \frac{1}{p!}\mathrm{Tr}[\hat{x}^{a_1},\dots,\hat{x}^{a_{p+1}}][\hat{x}_{a_1},\dots,\hat{x}_{a_{p+1}}]. \tag{35}$$

There have been several attempts to find a consistent quantization of Nambu–Poisson structures. One of these [15] is in fact suitable for our purposes (at least in the case $p = 2$): It is an approach based on nonassociative star product algebras on phase space, whose Jacobiator defines a quantized Nambu–Poisson bracket on configuration space. Let us mention without going into details that this approach can be adapted to provide a consistent quantization of the Nambu–Poisson gauge theory described in this letter, including a quantization of the generalized Seiberg–Witten maps. Details of this construction are beyond the scope of the present letter and will be reported elsewhere.

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