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DOCTORAL THESIS

Poisson–Lie T-duality and its applications

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Abstrakt

V této dizertační práci podrobně analyzujeme Poissonovu–Lieovu T-dualitu, která působí jako transformace svazující dvoudimenzionální sigma modely v různých (zakřivených) pozadích. Získané poznatky využíváme mimo jiné k nalezení řešení klasických pohybových rovnic sigma modelů v některých fyzikálně zajímavých pozadích.

Nejprve rekapitulujeme známé poznatky týkající se Poissonovy–Lieovy T-duality. Pro přehlednost připomeneme, jak lze pomocí T-duality svázat zdánlivě různé sigma modely na základě symetrií jejich pozadí, a zmíníme problémy, které vyvstávají při pokusech o rozšíření konceptu duality na grupy nekomutujících symetrií. Studium Noetherovských proudů indukovaných akcí grupy symetrií vede ke zjištění, že geometrickou strukturou na níž je třeba koncept duality vybudovat je Drinfeldův double. To umožňuje zavést Poissonovu–Lieovu T-dualitu a Poissonovu–Lieovu T-pluralitu, které T-dualitu zobecňují. V první části práce Drinfeldův double rozšíříme, abychom do standardního popisu zahrnuli i takzvané přihlížející proměnné. Prostudujeme také transformaci okrajových podmínek a uvedené postupy demonstrujeme na několika konkrétních příkladech.

Ve druhé části práce se zaměříme na dualitu sigma modelů v pozadích označovaných jako rovinné vlny. Tyto modely nalézají v teorii strun široké uplatnění. Vzhledem k bohaté struktuře symetrií rovinných vln jsou navíc vhodnými kandidáty pro aplikaci T-duality. Díky Poissonově–Lieově T-dualitě nalezneme řešení pohybových rovnic pro několik modelů v pozadí rovinných vln. Ukážeme také, že některé známé exaktní sigma modely jsou ve skutečnosti duální k modelu žijícímu v plochém pozadí.

Abstract

The Poisson–Lie T-duality transformation relating two-dimensional sigma models in different (curved) backgrounds is thoroughly studied and applied to obtain solutions of classical equations of motion of several sigma models in physically interesting backgrounds.

In the beginning we summarize well-known facts concerning Poisson–Lie T-duality. We recapitulate how seemingly different sigma models whose backgrounds have local symmetries can be related via T-duality, and mention the obstacles which appear when non-Abelian groups of symmetries are taken into account. The study of Noether currents induced by an action of a symmetry group reveals that the relevant geometric structure underlying T-duality is the Drinfel'd double. This concept can be used to generalize the notion of T-duality to Poisson–Lie T-duality and Poisson–Lie T-plurality. In the first part of the thesis we extend the Drinfel'd double to accommodate Poisson–Lie T-duality/plurality with spectator fields. We also find a formula that realizes the transformation of boundary conditions, and demonstrate the notion of Poisson–Lie T-duality using specific examples.

In the second part of the thesis we focus on duality of sigma models in plane wave backgrounds. These models find various applications in string theory. Moreover, due to the rich structure of groups of symmetries of plane wave backgrounds, these sigma models are appropriate to study implications of T-duality. Applying the Poisson–Lie T-duality transformation, we find solutions of equations of motion of several sigma models in plane wave backgrounds. We also prove that some of the exact sigma models mentioned in the literature are in fact dual to the flat background.

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Contents

List of symbols

$\mathscr{M}$ .... spacetime manifold, target space
Σ .... string worldsheet
$C^\infty(\mathscr{M})$ .... smooth functions on $\mathscr{M}$
$\mathcal{T}^p_q(\mathscr{M})$ .... tensor fields on $\mathscr{M}$, p-times contravariant, q-times covariant
$\mathfrak{X}(\mathscr{M})$ .... vector fields on $\mathscr{M}$
$\Omega^p(\mathscr{M})$ .... differential p-forms on $\mathscr{M}$
$\eta_{\mu\nu}$ .... (components of) Minkowski metric
$G_{\mu\nu}$ .... (components of) metric tensor
$\Gamma^\lambda_{\mu\nu}$ .... Christoffel symbols
$R^\kappa{}_{\lambda\mu\nu}$ .... (components of) Riemann curvature tensor
$R_{\mu\nu}$ .... (components of) Ricci tensor
R .... Ricci (curvature) scalar
$C_{\lambda\mu\nu\kappa}$ .... (components of) Weyl tensor
τ, σ .... worldsheet coordinates
σ_-, σ_+ .... light-cone worldsheet coordinates
$T_{\alpha\beta}$ .... (components of) worldsheet energy-momentum tensor
$B_{\mu\nu}$ .... (components of) B-field (torsion potential)
$H_{\mu\nu\lambda}$ .... (components of) torsion
Φ .... dilaton field
$F_{\mu\nu}$ .... (components of) sigma model background
$\nabla_{\mathcal{K}}$ .... covariant derivative w.r.t. a vector field $\mathcal{K}$
$\mathfrak{g}$ .... Lie algebra
$\mathscr{G}$ .... Lie group
c^c_{ab} .... structure constants of a Lie algebra
$\Pi^{\mu\nu}$ .... (components of) Poisson bivector
$\mathcal{L}_{\mathcal{K}}$ .... Lie derivative w.r.t. a vector field $\mathcal{K}$
v^μ_a .... components of left-invariant vector fields
e^a_μ .... components of right-invariant forms
$\mathcal{R}$ .... gluing matrix

Introduction

Preface

There is a principle that can be tracked over centuries of history of science. It is a strong driving force expressing our desire to understand even the most complex phenomena in terms of a few laws of nature, or even in terms of one all-encompassing unified theory. However, this was hardly conceivable even in Newton's era, and a plethora of new phenomena were yet to be discovered. During the 19th and 20th century the scientific community grew and so did the understanding of the laws of nature. In 1905 the discrepancy between Newtonian mechanics and Maxwell's electrodynamics was resolved by Einstein, who introduced special relativity and replaced Galilean invariance by Lorentz invariance. Having introduced general relativity in his 1916 paper, Einstein spent most of his life trying to find the theory that would unify the known forces of nature.

Meanwhile, from the work of Planck, Bohr, de Broglie, Schrödinger, Heisenberg, Dirac and others a completely different physics emerged. Quantum mechanics and quantum field theory can be understood as frameworks, from which the most successful physical theory – the Standard Model of particle physics – was born. The unification of electromagnetism and the weak force by Weinberg, Salam and Glashow in early 1970s was followed by the development of quantum chromodynamics describing the strong force, and the three forces were established in one framework based on the gauge theory with $SU(3) \times SU(2) \times U(1)$ gauge group. Within the Standard Model, the forces are mediated by bosons: eight gluons, W^+, W^-, Z^0 and the photon. These interact with fermions that constitute matter. Within the theory, all matter particles are arranged into multiplets on which the gauge group acts. All the particles then receive mass through the Higgs mechanism, i.e. via spontaneous symmetry breaking and via interaction with the Higgs boson.

In spite of the fact that the Standard Model is based on many ingredients and many parameters have to be determined experimentally, its theoretical construction is splendid, and its predictions were confirmed with an unprecedented accuracy. However, there are phenomena that do not fit in the framework, and the need for physics beyond the Standard Model is obvious. The most disruptive fact is that while three of the forces of nature were unified, gravitation stands aside, and any attempt to devise a quantum theory of gravitation fails. While the general relativity rules the physics at large scales, the quantum theory rules the physics at micro scales, and it is clear that without a consistent theory connecting both theories we can hardly understand processes where both should be applied, such as the cosmology of early universe or the phenomena related to black holes and their dense matter.

During the last few decades several candidates for a unified theory appeared. Perhaps the most promising was the string theory and its descendants, and many physicists were attracted to this field. The simple idea that elementary particles should be treated as one-dimensional objects rather than point particles gives rise to a framework from which particles emerge as oscilations of open or closed strings. String theory is a quantum theory, and since it naturally incorporates a massless spin-2 particle interpreted as graviton mediating the force of gravity, a lot of effort has been put into its development as a unifying theory. Soon it became clear that this is a long-distance run. It has been shown that in order to build a consistent theory, the dimension of spacetime has to be fixed to 26 for theories including only bosons and to 10 for more realistic superstring theories including bosons and fermions. To be able to describe the real world, string theory has to incorporate supersymmetry as well.

In the 1980s there was not only one plausible superstring theory, but five of them, with the elusive eleven-dimensional M-theory to be discovered yet. In this confusing situation some relations between different theories were discovered, and since then it is believed that the existing theories can be regarded as facets of a single truly unified theory. Using these relations, known as dualities, researchers found that the same physics can rise from apparently different settings. If a problem turns out to be too difficult to solve within one string theory, there is a chance that it can be solved within another one.

The idea of extra dimensions is actually rather old. Soon after the publication of general relativity, Kaluza and Klein tried to unify gravitation and electromagnetism by introducing a five-dimensional theory represented by a metric $G_{\mu\nu}$. The fields $G_{\mu\nu},\ \mu,\nu = 0,\dots,3$, would stand for the four-dimensional metric, $G_{\mu 4},\ \mu = 0,\dots,3$, for the photon and G_{44} for some predicted scalar field. Since the extra dimensions were never detected, spacetime was constructed as a product $\mathscr{M} \times S^1$ of a four-dimensional manifold $\mathscr{M}$ and a circle S^1 of small radius R representing the fifth compactified dimension.

String theory approaches the extra dimensions in the same manner describing the fields in lower dimension as descendants of fields in higher dimension. Compared to the Standard Model and its supply of parameters, string theories have only one parameter expressing the tension of the string. However, the number of possibilities how the extra dimensions can be compactified and how the spacetime manifold can be formed is enormous, and if different configurations may give the same physics, the hunt for inequivalent string vacua seems hopeless.

To make any calculations tractable, the extra dimensions are usually compactified to give an n-dimensional torus T_n (for instance T_6 for superstrings). In the discussion of the stability of the compactified space in Ref. [1] the string effective potential V was studied. V receives not only contributions from the discrete momenta along the ith compactified dimension, but contains also contributions coming from the tension energy of the string, which may wind around the ith circumference. Considering the limit $R_i \to 0$ of the radii of compactified dimensions, the authors found that the potential has its minimum in $\sqrt{\alpha'}/R_i = 1$, where α' is the Regge slope, the only parameter appearing in string theory. While the string winding modes tend to reduce their energy and the radii of the torus, the kinetic energy increases as $R_i \to 0$. The balance between these two phenomena keeps the compactification stable. Moreover, a peculiar feature of strings was discovered because it turned out that the potential is invariant under the interchange

$$\frac{\sqrt{\alpha'}}{R_i} \to \frac{R_i}{\sqrt{\alpha'}}.$$

This behavior is known as T-duality (the "T" standing for toroidal) and nowadays it is a standard topic covered in textbooks, see [2], [3] and references mentioned there.

Although compactification is inevitable to acquire a practical theory, T-duality itself can be abstracted from global issues and treated rather as a transformation relating equations of motion of strings propagating in the spacetime manifold. String dynamics is given by a sigma model. In his paper [4] Buscher showed that T-duality relates sigma models with different target space geometries (the "T" may also stand for target space duality). As the transformation changes the Riemann tensor, duality may relate sigma models in curved backgrounds to a sigma model in the flat background, thus allowing us to find solutions of their field equations. Taking care of determinants appearing in the path-integral formulation of the duality transformation, see Ref. [5], Busher proved that T-duality preserves one-loop conformal invariance provided the dilaton field is shifted properly. Buscher's duality is similar to the notorious duality between electric and magnetic fields. A gauge symmetry-based approach was presented in Ref. [6], where seemingly distinct dual sigma models were proven to be equivalent as conformal field theories (CFT).

Both procedures were executable only when a symmetry of the background was present, and T-duality has been easily extended to Abelian groups of symmetries. In a series of subsequent papers various aspects of T-duality were investigated. In [7] the group of discrete symmetries (including T-duality) of CFTs in backgrounds having d-dimensional Abelian groups of symmetries was found to be isomorphic to $O(d, d, \mathbb{Z})$. A covariant formulation of the transformation allowed authors of [8] to study global aspects of T-duality, and the procedure was recognized as a canonical transformation in [9].

In the attempt to extend T-duality to non-Abelian groups of symmetries several fundamental obstacles were encountered. The technique introduced in Ref. [10] relied on the presence of symmetries, but it turned out that they are not preserved under non-Abelian T-duality. The dual model lacked the symmetries, and duality was a misnomer to some extent as it was impossible to revert the transformation. Klimčík and Ševera overcame this obstacle by embedding both Abelian and non-Abelian T-duality into an algebraic framework based on Drinfel'd double, and introduced Poisson–Lie T-duality in Ref. [11]. Within their approach, a clear geometric description of duality as a canonical transformation was given, see also [12], and a lot of insight into the relation between the dual sigma models was obtained.

Soon a path integral formulation of Poisson–Lie T-duality was introduced in Refs. [13], [14], and one-loop renormalizability of dual models was investigated in the attempt to establish their equivalence at the quantum level, see [15] or more recent papers [16], [17]. Although Poisson–Lie T-duality was originally formulated for closed bosonic strings, it has been extended to incorporate duality of open strings and d-branes on which string endpoints lie. A first probe into the global issues was made in Ref. [18], and a compact formula for the transformation of boundary conditions of open strings under Poisson–Lie T-duality was given in [19] in terms of gluing matrices, which encode the properties of d-branes.

The introduction of Poisson–Lie T-duality led to a paradigm shift because the isometries of the background were no longer necessary in order to perform T-duality. Instead, the possibility to implement the technique relied on the presence of the solid algebraic structure of the Drinfel'd double. Due to the classification of all four- and six-dimensional real Drinfel'd doubles executed in [20], [21], numerous examples of mutually dual two- and three-dimensional sigma models could be constructed and various properties of T-duality further tested. The classification also showed that some Drinfel'd doubles can be decomposed in several ways, thus accommodating not only duality, but rather Poisson–Lie T-plurality discussed in [22]. In higher dimension, however, the classification is missing, and only a handful of examples of Drinfel'd doubles is known. To be able to investigate physically interesting models, we extend the framework by the so-called spectator fields, which do not participate in the duality transformation.

If any physical framework aims to represent something more than just a theoretical construct, it is necessary that it provides realistic examples where its implications can be tested. Unfortunately, even the simplest nontrivial models in string theory are usually difficult to grasp. When we analyze the behavior of strings propagating in curved backgrounds, our first step might consist of finding the classical solution of equations governing the motion of the string. This is often very complicated, not to say impossible, which is why any solvable case attracts considerable attention. However, examples presented for instance in [23] show that Poisson–Lie T-duality can be utilized to find solutions of several sigma models in curved backgrounds dual to the flat spacetime.

Due to its physical relevance and rather simple equations of motion, a prominent class of sigma models in plane wave backgrounds appears repeatedly in literature. Plane waves provide not only solvable models [24], [25], but occur also in the study of string behavior in the presence of spacetime singularities [26], [27], and give some of the exact conformal sigma models [28]. Moreover, the rich structure of groups of symmetries of these backgrounds allows us to test Poisson–Lie T-duality on these models, to search for their solutions, for solutions of their duals and generate other solvable models.

In this thesis we mostly concentrate on T-duality as a means of obtaining new string sigma models and solutions of their classical equations. From a broader perspective, T-duality turned out to be a valuable tool when it comes to disentangling the nature of strings and superstrings because it may also help us understand non-perturbative aspects of the emerging physics. The field evolved rapidly over the last two decades, and the notions of T-duality and Poisson–Lie T-duality were extended in various directions, including fermionic T-duality [29] and duality of sigma models with worldsheet supersymmetry [30], [31]. Nevertheless, while Poisson–Lie T-duality handles nicely the transformation of the spacetime metric and the B-field, analogous framework describing transformation of other fields, such as Ramond fields [32], [33] is missing, and there is still an ongoing research in this area.

Any theory has to withstand criticism, especially if it lacks evidence. The unification of the four forces of nature within a compact framework is a goal worth pursuing, and we started our discussion boldly. However, despite all the achievements and all the effort invested into the research in the field of string theory, it is so complex that with each resolved problem many new questions appear. The effects of gravity at small scales are so tiny that there is only a little hint based on the experiment, and the only lead that we have are the requirements of self-consistency of the theory and the fact that in some low-energy limit the theory has to reproduce the knowledge we already have. It is questionable when, or whether, string theory will be able to give measurable predictions. But even if it eventually failed as a unifying theory, its contributions to various fields of physics, such as the role of ADS/CFT correspondence in condensed matter physics, should not be forgotten. String-theoretical research also led to developments in mathematics. The investigation of Calabi–Yau manifolds representing spacetime led to the invention of generalized geometry [34] that seems to provide the appropriate framework to formulate T-duality [35]. Moreover, T-duality motivated the growth of new concepts in differential geometry, such as the double field theory with its T-folds [36]. Finally, even if we forget about what was said earlier, the Poisson–Lie T-duality transformation presented in this thesis can still be considered to be a useful way of obtaining a solution of a system of partial differential equations.

Author's contribution

The main purpose of this doctoral thesis is to present a compact overview of the Poisson–Lie T-duality transformation as a method of generating new string sigma models and their classical solutions. The thesis is based on the following papers written in collaboration with Prof. RNDr. Ladislav Hlavatý, DrSc. and Ing. Vojtěch Štěpán:

A) L. Hlavatý, I. Petr, and V. Štěpán. Poisson–Lie T-plurality with spectators. *Journal of Mathematical Physics*, 50(4), 2009.

B) L. Hlavatý and I. Petr. New solvable sigma models in plane-parallel wave background. *International Journal of Modern Physics A*, 29(02), 2014.

C) L. Hlavatý and I. Petr. Plane-parallel waves as duals of the flat background. *Classical and Quantum Gravity*, 32(3), 2015.

Short excerpts were made from author's contributions to conference proceedings:

a) I. Petr. From Buscher duality to Poisson–Lie T-plurality on supermanifolds. *AIP Conference Proceedings*, 1307, 2010.

b) I. Petr. Poisson sigma models and Lie bialgebras, *J. Phys. Conf. Ser.*, 343, 2012.

The three published papers constitute the body of chapters 5, 6, 8 and 9. We present them here with minor changes. These changes were forced by our commitment to write a homogeneous text and present the issue of Poisson–Lie T-duality in a compact manner. Therefore the introductory texts and sections covering the Poisson–Lie T-duality transformation, which were common to those papers, were extracted, put separately into chapter 4, and we do not repeat them over and over again.

In the first paper we introduced spectator fields into the framework of Poisson–Lie T-plurality, and subsequently studied the transformation of open string boundary conditions in the presence of spectators. These two topics were divided and form chapters 5 and 6 of the thesis. The results of [37] were further elaborated in our second article, where non-Abelian duals of a homogeneous isotropic plane wave were identified as plane waves and their classical solutions were found. This represents the content of chapter 8. Last but not least, chapter 9 presents the results published in our third paper, where we carried out the classification of non-Abelian T-duals of the flat metric in four dimensions with respect to four-dimensional continuous subgroups of the Poincaré group. Majority of the dual models was identified as conformal sigma models in plane wave backgrounds and their solutions were found. Compared to the original articles, some explanations were expanded, and partial results were added where the presentation was sketchy. All the changes are explicitly mentioned at the beginning of each particular chapter.

Outline

The thesis is composed of two major parts, so that the general framework of Poisson–Lie T-duality developed in Part I can be presented separately from its applications that we defer to Part II.

To get acquainted with the sigma model and other objects of our study, we first summarize basics of string theory in chapter 1. In chapter 2 we present the essential notions of Poisson–Lie groups and Drinfel'd doubles, and develop the algebraic background underlying Poisson–Lie T-duality. Chapter 3 covers Abelian and non-Abelian T-duality that might seem slightly outdated. Nevertheless, these topics are still actual and should not be omitted. We use these frameworks to demonstrate various aspects of the more intricate Poisson–Lie T-duality, which is introduced in chapter 4. A suitable extension of the Drinfel'd double allows us to study Poisson–Lie T-duality in the presence of spectator fields in chapter 5. The Poisson–Lie T-duality transformation of open string boundary conditions is then discussed in chapter 6.

Part II of our work is devoted to the study of sigma models in plane wave backgrounds. To explain their physical importance, we recapitulate relevant characteristics of these backgrounds in chapter 7. The sigma model in a homogeneous isotropic plane wave background is studied and dualized in chapter 8, and duals to the sigma model in the flat Minkowski background are investigated in chapter 9. Finally, results of our work and concluding remarks are summarized in chapter 10.

Conventions on pseudo-Riemannian geometry

For reader's convenience we shall start with a short summary of basic objects arising in the field of pseudo-Riemannian geometry. Sadly, it is common that different authors use different conventions. We shall mostly stick to those used in Weinberg's book [38]. The only change will occur in the sign of the Riemann curvature tensor.

In general, we shall study the geometry of D-dimensional smooth manifold $\mathscr{M}$ equipped with a *metric* $G \in \mathcal{T}_2^0(\mathscr{M})$ with Lorentzian signature. Our convention for the *flat Minkowski metric* in D dimensions shall be

$$\eta = \operatorname{diag}\underbrace{(-1, 1, \ldots, 1)}_{D\text{-terms}}.$$

Unless stated otherwise, we use Einstein summation rule for repeated indices. For example, in local coordinates x^μ the line element is given by

$$ds^2 = G_{\mu\nu}\, dx^\mu dx^\nu.$$

The *Christoffel symbols* of the Levi-Civita connection are calculated as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} G^{\lambda\sigma} \left(\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\mu\sigma} - \partial_\sigma G_{\mu\nu} \right),$$

where we use the inverse $G^{\lambda\sigma}$ of the matrix $G_{\lambda\sigma}$ to raise the index. The motion of a freely falling particle traveling on a trajectory $c^\mu(\tau)$ in a background given by some metric G is governed by the *geodesic equation*

$$\frac{d^2 c^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dc^\nu}{d\tau} \frac{dc^\lambda}{d\tau} = 0.$$

The components of the *Riemann curvature tensor* are expressed in terms of the Christoffel symbols and its derivatives as

$$R^\mu{}_{\nu\kappa\lambda} = \partial_\kappa \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\kappa} + \Gamma^\mu_{\kappa\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\mu_{\lambda\rho} \Gamma^\rho_{\nu\kappa}.$$

Contracting the indices in the components of the curvature tensor, we obtain the *Ricci tensor*

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu},$$

and contracting even further, we get the *Ricci (curvature) scalar*

$$R = G^{\mu\nu} R_{\mu\nu}.$$

Our sign convention for the curvature tensor was chosen in order to render positive curvature scalar of a sphere embedded in Euclidean space. The traceless part of the curvature tensor – the *Weyl tensor* – then has components

$$\begin{aligned} C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} &+ \frac{1}{D-2} (G_{\lambda\kappa} R_{\nu\mu} - G_{\lambda\nu} R_{\kappa\mu} + G_{\mu\nu} R_{\kappa\lambda} - G_{\mu\kappa} R_{\nu\lambda}) \\ &+ \frac{R}{(D-1)(D-2)} (G_{\lambda\nu} G_{\mu\kappa} - G_{\lambda\kappa} G_{\mu\nu}). \end{aligned}$$

Part I

Poisson–Lie T-duality

Chapter 1

Sigma models in string theory

In this chapter we introduce the sigma model, summarize relevant facts concerning sigma model action, and stress its significance in the formulation of bosonic string theory. The emphasis is put on the derivation of conditions that a string has to satisfy in order to give a viable field theory. We start with the description of a free non-interacting bosonic string evolving in the flat Minkowski background, and continue with the study of strings in general background. This necessary introductory discussion is common to most textbooks and we partially follow the standard explanation given in [39]. For more elaborate discussion see also Refs. [2], [3].

1.1 Strings moving in the flat background

A string is a one-dimensional object – a curve – whose points are labeled by values of a parameter σ. To consider its evolution, we also introduce a timelike parameter τ. The string propagating in a spacetime manifold $\mathscr{M}$ – called the *target space* – then sweeps out a two-dimensional surface in $\mathscr{M}$ called the *worldsheet*.

We introduce coordinates $\sigma^\alpha = (\tau, \sigma),\ \alpha = 0, 1$, on the worldsheet Σ and a dynamical field $X : \Sigma \mapsto \mathscr{M}$, which embeds the worldsheet into the D-dimensional manifold $\mathscr{M}$. We also adopt local coordinates $x^\mu : \mathscr{M} \mapsto \mathbb{R},\ \mu = 0, 1, \ldots, D-1$, on some neighborhood $U \subset \mathscr{M}$, so $x^\mu(X(\tau,\sigma)) = X^\mu(\tau,\sigma)$ and $X^\mu : \Sigma \mapsto \mathbb{R}$. Dynamics of the string is given by the requirement that classical trajectories extremize the *Nambu-Goto action*

$$S_{NG}[X] = -T \int_\Sigma d\tau d\sigma\ \sqrt{-\det\left(\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right)}, \tag{1.1}$$

where $\eta_{\mu\nu}$ are the components of the D-dimensional Minkowski metric. The parameter σ shall run from 0 to π for both closed and open strings including end points. τ runs through an interval $\langle \tau_1, \tau_2 \rangle$.

The term $\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ is actually the induced (pullback) metric $X^*(\eta)$ on the worldsheet. The action (1.1) is therefore a straightforward generalization of the action of a massive point particle because it is proportional to the area of the worldsheet embedded in spacetime. The constant of proportionality T is interpreted as the tension of the string, for historical reasons also denoted by $T = \frac{1}{2\pi\alpha'}$, α' being called the *Regge slope*. If we adopt the usual convention $\hbar = c = 1$, the dimension of T is $(mass)^2$. Classical equations of motion for the free string are obtained from (1.1) as Euler–Lagrange equations.

Despite its nice geometrical meaning, the action (1.1) is rather hard to work with due to the occurrence of the square root in the Lagrangian. Especially if one wants to apply the path integral formalism, it is much more convenient to work with a quadratic Lagrangian. Therefore, we introduce new dynamical field $h_{\alpha\beta}$ representing a metric tensor on the worldsheet, and form the *Polyakov action*

$$S_P[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\ \sqrt{-h}h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1.2}$$

where $h = \det h_{\alpha\beta}$, $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$, and $d^2\sigma$ is an abbreviation for $d\tau d\sigma$. The action (1.2) is classically equivalent to (1.1) in the sense that calculating the equation of motion for $h^{\alpha\beta}$

$$T_{\alpha\beta} := -\frac{2}{T\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}h_{\alpha\beta}h^{\alpha'\beta'}\eta_{\mu\nu}\partial_{\alpha'}X^\mu\partial_{\beta'}X^\nu = 0, \tag{1.3}$$

and plugging its solution back to (1.2), we restore (1.1).

The Polyakov action is clearly invariant under general changes of coordinates on Σ. As this coordinate transformation includes two free functions, two of the three independent components of $h_{\alpha\beta}$ can be eliminated. A standard choice of parametrization is the *conformal gauge*, where we introduce the two-dimensional Minkowski metric $\eta_{\alpha\beta}$ and set $h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$. However, there is one more local symmetry – the celebrated *Weyl scaling symmetry*. Due to the structure of (1.2), the conformal parameter e^ϕ drops out, and we end up with a free field action

$$S_{P_{CG}}[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\ \eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1.4}$$

giving equations of motion for the scalar fields $X^\mu(\tau,\sigma)$ in the form of wave equations

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu(\tau,\sigma) = (\partial_\sigma^2 - \partial_\tau^2)X^\mu(\tau,\sigma) = 0. \tag{1.5}$$

This way we successfully gauged the $h_{\alpha\beta}$-dependence away. However, we must not forget about the equations (1.3), which now have to be imposed as constraints on the allowed trajectories. The vanishing of the *worldsheet energy-momentum tensor* $T_{\alpha\beta}$ then implies

$$0 = T_{10} = T_{01} = \eta_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu, \tag{1.6}$$

$$0 = T_{00} = T_{11} = \frac{1}{2}\left(\eta_{\mu\nu}\partial_\tau X^\mu\partial_\tau X^\nu + \eta_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu\right). \tag{1.7}$$

The fact that $T_{\alpha\beta}$ is traceless, which is here expressed by the statement $T_{00} = T_{11} = 0$, has an important consequence as it implies that the theory is *conformally invariant*.

The general solution of the wave equations (1.5) can be written in terms of left- and right-moving modes

$$X^\mu(\tau,\sigma) = X^\mu_R(\sigma_-) + X^\mu_L(\sigma_+), \tag{1.8}$$

where X_R^μ, X_L^μ are arbitrary functions, and we denoted

$$\sigma_- = \frac{1}{\sqrt{2}}(\tau - \sigma), \qquad \sigma_+ = \frac{1}{\sqrt{2}}(\tau + \sigma)$$

as the *light-cone coordinates on the worldsheet.* The corresponding derivatives read

$$\partial_- = \frac{1}{\sqrt{2}}(\partial_\tau - \partial_\sigma), \qquad \partial_+ = \frac{1}{\sqrt{2}}(\partial_\tau + \partial_\sigma).$$

The worldsheet metric in these coordinates has the form

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and (1.5) becomes

$$\partial_+\partial_- X^\mu = 0, \tag{1.9}$$

which is obviously solved by (1.8). It may be useful to transform the tensor $T_{\alpha\beta}$ to the light-cone coordinates. Then we obtain the equations (1.6), (1.7) as

$$0 = T_{--} = T_{00} - T_{01} = \eta_{\mu\nu}\partial_- X^\mu \partial_- X^\nu,$$
$$0 = T_{++} = T_{00} + T_{01} = \eta_{\mu\nu}\partial_+ X^\mu \partial_+ X^\nu.$$

The components $T_{+-} = T_{-+}$ automatically vanish as the consequence of the tracelessness of $T_{\alpha\beta}$.

From the topological point of view, the closed string worldsheet is a cylinder. To ensure the stationarity of the action for closed strings, the Euler–Lagrange equations (1.5) must be supplemented by the condition of *periodicity* $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$. To fulfill the periodicity requirement, we expand functions X_L^μ and X_R^μ into Fourier modes. The general solution contains several adjustable constants. For later convenience, we take the solution to be

$$X_R^\mu(\tau,\sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau - \sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)},$$
$$X_L^\mu(\tau,\sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau + \sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}.$$

The x^μ and p^μ are interpreted as the position of the center of mass and the momentum of the string. We require X^μ to be real functions, hence x^μ and p^μ are real, and the Fourier components satisfy

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger, \qquad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^\dagger.$$

Finally, the solution of (1.5) for the closed string can be written as

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}e^{-2in\tau}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right). \tag{1.10}$$

This solution has to fulfill (1.6), (1.7) as well, so the modes α_n^μ, $\tilde{\alpha}_n^\mu$ are further restricted.

The appropriate *boundary conditions* for open strings follow from the vanishing of the surface term

$$-T\int_{\tau_1}^{\tau_2} d\tau\, [\eta_{\mu\nu}\partial_\sigma X^\mu \delta X^\nu]_{\sigma=0}^{\sigma=\pi} = 0 \tag{1.11}$$

that appears in the variation of the action with respect to a change $X^\mu \to X^\mu + \delta X^\mu$, where $\delta X^\mu|_{\tau=\tau_1} = \delta X^\mu|_{\tau=\tau_2} = 0$. There are two ways to fulfill this requirement representing two types of boundary conditions:

$$\begin{aligned} \partial_\sigma X^\mu|_{\sigma=0} &= 0 = \partial_\sigma X^\mu|_{\sigma=\pi} && \text{Neumann boundary conditions,} \\ \delta X^\mu|_{\sigma=0} &= 0 = \delta X^\mu|_{\sigma=\pi} && \text{Dirichlet boundary conditions.} \end{aligned}$$

Neumann boundary conditions represent free motion of the endpoints of the string. The general solution of the equations of motion of an open string subject to Neumann boundary conditions is

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{1.12}$$

Again, the constraints (1.6), (1.7) have to be applied. On the other hand, *Dirichlet boundary* conditions fix the ends of the string to a hyperplane $\mathscr{M}' \subset \mathscr{M}$, whose dimension is determined by the number of Neumann boundary conditions. We shall return to the question of boundary conditions in detail in chapter 6, where we formulate them for open strings propagating in a general background and where we find their transformation under Poisson–Lie T-duality.

1.2 Strings moving in a general background

To be able to study the effects of gravitation, it is not sufficient to stick to the rigid Minkowski spacetime. We need to put strings into a general, possibly curved, background. As it turns out, there is a natural way to generalize previous notions to a general manifold $\mathscr{M}$ endowed with a metric tensor $G \in \mathcal{T}_2^0(\mathscr{M})$. In order to do that, we start with the Polyakov action (1.2) and replace $\eta_{\mu\nu}$ with $G_{\mu\nu}$ to obtain

$$S_G[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \sqrt{-h}h^{\alpha\beta}G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1.13}$$

where $G_{\mu\nu}$ are the components of G in local coordinates $x^\mu : \mathscr{M} \to \mathbb{R}$. This action is invariant under a general change of coordinates on Σ, hence it is possible to choose the conformal gauge, where $h_{\mu\nu} = \eta_{\mu\nu}$. However, the gauge-fixed action

$$S_{G_{CG}}[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \eta^{\alpha\beta}G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu$$

is not the free field theory (1.4), but a nontrivial interacting two-dimensional field theory known as the two-dimensional *nonlinear sigma model*. This way we have introduced an interaction of the string with gravitation.

A charged point particle couples to an external electromagnetic field given by a gauge potential one-form $A \in \Omega^1(\mathscr{M})$ through the pull-back $X^*(A)$ of A onto the worldline of the particle. Since the worldsheet Σ of the string is two-dimensional, we can mimic this coupling using the pull-back of an antisymmetric tensor field $B \in \Omega^2(\mathscr{M})$, which goes under the names of the *Kalb–Ramond field*, *NS–NS two-form* (the "NS" standing for Neveu–Schwarz), or simply the *B-field*. The corresponding action term is constructed from the pull-back $X^*(B)$ as

$$S_B[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\ \epsilon^{\alpha\beta}B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1.14}$$

where $\epsilon^{01} = -1 = -\epsilon^{10}$. Note that under the transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \delta B_{\mu\nu}, \qquad \delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu,$$

the Lagrangian in the action S_B changes by a total divergence, and the change does not affect the equations of motion. Such a transformation is considered to be a gauge transformation. Similarly to point particle electrodynamics, we find the gauge-invariant object to be the *torsion* three-form $H = dB \in \Omega^3(\mathscr{M})$, and refer to the B-field as to the *torsion potential*. S_B is also invariant under a general change of coordinates on Σ, since

$$d^2\sigma\ \epsilon^{\alpha\beta} = d^2\sigma\ \sqrt{-h}\frac{\epsilon^{\alpha\beta}}{\sqrt{-h}},$$

where $d^2\sigma\sqrt{-h}$ is the invariant volume and $\frac{\epsilon^{\alpha\beta}}{\sqrt{-h}}$ transforms as a tensor.

The starting point for our considerations in the rest of the work will be the sum of S_G and S_B from (1.13) and (1.14)

$$S_G + S_B = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\ \left(\sqrt{-h}h^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right). \tag{1.15}$$

The invariance with respect to general coordinate changes on $\mathscr{M}$ is manifest, and we have seen that it is also invariant with respect to reparametrizations on Σ. It is also Weyl invariant, so we can choose the conformal gauge to have the action of the sigma model in the form

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\ \left(\eta^{\alpha\beta}G_{\mu\nu}(X) + \epsilon^{\alpha\beta}B_{\mu\nu}(X)\right)\partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.16}$$

When we introduce the *Christoffel symbols*

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}G^{\lambda\sigma}\left(\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\mu\sigma} - \partial_\sigma G_{\mu\nu}\right)$$

and components of the torsion

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu},$$

the Euler–Lagrange equations following from (1.16) can be written as

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\lambda + \Gamma^\lambda_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}H^\lambda_{\ \mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu = 0,$$

where the spacetime index has been raised with $G^{\sigma\lambda}$. For the moment we shall ignore the boundary terms rising from the variation principle, and defer the discussion of boundary conditions to chapter 6.

For future convenience, we write these expressions explicitly in the worldsheet coordinates (τ,σ). The action reads

$$\begin{aligned} S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \Big(&- \partial_\tau X^\mu G_{\mu\nu}(X)\partial_\tau X^\nu + \partial_\sigma X^\mu G_{\mu\nu}(X)\partial_\sigma X^\nu \\ &- 2\partial_\tau X^\mu B_{\mu\nu}(X)\partial_\sigma X^\nu\Big), \end{aligned} \tag{1.17}$$

with the equations of motion for the fields $X^\lambda(\tau,\sigma)$ given by

$$0 = -\partial_\tau^2 X^\lambda + \partial_\sigma^2 X^\lambda + \Gamma^\lambda_{\mu\nu}\left(-\partial_\tau X^\mu\partial_\tau X^\nu + \partial_\sigma X^\mu\partial_\sigma X^\nu\right) + H^\lambda_{\ \mu\nu}\partial_\tau X^\mu\partial_\sigma X^\nu. \tag{1.18}$$

Similarly to the case of the flat background, the choice of the conformal gauge has its price, and (1.18) must be supplemented by the constraints following from the equations of motion for $h^{\alpha\beta}$. The second term in (1.15) does not contribute, and the constraints (1.6), (1.7) generalize to equations

$$0 = T_{10} = T_{01} = G_{\mu\nu}\partial_\tau X^\mu\partial_\sigma X^\nu, \tag{1.19}$$

$$0 = T_{00} = T_{11} = \frac{1}{2}\left(G_{\mu\nu}\partial_\tau X^\mu\partial_\tau X^\nu + G_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu\right). \tag{1.20}$$

In the description of duality in the next chapter it will prove useful to rewrite the above formulas also in the light-cone coordinates (σ_-,σ_+) on the worldsheet. We form a tensor $F_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$, with the metric G as the symmetric and the B-field as the antisymmetric part of F,

$$G_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + F_{\nu\mu}), \qquad B_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu}).$$

Then the action (1.16) simplifies to

$$S[X] = \frac{1}{2\pi\alpha'}\int_\Sigma d\sigma_+ d\sigma_- \ \left(\partial_- X^\mu F_{\mu\nu}(X)\partial_+ X^\nu\right), \tag{1.21}$$

with Euler–Lagrange equations

$$\begin{aligned} 0 &= \partial_+\partial_- X^\lambda + \frac{1}{2}G^{\lambda\sigma}\left(\partial_\mu F_{\sigma\nu} + \partial_\nu F_{\mu\sigma} - \partial_\sigma F_{\mu\nu}\right)\partial_- X^\mu\partial_+ X^\nu \\ &= \partial_+\partial_- X^\lambda + \Gamma^\lambda_{\mu\nu}\partial_- X^\mu\partial_+ X^\nu - \frac{1}{2}H^\lambda_{\ \mu\nu}\partial_- X^\mu\partial_+ X^\nu, \end{aligned} \tag{1.22}$$

supplemented by constraints

$$0 = T_{--} = G_{\mu\nu}\partial_- X^\mu \partial_- X^\nu, \tag{1.23}$$
$$0 = T_{++} = G_{\mu\nu}\partial_+ X^\mu \partial_+ X^\nu. \tag{1.24}$$

Contrary to the Euler–Lagrange equations (1.5) or (1.9), the expressions (1.18) and (1.22) are quite involved and can be hard, or even impossible, to solve. Nevertheless, in the second part of our work we mostly concentrate on looking for solutions of these equations via Poisson–Lie T-duality. It is customary to introduce the *non-symmetric connection* with torsion

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2}H^\lambda{}_{\mu\nu} = \frac{1}{2}G^{\lambda\sigma}\left(\partial_\mu F_{\sigma\nu} + \partial_\nu F_{\mu\sigma} - \partial_\sigma F_{\mu\nu}\right),$$

and rewrite the equations of motion using $\widetilde{\Gamma}^\lambda_{\mu\nu}$. For vanishing $\widetilde{\Gamma}^\lambda_{\mu\nu}$ the expression (1.22) reduces again to the easily solvable wave equation.

Under a change of coordinates the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ transform according to the rule

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\kappa}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\kappa_{\rho\sigma} + \frac{\partial x'^\lambda}{\partial x^\kappa}\frac{\partial^2 x^\kappa}{\partial x'^\mu \partial x'^\nu}, \tag{1.25}$$

and in principle it might be possible to transform $\widetilde{\Gamma}^\lambda_{\mu\nu}$ away by adopting a coordinate system where the contributions of the non-tensorial and tensorial part cancel. An obvious example for which the equations of motion can be reduced to wave equations is the case of a sigma model in the flat background (i.e. the metric G is just η written in awkward coordinates) with a B-field whose torsion vanishes. In such a background one can use (1.25) to find coordinates $x'^\nu(x^\mu)$ such that $\Gamma'^\lambda_{\mu\nu}$ vanishes. In these coordinates the equations are easily solvable. Although this possibility does not seem to give anything besides the results which we already have for the free field theory (1.4), the opposite is true. In the following chapters we will learn that T-duality relates curved backgrounds to flat ones, and show that the possibility to solve the equations of the model in the flat background also opens the possibility to solve the dual model living in some curved spacetime. However, to perform the T-duality transformation, we usually operate with coordinates in which the Christoffel symbols for the flat spacetime do not vanish, and an additional transformation of coordinates, which can be found from (1.25), is necessary to make $\Gamma'^\lambda_{\mu\nu} = 0$.

Besides the two action terms S_G and S_B, we should also consider a third term which involves the spacetime scalar field Φ called *dilaton*. The dilaton expectation value is crucial to determine the fine structure constant after quantizing the theory. It turns out that the right term to add to the action is proportional to the two-dimensional worldsheet Ricci scalar $R^{(2)}$:

$$S_\Phi[X] = \frac{1}{4\pi}\int_\Sigma d^2\sigma \sqrt{-h}\,\Phi(X) R^{(2)}. \tag{1.26}$$

Surprisingly, this term does not contribute to the equations of motion for $h^{\alpha\beta}$ since the variation of (1.26) is proportional to $R^{(2)}_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}R^{(2)}$, which automatically vanishes in two dimensions, where Einstein's equations are trivially satisfied. As we discuss below, S_Φ also does not contribute to the classical equations of motion (1.18) or (1.22).

So far we have only discussed conditions following from classical physics, mostly from the variational principle. Proper quantization of the theory is much more subtle, but several consequences display also at the classical level. Firstly, for a bosonic string it is inevitable to fix the spacetime dimension to $D = 26$. This is a necessary condition to build a positive-definite Hilbert space free from negative-norm (ghost) states. Secondly, we want the theory to be renormalizable. Applying the dimensional regularization technique and taking the theory into $(2+\epsilon)$ worldsheet dimensions, the Weyl invariance is destroyed. To restore it for $\epsilon \to 0$, a set of conditions restricting the background fields must be satisfied. Expanding these in powers of the α' parameter, one gets in the lowest nontrivial order

$$0 = R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda}, \tag{1.27}$$

$$0 = \nabla^\mu \Phi H_{\mu\kappa\lambda} + \nabla^\mu H_{\mu\kappa\lambda}\,, \tag{1.28}$$

$$0 = R - 2\nabla_\mu \nabla^\mu \Phi - \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} H_{\mu\kappa\lambda} H^{\mu\kappa\lambda}. \tag{1.29}$$

The covariant derivatives ∇_μ, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R in these equations are all calculated from the metric $G_{\mu\nu}$ that is also used to lower and raise indices. We already assumed $D = 26$, otherwise the third equation would receive an additional term. In string theory, the Weyl invariance implies vanishing of the renormalization group β function and therefore cancellation of the ultraviolet divergences in Feynman diagrams. Consequently, satisfaction of conditions (1.27)–(1.29) is necessary to have a renormalizable theory. Equations (1.27)–(1.29) generalize the vacuum Einstein's equations to backgrounds with nontrivial torsion and dilaton.

Note that contrary to (1.13) and (1.14), the term (1.26) is not Weyl invariant at the classical level. Due to the absence of α' in the prefactor, we may already think of S_Φ as a first order correction to $S_G + S_B$. This is the reason why we treat the dilaton term a bit separately from the first two; it is only the quantized string that feels its effects. Therefore, S_Φ also does not contribute to the classical equations of motion derived from $S_G + S_B$. Equations (1.27)–(1.29) can be deduced from the *string effective action*, which was originally calculated from the string S-matrix. Despite the fact that it is known only perturbatively, it was shown that extrema of the effective action correspond to Weyl-invariant sigma models. Higher order terms in the α'-expansion would result in further corrections, and it is an important question whether we can find backgrounds which satisfy the conditions of Weyl invariance to all orders. We shall meet such solutions in the second part of our work when dealing with plane waves.

Chapter 2

Poisson–Lie groups and Drinfel'd doubles

To keep the presentation of Poisson–Lie T-duality in the following chapters as clear as possible, we introduce the underlying algebraic and geometric structures here in this separate chapter. Namely, we define a Lie bialgebra, its dual bialgebra and a Manin triple. Then we focus on Poisson manifolds. We define the Poisson–Lie group, the Drinfel'd double, and point out the relation between Lie bialgebras and Poisson–Lie groups. The material summarized in this chapter will help us understand the elements of Poisson–Lie T-duality. Moreover, once the algebraic setting will be fixed, we will be able to construct mutually dual sigma models.

The topics addressed here form the mathematical foundation of classical mechanics and the theory of integrable systems. They also play a prominent role in the discussion of quantization. Poisson geometry has experienced an enormous growth, and we only pick those notions that are absolutely necessary for our further studies. The content of this chapter is based on great lecture notes by Kosmann-Schwarzbach [40], from which we take over only short excerpts without going into details or proofs.

2.1 Lie bialgebras

Before introducing Lie bialgebras, we have to briefly recap definitions of several fundamental objects coming from the theory of Lie-algebra cohomology. Because infinite-dimensional cases are not of interest to us, we shall keep the discussion simple, and restrict our considerations to finite-dimensional Lie algebras.

First of all, let us have a Lie algebra $\mathfrak{g}$ equipped with a Lie bracket $[.,.]$, and let V be a vector space. A linear map $\rho : \mathfrak{g} \mapsto End(V)$, for which

$$\rho([x,y]) = [\rho(x), \rho(y)] := \rho(x) \cdot \rho(y) - \rho(y) \cdot \rho(x)$$

holds for all $x, y \in \mathfrak{g}$, is called a *representation* of $\mathfrak{g}$ on V. The $\cdot$ here denotes the map composition in the space $End(V)$ of endomorphisms on V. Clearly, each Lie algebra $\mathfrak{g}$

acts on itself by the *adjoint representation* $\rho(x) = ad_x \in End(\mathfrak{g})$, defined by $ad_x(y) = [x, y]$. The adjoint representation can be generalized to act on any tensor product $\otimes^p \mathfrak{g}$ of $\mathfrak{g}$ with itself. For our purposes is enough to define $ad^{(2)} : \mathfrak{g} \mapsto End(\mathfrak{g} \otimes \mathfrak{g})$. Any element $y \in \mathfrak{g} \otimes \mathfrak{g}$ can be decomposed using a basis $(T_1, \dots, T_n)$ of $\mathfrak{g}$ as $y = y^{ab} T_a \otimes T_b$. Therefore, we define $ad^{(2)}$ for every $x \in \mathfrak{g},\ y \in \mathfrak{g} \otimes \mathfrak{g}$ as

$$ad^{(2)}_x(y) := y^{ab} \left(ad_x(T_a) \otimes T_b + T_a \otimes ad_x(T_b) \right).$$

A linear map acting on a vector space V naturally induces a linear map on the dual space V^*. In the following we shall see how it can be used to induce a Lie algebra structure on the dual space to a Lie algebra $\mathfrak{g}$. In general, let us have two vector spaces V, W, a linear mapping $A : V \mapsto W$, and let $\langle ., . \rangle$ denote the canonical pairing between V and V^*, or W and W^* respectively. The *transpose of* A is the map $A^* : W^* \mapsto V^*$, for which the relation

$$\langle A^*(\xi), x \rangle = \langle \xi, A(x) \rangle$$

holds for any $x \in V, \xi \in W^*$. An important example that should be mentioned is the *coadjoint representation* ad^*, which is a representation of $\mathfrak{g}$ on $\mathfrak{g}^*$, given by

$$\langle ad^*_x(\xi), y \rangle := -\langle \xi, ad_x(y) \rangle,$$

for $x, y \in \mathfrak{g}, \xi \in \mathfrak{g}^*$. Now we can introduce a Lie bialgebra by adding some additional data to a Lie algebra $\mathfrak{g}$.

Definition 1: A *Lie bialgebra* $(\mathfrak{g}, \delta)$ is a Lie algebra $\mathfrak{g}$ equipped with a linear map $\delta : \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$, such that

1. $\delta^* : (\mathfrak{g}^* \otimes \mathfrak{g}^*) \mapsto \mathfrak{g}^*$ defines a Lie bracket $[., .]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$,
2. for each $x, y \in \mathfrak{g}$ the mapping δ satisfies the condition

$$ad^{(2)}_x \left(\delta(y) \right) - ad^{(2)}_y \left(\delta(x) \right) - \delta([x, y]) = 0. \tag{2.1}$$

The mapping δ is sometimes called the *cocommutator*.

The definition deserves a few comments. First, we were only allowed to replace $(\mathfrak{g} \otimes \mathfrak{g})^*$ by $(\mathfrak{g}^* \otimes \mathfrak{g}^*)$ because we restricted ourselves to finite-dimensional algebras. Second, from the broader perspective of the theory of Lie-algebra cohomology, the second condition demands that δ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. Our approach is more pedestrian because we do not want to introduce unnecessary framework. Anyway, the Lie bracket $[., .]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is given by

$$[\xi, \eta]_{\mathfrak{g}^*} = \delta^*(\xi \otimes \eta), \qquad \xi, \eta \in \mathfrak{g}^*.$$

When the dual basis $(\widetilde{T}^1, \dots, \widetilde{T}^n)$ to $(T_1, \dots, T_n)$ is found in $\mathfrak{g}^*$ and the structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$ are introduced,

$$[T_a, T_b] = c^c_{ab} T_c, \qquad [\widetilde{T}^a, \widetilde{T}^b]_{\mathfrak{g}^*} = \widetilde{c}^{ab}_c \widetilde{T}^c, \tag{2.2}$$

δ can be expressed as

$$\delta(T_a) = \widetilde{c}_a^{bc} T_b \otimes T_c,$$

and the equation (2.1) gives a condition

$$\widetilde{c}_b^{fd} c_{af}^e + \widetilde{c}_b^{ef} c_{af}^d - \widetilde{c}_a^{fd} c_{bf}^e - \widetilde{c}_a^{ef} c_{bf}^d - \widetilde{c}_f^{ed} c_{ab}^f = 0 \tag{2.3}$$

relating the structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$. However, it is possible to rewrite the condition (2.1) in a different way emphasizing the symmetric role of $\mathfrak{g}$ and $\mathfrak{g}^*$. Since $\mathfrak{g}^*$ is now a Lie algebra, we can define its adjoint representation on itself via

$$ad_\xi(\eta) = [\xi, \eta]_{\mathfrak{g}^*},$$

as well as its coadjoint representation

$$\langle \eta, ad_\xi^*(x) \rangle := -\langle ad_\xi(\eta), x \rangle$$

acting on $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. The condition (2.1) now reads

$$\begin{aligned} \langle [\xi, \eta]_{\mathfrak{g}^*}, [x, y] \rangle &+ \langle ad_x^*(\xi), ad_\eta^*(y) \rangle - \langle ad_x^*(\eta), ad_\xi^*(y) \rangle \\ &- \langle ad_y^*(\xi), ad_\eta^*(x) \rangle + \langle ad_y^*(\eta), ad_\xi^*(x) \rangle = 0. \end{aligned} \tag{2.4}$$

Both algebras now play a symmetric role, and we may wonder whether we can find a Lie bialgebra based on $\mathfrak{g}^*$ that is somehow related to $(\mathfrak{g}, \delta)$. It is indeed possible when we realize that the mapping $\mu : \mathfrak{g} \otimes \mathfrak{g} \mapsto \mathfrak{g}$ defining the Lie bracket on $\mathfrak{g}$ has a transpose $\mu^* : \mathfrak{g}^* \mapsto \mathfrak{g}^* \otimes \mathfrak{g}^*$ satisfying a condition analogous to (2.1). To prove this statement, one can rewrite (2.4) using $[x, y] = \mu(x \otimes y)$. The Lie bialgebra $(\mathfrak{g}^*, \mu^*)$ is therefore called the *dual of a Lie bialgebra* $(\mathfrak{g}, \delta)$.

Now that we are familiar with the concept of a Lie bialgebra and its dual, we may combine them into a single object. Note that having a vector space V over a field $\mathbb{T}$, we can form a *natural inner product*

$$\langle ., . \rangle_{V \oplus V^*} : (V \oplus V^*) \times (V \oplus V^*) \mapsto \mathbb{T}$$

as a symmetric bilinear map given by

$$\langle x + \xi, y + \eta \rangle_{V \oplus V^*} = \langle \eta, x \rangle + \langle \xi, y \rangle, \qquad x, y \in V, \quad \xi, \eta \in V^*.$$

Let us now have a Lie bialgebra $(\mathfrak{g}, \delta)$ with its dual bialgebra $(\mathfrak{g}^*, \mu^*)$. Then we may uniquely define a Lie bracket $[., .]_{\mathfrak{d}}$ on the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras of $\mathfrak{d}$, and the natural inner product $\langle ., . \rangle_{\mathfrak{d}}$ is *ad-invariant*, meaning that

$$\langle ad_X(Y), Z \rangle_{\mathfrak{d}} + \langle Y, ad_X(Z) \rangle_{\mathfrak{d}} = 0 \tag{2.5}$$

for all $X, Y, Z \in \mathfrak{d}$. This Lie algebra structure on $\mathfrak{d}$ is fixed by the requirement that $\mathfrak{g}$ and $\mathfrak{g}^*$ form subalgebras, i.e.

$$[x, y]_{\mathfrak{d}} = [x, y], \qquad [\xi, \eta]_{\mathfrak{d}} = [\xi, \eta]_{\mathfrak{g}^*}, \tag{2.6}$$

and by the requirement of ad-invariance of the inner product. The latter condition can be exploited to calculate the mixed Lie bracket

$$[x,\xi]_{\mathfrak{d}} = -ad^*_{\xi}(x) + ad^*_{x}(\xi), \tag{2.7}$$

since

$$\langle y,[x,\xi]_{\mathfrak{d}}\rangle_{\mathfrak{d}} = \langle [y,x]_{\mathfrak{d}},\xi\rangle_{\mathfrak{d}} = \langle \xi,[y,x]\rangle = \langle ad^*_x(\xi),y\rangle = \langle y, ad^*_x(\xi)\rangle_{\mathfrak{d}},$$

and a similar computation gives $\langle \eta,[x,\xi]_{\mathfrak{d}}\rangle_{\mathfrak{d}} = -\langle \eta, ad^*_{\xi}x\rangle_{\mathfrak{d}}$. The role of the condition (2.1) is crucial, because $[.,.]_{\mathfrak{d}}$ fulfills Jacobi identities if and only if (2.3) holds. Using the dual bases and structure constants, the mixed bracket (2.7) can be rewritten as

$$[T_a,\widetilde{T}^b]_{\mathfrak{d}} = c^b_{ca}\widetilde{T}^c + \tilde{c}^{bc}_{a}T_c. \tag{2.8}$$

Definition 2: Let $(\mathfrak{g},\delta)$ be a Lie bialgebra. The vector space $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}^*$ equipped with the Lie bracket $[.,.]_{\mathfrak{d}}$ given by (2.6) and (2.7) is called the *double of a Lie bialgebra* $\mathfrak{g}$, and denoted by $\mathfrak{g}\bowtie\mathfrak{g}^*$ or simply by $\mathfrak{d}$.

A brief inspection reveals that both $\mathfrak{g}$, $\mathfrak{g}^*$ are isotropic subspaces of $\mathfrak{d}$ with respect to $\langle .,.\rangle_{\mathfrak{d}}$ and that $\mathfrak{d}$ is also the double of $\mathfrak{g}^*$. All the data necessary to construct the double were encoded in the Lie bialgebra structure of $(\mathfrak{g},\delta)$. However, due to the one-to-one correspondence between finite-dimensional Lie bialgebras and finite dimensional Manin triples, there is also a different way to recover these results.

Definition 3: A *Manin triple* is a triple of Lie algebras $(\mathfrak{c},\mathfrak{a},\mathfrak{b})$, where $\mathfrak{c}$ is equipped with an ad-invariant, non-degenerate, symmetric bilinear form $\langle .,.\rangle_{\mathfrak{c}}$, $\mathfrak{c}$ decomposes as $\mathfrak{c}=\mathfrak{a}\oplus\mathfrak{b}$, and $\mathfrak{a}$, $\mathfrak{b}$ are subalgebras of $\mathfrak{c}$ that are isotropic with respect to $\langle .,.\rangle_{\mathfrak{c}}$.

Clearly, having a Lie bialgebra $(\mathfrak{g},\delta)$, the corresponding Manin triple is $(\mathfrak{d},\mathfrak{g},\mathfrak{g}^*)$. Conversely, having a Manin triple $(\mathfrak{c},\mathfrak{a},\mathfrak{b})$, we may choose $\mathfrak{g}=\mathfrak{a}$, and show that $\mathfrak{b}$ is isomorphic to $\mathfrak{g}^*$. The Lie bracket on $\mathfrak{b}$ then induces a Lie bracket on $\mathfrak{g}^*$, from which $\delta^*:\mathfrak{g}^*\otimes\mathfrak{g}^*\mapsto\mathfrak{g}^*$ rises, whose transpose δ satisfies (2.1) due to Jacobi identities of $\mathfrak{c}$ and the ad-invariance of $\langle .,.\rangle_{\mathfrak{c}}$. Having established this correspondence, we shall specify the Lie bialgebra doubles by particular Manin triples.

To conclude this section, we note that for a connected Lie group $\mathscr{D}$ with a Lie algebra $\mathfrak{d}$ the ad-invariance (2.5) is equivalent to

$$\langle Ad_g(X), Ad_g(Y)\rangle_{\mathfrak{d}} = \langle X,Y\rangle_{\mathfrak{d}} \tag{2.9}$$

for all $X,Y\in\mathfrak{d}, g\in\mathscr{D}$. A form $\langle .,.\rangle_{\mathfrak{d}}$ satisfying (2.9) is called *Ad-invariant.*

2.2 Poisson–Lie groups and Drinfel'd doubles

The time has come to turn our attention to fundamentals of Poisson geometry and to show its interplay with the algebraic concepts presented above.

Definition 4: A smooth manifold $\mathscr{M}$ equipped with a bilinear map $\{.,.\}: C^\infty(\mathscr{M})\times C^\infty(\mathscr{M})\mapsto C^\infty(\mathscr{M})$ satisfying

- antisymmetry $\{f,g\} = -\{g,f\}$,
- Leibnitz rule $\{fg,h\} = f\{g,h\} + \{f,h\}g$,
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$,

for all $f,g,h \in C^\infty(\mathscr{M})$, is called a *Poisson manifold* and denoted $(\mathscr{M},\{.,.\})$. The bracket $\{.,.\}$ defining a Lie algebra structure on $C^\infty(\mathscr{M})$ is called the *Poisson bracket.*

The notion of Poisson manifold is notorious since the Poisson bracket appears naturally in classical mechanics as the bracket of functions on the phase space. Alternatively, a Poisson bracket can be given by a bivector field $\Pi \in \mathcal{T}_0^2(\mathscr{M})$ via

$$\{f,g\} := \Pi(df,dg).$$

In order to satisfy the three conditions above, Π has to be a skew symmetric contravariant tensor field of second order satisfying an additional condition corresponding to the Jacobi identity. Using local coordinates x^μ on $\mathscr{M}$, the *Poisson bivector* is expressed as

$$\Pi(x) = \Pi^{\mu\nu}(x)\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu},$$

and the validity of the Jacobi identity for $\{.,.\}$ is equivalent to

$$\Pi^{\rho\mu}\frac{\partial \Pi^{\nu\kappa}}{\partial x^\rho} + \Pi^{\rho\nu}\frac{\partial \Pi^{\kappa\mu}}{\partial x^\rho} + \Pi^{\rho\kappa}\frac{\partial \Pi^{\mu\nu}}{\partial x^\rho} = 0. \tag{2.10}$$

The structure of a Poisson manifold is quite rich, but we shall consider a structure that is even more complex. If the manifold is also a Lie group, it seems natural to demand that the group multiplication and the Poisson structure are compatible in some sense. Having two Poisson manifolds $\mathscr{M}$ and $\mathscr{N}$ with Poisson brackets $\{.,.\}_\mathscr{M}$ and $\{.,.\}_\mathscr{N}$, we say that a smooth map $\alpha : \mathscr{M} \mapsto \mathscr{N}$ is a *Poisson map* if for all $f,g \in C^\infty(\mathscr{N})$

$$\{f,g\}_\mathscr{N} \circ \alpha = \{f\circ\alpha, g\circ\alpha\}_\mathscr{M},$$

i.e. if the map α preserves the Poisson brackets. We can also easily define a Poisson structure on the product $\mathscr{M}\times\mathscr{N}$ of two Poisson manifolds as

$$\{f,g\}_{\mathscr{M}\times\mathscr{N}}(x,y) = \{f(.,y),g(.,y)\}_\mathscr{M}(x) + \{f(x,.),g(x,.)\}_\mathscr{N}(y)$$

for all $f,g \in C^\infty(\mathscr{M}\times\mathscr{N})$ and every $(x,y) \in \mathscr{M}\times\mathscr{N}$. Now it is possible to merge the Poisson and the Lie group structures into a single object.

Definition 5: A Lie group $\mathscr{G}$ endowed with a Poisson structure is called a *Poisson-Lie group* if the group multiplication $\mu : \mathscr{G}\times\mathscr{G} \mapsto \mathscr{G}$, with $\mathscr{G}\times\mathscr{G}$ having the product Poisson structure, is a Poisson map.

Due to the correspondence between Poisson brackets and Poisson bivectors, the compatibility condition can be written in terms of the Poisson bivector Π. The Lie group is a Poisson–Lie group if and only if Π is *multiplicative*, that is if

$$\Pi(gh) = L_{g*}(\Pi(h)) + R_{h*}(\Pi(g))$$

for all $g, h \in \mathscr{G}$, where L_{g*} and R_{g*} denote the tangent maps to left- and right-translations on $\mathscr{G}$. Clearly, a Poisson bivector on a Poisson–Lie group always vanishes in the unit of the group $e \in \mathscr{G}$. This allows us to take the *intrinsic derivative* of Π at e, which is a map $D\Pi : \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$, defined for every $x \in \mathfrak{g}$ and any vector field satisfying $X(e) = x$ as

$$D\Pi(x) := (\mathcal{L}_X \Pi)(e).$$

The particular choice of X is irrelevant here because $\Pi(e) = 0$. Therefore, the intrinsic derivative does not depend on X and only takes x into account. This map will help us to find the connection between Lie bialgebras and Poisson–Lie groups. Namely, having a Poisson–Lie group $\mathscr{G}$ with a Lie algebra $\mathfrak{g}$ and a Poisson bivector Π, the mapping $\delta = D\Pi$ satisfies (2.1) and its transpose δ^* induces a Lie bracket on $\mathfrak{g}^*$. The Lie bialgebra structure $(\mathfrak{g}, \delta)$ thus appears as the infinitesimal counterpart of the Poisson–Lie group $(\mathscr{G}, \Pi)$. This unique Lie bialgebra $(\mathfrak{g}, \delta)$ is called the *tangent Lie bialgebra* of $(\mathscr{G}, \Pi)$. On the other hand, it can be shown, see [40], that if $(\mathfrak{g}, \delta)$ is a Lie bialgebra, then there exists a unique connected and simply connected Poisson–Lie group $(\mathscr{G}, \Pi)$, such that $(\mathfrak{g}, \delta)$ is its tangent Lie bialgebra.

Before showing how the Poisson bivector on $\mathscr{G}$ is constructed from the algebraic data contained in the Lie bialgebra $(\mathfrak{g}, \delta)$, we briefly discuss the Lie group counterparts of a dual and a double of a Lie algebra. In the last section we have seen that every Lie bialgebra has its dual bialgebra. With the one-to-one correspondence between tangent Lie bialgebras and Poisson–Lie manifolds mentioned above, the dual of a Poisson–Lie group can be found.

Definition 6: Let $(\mathscr{G}, \Pi)$ be a Poisson–Lie group with a tangent Lie bialgebra $(\mathfrak{g}, \delta)$, and let $(\mathfrak{g}^*, \mu^*)$ be the dual of $(\mathfrak{g}, \delta)$. Then the connected and simply connected Poisson–Lie group $(\widetilde{\mathscr{G}}, \widetilde{\Pi})$ having tangent Lie bialgebra $(\mathfrak{g}^*, \mu^*)$ is called the *dual of a Poisson–Lie group* $(\mathscr{G}, \Pi)$.

Note that if $\mathscr{G}$ is connected and simply connected, then $(\mathscr{G}, \Pi)$ is the dual of $(\widetilde{\mathscr{G}}, \widetilde{\Pi})$ and the relation between these two Poisson–Lie groups is symmetric. Besides its dual, every Lie bialgebra also has its double $\mathfrak{d}$ that can be expressed in terms of a Manin triple. Now we shall define its Lie group counterpart and see how the mutually dual Poisson–Lie groups $(\mathscr{G}, \Pi)$ and $(\widetilde{\mathscr{G}}, \widetilde{\Pi})$ are embedded into it.

Definition 7: Let $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$ be a Manin triple corresponding to a Lie bialgebra $(\mathfrak{g}, \delta)$, where $\mathfrak{d} = \mathfrak{g} \oplus \widetilde{\mathfrak{g}}$ and the subalgebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ are maximally isotropic with respect to ad-invariant, non-degenerate, symmetric bilinear form $\langle ., . \rangle_{\mathfrak{d}}$. The connected and simply connected Lie group $\mathscr{D}$ whose Lie algebra is $\mathfrak{d}$ is called the *Drinfel'd double*.

Since $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ are subalgebras of $\mathfrak{d}$, the corresponding connected and simply connected groups $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ are subgroups of $\mathscr{D}$. Moreover, both subgroups can be equipped with a Poisson structure to be Poisson–Lie groups. So far we have only proposed that there exists a unique Poisson–Lie group structure for every Lie bialgebra. Now we shall finally construct it.

Let us have a Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$, where $\dim \mathfrak{g} = \dim \widetilde{\mathfrak{g}} = n$ and $\dim \mathfrak{d} = 2n$. We choose a basis $\mathcal{X} = (T_a, \widetilde{T}^b)$ in $\mathfrak{d}$ composed of basis vectors $T_a \in \mathfrak{g}$ and $\widetilde{T}^a \in \widetilde{\mathfrak{g}}$, such that

$$\langle T_a, \widetilde{T}^b \rangle_{\mathfrak{d}} = \delta_a^b.$$

Since $\mathfrak{g}^*$ and $\widetilde{\mathfrak{g}}$ are isomorphic, the commutation relations (2.2), (2.8) hold also for the elements of $\mathcal{X}$. For each $g \in \mathscr{G}$ the matrix of the adjoint representation $Ad_{g^{-1}}$ of $\mathscr{G}$ on $\mathfrak{d}$ expressed in the basis $\mathcal{X}$ has the block form

$$^{\mathcal{X}}(Ad_{g^{-1}}) = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix},$$

where $a(g), b(g), d(g)$ are $n \times n$ g-dependent matrices coming from

$$Ad_{g^{-1}}(T_a) = a(g)_a^b T_b, \qquad Ad_{g^{-1}}(\widetilde{T}^a) = b(g)^{ab} T_b + d(g)_b^a \widetilde{T}^b.$$

The bottom-left corner of $^{\mathcal{X}}(Ad_{g^{-1}})$ vanishes because $\mathscr{G}$ is a subgroup of $\mathscr{D}$. In agreement with [12] we may define a matrix Π as a product

$$\Pi(g) = b(g) \cdot a(g)^{-1}.$$

The components of Π, i.e. the functions $\left(^{\mathcal{X}}\Pi(g)\right)^{ab}$, can be used to define a multiplicative Poisson bivector on $\mathscr{G}$. Indeed, $\Pi^{ab}(g)$ are the components of the Poisson–Lie bivector expressed in the frame of right-invariant vector fields on $\mathscr{G}$. To find the expression in the coordinate frame, we must multiply it by the components $^R v_a^\mu(g)$ of the right-invariant vector fields

$$^R v_a(g) = {^R v_a^\mu}(g) \frac{\partial}{\partial x^\mu}$$

according to the relation

$$\Pi^{\mu\nu}(g) = {^R v_a^\mu}(g) \Pi^{ab}(g) {^R v_b^\nu}(g).$$

This tensor field can be verified to be a multiplicative bivector satisfying (2.10), whose intrinsic derivative restores the tangent Lie bialgebra corresponding to the Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$ that we started with. In chapters 4 and 5 we give particular examples demonstrating this construction.

Chapter 3

T-duality

After the short mathematical detour that we made in the last chapter we shall begin our investigation of T-duality and its role in string theory. As we already mentioned in the introduction, the existence of dualities relating apparently different sigma models and emerging field theories is one of the most striking discoveries made in the field of string theory. In its original appearance the target space duality was understood in the context of closed strings with spacetime compactified on a torus, where it acted by switching the radius of the compactified dimension as $R \to 1/R$. Nevertheless, we shall adopt the point of view proposed in [4] and [6], where it was shown that any string background allowing continuous symmetry can be dualized as well. In the second part of this chapter we focus on the generalization of T-duality to non-Abelian groups of symmetries in the sense of [10], and discuss serious problems that one inevitably encounters when dealing with non-Abelian groups. This will motivate the introduction of Poisson–Lie T-duality in the following chapters.

3.1 Abelian T-duality

An important step to understand T-duality of sigma models in geometrically different backgrounds was made by Buscher in [4]. We summarize the procedure shortly here because it poses the first step to introduce Poisson–Lie T-duality. It will also help us demonstrate the upcoming notions since it represents the simplest special case of the duality transformation.

Let us return to the sigma model action (1.21)

$$S[X] = \frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \ \left(\partial_- X^\mu F_{\mu\nu}(X) \partial_+ X^\nu\right).$$

First of all, we shall make a crucial assumption that there is a symmetry of the background represented by a Killing vector $\mathcal{K} \in \mathfrak{X}(\mathscr{M})$ satisfying

$$\mathcal{L}_\mathcal{K} F = \mathcal{L}_\mathcal{K} G = \mathcal{L}_\mathcal{K} B = 0. \tag{3.1}$$

We admit that in general this condition turns out to be too stringent. We may relax it slightly by allowing the B-field to change as $\mathcal{L}_\mathcal{K} B = d\omega,\ \omega \in \Omega^1(\mathscr{M})$, because such a change does not affect the torsion and the equations of motion. However, for the sake of clarity, we shall consider only the symmetries fulfilling (3.1). The general case is treated e.g. in [6] or [8].

Having a non-vanishing vector field $\mathcal{K}$, we can find a set of local coordinates x^μ, $\mu = 0, 1, \ldots, D-1$, such that $\mathcal{K} = \partial_{x^0}$ and the background fields are independent of x^0. The action is then invariant under a constant shift

$$X^0 \to X^0 + \epsilon, \qquad X^\mu \to X^\mu, \qquad \mu \in 1, \ldots, D-1.$$

We shall construct the so-called *parent action*, from which both mutually dual sigma models can be obtained. In order to do that, we replace $\partial_\pm X^0$ in $S[X]$ by a pair of independent fields $(A_+,\ A_-)$, and add a term containing a Lagrange multiplier field $\widetilde{X}^0(\tau, \sigma)$ to obtain

$$\begin{aligned} S[X, A_\pm, \widetilde{X}] = \frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \, \Big(& A_- G_{00}(X) A_+ + A_- F_{0\nu}(X) \partial_+ X^\nu \\ & + \partial_- X^\mu F_{\mu 0}(X) A_+ + \partial_- X^\mu F_{\mu\nu}(X) \partial_+ X^\nu \\ & + \widetilde{X}^0 (\partial_- A_+ - \partial_+ A_-) \Big). \end{aligned} \tag{3.2}$$

This is an action describing a $(D+1)$-dimensional sigma model. Varying the parent action with respect to the Lagrange multiplier first, we force

$$\partial_- A_+ - \partial_+ A_- = 0. \tag{3.3}$$

Locally, or in topologically trivial worldsheets, a field X^0 can be found, such that

$$A_- = \partial_- X^0, \qquad A_+ = \partial_+ X^0.$$

Plugging it back to (3.2), we recover the original model.

However, if $G_{00} \neq 0$, we can solve the equations obtained by variations of the parent action with respect to $(A_+,\ A_-)$. We find

$$A_+ = G_{00}^{-1} \left(-\partial_+ \widetilde{X}^0 - F_{0\nu} \partial_+ X^\nu \right), \qquad A_- = G_{00}^{-1} \left(\partial_- \widetilde{X}^0 - \partial_- X^\mu F_{\mu 0} \right). \tag{3.4}$$

Using the relations (3.4) in the parent action (3.2), we obtain an action of a new sigma model, which in coordinates $\widetilde{X} = (\widetilde{X}^0, X^\mu)$, $\mu \in 1, \ldots, D-1$, reads

$$\begin{aligned} \widetilde{S}[\widetilde{X}] = \frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \, & \Big(\partial_- \widetilde{X}^0 G_{00}^{-1} \partial_+ \widetilde{X}^0 + \partial_- \widetilde{X}^0 G_{00}^{-1} F_{0\nu} \partial_+ X^\nu \\ & - \partial_- X^\mu F_{\mu 0} G_{00}^{-1} \partial_+ \widetilde{X}^0 + \partial_- X^\mu \left(F_{\mu\nu} - G_{00}^{-1} F_{\mu 0} F_{0\nu} \right) \partial_+ X^\nu \Big) \end{aligned}$$

The action of this model,

$$\widetilde{S}[\widetilde{X}] = \frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \, \left(\partial_- \widetilde{X}^\mu \widetilde{F}_{\mu\nu}(\widetilde{X}) \partial_+ \widetilde{X}^\nu \right), \tag{3.5}$$

$$\begin{tikzcd}
& S[X, A_\pm, \widetilde{X}] \arrow[dl, "\delta \widetilde{X}^0"'] \arrow[dr, "\delta A_+, \delta A_-"] & \\
S[X] & & \widetilde{S}[\widetilde{X}] \\
& \widetilde{S}[\widetilde{X}, \widetilde{A}_\pm, X] \arrow[ul, "\delta \widetilde{A}_+, \delta \widetilde{A}_-"] \arrow[ur, "\delta X^0"'] &
\end{tikzcd}$$

Figure 3.1: Buscher's duality transformation

is therefore given by a tensor $\widetilde{F}$, related to the original F through

$$\widetilde{G}_{00} = G_{00}^{-1}, \qquad \widetilde{F}_{0\nu} = G_{00}^{-1} F_{0\nu}, \qquad \widetilde{F}_{\mu 0} = -G_{00}^{-1} F_{\mu 0},$$
$$\widetilde{F}_{\mu\nu} = F_{\mu\nu} - G_{00}^{-1} \left(F_{\mu 0} F_{0\nu} \right). \tag{3.6}$$

The new model is called *T-dual* to the one given by F. We see that the Lagrange multiplier $\widetilde{X}^0$ became a dynamical field in context of the dual model. Due to (3.6), the components of $\widetilde{F}$ are independent of $\tilde{x}^0$. Thus, there is again a symmetry generated by $\partial_{\tilde{x}^0}$ and the procedure can be repeated on the dual model to restore the original one. We sketch the procedure in figure 3.1.

We note several important facts. First, the inspection of (3.6) reveals that even if the original model was given by metric tensor only, i.e. $F = G$, the dual model may have a nontrivial torsion potential

$$\widetilde{B}_{0\nu} = -\widetilde{B}_{\nu 0} = G_{00}^{-1} G_{0\nu},$$

so a model with torsion can be generated via Buscher's duality. Second, calculations made in Ref. [4] show that the curvature properties may change when the duality transformation is performed. Indeed, as we demonstrate later using specific examples, the flat metric can be transformed to a curved background and vice versa. T-duality therefore relates sigma models in backgrounds with different geometric properties. To give a plausible theory, the dual sigma model has to satisfy the β equations (1.27)–(1.29) and the duality transformation must be followed by a proper shift in the dilaton field. This is a quantum effect that can be deduced if the dualization is performed using path integral techniques [5]. The new dilaton is found to be

$$\widetilde{\Phi} = \Phi - \frac{1}{2} \ln G_{00}. \tag{3.7}$$

Another important property concerns the relation between equations of motion of the original and the dual model. During the construction of the parent action we denoted $A_+ = \partial_+ X^0$, $A_- = \partial_- X^0$. These fields can be regarded as components of a one-form $A = dX^0 = A_+ d\sigma_+ + A_- d\sigma_-$. The term containing the Lagrange multiplier that has been added to the action is in fact proportional to the *Bianchi identity* (3.3). It can be also written as $dA = d(dX^0) = 0$. The variation of the dual action $\widetilde{S}[\widetilde{X}]$ with respect to the field $\widetilde{X}^0$ gives the Euler–Lagrange equation

$$\partial_- \left(G_{00}^{-1} \left(-\partial_+ \widetilde{X}^0 - F_{0\nu} \partial_+ X^\nu \right) \right) - \partial_+ \left(G_{00}^{-1} \left(\partial_- \widetilde{X}^0 - \partial_- X^\mu F_{\mu 0} \right) \right) = 0. \qquad (3.8)$$

Clearly, upon (3.4), this equation of motion following from the dual model is given by the Bianchi identity of the original model (3.3). T-duality is revertible and treats both models symmetrically. Therefore, we conclude that duality swaps field equations and Bianchi identities. Note that being able to solve the dual model, we can use (3.4) to solve the original one.

The duality transformation can be easily generalized to backgrounds having n-dimensional groups of symmetries $\mathscr{G}$ provided these commute. Before rushing into the higher-dimensional case, we would like to mention a different technique introduced by Roček and Verlinde in [6]. Although it gives the same results, we describe it here shortly because the method is suitable for further generalization of T-duality to non-Abelian groups of symmetries.

Suppose again that there exists a global symmetry, which in suitable coordinates acts as a constant shift

$$X^0 \to X^0 + \epsilon, \qquad X^\mu \to X^\mu, \qquad \mu \in 1, \ldots, D-1.$$

First, we gauge this symmetry by introducing gauge fields $A_\pm$ representing components of a *connection one-form* $A = A_+ d\sigma_+ + A_- d\sigma_-$, and replace the partial derivatives of X^0 with their covariant counterparts:

$$\partial_+ X^0 \to D_+ X^0 = \partial_+ X^0 + A_+, \qquad \partial_- X^0 \to D_- X^0 = \partial_- X^0 + A_-.$$

Second, we calculate the *field strength* $\mathcal{F} = dA$ corresponding to A, and add it as the Lagrange multiplier term to the action, thus creating a parent action

$$\begin{aligned} S[X, A_\pm, \widetilde{X}] = \frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \, \Big(& D_- X^0 G_{00}(X) D_+ X^0 + D_- X^0 F_{0\nu}(X) \partial_+ X^\nu \\ & + \partial_- X^\mu F_{\mu 0}(X) D_+ X^0 + \partial_- X^\mu F_{\mu\nu}(X) \partial_+ X^\nu \\ & + \widetilde{X}^0 \left(\partial_- A_+ - \partial_+ A_- \right) \Big). \end{aligned} \qquad (3.9)$$

Clearly, the parent action is invariant with respect to a local ($\epsilon = \epsilon(\tau, \sigma)$) symmetry

$$X^0 \to X^0 + \epsilon, \quad A_\pm \to A_\pm - \partial_\pm \epsilon, \quad X^\mu \to X^\mu, \quad \mu \in 1, \ldots, D-1. \qquad (3.10)$$

The new term in (3.9) guarantees that A is a pure gauge because varying (3.9) with respect to $\widetilde{X}^0$ forces the field strength to vanish. Then $A = d\lambda$ and $A_\pm = \partial_\pm \lambda$ in

$$\begin{tikzcd}
S[X] \arrow[rr, bend right, "{\partial \to D,\ +\int \widetilde{X}^0 F}"'] & & S[X, A_\pm, \widetilde{X}] \arrow[ll, bend right, "\delta \widetilde{X}^0"'] \arrow[r, "{\delta A_+, \delta A_-,\ X^0 = 0}"'] & \widetilde{S}[\widetilde{X}]
\end{tikzcd}$$

Figure 3.2: Dualization via gauge symmetry approach

topologically trivial worldsheets. The original action is recovered either by choosing the gauge fields $A_\pm$ to vanish through (3.10), or by redefining $X^0 + \lambda \to X^0$. On the other hand, varying (3.9) with respect to $A_\pm$, we obtain

$$\begin{aligned}
A_+ &= G_{00}^{-1} \left(-\partial_+ \widetilde{X}^0 - F_{0\nu} \partial_+ X^\nu \right) - \partial_+ X^0, \\
A_- &= G_{00}^{-1} \left(\partial_- \widetilde{X}^0 - \partial_- X^\mu F_{\mu 0} \right) - \partial_- X^0.
\end{aligned} \tag{3.11}$$

Plugging these relations back to (3.9), we recover the dual action (3.5) together with an additional term

$$\frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \ \left(\partial_- \widetilde{X}^0 \partial_+ X^0 - \partial_+ \widetilde{X}^0 \partial_- X^0 \right),$$

which we eliminate if we fix the gauge to $X^0 = 0$. Note that the equations (3.11) are invariant with respect to (3.10), so the gauge choice is possible. This way we restored the dual model (3.5) as well as the rules (3.6). In the present framework the equation of motion of the dual model corresponds to the requirement that the connection A is flat, i.e. its field strength vanishes. Again, the dual model has the required symmetry, and the whole procedure can be repeated on the dual side. The procedure is depicted in figure 3.2.

Both methods described above assumed that there was a one-dimensional group of symmetries. The fact that we had to choose adapted coordinates in which the symmetry manifests itself as a shift is a bit confusing and contradicts the requirement of general covariance. A modification of the gauge symmetry approach given in Ref. [8] solved this issue. The authors obtained formulas (3.6), (3.7) in forms valid in any coordinate system, which also allowed them to address global aspects of duality.

To generalize the previous results, let us consider a sigma model living in a background F which has an n-dimensional Abelian group of symmetries $\mathscr{G}$. Also, let $\mathscr{G}$ be generated by Killing vectors $\mathcal{K}_i \in \mathfrak{X}(\mathscr{M}),\ i = 1, \ldots, n$, for which (3.1) holds. We again choose a set of coordinates $(x^a, x^\mu),\ a = 0, \ldots, n-1,\ \mu = n, \ldots, D-1$, such that the symmetry acts by constant shifts. Decomposing the background tensor into blocks as

$$F = \begin{pmatrix} F_{ab} & F_{a\nu} \\ F_{\mu b} & F_{\mu\nu} \end{pmatrix} =: \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}, \tag{3.12}$$

and introducing fields $A^a_\pm$, we can either gauge the symmetry or add the Bianchi identities to form the parent action. If the block $F_{ab} = F_1$ constitutes an invertible matrix, we can express $A^a_\pm$ similarly as in (3.4) or (3.11), use it in the parent action and obtain the dual sigma model (3.5). The best way to write this down is with the help of matrix notation. For future use we arrange the components $\partial_\pm X^\mu$, $\partial_\pm \widetilde{X}^a$ and $A^a_\pm$ into row vectors $\partial_\pm X$, $\partial_\pm \widetilde{X}$ and $A_\pm$. Variation of Buscher's parent action with respect to $A_\pm$ gives us relations

$$\begin{aligned} A_+^T &= (F_1)^{-1} \cdot \left(-(\partial_+ \widetilde{X})^T - F_2 \cdot (\partial_+ X)^T \right), \\ A_- &= \left((\partial_- \widetilde{X}) - (\partial_- X) \cdot F_3 \right) \cdot (F_1)^{-1}, \end{aligned} \tag{3.13}$$

where $\cdot$ denotes matrix multiplication and ^{T} is transposition. The dual background is then given by *Buscher rules*

$$\begin{gathered} \widetilde{F}_1 = (F_1)^{-1}, \qquad \widetilde{F}_2 = (F_1)^{-1} \cdot F_2, \qquad \widetilde{F}_3 = -F_3 \cdot (F_1)^{-1}, \\ \widetilde{F}_4 = F_4 - \left(F_3 \cdot (F_1)^{-1} \cdot F_2 \right), \end{gathered} \tag{3.14}$$

while the dilaton transforms as

$$\widetilde{\Phi} = \Phi - \frac{1}{2} \ln\left(\det(F_1)\right). \tag{3.15}$$

When the Abelian group of symmetries $\mathscr{G}$ is D-dimensional and acts freely and transitively, it is possible to identify $\mathscr{G}$ and the target manifold $\mathscr{M}$ itself. Buscher rules (3.14) then simplify dramatically. We refer to this case ($n = D$) as to the *atomic duality* and use this term even for non-Abelian versions of T-duality. If $n < D$, the fields X^μ that do not participate in the duality transformation are called *spectator fields* or simply *spectators*. Clearly, we do not have to dualize with respect to the whole n-dimensional symmetry group $\mathscr{G}$. It is possible to consider only its d-dimensional subgroups $\mathscr{H} \subset \mathscr{G}$. To perform the transformation on a subset of the fields X^a, we simply treat the rest as spectators and take into account only a subset of the Lagrange multipliers $\widetilde{X}^a$ and corresponding fields $A^a_\pm$. Expressions (3.14) and (3.15) still hold, with the matrices $F_1, \ldots, F_4$ representing the appropriate blocks of F, with indices a, b in (3.12) running only over the dualized variables.

3.2 Non-Abelian T-duality

It would be insufficient to deal only with backgrounds having Abelian groups of symmetries. The motivation is clear. Many of the physically interesting backgrounds have non-Abelian groups of symmetries, notorious examples being the Schwarzschild metric or cosmological Friedman–Robertson–Walker solutions of Einstein's equations, which all have the $SO(3)$ spherical symmetry. An attempt to extend T-duality to sigma models with non-Abelian groups of symmetries was made in Ref. [10] by de la Ossa and Quevedo, who based their method on the gauge symmetry approach of Ref. [6]. Although the generalization seems straightforward, we will see that one necessarily runs into serious problems which need to be dealt with within some more general framework.

Suppose that there is an n-dimensional non-Abelian group $\mathscr{G}$ of global symmetries of the background and that the group acts for $g \in \mathscr{G}$ as

$$X^a \to g^a{}_b X^b, \quad a = 0, \dots, m-1, \qquad X^\mu \to X^\mu, \quad \mu = m, \dots, D-1. \tag{3.16}$$

This time the background tensor F depends explicitly on X^a and we do not assume the existence of any preferred coordinate system. To gauge the symmetry, we introduce a connection one-form A, which takes values in the Lie algebra $\mathfrak{g}$ of the group $\mathscr{G}$,

$$A = A^\alpha T_\alpha = (A^\alpha_+ d\sigma_+ + A^\alpha_- d\sigma_-)T_\alpha = A_+ d\sigma_+ + A_- d\sigma_-,$$

where $\alpha = 1, \dots, n$ and T_α are the generators of the Lie algebra $\mathfrak{g}$ in the adjoint representation of $\mathscr{G}$. The matrices T_α obey the commutation relations

$$[T_\alpha, T_\beta] = c^\gamma_{\alpha\beta} T_\gamma,$$

with $c^\gamma_{\alpha\beta}$ being the structure constants of the Lie algebra $\mathfrak{g}$. The partial derivatives of X^a occurring in the action need to be replaced by covariant derivatives, which have the form

$$\partial_\pm X^a \to D_\pm X^a = \partial_\pm X^a + A^\alpha_\pm (T_\alpha)^a{}_b X^b.$$

This way of gauging, however, may not be valid for all sigma models and all symmetry groups because of topological obstructions [41].

In the next step we form the parent action. We introduce the field strength

$$\mathcal{F} = (\partial_- A_+ - \partial_+ A_- + [A_-, A_+]) d\sigma_+ \wedge d\sigma_-,$$
$$\mathcal{F} = (F_{+-}{}^\alpha)(T_\alpha) d\sigma_+ \wedge d\sigma_-, \qquad \mathcal{F}_{+-}{}^\alpha = \partial_- A^\alpha_+ - \partial_+ A^\alpha_- + c^\alpha_{\beta\gamma} A^\beta_- A^\gamma_+$$

and Lagrange multipliers $\widetilde{X} = \widetilde{X}^\alpha T_\alpha$ into the action through a term

$$\frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \; Tr(\widetilde{X}\mathcal{F}),$$

so the gauged parent action has the form

$$\begin{aligned} S[X, A_\pm, \widetilde{X}] = \frac{1}{2\pi\alpha'} \int_\Sigma d\sigma_+ d\sigma_- \; \Big(& D_- X^a F_{ab}(X) D_+ X^b + D_- X^a F_{a\nu}(X) \partial_+ X^\nu \\ & + \partial_- X^\mu F_{\mu b}(X) D_+ X^b + \partial_- X^\mu F_{\mu\nu}(X) \partial_+ X^\nu \\ & + Tr(\widetilde{X}\mathcal{F}) \Big). \end{aligned} \tag{3.17}$$

Let us denote $m_{\alpha\beta} := T_\alpha \cdot T_\beta$, so that $Tr(T_\alpha \cdot T_\beta) = Tr(m_{\alpha\beta})$. The last term of the parent action can be thus written as $Tr(\widetilde{X}\mathcal{F}) = \widetilde{X}^\alpha \mathcal{F}^\beta Tr(m_{\alpha\beta})$. When the transformation (3.16) is made local with $g = g(\sigma_+, \sigma_-)$, it has to be followed by a change in the gauge field $A \to gAg^{-1} + g(dg^{-1})$. The field strength transforms as $\mathcal{F} \to g\mathcal{F}g^{-1}$, so the Lagrange multipliers have to transform in the adjoint representation in order to keep the parent action invariant. The full gauge transformation leaving the parent action invariant therefore has the form

$$X \to gX, \qquad A \to gAg^{-1} + g(dg^{-1}), \qquad \widetilde{X} \to g\widetilde{X}g^{-1}, \qquad g \in \mathscr{G}. \tag{3.18}$$

The first serious problem with non-Abelian duality is encountered when we try to return to the original action. For semisimple algebras $\mathfrak{g}$ the Killing form is non-degenerate and the variation with respect to Lagrange multipliers $\widetilde{X}^\alpha$ restricts $\mathcal{F}^\alpha = 0$, so the field strength vanishes. With $\mathcal{F} = 0$ we conclude that the gauge fields are a pure gauge, and an element $h \in \mathscr{G}$ exists, such that

$$A_+ = h^{-1}\partial_+ h, \qquad A_- = h^{-1}\partial_- h, \qquad h \in \mathscr{G}.$$

As in the Abelian case, we may now use the gauge transformation (3.18) to adopt the gauge in which $A_\pm = 0$. This reduces the parent action to the original action. However, this is not possible if the algebra corresponding to the symmetry group is not semisimple. For non-semisimple algebras the Killing form is degenerate and the components $\mathcal{F}^\alpha$ are not necessarily zero. In such a case the original theory is not restored. This is a serious limitation.

Nevertheless, we shall try to find the dual theory. We label

$$\begin{aligned}
f_{\alpha\beta} &:= c^\gamma_{\beta\alpha}\widetilde{X}^\delta Tr(m_{\gamma\delta}) + X^c (T_\beta)^a{}_c F_{ab} (T_\alpha)^b{}_d X^d, \\
h_{\alpha+} &:= \partial_+\widetilde{X}^\beta Tr(m_{\alpha\beta}) + X^c (T_\alpha)^a{}_c \left(F_{ab}\partial_+ X^b + F_{a\nu}\partial_+ X^\nu\right), \\
h_{\alpha-} &:= -\partial_-\widetilde{X}^\beta Tr(m_{\beta\alpha}) + \left(\partial_- X^a F_{ab} + \partial_- X^\mu F_{\mu b}\right)(T_\alpha)^b{}_c X^c.
\end{aligned} \tag{3.19}$$

Then the parent action (3.17) can be written in a compact form as

$$S[X, A_\pm, \widetilde{X}] = S[X] + \frac{1}{2\pi\alpha'}\int_\Sigma d\sigma_+ d\sigma_- \left(A^\alpha_- f_{\alpha\beta} A^\beta_+ + A^\alpha_- h_{\alpha+} + h_{\alpha-} A^\alpha_+\right).$$

Variation with respect to the gauge fields $A^\alpha_\pm$ gives us non-Abelian versions of (3.13), which in matrix form read

$$A^T_+ = -(f_{\alpha\beta})^{-1} \cdot h^T_+, \qquad A_- = -h_- \cdot (f_{\alpha\beta})^{-1}.$$

Using these relations, we end up with the dual action

$$\widetilde{S}[X, \widetilde{X}] = S[X] - \frac{1}{2\pi\alpha'}\int d\sigma_+ d\sigma_- \left(h_- \cdot (f_{\alpha\beta})^{-1} \cdot h_+\right).$$

To finish our work, it is necessary to eliminate the extra degrees of freedom remaining from the original gauged action. If the orbit generated by the action of the n-dimensional group $\mathscr{G}$ is n-dimensional, we use the transformation (3.18) to get rid of X^a that obviously can not occur in the dual action. If the dimension of the orbit is d, $d < n$, the situation is trickier. In such a case the duality is in fact performed in d-directions, not n, so after gauge fixing of the m fields X^a, one has to use the remaining gauge freedom to gauge away the extra $(m-d)$ Lagrange multipliers introduced in the procedure. This is best illustrated by the example of $SO(N)$ symmetry studied in [10].

It is not possible to give an explicit prescription for the dual background. Nevertheless, we can make several conclusions that hold in general. First, the non-Abelian technique can be applied directly to the sigma model with commutative symmetries. In fact, following the procedure, one does not even need to find the adapted coordinates in which the symmetry acts as shifts of coordinates. For Abelian groups of symmetries the structure constants in (3.19) vanish, meaning that the dual coordinates $\widetilde{X}^a$ appear in the dual action only through derivatives. Therefore, the dual background again possesses Abelian symmetries associated to shifts in $\widetilde{X}^a$, which allow us to return to the original action.

Commutative symmetries may, however, disappear when we dualize with respect to a non-commutative group containing an Abelian symmetry subgroup because $f_{\alpha\beta}$ may depend explicitly on $\widetilde{X}$. Examples show an alarming fact that dualizing with respect to non-Abelian groups of symmetries, we may even obtain a background which has no continuous symmetries at all. This is a serious setback since it prevents us from returning to the original model using the duality transformation. Starting with a sigma model with no symmetries, we would never realize that there is a dual model.

Duality transformation again has to be accompanied by a change in the dilaton field. Precise calculation of the measure in the path integral conducted in [10] gives a formula

$$\widetilde{\Phi} = \Phi - \frac{1}{2} \ln \left(\det(f_{\alpha\beta}) \right). \tag{3.20}$$

We described the generalization of T-duality to non-Abelian groups of symmetries, and pointed out several limitations of the technique. We noted that when the algebra $\mathfrak{g}$ corresponding to the symmetry group is not semisimple, it is not possible to regain $S[X]$ from the parent action. Semisimple algebras are also prominent from another point of view. In Ref. [42] the trace of the worldsheet energy-momentum tensor of the dual model was calculated. It was shown that even with (3.20) it vanishes only if all the matrices of the generators T_α in the adjoint representation have a vanishing trace. For non-semisimple groups this does not hold. The dual model is not conformal in such a case and anomalies occur in the emerging quantum field theory. But even more striking fact was that the group of symmetries is not preserved under the duality transformation, making T-duality a one-way path. All these revelations suggest that T-duality should be understood within a more general framework which does not rely on the existence of local symmetries, but which incorporates both Buscher and non-Abelian T-duality. This is the topic of the following chapter.

Chapter 4

Atomic Poisson–Lie T-duality

Being familiar with the drawbacks of non-Abelian T-duality, we have a clear motivation to do the next step and discuss the notion of Poisson–Lie T-duality as it was introduced in 1995 by Klimčík and Ševera in [11].

The geometrical structure on which the construction relies is the Drinfel'd double that we defined in chapter 2. To identify dualizable sigma models, we first study Noether currents induced by symmetries of the background and calculate the variation of the sigma model action generated by these symmetries. Having this knowledge, we find the condition for dualizability of the sigma model. Investigating the self-consistency of this condition, we reveal a Lie bialgebra structure. This will allow us to construct sigma models on subgroups of the corresponding Drinfel'd double that will be mutually dual in the sense of Poisson–Lie T-duality.

Already in the original papers [11], [12] the authors considered the possibility that a Drinfel'd double can be decomposed in several ways. However, first explicit formulas and examples of what is now called Poisson–Lie T-plurality appeared in 2002 in [22]. Due to its importance as a tool for generating solvable sigma models, we describe the plurality here as well. Other examples can be found e.g. in [43].

For the sake of clarity, the discussion of Poisson–Lie T-duality which we carry out in this chapter focuses on atomic duality. Duality with spectators will be discussed later. A significant part of this chapter is based on excerpts from papers which the current author coauthored. Namely, sections 4.3, 4.4 and 4.5 follow the corresponding section of [44] and collect the common background concerning the construction of dual models presented also in [45], [46]. In section 4.3 we add a subsection proving that the construction gives solutions to the condition of dualizability. The material covered in sections 4.1 and 4.2 concerning Noether currents and the condition (4.13) also originates from [44], but it has been extended to shed some light on the origins of Poisson–Lie T-duality. We conclude giving an example in section 4.6.

Throughout this and the following chapters we assume that there is a Lie group $\mathscr{G}$ acting freely on $\mathscr{M}$, and that coordinates on the target manifold can be chosen in such a way that part of them (x^a) represent coordinates on the group $\mathscr{G}$, while the others (x^α) label the orbits of $\mathscr{G}$ in $\mathscr{M}$. In other words, we assume that locally $\mathscr{M} \approx \mathscr{N} \times \mathscr{G}$, so that coordinates on $\mathscr{M}$ can be chosen as

$$x^\mu = (x^\alpha, x^a), \qquad \alpha = 1, \dots, \dim \mathscr{N}, \quad a = 1, \dots, \dim \mathscr{G},$$
$$\dim \mathscr{M} = \dim \mathscr{N} + \dim \mathscr{G}.$$

As we shall see in the following, the coordinates x^α do not participate in the Poisson–Lie T-duality transformation and represent spectators. To begin with the study of Poisson–Lie T-duality, this chapter avoids spectators and describes only the simpler case when the group $\mathscr{G}$ acts freely and transitively on $\mathscr{M}$, i.e. when $\mathscr{M} \approx \mathscr{G}$. Spectator fields shall be included in the next chapter, when the atomic duality is thoroughly understood.

4.1 Noether currents and Poisson–Lie condition

The central point of our consideration is again the sigma model given by (1.17) and (1.21). Since we are interested in classical equations of the sigma model, it is convenient to set the overall factor $\frac{1}{2\pi\alpha'} = 1$, so that the action reads

$$S[X] = \int_\Sigma d\sigma_+ d\sigma_- \, L = - \int_\Sigma d\tau d\sigma \, \bar{L},$$

with Lagrangian densities

$$L = \partial_- X^\mu F_{\mu\nu}(X) \partial_+ X^\nu, \tag{4.1}$$

$$\begin{aligned} \bar{L} = & -\frac{1}{2}\partial_\tau X^\mu G_{\mu\nu}(X)\partial_\tau X^\nu + \frac{1}{2}\partial_\sigma X^\mu G_{\mu\nu}(X)\partial_\sigma X^\nu \\ & - \partial_\tau X^\mu B_{\mu\nu}(X)\partial_\sigma X^\nu. \end{aligned} \tag{4.2}$$

According to the *Noether theorem*, whenever there is a global symmetry of a field theory, there is also a conserved current and a charge corresponding to it. In the Lagrangian formalism the conserved quantities can be found via standard procedure known in classical field theory as the *Noether method*. We apply it now on the two dimensional field theory given by our sigma model. For a global symmetry transformation of the dynamical fields in the form $X(\tau, \sigma) \to X(\tau, \sigma) + \epsilon \, \delta X(\tau, \sigma)$, with ϵ being a constant infinitesimal parameter, the change in the action has to vanish, i.e. $\delta S = 0$. Of course, the action is stationary if the field equations are satisfied. In such a case, the condition $\delta S = 0$ holds for any variations of X. Therefore, we consider a transformation

$$X(\tau, \sigma) \to X(\tau, \sigma) + \epsilon(\tau, \sigma) \, \delta X(\tau, \sigma),$$

where the parameter $\epsilon(\tau, \sigma)$ is a non-constant function on the worldsheet. Then the variation of the action is proportional to the derivative of ϵ

$$\delta S = \int_\Sigma d^2\sigma \; J^\alpha \partial_\alpha \epsilon,$$

which indeed vanishes if ϵ is constant. Quantities $J^\alpha(\tau, \sigma)$ are the *Noether currents*. When the field equations are imposed, the action has to be stationary for any variations, i.e. $\delta S = 0$ for any ϵ, and integrating by parts in the last equation we find that the currents have to be conserved,

$$\partial_\alpha J^\alpha = 0. \tag{4.3}$$

The conserved charge is then computed as an integral over σ in $\tau = 0$:

$$Q = \int_\sigma J^\tau d\sigma.$$

Noether currents can be very useful in the description of T-duality. If we return to the easiest case of a sigma model with global shift symmetry in X^0, we can make this symmetry local,

$$X^0(\tau,\sigma) \to X^0(\tau,\sigma) + \epsilon(\tau,\sigma), \qquad X^\mu(\tau,\sigma) \to X^\mu(\tau,\sigma), \quad \mu \in 1,\ldots,D-1.$$

Discarding terms of second order in derivatives of ϵ, we get

$$\begin{aligned}\delta S &= S[X^0+\epsilon, X^\mu] - S[X] = \\ &= \int d\sigma^2 \, \partial_-\epsilon \underbrace{\left(G_{00}\partial_+X^0 + F_{0\nu}\partial_+X^\nu\right)}_{-J^+} + \underbrace{\left(\partial_-X^0G_{00} + \partial_-X^\mu F_{\mu 0}\right)}_{-J^-}\partial_+\epsilon.\end{aligned}$$

One can easily see that the equation (4.3) expressing the conservation of the current

$$\partial_+J^- + \partial_-J^+ = 0$$

is actually the equation of motion obtained by variation of the action with respect to X^0. In addition to this, we also have the identity

$$\partial_-\partial_+X^0 - \partial_+\partial_-X^0 = 0$$

that we wrote earlier as $\partial_-A_+ - \partial_+A_- = 0$ in (3.3). On the dual side this identity gives us the field equation (3.8) of the sigma model (3.5) through the identification (3.4). Moreover, the dual sigma model has a shift symmetry in $\widetilde{X}^0$. Calculating the corresponding Noether currents

$$\delta\widetilde{S} = \int d\sigma^2 \, \partial_-\epsilon \underbrace{\left(G_{00}^{-1}\left(\partial_+\widetilde{X}^0 + F_{0\nu}\partial_+X^\nu\right)\right)}_{-\widetilde{J}^+} + \underbrace{\left(G_{00}^{-1}\left(\partial_-\widetilde{X}^0 - \partial_-X^\mu F_{\mu 0}\right)\right)}_{-\widetilde{J}^-}\partial_+\epsilon,$$

we realize that according to (3.4) the following identification can be made,

$$A_+ = \widetilde{J}^+, \qquad A_- = -\widetilde{J}^-,$$

and that the Bianchi identity is in fact the current conservation condition for the dual currents

$$\partial_-A_+ - \partial_+A_- = \partial_-\widetilde{J}^+ + \partial_+\widetilde{J}^- = 0. \tag{4.4}$$

Analogous relations of course hold for the dual model.

The study of Noether currents expressed in general coordinates further allowed the authors of Ref. [8] to give a covariant description of T-duality. It also let them address global issues, which would be hardly possible if they had to work in adapted coordinates. We shall focus on Noether currents in non-Abelian theories. Our purpose is to demonstrate how they motivate the introduction of Poisson–Lie T-duality.

Suppose that there is a Lie group $\mathscr{G}$ acting freely and transitively on the target manifold $\mathscr{M}$. Then the group can be identified with the manifold itself, and coordinates x^μ may be thought of as coordinates on the group. Let also $v_a,\ a = 1, \dots, D = \dim \mathscr{G}$, be the left-invariant vector fields on $\mathscr{G}$. If $\mathscr{G}$ is a group of symmetries, we can compute the variation of S under the corresponding infinitesimal symmetry and find Noether currents. Nevertheless, the currents can be defined also when $\mathscr{G}$ does not represent the symmetry of the target manifold. An infinitesimal change of the fields X^μ generated by left-invariant vector fields as

$$X^\mu(\tau,\sigma) \to X^\mu(\tau,\sigma) + \epsilon^a(\tau,\sigma)\, v_a^\mu(\tau,\sigma), \tag{4.5}$$

then leads to a variation

$$\begin{aligned}
\delta S[X] &= S[X + \epsilon^a v_a] - S[X] && (4.6)\\
&= \int d\sigma^2\ \partial_-\left(X^\mu + \epsilon^a v_a^\mu\right) F_{\mu\nu} \left(X^\lambda + \epsilon^a v_a^\lambda\right) \partial_+\left(X^\nu + \epsilon^a v_a^\nu\right) - S\left[X\right] \\
&= \int d\sigma^2\ \Big(\partial_-\epsilon^a \underbrace{\left(v_a^\mu F_{\mu\nu}\partial_+ X^\nu\right)}_{-J_a^+} + \underbrace{\left(\partial_- X^\mu F_{\mu\nu} v_a^\nu\right)}_{-J_a^-} \partial_+\epsilon^a\Big) && (4.7)\\
&+ \int d\sigma^2\ \epsilon^a \partial_- X^\mu \underbrace{\left(v_a^\kappa \partial_\kappa F_{\mu\nu} + \partial_\mu v_a^\kappa F_{\kappa\nu} + \partial_\nu v_a^\kappa F_{\mu\kappa}\right)}_{(\mathcal{L}_{v_a}F)_{\mu\nu}} \partial_+ X^\nu. && (4.8)
\end{aligned}$$

We see that if (4.5) is a symmetry of the background, the term (4.8) containing the Lie derivative vanishes,

$$(\mathcal{L}_{v_a}F)_{\mu\nu} = 0.$$

Integrating by parts in (4.7), we again find that the currents are conserved,

$$\partial_+ J_a^- + \partial_- J_a^+ = 0.$$

To express the conservation of J_a using the language of differential geometry, we associate a *Noether one-form*

$$J_a = (v_a^\mu F_{\mu\nu}\partial_+ X^\nu)d\sigma_+ - (\partial_- X^\mu F_{\mu\nu} v_a^\nu)d\sigma_- \tag{4.9}$$

to each current. The current conservation property then translates as

$$dJ_a = 0, \tag{4.10}$$

i.e. the forms (4.9) are closed. Remember that when we dualized with respect to one-dimensional symmetry, we realized that the condition for current conservation represents the condition (4.4) for vanishing of the field strength $\widetilde{\mathcal{F}} = d\widetilde{A} = 0$ of the connection associated with the gauge symmetry of the dual model. Equation (4.10) generalizes this result to D-dimensional groups of commuting symmetries acting freely and transitively on the target manifold. Abelian duality preserves the symmetry group. When the symmetries of the dual theory are gauged, a Lie-algebra valued connection one form $\widetilde{A} = \widetilde{A}_\alpha \widetilde{T}^\alpha$ is introduced whose components stand for the gauge fields $\widetilde{A}_{\alpha\pm}$. Through (4.10) we state that the field strength $\widetilde{\mathcal{F}}$ associated to $\widetilde{A}$ vanishes, $\widetilde{A}$ is a flat connection and the gauge fields are a pure gauge.

The crucial question is how to generalize these results to non-Abelian groups. We may try to relax the condition (4.10), but hold on to the requirement that the field strength $\widetilde{\mathcal{F}}$ corresponding to $\widetilde{A}$ vanishes. In components $\widetilde{\mathcal{F}}$ reads

$$\widetilde{\mathcal{F}} = \left(\partial_- \widetilde{A}_+ - \partial_+ \widetilde{A}_- + [\widetilde{A}_-, \widetilde{A}_+]\right) d\sigma_+ \wedge d\sigma_-,$$

$$\widetilde{\mathcal{F}} = \left(\widetilde{\mathcal{F}}_{\alpha+-}\right) (\widetilde{T}^\alpha) d\sigma_+ \wedge d\sigma_-, \qquad \widetilde{\mathcal{F}}_{\alpha+-} = \partial_- \widetilde{A}_{\alpha+} - \partial_+ \widetilde{A}_{\alpha-} + \tilde{c}_\alpha^{\beta\gamma} \widetilde{A}_{\beta-} \widetilde{A}_{\gamma+}.$$

Following [11], we infer that in order to have a flat connection, the one-forms J_a have to satisfy the *Maurer–Cartan condition*

$$dJ_a + \frac{1}{2} \tilde{c}_a^{bc} J_b \wedge J_c = 0, \tag{4.11}$$

with $\tilde{c}_a^{bc}$ being structure constants of a Lie algebra $\tilde{\mathfrak{g}}$ of some Lie group $\widetilde{\mathscr{G}}$. When we plug the expressions for J_a into (4.11), we find that

$$\begin{aligned} dJ_a &= \left(\partial_+ J_a^- + \partial_- J_a^+\right) d\sigma_+ \wedge d\sigma_- \\ &= -\left(\partial_- X^\mu (F_{\mu\kappa}\, v_b^\kappa\, \tilde{c}_a^{bc}\, v_c^\lambda\, F_{\lambda\nu}) \partial_+ X^\nu\right) d\sigma_+ \wedge d\sigma_-. \end{aligned}$$

In the previous chapter we met a disturbing feature of non-Abelian duality when we realized that symmetries are not conserved by the duality transformation. Therefore, we relax our requirement that $\mathscr{G}$ represents symmetries of the background. However, the variation of S in (4.6) under the action of $\mathscr{G}$ still has to vanish on-shell since S has to be stationary under any variation. This is achieved if the two terms (4.7) and (4.8) cancel each other, i.e. when

$$\partial_+ J_a^- + \partial_- J_a^+ + \partial_- X^\mu (\mathcal{L}_{v_a} F)_{\mu\nu} \partial_+ X^\nu = 0. \tag{4.12}$$

Collecting this and the preceding equation we find a condition for the tensor $F_{\mu\nu}$

$$(\mathcal{L}_{v_a} F)_{\mu\nu} = F_{\mu\kappa}\, v_b^\kappa\, \tilde{c}_a^{bc}\, v_c^\lambda\, F_{\lambda\nu}. \tag{4.13}$$

If the equation (4.13) is satisfied, we say that $F_{\mu\nu}$ is *$\mathscr{G}$-Poisson-Lie symmetric with respect to $\widetilde{\mathscr{G}}$*, or that it has *generalized symmetries*. We have arrived at a crucial point because we see that Poisson–Lie T-dual sigma model to the model living on $\mathscr{G}$ can be found if and only if (4.13) holds. However, starting with a sigma model given by $F_{\mu\nu}$, it does not have to be easy to find its dual since we have to find $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ such that (4.13) is fulfilled. Only then the original sigma model is dualizable in the sense of Poisson–Lie T-duality.

4.2 Algebraic structure of Poisson–Lie T-duality

Identification of $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ in the case of genuine Poisson–Lie T-duality, i.e. when both these groups are non-commutative, is hard indeed. Nevertheless, a quick examination of equation (4.13) shows that it simplifies significantly if the left-invariant vector fields v_a generate a symmetry of the background. Then the left-hand side vanishes:

$$(\mathcal{L}_{v_a} F)_{\mu\nu} = 0 = F_{\mu\kappa}\, v_b^\kappa\, \widetilde{c}_a^{bc}\, v_c^\lambda\, F_{\lambda\nu}. \tag{4.14}$$

To fulfill this condition, one simply chooses $\widetilde{\mathscr{G}}$ Abelian, i.e. with $\widetilde{c}_a^{bc} = 0$. This would be the case of Abelian and non-Abelian T-duality. Whenever there is a group of symmetries of the background given by $F_{\mu\nu}$, the model is dualizable with the dual group being Abelian. This will be demonstrated thoroughly in Part II of the thesis, where the non-Abelian T-duality transformation is carried out using the techniques of Poisson–Lie T-duality.

It should be stressed that the Poisson–Lie condition is manifestly dual. Starting with a model given by $\widetilde{F}_{\mu\nu}$, it is dualizable if and only if

$$(\mathcal{L}_{\tilde{v}_a} \widetilde{F})_{\mu\nu} = \widetilde{F}_{\mu\kappa}\, \widetilde{v}^{b\kappa}\, c_{bc}^a\, \widetilde{v}^{c\lambda}\, \widetilde{F}_{\lambda\nu} \tag{4.15}$$

holds. This justifies the fact that symmetries are not preserved by non-Abelian T-duality. If $\mathscr{G}$ acts as a symmetry, (4.14) holds and $\widetilde{\mathscr{G}}$ can be chosen Abelian. But the right-hand side of (4.15) is nontrivial for nonvanishing c_{bc}^a. Then $\widetilde{\mathscr{G}}$ does not represent the group of symmetries of the dual model, preventing us from dualizing it back using the technique explained in section 3.2.

The algebraic structure underlying Poisson–Lie T-duality can be deduced from the self-consistency condition of the Lie derivative:

$$\mathcal{L}_{[v_a, v_b]} = [\mathcal{L}_{v_a}, \mathcal{L}_{v_b}]. \tag{4.16}$$

Using the equation (4.13), we express the left-hand side in terms of $F_{\mu\nu}$, c_{ab}^c and $\widetilde{c}_a^{bc}$ as

$$\mathcal{L}_{[v_a, v_b]} = \mathcal{L}_{c_{ab}^c v_c} = c_{ab}^c\, \mathcal{L}_{v_c}, \qquad (\mathcal{L}_{[v_a, v_b]} F)_{\mu\nu} = c_{ab}^c\, F_{\mu\kappa}\, v_d^\kappa\, \widetilde{c}_c^{de}\, v_e^\lambda\, F_{\lambda\nu}.$$

After a few manipulations the right-hand side reads

$$\begin{aligned} ([\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] F)_{\mu\nu} &= (\mathcal{L}_{v_a} \mathcal{L}_{v_b} F)_{\mu\nu} - (\mathcal{L}_{v_b} \mathcal{L}_{v_a} F)_{\mu\nu} \\ &= \left(-\widetilde{c}_b^{fd} c_{af}^e - \widetilde{c}_b^{ef} c_{af}^d + \widetilde{c}_a^{fd} c_{bf}^e + \widetilde{c}_a^{ef} c_{bf}^d \right) F_{\mu\kappa} v_d^\kappa v_e^\lambda F_{\lambda\nu}. \end{aligned}$$

The condition (4.16) therefore implies

$$\widetilde{c}_b^{fd} c_{af}^e + \widetilde{c}_b^{ef} c_{af}^d - \widetilde{c}_a^{fd} c_{bf}^e - \widetilde{c}_a^{ef} c_{bf}^d - \widetilde{c}_f^{ed} c_{ab}^f = 0,$$

which is exactly the one-cocycle condition (2.3) for a Lie bialgebra $(\mathfrak{g}, \delta)$ corresponding to a Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$ composed of D-dimensional Lie algebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ and a $2D$-dimensional Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \widetilde{\mathfrak{g}}$. $\mathfrak{d}$ is naturally equipped with a symmetric, nondegenerate ad-invariant bilinear form $\langle ., . \rangle_{\mathfrak{d}}$, and the subalgebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ are subspaces of $\mathfrak{d}$ that are maximally isotropic with respect to $\langle .,. \rangle_{\mathfrak{d}}$. The connected and simply connected Lie group $\mathscr{D}$ – the Drinfel'd double – corresponding to the Lie algebra $\mathfrak{d}$ contains connected and simply connected Lie groups $\mathscr{G}, \widetilde{\mathscr{G}}$ corresponding to $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ as subgroups.

Let us return for a moment to the Maurer–Cartan condition (4.11). If satisfied, it says that the connection one-form $J_a \widetilde{T}^a$ has a vanishing field strength and is integrable. Imposing (4.11) on-shell, we claimed that to each solution $g(\sigma_+, \sigma_-) \in \mathscr{G}$ of the field equations of the sigma model on $\mathscr{G}$ we may associate a mapping $\tilde{h}(\sigma_+, \sigma_-)$ from the worldsheet to $\widetilde{\mathscr{G}}$, such that

$$J_a \widetilde{T}^a = -d\tilde{h}\tilde{h}^{-1}.$$

Hence, a solution g of the sigma model on $\mathscr{G}$ can be considered as a hyperplane in the Drinfel'd double $\mathscr{D}$ obtained through

$$g(\sigma_+, \sigma_-)\tilde{h}(\sigma_+, \sigma_-) = l(\sigma_+, \sigma_-) \in \mathscr{D},$$

where the multiplication is taken in $\mathscr{D}$. According to [47], any element in the vicinity of the unit $e \in \mathscr{D}$ of the Drinfel'd double can be decomposed in two ways

$$g(\sigma_+, \sigma_-)\tilde{h}(\sigma_+, \sigma_-) = l(\sigma_+, \sigma_-) = \tilde{g}(\sigma_+, \sigma_-)h(\sigma_+, \sigma_-), \qquad (4.17)$$

where $g, h \in \mathscr{G}$, $\tilde{g}, \tilde{h} \in \widetilde{\mathscr{G}}$. In the following, we shall advocate the idea that the relation (4.17) is in fact the Poisson–Lie T-duality transformation, and that if $g(\sigma_+, \sigma_-)$ is a solution of a sigma model on the Lie group $\mathscr{G}$, then $\tilde{g}(\sigma_+, \sigma_-)$ is the solution of the dual sigma model living on $\widetilde{\mathscr{G}}$.

4.3 Construction of dual sigma models

We have already noted that when a sigma model in a background $F_{\mu\nu}$ is given, it may be hard to find the appropriate Drinfel'd double to fulfill the condition of Poisson–Lie symmetry (4.13). In this section we shall proceed from the opposite side and construct mutually dual theories using the technique presented in [11].

The foundation stone for our considerations will be a Drinfel'd double $\mathscr{D}$ containing subgroups $\mathscr{G}$ and $\widetilde{\mathscr{G}}$, whose Lie algebra $\mathfrak{d}$ decomposes as a direct sum of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. For convenience, we choose bases $T_a \in \mathfrak{g}$, $\widetilde{T}^a \in \widetilde{\mathfrak{g}}$, $a = 1, \ldots, D$, in the two algebras in such a way that

$$\langle T_a, T_b \rangle_{\mathfrak{d}} = 0, \qquad \langle \widetilde{T}^a, \widetilde{T}^b \rangle_{\mathfrak{d}} = 0, \qquad \langle T_a, \widetilde{T}^b \rangle_{\mathfrak{d}} = \delta_a^b, \qquad (4.18)$$

i.e. the bases are dual with respect to the symmetric, non-degenerate ad-invariant bilinear form $\langle .,. \rangle_{\mathfrak{d}}$ on $\mathfrak{d}$. Moreover, the basis vectors satisfy

$$[T_a, T_b] = c_{ab}^c \, T_c, \qquad [\widetilde{T}^a, \widetilde{T}^b] = \widetilde{c}_c^{ab} \, \widetilde{T}^c, \qquad [T_a, \widetilde{T}^b] = c_{ca}^b \, \widetilde{T}^c + \widetilde{c}_a^{bc} \, T_c, \qquad (4.19)$$

where c_{ab}^c and $\widetilde{c}_c^{ab}$ are the structure constants of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. We recall that elements $l \in \mathscr{D}$ of the Drinfel'd double can be decomposed in two ways as

$$g\tilde{h} = l = \tilde{g}h, \qquad g, h \in \mathscr{G}, \ \tilde{g}, \tilde{h} \in \widetilde{\mathscr{G}}.$$

Let us now focus on the tangent space $T_e\mathscr{D} \simeq \mathfrak{d}$ at the unit element $e \in \mathscr{D}$, which at the same time represents the unit element in $\mathscr{G}$ and $\widetilde{\mathscr{G}}$. We may define a regular linear mapping $E : \mathfrak{g} \mapsto \widetilde{\mathfrak{g}}$, given by an invertible matrix $E(e)$ denoted also as E_0. The graph of this mapping,

$$\mathcal{E}^+ := Span\big[(T_a + E(e)_{ab}\widetilde{T}^b)_{a\in\widehat{m}}\big], \qquad \widehat{m} := \{1, \dots, \dim\mathscr{G}\}, \tag{4.20}$$

together with a subspace

$$\mathcal{E}^- := Span\big[(T_a - E(e)_{ba}\widetilde{T}^b)_{a\in\widehat{m}}\big], \qquad \widehat{m} := \{1, \dots, \dim\mathscr{G}\}, \tag{4.21}$$

form two D-dimensional subspaces $\mathcal{E}^\pm \subset \mathfrak{d}$ which are orthogonal with respect to $\langle .,.\rangle_{\mathfrak{d}}$. This can be checked by direct calculation, since for any two vectors $v \in \mathcal{E}^+, w \in \mathcal{E}^-$ the following equation

$$\begin{aligned} 0 &= \langle v^a(T_a + E(e)_{ab}\widetilde{T}^b), w^c(T_c - E(e)_{dc}\widetilde{T}^d)\rangle_{\mathfrak{d}} \\ &= v^a w^c\Big(\underbrace{\langle T_a, T_c\rangle_{\mathfrak{d}}}_{0} - E(e)_{ab}E(e)_{dc}\underbrace{\langle \widetilde{T}^b, \widetilde{T}^d\rangle_{\mathfrak{d}}}_{0}\Big) \\ &+ v^a w^c\Big(E(e)_{ab}\underbrace{\langle \widetilde{T}^b, T_c\rangle_{\mathfrak{d}}}_{\delta^b_c} - E(e)_{dc}\underbrace{\langle T_a, \widetilde{T}^d\rangle_{\mathfrak{d}}}_{\delta^d_a}\Big) \end{aligned} \tag{4.22}$$

holds due to (4.18).

Construction of the sigma model on $\mathscr{G}$

To build a sigma model on the Lie group $\mathscr{G}$, we consider two elements $\partial_\pm l l^{-1} \in \mathfrak{d}$, where $l = g\tilde{h},\ g \in \mathscr{G},\ \tilde{h} \in \widetilde{\mathscr{G}}$. According to [11], the dual theories can be deduced from the requirement

$$\partial_\mp l l^{-1} \in \mathcal{E}^\pm, \tag{4.23}$$

or from the equivalent equation

$$\langle\, \partial_\pm l l^{-1}, \mathcal{E}^\pm \,\rangle_{\mathfrak{d}} = 0. \tag{4.24}$$

Plugging the decomposition $l = g\tilde{h}$ into this equation, the expression can be further processed as

$$\begin{aligned} \langle\, \partial_\pm l l^{-1}, \mathcal{E}^\pm \,\rangle_{\mathfrak{d}} &= \langle\, \partial_\pm(g\tilde{h})(g\tilde{h})^{-1}, \mathcal{E}^\pm \,\rangle_{\mathfrak{d}} = \langle\, \partial_\pm g g^{-1} + g(\partial_\pm\tilde{h}\tilde{h}^{-1})g^{-1}, \mathcal{E}^\pm \,\rangle_{\mathfrak{d}} \\ &= \langle\, g^{-1}\partial_\pm g + \partial_\pm\tilde{h}\tilde{h}^{-1}, g^{-1}\mathcal{E}^\pm g \,\rangle_{\mathfrak{d}} = 0. \end{aligned} \tag{4.25}$$

In the last step the Ad-invariance (equivalent to ad-invariance) of $\langle .,.\rangle_{\mathfrak{d}}$ has been used. The term $g^{-1}\mathcal{E}^\pm g$ denotes the action of $Ad_{g^{-1}}$ on $\mathcal{E}^\pm$.

Several steps now follow the construction of the Poisson bivector on a Poisson–Lie group that we discussed in chapter 2. We denote $a(g)$, $b(g)$ and $d(g)$ the submatrices of the adjoint representation of $\mathscr{G}$ on $\mathfrak{d}$ in the basis $\mathcal{X} = (T_a, \widetilde{T}^b)$ as

$$^{\mathcal{X}}(Ad_{g^{-1}})^T = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}, \tag{4.26}$$

that is

$$g^{-1}T_a g = a(g)_a^b T_b, \qquad g^{-1}\widetilde{T}^a g = b(g)^{ab} T_b + d(g)_b^a \widetilde{T}^b.$$

The subspace $g^{-1}\mathcal{E}^+ g$ is spanned by vectors

$$Ad_{g^{-1}}(T_a + E(e)_{ab}\widetilde{T}^b) = \left(\left(a(g)_a^c + E(e)_{ab} b(g)^{bc}\right) T_c + E(e)_{ab} d(g)_c^b \widetilde{T}^c\right).$$

Therefore, if the matrix $(a(g) + E(e)b(g))$ is invertible, we may express the subspace $g^{-1}\mathcal{E}^+ g$ as

$$\begin{aligned} g^{-1}\mathcal{E}^+ g &= Span\left[\left((a(g) + E(e)b)_a^c T_c + E(e)_{ab} d(g)_c^b \widetilde{T}^c\right)\right] \\ &= Span\left[(T_a + E(g)_{ab}\widetilde{T}^b)\right], \end{aligned} \tag{4.27}$$

where we defined $E(g)_{ab}$ as the components of a matrix

$$E(g) = \left(a\left(g\right) + E(e)\cdot b(g)\right)^{-1} \cdot E(e)\cdot d(g). \tag{4.28}$$

These steps can be repeated for the subspace $g^{-1}\mathcal{E}^- g$, which can be written with the help of $E(g)$ as

$$g^{-1}\mathcal{E}^- g = Span\left[(T_a - E(g)_{ba}\widetilde{T}^b)\right]. \tag{4.29}$$

Calculation similar to (4.22) proves that $g^{-1}\mathcal{E}^\pm g$ remain orthogonal with respect to $\langle .,. \rangle_{\mathfrak{d}}$.

Now we can return to the equations (4.25). Since the vectors of $g^{-1}\mathcal{E}^+ g$ are generated from the basis via (4.27), the first set of equations written in components gives

$$\begin{aligned} 0 &= \langle\, (g^{-1}\partial_+ g)^a T_a + (\partial_+ \tilde{h}\tilde{h}^{-1})_a \widetilde{T}^a, T_b + E(g)_{bc}\widetilde{T}^c \rangle_{\mathfrak{d}} \\ &= E(g)_{ba}(g^{-1}\partial_+ g)^a + (\partial_+ \tilde{h}\tilde{h}^{-1})_b, \end{aligned}$$

while the second set gives, using (4.29), the relation

$$\begin{aligned} 0 &= \langle\, (g^{-1}\partial_- g)^a T_a + (\partial_- \tilde{h}\tilde{h}^{-1})_a \widetilde{T}^a, T_b - E(g)_{cb}\widetilde{T}^c \rangle_{\mathfrak{d}} \\ &= -(g^{-1}\partial_- g)^a E(g)_{ab} + (\partial_- \tilde{h}\tilde{h}^{-1})_b. \end{aligned}$$

In the end of our calculations we have found that (4.24) leads to a set of equations

$$\left(\partial_+ \tilde{h}\tilde{h}^{-1}\right)_a = -E(g)_{ab}\left(g^{-1}\partial_+ g\right)^b, \tag{4.30}$$

$$\left(\partial_- \tilde{h}\tilde{h}^{-1}\right)_a = \left(g^{-1}\partial_- g\right)^b E(g)_{ba}. \tag{4.31}$$

Let us briefly return to the Noether currents and forms defined in the previous section. Comparing (4.9) against (4.30) and (4.31), we may assign

$$\begin{aligned} J_a^+ &:= -\left(\partial_+ \tilde{h}\tilde{h}^{-1}\right)_a = E(g)_{ab}\left(g^{-1}\partial_+ g\right)^b, \\ J_a^- &:= -\left(\partial_- \tilde{h}\tilde{h}^{-1}\right)_a = -\left(g^{-1}\partial_- g\right)^b E(g)_{ba}. \end{aligned}$$

These Noether currents correspond to a sigma model on the Lie group $\mathscr{G}$ whose dynamics is specified by a tensor field F given by

$$F_{\mu\nu} = {}^{L}e^{a}_{\mu}(g)\, E(g)_{ab}\, {}^{L}e^{b}_{\nu}(g), \tag{4.32}$$

where $g \in \mathscr{G}$, ${}^{L}e^{a}_{\mu}(g)$ are the components of the left-invariant Maurer–Cartan form $g^{-1}dg$, and $E(g)_{ab}$ are matrix elements of the nondegenerate bilinear form $E(g)$ defined by the matrix (4.28). The action of the sigma model on $\mathscr{G}$ then reads

$$\begin{aligned} S_F[g] &= \int_\Sigma d^2\sigma\ \left(g^{-1}\partial_+ g\right)^a E(g)_{ab} \left(g^{-1}\partial_- g\right)^b \\ &= \int_\Sigma d^2\sigma\ \partial_- X^\mu \left({}^{L}e^{a}_{\mu}(g)\, E(g)_{ab}\, {}^{L}e^{b}_{\nu}(g)\right) \partial_+ X^\nu, \end{aligned} \tag{4.33}$$

with the components of the left-invariant form given by

$$(g^{-1}\partial_\pm g)^a = \partial_\pm X^\mu\, {}^{L}e^{a}_{\mu}(g), \qquad (g^{-1}\partial_\pm g) = (g^{-1}\partial_\pm g)^a T_a.$$

One one hand, the Maurer–Cartan identity (4.11) guarantees that the connection one-form $J_a\widetilde{T}^a$ is integrable and its components can be written as $J^{\pm}_a = -(\partial_\pm \tilde{h}\tilde{h}^{-1})_a$. On the other hand, substituting for $J^{\pm}_a$ we obtain the identity in the form

$$\partial_-(\partial_+\tilde{h}\tilde{h}^{-1})_a - \partial_+(\partial_-\tilde{h}\tilde{h}^{-1})_a + \tilde{c}^{bc}_a(\partial_+\tilde{h}\tilde{h}^{-1})_b(\partial_-\tilde{h}\tilde{h}^{-1})_c = 0. \tag{4.34}$$

In (4.12) we found the equations of motion of a sigma model on whose target manifold a free and transitive action of a Lie group $\mathscr{G}$ is defined. When (4.30), (4.31) are inserted into (4.34), we find that for dualizable sigma models, i.e. when (4.13) holds, the Maurer–Cartan identity (4.34) represents the equations of motion for the sigma model (4.33). Thus, our construction led to a similar situation that we had for Abelian duality. Again, the equations of motion can be formulated in terms of Bianchi identities, or to be more precise, in terms of Maurer–Cartan condition declaring that connection one-form is flat and integrable.

The construction of the sigma model (4.33) carried out above followed the original paper [11] and employed the left-invariant fields $(g^{-1}\partial_\pm g)^a$. For future discussions, it will be convenient to rewrite it using right-invariant fields. First, we note that as a consequence of the definition of the adjoint representation of $\mathscr{G}$ on $\mathfrak{g}$ the following relation holds for the matrices of left-invariant vielbeins ${}^{L}e(g)$ and right-invariant vielbeins $e(g) := {}^{R}e(g)$:

$${}^{L}e(g) = e(g)\cdot a(g).$$

Second, from the Ad-invariance of $\langle .,.\rangle_\mathfrak{d}$ we deduce

$$\begin{aligned} \delta^b_a = \langle T_a, \widetilde{T}^b\rangle_\mathfrak{d} &= \langle Ad_{g^{-1}}T_a, Ad_{g^{-1}}\widetilde{T}^b\rangle_\mathfrak{d} \\ &= \langle a(g)^c_a T_c, b(g)^{bc}T_c + d(g)^b_d \widetilde{T}^d\rangle_\mathfrak{d} = a(g)^c_a d(g)^b_c, \end{aligned}$$

such that in matrix notation we have

$$d(g) = a^{-T}(g). \tag{4.35}$$

With these relations we may rewrite the tensor F given in (4.32) and (4.28) into the form

$$\begin{aligned} F &= {}^{L}e(g) \cdot E(g) \cdot {}^{L}e^{T}(g) \\ &= {}^{L}e(g) \cdot [a(g) + E(e) \cdot b(g)]^{-1} \cdot E(e) \cdot d(g) \cdot {}^{L}e^{T}(g) \\ &= {}^{L}e(g) \cdot a^{-1}(g) \cdot \Big[E^{-1}(e) + b(g) \cdot a^{-1}(g)\Big]^{-1} \cdot a^{-T}(g) \cdot {}^{L}e^{T}(g) \\ &= e(g) \cdot {}^{R}E(g) \cdot e^{T}(g) \end{aligned}$$

From now on, we shall use only the right-invariant fields in the construction of the dual sigma models, so we drop the superscript ^{R} and refer to ${}^{R}E(g)$ simply as to $E(g)$. The tensor field F defining the dualizable sigma model on the Lie group $\mathscr{G}$ therefore is

$$F_{\mu\nu} = e^{a}_{\mu}(g) E(g)_{ab} e^{b}_{\nu}(g), \tag{4.36}$$

where $e^{a}_{\mu}(g)$ are the components of the right-invariant Maurer–Cartan form dgg^{-1}, and matrices $E(g)$ are of the form

$$E(g) = \Big(E^{-1}(e) + \Pi(g)\Big)^{-1}, \qquad \Pi(g) = b(g) \cdot a(g)^{-1}. \tag{4.37}$$

The constant invertible matrix $E(e)$ is the only necessary input besides the algebraic structure of the Drinfel'd double. Translating the construction into the language of right-invariant fields, we have revealed an important structure living on $\mathscr{G}$ – the Poisson–Lie structure defined by Π. The action of the sigma model now reads

$$S_E[g] = \int_{\Sigma} d^2\sigma \; \rho_{-}(g) \cdot E(g) \cdot \rho_{+}(g)^{T},$$

where the right–invariant fields $\rho_{\pm}(g)$ are given by

$$\rho^{a}_{\pm}(g) \equiv \Big(\partial_{\pm} g g^{-1}\Big)^{a} = \partial_{\pm} X^{\mu}\, e^{a}_{\mu}(g), \qquad \Big(\partial_{\pm} g g^{-1}\Big) = \rho_{\pm}(g) \cdot T. \tag{4.38}$$

Using the same notation for the right-invariant fields on $\widetilde{\mathscr{G}}$, we get

$$\widetilde{\rho}_{a\pm}(\tilde{g}) \equiv (\partial_{\pm}\tilde{g}\tilde{g}^{-1})_{a} = \partial_{\pm}\widetilde{X}^{\mu}\, \widetilde{e}_{a\mu}(\tilde{g}), \qquad (\partial_{\pm}\tilde{g}\tilde{g}^{-1}) = \widetilde{\rho}_{\pm}(\tilde{g}) \cdot \widetilde{T},$$

and the equations (4.30) and (4.31) acquire the form

$$\widetilde{\rho}_{+}(\tilde{h}) = -\rho_{+}(g) \cdot E(g)^{T} \cdot a^{-T}(g), \tag{4.39}$$

$$\widetilde{\rho}_{-}(\tilde{h}) = \rho_{-}(g) \cdot E(g) \cdot a^{-T}(g). \tag{4.40}$$

Construction of the sigma model on $\widetilde{\mathscr{G}}$

In order to obtain a sigma model on the Lie group $\widetilde{\mathscr{G}}$, we again have to work with subspaces $\mathcal{E}^{\pm}$. Because $E(e)$ was invertible, the subspaces can be given by

$$\begin{aligned} \mathcal{E}^{+} &= Span\big[(\widetilde{T}^{a} + \widetilde{E}(e)^{ab} T_{b})_{a\in\widehat{m}}\big], \qquad \widehat{m} := \{1, \ldots, \dim\widetilde{\mathscr{G}}\}, \\ \mathcal{E}^{-} &= Span\big[(\widetilde{T}^{a} - \widetilde{E}(e)^{ba} T_{b})_{a\in\widehat{m}}\big], \qquad \widehat{m} := \{1, \ldots, \dim\widetilde{\mathscr{G}}\}, \end{aligned}$$

where obviously

$$\widetilde{E}(e) = E^{-1}(e).$$

Starting from (4.23), each step of the preceding calculation can be repeated using the other decomposition of the element of the Drinfel'd double. This time we plug in the decomposition $l = \tilde{g}h,\ h \in \mathscr{G},\ \tilde{g} \in \widetilde{\mathscr{G}}$. Similarly to (4.26), we define submatrices $\widetilde{a}(\tilde{g}), \widetilde{b}(\tilde{g}), \widetilde{d}(\tilde{g})$ of the adjoint representation of $\widetilde{\mathscr{G}}$ on $\mathfrak{d}$ and use them to express $\tilde{g}^{-1}\mathcal{E}^{\pm}\tilde{g}$ in terms of basis vectors $(\widetilde{T}^a, T_a)$. Finally, we arrive at a sigma model

$$S_{\widetilde{E}}[\tilde{g}] = \int_{\Sigma} d^2\sigma\, \widetilde{\rho}_-(\tilde{g}) \cdot \widetilde{E}(\tilde{g}) \cdot \widetilde{\rho}_+^T(\tilde{g}),$$

where

$$\widetilde{E}(\tilde{g}) = \Big(E(e) + \widetilde{\Pi}(\tilde{g})\Big)^{-1}, \qquad \widetilde{\Pi}(\tilde{g}) = \widetilde{b}(\tilde{g}) \cdot \widetilde{a}(\tilde{g})^{-1}. \tag{4.41}$$

Conditions analogous to (4.39) and (4.40) also follow just by changing tilded quantities to untilded and vice versa.

Correctness of the construction

There is one thing which remains to be shown. Authors of [11] claim that the dual models on $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ constructed above give the general solution to (4.13). To advocate this idea, we have to return to the concepts presented in chapter 2. In particular, we have defined the intrinsic derivative of a Poisson bivector $D\Pi(x) = (\mathcal{L}_X\Pi)(e)$ for $X(e) = x$, and noticed that $\delta = D\Pi$ is the map whose transpose specifies the Lie algebra structure on $\mathfrak{g}^*$. Expressing the components of $\mathcal{L}_{v_a}\Pi$ in the frame of left-invariant fields on $\mathscr{G}$, we have $(\mathcal{L}_{v_a}\Pi)^{bc} = -\tilde{c}_a^{bc}$ since Π is skew-symmetric. Dualizability condition (4.13) can be thus written as

$$(\mathcal{L}_{v_a}F)_{\mu\nu} = -F_{\mu\kappa}\,(\mathcal{L}_{v_a}\Pi)^{\kappa\lambda}\,F_{\lambda\nu}.$$

In matrix form this reads

$$\mathcal{L}_{v_a}F = -F \cdot (\mathcal{L}_{v_a}\Pi) \cdot F.$$

Multiplying the last equation from both sides by F^{-1}, we receive a formula $F^{-1} \cdot (\mathcal{L}_{v_a}F) \cdot F^{-1} = -\mathcal{L}_{v_a}\Pi$. On the left-hand side we already recognize the prescription for the Lie derivative $\mathcal{L}_{v_a}(F^{-1}) = -F^{-1} \cdot (\mathcal{L}_{v_a}F) \cdot F^{-1}$, and we conclude that (4.13) can be rewritten as

$$\mathcal{L}_{v_a}(F^{-1} - \Pi) = 0.$$

The Lie derivative of the tensor $(F^{-1} - \Pi)$ with respect to each left-invariant vector field vanishes, hence it has to be a right-invariant tensor field on the connected Lie group $\mathscr{G}$. Since Π is a Poisson–Lie bivector field, it vanishes in e, and $(F^{-1} - \Pi)(e) = E^{-1}(e)$ for some right-invariant tensor field E, where we denoted $E^{-1} := (F^{-1} - \Pi)$. But E is constant in the frame of right-invariant fields, so we may write

$$(F^{-1})^{ab} = E^{ab}(e) + \Pi^{ab},$$

which leads to the prescription (4.37). Finally, rewriting (4.37) from the frame of right-invariant fields, we get (4.36). Therefore, we conclude that the construction presented in the previous section is valid and gives the general solution of the condition of dualizability of sigma models.

4.4 Poisson–Lie T-duality transformation

We have successfully constructed two two-dimensional non-linear sigma models living on Lie groups $\mathscr{G}$, $\widetilde{\mathscr{G}}$. Both these groups are now endowed with background tensors F and $\widetilde{F}$ and carry Poisson–Lie structures Π and $\widetilde{\Pi}$. The models are dual in the sense of Poisson–Lie T-duality, i.e. if $g(\sigma_+,\sigma_-)$ is the solution of the equations of the sigma model on $\mathscr{G}$ and $\tilde{g}(\sigma_+,\sigma_-)$ is the solution of the dual sigma model living on $\widetilde{\mathscr{G}}$, then these solutions are related by the Poisson–Lie T-duality transformation

$$g(\sigma_+,\sigma_-)\tilde{h}(\sigma_+,\sigma_-) = \tilde{g}(\sigma_+,\sigma_-)h(\sigma_+,\sigma_-), \qquad g,h \in \mathscr{G},\ \tilde{g},\tilde{h} \in \widetilde{\mathscr{G}}. \tag{4.42}$$

If a solution g is known, the auxiliary fields $\tilde{h}$ can be found as solutions of the set of PDEs (4.39), (4.40). The essence of the Poisson–Lie T-duality transformation then lies in the possibility to express the element of the Drinfel'd double in two different ways as in the equation (4.42). The solution of the original sigma model is mapped to the solution of the dual model through the change of decomposition (4.42).

Clearly, the steps which one has to perform in order to find the solution of the dual model are not trivial. Even if the solution g is known, one has to solve a set of PDEs to find the auxiliary fields $\tilde{h}$. Another non-trivial task is to find how the change of decomposition can be realized. To keep the ongoing presentation clear, we defer the discussion of these problems to Part II of the thesis, where specific examples will be presented. Nevertheless, we have achieved a precious generalization of T-duality. The special cases of Abelian and non-Abelian T-duality are simply obtained if one chooses one or both of the Lie groups $\mathscr{G}$, $\widetilde{\mathscr{G}}$ Abelian.

4.5 Poisson–Lie T-plurality

We have seen that Poisson–Lie T-dual sigma models live in complementary subgroups of a Drinfel'd double. The Poisson–Lie T-duality transformation relies heavily on the decompositions (4.42). However, there might be several different decompositions of the particular Drinfel'd double $\mathscr{D}$. The first thorough study of this possibility was carried out in [22], where the name *Poisson–Lie T-plurality* was coined. Later, in [48] it was shown that, similarly to Poisson–Lie T-duality, the plurality is a canonical transformation.

The tangent space $\mathfrak{d}$ of the Drinfel'd double can be in principle decomposed into isotropic subalgebras in several ways, each corresponding to different Manin triple. Let $\hat{\mathfrak{g}}\oplus\bar{\mathfrak{g}} = \mathfrak{d}$ be another decomposition of the Lie algebra $\mathfrak{d}$. The pairs of mutually dual bases $T_a \in \mathfrak{g}$, $\widetilde{T}^a \in \tilde{\mathfrak{g}}$ and $\widehat{T}_a \in \hat{\mathfrak{g}}$, $\bar{T}^a \in \bar{\mathfrak{g}}$, $a = 1,\dots,D$, are related by a linear transformation

$$\begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} = \begin{pmatrix} P & Q \\ K & S \end{pmatrix} \cdot \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix}, \tag{4.43}$$

where $P,\ Q,\ K,\ S$ are $(D \times D)$-matrices. Since we want the new bases to be dual to each other with respect to $\langle .,.\rangle_{\mathfrak{d}}$, i.e. we demand that

$$\langle \widehat{T}_a, \widehat{T}_b \rangle_{\mathfrak{d}} = 0, \qquad \langle \bar{T}^a, \bar{T}^b \rangle_{\mathfrak{d}} = 0, \qquad \langle \widehat{T}_a, \bar{T}^b \rangle_{\mathfrak{d}} = \delta_a^b,$$

the submatrices of the linear operator have to fulfill

$$\begin{pmatrix} P & Q \\ K & S \end{pmatrix}^{-1} = \begin{pmatrix} S^T & Q^T \\ K^T & P^T \end{pmatrix},$$

or in other words

$$\begin{aligned} P \cdot S^T + Q \cdot K^T &= \mathbf{1}, \\ P \cdot Q^T + Q \cdot P^T &= 0, \\ K \cdot S^T + S \cdot K^T &= 0. \end{aligned} \tag{4.44}$$

The construction of models connected via Poisson–Lie T-plurality is quite similar to the construction of dual models. To form a sigma model on the Lie group $\widehat{\mathscr{G}}$, we again return to the subspaces $\mathcal{E}^{\pm}$ defined in (4.20) and (4.21), and express them in terms of the bases $\widehat{T}_a,\ \bar{T}^a$. Using (4.43), we have

$$\begin{aligned} \mathcal{E}^+ &= Span\big[(T_a + E(e)_{ab}\widetilde{T}^b)\big] \\ &= Span\big[(P_a^b\widehat{T}_b + Q_{ab}\bar{T}^b + E(e)_{ab}(K^{bc}\widehat{T}_c + S_c^b\bar{T}^c))\big]. \end{aligned}$$

If the matrix $(P + E(e) \cdot K)$ is invertible, we may continue and write

$$\mathcal{E}^+ = Span\big[(\widehat{T}_a + \widehat{E}(e)_{ab}\bar{T}^b)\big],$$

where

$$\widehat{E}(e) = (P + E(e) \cdot K)^{-1} \cdot (Q + E(e) \cdot S)\,. \tag{4.45}$$

Adequate results are obtained when dealing with $\mathcal{E}^-$, which reads

$$\mathcal{E}^- = Span\big[(\widehat{T}_a - \widehat{E}(e)_{ba}\bar{T}^b)\big].$$

The method described in the preceding section can be followed when we insert $l = \hat{g}\bar{h},\ \hat{g} \in \widehat{\mathscr{G}},\ \bar{h} \in \bar{\mathscr{G}}$ into (4.24). When the submatrices $\widehat{a}(\hat{g}),\ \widehat{b}(\hat{g}),\ \widehat{d}(\hat{g})$ of the adjoint representation of $\widehat{\mathscr{G}}$ on $\mathfrak{d}$ are defined similarly to (4.26), the sigma model obtained by the plurality transformation is given analogously to the original one, namely by substituting

$$\widehat{E}(\hat{g}) = \left(\widehat{E}^{-1}(e) + \widehat{\Pi}(\hat{g})\right)^{-1}, \qquad \widehat{\Pi}(\hat{g}) = \widehat{b}(\hat{g}) \cdot \widehat{a}(\hat{g})^{-1} \tag{4.46}$$

into

$$S_{\widehat{E}}[\hat{g}] = \int_{\Sigma} d^2\sigma\ \widehat{\rho}_-(\hat{g}) \cdot \widehat{E}(\hat{g}) \cdot \widehat{\rho}_+^T(\hat{g}).$$

Solutions of the sigma models living in $\mathscr{G}$ and $\widehat{\mathscr{G}}$ are related by two possible decompositions of $l \in D$, in particular by

$$l(\sigma_+, \sigma_-) = g(\sigma_+, \sigma_-)\tilde{h}(\sigma_+, \sigma_-) = \hat{g}(\sigma_+, \sigma_-)\bar{h}(\sigma_+, \sigma_-).$$

Similarly, if the matrix $(Q + E(e) \cdot S)$ is invertible, we can build a sigma model on the Lie group $\bar{\mathscr{G}}$ and try to solve its equations using analogous decomposition. While Poisson–Lie T-duality switched the role of two sigma models, Poisson–Lie T-plurality offers a spectrum of related models. It is easily seen that plurality generalizes duality because we may choose $\widehat{\mathscr{G}} = \widetilde{\mathscr{G}}$, $\bar{\mathscr{G}} = \mathscr{G}$, so that

$$P = S = 0, \qquad Q = K = \mathbf{1}$$

in (4.43), so that (4.45) reduces to $\widehat{E}(e) = E^{-1}(e)$ and (4.46) gives (4.41).

It is worth mentioning that while Poisson–Lie T-duality required $E(e)$ to be invertible, thus leading to invertible $\widetilde{E}(e)$, the $\widehat{E}(e)$ in (4.45) is not invertible in general. Indeed, when $(Q + E(e) \cdot S)$ is singular, it is still possible to form a sigma model on $\widehat{\mathscr{G}}$ because the background tensor field can be computed using left-invariant fields as in (4.28), (4.33). Then, however, $\widehat{F}$ is degenerate. Authors of [13] touch the issue of Poisson–Lie T-duality of a degenerate background and show that it renders a dual sigma model with constraints. Nevertheless, it would be interesting to investigate this matter in more detail.

4.6 Example – low-dimensional Drinfel'd doubles

Having summarized the general formulas realizing the atomic Poisson–Lie T-duality and Poisson–Lie T-plurality, we shall demonstrate several steps of the construction of dual/plural models in more detail. Similar computations need to be carried out also in the following chapters in the presence of spectators. However, for the sake of clarity, we rather present these details here in the simpler case of atomic duality acting on low-dimensional sigma models.

Knowing that the procedure summarized in section 4.3 generates all the solutions of the condition of dualizability, we may not only give examples of dual sigma models, but we may also attempt to give a classification of dual sigma models. For one-dimensional $\mathscr{G}, \widetilde{\mathscr{G}}$ there is nothing really interesting, since even after adding spectator fields, we get only Buscher duality.

Situation is much more interesting for two-dimensional groups. In [20] the four-dimensional Drinfel'd doubles were classified, and it turned out that there are four types of corresponding Manin triples. Obviously, one of them is the case where both $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ are Abelian. This would again lead to Abelian duality. The relation between dual backgrounds would be given by $\widetilde{F} = F^{-1}$ in the atomic case, while the formulas (3.14) would apply in the presence of spectators.

The second four-dimensional Manin triple is semi-Abelian, where $\mathfrak{g}$ is commutative, while the commutation relations for $\widetilde{\mathfrak{g}}$ and nontrivial commutation relations in $\mathfrak{d}$ are

$$[\widetilde{T}^1, \widetilde{T}^2] = \widetilde{T}^2, \qquad [T_2, \widetilde{T}^1] = T_2, \qquad [T_2, \widetilde{T}^2] = -T_1.$$

Interestingly, this Manin triple is isomorphic to another one, where $\mathfrak{d}$ decomposes into $\hat{\mathfrak{g}} \oplus \bar{\mathfrak{g}}$ with both subalgebras being non-Abelian. The commutation relations read:

$$[\widehat{T}_1, \widehat{T}_2] = \widehat{T}_2, \qquad [\bar{T}^1, \bar{T}^2] = \bar{T}^1,$$

and

$$[\widehat{T}_1, \bar{T}^1] = \widehat{T}_2, \qquad [\widehat{T}_1, \bar{T}^2] = -\widehat{T}_1 - \bar{T}^2, \qquad [\widehat{T}_2, \bar{T}^2] = \bar{T}^1.$$

The isomorphism (4.43) between the dual bases $(T_i, \widetilde{T}^j)$ and $(\widehat{T}_i, \bar{T}^j)$ is given by

$$\begin{pmatrix} T_1 \\ T_2 \\ \widetilde{T}^1 \\ \widetilde{T}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \widehat{T}_1 \\ \widehat{T}_2 \\ \bar{T}^1 \\ \bar{T}^2 \end{pmatrix}.$$

In a sufficiently small neighborhood of $e \in \mathscr{D}$ we may parametrize the group elements of $\mathscr{G}$, $\widetilde{\mathscr{G}}$, $\widehat{\mathscr{G}}$ and $\bar{\mathscr{G}}$ using *one-parametric subgroups*:

$$\begin{aligned} g &= e^{x^1 T_1} e^{x^2 T_2}, & \tilde{g} &= e^{\tilde{x}_1 \widetilde{T}^1} e^{\tilde{x}_2 \widetilde{T}^2}, \\ \hat{g} &= e^{\hat{x}^1 \widehat{T}_1} e^{\hat{x}^2 \widehat{T}_2}, & \bar{g} &= e^{\bar{x}_1 \bar{T}^1} e^{\bar{x}_2 \bar{T}^2}, \end{aligned} \tag{4.47}$$

where T, $\widetilde{T}$, $\widehat{T}$ and $\bar{T}$ represent the generators of the corresponding groups and x^i, $\tilde{x}_i$, $\hat{x}^i$ and $\bar{x}_i$ serve as local coordinates.

In order to construct a dualizable sigma model on $\mathscr{G}$, we choose the constant matrix $E(e)$ as

$$E(e) = \begin{pmatrix} t & u \\ v & w \end{pmatrix},$$

and continue by calculating ${}^{\mathcal{X}}(Ad_{g^{-1}})^T$ to get the matrices $a(g)$, $b(g)$, $d(g)$ from (4.26). Due to the definition of the adjoint representation, the following relation

$${}^{\mathcal{X}}(Ad_{(g_1 g_2)}) = {}^{\mathcal{X}}(Ad_{g_1}) \cdot {}^{\mathcal{X}}(Ad_{g_2})$$

clearly holds. Since g is parametrized as in (4.47) and the matrix of the adjoint representation can be written as a matrix exponential

$${}^{\mathcal{X}}(Ad_{e^{x \cdot T}}) = e^{(x \cdot {}^{\mathcal{X}}(ad_T))},$$

we learn that ${}^{\mathcal{X}}(Ad_{g^{-1}})^T$ is given by

$${}^{\mathcal{X}}(Ad_{g^{-1}})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x^2 & 1 & 0 \\ x^2 & 0 & 0 & 1 \end{pmatrix}.$$

The components of the Poisson bivector on the Poisson–Lie group $\mathscr{G}$ expressed in the frame of right-invariant fields were defined in (4.37). Since $a(g) = \mathbf{1}$ as expected, we see that

$$\Pi(g) = b(g) = \begin{pmatrix} 0 & -x^2 \\ x^2 & 0 \end{pmatrix}.$$

From the first formula in (4.37) we obtain

$$E(g) = \begin{pmatrix} \frac{t}{1+x^2(u-v-uvx^2+twx^2)} & \frac{u-uvx^2+twx^2}{1+x^2(u-v-uvx^2+twx^2)} \\ \frac{v+uvx^2-twx^2}{1+x^2(u-v-uvx^2+twx^2)} & \frac{w}{1+x^2(u-v-uvx^2+twx^2)} \end{pmatrix}. \tag{4.48}$$

There is no need to calculate the components of right-invariant fields since $\mathscr{G}$ is Abelian. We have $e_\mu^a(g) = \mathbf{1}$, so the sigma model background tensor is given by $F(g) = E(g)$, i.e. by (4.48).

The dual sigma model is found when the roles of $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ are interchanged. First, we find the matrix of the adjoint representation

$$^{\mathcal{X}}(Ad_{\tilde{g}^{-1}})^T = \begin{pmatrix} 1 & \tilde{x}_2 & 0 & 0 \\ 0 & e^{-\tilde{x}_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\tilde{x}_2 e^{\tilde{x}_1} & e^{\tilde{x}_1} \end{pmatrix}.$$

We see that $\widetilde{b}(\tilde{g})$ vanishes and so does $\widetilde{\Pi}(\tilde{g})$. This is no surprise because $\mathscr{G}$ is Abelian. Applying (4.41), the matrix $\widetilde{E}(\tilde{g})$ therefore equals $E^{-1}(e)$.

We still know only the algebraic structure of $\mathfrak{g}$, but not the multiplication laws in $\widetilde{\mathscr{G}}$. Therefore, it does not have to be obvious how one finds the components $e_\mu^a(g)$. To find the matrix $e(g)$ which fulfills

$$\frac{\partial}{\partial x^\mu}\Big|_g = e_\mu^a(g) R_{g*}(T_a), \qquad \mu = 1, \ldots, D, \tag{4.49}$$

with $\frac{\partial}{\partial x^\mu}$ being the coordinate basis vectors, we take a look at the integral curves $\phi^\mu(t)$ of $\frac{\partial}{\partial x^\mu}\big|_g$. If g is parametrized using one-parametric subgroups as in (4.47), we have

$$\phi^\mu(t) = e^{x^1 T_1} \ldots e^{(x^\mu + t)T_\mu} \ldots e^{x^D T_D}$$

for a curve starting from a point $g = e^{x^1 T_1} \ldots e^{x^D T_D}$. Then

$$\begin{aligned} R_{g^{-1}*}\left(\frac{\partial}{\partial x^\mu}\Big|_g\right) &= \frac{d}{dt}\Big|_{t=0} R_{g^{-1}}(\phi^\mu(t)) \\ &= \frac{d}{dt}\Big|_{t=0}\left(e^{x^1 T_1} \ldots e^{(x^\mu+t)T_\mu} \ldots e^{x^D T_D} e^{-x^D T_D} \ldots e^{-x^1 T_1}\right) \\ &= Ad_{\left(e^{x^1 T_1} \ldots e^{x^{\mu-1} T_{\mu-1}}\right)}(T_\mu), \end{aligned}$$

and defining $e_\mu^a(g)$ via

$$e_\mu^a(g) := \langle \widetilde{T}^a, Ad_{\left(e^{x^1 T_1} \ldots e^{x^{\mu-1} T_{\mu-1}}\right)}(T_\mu) \rangle_{\mathfrak{d}}, \tag{4.50}$$

we have a matrix $e(g)$ whose components satisfy

$$R_{g^{-1}*}\left(\frac{\partial}{\partial x^\mu}\Big|_g\right) = e_\mu^a(g)(T_a), \qquad \mu = 1, \ldots, D,$$

which is equivalent to (4.49).

We may now proceed in the construction of the model on $\widetilde{\mathscr{G}}$. From (4.50) we find

$$\widetilde{e}(\tilde{g}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\tilde{x}_1} \end{pmatrix},$$

so the model dual to (4.48) is given by

$$\widetilde{F}(\tilde{g}) = \begin{pmatrix} \frac{w}{-uv+tw} & \frac{e^{\tilde{x}_1}u}{uv-tw} \\ \frac{e^{\tilde{x}_1}v}{uv-tw} & \frac{e^{2\tilde{x}_1}t}{-uv+tw} \end{pmatrix}.$$

Note that our choice of groups $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ was such that we actually dualized in the opposite direction than we would if we performed standard non-Abelian duality.

To find the model on $\widehat{\mathscr{G}}$, we apply the formulas (4.45), (4.46). With the matrix of the adjoint representation given by

$$^{\mathcal{X}}(Ad_{\hat{g}^{-1}})^T = \begin{pmatrix} 1 & \hat{x}^2 & 0 & 0 \\ 0 & e^{-\hat{x}^1} & 0 & 0 \\ 0 & -1+e^{-\hat{x}^1} & 1 & 0 \\ -1+e^{\hat{x}^1} & \left(-1+e^{\hat{x}^1}\right)\hat{x}^2 & -e^{\hat{x}^1}\hat{x}^2 & e^{\hat{x}^1} \end{pmatrix},$$

and

$$\widehat{E}(e) = \begin{pmatrix} \frac{1}{w} & \frac{v+w}{w} \\ -\frac{u+w}{w} & t-\frac{uv}{w} \end{pmatrix}, \qquad \widehat{e}(\hat{g}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\hat{x}^1} \end{pmatrix},$$

we finally find a bit more complicated expression

$$\widehat{F}(\hat{g}) = \begin{pmatrix} \frac{1}{t-e^{\hat{x}^1}(2t+u+v)+e^{2\hat{x}^1}(t+u+v+w)} & \frac{-e^{\hat{x}^1}(t+u)+e^{2\hat{x}^1}(t+u+v+w)}{t-e^{\hat{x}^1}(2t+u+v)+e^{2\hat{x}^1}(t+u+v+w)} \\ \frac{e^{\hat{x}^1}\left(t+v-e^{\hat{x}^1}(t+u+v+w)\right)}{t-e^{\hat{x}^1}(2t+u+v)+e^{2\hat{x}^1}(t+u+v+w)} & \frac{e^{2\hat{x}^1}(-uv+tw)}{t-e^{\hat{x}^1}(2t+u+v)+e^{2\hat{x}^1}(t+u+v+w)} \end{pmatrix}.$$

The complete classification of real six-dimensional Drinfel'd doubles carried out in [21] allows us to construct plethora of other examples of dual/plural sigma models. We shall meet one in chapter 5. In higher dimension, however, the classification is missing and only a few examples are known, see chapters 8 and 9.

Chapter 5

Poisson–Lie T-plurality with spectators

In the previous chapter we introduced Poisson–Lie T-duality and plurality, but intentionally limited our exposition to the case of the atomic Poisson–Lie T-duality. This is, however, unsatisfactory since even the simplest possible case of Buscher duality included spectators. We shall broaden our scope and include spectator fields into our considerations in this chapter.

Admittedly, formulas for the Poisson–Lie T-duality transformation of sigma models with spectators were already given in [11], but these hold only for the case of duality. On the other hand, examples presented in Ref. [22] intuitively include a spectator t, but the formulas used there for Poisson–Lie T-plurality hold only for the atomic case.

To handle sigma models whose group of generalized isometries does not act transitively on the target manifold, we extend the Drinfel'd double and derive formulas for the Poisson–Lie T-duality and plurality transformation with spectators. This chapter follows corresponding sections of the paper [44], which the author of this thesis coauthored. Minor changes were made in order to unify the notation used throughout the thesis.

5.1 Sigma models without duality

As a warm up, we investigate the complementary case to the one treated in the last chapter. Namely, we consider $\mathscr{G}$ to be trivial, so that $\mathscr{M} = \mathscr{N}$ and the coordinates on the target manifold are chosen to be $x^\alpha,\ \alpha = 1, \ldots, D$. Our strategy is to build an algebraic structure similar to the one we used when we were dealing with atomic duality. We shall again try to write down the equations of motion by virtue of a bilinear form hoping that this will enable us to derive the transformation formulas for Poisson–Lie T-plurality of sigma models with spectators.

Let $T_m\mathscr{M}$ and $T^*_m\mathscr{M}$ be the D-dimensional tangent and cotangent spaces at $m \in \mathscr{M}$

with mutually dual coordinate bases

$$\left\{Y_\alpha = \frac{\partial}{\partial x^\alpha}\right\}_{\alpha\in\widehat{D}}, \qquad \left\{\widetilde{Y}^\alpha = dx^\alpha\right\}_{\alpha\in\widehat{D}}, \qquad \langle\widetilde{Y}^\beta, Y_\alpha\rangle = \delta^\beta_\alpha,$$

and let us introduce a bilinear form $\langle .,.\rangle_T$ on $T_m\mathscr{M} \oplus T^*_m\mathscr{M}$ using the canonical pairing $\langle .,.\rangle$ between vectors and covectors, i.e. via

$$\langle(u,u'),(v,v')\rangle_T := \langle v',u\rangle + \langle u',v\rangle$$

for all $u, v \in T_m\mathscr{M},\ u', v' \in T^*_m\mathscr{M}$. This form is naturally symmetric and nondegenerate. Moreover, the vector spaces $T_m\mathscr{M}$ and $T^*_m\mathscr{M}$ are maximally symmetric with respect to $\langle .,.\rangle_T$.

Consider a nondegenerate second order covariant tensor field F on $\mathscr{M}$. Such a tensor field defines a mapping $F(m) : T_m\mathscr{M} \mapsto T^*_m\mathscr{M}$ at every $m \in \mathscr{M}$. Similarly to (4.20) and (4.21), we can define subspaces of $T_m\mathscr{M} \oplus T^*_m\mathscr{M}$ that are orthogonal with respect to $\langle .,.\rangle_T$ as

$$\mathcal{E}^+ := Span\big[(Y_\alpha + F(x)_{\alpha\beta}\widetilde{Y}^\beta)_{\alpha\in\widehat{D}}\big], \tag{5.1}$$

$$\mathcal{E}^- := Span\big[(Y_\alpha - F(x)_{\beta\alpha}\widetilde{Y}^\beta)_{\alpha\in\widehat{D}}\big]. \tag{5.2}$$

As we shall see, the bilinear form $\langle .,.\rangle_T$ and the decomposition of $T_m\mathscr{M} \oplus T^*_m\mathscr{M}$ allow us to understand elements of $T^*_m\mathscr{M}$ as decompositions of the *canonical momentum*. Indeed, let $m = X(\sigma_+,\sigma_-)$ and let $X_*(\partial_\pm) = \partial_\pm X \in T_m\mathscr{M}$, $p^{(\pm)} \in T^*_m\mathscr{M}$ satisfy

$$(\partial_\mp X,\ \pm p^{(\pm)}) \in \mathcal{E}^\pm, \tag{5.3}$$

or in other words

$$\langle(\partial_\mp X,\ \pm p^{(\pm)}), \mathcal{E}^\mp\rangle_T = 0. \tag{5.4}$$

Inserting (5.1), (5.2) into (5.4), we find that

$$p^{(+)}_\alpha = \partial_- X^\beta F_{\beta\alpha}, \qquad p^{(-)}_\alpha = F_{\alpha\beta}\partial_+ X^\beta.$$

Then we observe that

$$p^{(\pm)}_\alpha = \frac{\partial L}{\partial(\partial_\pm X^\alpha)}$$

for a Lagrangian $L = \partial_- X^\alpha F_{\alpha\beta}(X)\partial_+ X^\beta$. The Euler–Lagrange equations following from L read

$$\partial_+ p^{(+)}_\alpha + \partial_- p^{(-)}_\alpha - \frac{\partial L}{\partial X^\alpha} = 0. \tag{5.5}$$

Moreover, the canonical momentum (found from the Lagrangian (4.2)) is

$$p^{(\tau)}_\alpha = \frac{\partial \bar{L}}{\partial(\partial_\tau X^\alpha)} = p^{(+)}_\alpha + p^{(-)}_\alpha,$$

and the equations of motion can be also written as

$$\partial_\tau p_\alpha^{(\tau)} + \partial_\sigma p_\alpha^{(\sigma)} - \frac{\partial \bar{L}}{\partial X^\alpha} = 0,$$

where

$$p_\alpha^{(\sigma)} = \frac{\partial \bar{L}}{\partial(\partial_\sigma X^\alpha)} = p_\alpha^{(+)} - p_\alpha^{(-)}.$$

As there is no Lie group acting on the target manifold, there is no duality as well. Nevertheless, this important intermezzo showed us that we can treat spectator fields in the same manner as the coordinates participating in the Poisson–Lie T-duality transformation.

5.2 Dualizable sigma models with spectators

Let us now consider the most interesting case when the action of the group $\mathscr{G}$ on the target manifold $\mathscr{M}$ is free but not necessarily transitive. We assume that locally $\mathscr{M}$ splits as $\mathscr{N} \times \mathscr{G}$ with both $\mathscr{N}$ and $\mathscr{G}$ nontrivial. The goal of this section is to form mutually dual sigma models on $\mathscr{N} \times \mathscr{G}$ and $\mathscr{N} \times \widetilde{\mathscr{G}}$, where $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ are subgroups of an appropriate Drinfel'd double. We will not repeat the discussion of Noether currents which is essentially the same as in chapter 4, but proceed directly to the construction of dual models combining the techniques used above.

We recall that the coordinates on $\mathscr{M}$ are chosen in such a way that

$$x = (x^\alpha, x^a), \qquad \alpha = 1, \ldots, \dim \mathscr{N}, \quad a = 1, \ldots, \dim \mathscr{G}.$$

Let us again denote $Y_\alpha = \frac{\partial}{\partial x^\alpha}$, resp. $\widetilde{Y}^\beta = dx^\beta$, the vectors of dual bases in $T_n\mathscr{N}$, resp. $T_n^*\mathscr{N}$. We also introduce a Drinfel'd double $\mathscr{D}$ with a Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \widetilde{\mathfrak{g}}$, and choose T_a, resp. $\widetilde{T}^a$, as vectors of dual bases in $\mathfrak{g}$, resp. $\widetilde{\mathfrak{g}}$, such that

$$\langle \widetilde{Y}^\beta, Y_\alpha \rangle = \delta_\alpha^\beta, \qquad \langle T_a, \widetilde{T}^b \rangle_{\mathfrak{d}} = \delta_a^b.$$

In order to include spectator fields into the concept of Poisson–Lie T-duality, we extend the algebra of the Drinfel'd double to a vector space given in each $n \in \mathscr{N}$ by

$$\mathfrak{d}_E = T_n\mathscr{N} \oplus T_n^*\mathscr{N} \oplus \mathfrak{d}.$$

The extended space naturally carries a bilinear form, given for arbitrary $u, v \in T_n\mathscr{N}$, $u', v' \in T_n^*\mathscr{N}$ and $p, q \in \mathfrak{d}$ as

$$\langle (u, u', p), (v, v', q) \rangle_{\mathfrak{d}_E} = \langle v', u \rangle + \langle u', v \rangle + \langle p, q \rangle_{\mathfrak{d}}.$$

This form is obviously nondegenerate, symmetric and ad-invariant in the third component. Moreover, with respect to $\langle ., . \rangle_{\mathfrak{d}_E}$, the subspaces $(T_n\mathscr{N} \oplus \mathfrak{g})$ and $(T_n^*\mathscr{N} \oplus \widetilde{\mathfrak{g}})$ are maximally symmetric subspaces of $\mathfrak{d}_E$.

Combining the techniques presented in sections 4.3 and 5.1, we define a nondegenerate linear map E:

$$E(x^\alpha) : (T_n\mathscr{N} \oplus \mathfrak{g}) \mapsto (T_n^*\mathscr{N} \oplus \widetilde{\mathfrak{g}})$$

for each $n \in \mathscr{N}$. The dependence of E on n is explicitly marked as the dependence on coordinates x^α. It is convenient to denote the bases of $(T_n\mathscr{N} \oplus \mathfrak{g})$ and $(T_n^*\mathscr{N} \oplus \widetilde{\mathfrak{g}})$ collectively as

$$X_i := (Y_\alpha, T_a), \qquad \widetilde{X}^j := (\widetilde{Y}^\beta, \widetilde{T}^b), \qquad i,j = 1,\ldots,D,$$

so that

$$\langle X_i, X_j\rangle_{\mathfrak{d}_E} = 0, \qquad \langle \widetilde{X}^i, \widetilde{X}^j\rangle_{\mathfrak{d}_E} = 0, \qquad \langle X_i, \widetilde{X}^j\rangle_{\mathfrak{d}_E} = \delta_i^j.$$

In a point $n \in \mathscr{N}$ the graph of $E(x^\alpha)$ and its orthogonal complement with respect to $\langle .,.\rangle_{\mathfrak{d}_E}$ are subspaces given by

$$\begin{aligned}\mathcal{E}^+ &:= Span\big[(X_i + E(x^\alpha)_{ij}\widetilde{X}^j)_{i\in\widehat{D}}\big], \\ \mathcal{E}^- &:= Span\big[(X_i - E(x^\alpha)_{ji}\widetilde{X}^j)_{i\in\widehat{D}}\big].\end{aligned} \tag{5.6}$$

Motivated by formulas (4.23) and (5.3) we consider two elements of $\mathfrak{d}_E$, namely

$$V_\pm := \partial_\pm X^\alpha\, Y_\alpha \mp p_\alpha^{(\mp)}\, \widetilde{Y}^\alpha + \partial_\pm l\, l^{-1}.$$

Inserting a decomposition $l = g\tilde{h}$ of an element of the Drinfel'd double into the expression for $V_\pm$, we find

$$\begin{aligned}V_\pm &= \partial_\pm X^\alpha\, Y_\alpha \mp p_\alpha^{(\mp)}\, \widetilde{Y}^\alpha + (\partial_\pm g g^{-1})^a\, T_a + g\left((\partial_\pm\tilde{h}\tilde{h}^{-1})_a\, \widetilde{T}^a\right) g^{-1} \\ &= \partial_\pm X^\alpha\, Y_\alpha + (\partial_\pm g g^{-1})^a\, T_a \mp p_\alpha^{(\mp)}\, \widetilde{Y}^\alpha + (\partial_\pm\tilde{h}\tilde{h}^{-1})_a\, (g\,\widetilde{T}^a g^{-1}).\end{aligned} \tag{5.7}$$

Remember that in (4.26) we defined matrices $a(g)$, $b(g)$, $d(g)$ as the submatrices of $Ad_{g^{-1}}$. Besides (4.35), other relations can be found using the properties of $\langle .,.\rangle_{\mathfrak{d}}$. Now we are interested in the action of Ad_g, so we need to find $a(g^{-1})$, $b(g^{-1})$, $d(g^{-1})$. Obviously,

$$a(g^{-1}) = a(g)^{-1},$$

which implies

$$d(g^{-1}) = a^T(g)$$

due to (4.35). The Ad-invariance of $\langle .,.\rangle_{\mathfrak{d}}$ also implies

$$b(g^{-1}) = b^T(g)$$

since we have

$$\begin{aligned}\langle Ad_g\widetilde{T}^b, \widetilde{T}^a\rangle_{\mathfrak{d}} &= \langle b(g^{-1})^{bc}T_c + d(g^{-1})^b_c\widetilde{T}^c, \widetilde{T}^a\rangle_{\mathfrak{d}} = b(g^{-1})^{ba}, \\ \langle Ad_g\widetilde{T}^b, \widetilde{T}^a\rangle_{\mathfrak{d}} &= \langle \widetilde{T}^b, Ad_{g^{-1}}\widetilde{T}^a\rangle_{\mathfrak{d}} = \langle \widetilde{T}^b, b(g)^{ac}T_c + d(g)^a_d\widetilde{T}^d\rangle_{\mathfrak{d}} = b(g)^{ab}.\end{aligned}$$

Therefore, we may use

$$g\,\widetilde{T}g^{-1} = Ad_g\widetilde{T} = b(g)^T \cdot T + a(g)^T \cdot \widetilde{T}$$

in the last term of (5.7). Note that the expression for $V_\pm$ can be understood better in terms of bases X_i and $\widetilde{X}^j$. First, let us denote extended vectors of components of right-invariant forms as

$$R^i_\pm = (\partial_\pm X^\alpha, (\partial_\pm g g^{-1})^a), \qquad \widetilde{R}_{\pm i} = (\mp p^{(\mp)}_\alpha, (\partial_\pm \tilde{h}\tilde{h}^{-1})_a).$$

Then the expression for $V_\pm$ can be written in a compact form

$$V_\pm = R_\pm \cdot X + \widetilde{R}_\pm \cdot (\mathcal{B}^T(g) \cdot X + \mathcal{A}^T(g) \cdot \widetilde{X}), \tag{5.8}$$

where

$$\mathcal{A}(g) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & a(g) \end{pmatrix}, \qquad \mathcal{B}(g) = \begin{pmatrix} 0 & 0 \\ 0 & b(g) \end{pmatrix}.$$

To find the sigma model background and its field equations, we impose analogues of the conditions (4.23) and (5.3), namely

$$V_\pm \in \mathcal{E}^\mp, \tag{5.9}$$

i.e. we demand that

$$\langle V_+, X_i + E(x^\alpha)_{ij}\widetilde{X}^j\rangle_{\mathfrak{d}_E} = 0, \qquad \langle V_-, X_i - E(x^\alpha)_{ji}\widetilde{X}^j\rangle_{\mathfrak{d}_E} = 0.$$

Using the formulas for $V_\pm$, the first equation reads

$$\begin{aligned} 0 &= \langle R^k_+ X_k + \widetilde{R}_{+l}\left(\mathcal{B}(g)^{ml} X_m + \mathcal{A}(g)^l_m \widetilde{X}^m\right), X_i + E(x^\alpha)_{ij}\widetilde{X}^j\rangle_{\mathfrak{d}_E} \\ &= R^j_+ E(x^\alpha)_{ij} + \widetilde{R}_{+l}\left(\mathcal{B}(g)^{jl} E(x^\alpha)_{ij} + \mathcal{A}(g)^l_i\right), \end{aligned}$$

so that in matrix notation we write

$$\begin{aligned} \widetilde{R}_+ &= -R_+ \cdot E^T \cdot \left(\mathcal{B}^T \cdot E^T + \mathcal{A}^T\right)^{-1} = -R_+ \cdot \left(\mathcal{B}^T + \mathcal{A}^T \cdot E^{-T}\right)^{-1} \\ &= -R_+ \cdot \left(\mathcal{A}^{-T} \cdot \mathcal{B}^T + E^{-T}\right)^{-1} \cdot \mathcal{A}^{-T} = -R_+ \cdot \left(\mathcal{B} \cdot \mathcal{A}^{-1} + E^{-1}\right)^{-T} \cdot \mathcal{A}^{-T}. \end{aligned}$$

An analogous computation can be carried out for V_-, and we conclude that (5.9) leads to a set of equations

$$\widetilde{R}_+ = -R_+ \cdot E^T(x^\alpha, g) \cdot \mathcal{A}^{-T}(g), \tag{5.10}$$

$$\widetilde{R}_- = R_- \cdot E(x^\alpha, g) \cdot \mathcal{A}^{-T}(g), \tag{5.11}$$

where

$$E(x^\alpha, g) = \left(E(x^\alpha)^{-1} + \Pi(g)\right)^{-1}, \tag{5.12}$$

and

$$\Pi(g) = \mathcal{B}(g) \cdot \mathcal{A}(g)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & b(g) \cdot a(g)^{-1} \end{pmatrix}. \tag{5.13}$$

These equations naturally generalize (4.39) and (4.40), which we would obtain if there were no spectators. Written in components, the equations (5.10), (5.11) read

$$p_\alpha^{(+)} = \partial_- X^\beta E_{\beta\alpha}(x^\gamma, g) + (\partial_- g g^{-1})^b E_{b\alpha}(x^\gamma, g),$$
$$p_\alpha^{(-)} = E_{\alpha\beta}(x^\gamma, g)\partial_+ X^\beta + E_{\alpha b}(x^\gamma, g)(\partial_+ g g^{-1})^b,$$

for the spectator part, and

$$(\partial_+ \tilde{h}\tilde{h}^{-1})_d = -\left(a^{-1}(g)\right)_d^{\ c} \left[E_{c\alpha}(x^\gamma, g)\partial_+ X^\alpha + E_{ca}(x^\gamma, g)(\partial_+ g g^{-1})^a\right], \tag{5.14}$$
$$(\partial_- \tilde{h}\tilde{h}^{-1})_d = \left[\partial_- X^\alpha E_{\alpha c}(x^\gamma, g) + (\partial_- g g^{-1})^a E_{ac}(x^\gamma, g)\right] \left(a^{-1}(g)\right)_d^{\ c}, \tag{5.15}$$

for the components of right-invariant fields on the Drinfel'd double. Comparing these formulas with (4.39) and (4.40), we conclude that the tensor F specifying the background for a sigma model on $\mathscr{N} \times \mathscr{G}$ shall be chosen as

$$\begin{pmatrix} F_{\alpha\beta}(x^\gamma, g) & F_{\alpha b}(x^\gamma, g) \\ F_{a\beta}(x^\gamma, g) & F_{ab}(x^\gamma, g) \end{pmatrix} = \begin{pmatrix} E_{\alpha\beta}(x^\gamma, g) & E_{\alpha c}(x^\gamma, g) e_b^c(g) \\ e_a^c(g) E_{c\beta}(x^\gamma, g) & e_a^c(g) E_{cd}(x^\gamma, g) e_b^d(g) \end{pmatrix},$$

or equivalently as

$$F(x^\gamma, g) = \mathcal{E}(g) \cdot E(x^\gamma, g) \cdot \mathcal{E}^T(g), \tag{5.16}$$

where

$$\mathcal{E}(g) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & e(g) \end{pmatrix},$$

and $e(g)$ are the matrices defined in (4.38). The action for the sigma model living on $\mathscr{N} \times \mathscr{G}$ is then given by

$$S_E = \int_\Sigma d^2\sigma \ R_- \cdot E(x^\gamma, g) \cdot R_+^T, \tag{5.17}$$

and the Lagrangian has the form

$$\begin{aligned} L =& E_{\alpha\beta}(x^\gamma, g)\, \partial_- X^\alpha \, \partial_+ X^\beta + E_{\alpha b}(x^\gamma, g)\, \partial_- X^\alpha \, (\partial_+ g g^{-1})^b \\ &+ E_{a\beta}(x^\gamma, g)(\partial_- g g^{-1})^a \, \partial_+ X^\beta + E_{ab}(x^\gamma, g)(\partial_- g g^{-1})^a (\partial_+ g g^{-1})^b. \end{aligned} \tag{5.18}$$

In sections 4.3 and 5.1 we discussed two special cases of the situation that we are trying to handle now. We saw that equations of motion were given either by (5.5) or by (4.34). The complete set of field equations for the present case is the combination of both. In particular, the variation of the action with respect to spectator fields X^α results in equations

$$\partial_+ p_\alpha^{(+)} + \partial_- p_\alpha^{(-)} - \frac{\partial L}{\partial X^\alpha} = 0. \tag{5.19}$$

To finish the job, we need to rewrite the rest of the field equations as in (4.34). To do that, we need to calculate the Noether currents and discuss the generalization of the condition (4.13). Repeating the computation (4.6) of the change in the sigma model action under the action of $\mathscr{G}$, one obtains Noether forms

$$J_a = (v_a^c F_{c\alpha}\partial_+ X^\alpha + v_a^c F_{cb}\partial_+ X^b)d\sigma_+ - (\partial_- X^\alpha F_{\alpha c} v_a^c + \partial_- X^b F_{bc} v_a^c)d\sigma_-.$$

When constructing the sigma model, we have followed a slightly different way than in section 4.3, and obtained the equations (5.14) and (5.15) directly in terms of right-invariant fields. Nevertheless, it is no problem to rewrite them using left-invariant fields and reveal that the right-hand sides in fact correspond to the components of Noether currents. Then it is easy to see that the Maurer–Cartan identity

$$\partial_-(\partial_+\tilde{h}\tilde{h}^{-1})_a - \partial_+(\partial_-\tilde{h}\tilde{h}^{-1})_a + \tilde{c}_a^{bc}(\partial_+\tilde{h}\tilde{h}^{-1})_b(\partial_-\tilde{h}\tilde{h}^{-1})_c = 0$$

gives the rest of the equations of motion for (5.17) provided the generalized version of (4.13) holds, namely that

$$\begin{pmatrix} F_{\alpha\beta,l}v_a^l & F_{\alpha l}v_{a,j}^l + F_{\alpha j,l}v_a^l \\ v_{a,i}^l F_{l\beta} + F_{i\beta,l}v_a^l & v_{a,i}^l F_{lj} + F_{il}v_{a,j}^l + F_{ij,l}v_a^l \end{pmatrix} = \tag{5.20}$$
$$= \begin{pmatrix} F_{\alpha k}v_b^k \tilde{c}_a^{bc} v_c^l F_{l\beta} & F_{\alpha k}v_b^k \tilde{c}_a^{bc} v_c^l F_{lj} \\ F_{ik}v_b^k \tilde{c}_a^{bc} v_c^l F_{l\beta} & F_{ik}v_b^k \tilde{c}_a^{bc} v_c^l F_{lj} \end{pmatrix}.$$

The tensor F constructed above in (5.16) satisfies this condition, thus the sigma model with spectators described by the Lagrangian (5.18) is dualizable.

To get the dual sigma model, one proceeds in an analogous way only switching $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. The subspaces $\mathcal{E}^\pm$ are expressed as

$$\begin{aligned} \mathcal{E}^+ &= Span\big[(\widetilde{Z}_i + \widetilde{E}(x^\gamma)_{ij}Z^j)_{i\in\widehat{D}}\big], \\ \mathcal{E}^- &= Span\big[(\widetilde{Z}_i - \widetilde{E}(x^\gamma)_{ji}Z^j)_{i\in\widehat{D}}\big], \end{aligned} \tag{5.21}$$

where the dual bases of $\mathfrak{d}_E$ are defined as $Z^j := (\widetilde{Y}^\alpha, T_a)$, $\widetilde{Z}_i := (Y_\beta, \widetilde{T}^b)$. Comparing (5.6) and (5.21), we find the transformation for $E(x^\gamma)$:

$$\widetilde{E}(x^\gamma) = \big(A + E(x^\gamma)\cdot B\big)^{-1}\cdot\big(B + E(x^\gamma)\cdot A\big), \tag{5.22}$$

where the auxiliary matrices A and B are

$$A = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

The equations of motion for x^γ and $\widetilde{g}$ that follow from (5.9) correspond to the Lagrangian (4.1) with $\widetilde{F}$ given by

$$\widetilde{F}(x^\gamma, \tilde{g}) = \widetilde{\mathcal{E}}(\tilde{g})\cdot\widetilde{E}(x^\gamma, \tilde{g})\cdot\widetilde{\mathcal{E}}^T(\tilde{g}), \tag{5.23}$$

where

$$\widetilde{\mathcal{E}}(\tilde{g}) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \widetilde{e}(\tilde{g}) \end{pmatrix}$$

contains the right-invariant vielbein of $\widetilde{\mathscr{G}}$ and

$$\widetilde{E}(x^\gamma, \widetilde{g}) = (\widetilde{E}^{-1}(x^\gamma) + \widetilde{\Pi}(\widetilde{g}))^{-1}, \tag{5.24}$$

with

$$\widetilde{\Pi}(\widetilde{g}) = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{b}(\widetilde{g}) \cdot \widetilde{a}(\widetilde{g})^{-1} \end{pmatrix}.$$

Using the technique explained above, we have formed two sigma models given by tensor fields (5.16) and (5.23). Solutions of Euler–Lagrange equations of these models are related by the Poisson–Lie T-duality transformation (4.42). This time Poisson–Lie T-duality can be used to solve the field equations only partially, since the equations (5.19) are untouched by the transformation.

This formalism naturally extends Poisson–Lie T-duality introduced earlier. We can also recognize that it naturally incorporates Abelian duality, where spectators were present from the beginning. Indeed, for Abelian groups of symmetries the matrices $a(g), e(g)$ and $b(g)$ are

$$a(g) = e(g) = \mathbf{1}, \qquad b(g) = 0,$$

so that $F(x^\gamma, g) = E(x^\gamma, e)$ is independent of the group coordinates, and the fact that we worked with the splitting $\mathscr{M} \approx \mathscr{N} \times \mathscr{G}$ corresponds to the fact that Buscher's duality was carried out in adapted coordinates. F satisfies dualizability conditions (5.20) if the dual group is also chosen Abelian. The dual background is given by $\widetilde{F}(x^\gamma, \tilde{g}) = \widetilde{E}(x^\gamma, e)$, i.e. by (5.22), thus the rules (3.14) are restored. We may also recognize the components of Noether currents in equations (5.14) and (5.15).

Finally, we should add a comment on the geometric structure of T-duality with spectator fields. Both manifolds $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ which accommodate mutually dual sigma models are principal bundles over the same base manifold $\mathscr{N} = \mathscr{M}/\mathscr{G} = \widetilde{\mathscr{M}}/\widetilde{\mathscr{G}}$ and differ only in their fibers $\mathscr{G}$ and $\widetilde{\mathscr{G}}$. To have a duality transformation, they should be understood as being embedded into a manifold $\mathscr{E}$ which has a structure of a principal bundle over $\mathscr{N} = \mathscr{M}/\mathscr{G} = \widetilde{\mathscr{M}}/\widetilde{\mathscr{G}}$ with fiber being the Drinfel'd double $\mathscr{D}$.

5.3 Poisson–Lie T-plurality with spectators

In section 4.5 we saw that duality does not have to be the end of the story when several decompositions of the particular Drinfel'd double exist. The final step that we shall make is to generalize the transformation formulas (4.45), (4.46) and (5.22), (5.24) to incorporate Poisson–Lie T-plurality of sigma models with spectators. To achieve this goal, we shall employ the formalism introduced in section 5.2.

Our starting point is again the vector space $\mathfrak{d}_E = T_n\mathscr{N} \oplus T_n^*\mathscr{N} \oplus \mathfrak{d}$ with the bilinear form $\langle .,. \rangle_{\mathfrak{d}_E}$. Suppose that the Lie algebra of the Drinfel'd double can be decomposed in two ways as $\mathfrak{g} \oplus \tilde{\mathfrak{g}} = \mathfrak{d} = \hat{\mathfrak{g}} \oplus \bar{\mathfrak{g}}$. We choose pairs of mutually dual bases

$$\begin{aligned} &Y_\alpha, \widehat{Y}_\alpha \in T_n\mathscr{N}, && \widetilde{Y}^\beta, \bar{Y}^\beta \in T_n^*\mathscr{N}, && \langle \widetilde{Y}^\beta, Y_\alpha\rangle = \langle \bar{Y}^\beta, \widehat{Y}_\alpha\rangle = \delta_\alpha^\beta, \\ &T_a \in \mathfrak{g},\ \widetilde{T}^a \in \tilde{\mathfrak{g}}, && \widehat{T}_a \in \hat{\mathfrak{g}},\ \bar{T}^a \in \bar{\mathfrak{g}}, && \langle T_a, \widetilde{T}^b\rangle_{\mathfrak{d}} = \langle \widehat{T}_a, \bar{T}^b\rangle_{\mathfrak{d}} = \delta_a^b, \end{aligned}$$

and form $X, \widetilde{X}$ and $\widehat{X}, \bar{X}$ such that

$$X := \begin{pmatrix} Y_\alpha \\ T_a \end{pmatrix}, \quad \widetilde{X} := \begin{pmatrix} \widetilde{Y}^\beta \\ \widetilde{T}^b \end{pmatrix}, \quad \widehat{X} := \begin{pmatrix} \widehat{Y}_\alpha \\ \widehat{T}_a \end{pmatrix}, \quad \bar{X} := \begin{pmatrix} \bar{Y}^\beta \\ \bar{T}^b \end{pmatrix}.$$

These obviously satisfy

$$\langle X_i, X_j\rangle_{\mathfrak{d}_E} = \langle \widetilde{X}^i, \widetilde{X}^j\rangle_{\mathfrak{d}_E} = \langle \hat{X}_i, \hat{X}_j\rangle_{\mathfrak{d}_E} = \langle \bar{X}^i, \bar{X}^j\rangle_{\mathfrak{d}_E} = 0,$$

and

$$\langle X_i, \widetilde{X}^j\rangle_{\mathfrak{d}_E} = \delta_i^j, \qquad \langle \widehat{X}_i, \bar{X}^j\rangle_{\mathfrak{d}_E} = \delta_i^j.$$

Similarly to (4.43), there must be a linear transformation (possibly dependent on $n \in \mathscr{N}$) between the two dual bases $(X, \widetilde{X})$ and $(\widehat{X}, \bar{X})$. Written in the block form it shall read

$$\begin{pmatrix} X \\ \widetilde{X} \end{pmatrix} = \begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{K} & \mathcal{S} \end{pmatrix} \cdot \begin{pmatrix} \widehat{X} \\ \bar{X} \end{pmatrix}, \tag{5.25}$$

where $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{K}$, $\mathcal{S}$ are $D \times D$ matrices. The requirement that the coordinates x^β are spectators which do not participate in the Poisson–Lie transformation means that $Y_\beta = \widehat{Y}_\beta$, $\widetilde{Y}^\beta = \bar{Y}^\beta$, which restricts the general form of this transformation to

$$\mathcal{P} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & P \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & S \end{pmatrix}, \tag{5.26}$$

with matrices P, Q, K, S being constant $\dim\mathscr{G} \times \dim\mathscr{G}$ matrices satisfying conditions (4.44).

The sigma model on $\mathscr{N} \times \widehat{\mathscr{G}}$ obtained through the plurality transformation of (5.17) is then defined by $\widehat{F}$, which we calculate using a formula

$$\widehat{F}(x^\gamma, \hat{g}) = \widehat{\mathcal{E}}(\hat{g}) \cdot \widehat{E}(x^\gamma, \hat{g}) \cdot \widehat{\mathcal{E}}^T(\hat{g}), \tag{5.27}$$

where the matrix

$$\widehat{\mathcal{E}}(\hat{g}) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \hat{e}(\hat{g}) \end{pmatrix}$$

contains the right-invariant vielbein of $\widehat{\mathscr{G}}$. $\widehat{E}(x^\gamma, \hat{g})$ is found through the same process that led to (5.24) and reads

$$\widehat{E}(x^\gamma, \hat{g}) = (\widehat{E}(x^\gamma)^{-1} + \widehat{\Pi}(\hat{g}))^{-1}, \tag{5.28}$$

with

$$\widehat{\Pi}(\hat{g}) = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{b}(\hat{g}) \cdot \widehat{a}^{-1}(\hat{g}) \end{pmatrix}. \tag{5.29}$$

To extract $\widehat{E}(x^\gamma)$, the subspaces $\mathcal{E}^\pm$ defined in (5.6) need to be expressed in terms of $(\widehat{X},\, \bar{X})$. If the matrix $(\mathcal{P} + E(x^\gamma)\cdot\mathcal{K})$ is invertible, we calculate $\widehat{E}(x^\gamma)$ as

$$\widehat{E}(x^\gamma) = (\mathcal{P} + E(x^\gamma)\cdot\mathcal{K})^{-1}\cdot(\mathcal{Q} + E(x^\gamma)\cdot\mathcal{S}). \tag{5.30}$$

These are the formulas describing Poisson–Lie T-plurality of sigma models with spectators. Solutions of equations of motion of the two plural sigma models are again related by two possible decompositions of $l \in D$

$$l = g\tilde{h} = \hat{g}\bar{h}.$$

The above given formulas cover all the cases discussed earlier. The Poisson–Lie T-duality with spectators obtained in the previous section is a special case, which can be restored when $P = S = 0,\ Q = K = \mathbf{1}$. For $\mathscr{M} \approx G$ the transformation (5.25) simplifies to (4.43) and we get the atomic Poisson–Lie T-plurality obtained in section 4.5.

Investigating the formulas above, we find that if the matrix $E(x^\gamma)$ is block diagonal, more precisely if $E(x^\gamma)_{a\alpha} = E(x^\gamma)_{\alpha a} = 0$, then $E(x^\gamma)_{\alpha\beta}$ and $F_{\alpha\beta}$ remain invariant under Poisson–Lie T-plurality, and $E(x^\gamma)_{ab}$ and F_{ab} transform according to formulas found already for the atomic duality/plurality. This fact was intuitively used in [22], however, for general $E(x^\gamma)$ this does not hold, and $\widehat{F}_{\alpha\beta}$ may depend on the group coordinates even if $F_{\alpha\beta}$ depends only on the spectators.

5.4 Example

To illustrate the above explained notions, we present the transformation of a sigma model with four-dimensional target space endowed with a background F given in coordinates (t, x^1, x^2, x^3) as

$$F(t,x) = \begin{pmatrix} -1 & 0 & b\,e^{-x^1} & 0 \\ 0 & t^2 & 0 & 0 \\ -b\,e^{-x^1} & 0 & t^2 e^{-2x^1} & 0 \\ 0 & 0 & 0 & t^2 e^{-2x^1} \end{pmatrix}, \tag{5.31}$$

where b is a constant giving rise to nontrivial B-field and torsion. The metric G is in fact flat. For $b = 0$ this background gives the model considered in [22]. There it was used to illustrate the possibility that plurality may render a conformal sigma model even if duality leads to a model which is not conformal as a consequence of nonvanishing trace of the structure constants of the group of symmetries, see also [49]. Here the Kalb–Ramond B-field was added, so that F is not of the (1+3)-block diagonal form.

This background has a three-dimensional non-Abelian group of symmetries $\mathscr{G}$ generated by rotation and translations in the (x^2, x^3)-plane. Its Lie algebra $\mathfrak{g}$ generated by T_a has commutation relations

$$[T_1, T_2] = -T_2, \qquad [T_2, T_3] = 0, \qquad [T_3, T_1] = T_3. \tag{5.32}$$

We shall dualize this background with respect to $\mathscr{G}$ while treating the coordinate t as a spectator. As $\mathscr{G}$ represents the symmetries of F, we choose the dual group $\widetilde{\mathscr{G}}$ Abelian to satisfy the condition (5.20). According to (4.19), the nonvanishing commutation relations of the basis vectors of $\mathfrak{d}$ are

$$[T_1, \widetilde{T}^2] = \widetilde{T}^2, \qquad [T_2, \widetilde{T}^2] = -\widetilde{T}^1,$$
$$[T_1, \widetilde{T}^3] = \widetilde{T}^3, \qquad [T_3, \widetilde{T}^3] = -\widetilde{T}^1.$$

Next, we represent group elements using one-parametric subgroups as

$$g = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3}, \qquad \tilde{g} = e^{\tilde{x}_1 \widetilde{T}^1} e^{\tilde{x}_2 \widetilde{T}^2} e^{\tilde{x}_3 \widetilde{T}^3}.$$

The tensor $F(t,x)$ is constructed via formulas (5.12), (5.13) and (5.16) from the matrix

$$E(t) = \begin{pmatrix} -1 & 0 & b & 0 \\ 0 & t^2 & 0 & 0 \\ -b & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2 \end{pmatrix},$$

and matrices $\mathcal{E}(g)$ containing the right-invariant vielbeins

$$e(g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-x^1} & 0 \\ 0 & 0 & e^{-x^1} \end{pmatrix}.$$

The Poisson–Lie structure $\Pi(g)$ vanishes because the Drinfel'd double is semi-Abelian, and the submatrices of the adjoint representation of $\mathscr{G}$ on $\mathfrak{d}$ read

$$a(g) = \begin{pmatrix} 1 & -x^2 & -x^3 \\ 0 & e^{x^1} & 0 \\ 0 & 0 & e^{x^1} \end{pmatrix}, \qquad b(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Adopting the notation of [21], where the classification of non-equivalent six-dimensional real Drinfel'd doubles was given, the double used in our construction is labeled as $(5|1)$, where the numbers refer to the *Bianchi classification* of three-dimensional Lie algebras. In particular, $(5|1)$ means that $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$, where $\mathfrak{g}$ is in the class Bianchi 5 with commutation relations (5.32), while $\tilde{\mathfrak{g}}$ is Abelian and corresponds to Bianchi 1.

There exist four possible decompositions of $(5|1)$. According to [21] it splits as

$$(5|1) \simeq (5|2i) \simeq (6_0|1) \simeq (6_0|5ii).$$

To demonstrate the concept of Poisson–Lie T-plurality, we choose the decomposition $(5|2i)$ and build a sigma model on $\mathscr{N} \times \widehat{\mathscr{G}}$, with $\hat{\mathfrak{g}}$ being again from the class Bianchi 5. The nonzero commutation relations of basis elements are

$$[\widehat{T}_1, \widehat{T}_2] = -\widehat{T}_2, \qquad [\widehat{T}_3, \widehat{T}_1] = \widehat{T}_3, \qquad [\bar{T}^2, \bar{T}^3] = \bar{T}^1,$$
$$[\widehat{T}_1, \bar{T}^2] = \widehat{T}_3 + \bar{T}^2, \qquad [\widehat{T}_2, \bar{T}^2] = -\bar{T}^1, \qquad [\widehat{T}_1, \bar{T}^3] = -\widehat{T}_2 + \bar{T}^3,$$
$$[\widehat{T}_3, \bar{T}^3] = -\bar{T}^1.$$

We again represent $\hat{g}$ and $\bar{g}$ using one-parameter subgroups as

$$\hat{g} = e^{\hat{x}^1 \widehat{T}_1} e^{\hat{x}^2 \widehat{T}_2} e^{\hat{x}^3 \widehat{T}_3}, \qquad \bar{g} = e^{\bar{x}_1 \bar{T}^1} e^{\bar{x}_2 \bar{T}^2} e^{\bar{x}_3 \bar{T}^3}.$$

The algebra $\hat{\mathfrak{g}}$ has the same structure constants as $\mathfrak{g}$, so the matrices $\widehat{e}(\hat{g})$, $\widehat{a}(\hat{g})$ have the same form as $e(g)$, $a(g)$ and we only need to relabel the coordinates. Nevertheless, the decomposition of the double is not semi-Abelian any more and $\widehat{b}(\hat{g})$ does not vanish. Instead, it equals to

$$\widehat{b}(\hat{g}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sinh(\hat{x}^1) \\ 0 & \sinh(\hat{x}^1) & 0 \end{pmatrix}.$$

The change from $(5|1)$ to $(5|2i)$ is realized by a linear transformation (5.25). The matrices P, Q, K and S were found in Ref. [21] to be

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the formulas (5.26)–(5.30) for the Poisson–Lie transformation, we get the tensor field specifying the Poisson–Lie T-plural background in the form

$$\widehat{F}(t, \hat{x}) = \begin{pmatrix} -1 + \frac{4b^2 e^{4\hat{x}^1} t^2}{4e^{4\hat{x}^1} t^4 + 1} & 0 & -\frac{4be^{3\hat{x}^1} t^2}{4e^{4\hat{x}^1} t^4 + 1} & \frac{2be^{\hat{x}^1}}{4e^{4\hat{x}^1} t^4 + 1} \\ 0 & t^2 & 0 & 0 \\ -\frac{4be^{3\hat{x}^1} t^2}{4e^{4\hat{x}^1} t^4 + 1} & 0 & \frac{4e^{2\hat{x}^1} t^2}{4e^{4\hat{x}^1} t^4 + 1} & -\frac{2}{4e^{4\hat{x}^1} t^4 + 1} \\ -\frac{2be^{\hat{x}^1}}{4e^{4\hat{x}^1} t^4 + 1} & 0 & \frac{2}{4e^{4\hat{x}^1} t^4 + 1} & \frac{4e^{2\hat{x}^1} t^2}{4e^{4\hat{x}^1} t^4 + 1} \end{pmatrix}. \tag{5.33}$$

Note that the block $F_{11} = -1$ corresponding to spectator fields is transformed to $\widehat{F}_{11} = -1 + \frac{4b^2 e^{4\hat{x}^1} t^2}{4e^{4\hat{x}^1} t^4 + 1}$ and depends on the coordinate $\hat{x}^1$ of the group $\widehat{\mathscr{G}}$.

Chapter 6

Transformation of boundary conditions

When investigating the sigma model action in Chapter 1, we have only discussed the equations of motion following from the action principle and ignored the issue of boundary conditions. We shall address this issue in this chapter.

Returning once again to the action (1.17), we find the equations of motion by taking the variation with respect to dynamical fields X^μ. In addition to that, we have to impose the periodicity condition $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$ when dealing with closed strings. For open strings the situation is more complicated though. The string endpoints are confined to propagate in some submanifold of the target space. We show that the boundary conditions (and the corresponding submanifold) can be represented using the so-called gluing matrix $\mathcal{R}$ introduced in [50], [51], and summarize what conditions this matrix has to satisfy.

In the preceding chapters we have developed a powerful tool of Poisson–Lie T-duality. We were able to transform a solution of a sigma model to a solution of its dual. One may thus ask how the boundary conditions transform, and whether the transformed gluing matrix satisfies the appropriate consistency conditions. The topic was already studied for atomic duality/plurality in several papers including [52], [53], where it was shown that the transformed boundary conditions can be well defined only if we also allow the possibility that the string endpoints are electrically charged.

Our goal is to derive a formula for the transformation of gluing matrices under Poisson–Lie T-plurality with spectators, and to check, whether the transformed matrices give well defined boundary conditions. When the formula is found, we develop further the example presented in the previous chapter, and demonstrate how gluing matrices specifying branes in the background (5.31) transform.

The content of the chapter follows the corresponding sections of author's coauthored paper [44]. Some changes were made at the beginning of the chapter in sections 6.1 and 6.2, where we added comments concerning the origin of the gluing matrix and the role that it plays in the formulation of consistent boundary conditions.

6.1 Boundary conditions and Dp-branes

Under general variations with $\delta X^\mu|_{\tau=\tau_1} = \delta X^\mu|_{\tau=\tau_2} = 0$ the action (1.17) changes by two terms. One of them vanishes if the equations of motion (1.18) are satisfied. The second is the surface term, which vanishes if

$$\delta X^\mu \left(G_{\mu\nu}\partial_\sigma X^\nu + B_{\mu\nu}\partial_\tau X^\nu \right)|_{\sigma=0,\pi} = 0.$$

This straightforwardly generalizes the expression (1.11) to curved backgrounds and strings coupled to an antisymmetric field B. In terms of worldsheet lightcone coordinates we have

$$\delta X^\mu \left(F_{\mu\nu}\partial_+ X^\nu - \partial_- X^\nu F_{\nu\mu} \right)|_{\sigma=0,\pi} = 0. \tag{6.1}$$

This condition can be further processed in terms of Dp-branes and gluing matrices.

The Dirichlet conditions fix the string endpoint to a submanifold $\mathscr{M}' \subset \mathscr{M}$ called a *Dp-brane*. Its dimension p is given by the number of $(p+1)$ independent Neumann boundary conditions (from the physical point of view one coordinate has to remain free). With this convention the $D0$-brane represents a point in the target manifold. Let us now choose coordinates on $\mathscr{M}$ in such a way that $(p+1)$ of them give a local coordinate system on $\mathscr{M}' \subset \mathscr{M}$, while the remaining $(D-(p+1))$ coordinates run perpendicular to the brane and represent the directions in which the Dirichlet conditions apply. Such coordinates y^μ are called the *adapted coordinates* for the Dp-brane. In terms of y^μ the Dirichlet condition for fixed endpoints can be written as

$$\partial_\tau Y^\mu|_{\sigma=0,\pi} = 0, \qquad \text{Dirichlet b. c.} \qquad \partial_- Y^\mu|_{\sigma=0,\pi} = -\partial_+ Y^\mu|_{\sigma=0,\pi}. \tag{6.2}$$

In the absence of the B-field the free endpoint boundary conditions would be similar to Neumann conditions in flat space:

$$\partial_\sigma Y^\mu|_{\sigma=0,\pi} = 0, \qquad \text{Neumann b. c.} \qquad \partial_- Y^\mu|_{\sigma=0,\pi} = \partial_+ Y^\mu|_{\sigma=0,\pi}. \tag{6.3}$$

However, our description should handle nontrivial B-field and should be valid in arbitrary coordinates. This is achieved with the help of the *gluing matrix* $\mathcal{R}(Y)$ representing the *gluing operator* $\mathfrak{R}$.

6.2 Gluing matrices

To capture the essence of (6.2) and (6.3), we shall write the relation between left- and right- derivatives of dynamical fields expressed in general coordinates x^μ as

$$\partial_- X^\mu|_{\sigma=0,\pi} = \partial_+ X^\nu \, \mathcal{R}^\mu_\nu(X)|_{\sigma=0,\pi}. \tag{6.4}$$

Let us examine this defining relation closer. In accordance with (6.2), the geometrical multiplicity of the eigenvalue -1 of $\mathfrak{R}$ encodes the number of Dirichlet directions and specifies the dimension of the Dp-brane. The particular form of $\mathcal{R}$ tells us how the brane $\mathscr{M}'$ is embedded into the target manifold $\mathscr{M}$. In the adapted coordinates the gluing matrix has the form

$$\mathcal{R}(Y) = \begin{pmatrix} \mathcal{R}_\rho^\sigma(Y) & 0 \\ 0 & -\delta_\kappa^\lambda \end{pmatrix} \qquad \rho,\sigma = 1,...,p+1; \quad \kappa,\lambda = p+2,...,D,$$

where $\mathcal{R}_\rho^\sigma(Y)$ is to be determined. Since the brane $\mathscr{M}'$ given by $\mathcal{R}$ should be a well-defined submanifold of $\mathscr{M}$, not every matrix can represent a gluing matrix, and several requirements have to be satisfied.

Since $\mathscr{M}'$ is a submanifold of $\mathscr{M}$, the tangent space $T_p\mathscr{M}'$ at $p \in \mathscr{M}'$ is a subspace of $T_p\mathscr{M}$. Let us define the *Neumann projector* $\mathfrak{N}$ as a projector on the tangent space of the brane $\mathfrak{N} : T_p\mathscr{M} \mapsto T_p\mathscr{M}'$. No matter what coordinates we use for the description of boundary conditions, the string endpoint has to move in the direction tangent to the brane, meaning that $\partial_\tau X^\mu \mid_{\sigma=0,\pi} \in T_p\mathscr{M}'$. Denoting the matrix of the Neumann projector as $\mathcal{N}$, we write this as an equation

$$\partial_\tau X|_{\sigma=0,\pi} \cdot \mathcal{N} = \partial_\tau X|_{\sigma=0,\pi}.$$

Rewriting the equation once again in terms of $\partial_\pm$ and using (6.4), we obtain

$$(\mathbf{1} + \mathcal{R}) \cdot \mathcal{N} = \mathbf{1} + \mathcal{R}.$$

The images of $\mathfrak{N}$ and $(\mathbf{1} + \mathfrak{R})$ are thus the same, and we see that $\mathfrak{N}$ specifies the tangent space of the Dp-brane.

Similarly, we are only allowed to perform such variations of the action which fulfill $\delta X^\mu \mid_{\sigma=0,\pi} \in T_p\mathscr{M}'$. This leads to an equation

$$\delta X|_{\sigma=0,\pi} \cdot \mathcal{N} = \delta X|_{\sigma=0,\pi}.$$

Now we are able to process the boundary conditions (6.1). Using the above given relations, we get

$$\delta X^\lambda \mathcal{N}_\lambda^\mu (F_{\mu\kappa} - \mathcal{R}_\kappa^\nu F_{\nu\mu}) \partial_+ X^\kappa \mid_{\sigma=0,\pi} = 0,$$

where δX and $\partial_+ X$ are now arbitrary. The final form of the boundary conditions for our sigma model is

$$\mathcal{N} \cdot (F - F^T \cdot \mathcal{R}^T) = 0.$$

The projector $\mathfrak{N}$ on $T_p\mathscr{M}'$ specifies a subspace of $T_p\mathscr{M}$ at every point $p \in \mathscr{M}'$. Since the brane $\mathscr{M}'$ is supposed to be a well-defined manifold, the distribution defined by $\mathfrak{N}$ has to be *in involution*. That leads to another condition on $\mathcal{N}$:

$$\mathcal{N}_\kappa^\mu \mathcal{N}_\lambda^\nu \partial_{[\mu} \mathcal{N}_{\nu]}^\rho = 0.$$

This is, however, not the last condition that has to be taken into account.

Since we focus on sigma models with application in string theory, we require that our model is conformally invariant. This can be achieved only if the gluing matrix is orthogonal with respect to the spacetime metric, i.e. if

$$\mathcal{R} \cdot G \cdot \mathcal{R}^T = G.$$

A formula realizing the transformation of gluing matrices under atomic Poisson–Lie T-plurality was found in [52], but it was shown that some of the constraints considered there are not preserved. As it turned out, a possible solution is to assume that the endpoints of the string are electrically charged as suggested in [18]. This can be realized by addition of boundary terms to the action:

$$S_F[X] \to S_F[X] + S_{bnd}[X],$$

where

$$S_{bnd}[X] = q_0 \int_{\sigma=0} A_\mu \frac{\partial X^\mu}{\partial \tau} d\tau - q_0 \int_{\sigma=\pi} A_\mu \frac{\partial X^\mu}{\partial \tau} d\tau.$$

According to [53], well-defined boundary conditions are given by a gluing matrix $\mathcal{R}$, for which a matrix $\mathcal{N}$ can be found, such that the following conditions hold:

$$\mathcal{R} \cdot G \cdot \mathcal{R}^T = G, \tag{6.5}$$

$$(\mathcal{R} + \mathbf{1}) \cdot \mathcal{N} = (\mathcal{R} + \mathbf{1}), \tag{6.6}$$

$$\mathcal{N}^2 = \mathcal{N}, \tag{6.7}$$

$$\operatorname{rank} \mathcal{N} = \operatorname{rank}(\mathcal{R} + \mathbf{1}), \tag{6.8}$$

$$\mathcal{N}^\mu_\kappa \mathcal{N}^\nu_\lambda \partial_{[\mu} \mathcal{N}^\rho_{\nu]} = 0, \tag{6.9}$$

$$\mathcal{N} \cdot \Big((F + \Delta) - (F + \Delta)^T \cdot \mathcal{R}^T \Big) = 0, \tag{6.10}$$

$$\mathcal{N} \cdot \Delta \cdot \mathcal{N}^T = \Delta, \tag{6.11}$$

$$\mathcal{N}_\kappa{}^\nu \mathcal{N}_\lambda{}^\rho \mathcal{N}_\mu{}^\sigma \partial_{[\nu} \Delta_{\rho\sigma]} = 0, \tag{6.12}$$

where G is the symmetric part of F and Δ written in local coordinates y^μ adapted to the brane $\mathscr{M}'$ reads

$$\Delta_{\mu\nu} = \frac{1}{2} \left(\frac{\partial A_\nu}{\partial y^\mu} - \frac{\partial A_\mu}{\partial y^\nu} \right), \quad \mu, \nu = 1, \ldots, \dim \mathscr{M}'.$$

It was shown in [53] that this set of constraints is invariant under the atomic Poisson–Lie transformation. Our next task is to derive a formula for the transformation of gluing matrices under Poisson–Lie T-plurality with spectators, and to check the invariance of the constraints.

6.3 Transformation of gluing matrices

To obtain the formula for the transformation of gluing matrices under Poisson–Lie T-plurality with spectator fields, we can apply a procedure analogous to the one used for the atomic case in [52]. To adapt it to the case with spectators, we write the relation (6.4) in the block form

$$\Big(\partial_- X^\alpha, \ \rho_-^a(g) \Big)\Big|_{\sigma=0,\pi} = \Big(\partial_+ X^\beta, \ \rho_+^b(g) \Big) \cdot \begin{pmatrix} R^\alpha_\beta & R^a_\beta \\ R^\alpha_b & R^a_b \end{pmatrix} \Big|_{\sigma=0,\pi}, \tag{6.13}$$

where $\rho^a_\pm(g) = \partial_\pm X^b e^a_b(g) = (\partial_\pm g g^{-1})^a$ and the blocks R are defined in the frame of right-invariant fields as

$$R = \begin{pmatrix} R^\alpha_\beta & R^a_\beta \\ R^\alpha_b & R^a_b \end{pmatrix} := \begin{pmatrix} \mathcal{R}^\alpha_\beta & \mathcal{R}^c_\beta (e^{-1})^a_c \\ (e^{-1})^c_b \mathcal{R}^\alpha_c & (e^{-1})^c_b \mathcal{R}^d_c e^a_d \end{pmatrix}. \tag{6.14}$$

Note that in the last chapter we have found the relations (5.10), (5.11) between $R^i_\pm = (\partial_\pm X^\alpha, \rho^a_\pm(g))$ and its dual counterpart. Our strategy thus shall be to look for the transformation of $(\partial_\pm X^\alpha, \rho^a_\pm(g))$ under Poisson–Lie T-plurality. Inserting (5.10) and (5.11) into (5.8), we find

$$V_+ = R_+ \cdot E(x^\gamma, g)^T \cdot \left(E(x^\gamma)^{-T} \cdot X - \widetilde{X} \right).$$

Repeating this step for the decomposition $\hat{g}\bar{h} = l \in \mathscr{D}$ and using the linear relation (5.25) between the bases $(X, \widetilde{X})$ and $(\widehat{X}, \bar{X})$, we get the transformation of R_+ in the form

$$\widehat{R}_+ = R_+ \cdot E(x^\gamma, g)^T \cdot M_- \cdot \widehat{E}(x^\gamma, \hat{g})^{-T},$$

where

$$M_- = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & S \end{pmatrix} - E(x^\gamma)^{-T} \cdot \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}.$$

Similarly

$$\widehat{R}_- = R_- \cdot E(x^\gamma, g) \cdot M_+ \cdot \widehat{E}(x^\gamma, \hat{g})^{-1},$$

where

$$M_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & S \end{pmatrix} + E(x^\gamma)^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}.$$

Note that inspecting the equations (5.12), (5.13), (5.26), (5.28)–(5.29), we learn that $E(x^\gamma, g)^T \cdot M_- \cdot \widehat{E}(x^\gamma, \hat{g})^{-T}$ and $E(x^\gamma, g) \cdot M_+ \cdot \widehat{E}(x^\gamma, \hat{g})^{-1}$ are block matrices of the form

$$\begin{pmatrix} \mathbf{1} & \times \\ 0 & \times \end{pmatrix}$$

so we have $\partial_\pm \widehat{X}^\alpha = \partial_\pm X^\alpha$, which is consistent with our interpretation of the coordinates x^α as spectators.

Inserting the formulas given above into the relation between $\widehat{R}_-, \widehat{R}_+$, i.e. into

$$\left(\partial_- X^\alpha, \hat{\rho}^a_-(\hat{g}) \right)\Big|_{\sigma=0,\pi} = \left(\partial_+ X^\beta, \hat{\rho}^b_+(\hat{g}) \right) \cdot \begin{pmatrix} \widehat{R}^\alpha_\beta & \widehat{R}^a_\beta \\ \widehat{R}^\alpha_b & \widehat{R}^a_b \end{pmatrix}\Big|_{\sigma=0,\pi},$$

we get the formula for the transformation of gluing matrices R under Poisson–Lie T-plurality with spectators in the form

$$\widehat{R} = \widehat{E}(x^\gamma, \hat{g})^T \cdot M_-^{-1} \cdot E(x^\gamma, g)^{-T} \cdot R \cdot E(x^\gamma, g) \cdot M_+ \cdot \widehat{E}(x^\gamma, \hat{g})^{-1}. \tag{6.15}$$

The gluing matrix $\widehat{\mathcal{R}}$ expressed in the coordinate frame is then obtained from $\widehat{R}$ via a formula analogous to (6.14).

From the form of the transformation matrices (5.26) and the formula (6.15) it follows that for block diagonal $E(x^\gamma)$, i.e. for $E(x^\gamma)_{a\alpha} = E(x^\gamma)_{\alpha a} = 0$, the spectator part of the gluing matrix $R_{\alpha\beta}$ remains invariant, and block diagonal gluing matrices remain block diagonal. Sadly, as in the atomic case, the transformed gluing matrix $\widehat{R}$ may depend not only on $\hat{g}$ but also on $\bar{g}$, which may enter when we express g in terms of $\hat{g}, \bar{g}$ when changing the decomposition in (4.42). This contradicts any reasonable geometric interpretation of the transformed boundary conditions. That is why we restrict our considerations to gluing matrices of the form

$$R = R(x^\gamma, g) = E(x^\gamma, g)^T \cdot C(x^\gamma) \cdot E(x^\gamma, g)^{-1},$$

where the matrix C does not depend on the group coordinates.

6.4 Example – D2 brane

We will apply the above derived formulas on a very simple block diagonal and spectator-independent gluing matrix

$$\mathcal{R}(t,x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.16}$$

to show that for the background tensor (5.31), which has nontrivial mixed elements $F_{a\alpha}$ and $F_{\alpha a}$, the transformed gluing matrix is no longer block diagonal. The coordinates (t, x^1, x^2, x^3) shall stand for the coordinates adapted to the brane. The gluing matrix (6.16) satisfies the conditions (6.5)–(6.12) for the background (5.31) and defines a D2-brane in this background. The projector $\mathcal{N}$ in this case can be chosen as $\mathcal{N} = \mathrm{diag}(1,0,1,1)$, while Δ is given by

$$\Delta = \begin{pmatrix} 0 & 0 & -b\, e^{-x^1} & 0 \\ 0 & 0 & 0 & 0 \\ b\, e^{-x^1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The transformed gluing matrix obtained from the formula (6.15) for the sigma model with the background (5.33) becomes block non-diagonal and both spectator and group dependent

$$\widehat{\mathcal{R}} = \begin{pmatrix} 1 & 0 & \frac{8be^{5\hat{x}^1}t^4}{4e^{4\hat{x}^1}t^4+1} & \frac{4be^{3\hat{x}^1}t^2}{4e^{4\hat{x}^1}t^4+1} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1-4e^{4\hat{x}^1}t^4}{4e^{4\hat{x}^1}t^4+1} & -\frac{4e^{2\hat{x}^1}t^2}{4e^{4\hat{x}^1}t^4+1} \\ 0 & 0 & \frac{4e^{2\hat{x}^1}t^2}{4e^{4\hat{x}^1}t^4+1} & \frac{1-4e^{4\hat{x}^1}t^4}{4e^{4\hat{x}^1}t^4+1} \end{pmatrix}.$$

It satisfies the conditions (6.5)–(6.12) for $\widehat{\mathcal{N}} = \mathrm{diag}(1,0,1,1)$ and

$$\widehat{\Delta} = \begin{pmatrix} 0 & 0 & 0 & -2be^{\hat{x}^1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2be^{\hat{x}^1} & 0 & -2 & 0 \end{pmatrix},$$

so it defines a D2-brane for the Poisson–Lie transformed model.

6.5 Example – D0 brane

To conclude our discussion, we shall present the transformation of a class of block non-diagonal spectator-dependent gluing matrices satisfying the conditions (6.5)–(6.12) for the symmetrized background (5.31) with $b = 0$.

As an example we consider matrices

$$\mathcal{R} = \begin{pmatrix} -\frac{t^2}{4\gamma(t)} - \frac{\gamma(t)}{t^2} & 0 & e^{x^1}\left(\frac{\gamma(t)}{t^3} - \frac{t}{4\gamma(t)}\right) & 0 \\ 0 & -1 & 0 & 0 \\ e^{-x^1}\left(\frac{t^3}{4\gamma(t)} - \frac{\gamma(t)}{t}\right) & 0 & \frac{t^2}{4\gamma(t)} + \frac{\gamma(t)}{t^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where γ is an arbitrary function. The rank of $\mathcal{R} + \mathbf{1}$ is 1, so these matrices define a class of D0-branes.

The transformed gluing matrix for the sigma model given by (5.33) with $b = 0$ is again obtained from the formula for the Poisson–Lie transformation (6.15), and reads

$$\widehat{\mathcal{R}} = \begin{pmatrix} -\frac{t^2}{4\gamma(t)} - \frac{\gamma(t)}{t^2} & 0 & -\frac{e^{\hat{x}^1}\left(t^4-4\gamma(t)^2\right)}{4t\gamma(t)} & -\frac{e^{-\hat{x}^1}\left(t^4-4\gamma(t)^2\right)}{8t^3\gamma(t)} \\ 0 & -1 & 0 & 0 \\ -\frac{e^{3\hat{x}^1}t\left(t^4-4\gamma(t)^2\right)}{\left(4e^{4\hat{x}^1}t^4+1\right)\gamma(t)} & 0 & -\frac{e^{4\hat{x}^1}t^6+4e^{4\hat{x}^1}\gamma(t)^2t^2+\gamma(t)}{4e^{4\hat{x}^1}\gamma(t)t^4+\gamma(t)} & -\frac{e^{2\hat{x}^1}\left(t^2-2\gamma(t)\right)^2}{2\left(4e^{4\hat{x}^1}t^4+1\right)\gamma(t)} \\ \frac{e^{\hat{x}^1}t^4-4e^{\hat{x}^1}\gamma(t)^2}{8e^{4\hat{x}^1}\gamma(t)t^5+2\gamma(t)t} & 0 & \frac{e^{2\hat{x}^1}\left(t^2-2\gamma(t)\right)^2}{2\left(4e^{4\hat{x}^1}t^4+1\right)\gamma(t)} & \frac{16e^{4\hat{x}^1}\gamma(t)t^6+t^4+4\gamma(t)^2}{16e^{4\hat{x}^1}\gamma(t)t^6+4\gamma(t)t^2} \end{pmatrix}.$$

This time is defines a class of D2-branes, and we see that the dimension of the brane changes after the Poisson–Lie transformation.

A more complicated spectator-dependent gluing matrix would be

$$\mathcal{R} = \begin{pmatrix} -\frac{2t^2\alpha(t)^2-2t\beta(t)\alpha(t)+\beta(t)^2+1}{2t^2\alpha(t)^2-2t\alpha(t)\beta(t)} & -\frac{1}{t^2\alpha(t)} & \frac{e^{x^1}\left(-\beta(t)^2+2t\alpha(t)\beta(t)+1\right)}{2t^2\alpha(t)(t\alpha(t)-\beta(t))} & 0 \\ \frac{t}{t\alpha(t)-\beta(t)} & 1 & \frac{e^{x^1}}{\beta(t)-t\alpha(t)} & 0 \\ \frac{e^{-x^1}\left(\beta(t)^2-2t\alpha(t)\beta(t)+1\right)}{2\alpha(t)(t\alpha(t)-\beta(t))} & \frac{e^{-x^1}}{t\alpha(t)} & -\frac{-2t^2\alpha(t)^2+2t\beta(t)\alpha(t)-\beta(t)^2+1}{2t^2\alpha(t)^2-2t\alpha(t)\beta(t)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Such $\mathcal{R}$ defines D2-branes, and satisfies the conditions (6.5)–(6.12) for the background (5.31) with $b = 0$ if and only if

$$\alpha(t) = \frac{\beta(t)^2 + t\,\beta'(t)}{2t\,\beta(t)}.$$

The gluing matrix $\widehat{R}$ for the background (5.33) is rather complicated to display, but it defines D2-branes and satisfies the conditions (6.5)–(6.12) as well.

6.6 Conclusions

This chapter concludes the first part of the thesis. Step-by-step we have developed the apparatus of Poisson–Lie T-plurality. Based on the concepts previously presented by other authors, who focused mostly on the atomic Poisson–Lie transformation, we have extended the Drinfel'd double to obtain a compact formulation of the Poisson–Lie T-plurality transformation formulas in the presence of spectators. These formulas handle not only the transformation of the backgrounds of the corresponding sigma models, but cover also the transformation of gluing matrices defining the boundary conditions. The formulas have turned out to be rather similar to those valid in the atomic case, where the group of generalized isometries acts freely and transitively on the target manifold. Applying the formulas, we have found that spectator parts of backgrounds and gluing matrices are not invariant in general. Instead, they can transform to forms that depend on the group coordinates. Examples of such cases were presented as well.

Part II

T-duality and plane-parallel waves

Chapter 7

Properties of plane-parallel waves

In the first part of the thesis we focused on the development of the framework of Poisson–Lie T-duality, and emphasized its ability to relate sigma models describing bosonic strings living in geometrically different backgrounds. In the examples presented above we explained how mutually dual sigma models can be constructed, and we have even found the formula that realizes the transformation of boundary conditions of a sigma model describing an open string. Despite our claim that the Poisson–Lie transformation can be employed to find solutions of equations of a particular model once its dual is solved, we have not presented any example so far. Admittedly, in general it is not easy to perform all the steps of the transformation. When a specific background is given, we may not even know what the particular Drinfel'd double should be. Even if we overcome this problem, we still have to solve sets of partial differential equations, find the transformation of coordinates originating from the change of the decomposition of an element of the Drinfel'd double and, first of all, solve Euler–Lagrange equations of one of the dual models. We intentionally deferred the detailed discussion of these technical steps to the second part of our work, where we apply the explained methods to a specific class of backgrounds called plane-parallel waves, which are particularly important from the physical point of view.

Gravitational plane waves were studied in general relativity for decades as a simplifying idealization. They were discussed mostly in the context of linearized gravity, where the spacetime metric splits as the sum $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ of the Minkowski metric and some small perturbation $h_{\mu\nu}$; see [54] for a review and a thorough discussion.

However, plane waves were also repeatedly investigated in the context of string theory. One of the reasons clarifying this renewed interest was the fact that they were recognized to give exact string theory backgrounds [28]. Moreover, the structure of Euler–Lagrange equations of a sigma model in a plane wave background often allows us to find exact solutions, such as in the case of a homogeneous plane wave background investigated in [24]. Generally, finding solutions of sigma models in curved and time-dependent backgrounds is often very complicated, not to say impossible, which is why every solvable case attracts considerable attention. Due to the rich structure of symmetries of plane waves, we will be able to apply the techniques of Poisson–Lie T-duality and find solutions of several nontrivial sigma models.

Besides all these properties, the intensive study of plane waves is also justified by their physical relevance. The general relativistic statement that any spacetime has a plane wave as a limit [55] was reformulated for string theories in [56]. Instead of studying the propagation of strings in complicated cosmological and p-brane backgrounds, one can take the Penrose limit and extract some information from the behavior of strings in the resulting plane wave background. As it turns out, plane waves also offer a suitable playground to study the propagation of strings near a spacetime singularity [26], [27].

Before presenting our results concerning the duality of plane waves, we start by summarizing the most important features of these backgrounds relevant to our discussion. Firstly, in the following two sections we introduce pp-wave and plane wave metrics in their standard forms in Brinkmann and Rosen coordinates, and show how these forms can be transformed to each other. In section 7.3 we focus on the exceptional curvature properties of plane waves. We shall use this knowledge later to identify these backgrounds. The description of groups of symmetries given in section 7.4 provides the necessary background to perform the Poisson–Lie transformation because we shall dualize with respect to subgroups of groups of symmetries.

The content of this chapter concerning change of coordinates, curvature properties and symmetries was inspired by the discussion given in Ref. [57]. The study of equations of motion of a string in a plane wave background is based on [25].

7.1 General plane wave metric

One of the defining features of a plane wave metric G is the existence of a covariantly constant null vector field $\mathcal{V}$ which does not vanish in any point of the spacetime manifold $\mathscr{M}$. Since $\mathcal{V}$ is covariantly constant, the covariant derivatives of its components $\mathcal{V}^\mu$ expressed in some coordinate system x^μ satisfy

$$\nabla_\mu \mathcal{V}^\nu = 0,$$

where the covariant derivative is taken with respect to the Levi–Civita connection. This is equivalent to a pair of equations

$$\nabla_\mu \mathcal{V}_\nu + \nabla_\nu \mathcal{V}_\mu = 0, \qquad \nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu = 0. \tag{7.1}$$

The first condition is the Killing equation saying that $\mathcal{V}$ represents a symmetry of G. $\mathcal{V}$ does not vanish, and using a parameter along its integral curve as a coordinate v we may write it as

$$\mathcal{V} = \partial_v.$$

In such a coordinate system, the components of $\mathcal{V}$ are $\mathcal{V}^\mu = \delta^\mu_v$, or $\mathcal{V}_\mu = G_{\mu v}$ when the index μ is lowered using the metric G. $\mathcal{V}$ has to be a null vector field, hence $\mathcal{V}^\mu G_{\mu\nu} \mathcal{V}^\nu = 0$, resulting in

$$G_{vv} = 0.$$

The Killing equation now restricts the components of the metric to be v-independent

$$\partial_v G_{\mu\nu} = 0.$$

Since the Levi–Civita connection is symmetric, the second condition in (7.1) is equivalent to

$$\partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu = 0,$$

and locally a function $u = u(x^\mu)$ exists, such that $\mathcal{V}_\mu = G_{\mu v} = \partial_\mu u$. In the local coordinate system (u, v, x^a) with d coordinates $x^a,\ a = 3, \ldots, D$ and $D = 2 + d$, the most general form of the plane wave metric gives the line element

$$ds^2 = -K(u, x^c)du^2 + 2dudv + 2A_a(u, x^c)dudx^a + G_{ab}(u, x^c)dx^a dx^b.$$

This can be simplified even more, since the remaining coordinate freedom can be used to absorb K and A_a in G_{ab}. However, being interested in a different form of the metric, we will not follow this way now.

Instead, we focus on a special class of plane wave metrics with $G_{ab} = \delta_{ab}$. For such metrics the wave fronts characterized by constant u are planar. They also possess the null vector $\mathcal{V}$ that is parallel transported. Such metrics are thus referred to as plane-fronted waves with parallel rays or *pp-waves* for short. The line element then reads

$$ds^2 = -K(u, x^c)du^2 + 2dudv + 2A_a(u, x^c)dudx^a + d\vec{x}^2, \tag{7.2}$$

with $d\vec{x}^2$ denoting the Euclidean metric in the so-called *transversal space* given by coordinates x^a. In the following chapters we will be interested in transformations of coordinates leading to standard forms of pp-wave metrics. It is therefore helpful to note that the transformation

$$v \to v - \Lambda(u, x^c) \tag{7.3}$$

leads to a change of the coefficients

$$-K \to -K + 2\partial_u \Lambda, \qquad A_a \to A_a + \partial_a \Lambda.$$

We shall restrict our considerations further to pp-wave metrics particularly relevant to string theory. These will have $A_a = 0$ and $K(u, x^c)$ at most quadratic in the transversal coordinates. Such pp-waves will be called *plane waves.* Our motivation to specialize in this kind of pp-waves follows from the fact that equations of motion of a string propagating in such a background reduce to particularly simple form allowing us to find solutions in several cases. Assuming that the wave profile is given by

$$K(u, x^c) = K_{ab}(u)x^a x^b + K_a(u)x^a + K(u), \tag{7.4}$$

we can always find a coordinate transformation which eliminates the first and zeroth order terms $K_a(u)$ and $K(u)$ while preserving the other components of $G_{\mu\nu}$. Clearly, the zeroth order term can be eliminated by x^c-independent shifts (7.3). The elimination of the term linear in x^a is obtained via u-dependent shifts of x^a followed by u-dependent transformations linear in x^a:

$$x^a \to x^a - f_a(u), \qquad v \to v + x^a f'_a(u) + g(u).$$

The functions $f_a(u), g(u)$ have to be determined from the particular form of $K(u, x^c)$ as a solution of a set of differential equations. The metric with

$$ds^2 = 2dudv - K_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \tag{7.5}$$

will be referred to as a *plane wave metric* in *Brinkmann coordinates*. Obviously, for vanishing $K_{ab}(u)$ the plane wave metric is in fact the D-dimensional Minkowski metric in light-cone coordinates u, v. A plane wave metric is fully characterized by the entries $K_{ab}(u)$. Without loss of generality we consider it to be a symmetric matrix with components given by $\frac{d(d+1)}{2}$ functions of u.

7.2 Brinkmann and Rosen coordinates

The plane wave metrics are usually expressed in two standard forms. For example, Brinkmann coordinates are well suited to study the curvature properties, which we discuss in the following section. To study other aspects of this kind of background, it may be appropriate to pass to coordinates in which the line element reads

$$ds^2 = 2dUdV + G_{ab}(U)dy^a dy^b. \tag{7.6}$$

This form of the metric is referred to as the plane wave metric in *Rosen coordinates.* Adopting such a coordinate system, it is obvious that, besides the isometry generated by the vector $\mathcal{V}$, there are also symmetries in the transversal coordinates that were otherwise hidden in Brinkmann coordinates. The symmetric matrix $G_{ab}(U)$ in (7.6) is positive definite and nondegenerate in the range where Rosen coordinates are valid.

Now we shall explain how the transformation between (7.5) and (7.6) can be found. According to [57], starting with the metric written in Rosen coordinates, it is necessary to transform the transverse metric $G_{ab}(U)$ to the Euclidean metric δ_{ab} in Brinkmann coordinates. This can be done only via

$$x^a = \hat{Q}^a{}_b(U)y^b,$$

where the matrix $\hat{Q}^a{}_b(U)$ is the vielbein for the transverse metric, i.e.

$$G_{ab}(U) = \delta_{cd}\hat{Q}^c{}_a(U)\hat{Q}^d{}_b(U).$$

Denoting the transposed inverse vielbein $Q = \hat{Q}^{-T}$ as $Q^c{}_a(U)$, we find that the change of coordinates $U = u,\ y^a = Q^a{}_b(u)x^b$ produces in the line element also terms proportional to $dudx^a$. These cross-terms can be eliminated by a shift in V if the following symmetry condition holds

$$G_{ab}(U)Q'^a_c(U)Q^b{}_d(U) = G_{ab}(U)Q^a{}_c(U)Q'^b_d(U),$$

where the prime means the derivative with respect to U. Such $Q(U)$ satisfying this condition together with

$$G_{ab}(U)Q^a{}_c(U)Q^b{}_d(U) = \delta_{cd},$$

can always be found, see [58], and is unique up to U-independent transformations orthogonal with respect to G_{ab}. The resulting change of coordinates

$$\begin{aligned} U &= u, \\ V &= v - \frac{1}{2}G_{ab}(u)Q'^a{}_c(u)Q^b{}_d(u)x^c x^d, \\ y^a &= Q^a{}_b(u)x^b \end{aligned} \tag{7.7}$$

brings the metric (7.6) to Brinkmann form (7.5) with

$$K_{ab} = \left(G_{cd}Q'^d{}_a\right)' Q^c{}_b.$$

The preceding conditions on Q can be written in a compact form using matrix notation:

$$Q^T \cdot G \cdot Q = \mathbf{1}, \qquad Q'^T \cdot G \cdot Q = Q^T \cdot G \cdot Q'. \tag{7.8}$$

The matrix K_{ab} is then given by

$$K = \left(Q'^T \cdot G\right)' \cdot Q. \tag{7.9}$$

As we shall see, it might be harder to go in the opposite direction and find the transformation from Brinkmann to Rosen coordinates because it involves solution of a set of second order ODEs.

In the following chapters we will be mostly interested in transformations between metrics (7.5) and (7.6) with diagonal K_{ab} and G_{ab}, that is with

$$K_{ab}(u) = K_a(u)\delta_{ab}, \qquad G_{ab}(U) = e_a^2(U)\delta_{ab}. \tag{7.10}$$

Choosing the vielbein $\hat{Q}$ diagonal, the procedure simplifies dramatically because the relation (7.9) can be solved by an ansatz

$$Q^a{}_b(U) = e_a^{-1}(U)\delta_b^a, \tag{7.11}$$

leading to

$$K_{ab} = -\frac{e_a''}{e_a}\delta_{ab}.$$

If we are given $K_{ab}(u) = K_a(u)\delta_{ab}$, we may find $e_a(u)$ as the solution of an equation

$$e_a''(u) = -K_a(u)e_a(u). \tag{7.12}$$

The metric in Rosen coordinates is then given by the formula (7.10), and the transformation from Brinkmann to Rosen coordinates is given by the inverse of (7.7) with Q given as in (7.11). The solution of the equation (7.12) depends on integration constants, so the vielbein, as well as the coordinate transformation, is not given uniquely. Therefore, there will be more ambiguity in the description of plane waves in Rosen coordinates then in Brinkmann coordinates.

For non-diagonal K_{ab} and G_{ab} we again need to find the relation between K_{ab} and the vielbein $\hat{Q}$. The matrix G_{ab} is given by

$$G = \hat{Q}\hat{Q}^T.$$

Using the symmetry conditions (7.8) repeatedly in (7.9), see [57] for details, it is possible to eliminate G. The outcome that generalizes the equation (7.12) represents a set of ODEs given by an equation

$$\hat{Q}''(u) = -\hat{Q}(u) \cdot K(u) \tag{7.13}$$

for the vielbein $\hat{Q}$. In addition to this, the symmetry condition (7.8) has to be fulfilled, i.e. $\hat{Q}$ has to satisfy

$$\hat{Q}' \cdot \hat{Q}^T = \hat{Q} \cdot \hat{Q}'^T.$$

The plane wave metric in Rosen coordinates can be deduced directly and the coordinate transformation is again given by the inverse of (7.7), where $Q = \hat{Q}^{-T}$. We note that an interesting discussion of the connection between solutions of (7.13) and Killing vectors of the metric can be found in the appendices of Ref. [57] together with an explanation of the role of the symmetry condition.

7.3 Curvature properties of plane wave metrics

The simple form of a pp-wave metric in Brinkmann coordinates leads to particularly simple curvature properties of the background. Even if we do not make the assumption that $K(u, x^c)$ in (7.2) is at most quadratic in x^c, for vanishing $A_a(u, x^c)$ the Ricci tensor has only one nonzero component given by

$$R_{uu} = \frac{1}{2}\left(\partial_3^2 K + \ldots + \partial_D^2 K\right),$$

and the scalar curvature R vanishes. For a plane wave (7.5) this further simplifies as the only nonvanishing components of the Riemann curvature tensor are determined by

$$R_{uaub} = -K_{ab}(u).$$

The other components are obtained through the (anti)symmetry of $R_{\mu\nu\kappa\lambda}$ upon permutation of the indices. This proves that the background is flat if and only if all the entries in the matrix $K_{ab}(u)$ vanish. Calculation of the Ricci tensor for $K(u, x^c)$ given by (7.4) shows that the only nonzero component reads

$$R_{uu} = K_{33} + \ldots + K_{DD} = Tr(K_{ab})(u).$$

Note that having $R = 0$, the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = 0$$

reduce to the Ricci-flatness condition that for pp-waves reads $R_{uu} = 0$. Plane waves thus have to have traceless $K_{ab}(u)$ in order to give a vacuum solution of Einstein's equations. On the other hand, the Weyl tensor containing the traceless part of the Riemann tensor has components

$$C_{uaub} = -K_{ab}(u) + \frac{1}{d}\delta_{ab}Tr(K_{ab})(u).$$

Therefore, the Weyl tensor vanishes, and the plane wave metric is conformally flat, if and only if

$$K_{ab}(u) = \frac{1}{d}\delta_{ab}Tr(K_{ab})(u),$$

i.e. when the components of $K_{ab}(u)$ are

$$K_{ab}(u) = \widetilde{K}(u)\delta_{ab}.$$

Remarkably, the simple form of the Riemann tensor implies not only the vanishing of the scalar curvature, it also says that all the *curvature invariants* are zero. This can be proven by an argument presented in [59]. Any curvature invariant is a scalar constructed from the metric, the Riemann tensor and its covariant derivatives. It is a function of products of covariant derivatives $\nabla_{\mu_1} \ldots \nabla_{\mu_k} R^{\lambda}{}_{\mu\nu\kappa}$ multiplied by an appropriate number of factors of the inverse metric in such a way that we obtain a scalar. For instance, the well known Kretschmann scalar is defined as the contraction

$$K_1 = R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa}.$$

Assume that there is a coordinate transformation which is not an isometry and which results in a constant rescaling of the metric, i.e. it is a *homothety*. The Christoffel symbols are invariant under such rescaling and so is the Riemann tensor $R^{\lambda}{}_{\mu\nu\kappa}$ and its covariant derivatives. However, to construct a scalar, these have to be multiplied by the inverse metric, so the curvature invariants are not preserved under such rescalings. Now, in a point x_0 which is a fixed point of the homothety any curvature invariant has to remain invariant under this transformation, while at the same time it transforms nontrivially as the metric scales. Therefore, curvature invariants have to vanish in the points x_0, where a nontrivial homothety exists with a fixed point x_0.

To finish the proof, it has to be shown that such nontrivial homotheties can be constructed in any point x_0 of the manifold. Writing the metric in Rosen coordinates, we see that there is a translation symmetry in the coordinates V and y^a. Any point can be thus given by coordinates $(U, 0, 0)$. This is a fixed point of a transformation

$$(U, V, y^a) \to (U, \lambda^2 V, \lambda y^a),$$

under which the line element scales as

$$ds^2 \to \lambda^2 ds^2.$$

Due to the translation invariance of the metric, the vanishing of curvature invariants in the point $(U, 0, 0)$ implies that they vanish everywhere.

The notion of spacetime singularities in general relativity is tricky. Since the components of the metric tensor depend on the choice of coordinates, one can not simply take the spacetime manifold to be the collection of points where the metric is sufficiently differentiable. To distinguish a true curvature singularity from apparent singularities resulting from poor choices of coordinates, one usually calculates the curvature invariants. A singularity in curvature invariants in a point p indicates a singularity in p. We have seen that all the invariants of a plane wave vanish. Nevertheless, this does not mean that there are no singularities. A thorough discussion of singularities arising in general relativity is carried out for example in [60]. The authors define the spacetime to have a singularity if it is timelike or null *geodesically incomplete*, that is when there exist a timelike or null geodesic $p(t)$ which can not be extended for all the values of its affine parameter t. For timelike geodesics this requirement is natural since the incompleteness would mean that there are freely falling observers whose histories do not exist before/after a finite interval of proper time. The incompleteness of a null geodesic seemingly does not represent such a threat because the affine parameter does not have the same meaning as the proper time. But null geodesics represent histories of massless particles, so we shall avoid geodesical incompleteness as well. Also, there are examples of metrics which one wants to consider to be singular, such as the Reissner–Nordström solution, where timelike geodesics are complete, but null are not.

Ref. [60] mentions several criteria according to which the singularities can be detected, and shows that an observer traveling on an incomplete geodesic would feel infinite tidal forces. Instead of inspecting the geodesical completeness, we will examine directly the *tidal forces* experienced by nearby freely falling particles. The effects of curvature can be explicitly calculated for a plane wave, thus giving $K_{ab}(u)$ a physical meaning. As in [38], we consider two freely falling particles traveling on nearby trajectories $x^\mu(\tau)$ and $x'^\mu(\tau) = x^\mu(\tau) + \delta x^\mu(\tau)$, where $\delta x^\mu(\tau)$ is the separation vector between the nearby geodesics and τ is the affine parameter along the curves. Both trajectories satisfy geodesic equations

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\dot{x}^\nu \dot{x}^\lambda = 0, \qquad \frac{d^2 x'^\mu(\tau)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x'(\tau))\dot{x}'^\nu \dot{x}'^\lambda = 0.$$

Expanding the second one, taking their difference to the first order in δx and expressing it in terms of the covariant derivative $\frac{D}{D\tau}$ along the curve $x^\mu(\tau)$, one arrives at the *geodesic deviation equation*

$$\frac{D^2}{D\tau^2}\delta x^\lambda = R^\lambda{}_{\mu\nu\kappa}\dot{x}^\mu \dot{x}^\nu \delta x^\kappa,$$

which shows that the effects of gravitation can be detected if the relative difference between two nearby freely falling particles is measured. To get to the point, we choose a particular class of geodesics given in Brinkmann coordinates as

$$u = p_v \tau, \qquad v = const., \qquad x^a = 0.$$

The geodesic equation is satisfied because for $x^a = 0$ all the Christoffel symbols vanish. The vanishing of G_{11} implies that these geodesics are null. From this class we choose two geodesics with a separation vector δx^μ connecting points with the same value of the parameter τ. Due to the structure of the Riemann tensor, we have $\delta u = 0,\ \delta v = v_2 - v_1$ constantly, while the geodesic equation for the transversal components becomes

$$\frac{d^2}{d\tau^2}\delta x^a = -K_{ab}(u)\delta x^b.$$

This second-order differential equation for the transversal components of the separation vector shows that $K_{ab}(u)$ carries information about the tidal forces. First, we see that the forces may be attractive or repulsive depending on the sign of the eigenvalues of $K_{ab}(u)$. Second, the forces become infinite in the points where $K_{ab}(u)$ diverges. We call these points *singularities*. If such a singularity occurs at $u = u_0$, i.e. at some value of the parameter τ_0, a geodesic starting at a finite value τ_1 will reach the singularity in a finite value $\tau_0 - \tau_1$. There it has to end rendering the spacetime geodesically incomplete.

Singularities, as we described them in this section, were understood from the point of view of general relativity, where the effects of gravitation are studied on test point particles. However, the behavior of strings in the presence of singularities might be of a different nature. As was pointed out in Ref. [26], the criteria for singularity recognition should be different in string theory than in general relativity. Examples show that string evolution is well defined on orbifolds (locally flat manifolds with conical singularities), while the string interaction with an axion field leads to excitations causing the mass squared of the string to diverge even on a geodesically complete manifold. Nevertheless, our definition of singularity seems appropriate for strings as the tidal forces cause excitations in the propagating string, and these excitations become infinite when the forces diverge, see Ref. [26]. It turns out that due to the structure of their equations of motion, the plane waves are particularly useful in the study of string behavior near spacetime singularities, see e.g. [26], [27].

7.4 Symmetries of a plane wave background

Considerations made in Part I of this thesis demonstrate that in order to apply (non-)Abelian duality or Poisson–Lie T-duality to any background, the question of symmetries is of utmost importance. In this section we focus on finding the symmetry groups of plane wave backgrounds, thus showing that these backgrounds offer an interesting set of examples that can be used to study the effects of duality transformation.

As we already mentioned, some of the symmetries of plane waves are revealed when Rosen coordinates are adopted. We see that the metric (7.6) is manifestly invariant with respect to d translations in the transversal coordinates x^a. There are, however, another d symmetries which are not manifest, and there is also the shift in v generated by the covariantly constant null vector $\mathcal{V}$. Generically, a plane wave has $(2d + 1)$-dimensional group of symmetries, which might be further extended by symmetries following from the particular form of $G_{ab}(U)$. However, it might be sometimes easier to detect these extra symmetries in Brinkmann form, so we shall focus on the problem of determining the symmetries of (7.5) instead of (7.6).

To find the symmetries, it is necessary to solve the Killing equation or, equivalently, the condition for vanishing of the Lie derivative of $G_{\mu\nu}$ with respect to a general vector field given by

$$\mathcal{K}(u,v,x^c) = \mathcal{K}^u(u,v,x^c)\partial_u + \mathcal{K}^v(u,v,x^c)\partial_v + \mathcal{K}^a(u,v,x^c)\partial_a.$$

We follow the latter formulation of the problem, and solve the equations following from

$$\left(\mathcal{L}_{\mathcal{K}}G\right)_{\mu\nu} = 0, \qquad \mu,\nu \in 1,\ldots,D. \tag{7.14}$$

Equation (7.14) leads to a series of PDEs for the components of $\mathcal{K}$ that might be quite hard to solve for a general metric. Nevertheless, for a plane wave metric (7.5) the vanishing of the Lie derivative

$$(\mathcal{L}_{\mathcal{K}}G)_{\mu\nu} = \mathcal{K}^\lambda\partial_\lambda G_{\mu\nu} + \partial_\mu\mathcal{K}^\lambda G_{\lambda\nu} + \partial_\nu\mathcal{K}^\lambda G_{\mu\lambda} = 0$$

implies that the following series of equations has to hold:

$$\begin{aligned}
0 =& -K'_{cd}(u)x^cx^d\mathcal{K}^u - \left(K_{ad}(u) + K_{da}(u)\right)x^d\mathcal{K}^a && (7.15)\\
&+ 2\partial_u\mathcal{K}^v - 2K_{cd}(u)x^cx^d\partial_u\mathcal{K}^u, &&\\
0 =& \ \partial_u\mathcal{K}^u + \partial_v\mathcal{K}^v, && (7.16)\\
0 =& \ \partial_u\mathcal{K}^a + \partial_a\mathcal{K}^v - K_{cd}(u)x^cx^d\partial_a\mathcal{K}^u, && (7.17)\\
0 =& \ \partial_v\mathcal{K}^u, && (7.18)\\
0 =& \ \partial_v\mathcal{K}^a + \partial_a\mathcal{K}^u, && (7.19)\\
0 =& \ \partial_a\mathcal{K}^b + \partial_b\mathcal{K}^a. && (7.20)
\end{aligned}$$

Analyzing equations (7.18) and (7.20) for $a = b$, we learn that $\mathcal{K}^u$ is v-independent and that $\mathcal{K}^a$ is x^a-independent. Taking the derivative of (7.16) with respect to v and using (7.18), we conclude that $\mathcal{K}^v$ is at most linear in v. A similar argument can be used to show that $\mathcal{K}^a$ is at most linear in v, a fact that follows from the ∂_v derivative of (7.19) and (7.18). Also, taking the ∂_v derivative of (7.20) and using (7.19), we learn that $\partial_a\partial_b\mathcal{K}^u = 0$, so $\mathcal{K}^u$ is at most linear in x^a. Taking the ∂_a derivative of (7.19) and using the last fact, we find that $\partial_a\partial_v\mathcal{K}^b = 0$ and that $\mathcal{K}^b$ can be split into a sum of two terms, where the first is linear in v and the second contains the dependency on transversal coordinates. Last but not least, for $a \neq b$, the equation (7.20) implies that $\mathcal{K}^a$ contains only such terms, where the transversal coordinate x^b appears at most linearly (mixed terms proportional to products of $x^{b_i},\ b_i \neq a$ are allowed, where each x^{b_i} can appear at most in the first power). In fact, taking the ∂_c derivative of (7.20) and using (7.20) for $\partial_c\mathcal{K}^b$ and $\partial_c\mathcal{K}^a$, we find $\partial_a\partial_b\mathcal{K}^c = 0$, so the terms in $\mathcal{K}^a$ of higher order in transversal coordinates vanish. Parametrizing $\mathcal{K}^u$ as

$$\mathcal{K}^u = k_a(u)x^a + k_u(u),$$

the equations (7.16) and (7.19) dictate that

$$\begin{aligned}\mathcal{K}^v &= -(k_a'(u)x^a + k_u'(u))v + k_v(u, x^c),\\ \mathcal{K}^a &= \alpha_a(u) - k_a(u)v + \sum_{b\neq a}\gamma_{ab}(u)x^b,\end{aligned}$$

while (7.20) demands that γ_{ab} is antisymmetric. Using this form of $\mathcal{K}$, we learn that the only term in (7.17) proportional to v is $k_a'(u)$, meaning $k_a = C_a =$ const. $\mathcal{K}$ thus has the form

$$\begin{aligned}\mathcal{K}^u &= C_a x^a + k_u(u),\\ \mathcal{K}^v &= -k_u'(u)v + k_v(u, x^c),\\ \mathcal{K}^a &= \alpha_a(u) - C_a v + \sum_{b\neq a}\gamma_{ab}(u)x^b.\end{aligned} \tag{7.21}$$

It is possible to proceed in a similar manner even further, see Ref. [57], discovering the ubiquitous structure of the algebra of Killing vectors of a general plane wave. However, for the particular examples discussed in the following chapters this general discussion in not necessary. Therefore, we shall restrict our considerations to plane wave metrics in four dimensions uncovering in detail also the extra symmetries following from the particular form of the matrix $K_{ab}(u)$. In the end we shall see that the general structure of symmetries will be revealed anyway.

Consider a four-dimensional plane wave metric given in Brinkmann coordinates (u, v, x^3, x^4) by a matrix

$$G = \begin{pmatrix} -\left(K_{33}(u)(x^3)^2 + K_{44}(u)(x^4)^2\right) & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

i.e. by diagonal $K_{ab}(u)$. This will cover all the cases which we will treat in the following chapters, but still allows us to extend our results to the general case. Note that for u-independent matrix K_{ab}, the metric can always be brought into this form by u-independent orthogonal transformations of the transversal coordinates. Such a transformation can not be utilized for u-dependent $K_{ab}(u)$ as it, besides diagonalization of $K_{ab}(u)$, also introduces terms $A_a(u, x^c)$ in the metric (7.2). As the interchange of the functions $K_{33}(u) \leftrightarrow K_{44}(u)$ leads only to relabeling of the transversal coordinates, we will not discuss all the possibilities in detail, the rest is obtained with $x^3 \leftrightarrow x^4$.

When we use the solution (7.21) in the equation (7.15), the term proportional to v is

$$C_3K_{33}(u)x^3 + C_4K_{44}(u)x^4 - k_u''(u),$$

and each of these terms has to vanish separately. $k_u(u)$ is therefore at most linear in u. It is also possible to integrate one of the equations (7.17) to obtain a more specific expression for k_v, but we can not move much further without specifying the functions

$K_{33}(u), K_{44}(u)$. All the equations following from (7.14) are now satisfied except for (7.15) and one of the equations (7.17). These equations are at most quadratic polynomials in the transversal coordinates. When trying to find the most general solution, it is enough to compare terms with corresponding powers of x^3 and x^4. With $K_{33}(u), K_{44}(u)$ given, we distinguish several cases.

1. First, the trivial case $K_{33}(u) = K_{44}(u) = 0$ gives the flat metric $G = \eta$ in ligth-cone coordinates with ten-dimensional Poincaré algebra of symmetries. The general Killing vector is in the form

$$\mathcal{K} = \begin{pmatrix} C_0 + C_1 u + C_3 x^3 + C_4 x^4 \\ C_2 - C_1 v + C_5 x^3 + C_6 x^4 \\ C_7 - C_5 u - C_3 v + C_9 x^4 \\ C_8 - C_6 u - C_4 v - C_9 x^3 \end{pmatrix}.$$

However, this degenerate case is not of much importance for us.

2. On the other hand, for $K_{33}(u) \neq 0, K_{44}(u) \neq 0$, the general Killing vector $\mathcal{K}$ is restricted to

$$\mathcal{K} = \begin{pmatrix} C_0 + C_1 u \\ C_2 - C_1 v - x^3 f_3'(u) - x^4 f_4'(u) \\ f_3(u) + C_9 x^4 \\ f_4(u) - C_9 x^3 \end{pmatrix}, \tag{7.22}$$

where the functions f_3, f_4 satisfy second order ODEs

$$\begin{aligned} 0 &= K_{33}(u) f_3(u) + f_3''(u), \\ 0 &= K_{44}(u) f_4(u) + f_4''(u), \end{aligned} \tag{7.23}$$

and the rotational symmetry ($C_9 \neq 0$) is present only when $K_{33}(u) = K_{44}(u)$. In addition to this, the constants C_0, C_1 have to fulfill

$$\begin{aligned} 0 &= 2C_1 K_{33}(u) + (C_0 + C_1 u)\, K_{33}'(u), \\ 0 &= 2C_1 K_{44}(u) + (C_0 + C_1 u)\, K_{44}'(u). \end{aligned} \tag{7.24}$$

From a different perspective, (7.24) can be also treated as conditions restricting $K_{33}(u), K_{44}(u)$. The Killing vector has a nontrivial component in the u-direction if and only if

$$K_{33}(u) = \frac{c_3}{(C_0 + C_1 u)^2}, \qquad K_{44}(u) = \frac{c_4}{(C_0 + C_1 u)^2} \tag{7.25}$$

for c_3, c_4 constant, otherwise both C_0 and C_1 have to vanish. In the event of $C_1 = 0$, this leads to K_{33}, K_{44} constant, a case which we shall discuss in detail later. If $C_1 \neq 0$, we can perform a transformation of coordinates

$$C_0 + C_1 u \to u, \quad v \to \frac{v}{C_1}$$

learning that it is enough to consider K_{33} and K_{44} in the form

$$K_{33}(u) = \frac{c_3}{u^2}, \qquad K_{44}(u) = \frac{c_4}{u^2}. \tag{7.26}$$

If the condition (7.25) is not met, the only solution to (7.24) is trivial ($C_0 = C_1 = 0$). Plane waves with nontrivial symmetries in the u-direction were called *homogeneous* plane waves in Ref. [57]. We see that such symmetries are allowed only if K_{aa} is constant or proportional to the inverse square of u.

Since (7.23) are ODEs of second order, there are two independent solutions for each $f_3(u)$ and $f_4(u)$. The algebra of symmetries is thus 5-dimensional in general, given by four solutions of (7.23) and by the shift in the coordinate v. However, other symmetries emerge from the particular form of $K_{ab}(u)$.

- The Killing algebra is particularly rich in the special case when K_{33} and K_{44} are equal, both being a nonzero constant. This constant can be set to $K_{33} = K_{44} = \pm 1$ by a coordinate transformation (rescaling of u and v), and $\mathcal{K}$ looks like in (7.22) with $C_1 = 0$. Conditions (7.23) can be solved in terms of a linear combination of functions $\cos u$, $\sin u$ for $K_{33} = 1$, or $\cosh u$, $\sinh u$ for $K_{33} = -1$ respectively, and the algebra of Killing vectors is 7-dimensional.
- If K_{33}, K_{44} are different nonzero constants, the rotational symmetry is not present and the algebra of symmetries is 6-dimensional. The conditions (7.23) are solved similarly as in the previous case in terms of functions $\cos(\sqrt{K_{33}}u)$ etc.
- If K_{33}, K_{44} fulfill (7.26), the general Killing vector is (7.22) with $C_0 = 0$, and the conditions (7.23) are solved as

$$\begin{aligned} f_3(u) &= C_3 u^{\frac{1}{2}\left(1-\sqrt{1-4c_3}\right)} + C_4 u^{\frac{1}{2}\left(1+\sqrt{1-4c_3}\right)}, \\ f_4(u) &= C_5 u^{\frac{1}{2}\left(1-\sqrt{1-4c_4}\right)} + C_6 u^{\frac{1}{2}\left(1+\sqrt{1-4c_4}\right)}, \end{aligned} \tag{7.27}$$

  where C_3, C_4, C_5, C_6 are arbitrary constants. The algebra is 6-dimensional. For $c_3 = c_4$ there is again the additional rotational symmetry and the algebra is 7-dimensional.
- Combination of the two previous cases $K_{33} = \frac{c_3}{u^2}, K_{44} = c_4 \neq 0$, yields $C_0 = C_1 = 0$ destroying the symmetry in u. The algebra of symmetries is only 5-dimensional.

3. Next, we consider $K_{33} \neq 0$ and vanishing K_{44}. The second condition in (7.24) is now trivial, while the first holds. The second equation in (7.23) gives $f_4(u)$ at most linear in u, so the most general form of a Killing vector is

$$\mathcal{K} = \begin{pmatrix} C_0 + C_1 u \\ C_2 - C_1 v + C_6 x^4 - x^3 f_3'(u) \\ f_3(u) \\ C_8 - C_6 u \end{pmatrix}.$$

The function $f_3(u)$ satisfies (7.23) and (7.24), and the algebra of symmetries is 5-dimensional in general. The same restrictions that we already discussed apply to C_0 and C_1.

- For $K_{44} = 0$ and K_{33} a nonzero constant, this constant can be set to $K_{33} = \pm 1$ by a coordinate transformation. The algebra of symmetries is 6-dimensional, and the allowed Killing vector is

$$\mathcal{K} = \begin{pmatrix} C_0 \\ C_2 + C_6 x^4 - x^3 f_3'(u) \\ f_3(u) \\ C_8 - C_6 u \end{pmatrix},$$

  where f_3 is found as a solution of (7.23) as a linear combination of $\cos u$, $\sin u$, or $\cosh u$, $\sinh u$ respectively.

- For $K_{44} = 0$ and $K_{33} = \frac{c_3}{u^2}$, the algebra of Killing vectors is 6-dimensional and the allowed Killing vector is

$$\mathcal{K} = \begin{pmatrix} C_1 u \\ C_2 - C_1 v + C_6 x^4 - x^3 f_3'(u) \\ f_3(u) \\ C_8 - C_6 u \end{pmatrix},$$

  where $f_3(u)$ is found as a solution of (7.23) as

$$f_3(u) = C_3 u^{\frac{1}{2}\left(1-\sqrt{1-4c_3}\right)} + C_4 u^{\frac{1}{2}\left(1+\sqrt{1-4c_3}\right)}.$$

In chapters 8 and 9 we obtain most of these cases as a result of duality transformation. Having found the symmetries here, we do not repeat the discussion later for each background separately as particular results can be easily restored from the general discussion.

The results mentioned so far can be generalized to spacetime of arbitrary dimension. Consider now a plane wave in $D = 2 + d$ dimensions, where d denotes the number of transversal coordinates. The metric with diagonal $K_{ab}(u)$ in Brinkmann coordinates $\left(u, v, x^3, \ldots, x^D\right)$ then has the block form

$$G = \begin{pmatrix} -\sum_{a=1}^{d} K_{aa}(u)(x^a)^2 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1}$ is the $(d \times d)$ unit matrix and $\mathbf{0}$ denotes vectors of 0's. Then for $K_{aa} \neq 0$ the vector $\mathcal{K}$ has similar form to (7.22) with additional terms containing other transversal coordinates. In particular, there are d vectors

$$\mathcal{K}_{f_a} = -x^a f_a'(u)\partial_v + f_a(u)\partial_a$$

with $f_a(u)$ satisfying second order ODEs

$$0 = K_{aa}(u) f_a(u) + f_a''(u), \tag{7.28}$$

which lead to $2d$ independent Killing vectors $\mathcal{K}_{f_{a_1}}$, $\mathcal{K}_{f_{a_2}}$ corresponding to solutions f_{a_1} and f_{a_2}. Note that we have already met this equation when trying to pass from Brinkmann to Rosen coordinates in (7.12). Also the symmetry given by $\mathcal{V} = \partial_v$ is always present. A direct calculation of the commutator reveals $2d$-dimensional *Heisenberg algebra* with the central element $\mathcal{V}$ because

$$[\mathcal{K}_{f_{i_j}}, \mathcal{V}] = 0, \qquad [\mathcal{K}_{f_{i_l}}, \mathcal{K}_{f_{j_m}}] = -\delta_{ij} W(f_{i_l}, f_{j_m})\mathcal{V},$$

where $W(.,.)$ is the Wronskian

$$W(f_{i_l}, f_{j_m}) = f_{i_l} f'_{j_m} - f'_{i_l} f_{j_m}.$$

To check that such vectors form an algebra, one has to show that the Wronskian is constant. This indeed holds since the derivative of $W(f_{i_l}, f_{j_m})$ with respect to u vanishes due to (7.28). The presence of the Heisenberg algebra of Killing vectors is a general feature of any plane wave. When the general plane wave metric is examined (allowing also non-diagonal K_{ab}), all the Killing vectors K_{f_a} are present. What changes is the structure of the set of ODEs (7.28) which does not separate:

$$0 = K_{ab}(u) f_b(u) + f_a''(u).$$

As we saw, other symmetries may arise from the particular properties of functions $K_{ab}(u)$. When investigating metrics with diagonal $K_{ab}(u)$, we saw that whenever $K_{aa}(u) = K_{bb}(u)$ holds for $a \neq b$, there is an additional Killing vector representing rotation $X_{rot} = x_b \partial_a - x_a \partial_b$. In order to have a nontrivial symmetry in the u-direction, the matrix $K_{ab}(u)$ has to be constant or satisfy $K_{ab}(u) = \frac{c_{ab}}{u^2}$. In both these cases $K_{ab}(u)$ can be diagonalized by a u-independent transformation of the transversal coordinates. In the first case we have the shift in u represented by the Killing vector $X_{sh} = \partial_u$. In the second case the additional Killing vector $X_{sc} = u\partial_u - v\partial_v$ represents scaling. Then $\mathcal{V}$ is no longer the central element of the algebra of symmetries because $[X_{sc}, \mathcal{V}] = \mathcal{V}$. Such plane waves appear in the study of spacetime singularities and we shall focus on them in the next chapter.

We have learned that the group of symmetries of plane waves has a rich structure. Knowing that symmetry is essential when one wants to perform duality transformation on a particular background, we conclude that this makes these backgrounds great candidates to test the implications of Poisson–Lie T-duality. Starting with a plane wave background, we can perform dualization with respect to some subgroup of the group of symmetries and find the dual sigma model. Moreover, we shall see that dual models also often live in a plane wave background allowing us to dualize even further if we want.

7.5 Sigma models in plane wave backgrounds

We have mentioned another compelling property of plane waves which is the particularly simple structure of Euler–Lagrange equations following from the action. According to our previous discussion, classical string solutions have to fulfill conditions (1.19), (1.20), which ensure the vanishing of the energy-momentum tensor, as well as the equations of motion (1.18). These are highly non-linear coupled equations that might be quite hard to solve. However, for strings in a plane wave background expressed in Brinkmann coordinates (u, v, x^a) there is a standard procedure that can sometimes give a solution.

First, we notice that the equations of motion can be easily deduced directly from the action. Namely, as $B = 0$, the variation of (1.17) with respect to $V(\tau, \sigma)$ always gives the wave equation for U:

$$(-\partial_\tau^2 + \partial_\sigma^2)U(\tau, \sigma) = 0.$$

The standard way how we can proceed is to choose the *light-cone gauge*, i.e. choose

$$U(\tau, \sigma) = \kappa\tau, \qquad \kappa := 2\alpha' p^u.$$

In fact, it was shown in [26] that the only curved background for which the light-cone gauge can be chosen is the pp-wave background. The main asset of choosing U in such a way is that the equations for transversal fields X^a become linear:

$$(-\partial_\tau^2 + \partial_\sigma^2)X^a(\tau, \sigma) = \kappa^2 K_{ab}(\kappa\tau)X^b(\tau, \sigma). \tag{7.29}$$

Now it is possible to mimic the procedure used already in section 1.1 in the case of the flat background, expand $X^a(\tau, \sigma)$ in Fourier modes

$$X^a(\tau, \sigma) = \sum_n X_n^a(\tau)e^{in\sigma},$$

and try to solve the resulting ODEs for individual modes

$$\partial_\tau^2 X_n^a(\tau) = -\left(\kappa^2 K_{ab}(\kappa\tau) + n^2\delta_{ab}\right) X_n^b(\tau). \tag{7.30}$$

The dynamical equation containing the field $V(\tau, \sigma)$ is obtained when the action is varied with respect to $U(\tau, \sigma)$. It is more involved than the equations above, but it turns out that it can be substituted by the string constraints (1.19), (1.20), which in the light-cone gauge read

$$\begin{aligned}\kappa\partial_\sigma V &= -\partial_\tau X^a \partial_\sigma X^a,\\ 2\kappa\partial_\tau V &= \kappa^2 K_{ab}X^a X^b - \partial_\tau X^a\partial_\tau X^a - \partial_\sigma X^a \partial_\sigma X^a.\end{aligned}$$

The first condition can be integrated to give

$$V(\tau, \sigma) = v(\tau) - \frac{1}{\kappa}\int_\sigma d\sigma(\partial_\tau X^a \partial_\sigma X^a),$$

which solves the second constraint as well provided the equations (7.29) for $X^a(\tau, \sigma)$ hold. In summary, we see that if we succeed in finding the solutions of (7.30), we obtain the solution of classical equations of our sigma model in terms of Fourier modes. The expansions can be then considered as a starting point towards quantization of strings in the light-cone gauge and other developments.

Chapter 8

T-duality of a homogeneous isotropic plane wave

Authors of [24] investigate a sigma model in a homogeneous isotropic plane wave background, and show that its classical equations can be explicitly solved in terms of Bessel functions when the method explained in section 7.5 is applied. We use their result as a starting point, and explicitly solve classical equations of motion for strings in backgrounds obtained as non-Abelian T-duals of this homogeneous isotropic plane wave.

To construct the dual backgrounds, we shall use semi-Abelian Drinfel'd doubles which contain subgroups of the isometry group of the homogeneous plane wave metric. Admittedly, this duality transformation can be described within the framework of non-Abelian T-duality introduced in Ref. [10]. Nevertheless, we shall understand non-Abelian T-duality as a special case of Poisson–Lie T-duality. Applying the techniques of Poisson–Lie T-duality, we will find solutions of the dual sigma models via the Poisson–Lie transformation of the explicit solution of the original homogeneous plane wave background.

The backgrounds resulting from the Poisson–Lie T-duality are expressed in coordinates that may hide their commonly used forms. Investigating their Killing vectors, we shall find that the dual backgrounds can be transformed to the form of a plane wave.

In the first section of the chapter we review relevant results concerning homogeneous isotropic plane wave, and summarize the solution of classical string equations of motion obtained in [24]. Section 8.2 describes two particular Drinfel'd doubles found in [37], which are used in sections 8.3 and 8.4 to construct dual models. In these sections we also solve the dual models and study their symmetries to reveal the Brinkmann/Rosen form of the metric through an appropriate coordinate transformation.

This chapter is based on the paper [45] that was coauthored by the author of this thesis. Compared to the original paper, we explicitly mention the form of left-invariant vector fields on $\mathscr{G}$ in sections 8.3 and 8.4, and pay more attention to coordinate transformation to Brinkmann form of the metric (the case $|\tilde{x}_1| > 1$ was originally not presented and metrics (8.45), (8.63) were not mentioned). Otherwise only minor changes were made in the notation.

8.1 Properties of a homogeneous isotropic plane wave metric

We recall that a plane wave in $(D = d + 2)$-dimensions was defined in (7.5) by a metric of the following form [24, 57, 58]:

$$ds^2 = 2dudv - K_{ab}(u)x^a x^b du^2 + d\vec{x}^{\,2}, \tag{8.1}$$

where $d\vec{x}^{\,2}$ is the standard metric in the Euclidean space $\mathbf{E}^d$ and $\vec{x} \in \mathbf{E}^d$. The form of this metric seems to be simple, but explicit solution of the equations of motion of the corresponding sigma model can be complicated. Therefore, the authors of Ref. [24] restricted themselves to the special case of an *isotropic homogeneous plane wave metric*, where

$$K_{ab}(u) = \lambda(u)\delta_{ab}, \quad \lambda(u) = \frac{k}{u^2}, \quad k = \nu(1-\nu) = \text{const.} > 0. \tag{8.2}$$

We have already met $K_{ab}(u)$ with this structure when discussing symmetries of a general plane wave background. Besides the Heisenberg algebra of Killing vectors it also has a scaling and rotational symmetry justifying the name isotropic homogeneous. In the following, we shall investigate the case $d = 2$, i.e. the dimension of the spacetime will be four, which seems to be the simplest physically interesting background. The study of higher-dimensional cases is possible either by employing spectator fields, or by dualizing the sigma model with respect to up to $(d+2)$-dimensional subalgebras of the $\left(2 + \frac{1}{2}d(d+3)\right)$-dimensional algebra of Killing vectors.

The metric tensor in Brinkmann coordinates (u, v, x, y) has components

$$G_{\mu\nu}(u, v, x, y) = \begin{pmatrix} \frac{-k(x^2+y^2)}{u^2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.3}$$

This metric is not flat, but its scalar curvature vanishes, and the only nontrivial component of the Ricci tensor is R_{uu}. It has a singularity in $u = 0$, but tends to the flat metric for $u \to \infty$. G by itself does not satisfy Einstein's equations, but with a proper choice of dilaton it may fulfill the equations (1.27)–(1.29). The torsion potential B, as well as the torsion H, in this case vanishes, and the equations (1.27)–(1.29) simplify dramatically, giving the dilaton field as

$$\Phi(u) = \Phi_0 - c\,u + 2\nu(\nu - 1)\ln\, u. \tag{8.4}$$

Let us look closely on the symmetries of the background. The metric admits symmetries generated by the following Killing vectors:

$$\begin{aligned}
\mathcal{K}_1 &= \partial_v, \\
\mathcal{K}_2 &= u^\nu \partial_x - \nu u^{\nu-1} x \partial_v, \\
\mathcal{K}_3 &= u^\nu \partial_y - \nu u^{\nu-1} y \partial_v, \\
\mathcal{K}_4 &= u^{1-\nu} \partial_x - (1-\nu) u^{-\nu} x \partial_v, \\
\mathcal{K}_5 &= u^{1-\nu} \partial_y - (1-\nu) u^{-\nu} y \partial_v, \\
\mathcal{K}_6 &= u\partial_u - v\partial_v, \\
\mathcal{K}_7 &= x\partial_y - y\partial_x.
\end{aligned} \tag{8.5}$$

One can easily check that the Lie algebra spanned by these vectors is the semidirect sum $\mathcal{S} \ltimes \mathcal{N}$ of

$$\mathcal{S} = Span[\mathcal{K}_6, \mathcal{K}_7]$$

and an ideal

$$\mathcal{N} = Span[\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5].$$

The algebra $\mathcal{S}$ is Abelian and its generators can be interpreted as dilation in u, v and rotation in x, y. The generators of the algebra $\mathcal{N}$ commute as the two-dimensional Heisenberg algebra with the center $\mathcal{K}_1$. Since (8.2) implies

$$\nu = \frac{1}{2}\left(1 \pm \sqrt{1-4k}\right),$$

this agrees with (7.27) and other results presented in section 7.4.

The equations of motion for $X^\mu(\tau,\sigma) = (U(\tau,\sigma), V(\tau,\sigma), X(\tau,\sigma), Y(\tau,\sigma))$ are given by the sigma model action and read

$$\begin{aligned}
(\partial_\sigma^2 - \partial_\tau^2)U &= 0, \\
(\partial_\sigma^2 - \partial_\tau^2)X + \frac{k}{U^2}\,\partial_a U\,\partial^a U\,X &= 0, \\
(\partial_\sigma^2 - \partial_\tau^2)Y + \frac{k}{U^2}\,\partial_a U\,\partial^a U\,Y &= 0, \\
(\partial_\sigma^2 - \partial_\tau^2)V + \frac{k}{U^3}\,\partial_a U\,\partial^a U\,(X^2+Y^2) - \frac{2k}{U^2}\,\partial_a U\,(\partial^a X\,X + \partial^a Y\,Y) &= 0.
\end{aligned}$$

The plane wave metric (8.3) allows to adopt the light-cone gauge

$$U(\tau,\sigma) = \kappa\tau, \qquad \kappa := 2\alpha' p^u, \tag{8.6}$$

in which the preceding equations of motion simplify and acquire the form

$$(\partial_\sigma^2 - \partial_\tau^2)X - \frac{k}{\tau^2}X = 0, \tag{8.7}$$

$$(\partial_\sigma^2 - \partial_\tau^2)Y - \frac{k}{\tau^2}Y = 0, \tag{8.8}$$

$$(\partial_\sigma^2 - \partial_\tau^2)V - \frac{k}{\kappa\tau^3}(X^2+Y^2) + \frac{2k}{\kappa\tau^2}(\partial_\tau X\,X + \partial_\tau Y\,Y) = 0. \tag{8.9}$$

Solution of the equations (8.7), (8.8) that for $\tau \to \infty$ tends to the free string solution (1.10) was given in Ref. [24] in terms of Fourier modes as

$$X^i(\sigma,\tau) = x_0^i(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\left[X_n^i(\tau,\sigma) - X_n^{i*}(\tau,\sigma)\right],$$

where $i = 2,3,\ X = X^2,\ Y = X^3,\ \nu = \frac{1}{2}\left(1+\sqrt{1-4k}\right)$, and the zero modes are

$$\begin{aligned} x_0^i(\tau) &= \frac{1}{\sqrt{2\nu-1}}\left(\tilde{x}^i\,\tau^{1-\nu} + 2\alpha'\tilde{p}^i\,\tau^{\nu}\right), && \text{for } k \neq \frac{1}{4}, \\ x_0^i(\tau) &= \sqrt{\tau}\left(\tilde{x}^i + 2\alpha'\tilde{p}^i\ \ln\tau\right), && \text{for } k = \frac{1}{4}. \end{aligned} \tag{8.10}$$

The higher modes are expanded as

$$X_n^i(\tau,\sigma) = Z(2n\tau)\left(\alpha_n^i e^{2in\sigma} + \tilde{\alpha}_n^i e^{-2in\sigma}\right),$$

$$Z(2n\tau) := e^{-i\frac{\pi}{2}\nu}\sqrt{\pi n\tau}\ H^{(2)}_{\nu-\frac{1}{2}}(2n\tau),$$

and $H^{(2)}_{\nu-\frac{1}{2}}$ is the Hankel function of the second kind

$$H^{(2)}_{\nu-\frac{1}{2}}(t) = \left[J_{\nu-\frac{1}{2}}(t) - i\,Y_{\nu-\frac{1}{2}}(t)\right],$$

given by Bessel functions $J_{\nu-\frac{1}{2}}(t)$ and $Y_{\nu-\frac{1}{2}}(t)$.

Being interested in string solutions of the sigma model, we have to add supplementary string conditions ensuring the vanishing of the two-dimensional energy-momentum tensor (1.19) and (1.20). In the light-cone gauge (8.6) these read

$$\kappa\partial_\sigma V + \partial_\tau X\partial_\sigma X + \partial_\tau Y\partial_\sigma Y = 0,$$

$$2\kappa\partial_\tau V - \frac{k}{\tau^2}(X^2+Y^2) + \partial_\tau X\partial_\tau X + \partial_\sigma X\partial_\sigma X + \partial_\tau Y\partial_\tau Y + \partial_\sigma Y\partial_\sigma Y = 0.$$

Compatibility of these two equations is guaranteed by the equations of motion (8.7) and (8.8) for X and Y . The first condition can be integrated giving also the solution to the second one:

$$V(\tau,\sigma) = v(\tau) - \frac{1}{\kappa}\int d\sigma\left(\partial_\tau X\partial_\sigma X + \partial_\tau Y\partial_\sigma Y\right), \tag{8.11}$$

where $v(\tau)$ is an arbitrary function. The field equation (8.9) is solved by (8.11) provided the functions $X,\ Y$ satisfy (8.7) and (8.8).

8.2 Data for the construction of dual backgrounds

Having summarized the properties and the solution of the isotropic homogeneous plane wave, we would like to find dual backgrounds and solve also these duals.

The concept used for the construction of mutually dual sigma models was presented in section 4.3. We saw that the relevant structure is a Drinfel'd double – a Lie group that splits into two equally dimensional subgroups $\mathscr{G},\widetilde{\mathscr{G}}$ of the Drinfel'd double $\mathscr{D}$, such that the corresponding Lie subalgebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ are isotropic subspaces of the Lie algebra $\mathfrak{d}$ of the Drinfel'd double. The Drinfel'd double suitable for a given sigma model living in a curved background can sometimes be found from the knowledge of the symmetry group of the metric. In the case of the plane wave (8.3) the metric has the sufficient number of independent Killing vectors, and the subgroups of the isometry group can be taken as one of the subgroups of the Drinfel'd double. The other group then has to be chosen Abelian in order to satisfy the conditions of dualizability (4.14). We shall focus on the case when the isometry subgroup acts freely and transitively on the manifold, i.e. the atomic duality.

To get the metric (8.3), the Lie algebra of the Drinfel'd double can be composed from a four-dimensional Lie subalgebra of the algebra of Killing vectors (8.5) and the four-dimensional Abelian algebra. According to [37], there are six classes of four-dimensional subalgebras of the seven-dimensional isometry algebra. However, to have $\mathscr{M} \approx \mathscr{G}$, the four-dimensional subgroup of isometries has to act freely and transitively on the target manifold $\mathscr{M}$ where the metric is defined. There are two such four-dimensional subgroups that were found in Ref. [37]. They are generated by

$$
\begin{aligned}
\mathfrak{g}_1 &= Span\,[\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_5, \mathcal{K}_6]\,,\\
\mathfrak{g}_2 &= Span\,[\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_6 + \rho\,\mathcal{K}_7]\,.
\end{aligned}
$$

The non-vanishing commutation relations are

$$
\begin{aligned}
[\mathcal{K}_6, \mathcal{K}_1] &= \mathcal{K}_1,\\
[\mathcal{K}_6, \mathcal{K}_2] &= \nu\,\mathcal{K}_2,\\
[\mathcal{K}_6, \mathcal{K}_5] &= (1-\nu)\,\mathcal{K}_5,
\end{aligned}
\tag{8.12}
$$

for $\mathfrak{g}_1$, and

$$
\begin{aligned}
[\mathcal{K}_6 + \rho\mathcal{K}_7, \mathcal{K}_1] &= \mathcal{K}_1,\\
[\mathcal{K}_6 + \rho\mathcal{K}_7, \mathcal{K}_2] &= \nu\,\mathcal{K}_2 - \rho\,\mathcal{K}_3,\\
[\mathcal{K}_6 + \rho\mathcal{K}_7, \mathcal{K}_3] &= \nu\,\mathcal{K}_3 + \rho\,\mathcal{K}_2,
\end{aligned}
\tag{8.13}
$$

for $\mathfrak{g}_2$ respectively, where the parameter ν was given in (8.2) and ρ is an arbitrary real parameter. In the following, we shall find metrics dual to (8.3) that are constructed from the Drinfel'd doubles with $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{a}$, where $\mathfrak{g}$ is either $\mathfrak{g}_1$ or $\mathfrak{g}_2$ and $\mathfrak{a}$ is the four-dimensional Abelian algebra.

Our first goal is to construct the background tensor (8.3) via the techniques presented in section 4.3, namely via equations (4.36) and (4.37)

$$F_{\mu\nu}(x) = e_{\mu}^{a}(g)\, E_{ab}(g)\, e_{\nu}^{b}(g)\,,$$

$$E(g) = \left[E^{-1}(e) + \Pi(g)\right]^{-1}, \qquad \Pi(g) = b(g) \cdot a(g)^{-1}.$$

As $\widetilde{\mathscr{G}}$ was chosen Abelian, the matrix $b(g)$ in the adjoint representation vanishes and so does $\Pi(g)$, but $E(e)$ is unknown.

We have identified points of the manifold with points of the group. However, the metric (8.3) and the $F_{\mu\nu}$ resulting from the construction may differ by a change of coordinates. When trying to find this coordinate transformation directly from the transformation properties of a second order covariant tensor field, one would encounter serious problems solving a set of PDEs which is quadratic in derivatives. Instead, it is much easier to compare the left-invariant vector fields of $\mathscr{G}$ with the Killing vectors of the desired metric. Then the set of PDEs contains only first powers of the derivatives, and is thus much simpler to solve. Then it is easy to transform the metric to coordinates (x^1, x^2, x^3, x^4) parametrizing the group elements, and to find the appropriate matrix $E(e)$ as the value of $F_{\mu\nu}(g)$ in the unit of the group. Choosing the parametrization of the elements of the group $\mathscr{G}$ as

$$g(x) = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, \tag{8.14}$$

where T_1, T_2, T_3 and T_4 are generators of the group $\mathscr{G}$, the matrix $E(e)$ was found in Ref. [37] to be

$$E(e) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{8.15}$$

for both Drinfel'd doubles. Dual backgrounds on the group $\widetilde{\mathscr{G}}$ with elements

$$\tilde{g}(\tilde{x}) = e^{\tilde{x}_1 \widetilde{T}^1} e^{\tilde{x}_2 \widetilde{T}^2} e^{\tilde{x}_3 \widetilde{T}^3} e^{\tilde{x}_4 \widetilde{T}^4},$$

as well as classical solutions of their sigma models

$$\tilde{g}(\tau,\sigma) = e^{\tilde{x}_1(\tau,\sigma) \widetilde{T}^1} e^{\tilde{x}_2(\tau,\sigma) \widetilde{T}^2} e^{\tilde{x}_3(\tau,\sigma) \widetilde{T}^3} e^{\tilde{x}_4(\tau,\sigma) \widetilde{T}^4},$$

will be found in the following sections. The dual model can be obtained through (4.41), i.e. by the exchange

$$\mathscr{G} \leftrightarrow \widetilde{\mathscr{G}}, \qquad \mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}, \qquad \Pi(g) \leftrightarrow \widetilde{\Pi}(\tilde{g}), \qquad E(e) \leftrightarrow E(e)^{-1}.$$

We recall that the relation between the solution $X^{\mu}(\tau,\sigma)$ of the equations of motion of the sigma model given by F and the solution $\widetilde{X}^{\mu}(\tau,\sigma)$ of the model given by $\widetilde{F}$ follows from two possible decompositions of elements of the Drinfel'd double

$$g(\tau,\sigma)\tilde{h}(\tau,\sigma) = \tilde{g}(\tau,\sigma) h(\tau,\sigma), \tag{8.16}$$

where $g, h \in \mathscr{G},\ \tilde{g}, \tilde{h} \in \widetilde{\mathscr{G}}$. The map $\tilde{h} : \mathbb{R}^2 \to \widetilde{\mathscr{G}}$ that we need for this transformation is the solution of the equations (4.30),(4.31), which in terms of (τ, σ) read

$$\left((\partial_\tau \tilde{h}).\tilde{h}^{-1}\right)_j = -v_j^\lambda \left[G_{\lambda\nu}\partial_\sigma X^\nu + B_{\lambda\nu}\partial_\tau X^\nu\right], \tag{8.17}$$

$$\left((\partial_\sigma \tilde{h}).\tilde{h}^{-1}\right)_j = -v_j^\lambda \left[G_{\lambda\nu}\partial_\tau X^\nu + B_{\lambda\nu}\partial_\sigma X^\nu\right]. \tag{8.18}$$

The equation (8.16) then defines the Poisson–Lie transformation between the solution of the equations of motion of the original sigma model and its dual. Its application may still be quite complicated. We recall that in order to use it for solution of the dual model, the following three steps must be done:

- One has to know the solution $X^\mu(\tau, \sigma)$ of the sigma model given by F.
- Given $X^\mu(\tau, \sigma)$, one has to find $\tilde{h}(\tau, \sigma)$, i.e. solve the system of PDEs (8.17), (8.18).
- Given $l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma) \in \mathscr{D}$, one has to find the dual decomposition $l(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma)$, where $\tilde{g}(\tau, \sigma) \in \widetilde{\mathscr{G}}$, $h(\tau, \sigma) \in \mathscr{G}$.

Since the solution $X^\mu(\tau, \sigma)$ is already known to us, we focus on the two remaining steps of the Poisson–Lie transformation, and solve the equations of motion for strings in backgrounds dual to the isotropic homogeneous plane wave metric (8.3).

8.3 Strings in the dual background obtained from $\mathfrak{g}_1$

Let us first consider the group generated by the Lie algebra

$$\mathfrak{g}_1 = Span[T_1, T_2, T_3, T_4]$$

with commutation relations (cf. (8.12))

$$[T_4, T_1] = T_1, \qquad [T_4, T_2] = \nu\, T_2, \qquad [T_4, T_3] = (1-\nu)T_3. \tag{8.19}$$

The transformation between coordinates (x^1, x^2, x^3, x^4) parametrizing group elements via (8.14) and coordinates (u, v, x, y) on $\mathscr{M}$ can be obtained by comparing the basis of left-invariant vector fields on the group $\mathscr{G}$ given by

$$\mathfrak{X}_1 = e^{x^4}\frac{\partial}{\partial x^1}, \qquad \mathfrak{X}_2 = e^{\nu x^4}\frac{\partial}{\partial x^2}, \qquad \mathfrak{X}_3 = e^{(1-\nu)x^4}\frac{\partial}{\partial x^3}, \qquad \mathfrak{X}_4 = \frac{\partial}{\partial x^4},$$

and the Killing vectors $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_5, \mathcal{K}_6$ of the metric (8.3). One gets

$$\begin{aligned} u &= e^{x^4}, \\ v &= \frac{1}{2}\left[2x^1 - \nu(x^2)^2 - (1-\nu)(x^3)^2\right] e^{-x^4}, \\ x &= x^2, \\ y &= x^3. \end{aligned} \tag{8.20}$$

The metric (8.3) expressed in group coordinates then acquires the form

$$G_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\nu x^2 \\ 0 & 0 & 1 & (\nu-1)x^3 \\ 1 & -\nu x^2 & (\nu-1)x^3 & \nu^2(x^2)^2 + (\nu-1)^2(x^3)^2 - 2x^1 \end{pmatrix},$$

and setting $x^1 = x^2 = x^3 = x^4 = 0$ we recover $E(e)$ as in (8.15).

The group $\widetilde{\mathscr{G}}$ is Abelian, and the right-hand sides of the equations (8.17), (8.18) are invariant with respect to coordinate transformations. That is why in Brinkmann coordinates (u, v, x, y) we can use just $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_5, \mathcal{K}_6$ as the left-invariant fields on $\mathscr{G}$. Moreover, the B-field vanishes in our model, so the equations (8.17), (8.18) for $\tilde{h}$ read

$$\partial_\tau \tilde{h} = -\begin{pmatrix} 0 \\ (\kappa\tau)^\nu \partial_\sigma X \\ (\kappa\tau)^{1-\nu}\partial_\sigma Y \\ \kappa\tau\,\partial_\sigma V \end{pmatrix}, \qquad \partial_\sigma \tilde{h} = -\begin{pmatrix} \kappa \\ (\kappa\tau)^\nu\left(\partial_\tau X - \frac{\nu}{\tau}X\right) \\ \kappa(\kappa\tau)^{-\nu}\left(\tau\partial_\tau Y + (\nu-1)Y\right) \\ \frac{\nu(\nu-1)}{\tau}(X^2+Y^2) + \kappa\tau\partial_\tau V - \kappa V \end{pmatrix},$$

and are solved by

$$\tilde{h}_1 = c_1 - \kappa\sigma, \tag{8.21}$$

$$\tilde{h}_2 = c_2 - (\kappa\tau)^\nu \int d\sigma \left(\partial_\tau X - \frac{\nu}{\tau}X\right), \tag{8.22}$$

$$\tilde{h}_3 = c_3 - (\kappa\tau)^{(1-\nu)} \int d\sigma \left(\partial_\tau Y - \frac{1-\nu}{\tau}Y\right), \tag{8.23}$$

$$\begin{aligned} \tilde{h}_4 &= c_4 + \int d\sigma \left[\frac{\nu(1-\nu)}{\tau}\left(X^2+Y^2\right) + \kappa\left(V - \tau\partial_\tau V\right)\right] \\ &= \int d\sigma \left[\frac{\nu(1-\nu)}{2\tau}(X^2+Y^2) + \frac{\tau}{2}\left[(\partial_\tau X)^2 + (\partial_\sigma X)^2 + (\partial_\tau Y)^2 + (\partial_\sigma Y)^2\right]\right] \\ &\quad - \int d\sigma \int d\sigma' (\partial_\tau X \partial_\sigma X + \partial_\tau Y \partial_\sigma Y) + c_4 + \kappa\,\sigma\, v(t), \end{aligned}$$

where $c_i, i = 1, \ldots, 4$ are arbitrary constants. Obviously, when solving the system (8.17), (8.18), the Abelian nature of $\widetilde{\mathscr{G}}$ was crucial.

The dual tensor on the group $\widetilde{\mathscr{G}}$ is calculated as

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} \frac{\nu^2\tilde{x}_2^2 + (1-\nu)^2\tilde{x}_3^2}{\tilde{x}_1^2 - 1} & \frac{\nu\,\tilde{x}_2}{1-\tilde{x}_1} & \frac{(1-\nu)\,\tilde{x}_3}{1-\tilde{x}_1} & \frac{1}{1-\tilde{x}_1} \\ -\frac{\nu\,\tilde{x}_2}{\tilde{x}_1+1} & 1 & 0 & 0 \\ \frac{(\nu-1)\,\tilde{x}_3}{\tilde{x}_1+1} & 0 & 1 & 0 \\ \frac{1}{\tilde{x}_1+1} & 0 & 0 & 0 \end{pmatrix}. \tag{8.24}$$

Even though it is not symmetric, its torsion is zero and the $\widetilde{B}$-field does not enter the equations of motion. It satisfies the conformal invariance conditions (1.27)–(1.29) with the dilaton field

$$\tilde{\Phi}(\tilde{x}_1) = \tilde{\Phi}_0 + C\ln\left(\frac{\tilde{x}_1 - 1}{\tilde{x}_1 + 1}\right) + \left(\nu - 1 - \nu^2\right)\ln\left(1 - \tilde{x}_1^2\right) \tag{8.25}$$

that we rederive later.

To obtain the solution of the sigma model on $\widetilde{\mathscr{G}}$ given by $\widetilde{F}$, we have to solve the equation (8.16) for $\tilde{x}_j$, where the group elements are parametrized as

$$g = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, \qquad \tilde{h} = e^{\tilde{h}_1 \widetilde{T}^1} e^{\tilde{h}_2 \widetilde{T}^2} e^{\tilde{h}_3 \widetilde{T}^3} e^{\tilde{h}_4 \widetilde{T}^4}, \tag{8.26}$$

$$\tilde{g} = e^{\tilde{x}_1 \widetilde{T}^1} e^{\tilde{x}_2 \widetilde{T}^2} e^{\tilde{x}_3 \widetilde{T}^3} e^{\tilde{x}_4 \widetilde{T}^4}, \qquad h = e^{h^1 T_1} e^{h^2 T_2} e^{h^3 T_3} e^{h^4 T_4}. \tag{8.27}$$

To accomplish this, we can either use the Baker–Campbell–Hausdorff formula, or employ a representation r of the semi-Abelian Drinfel'd double in the form of $(\dim \mathfrak{g} + 1) \times (\dim \mathfrak{g} + 1)$ block matrices, such that

$$r(g) = \begin{pmatrix} Ad_g & 0 \\ 0 & 1 \end{pmatrix}, \qquad r(\tilde{h}) = \begin{pmatrix} \mathbf{1} & 0 \\ v(\tilde{h}) & 1 \end{pmatrix},$$

where $v(\tilde{h}) = (\tilde{h}_1, \dots, \tilde{h}_{\dim \mathfrak{g}})$. From the equation (8.16) we get

$$r(l) = r(g\tilde{h}) = \begin{pmatrix} Ad_g & 0 \\ v(\tilde{h}) & 1 \end{pmatrix} = r(\tilde{g}h) = \begin{pmatrix} Ad_h & 0 \\ v(\tilde{g}) \cdot (Ad_h) & 1 \end{pmatrix}. \tag{8.28}$$

If the adjoint representation of the Lie algebra $\mathfrak{g}$ is faithful, which is the case of (8.19), then the representation r of the Drinfel'd double is faithful as well, and the relation (8.28) gives a system of equations for $\tilde{x}_j$ and h^j. If possible, we prefer this way of solving the problem instead of using the BCH formula.

Plugging (8.26) and (8.27) into (8.28) together with the the adjoint representation of the Lie algebra (8.19), we get the solution of (8.16) in the form

$$h^j = x^j,$$

$$\tilde{x}_1 = \mathrm{e}^{-x^4} \tilde{h}_1, \tag{8.29}$$

$$\tilde{x}_2 = \mathrm{e}^{-\nu x^4} \tilde{h}_2, \tag{8.30}$$

$$\tilde{x}_3 = \mathrm{e}^{(\nu-1) x^4} \tilde{h}_3, \tag{8.31}$$

$$\tilde{x}_4 = \mathrm{e}^{-x^4} x^1 \tilde{h}_1 + \nu \mathrm{e}^{-\nu x^4} x^2 \tilde{h}_2 + (1-\nu) \mathrm{e}^{(\nu-1)x^4} x^3 \tilde{h}_3 + \tilde{h}_4. \tag{8.32}$$

The expressions for $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are very simple, and it can be checked that, combined with (8.20), they give explicit solution of the equations of motion of the sigma model on $\widetilde{\mathscr{G}}$ in the background (8.24). Namely, inserting the light-cone gauge solution (8.6) into (8.20) and (8.29), we get

$$\widetilde{X}_1(\tau, \sigma) = \frac{c_1 - \kappa\sigma}{\kappa\tau} \tag{8.33}$$

from (8.21), which solves the dual equation of motion

$$\frac{\delta S_{\widetilde{F}}}{\delta \widetilde{X}_4} = \left(\partial_\sigma^2 \widetilde{X}_1 - \partial_\tau^2 \widetilde{X}_1\right)\left(1 - \widetilde{X}_1^2\right) + 2\,\widetilde{X}_1 \left((\partial_\sigma \widetilde{X}_1)^2 - (\partial_\tau \widetilde{X}_1)^2\right) = 0. \tag{8.34}$$

The other two equations then reduce to

$$\frac{\delta S_{\widetilde{F}}}{\delta \widetilde{X}_2} = (\partial_\sigma^2 - \partial_\tau^2)\widetilde{X}_2 + \frac{\nu(1+\nu)}{\tau^2}\widetilde{X}_2 = 0,$$
$$\frac{\delta S_{\widetilde{F}}}{\delta \widetilde{X}_3} = (\partial_\sigma^2 - \partial_\tau^2)\widetilde{X}_3 + \frac{(\nu-2)(\nu-1)}{\tau^2}\widetilde{X}_3 = 0,$$

and are solved in agreement with (8.30), (8.31) and (8.20), (8.22), (8.23) by

$$\widetilde{X}_2(\tau,\sigma) = c_2(\kappa\tau)^{-\nu} + \int d\sigma \left(\frac{\nu}{\tau}X - \partial_\tau X\right),$$
$$\widetilde{X}_3(\tau,\sigma) = c_3(\kappa\tau)^{\nu-1} + \int d\sigma \left(\frac{1-\nu}{\tau}Y - \partial_\tau Y\right),$$

where $X = X^2,\ Y = X^3$ were given in section 8.1. For $\nu \neq \frac{1}{2}$ we have

$$\widetilde{X}_2(\tau,\sigma) = \left(c_2\,\kappa^{-\nu} + \sqrt{2\nu-1}\,\tilde{x}\,\sigma\right)\tau^{-\nu} + \Sigma_2^x(\tau,\sigma), \tag{8.35}$$
$$\widetilde{X}_3(\tau,\sigma) = \left(c_3\,\kappa^{\nu-1} - \sqrt{2\nu-1}\;2\,\tilde{p}^y\alpha'\,\sigma\right)\tau^{\nu-1} + \Sigma_3^y(\tau,\sigma), \tag{8.36}$$

where we denoted

$$\begin{aligned}\Sigma_2^j(\tau,\sigma) :=& \frac{1}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\Big[W(2n\tau)\left(\alpha_n^j e^{2in\sigma} - \tilde{\alpha}_n^j e^{-2in\sigma}\right)\\ &+\ W^*(2n\tau)\left(\alpha_{-n}^j e^{-2in\sigma} - \tilde{\alpha}_{-n}^j e^{2in\sigma}\right)\Big],\\ \Sigma_3^j(\tau,\sigma) :=& \frac{1}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\Big[\widetilde{W}(2n\tau)\left(\alpha_n^j e^{2in\sigma} - \tilde{\alpha}_n^j e^{-2in\sigma}\right)\\ &+\widetilde{W}^*(2n\tau)\left(\alpha_{-n}^j e^{-2in\sigma} - \tilde{\alpha}_{-n}^j e^{2in\sigma}\right)\Big],\end{aligned} \tag{8.37}$$

and $W, \widetilde{W}$ stand for

$$W(2n\tau) := \frac{1}{2n}\left(\frac{\nu}{\tau}Z(2n\tau) - \partial_\tau Z(2n\tau)\right) = e^{-\frac{i\pi\nu}{2}}\sqrt{\pi n\tau}\;\;H^{(2)}_{\nu+\frac{1}{2}}(2n\tau),$$
$$\widetilde{W}(2n\tau) := \frac{1}{2n}\left(\frac{1-\nu}{\tau}Z(2n\tau) - \partial_\tau Z(2n\tau)\right) = -\,e^{-\frac{i\pi\nu}{2}}\sqrt{\pi n\tau}\;\;H^{(2)}_{\nu-\frac{3}{2}}(2n\tau).$$

For $\nu = \frac{1}{2}$ the expressions are a bit different and can be derived from (8.10). Finally, we get the solution of the last equation of motion of the dual model from the expression (8.32) as

$$\begin{aligned}\widetilde{X}_4(\tau,\sigma) = c_4 - \frac{1}{2}\frac{\sigma + c_1}{\tau}\left(2\kappa\tau\,V + \nu X^2 + (1-\nu)Y^2\right) + \nu X\widetilde{X}_2 + (1-\nu)Y\widetilde{X}_3\\ + \int d\sigma\left[\frac{\nu(1-\nu)}{\tau}(X^2+Y^2) + \kappa(V - \tau\partial_\tau V)\right].\end{aligned}$$

Killing vectors of the dual metric and its plane wave form

We have found the solution of the equations of motion of the dual model. The purpose of this subsection is to show that the metric corresponding to the dual tensor (8.24) is again a plane wave.

As the torsion $\widetilde{H} = d\widetilde{B}$ vanishes and does not influence neither the equations of motion nor the β equations (1.27)–(1.29), the relevant part of the tensor $\widetilde{F}_{\mu\nu}$ is only its symmetric part. The dual metric calculated from (8.24) is

$$\widetilde{G}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} \frac{\nu^2\tilde{x}_2^2+(1-\nu)^2\tilde{x}_3^2}{\tilde{x}_1^2-1} & \frac{\nu\,\tilde{x}_1\tilde{x}_2}{1-\tilde{x}_1^2} & \frac{(1-\nu)\tilde{x}_1\tilde{x}_3}{1-\tilde{x}_1^2} & \frac{1}{1-\tilde{x}_1^2} \\ \frac{\nu\,\tilde{x}_1\tilde{x}_2}{1-\tilde{x}_1^2} & 1 & 0 & 0 \\ \frac{(1-\nu)\tilde{x}_1\tilde{x}_3}{1-\tilde{x}_1^2} & 0 & 1 & 0 \\ \frac{1}{1-\tilde{x}_1^2} & 0 & 0 & 0 \end{pmatrix}. \tag{8.38}$$

For $\nu \notin \{0,1\}$, i.e. for $k \neq 0$, this metric has a five-dimensional algebra generated by Killing vectors

$$\begin{aligned} \widetilde{\mathcal{K}}_1 &= P_\nu(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_2} + \tilde{x}_2\left[(\nu+1)P_{\nu+1}(\tilde{x}_1) - \tilde{x}_1(1+2\nu)P_\nu(\tilde{x}_1)\right]\frac{\partial}{\partial\tilde{x}_4}, \\ \widetilde{\mathcal{K}}_2 &= Q_\nu(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_2} + \tilde{x}_2\left[(\nu+1)Q_{\nu+1}(\tilde{x}_1) - \tilde{x}_1(1+2\nu)Q_\nu(\tilde{x}_1)\right]\frac{\partial}{\partial\tilde{x}_4}, \\ \widetilde{\mathcal{K}}_3 &= P_{\nu-2}(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_3} + \tilde{x}_3(\nu-1)P_{\nu-1}(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_4}, \\ \widetilde{\mathcal{K}}_4 &= Q_{\nu-2}(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_3} + \tilde{x}_3(\nu-1)Q_{\nu-1}(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_4}, \\ \widetilde{\mathcal{K}}_5 &= -\frac{\partial}{\partial\tilde{x}_4}, \end{aligned} \tag{8.39}$$

where $P_\nu(z)$ and $Q_\nu(z)$ are the Legendre functions of the first and second kind. The commutators close to the Heisenberg algebra with the central element $\widetilde{\mathcal{K}}_5$

$$[\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2] = \widetilde{\mathcal{K}}_5, \qquad [\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4] = \widetilde{\mathcal{K}}_5$$

due to the identity

$$P_{1+\nu}(z)Q_\nu(z) - Q_{1+\nu}(z)P_\nu(z) = \frac{1}{1+\nu}.$$

The number of Killing vectors, as well as the fact that they close to the Heisenberg algebra, suggests (cf. Ref. [57]) that the metric (8.38) might be brought to the form of a plane wave. Trying to find new (Rosen) coordinates (z, w, y^1, y^2) such that the Killing vectors (8.39) acquire the standard form mentioned in Ref. [58]:

$$e_+ = \frac{\partial}{\partial w}, \qquad e_i = \frac{\partial}{\partial y^i}, \qquad e_i^* = y^i\frac{\partial}{\partial w} - \Gamma^{ij}(z)\frac{\partial}{\partial y^i}, \quad i,j \in \{1,2\}, \tag{8.40}$$

we can find the transformation

$$
\begin{aligned}
\tilde{x}_1 =& z, \\
\tilde{x}_2 =& y^1 P_\nu(z), \\
\tilde{x}_3 =& y^2 P_{\nu-2}(z), \\
\tilde{x}_4 =& -w + \frac{1}{2}\Big[(y^1)^2 P_\nu(z)((\nu+1)P_{\nu+1}(z) - z(1+2\nu)P_\nu(z)) \\
&+ (y^2)^2(\nu-1)P_{\nu-2}(z)P_{\nu-1}(z)\Big]
\end{aligned}
\tag{8.41}
$$

giving the functions $\Gamma^{ij}(z)$ as

$$
\Gamma^{11}(z) = -\frac{Q_\nu(z)}{P_\nu(z)}, \qquad \Gamma^{12}(z) = \Gamma^{21}(z) = 0, \qquad \Gamma^{22}(z) = -\frac{Q_{\nu-2}(z)}{P_{\nu-2}(z)}.
$$

In these coordinates (z, w, y^1, y^2) the dual metric (8.38) gives the line element

$$
ds^2 = \frac{2}{z^2-1}\, dz\, dw + (P_\nu(z))^2\, (dy^1)^2 + (P_{\nu-2}(z))^2\, (dy^2)^2.
$$

To get rid of the denominator in the first term, an additional transformation

$$
z = \begin{cases} -\tanh\,\tilde{z} & \text{for } |z| < 1, \\ -\coth\,\tilde{z} & \text{for } |z| > 1, \end{cases}
$$

can be performed, which brings the metric to the diagonal Rosen form.

Transition from Rosen to Brinkmann coordinates is obtained via relations (7.7), cf. [57], by virtue of the matrices $G_{ab}(\tilde{z})$ and $Q(\tilde{z})$, where $G_{ab}(\tilde{z})$ is given by Rosen form of the matrix. In the case $|z| < 1$ we have

$$
G_{ab}(\tilde{z}) = \begin{pmatrix} (P_\nu(-\tanh\,\tilde{z}))^2 & 0 \\ 0 & (P_{\nu-2}(-\tanh\,\tilde{z}))^2 \end{pmatrix},
$$

and the matrix $Q(\tilde{z})$ is the solution of equations (7.8)

$$
Q^T(\tilde{z}) \cdot G(\tilde{z}) \cdot Q(\tilde{z}) = \mathbf{1}, \qquad Q'^T(\tilde{z}) \cdot G(\tilde{z}) \cdot Q(\tilde{z}) = Q^T(\tilde{z}) \cdot G(\tilde{z}) \cdot Q(\tilde{z})'.
$$

In our case the solution can be chosen as

$$
Q(\tilde{z}) = \begin{pmatrix} (P_\nu(-\tanh\,\tilde{z}))^{-1} & 0 \\ 0 & (P_{\nu-2}(-\tanh\,\tilde{z}))^{-1} \end{pmatrix},
$$

and Brinkmann coordinates written in terms of Rosen coordinates (inverse of (7.7)) read

$$
\begin{aligned}
x^+ &= \tilde{z}, \\
x^- &= w - \frac{1}{2}(\nu+1)P_\nu(-\tanh\,\tilde{z})\Big[P_{\nu+1}(-\tanh\,\tilde{z}) + \tanh\tilde{z}\; P_\nu(-\tanh\,\tilde{z})\Big](y^1)^2 \\
&\quad -\frac{1}{2}(\nu-1)P_{\nu-2}(-\tanh\,\tilde{z})\Big[P_{\nu-1}(-\tanh\,\tilde{z}) + \tanh\tilde{z}\; P_{\nu-2}(-\tanh\,\tilde{z})\Big](y^2)^2, \\
z^1 &= y^1 P_\nu(-\tanh\,\tilde{z}), \\
z^2 &= y^2 P_{\nu-2}(-\tanh\,\tilde{z}).
\end{aligned}
$$

With the help of the inverse of the transformation of coordinates (8.41) we get directly

$$
\begin{aligned}
x^+ &= -\operatorname{arctanh}(\tilde{x}_1), \\
x^- &= -\tilde{x}_4 + \frac{1}{2}\tilde{x}_1 \left(\tilde{x}_3^2(\nu-1) - \tilde{x}_2^2\nu\right), \\
z^1 &= \tilde{x}_2, \\
z^2 &= \tilde{x}_3.
\end{aligned}
\tag{8.42}
$$

The dual metric (8.38) is then transformed to the form of the plane wave (8.1), where

$$
K_{ab}(x^+) = \frac{1}{(\cosh x^+)^2} \begin{pmatrix} \nu(\nu+1) & 0 \\ 0 & (\nu-2)(\nu-1) \end{pmatrix}. \tag{8.43}
$$

It means that non-Abelian T-duality transforms the plane wave metric (8.3) to another plane wave metric that is again solvable. The solutions of classical equations of motion in the dual background can be obtained by the ansatz

$$
X^+(\tau,\sigma) = -\operatorname{arctanh}\left(\frac{c_1 - \kappa\sigma}{\kappa\tau}\right) \tag{8.44}
$$

that follows from the duality transformation and coordinate transformation of the light-cone gauge of the original background. From the transformation (8.42) and (8.35), (8.36) one can see that the transversal components $Z^1(\tau,\sigma), Z^2(\tau,\sigma)$ of the classical solutions of the dual model are again expressed in terms of the Hankel function. The conformal invariance conditions (1.27)–(1.29) written in Brinkmann coordinates result in a very simple equation for the dilaton

$$
\tilde{\Phi}''(x^+) = \frac{2(1-\nu+\nu^2)}{\cosh^2(x^+)},
$$

which is solved in agreement with (8.25) by

$$
\tilde{\Phi}(x^+) = C_0 + C_1\, x^+ + 2(1-\nu+\nu^2)\ln(\cosh x^+).
$$

The transformation (8.42) is valid for $|\tilde{x}_1| < 1$. For $|\tilde{x}_1| > 1$ we have to replace tanh by coth, obtaining the plane wave metric in Brinkmann form with

$$
K_{ab}(x^+) = -\frac{1}{(\sinh x^+)^2} \begin{pmatrix} \nu(\nu+1) & 0 \\ 0 & (\nu-2)(\nu-1) \end{pmatrix}. \tag{8.45}
$$

Due to the singularity of $K_{ab}(x^+)$ in $x^+ = 0$, we claim that this metric gives a singular background. Nevertheless, the string β equations again reduce to a single equation for the dilaton, which can be solved by

$$
\tilde{\Phi}(x^+) = C_0 + C_1\, x^+ + 2(1-\nu+\nu^2)\ln(\sinh x^+).
$$

Solution of classical equations of this sigma model is obtained by appropriate transformation of coordinates applied on the solutions $\widetilde{X}^1(\tau,\sigma), \ldots, \widetilde{X}^4(\tau,\sigma)$ found above.

8.4 Strings in the dual background obtained from $\mathfrak{g}_2$

Let us now consider the group generated by the Lie algebra

$$\mathfrak{g}_2 = Span[T_1, T_2, T_3, T_4]$$

with commutation relations (cf. (8.13))

$$[T_4, T_1] = T_1, \qquad [T_4, T_2] = \nu\, T_2 - \rho\, T_3, \qquad [T_4, T_3] = \nu\, T_3 + \rho\, T_2.$$

With the parametrization (8.26), the left-invariant vector fields are:

$$\begin{aligned}
\mathfrak{X}_1 &= e^{x^4}\frac{\partial}{\partial x^1}, & \mathfrak{X}_2 &= e^{\nu x^4}\cos(\rho\, x^4)\frac{\partial}{\partial x^2} - e^{\nu x^4}\sin(\rho\, x^4)\frac{\partial}{\partial x^3},\\
\mathfrak{X}_4 &= \frac{\partial}{\partial x^4}, & \mathfrak{X}_3 &= e^{\nu x^4}\cos(\rho\, x^4)\frac{\partial}{\partial x^2} + e^{\nu x^4}\cos(\rho\, x^4)\frac{\partial}{\partial x^3}.
\end{aligned}$$

The algebra of left-invariant vector fields now has to be compared with the algebra of Killing vectors $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_6 + \rho\,\mathcal{K}_7$. The resulting transformation between group coordinates (x^1, x^2, x^3, x^4) and coordinates (u, v, x, y) is

$$\begin{aligned}
u &= e^{x^4},\\
v &= \left[-\frac{1}{2}\nu\left((x^2)^2 + (x^3)^2\right) + x^1\right] e^{-x^4},\\
x &= x^2\cos(\rho\, x^4) - x^3\sin(\rho\, x^4),\\
y &= x^3\cos(\rho\, x^4) + x^2\sin(\rho\, x^4).
\end{aligned}$$

The metric (8.3) is then transformed into the form

$$G_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 1 & 0 & -\nu x^2 - \rho\, x^3\\ 0 & 0 & 1 & -\nu x^3 + \rho\, x^2\\ 1 & -\nu x^2 - \rho\, x^3 & -\nu x^3 + \rho\, x^2 & -2x^1 + (\nu^2+\rho^2)\left((x^2)^2 + (x^3)^2\right) \end{pmatrix}, \tag{8.46}$$

giving (8.15) at the unit of the group $\mathscr{G}$. The equations (8.17), (8.18) for $\tilde{h}$ now read

$$\partial_\tau \tilde{h} = -\begin{pmatrix} 0\\ (\kappa\tau)^\nu \partial_\sigma X\\ (\kappa\tau)^\nu \partial_\sigma Y\\ \kappa\tau\,\partial_\sigma V - \rho\, Y\partial_\sigma X + \rho\, X\partial_\sigma Y \end{pmatrix},$$

$$\partial_\sigma \tilde{h} = -\begin{pmatrix} \kappa\\ (\kappa\tau)^\nu\left(\partial_\tau X - \frac{\nu}{\tau}X\right)\\ (\kappa\tau)^\nu\left(\tau\partial_\tau Y - \frac{\nu}{\tau}Y\right)\\ \kappa\tau\partial_\tau V - \kappa V - \frac{\nu(1-\nu)}{\tau}(X^2+Y^2) - \rho\, Y\partial_\tau X + \rho\, X\partial_\tau Y \end{pmatrix},$$

and are solved by

$$\tilde{h}_1 = c_1 - \kappa\sigma, \tag{8.47}$$

$$\tilde{h}_2 = c_2 - (\kappa\tau)^\nu \int d\sigma \left(\partial_\tau X - \frac{\nu}{\tau} X\right), \tag{8.48}$$

$$\tilde{h}_3 = c_3 - (\kappa\tau)^\nu \int d\sigma \left(\partial_\tau Y - \frac{\nu}{\tau} Y\right), \tag{8.49}$$

$$\tilde{h}_4 = c_4 + \int d\sigma \left(\frac{\nu(1-\nu)}{\tau}(X^2+Y^2) + \kappa(V - \tau\partial_\tau V) + \rho\, Y\partial_\tau X - \rho\, X\partial_\tau Y\right),$$

where $c_i,\ i = 1, \ldots, 4$ are arbitrary constants.

The tensor dual to (8.46) was found in [37] as

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} \frac{(\nu^2+\rho^2)(\tilde{x}_2^2+\tilde{x}_3^2)}{\tilde{x}_1^2-1} & \frac{\nu\,\tilde{x}_2-\rho\,\tilde{x}_3}{1-\tilde{x}_1} & \frac{\nu\,\tilde{x}_3+\rho\,\tilde{x}_2}{1-\tilde{x}_1} & \frac{1}{1-\tilde{x}_1} \\ \frac{-\nu\,\tilde{x}_2+\rho\,\tilde{x}_3}{\tilde{x}_1+1} & 1 & 0 & 0 \\ \frac{-\nu\,\tilde{x}_3-\rho\,\tilde{x}_2}{\tilde{x}_1+1} & 0 & 1 & 0 \\ \frac{1}{\tilde{x}_1+1} & 0 & 0 & 0 \end{pmatrix}. \tag{8.50}$$

It has a nontrivial antisymmetric part (torsion potential) $\widetilde{B}_{\mu\nu}$ with torsion $\widetilde{H} = d\widetilde{B}$ equal to

$$\widetilde{H} = \frac{2\rho}{\tilde{x}_1^2 - 1}\, d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3. \tag{8.51}$$

The dual metric, which is the symmetric part of (8.50), does not solve Einstein's equations. Solving the conformal invariance conditions (1.27)–(1.29), we learn that the dual dilaton $\tilde{\Phi}$ is a function of $\tilde{x}_1$ only. For such $\tilde{\Phi}(\tilde{x}_1)$ the torsion does not enter equations (1.27)–(1.29), and we are left with one ODE solved by the dilaton field

$$\tilde{\Phi}(\tilde{x}_1) = C_0 + C_1 \ln\left(\frac{\tilde{x}_1 - 1}{\tilde{x}_1 + 1}\right) - \nu(\nu+1)\ln(1 - \tilde{x}_1^2). \tag{8.52}$$

The transformation between two dual decompositions of the element of the Drinfel'd double (8.16) can be found using the same faithful representation as in the previous section. From (8.28) we get the relation between the solutions of the dual sigma models as

$$h^j = x^j,$$

$$\tilde{x}_1 = e^{-x^4}\tilde{h}_1, \tag{8.53}$$

$$\tilde{x}_2 = e^{-\nu x^4}\left(\tilde{h}_3 \sin(\rho x^4) + \tilde{h}_2 \cos(\rho x^4)\right), \tag{8.54}$$

$$\tilde{x}_3 = e^{-\nu x^4}\left(\tilde{h}_3 \cos(\rho x^4) - \tilde{h}_2 \sin(\rho x^4)\right), \tag{8.55}$$

$$\begin{aligned}\tilde{x}_4 = {} & e^{-\nu x^4}(\nu x^3 - \rho x^2)\left(\tilde{h}_3 \cos(\rho x^4) - \tilde{h}_2 \sin(\rho x^4)\right) \\ & + e^{-\nu x^4}(\nu x^2 + \rho x^3)\left(\tilde{h}_3 \sin(\rho x^4) + \tilde{h}_2 \cos(\rho x^4)\right) \\ & + e^{-x^4} x^1 \tilde{h}_1 + \tilde{h}_4.\end{aligned}$$

The equation of motion

$$\frac{\delta S_{\widetilde{F}}}{\delta \widetilde{X}_4} = 0$$

has the same form (8.34) as in the previous section, and from (8.47) and (8.53) we again get its solution in the form

$$\widetilde{X}_1(\tau,\sigma) = \frac{c_1 - \kappa\sigma}{\kappa\tau}.$$

The other two equations then reduce to

$$\begin{aligned}
\frac{\delta S_{\widetilde{F}}}{\delta \widetilde{X}_2} &= (\partial_\sigma^2 - \partial_\tau^2)\widetilde{X}_2 + \frac{2\rho}{\tau}\partial_\tau \widetilde{X}_3 + \frac{1}{\tau^2}\left[(\nu + \nu^2 + \rho^2)\widetilde{X}_2 - \rho \widetilde{X}_3\right] = 0,\\
\frac{\delta S_{\widetilde{F}}}{\delta \widetilde{X}_3} &= (\partial_\sigma^2 - \partial_\tau^2)\widetilde{X}_3 - \frac{2\rho}{\tau}\partial_\tau \widetilde{X}_2 + \frac{1}{\tau^2}\left[(\nu + \nu^2 + \rho^2)\widetilde{X}_3 + \rho \widetilde{X}_2\right] = 0.
\end{aligned}$$

Their solution for $\nu \neq \frac{1}{2}$ follows from (8.54), (8.55) and (8.48), (8.49) in the form

$$\begin{aligned}
\widetilde{X}_2(\tau,\sigma) =& \cos\Big(\rho\, \ln(\kappa\tau)\Big)\Big[(c_2\, \kappa^{-\nu} + \sqrt{2\nu-1}\, \tilde{x}\, \sigma)\tau^{-\nu} + \Sigma_2^x(\tau,\sigma)\Big]\\
&+ \sin\Big(\rho\, \ln(\kappa\tau)\Big)\Big[(c_3\, \kappa^{-\nu} - \sqrt{2\nu-1}\, \tilde{y}\, \sigma)\tau^{-\nu} + \Sigma_2^y(\tau,\sigma)\Big],\\
\widetilde{X}_3(\tau,\sigma) =& - \sin\Big(\rho\, \ln(\kappa\tau)\Big)\Big[(c_2\, \kappa^{-\nu} + \sqrt{2\nu-1}\, \tilde{x}\, \sigma)\tau^{-\nu} + \Sigma_2^x(\tau,\sigma)\Big]\\
&+ \cos\Big(\rho\, \ln(\kappa\tau)\Big)\Big[(c_3\, \kappa^{-\nu} - \sqrt{2\nu-1}\, \tilde{y}\, \sigma)\tau^{-\nu} + \Sigma_2^y(\tau,\sigma)\Big],
\end{aligned}$$

where Σ_2^j are given by (8.37). The last equation of motion is solved by

$$\begin{aligned}
\widetilde{X}_4(\tau,\sigma) =& c_4 - \frac{\sigma + c_1}{\tau}x^1 + \ (\nu x^2 + \rho x^3)\ \widetilde{X}_2 + (\nu x^3 - \rho x^2)\ \widetilde{X}_3\\
&+ \int d\sigma \left[\frac{\nu(1-\nu)}{\tau}(X^2 + Y^2) + \kappa(V - \tau\partial_\tau V) + \rho Y \partial_\tau X - \rho X \partial_\tau Y\right].
\end{aligned}$$

It is worth mentioning that duality is not the only option here. It is possible to use another decomposition, [61], of $\mathscr{D}$ into groups $\widehat{\mathscr{G}}, \bar{\mathscr{G}}$ with the algebra of the Drinfel'd double $\mathfrak{d} = Span[\widehat{T}_1, \ldots, \widehat{T}_4, \bar{T}^1, \ldots, \bar{T}^4]$ given by basis vectors

$$\begin{aligned}
&\widehat{T}_1 = T_1 + T_4, && \widehat{T}_2 = \widetilde{T}^1 - \widetilde{T}^4, && \widehat{T}_3 = T_2, && \widehat{T}_4 = T_3,\\
&\bar{T}^1 = \frac{1}{2}(\widetilde{T}^1 + \widetilde{T}^4), && \bar{T}^2 = \frac{1}{2}(T_1 - T_4), && \bar{T}^3 = \widetilde{T}^2, && \bar{T}^4 = \widetilde{T}^3,
\end{aligned}$$

with commutation relations of the basis elements

$$\begin{aligned}
&[\widehat{T}_1, \widehat{T}_2] = -\widehat{T}_2, && [\bar{T}^1, \bar{T}^2] = -\frac{1}{2}\bar{T}^1,\\
&[\widehat{T}_1, \widehat{T}_3] = \nu\widehat{T}_3 - \rho\widehat{T}_4, && [\bar{T}^2, \bar{T}^3] = \frac{1}{2}\nu\bar{T}^3 + \frac{1}{2}\rho\bar{T}^4,\\
&[\widehat{T}_1, \widehat{T}_4] = \rho\widehat{T}_3 + \nu\widehat{T}_4, && [\bar{T}^2, \bar{T}^4] = -\frac{1}{2}\rho\bar{T}_3 + \frac{1}{2}\nu\bar{T}^4.
\end{aligned}$$

Then it is possible to construct sigma models on groups $\widehat{\mathscr{G}}$ or $\bar{\mathscr{G}}$ and perhaps solve their equations of motion by the Poisson–Lie T-plurality transformation, [22], which relates the solutions of the sigma models on $\mathscr{G}$ and $\widehat{\mathscr{G}}$, or $\bar{\mathscr{G}}$ respectively. However, all the calculations get very complicated as none of the algebras is Abelian.

Killing vectors of the dual metric and its plane wave form

Now we shall inspect the symmetry properties of the dual background (8.50). The torsion is not trivial in general, so the $\widetilde{B}$ field has to be taken into account, and there are only two linearly independent Killing vectors of the tensor $\widetilde{F}$ satisfying $\mathcal{L}_{\widetilde{\mathcal{K}}}\widetilde{F} = 0$, namely

$$\widetilde{\mathcal{K}}_1 = \tilde{x}_2\frac{\partial}{\partial\tilde{x}_3} - \tilde{x}_3\frac{\partial}{\partial\tilde{x}_2}, \qquad \widetilde{\mathcal{K}}_2 = -\frac{\partial}{\partial\tilde{x}_4},$$

which represent rotation in $\tilde{x}_2, \tilde{x}_3$ and shift in $\tilde{x}_4$.

However, for $\rho = 0$ the torsion (8.51) vanishes, and both the equations of motion and the β equations (1.27)–(1.29) for the sigma model are equivalent to those calculated from the symmetric part of (8.50) that now reads

$$\widetilde{G}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} \frac{(\tilde{x}_2^2+\tilde{x}_3^2)\nu^2}{\tilde{x}_1^2-1} & \frac{\nu\,\tilde{x}_1\tilde{x}_2}{1-\tilde{x}_1^2} & \frac{\nu\,\tilde{x}_1\tilde{x}_3}{1-\tilde{x}_1^2} & \frac{1}{1-\tilde{x}_1^2} \\ \frac{\nu\,\tilde{x}_1\tilde{x}_2}{1-\tilde{x}_1^2} & 1 & 0 & 0 \\ \frac{\nu\,\tilde{x}_1\tilde{x}_3}{1-\tilde{x}_1^2} & 0 & 1 & 0 \\ \frac{1}{1-\tilde{x}_1^2} & 0 & 0 & 0 \end{pmatrix}. \tag{8.56}$$

We shall show that this metric is again a plane wave. It has a six-dimensional algebra of Killing vectors generated by $\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2$ and

$$\begin{aligned}
\widetilde{\mathcal{K}}_3 &= P_\nu(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_2} + \tilde{x}_2\left[(\nu+1)P_{\nu+1}(\tilde{x}_1) - \tilde{x}_1(1+2\nu)P_\nu(\tilde{x}_1)\right]\frac{\partial}{\partial\tilde{x}_4},\\
\widetilde{\mathcal{K}}_4 &= Q_\nu(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_2} + \tilde{x}_2\left[(\nu+1)Q_{\nu+1}(\tilde{x}_1) - \tilde{x}_1(1+2\nu)Q_\nu(\tilde{x}_1)\right]\frac{\partial}{\partial\tilde{x}_4},\\
\widetilde{\mathcal{K}}_5 &= P_\nu(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_3} + \tilde{x}_3\left[(\nu+1)P_{\nu+1}(\tilde{x}_1) - \tilde{x}_1(1+2\nu)P_\nu(\tilde{x}_1)\right]\frac{\partial}{\partial\tilde{x}_4},\\
\widetilde{\mathcal{K}}_6 &= Q_\nu(\tilde{x}_1)\frac{\partial}{\partial\tilde{x}_3} + \tilde{x}_3\left[(\nu+1)Q_{\nu+1}(\tilde{x}_1) - \tilde{x}_1(1+2\nu)Q_\nu(\tilde{x}_1)\right]\frac{\partial}{\partial\tilde{x}_4}.
\end{aligned}$$

Their nonzero commutation relations are

$$\begin{aligned}
[\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_3] &= -\widetilde{\mathcal{K}}_5, & [\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_4] &= -\widetilde{\mathcal{K}}_6,\\
[\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_5] &= \widetilde{\mathcal{K}}_3, & [\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_6] &= \widetilde{\mathcal{K}}_4,\\
[\widetilde{\mathcal{K}}_3, \widetilde{\mathcal{K}}_4] &= \widetilde{\mathcal{K}}_2, & [\widetilde{\mathcal{K}}_5, \widetilde{\mathcal{K}}_6] &= \widetilde{\mathcal{K}}_2.
\end{aligned}$$

One can see that the Killing vectors $\widetilde{\mathcal{K}}_2$–$\widetilde{\mathcal{K}}_6$ form the Heisenberg algebra with the central element $\widetilde{\mathcal{K}}_2$. This opens the possibility that this metric might be again brought to the form of a plane wave. The transformation to coordinates (z, w, y^1, y^2) related to $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$ by

$$\begin{aligned}
\tilde{x}_1 &= z, \\
\tilde{x}_2 &= y^1 P_\nu(z), \\
\tilde{x}_3 &= y^2 P_\nu(z), \\
\tilde{x}_4 &= -w + \frac{1}{2}\left[(y^1)^2 + (y^2)^2\right] P_\nu(z)\left[(\nu+1)P_{\nu+1}(z) - z(1+2\nu)P_\nu(z)\right]
\end{aligned}$$

brings the Killing vectors $\widetilde{\mathcal{K}}_2$–$\widetilde{\mathcal{K}}_6$ to the form (8.40) and the metric to

$$ds^2 = \frac{2}{z^2-1}\,dz\,dw + (P_\nu(z))^2(dy^1)^2 + (P_\nu(z))^2(dy^2)^2.$$

The transition to Brinkmann coordinates is obtained similarly as in the section 8.3 through

$$\begin{aligned}
x^+ &= -\mathrm{arctanh}(\tilde{x}_1), \\
x^- &= -\tilde{x}_4 - \frac{1}{2}\,\nu\,\tilde{x}_1(\tilde{x}_2^2 + \tilde{x}_3^2), \\
z^1 &= \tilde{x}_2, \\
z^2 &= \tilde{x}_3
\end{aligned} \tag{8.57}$$

for $|\tilde{x}_1| < 1$. For $|\tilde{x}_1| > 1$ we again need to replace tanh by coth. The dual metric (8.56) is transformed to the form of a plane wave

$$ds^2 = 2dx^+dx^- - \frac{\nu(\nu+1)\left[(z^1)^2 + (z^2)^2\right]}{(\cosh x^+)^2}(dx^+)^2 + (dz^1)^2 + (dz^2)^2, \tag{8.58}$$

or

$$ds^2 = 2dx^+dx^- + \frac{\nu(\nu+1)\left[(z^1)^2 + (z^2)^2\right]}{(\sinh x^+)^2}(dx^+)^2 + (dz^1)^2 + (dz^2)^2, \tag{8.59}$$

both being is isotropic in z^1, z^2.

With the above given results we may now handle the general case. For $\rho \neq 0$ and $|\tilde{x}_1| < 1$ we can use a rotated version of (8.57):

$$\begin{aligned}
x^+ &= -\mathrm{arctanh}(\tilde{x}_1), \\
x^- &= -\tilde{x}_4 - \frac{1}{2}\,\nu\,\tilde{x}_1(\tilde{x}_2^2 + \tilde{x}_3^2), \\
z^1 &= \tilde{x}_2 \cos\Omega - \tilde{x}_3 \sin\Omega, \\
z^2 &= \tilde{x}_2 \sin\Omega + \tilde{x}_3 \cos\Omega,
\end{aligned} \tag{8.60}$$

where $\Omega = \rho\,\ln(\cosh\ x^+)$, to bring the dual metric derived from (8.50) to the form of a plane wave in Brinkmann coordinates

$$\begin{aligned}
ds^2 = &-\left[(z^1)^2 + (z^2)^2\right]\frac{2\nu(\nu+1) + \rho^2(1+\cosh(2x^+))}{2(\cosh x^+)^2}(dx^+)^2 \\
&+ 2dx^+dx^- + (dz^1)^2 + (dz^2)^2.
\end{aligned} \tag{8.61}$$

The torsion becomes constant in these coordinates

$$\widetilde{H} = 2\rho\, dx^+ \wedge dz^1 \wedge dz^2,$$

and the β equations (1.27)–(1.29) result in the ordinary differential equation for the dilaton

$$\widetilde{\Phi}''(x^+) = \frac{2\nu(\nu+1)}{\cosh^2(x^+)},$$

which is, in agreement with (8.52), solved by

$$\widetilde{\Phi}(x^+) = C_0 + C_1\, x^+ + 2\nu(\nu+1)\ln(\cosh x^+).$$

We can see that non-Abelian T-duality based on the semi-Abelian Drinfel'd double given by (8.13) transforms the plane wave metric (8.3) to another plane wave metric (8.61) and torsion.

Analogous results can be obtained for $|\tilde{x}_1| > 1$ using the transformation

$$\begin{aligned} x^+ &= -\mathrm{arccoth}(\tilde{x}_1), \\ x^- &= -\tilde{x}_4 - \frac{1}{2}\,\nu\,\tilde{x}_1(\tilde{x}_2^2 + \tilde{x}_3^2), \\ z^1 &= \tilde{x}_2\cos\Omega - \tilde{x}_3\sin\Omega, \\ z^2 &= \tilde{x}_2\sin\Omega + \tilde{x}_3\cos\Omega, \end{aligned} \tag{8.62}$$

where $\Omega = \rho\,\ln(\sinh\ x^+)$. The transformation (8.62) brings the metric into the form

$$\begin{aligned} ds^2 =& [(z^1)^2 + (z^2)^2]\frac{2\nu(\nu+1) + \rho^2(1-\cosh(2x^+))}{2(\sinh x^+)^2}(dx^+)^2 \\ &+ 2dx^+dx^- + (dz^1)^2 + (dz^2)^2, \end{aligned} \tag{8.63}$$

whereas the torsion and the dilaton read

$$\widetilde{H} = 2\rho\, dx^+ \wedge dz^1 \wedge dz^2, \qquad \widetilde{\Phi}(x^+) = C_0 + C_1\, x^+ + 2\nu(\nu+1)\ln(\sinh x^+).$$

8.5 Conclusions

The results presented in this chapter extend the number of exactly solved sigma models in curved backgrounds. We investigated non-Abelian T-duals of the isotropic homogeneous plane wave metric (8.3) that belongs to the class of string backgrounds given by a metric, B-field and dilaton

$$\begin{aligned} ds^2 &= 2dudv - K_{ab}(u)x^a x^b du^2 + d\vec{x}^2, \\ B &= \frac{1}{2}H_{ab}(u)x^a du \wedge dx^b, \\ \Phi &= \Phi(u), \end{aligned} \tag{8.64}$$

analyzed in Ref. [25]. We have found classical string solutions for the dual backgrounds, and we have also shown that by appropriate coordinate transformations the dual backgrounds can be brought again to the form (8.64). There are less Killing vectors of the dual models than those of (8.3), meaning the dual backgrounds are inevitably different from the initial one.

The dual metrics (8.38) and (8.56), obtained in the group coordinates by the procedure described in section 4.3, have five-dimensional algebras of Killing vectors commuting as the two-dimensional Heisenberg algebra. This suggests that by a change of coordinates they can be rewritten to diagonal Rosen form or Brinkmann form given by (8.43), (8.45) and (8.58), (8.59). By virtue of more general coordinate transformations (8.60) or (8.62) we have found that the metric obtained from the dual tensor (8.50) can be brought to the plane wave form (8.61) or (8.63) even for $\rho \neq 0$. The conformal invariance conditions (1.27)–(1.29) for the dual backgrounds in Brinkmann coordinates acquire simple form of a solvable ODE for the dilaton, cf. Ref. [25]:

$$\Phi''(u) = K_{aa}(u) + H_{ab}(u)H_{ab}(u),$$

and the dual backgrounds satisfy the conformal invariance as well.

The classical string solutions of the dual models, similarly as in the case of the initial metric (8.3), are given in terms of Hankel functions. An interesting point is that the equations of the dual models may not admit the usual light-cone gauge, and the light-cone gauge (8.6) is transformed to the solution (8.33) of the equation (8.34). On the other hand, the backgrounds (8.43), (8.45) and (8.61), (8.63) obtained by coordinate transformations of the dual backgrounds do admit the light-cone gauge. However, in these cases it leads to complicated equations for the transversal components. In order to get the solution, the use of the ansatz (8.44) is crucial.

Chapter 9

Pp-waves as duals of the flat background

Particular cases of sigma models in four-dimensional pp-wave backgrounds obtained from gauged WZW models were given in [62],[63],[28]. In Rosen coordinates are the backgrounds given by

$$ds^2 = dudv + \frac{g_1(u')}{g_1(u')g_2(u) + q^2}\, dx_1^2 + \frac{g_2(u)}{g_1(u')g_2(u) + q^2}\, dx_2^2, \tag{9.1}$$

$$B_{12} = \frac{q}{g_1(u')g_2(u) + q^2},$$

where $u' = au+d$, $a, d, q = const.$, and the functions g_i can take any pair of the following values

$$g(u) = 1, \quad u^2, \quad \tanh^2 u, \quad \tan^2 u, \quad u^{-2}, \quad \coth^2 u, \quad \cot^2 u. \tag{9.2}$$

As mentioned in [62], for $g_1 = 1,\ g_2 = u^2$ this background is dual to the flat space. We shall show that several other cases of these backgrounds are dual to the flat space as well. Moreover, we shall use this fact to find general solutions of classical sigma model field equations in these pp-wave backgrounds.

Therefore, we focus on the flat metric in $D = 4$ dimensions, and give the classification of its non-Abelian T-duals with respect to four-dimensional continuous subgroups of the Poincaré group. We identify majority of the dual models as conformal sigma models in plane wave backgrounds, most of them having torsion. Using suitable coordinate transformations, we give their form in Brinkmann coordinates. Beside pp-waves we also find several curved backgrounds with diagonalizable metrics with nonvanishing scalar curvature that resemble black hole [64] and cosmological [65] solutions. Due to non-Abelian T-duality, we also find general solutions of classical field equations of all the sigma models in terms of d'Alembert solutions of the wave equation.

We understand non-Abelian T-duality [10] as a special case of Poisson–Lie T-duality [11] based on the structure of Drinfel'd double. For technical reasons we shall restrict to four spacetime dimensions, but the discussion can be extended to higher dimension using spectator fields or subgroups of the Poincaré group in higher dimension. Investigation of conformal invariance of pp-waves in higher dimension can be found e.g. in [66],[67].

This chapter follows the paper [46], from which we removed the recap of Poisson–Lie T-duality. On the other hand, we extended the discussion of symmetries and dualization of the flat background in section 9.2, giving also some details of the computation. We start by adding a few comments concerning strings in the pp-wave background. In sections 9.2 and 9.3 we discuss the symmetries of the flat background and the way to dualize it using semi-Abelian Drinfel'd doubles. Detailed discussion of particular examples is given in sections 9.4, 9.5, 9.6. Sections 9.7, 9.8 summarize results of dualization with respect to other subgroups of the Poincaré group.

9.1 Strings in a plane wave background

In this chapter we will be again interested in the special subclass of backgrounds called pp-waves. The metric in Brinkmann coordinates $(u, v, z_3, z_4, \ldots, z_D)$ can be written as

$$ds^2 = 2dudv - K(u, \vec{z})du^2 + d\vec{z}^{\,2}, \tag{9.3}$$

where $d\vec{z}^{\,2}$ is the Euclidean metric in the transversal space in which we introduce coordinates $\vec{z} = (z_3, z_4, \ldots, z_D)$. We again denote the number of transversal coordinates by d, such that $D = 2 + d$. The NS–NS 2-form of particular interest to us has the form

$$B = B_j(u, \vec{z})du \wedge dz_j. \tag{9.4}$$

The metric (9.3) is distinguishable among others due to the fact that it has the covariantly constant null Killing vector $\mathcal{V} = \partial_v$ and particularly simple curvature properties. In section 7.3 we saw that the Ricci tensor has only one nonzero component

$$R_{uu} = \frac{1}{2}(\partial_3^2 K + \partial_4^2 K + \ldots + \partial_D^2 K)$$

and the scalar curvature vanishes.

The one-loop conformal invariance conditions (1.27)–(1.29) for a sigma model can be solved in some special cases. One of them is the model in the background resulting from the Penrose–Güven limit [55, 56], with

$$K(u, \vec{z}) = K_{ij}(u)z_i z_j, \tag{9.5}$$

and torsion

$$H = H_{ij}(u)du \wedge dz_i \wedge dz_j \tag{9.6}$$

that follows from the NS–NS 2-form (9.4) if $B_j(u, \vec{z})$ is linear in the transversal coordinates z_i. The one-loop conformal invariance conditions then simplify to a solvable differential equation for the dilaton $\Phi = \Phi(u)$

$$\Phi''(u) - K_{jj}(u) + \frac{1}{4}H_{ij}(u)H_{ij}(u) = 0. \tag{9.7}$$

We are going to show that sigma models in pp-wave backgrounds with special forms of functions K_{ij} in (9.5) and H_{ij} in (9.6) can be obtained as non-Abelian T-duals of sigma models in the flat background. As we have already noted, if $K(u,\vec{z})$ in (9.3) is at most quadratic in the transversal coordinates, one can find a transformation that brings it to the form (9.5). In the following, we will be able to bring the metrics of the resulting dual models with vanishing scalar curvature to the form (9.3), where

$$K(u,\vec{z}) = K_{ij}(u)z_i z_j = K_3(u)z_3^2 + K_4(u)z_4^2, \tag{9.8}$$

and the torsion is

$$H = H(u)\, du \wedge dz_3 \wedge dz_4.$$

We have met the classical field equations of a sigma model in such a plane wave background, and we know that they are most easily found directly by varying the sigma model action. The equation for $U(\tau,\sigma)$

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) U = 0, \tag{9.9}$$

allows us to introduce the lightcone gauge

$$U(\tau,\sigma) = \kappa\tau, \qquad \kappa := 2\alpha' p^u.$$

In the previous chapters we noted that the equations for the transversal coordinates separate when $K_{ab}(u)$ is diagonal as in (9.8). However, this property is spoiled when the torsion is not trivial, and until the lightcone gauge is used, the field equations read

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) Z_3 = K_3(U)\left[(\partial_\sigma U)^2 - (\partial_\tau U)^2\right] Z_3 - H(U)\left[\partial_\sigma Z_4\, \partial_\tau U - \partial_\tau Z_4\, \partial_\sigma U\right], \tag{9.10}$$

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) Z_4 = K_4(U)\left[(\partial_\sigma U)^2 - (\partial_\tau U)^2\right] Z_4 + H(U)\left[\partial_\sigma Z_3\, \partial_\tau U - \partial_\tau Z_3\, \partial_\sigma U\right], \tag{9.11}$$

$$\begin{aligned}\left(\partial_\tau^2 - \partial_\sigma^2\right) V = {} & H(U)\left[\partial_\sigma Z_4 \partial_\tau Z_3 - \partial_\sigma Z_3 \partial_\tau Z_4\right] \\ & + \sum_{j=3}^{4}\Big\{2K_j(U)Z_j\left[\partial_\tau Z_j\, \partial_\tau U - \partial_\sigma Z_j\, \partial_\sigma U\right] \\ & + (Z_j)^2\left[\frac{1}{2}K_j'(U)\left[(\partial_\tau U)^2 - (\partial_\sigma U)^2\right] + K_j(U)\left(\partial_\tau^2 U - \partial_\sigma^2 U\right)\right]\Big\}.\end{aligned}$$

For string backgrounds the last equation can be replaced by the conformal invariance conditions for vanishing of the two-dimensional energy-momentum tensor (1.19) and (1.20). For the plane wave with the function $K(u,\vec{z})$ given by (9.8) these conditions yield

$$2\partial_\tau U\, \partial_\tau V + \sum_{j=3}^{4}\left\{(\partial_\tau Z_j)^2 - (\partial_\tau U)^2 K_j(U)(Z_j)^2\right\} + (\tau \to \sigma) = 0,$$

$$\partial_\tau U \partial_\sigma V + \partial_\tau V \partial_\sigma U + \sum_{j=3}^{4}\left\{\partial_\tau Z_j \partial_\sigma Z_j - \partial_\tau U \partial_\sigma U K_j(U)(Z_j)^2\right\} = 0.$$

Compatibility of these two first order equations for $V = V(\tau, \sigma)$ is guaranteed by the equations (9.9)–(9.11).

For non-vanishing torsion both Z_3 and Z_4 appear in (9.10) and (9.11), so even in the light-cone gauge $U = \kappa\tau$ these equations do not separate, and it can be rather difficult to solve them the usual way using Fourier mode expansion. Nevertheless, T-duality offers a method to obtain the general solution.

9.2 Symmetries of the flat background

We are interested in dualization of the sigma model with a four-dimensional target space equipped solely with Minkowski metric. We discussed such a theory in the very beginning of our thesis in section 1.1, where the classical field equations (1.5) and the conformal invariance conditions (1.19), (1.20) were solved using the mode expansions (1.10) for closed strings or (1.12) for open strings. However, the knowledge of classical solutions of equations of motion in the flat Minkowski background allows us in principle to derive solutions of equations of motion of dual models. Such solutions are quite valuable if these duals prove to be physically interesting.

Killing vectors of the four-dimensional Minkowski metric form the well-known 10-dimensional Poincaré algebra. In order to perform the Poisson–Lie T-duality transformation of the flat metric, whose standard form in coordinates (t, x, y, z) is

$$\eta = \mathrm{diag}(-1, 1, 1, 1), \tag{9.12}$$

we need to find subalgebras of the Poincaré Lie algebra spanned by Killing vectors

$$P_0 = \partial_t, \qquad P_j = \partial_j, \qquad L_j = -\varepsilon_{ijk} x^j \partial_k, \qquad M_j = -x^j \partial_t - t\partial_j \tag{9.13}$$

representing translations, rotations and boosts. All the subalgebras were classified in Ref. [68] into classes of equivalence under proper ortochronous Poincaré transformations. We shall use these results in the following.

In particular, to apply the atomic non-Abelian T-duality on the sigma model in the flat background, we shall use four-dimensional subalgebras of the Poincaré Lie algebra classified in [68], Table IV. According to the classification, there are 16 types of non-isomorphic four-dimensional subalgebras, which further divide into 35 equivalence classes. Out of the 35 non-equivalent subalgebras we need to pick those, whose group acts freely and transitively on the target manifold.

The requirement that a subgroup acts transitively can be reformulated into infinitesimal version by saying that the corresponding four-dimensional subalgebra of Killing vectors generated by $\mathfrak{X}_i,\ i = 1, \ldots, 4$, forms a four-dimensional vector distribution. The basis of the subalgebra then forms the basis of the tangent space $T_p\mathscr{M}$ at every point p of the target manifold, and the vector fields ∂_t, ∂_x, ∂_y, ∂_z corresponding to coordinates (t, x, y, z) are related to the basis of the particular subalgebra $\mathfrak{X}_i$ by an invertible matrix.

The action of the subgroup has to be free as well, meaning that if there is any point $p \in \mathscr{M}$ in which the action generated by a vector of the corresponding Lie algebra vanishes, then the vector has to be the null vector. In other words, the four vectors which we choose for the basis of the Lie algebra have to be independent in any point of the manifold $\mathscr{M}$.

We see that both conditions on the action of the subgroup can be checked by the same criterion, namely by checking the regularity of the matrix relating $\partial_t, \partial_x, \partial_y, \partial_z$ and the basis $\mathfrak{X}_i$. For example, the group corresponding to the last subalgebra in [68], Table IV, given as $S_{35} = Span[L_3, M_3, L_2 + M_1, L_1 - M_2]$, does not act freely and transitively in any point of $\mathscr{M}$ since

$$\begin{pmatrix} L_3 \\ M_3 \\ L_2 + M_1 \\ L_1 - M_2 \end{pmatrix} = \begin{pmatrix} 0 & y & -x & 0 \\ -z & 0 & 0 & -t \\ -x & -t-z & 0 & x \\ y & 0 & t+z & -y \end{pmatrix} \cdot \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

The matrix of components of the vectors $\mathfrak{X}_i$ is singular in any point, and vectors $L_3, M_3, L_2 + M_1$ and $L_1 - M_2$ are not linearly independent. On the other hand, for the subalgebras S_1, S_6, S_{11}, S_{17}, S_{18}, S_{19}, S_{23} and S_{25} the groups do act freely and transitively in each point of the target manifold provided the parameters α, β, ϵ appearing in the classification are not zero. Moreover, subalgebras S_2, S_7, S_8, S_{26}, S_{27}, S_{28}, S_{29}, S_{31} and S_{33} satisfy both conditions away from points $t + z = 0$. For example, for $S_2 = Span[M_3, P_0 - P_3, P_1, P_2]$ one receives

$$\begin{pmatrix} M_3 \\ P_0 - P_3 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} -z & 0 & 0 & -t \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix},$$

which is clearly degenerate for $t + z = 0$. We shall avoid these fixed points of the action, but comment on them later when they reappear as the points where coordinate transformations realizing the identification $\mathscr{M} \approx \mathscr{G}$ are ill-defined.

Let us now summarize all four-dimensional Poincaré subalgebras whose groups act freely and transitively on the flat manifold. Numbering of the subalgebras follows from the order introduced in [68], Table IV.

$$\begin{aligned}
S_1 &= Span[P_0, P_1, P_2, P_3], \\
S_2 &= Span[M_3, P_0 - P_3, P_1, P_2], \\
S_6 &= Span[L_2 + M_1 - \frac{1}{2}(P_0 + P_3), P_1, P_0 - P_3, P_2], \\
S_7 &= Span[2M_3 + \alpha\, P_1, L_2 + M_1, P_0 - P_3, P_2], && \alpha > 0, \\
S_8 &= Span[M_3, L_2 + M_1, P_0 - P_3, P_2], \\
S_{11} &= Span[M_3 + \alpha\, P_2, P_0, P_3, P_1], && \alpha > 0, \\
S_{17} &= Span[L_3 + \epsilon\,(P_0 + P_3), P_1, P_2, (P_0 - P_3)], && \epsilon = \pm 1, \\
S_{18} &= Span[L_3 + \alpha\, P_0, P_1, P_2, P_3], && \alpha > 0, \\
S_{19} &= Span[L_3 + \alpha\, P_3, P_1, P_2, P_0], && \alpha \neq 0,
\end{aligned}$$

$$
\begin{aligned}
S_{23} &= Span[L_2 + M_1 - \frac{1}{2}(P_0 + P_3), L_1 - M_2 + \alpha\, P_1, P_0 - P_3, P_2], && \alpha \neq 0,\\
S_{25} &= Span[L_2 + M_1 - \epsilon P_2, P_0 + P_3, P_1, P_0 - P_3], && \epsilon = \pm 1,\\
S_{26} &= Span[M_3 + \alpha\, P_1, L_2 + M_1, L_1 - M_2, P_0 - P_3], && \alpha > 0,\\
S_{27} &= Span[M_3, L_2 + M_1, L_1 - M_2, P_0 - P_3], &&\\
S_{28} &= Span[L_3 - \beta\, M_3, L_2 + M_1, L_1 - M_2, P_0 - P_3], && \beta \neq 0,\\
S_{29} &= Span[L_3 - \beta\, M_3, P_0 - P_3, P_1, P_2], && \beta \neq 0,\\
S_{31} &= Span[M_3, P_1 + \beta\, P_2, P_0 - P_3, L_2 + M_1], && \beta \neq 0,\\
S_{33} &= Span[M_3 + \alpha\, P_2, P_1 + \beta\, P_2, P_0 - P_3, L_2 + M_1], \qquad \alpha > 0, && \beta \neq 0.
\end{aligned}
$$

9.3 Dual backgrounds and solution of their equations

Our goal is to dualize the sigma model in the flat background with respect to the 17 subgroups of the isometry group given above. To employ the atomic non-Abelian T-duality, we choose the Drinfel'd double as the semidirect product $\mathscr{G} \ltimes \widetilde{\mathscr{G}}$, where the group $\mathscr{G}$ is taken to be one of the four-dimensional subgroups mentioned in section 9.2. The group $\widetilde{\mathscr{G}}$ is chosen Abelian in order to satisfy the condition of dualizability (4.14).

With the semi-Abelian Drinfel'd double the tensor F specifying dynamics of the sigma model on $\mathscr{G}$ is symmetric, and can be written as

$$F_{\mu\nu}(x) = G_{\mu\nu}(x) = e_\mu^a(g) E(e)_{ab} e_\nu^b(g) \tag{9.14}$$

since $\Pi(g)$ in (4.37) vanishes. The metric and the torsion potential of the T-dual model are computed from the tensor $\widetilde{F}$

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = [E(e) + \widetilde{\Pi}(\tilde{g})]^{-1}, \tag{9.15}$$

where the matrix $\widetilde{\Pi}$ is given by the adjoint representation of the Abelian subgroup $\widetilde{\mathscr{G}}$ on the Lie algebra $\mathfrak{d}$ of the Drinfel'd double. Using mutually dual bases $T_a, \widetilde{T}^b$, we receive it as

$$^{\mathcal{X}}(Ad_{\tilde{g}^{-1}})^T = \begin{pmatrix} \mathbf{1} & 0 \\ \widetilde{\Pi}(\tilde{g}) & \mathbf{1} \end{pmatrix}.$$

The relation between the solution $X^\mu(\tau,\sigma)$ of the equations of motion of the sigma model given by F and the solution $\widetilde{X}^\mu(\tau,\sigma) := \tilde{x}^\mu(\tilde{g}(\tau,\sigma))$ of the sigma model given by $\widetilde{F}$ follows from two possible decompositions of elements l of the Drinfel'd double

$$g(\tau,\sigma)\tilde{h}(\tau,\sigma) = \tilde{g}(\tau,\sigma)h(\tau,\sigma),$$

where $g, h \in \mathscr{G}$, $\tilde{g}, \tilde{h} \in \widetilde{\mathscr{G}}$ and the mapping $\tilde{h}(\tau,\sigma)$ satisfies the equations (4.30), (4.31), or more exactly equations (8.17), (8.18) that in the absence of the B-field read

$$\partial_\tau \tilde{h}_j = -v_j^\lambda G_{\lambda\nu} \partial_\sigma X^\nu, \tag{9.16}$$
$$\partial_\sigma \tilde{h}_j = -v_j^\lambda G_{\lambda\nu} \partial_\tau X^\nu. \tag{9.17}$$

We recap the three steps that have to be done in order to use Poisson–Lie T-duality to find the solution of the dual model:

- Step 1: One has to find the solution $X^\mu(\tau, \sigma)$ of the sigma model given by $F_{\mu\nu}(x)$.
- Step 2: Given $X^\mu(\tau, \sigma)$, one has to find $\tilde{h}(\tau, \sigma)$, i.e. solve the system of PDEs (9.16), (9.17).
- Step 3: Given $l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma) \in \mathscr{D}$, one has to find the dual decomposition $l(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma)$, where $\tilde{g}(\tau, \sigma) \in \widetilde{\mathscr{G}}$, $h(\tau, \sigma) \in \mathscr{G}$. Functions $\widetilde{X}^\mu(\tau, \sigma) := \tilde{x}^\mu(\tilde{g}(\tau, \sigma))$ then solve the field equations of the dual sigma model.

As for the first step, it is easy to find solutions of equations following from the flat metric (9.12) in coordinates $x^I \in \{t, x, y, z\}$. The Christoffel symbols vanish, and the Euler–Lagrange equations (1.18) reduce to two-dimensional wave equations (1.5). Their solution was given in terms of left- and right-moving fields (1.8) that were otherwise arbitrary. For the sake of clarity, we denote the solution here simply as $W^J(\tau, \sigma)$, with the functions W^J satisfying

$$\partial_\tau^2 W^J - \partial_\sigma^2 W^J = 0, \qquad J = t, x, y, z. \tag{9.18}$$

However, we need to identify the group $\mathscr{G}$ with the manifold, i.e. find the appropriate coordinate transformation between (t, x, y, z) and the coordinates parametrizing the group. Choosing the parametrization of group elements as

$$g = g(x^\mu) = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, \tag{9.19}$$

where T_j form the basis of the Lie algebra of the group, we may calculate the algebra of left-invariant vector fields

$$v_j = v_j^\mu \frac{\partial}{\partial x^\mu}, \quad j = 1, \ldots, 4,$$

and compare it with the particular four-dimensional subalgebra of Killing vectors $\mathcal{K}_i$ of the flat metric in coordinates (t, x, y, z). To accomplish this goal, we must first identify which Killing vector shall be mapped to which left-invariant field, so that they generate the same Lie algebra. The transformation properties of vector fields then result in the relation

$$v_j^\mu = J_I^\mu \mathcal{K}_j^I$$

between the components of the j-th left-invariant field v_j^μ and the components of j-th Killing vector $\mathcal{K}_j^I$. Here J_I^μ stands for the Jacobi matrix

$$J_I^\mu = \frac{\partial x^\mu}{\partial x^I}$$

of the transformation. The comparison then gives the coordinate transformation $x^\mu = x^\mu(t, x, y, z)$ as a solution of a set of PDEs. Four integration constants appearing in the solution express the freedom of choice of $e \in \mathscr{G}$ in $\mathscr{M}$, and we can choose them arbitrarily as long as the transformation of coordinates is well defined.

For example, for the group corresponding to S_2 the appropriate transformation is found to be

$$\begin{aligned} t &= e^{x_1}C_1 + e^{-x_1}C_4 + x_2, & x &= C_2 + x_3, \\ z &= -e^{x_1}C_1 + e^{-x_1}C_4 - x_2, & y &= C_3 + x_4. \end{aligned}$$

The metric in coordinates x^μ reads

$$G(x^\mu) = \begin{pmatrix} 4C_1C_4 & 2e^{-x^1}C_4 & 0 & 0 \\ 2e^{-x^1}C_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and fixing $x^\mu = 0$ in $G(x^\mu)$, we obtain the matrix

$$E(e) = \begin{pmatrix} 4C_1C_4 & 2C_4 & 0 & 0 \\ 2C_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The constants C_1, C_2, C_3 can be chosen arbitrarily. However, it is necessary that $C_4 \neq 0$, otherwise the transformation would be ill-defined and the resulting tensor would be degenerate. We can see that the case $C_4 = 0$ corresponds to $t + z = 0$, where the action of the group corresponding to S_2 is neither free nor transitive. The dual tensor $\widetilde{F}$ resulting from (9.15) obviously depends on C_i. Nevertheless, explicit calculations show that the dependence can be eliminated by a coordinate transformation, so we can fix $C_1 = C_2 = C_3 = 0$ and $C_4 = 1$ before even computing the dual background. Since the same approach can be applied to other subgroups as well, we mention only coordinate transformations and matrices $E(e)$ where the choice of constants have already been made in order to simplify all expressions as much as possible.

The right-hand sides of the PDEs (9.16), (9.17), solved in step 2, are invariant with respect to coordinate transformations, so we can express them in terms of coordinates (t, x, y, z), and use the Killing fields $\mathcal{K}_j$ instead of the left-invariant fields on $\mathscr{G}$. The equations (9.16), (9.17) then acquire the form

$$\partial_\tau \tilde{h}_j = -\mathcal{K}_j^I \eta_{IJ} \partial_\sigma W^J, \tag{9.20}$$

$$\partial_\sigma \tilde{h}_j = -\mathcal{K}_j^I \eta_{IJ} \partial_\tau W^J, \tag{9.21}$$

where W^J are solutions of the wave equations (9.18). Solution of these equations will be demonstrated several times in the following sections.

In general, step 3 represents rather complicated problem related to the Baker–Campbell–Hausdorff formula. If the adjoint representation of the Lie algebra $\mathfrak{g}$ is faithful, we may circumvent the problem using the representation r of an element of the semi-Abelian Drinfel'd double introduced in section 8.3 in the form of block matrices

$$r(g) = \begin{pmatrix} Ad_g & 0 \\ 0 & 1 \end{pmatrix}, \quad r(\tilde{h}) = \begin{pmatrix} \mathbf{1} & 0 \\ v(\tilde{h}) & 1 \end{pmatrix},$$

where $v(\tilde{h}) = (\tilde{h}_1, \ldots, \tilde{h}_{\dim \mathfrak{g}})$. Using the parametrizations

$$g = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, \qquad \tilde{h} = e^{\tilde{h}_1 \widetilde{T}^1} e^{\tilde{h}_2 \widetilde{T}^2} e^{\tilde{h}_3 \widetilde{T}^3} e^{\tilde{h}_4 \widetilde{T}^4},$$
$$\tilde{g} = e^{\tilde{x}_1 \widetilde{T}^1} e^{\tilde{x}_2 \widetilde{T}^2} e^{\tilde{x}_3 \widetilde{T}^3} e^{\tilde{x}_4 \widetilde{T}^4}, \qquad h = e^{h^1 T_1} e^{h^2 T_2} e^{h^3 T_3} e^{h^4 T_4}$$

of group elements in the decompositions

$$l = g\tilde{h} = \tilde{g}h, \quad g, h \in \mathscr{G}, \quad \tilde{g}, \tilde{h} \in \widetilde{\mathscr{G}},$$

the variables $\tilde{x}_j, h^k$ can be expressed in terms of $x^j, \tilde{h}_k$ from a set of equations following from

$$r(l) = r(g\tilde{h}) = \begin{pmatrix} Ad_g & 0 \\ v(\tilde{h}) & 1 \end{pmatrix} = r(\tilde{g}h) = \begin{pmatrix} Ad_h & 0 \\ v(\tilde{g}) \cdot (Ad_h) & 1 \end{pmatrix}.$$

Note that the representation r of the Drinfel'd double is faithful if and only if the adjoint representation of the Lie algebra $\mathfrak{g}$ is faithful. Otherwise the system of equations for $\tilde{x}_j$ and h^k does not receive all the contributions. In such a case, we have to employ the BCH formula.

The application of BCH formula is tedious, yet under certain circumstances it simplifies a lot. One particular instance occurs when the elements of the Lie algebra $X, Y \in \mathfrak{d}$ satisfy

$$[X, [X, Y]] = [Y, [X, Y]] = 0.$$

Then the infinite series in the formula ends right after the third term, and we have

$$e^X e^Y = e^Y e^X e^{[X,Y]}.$$

Another tractable case occurs when the series does not end, but

$$[X, Y] = aY$$

for some non-zero constant a. Then the series can be summed to

$$e^X e^Y = e^{\exp(a) Y} e^X. \tag{9.22}$$

Fortunately, for all of the subalgebras encountered in the rest of the chapter one of the above discussed methods can be applied to permute the elements of $\mathscr{G}$ and $\widetilde{\mathscr{G}}$. To express the coordinates $\tilde{x}_j, h^k$ in terms of $x^j, \tilde{h}_k$, we use either the matrix representation r or the special instances of the BCH formula.

In the following sections we shall apply the above given three steps of the Poisson–Lie transformation to solve the sigma model field equations in curved backgrounds dual to the flat metric.

9.4 Example 1 – subalgebra S_{27}

We shall illustrate the above described methods of non-Abelian dualization of the flat metric in detail on the example of Killing vectors

$$\begin{aligned}
\mathcal{K}_1 &= M_3 = -z\partial_t - t\partial_z, \\
\mathcal{K}_2 &= L_2 + M_1 = -x\partial_t - (t+z)\partial_x + x\partial_z, \\
\mathcal{K}_3 &= L_1 - M_2 = y\partial_t + (t+z)\partial_y - y\partial_z, \\
\mathcal{K}_4 &= P_0 - P_3 = \partial_t - \partial_z
\end{aligned} \tag{9.23}$$

that span the subalgebra S_{27} (see section 9.2). Their non-vanishing commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = -\mathcal{K}_2, \qquad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4. \tag{9.24}$$

Duals of the flat metric

Using the parametrization (9.19) of the isometry subgroup $\mathscr{G}$, where T_a are elements of its Lie algebra commuting as in (9.24), we get the basis of left-invariant fields on $\mathscr{G}$ as

$$v_1 = \partial_1 + x^2\partial_2 + x^3\partial_3 + x^4\partial_4,$$

$$v_2 = \partial_2, \qquad v_3 = \partial_3, \qquad v_4 = \partial_4.$$

Identifying the Killing vectors (9.23) with these left-invariant fields, we get a transformation of coordinates on the flat manifold given by

$$\begin{aligned}
t &= \frac{1}{2}e^{-x^1}\left((x^2)^2 + (x^3)^2 + 1\right) + x^4, & x &= -e^{-x^1}x^2, \\
z &= -\frac{1}{2}e^{-x^1}\left((x^2)^2 + (x^3)^2 - 1\right) - x^4, & y &= e^{-x^1}x^3.
\end{aligned} \tag{9.25}$$

This gives the flat metric in the group coordinates x^μ as

$$G_{\mu\nu}(x) = F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & e^{-x^1} \\ 0 & e^{-2x^1} & 0 & 0 \\ 0 & 0 & e^{-2x^1} & 0 \\ e^{-x^1} & 0 & 0 & 0 \end{pmatrix}. \tag{9.26}$$

Setting $x^1 = 0$, we get the value of $F_{\mu\nu}$ in the unit of the group. Hence, $F_{\mu\nu}$ can be obtained from the formula (9.14) if one chooses

$$E(e) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The dual tensor $\widetilde{F}$ can be found from the formula (9.15) as

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1-\tilde{x}_4} \\ 0 & 1 & 0 & \frac{\tilde{x}_2}{1-\tilde{x}_4} \\ 0 & 0 & 1 & \frac{\tilde{x}_3}{1-\tilde{x}_4} \\ \frac{1}{\tilde{x}_4+1} & -\frac{\tilde{x}_2}{\tilde{x}_4+1} & -\frac{\tilde{x}_3}{\tilde{x}_4+1} & \frac{\tilde{x}_2^2+\tilde{x}_3^2}{\tilde{x}_4^2-1} \end{pmatrix}.$$

The scalar curvature corresponding to the metric obtained from the symmetric part of this tensor vanishes, and the Ricci tensor has only one non-vanishing component

$$\widetilde{R}_{44} = -\frac{4}{(\tilde{x}_4^2-1)}.$$

This suggests that the dual metric could be of the plane wave form. Indeed, the transformation of coordinates suitable for $|\tilde{x}_4| > 1$

$$\begin{aligned} \tilde{x}_1 &= v - \frac{1}{2}\left(z_3^2+z_4^2\right)\coth(u), & \tilde{x}_2 &= z_3, \\ \tilde{x}_4 &= \coth(u), & \tilde{x}_3 &= z_4, \end{aligned} \tag{9.27}$$

brings the components of the tensor $\widetilde{F}$ into the form

$$\widetilde{F} = \begin{pmatrix} 2\,\frac{z_3^2+z_4^2}{\sinh^2(u)} & 1-\coth(u) & \frac{z_3}{\sinh^2(u)} & \frac{z_4}{\sinh^2(u)} \\ 1+\coth(u) & 0 & 0 & 0 \\ -\frac{z_3}{\sinh^2(u)} & 0 & 1 & 0 \\ -\frac{z_4}{\sinh^2(u)} & 0 & 0 & 1 \end{pmatrix}. \tag{9.28}$$

The symmetric part yields a plane wave in Brinkmann form with the corresponding line element

$$ds^2 = 2dudv + 2\,\frac{z_3^2+z_4^2}{\sinh^2(u)}du^2 + dz_3^2 + dz_4^2. \tag{9.29}$$

The torsion obtained from the antisymmetric part vanishes, and the dilaton obtained as a solution of the equation (9.7) acquires rather simple form

$$\Phi(u) = c_2 + c_1\,u + 4\ln\left(\sinh(u)\right),$$

where c_1 and c_2 are arbitrary constants.

For $|\tilde{x}_4| < 1$ the transformation to Brinkmann coordinates is slightly different, and

$$\begin{aligned} \tilde{x}_1 &= v - \frac{1}{2}\left(z_3^2+z_4^2\right)\tanh(u), & \tilde{x}_2 &= z_3, \\ \tilde{x}_4 &= \tanh(u), & \tilde{x}_3 &= z_4, \end{aligned}$$

brings the tensor $\widetilde{F}$ into the form

$$\widetilde{F} = \begin{pmatrix} -2\,\frac{z_3^2+z_4^2}{\cosh^2(u)} & 1-\tanh(u) & -\frac{z_3}{\cosh^2(u)} & -\frac{z_4}{\cosh^2(u)} \\ 1+\tanh(u) & 0 & 0 & 0 \\ \frac{z_3}{\cosh^2(u)} & 0 & 1 & 0 \\ \frac{z_4}{\cosh^2(u)} & 0 & 0 & 1 \end{pmatrix}, \tag{9.30}$$

giving the plane wave metric

$$ds^2 = 2dudv - 2\,\frac{z_3^2 + z_4^2}{\cosh^2(u)}du^2 + dz_3^2 + dz_4^2. \tag{9.31}$$

The torsion again vanishes, and the dilaton has the form

$$\Phi(u) = c_2 + c_1\,u + 4\ln\left(\cosh(u)\right).$$

We have learned that non-Abelian T-duality with respect to the group corresponding to S_{27} produces two types of sigma models in plane wave backgrounds, one of them singular and the other regular. As we shall see, this result is obtained also from dualization with respect to several other subgroups of the Poincaré group. Both plane wave metrics have 6 dimensional algebras of Killing vectors. The symmetry group of $\widetilde{F}$, however, is only 4 dimensional. Yet, this still allows the background to be further dualized.

The solution of the classical equations of the dual sigma model

Our next goal is to write down the general solution of classical field equations in the backgrounds (9.28) and (9.30). As their torsions vanish, the antisymmetric parts do not contribute to the classical field equations and the β equations are also not affected by the B-field. The Lagrangian for the metric (9.29) can be written in the form (cf. (4.2))

$$\begin{aligned} L = &\left[\frac{Z_3^2 + Z_4^2}{\sinh^2(U)}(\partial_\sigma U)^2 + \partial_\sigma U\,\partial_\sigma V + \frac{1}{2}(\partial_\sigma Z_3)^2 + \frac{1}{2}(\partial_\sigma Z_4)^2\right] \\ &- \left[\frac{Z_3^2 + Z_4^2}{\sinh^2(U)}(\partial_\tau U)^2 + \partial_\tau U\,\partial_\tau V + \frac{1}{2}(\partial_\tau Z_3)^2 + \frac{1}{2}(\partial_\tau Z_4)^2\right]. \end{aligned}$$

As always, the field equation for $U(\tau,\sigma)$ is

$$\partial_\tau^2 U - \partial_\sigma^2 U = 0, \tag{9.32}$$

while the other equations read

$$\partial_\tau^2 Z_3 - \partial_\sigma^2 Z_3 = -\,2\left((\partial_\sigma U)^2 - (\partial_\tau U)^2\right)\frac{Z_3}{\sinh^2(U)}, \tag{9.33}$$

$$\partial_\tau^2 Z_4 - \partial_\sigma^2 Z_4 = -\,2\left((\partial_\sigma U)^2 - (\partial_\tau U)^2\right)\frac{Z_4}{\sinh^2(U)}, \tag{9.34}$$

$$\begin{aligned} \partial_\tau^2 V - \partial_\sigma^2 V = {} & 4\,\mathrm{csch}^2(U)\left[Z_3\,\left(\partial_\sigma U\,\partial_\sigma Z_3 - \partial_\tau U\,\partial_\tau Z_3\right)\right. \\ & \left. + Z_4\left(\partial_\sigma U\,\partial_\sigma Z_4 - \partial_\tau U\,\partial_\tau Z_4\right)\right] \\ & - 2\,\mathrm{csch}^3(U)\left[(Z_3)^2 + (Z_4)^2\right]\left[(-\partial_\sigma^2 U + \partial_\tau^2 U)\sinh(U)\right. \\ & \left. + ((\partial_\sigma U)^2 - (\partial_\tau U)^2)\cosh(U)\right]. \end{aligned} \tag{9.35}$$

To solve these field equations, we can follow steps 1–3 described in detail in section 9.3.

Step 1 starts with solution of the field equations in the flat background. In the coordinates (t, x, y, z) they are of the form (9.18) solved by

$$W^I(\tau,\sigma) = X_R^I(\sigma_-) + X_L^I(\sigma_+), \qquad I = t, x, y, z,$$

with X_R^I, X_L^I arbitrary functions. Subsequent transformation of this solution to the coordinates x^μ by formulas (9.25) produces functions

$$\begin{aligned} X^1(\tau,\sigma) &= -\ln(W^t + W^z), & X^2(\tau,\sigma) &= -\frac{W^x}{W^t + W^z}, \\ X^4(\tau,\sigma) &= \frac{(W^t)^2 - (W^x)^2 - (W^y)^2 - (W^z)^2}{2(W^t + W^z)}, & X^3(\tau,\sigma) &= \frac{W^y}{W^t + W^z} \end{aligned}$$

that solve the field equations of the sigma model living in the flat background in the coordinates x^μ, i.e. equations following from the metric (9.26).

Next, we have to perform step 2, consisting in solution of the PDEs (9.20), (9.21), with Killing fields (9.23) on the right-hand sides. The equations (9.20) in this case read

$$\begin{aligned} \partial_\tau \tilde{h}_1 =&\ W^t \partial_\sigma W^z - W^z \partial_\sigma W^t, \\ \partial_\tau \tilde{h}_2 =&\ -W^x \left(\partial_\sigma W^t + \partial_\sigma W^z\right) + (W^t + W^z)\partial_\sigma W^x, \\ \partial_\tau \tilde{h}_3 =&\ W^y \left(\partial_\sigma W^t + \partial_\sigma W^z\right) - (W^t + W^z)\partial_\sigma W^y, \\ \partial_\tau \tilde{h}_4 =&\ \partial_\sigma W^t + \partial_\sigma W^z, \end{aligned}$$

while the equations (9.21) are obtained by the exchange $\tau \leftrightarrow \sigma$. Compatibility of these two sets of PDEs is guaranteed by the wave equations for W^I. Their solution can be given in terms of the arbitrary functions W^I as

$$\begin{aligned} \tilde{h}_1(\tau,\sigma) &= \gamma_1 + \int \left(W^t \partial_\sigma W^z - W^z \partial_\sigma W^t\right) d\tau, \\ \tilde{h}_2(\tau,\sigma) &= \gamma_2 - \int \left(W^x \left(\partial_\sigma W^t + \partial_\sigma W^z\right) - (W^t + W^z)\partial_\sigma W^x\right) d\tau, \\ \tilde{h}_3(\tau,\sigma) &= \gamma_3 + \int \left(W^y \left(\partial_\sigma W^t + \partial_\sigma W^z\right) - (W^t + W^z)\partial_\sigma W^y\right) d\tau, \\ \tilde{h}_4(\tau,\sigma) &= \gamma_4 + \int \left(\partial_\sigma W^t + \partial_\sigma W^z\right) d\tau, \end{aligned} \tag{9.36}$$

where $\gamma_1, \ldots, \gamma_4$ are arbitrary integration constants.

To get the solution of field equations (9.32)–(9.35), we have to carry out step 3, that is we have to pass from one decomposition of the element of the Drinfel'd double to the dual one. One can easily check that the adjoint representation of the algebra (9.24) is faithful, so we can use the equation (8.28) to express the coordinates $\tilde{x}_\mu$ in terms of x^ν and $\tilde{h}_k$. We get

$$\tilde{x}_1 = \tilde{h}_1 - x^2\tilde{h}_2 - x^3\tilde{h}_3 - x^4\tilde{h}_4, \tag{9.37}$$

$$\tilde{x}_2 = e^{x^1}\tilde{h}_2, \qquad \tilde{x}_3 = e^{x^1}\tilde{h}_3, \qquad \tilde{x}_4 = e^{x^1}\tilde{h}_4.$$

Finally, we have to transform the coordinates $\tilde{x}_\mu$ into Brinkmann's. Composing the inverse of (9.25), (9.37) and the inverse of (9.27), we obtain Brinkmann coordinates (u, v, z_3, z_4) on $\widetilde{\mathscr{G}}$ as functions of the spacetime coordinates (t, x, y, z) on the initial flat manifold and coordinates $\tilde{h}_j$ on the subgroup $\widetilde{\mathscr{G}}$ of the Drinfel'd double:

$$u = \operatorname{arccoth}\left(\frac{\tilde{h}_4}{t+z}\right), \qquad z_3 = \frac{\tilde{h}_2}{t+z}, \qquad z_4 = \frac{\tilde{h}_3}{t+z}, \tag{9.38}$$

$$v = \frac{\left(2\tilde{h}_2 x - 2\tilde{h}_3 y + 2\tilde{h}_1(t+z) + \tilde{h}_4\left(-t^2+x^2+y^2+z^2\right)\right)}{2(t+z)} + \frac{\tilde{h}_4\tilde{h}_2^2 + \tilde{h}_3^2\tilde{h}_4}{2(t+z)^3}.$$

To get the general solution of the classical field equations (9.32)–(9.35) in the curved background with the metric (9.29), we just have to replace the coordinates (t, x, y, z) in (9.38) by solutions $W^I = W^I(\tau, \sigma)$ of the wave equations (9.18) and $\tilde{h}_\mu$ by the solutions (9.36) of the PDEs (9.20), (9.21). In the end of our calculations we have

$$U(\tau,\sigma) = \operatorname{arccoth}\left(\frac{\tilde{h}_4(\tau,\sigma)}{W^t(\tau,\sigma)+W^z(\tau,\sigma)}\right), \tag{9.39}$$

$$Z_3(\tau,\sigma) = \frac{\tilde{h}_2(\tau,\sigma)}{W^t(\tau,\sigma)+W^z(\tau,\sigma)}, \qquad Z_4(\tau,\sigma) = \frac{\tilde{h}_3(\tau,\sigma)}{W^t(\tau,\sigma)+W^z(\tau,\sigma)}.$$

The expression for the function $V(\tau, \sigma)$ is rather extensive, but can be easily read out of (9.38).

It is interesting to see how the string-type solutions emerge from the general solution of the field equations. To adopt the light-cone gauge and obtain the standard mode expansions (see e.g. [24],[45])

$$U(\tau,\sigma) = \kappa\tau, \qquad Z_a(\tau,\sigma) = \sum_{n=-\infty}^{\infty} Z_a^n(\tau) e^{2in\sigma}, \quad a = 3, 4, \tag{9.40}$$

one has to choose

$$W^t(\tau,\sigma) + W^z(\tau,\sigma) = e^{\kappa\sigma}\sinh(\kappa\tau),$$

and

$$W^x(\tau,\sigma) = \sinh(\kappa\tau) \sum_{n=-\infty}^{\infty} e^{2in\sigma}(2i\,n+\kappa) \int Z_3^n(\tau)\operatorname{csch}(\kappa\tau)d\tau,$$

$$W^y(\tau,\sigma) = \sinh(\kappa\tau) \sum_{n=-\infty}^{\infty} e^{2in\sigma}(2i\,n+\kappa) \int Z_4^n(\tau)\operatorname{csch}(\kappa\tau)d\tau,$$

where $Z_3^n(\tau)$ and $Z_4^n(\tau)$ solve a second order ODE

$$Z_a^{n\prime\prime}(\tau) + \left(4\,n^2 - 2\,\kappa^2\operatorname{csch}^2(\kappa\tau)\right) Z_a^n(\tau) = 0.$$

Obviously, all the calculations presented above can be repeated for the curved background with the metric (9.31). The solution of the classical field equations is obtained from the solution (9.39) when arccoth is replaced by arctanh.

9.5 Example 2 – subalgebra S_{17}

The second example will deal with the subalgebra

$$S_{17} = Span[\mathcal{K}_1 = L_3 + \epsilon\,(P_0 + P_3), \mathcal{K}_2 = P_1, \mathcal{K}_3 = P_2, \mathcal{K}_4 = P_0 - P_3], \quad \epsilon = \pm 1,$$

which produces a dual model with torsion, and whose representation is not faithful. The commutation relations of this subalgebra are

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_2.$$

The transformation of coordinates of the flat background

$$t = x^1\epsilon + x^4, \qquad x = x^2, \qquad y = x^3, \qquad z = x^1\epsilon - x^4, \tag{9.41}$$

yields the components of the flat metric in the group coordinates as

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & -2\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2\epsilon & 0 & 0 & 0 \end{pmatrix}.$$

In this case the dual background is given by

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2\epsilon} \\ 0 & 1 & 0 & \frac{\tilde{x}_3}{2\epsilon} \\ 0 & 0 & 1 & -\frac{\tilde{x}_2}{2\epsilon} \\ -\frac{1}{2\epsilon} & -\frac{\tilde{x}_3}{2\epsilon} & \frac{\tilde{x}_2}{2\epsilon} & -\frac{\tilde{x}_2^2+\tilde{x}_3^2}{4\epsilon^2} \end{pmatrix},$$

and the transformation to Brinkmann coordinates

$$\tilde{x}_1 = -v, \qquad \tilde{x}_2 = z_3, \qquad \tilde{x}_3 = z_4, \qquad \tilde{x}_4 = 2\epsilon\, u, \tag{9.42}$$

brings the dual metric to the homogeneous and isotropic form

$$ds^2 = 2dudv - (z_3^2 + z_4^2)\, du^2 + dz_3^2 + dz_4^2. \tag{9.43}$$

The torsion is constant in Brinkmann coordinates,

$$H = -2\, du \wedge dz_3 \wedge dz_4, \tag{9.44}$$

and the dilaton is

$$\Phi(u) = c_1 + c_2\, u.$$

To find the general solution of field equations of the dual sigma model with torsion, we have to express the coordinates $\tilde{x}_\mu$ in terms of x^ν and $\tilde{h}_k$. As the adjoint representation of S_{17} is not faithful, we have to use the formula (9.22) to solve the equation (8.16) for the coordinates of $\tilde{g}$. We get

$$\tilde{x}_1 = \tilde{h}_1 + x^2\tilde{h}_3 - x^3\tilde{h}_2, \qquad \tilde{x}_2 = \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1,$$
$$\tilde{x}_4 = \tilde{h}_4, \qquad \tilde{x}_3 = \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1.$$

Similarly as in the previous section, combining this with (9.42) and (9.41), we find the general solution of field equations of the sigma model with metric (9.43) and torsion (9.44) as

$$\begin{aligned}
U(\tau,\sigma) &= \frac{\tilde{h}_4(\tau,\sigma)}{2\epsilon},\\
V(\tau,\sigma) &= -\tilde{h}_1(\tau,\sigma) - \tilde{h}_3(\tau,\sigma)W^x(\tau,\sigma) + \tilde{h}_2(\tau,\sigma)W^y(\tau,\sigma),\\
Z_3(\tau,\sigma) &= \cos(\Omega(\tau,\sigma))\,\tilde{h}_2(\tau,\sigma) - \sin(\Omega(\tau,\sigma))\,\tilde{h}_3(\tau,\sigma),\\
Z_4(\tau,\sigma) &= \cos(\Omega(\tau,\sigma))\tilde{h}_3(\tau,\sigma) + \sin(\Omega(\tau,\sigma))\tilde{h}_2(\tau,\sigma),
\end{aligned}$$

where $W^I(\tau,\sigma)$ are solutions of the wave equations (9.18),

$$\Omega(\tau,\sigma) = \frac{W^t + W^z}{2\epsilon},$$

and $\tilde{h}_\mu$ are the solutions of the PDEs (9.20), (9.21):

$$\tilde{h}_1 = \gamma_1 + \int \left[\epsilon\left(\partial_\tau W^t - \partial_\tau W^z\right) + W^x\partial_\tau W^y - W^y\partial_\tau W^x\right] d\sigma,$$
$$\tilde{h}_2 = \gamma_2 - \int \partial_\tau W^x\, d\sigma, \qquad \tilde{h}_3 = \gamma_3 - \int \partial_\tau W^y\, d\sigma,$$
$$\tilde{h}_4 = \gamma_4 + \int \left(\partial_\tau W^t + \partial_\tau W^z\right) d\sigma.$$

String-type solutions in the light-cone gauge (9.40) are obtained if we choose

$$W^t(\tau,\sigma) + W^z(\tau,\sigma) = 2\,\epsilon\,\kappa\,\sigma,$$

and

$$\begin{aligned}
W^x(\tau,\sigma) = \sum_{n=-\infty}^{\infty} e^{2in\sigma} \int \Big(&Z_3^n(\tau)\,(\kappa\sin(\kappa\sigma) - 2i\,n\cos(\kappa\sigma))\\
&- Z_4^n(\tau)\,(\kappa\cos(\kappa\sigma) + 2i\,n\sin(\kappa\sigma))\Big)d\tau,
\end{aligned}$$

$$\begin{aligned}
W^y(\tau,\sigma) = \sum_{n=-\infty}^{\infty} e^{2in\sigma} \int \Big(&Z_3^n(\tau)\,(\kappa\cos(\kappa\sigma) + 2i\,n\sin(\kappa\sigma))\\
&+ Z_4^n(\tau)\,(\kappa\sin(\kappa\sigma) - 2i\,n\cos(\kappa\sigma))\Big)d\tau,
\end{aligned}$$

where $Z_3^n(\tau)$ and $Z_4^n(\tau)$ solve the system of differential equations

$$Z_3^{n\prime\prime}(\tau) + \left(4\,n^2 + \kappa^2\right) Z_3^n(\tau) - 4i\,n\,\kappa Z_4^n(\tau) = 0,$$
$$Z_4^{n\prime\prime}(\tau) + \left(4\,n^2 + \kappa^2\right) Z_4^n(\tau) + 4i\,n\,\kappa Z_3^n(\tau) = 0.$$

9.6 Example 3 – subalgebra S_{19}

The third example will deal with the subalgebra

$$S_{19} = Span[\mathcal{K}_1 = L_3 + \alpha P_3,\ \mathcal{K}_2 = P_1,\ \mathcal{K}_3 = P_2,\ \mathcal{K}_4 = P_0], \qquad \alpha \neq 0$$

that produces a diagonalizable dual metric with non-vanishing scalar curvature and torsion.

The commutation relations of this subalgebra

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_2$$

are equal to those in the previous example, but the groups of symmetries corresponding to these subalgebras cannot be transformed one into another by an element of the group of proper ortochronous Poincaré transformations, see [68], and the representations of the commutation relations in Killing vector fields on $\mathscr{M}$ are different. This leads to a different transformation of coordinates in the flat background, namely

$$x^1 = \frac{z}{\alpha}, \qquad x^2 = x, \qquad x^3 = y, \qquad x^4 = t. \tag{9.45}$$

Components of the flat metric in the group coordinates then read

$$F_{\mu\nu} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The dual background in this case is

$$\widetilde{F}_{\mu\nu} = \begin{pmatrix} \frac{1}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_3}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & -\frac{\tilde{x}_2}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & 0 \\ -\frac{\tilde{x}_3}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\alpha^2+\tilde{x}_2^2}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_2\tilde{x}_3}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & 0 \\ \frac{\tilde{x}_2}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_2\tilde{x}_3}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\alpha^2+\tilde{x}_3^2}{\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Its symmetric part gives a metric with non-vanishing scalar curvature

$$\widetilde{R} = -\frac{4\left(\tilde{x}_2^2 + \tilde{x}_3^2\right) - 10\alpha^2}{\left(\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2\right)^2},$$

and cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the form

$$ds^2 = -dy_1^2 + dy_2^2 + \frac{y_2^2\alpha^2}{y_2^2 + \alpha^2}\, dy_3^2 + \frac{1}{y_2^2 + \alpha^2}\, dy_4^2, \tag{9.46}$$

via the transformation

$$\tilde{x}_1 = y_4, \qquad \tilde{x}_2 = y_2 \cos y_3, \qquad \tilde{x}_3 = y_2 \sin y_3, \qquad \tilde{x}_4 = y_1. \tag{9.47}$$

The torsion then acquires the form

$$H = \frac{2y_2\alpha^2}{\left(y_2^2+\alpha^2\right)^2}\, dy_2 \wedge dy_3 \wedge dy_4, \tag{9.48}$$

and the dilaton satisfying β equations (1.27)–(1.29) is

$$\Phi = \ln(y_2^2+\alpha^2) + c,$$

where c is a constant.

To find the general solution of field equations of this dual sigma model, we have to express the coordinates $\tilde{x}_\mu$ in terms of x^ν and $\tilde{h}_k$. As the adjoint representation of S_{19} is not faithful, we have to use the formula (9.22) to solve the equation (8.16) for the coordinates of $\tilde{g}$. We get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2\tilde{h}_3 - x^3\tilde{h}_2, & \tilde{x}_2 &= \tilde{h}_2\cos x^1 - \tilde{h}_3 \sin x^1, \\ \tilde{x}_4 &= \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_2 \sin x^1 + \tilde{h}_3\cos x^1. \end{aligned}$$

Similarly as in the previous section, combining this with (9.47) and (9.45), we find the general solution of the field equations of the sigma model with metric (9.46) and torsion (9.48) as

$$\begin{aligned} Y_1(\tau,\sigma) =& \tilde{h}_4(\tau,\sigma), \\ Y_2(\tau,\sigma) =& \sqrt{\tilde{h}_2(\tau,\sigma)^2 + \tilde{h}_3(\tau,\sigma)^2}, \\ Y_3(\tau,\sigma) =& \arctan\left(\frac{\cos(\Omega(\tau,\sigma))\tilde{h}_3(\tau,\sigma) + \sin(\Omega(\tau,\sigma))\tilde{h}_2(\tau,\sigma)}{\cos(\Omega(\tau,\sigma))\tilde{h}_2(\tau,\sigma) - \sin(\Omega(\tau,\sigma))\tilde{h}_3(\tau,\sigma)}\right), \\ Y_4(\tau,\sigma) =& \tilde{h}_1(\tau,\sigma) + \tilde{h}_3(\tau,\sigma)W^x(\tau,\sigma) - \tilde{h}_2(\tau,\sigma)W^y(\tau,\sigma), \end{aligned}$$

where $W^I(\tau,\sigma)$ are the solutions of the wave equations (9.18),

$$\Omega(\tau,\sigma) = \frac{W^z(\tau,\sigma)}{\alpha},$$

and $\tilde{h}_\mu$ are solutions of the PDEs (9.20), (9.21) given by

$$\begin{aligned} \tilde{h}_1 &= \gamma_1 - \int (\alpha\, \partial_\tau W^z + W^y\partial_\tau W^x - W^x\partial_\tau W^y)\, d\sigma, \\ \tilde{h}_2 &= \gamma_2 - \int \partial_\tau W^x d\sigma, \\ \tilde{h}_3 &= \gamma_3 - \int \partial_\tau W^y d\sigma, \\ \tilde{h}_4 &= \gamma_4 + \int \partial_\tau W^t d\sigma. \end{aligned}$$

As this background is not of the pp-wave form, the light-cone gauge cannot be implemented, see Ref. [26]. Nevertheless, the field equations turned out to be solvable due to Poisson–Lie T-duality.

9.7 Results for other subalgebras I – plane waves

The classification of subalgebras of the Poincaré algebra in Ref. [68] was carried out up to the group of inner automorphisms of the connected component of the Poincaré group (proper orthochronous Poincaré transformations). There are 35 inequivalent four-dimensional subalgebras of the Poincaré algebra generated by Killing vectors (9.13). The examples in the two preceding sections demonstrated that quite different backgrounds are obtained even if the subalgebras are isomorphic. Therefore, we will examine all the cases one by one, and study the resulting dual sigma models.

Only the subgroups corresponding to the subalgebras S_1, S_2, S_6, S_7, S_8, S_{11}, S_{17}, S_{18}, S_{19}, S_{23}, $S_{25}-S_{29}$, S_{31}, S_{33}, listed in section 9.2, act transitively and freely on the flat spacetime and can be used for the atomic non-Abelian T-duality. Non-Abelian duals generated by the algebras S_1, S_2, S_6 give backgrounds with flat metric and vanishing torsion. We will not discuss them further as the duality in these cases represents merely a transformation of coordinates and no new background is obtained. Dual backgrounds obtained from duality with respect to the subalgebras S_{11}, S_{18}, S_{19} have nontrivial scalar curvature and we deal with them separately. The others are plane waves, most of them with nonzero torsion, as we shall see from the following list of results. We shall give the results for plane waves first. We do not repeat results for subalgebras S_{27}, S_{17} and S_{19} which were described previously in detail.

Subalgebras S_7, S_8

The non-isomorphic subalgebras

$$S_7 = Span[\mathcal{K}_1 = 2M_3 + \alpha\, P_1,\ \mathcal{K}_2 = L_2 + M_1,\ \mathcal{K}_3 = P_0 - P_3,\ \mathcal{K}_4 = P_2],$$
$$S_8 = Span[\mathcal{K}_1 = M_3,\ \mathcal{K}_2 = L_2 + M_1,\ \mathcal{K}_3 = P_0 - P_3,\ \mathcal{K}_4 = P_2],$$

differ only in the value of the parameter α: it is positive for S_7, while $\alpha = 0$ for S_8, see section 9.2. We start the discussion with dualization with respect to S_7, realizing in the end that the results hold for both subalgebras. The commutation relations of S_7 are

$$[\mathcal{K}_1, \mathcal{K}_2] = -2\mathcal{K}_2 - \alpha\mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_3] = -2\mathcal{K}_3.$$

The transformation of coordinates in the flat background

$$t = -\alpha x^1 x^2 + \frac{1}{2} e^{-2x^1}\left((x^2)^2 + 1\right) + x^3, \qquad x = \alpha x^1 - e^{-2x^1} x^2,$$
$$z = \alpha x^1 x^2 - \frac{1}{2} e^{-2x^1}\left((x^2)^2 - 1\right) - x^3, \qquad y = x^4,$$

gives components of the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} \alpha^2 & -e^{-2x^1}\alpha(2x^1+1) & 2e^{-2x^1} & 0 \\ -e^{-2x^1}\alpha(2x^1+1) & e^{-4x^1} & 0 & 0 \\ 2e^{-2x^1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background in this case is

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & \frac{1}{2-2\tilde{x}_3} & 0 \\ 0 & 1 & \frac{\tilde{x}_3\alpha+\alpha+2\tilde{x}_2}{2-2\tilde{x}_3} & 0 \\ \frac{1}{2\tilde{x}_3+2} & \frac{-\tilde{x}_3\alpha+\alpha-2\tilde{x}_2}{2\tilde{x}_3+2} & \frac{(2\tilde{x}_2+\alpha\tilde{x}_3)^2}{4\left(\tilde{x}_3^2-1\right)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the torsion vanishes.

The transformation to Brinkmann coordinates valid for $|\tilde{x}_3| < 1$

$$\begin{aligned} \tilde{x}_1 =& -2v - \frac{1}{4}\left(\alpha^2 u + 4z_3\alpha - (\alpha^2+4z_3^2)\tanh(u) + 2z_3\alpha\ln\left(1-\tanh^2(u)\right)\right) \\ &+ \frac{1}{16}\Big[\tanh(u)\left[\alpha^2\ln^2\left(1-\tanh^2(u)\right) + 4\alpha^2\ln\left(1-\tanh^2(u)\right)\right]\Big], \\ \tilde{x}_2 =& \, z_3 - \frac{1}{4}\alpha\tanh(u)\ln\left(1-\tanh^2(u)\right), \\ \tilde{x}_3 =& -\tanh(u), \\ \tilde{x}_4 =& \, z_4, \end{aligned} \tag{9.49}$$

brings the dual metric and dilaton to forms

$$\begin{aligned} ds^2 &= 2dudv - 2\,\frac{z_3^2}{\cosh^2(u)}du^2 + dz_3^2 + dz_4^2, \\ \Phi(u) &= c_1 + c_2\,u + 2\ln\left(\cosh(u)\right). \end{aligned} \tag{9.50}$$

The transformation of coordinates for $|\tilde{x}_3| > 1$, obtained from (9.49) by the replacement tanh $\to$ coth, gives the dual metric and dilaton in Brinkmann coordinates

$$\begin{aligned} ds^2 &= 2dudv + 2\,\frac{z_3^2}{\sinh^2(u)}du^2 + dz_3^2 + dz_4^2, \\ \Phi(u) &= c_1 + c_2\,u + 2\ln\left(\sinh(u)\right). \end{aligned} \tag{9.51}$$

These results are independent of α and valid for both S_7 and S_8, hence we can restrict our considerations to the simpler case of S_8. Even though the adjoint representation of S_8 is not faithful, we can solve the equation (8.16) for coordinates of $\tilde{g}$:

$$\tilde{x}_1 = \tilde{h}_1 - x^2\tilde{h}_2 - x^3\tilde{h}_3,$$

$$\tilde{x}_2 = e^{x^1}\tilde{h}_2, \qquad \tilde{x}_3 = e^{x^1}\tilde{h}_3, \qquad \tilde{x}_4 = \tilde{h}_4. \tag{9.52}$$

Like in the previous section, the transformations (9.49) and (9.52) enable us to find the general solution of field equations of the sigma models with metrics (9.50) and (9.51).

Subalgebra S_{23}

$$
\begin{aligned}
S_{23} = Span[\mathcal{K}_1 = L_2 + M_1 - \frac{1}{2}(P_0 + P_3),\ \mathcal{K}_2 = L_1 - M_2 + \alpha P_1,\\
\mathcal{K}_3 = P_0 - P_3,\ \mathcal{K}_4 = P_2], \qquad \alpha > 0.
\end{aligned}
$$

The commutation relations are

$$
[\mathcal{K}_1, \mathcal{K}_2] = \alpha\,\mathcal{K}_3 - \mathcal{K}_4, \qquad [\mathcal{K}_2, \mathcal{K}_4] = -\mathcal{K}_3, \qquad \alpha > 0.
$$

The transformation of coordinates on the flat background was found to be

$$
\begin{aligned}
t &= \frac{1}{6}\left(-(x^1)^3 - 3\left((x^2)^2 + 1\right)x^1 + 6x^3\right), & x &= x^2\alpha + \frac{(x^1)^2}{2},\\
z &= \frac{1}{6}\left((x^1)^3 + 3\left((x^2)^2 - 1\right)x^1 - 6x^3\right), & y &= x^4 - x^1 x^2.
\end{aligned}
$$

The flat metric in the group coordinates depends on α,

$$
F_{\mu\nu}(x) = \begin{pmatrix} 0 & \alpha x^1 & 1 & -x^2 \\ \alpha x^1 & \alpha^2 + (x^1)^2 & 0 & -x^1 \\ 1 & 0 & 0 & 0 \\ -x^2 & -x^1 & 0 & 1 \end{pmatrix},
$$

and the dual background is given by

$$
\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\alpha^2+\tilde{x}_3^2} & \frac{\tilde{x}_4-\alpha\tilde{x}_3}{\alpha^2+\tilde{x}_3^2} & -\frac{\tilde{x}_3}{\alpha^2+\tilde{x}_3^2} \\ 1 & \frac{\alpha\tilde{x}_3-\tilde{x}_4}{\alpha^2+\tilde{x}_3^2} & -\frac{(\tilde{x}_4-\alpha\tilde{x}_3)^2}{\alpha^2+\tilde{x}_3^2} & \frac{\tilde{x}_3(\tilde{x}_4-\alpha\tilde{x}_3)}{\alpha^2+\tilde{x}_3^2} \\ 0 & \frac{\tilde{x}_3}{\alpha^2+\tilde{x}_3^2} & \frac{\tilde{x}_3(\tilde{x}_4-\alpha\tilde{x}_3)}{\alpha^2+\tilde{x}_3^2} & \frac{\alpha^2}{\alpha^2+\tilde{x}_3^2} \end{pmatrix}.
$$

The dual metric in Brinkmann coordinates

$$
\begin{aligned}
\tilde{x}_1 =& \frac{1}{24\left(1+u^2\right)^{3/2}\alpha}\Bigg(-12\left(-4+u^2\right)\left(1+u^2\right)^2\alpha^2 z_3\\
&-12u\sqrt{1+u^2}\left(2+u^2\right)z_3^2 - 12u z_4^2\sqrt{1+u^2}\\
&+\sqrt{1+u^2}\left(\left(1+u^2\right)\left(24v + u\left(-48+28u^2-3u^4\right)\alpha^4\right)\right)\Bigg),\\
\tilde{x}_2 =& \sqrt{1+u^2}\,\alpha z_4,\\
\tilde{x}_3 =& u\alpha,\\
\tilde{x}_4 =& \frac{1}{2}u\left(-4+u^2\right)\alpha^2 + \sqrt{1+u^2}\,z_3
\end{aligned}
$$

then has the form

$$ds^2 = 2dudv + \frac{(2u^2-1)\, z_4^2 - 3z_3^2}{(u^2+1)^2} du^2 + dz_3^2 + dz_4^2,$$

while the torsion and the dilaton are

$$H = \frac{2}{1+u^2}\, du \wedge dz_3 \wedge dz_4, \qquad \Phi(u) = c_1 + c_2\, u + \ln(1+u^2).$$

To find the general solution of field equations of the dual sigma model, we have to express the coordinates $\tilde{x}_\mu$ in terms of x^ν and $\tilde{h}_k$. We get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2 \left(\alpha\, \tilde{h}_3 - \frac{1}{2} x^2 \tilde{h}_3 - \tilde{h}_4\right), & \tilde{x}_3 &= \tilde{h}_3, \\ \tilde{x}_2 &= \tilde{h}_2 - x^1 \left(\alpha\, \tilde{h}_3 - x^2 \tilde{h}_3 - \tilde{h}_4\right), & \tilde{x}_4 &= x^2 \tilde{h}_3 + \tilde{h}_4. \end{aligned}$$

Subalgebra S_{25}

$$S_{25} = Span[\mathcal{K}_1 = L_2 + M_1 - \epsilon\, P_2,\ \mathcal{K}_2 = P_0 + P_3, \mathcal{K}_3 = P_1,\ \mathcal{K}_4 = P_0 - P_3], \quad \epsilon = \pm 1.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = 2\, \mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_4.$$

The transformation of coordinates on the flat background

$$t = x^2 + x^4, \qquad x = x^3, \qquad y = -\epsilon\, x^1, \qquad z = x^2 - x^4$$

yields the flat metric in the group coordinates as

$$F_{\mu\nu}(x) = \begin{pmatrix} \epsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

The dual background is given by

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} \frac{1}{\tilde{x}_4^2+\epsilon^2} & 0 & \frac{\tilde{x}_4}{\tilde{x}_4^2+\epsilon^2} & -\frac{\tilde{x}_3}{\tilde{x}_4^2+\epsilon^2} \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{\tilde{x}_4}{\tilde{x}_4^2+\epsilon^2} & 0 & \frac{\epsilon^2}{\tilde{x}_4^2+\epsilon^2} & \frac{\tilde{x}_3\tilde{x}_4}{\tilde{x}_4^2+\epsilon^2} \\ \frac{\tilde{x}_3}{\tilde{x}_4^2+\epsilon^2} & -\frac{1}{2} & \frac{\tilde{x}_3\tilde{x}_4}{\tilde{x}_4^2+\epsilon^2} & -\frac{\tilde{x}_3^2}{\tilde{x}_4^2+\epsilon^2} \end{pmatrix}.$$

The transformation to Brinkmann coordinates

$$\begin{aligned} \tilde{x}_1 &= \epsilon\, \sqrt{u^2+1}\, z_4, & \tilde{x}_3 &= \sqrt{u^2+1}\, z_3, \\ \tilde{x}_2 &= \frac{1}{\epsilon(u^2+1)} \left(u\left(u^2+2\right) z_3^2 + u z_4^2\right) - 2\epsilon\, v, & \tilde{x}_4 &= \epsilon\, u \end{aligned}$$

gives the dual metric

$$ds^2 = 2dudv + \frac{(2u^2-1)\,z_4^2 - 3z_3^2}{(u^2+1)^2}du^2 + dz_3^2 + dz_4^2.$$

The torsion and the dilaton then read

$$H = -\frac{2}{1+u^2}\,du\wedge dz_3\wedge dz_4, \qquad \Phi(u) = c_1 + c_2\,u + \ln\left(1+u^2\right).$$

To find the general solution of field equations of the dual sigma model, we have to express the coordinates $\tilde{x}_\mu$ in terms of x^ν and $\tilde{h}_k$. We get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + 2x^2\tilde{h}_3 + x^3\tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_3 - x^1\tilde{h}_4,\\ \tilde{x}_2 &= \tilde{h}_2 - x^1\left(2\tilde{h}_3 - x^1\tilde{h}_4\right), & \tilde{x}_4 &= \tilde{h}_4. \end{aligned}$$

Subalgebras S_{26}, S_{27}

$$\begin{aligned} S_{26} =& Span[\mathcal{K}_1 = M_3 + \alpha P_1,\ \mathcal{K}_2 = L_2 + M_1,\ \mathcal{K}_3 = L_1 - M_2,\ \mathcal{K}_4 = P_0 - P_3],\\ S_{27} =& Span[\mathcal{K}_1 = M_3,\ \mathcal{K}_2 = L_2 + M_1,\ \mathcal{K}_3 = L_1 - M_2,\ \mathcal{K}_4 = P_0 - P_3]. \end{aligned}$$

The subalgebras S_{26} and S_{27} differ once again only by the value of the parameter α: it is positive for S_{26}, while $\alpha = 0$ for S_{27}, see section 9.2. Since the detailed discussion of dualization procedure with respect to S_{27} was given in section 9.4, we shall now focus on S_{26}, i.e. $\alpha > 0$. Their commutation relations are

$$[\mathcal{K}_1,\mathcal{K}_2] = -\mathcal{K}_2 - \alpha\mathcal{K}_4, \qquad [\mathcal{K}_1,\mathcal{K}_3] = -\mathcal{K}_3, \qquad [\mathcal{K}_1,\mathcal{K}_4] = -\mathcal{K}_4.$$

The transformation of coordinates in the flat background

$$\begin{aligned} t &= -\alpha\,x^1x^2 + \frac{1}{2}e^{-x^1}\left((x^2)^2 + (x^3)^2 + 1\right) + x^4, & x &= \alpha\,x^1 - e^{-x^1}x^2,\\ z &= \alpha\,x^1x^2 - \frac{1}{2}e^{-x^1}\left((x^2)^2 + (x^3)^2 - 1\right) - x^4, & y &= e^{-x^1}x^3 \end{aligned}$$

gives the flat metric in the group coordinates as

$$F_{\mu\nu}(x) = \begin{pmatrix} \alpha^2 & -e^{-x^1}\alpha(x^1+1) & 0 & e^{-x^1}\\ -e^{-x^1}\alpha(x^1+1) & e^{-2x^1} & 0 & 0\\ 0 & 0 & e^{-2x^1} & 0\\ e^{-x^1} & 0 & 0 & 0 \end{pmatrix}.$$

In the dual background

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1-\tilde{x}_4}\\ 0 & 1 & 0 & \frac{\tilde{x}_4\alpha+\alpha+\tilde{x}_2}{1-\tilde{x}_4}\\ 0 & 0 & 1 & \frac{\tilde{x}_3}{1-\tilde{x}_4}\\ \frac{1}{\tilde{x}_4+1} & \frac{-\tilde{x}_4\alpha+\alpha-\tilde{x}_2}{\tilde{x}_4+1} & -\frac{\tilde{x}_3}{\tilde{x}_4+1} & \frac{\tilde{x}_2^2+2\alpha\tilde{x}_4\tilde{x}_2+\tilde{x}_3^2+\alpha^2\tilde{x}_4^2}{\tilde{x}_4^2-1} \end{pmatrix}$$

the torsion vanishes.

The transformation to Brinkmann coordinates

$$\begin{aligned}
\tilde{x}_1 =& - v + \frac{1}{8}\Big(- 4u\alpha^2 + 4\tanh(u)\left(z_3^2 + z_4^2 + \alpha^2\right) \\
&+ \alpha^2 \tanh(u) \ln\left(1 - \tanh^2(u)\right)\left(\ln\left(1 - \tanh^2(u)\right) + 4\right) \\
&- 4z_3\alpha\left(\ln\left(1 - \tanh^2(u)\right) + 2\right)\Big), \\
\tilde{x}_2 =& z_3 - \frac{1}{2}\alpha\tanh(u)\ln\left(1 - \tanh^2(u)\right), \\
\tilde{x}_3 =& z_4, \\
\tilde{x}_4 =& - \tanh(u),
\end{aligned}$$

valid for $|\tilde{x}_1| < 1$ brings the dual metric and dilaton to forms independent of α:

$$\begin{aligned}
ds^2 &= 2dudv - 2\,\frac{z_3^2 + z_4^2}{\cosh^2(u)}du^2 + dz_3^2 + dz_4^2, \\
\Phi(u) &= c_1 + c_2\, u + 4\ln(\cosh(u)).
\end{aligned}$$

A similar transformation (see section 9.4) yields the dual metric and dilaton for $|\tilde{x}_1| > 1$ in Brinkmann coordinates as

$$\begin{aligned}
ds^2 &= 2dudv + 2\,\frac{z_3^2 + z_4^2}{\sinh^2(u)}du^2 + dz_3^2 + dz_4^2, \\
\Phi(u) &= c_1 + c_2\, u + 4\ln(\sinh(u)).
\end{aligned}$$

The backgrounds resulting from Poisson–Lie T-duality are independent on α, and we may use the results presented for S_{27} also for S_{26}. The solution of the field equations of both dual sigma models was found in section 9.4.

Subalgebra S_{28}

$$S_{28} = Span[\mathcal{K}_1 = L_3 - \beta M_3, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = L_1 - M_2, \mathcal{K}_4 = P_0 - P_3], \qquad \beta \neq 0.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = \beta\,\mathcal{K}_2 - \mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_2 + \beta\,\mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_4] = \beta\,\mathcal{K}_4.$$

The transformation of coordinates in the flat background

$$\begin{aligned}
t &= \frac{1}{2}\left((x^2)^2 + (x^3)^2 + 1\right)e^{x^1\beta} + x^4, & x &= -x^2 e^{x^1\beta}, \\
z &= -\frac{1}{2}\left((x^2)^2 + (x^3)^2 - 1\right)e^{x^1\beta} - x^4, & y &= x^3 e^{x^1\beta},
\end{aligned}$$

gives the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & -e^{\beta x^1}\beta \\ 0 & e^{2\beta x^1} & 0 & 0 \\ 0 & 0 & e^{2\beta x^1} & 0 \\ -e^{\beta x^1}\beta & 0 & 0 & 0 \end{pmatrix}.$$

After the transformation

$$\begin{aligned}
\tilde{x}_1 &= \frac{1}{2}\beta\left(2v - \tanh(u)\left(z_3^2 + z_4^2\right)\right), \\
\tilde{x}_2 &= z_3 \cos\left(\frac{\ln(\cosh(u))}{\beta}\right) + z_4 \sin\left(\frac{\ln(\cosh(u))}{\beta}\right), \\
\tilde{x}_3 &= z_4 \cos\left(\frac{\ln(\cosh(u))}{\beta}\right) - z_3 \sin\left(\frac{\ln(\cosh(u))}{\beta}\right), \\
\tilde{x}_4 &= -\tanh(u)
\end{aligned}$$

of the dual background

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\beta(\tilde{x}_4-1)} \\ 0 & 1 & 0 & \frac{\beta\tilde{x}_2-\tilde{x}_3}{\beta-\beta\tilde{x}_4} \\ 0 & 0 & 1 & \frac{\tilde{x}_2+\beta\tilde{x}_3}{\beta-\beta\tilde{x}_4} \\ -\frac{1}{\tilde{x}_4\beta+\beta} & \frac{\tilde{x}_3-\beta\tilde{x}_2}{\beta(\tilde{x}_4+1)} & -\frac{\tilde{x}_2+\beta\tilde{x}_3}{\tilde{x}_4\beta+\beta} & \frac{\left(\beta^2+1\right)\left(\tilde{x}_2^2+\tilde{x}_3^2\right)}{\beta^2\left(\tilde{x}_4^2-1\right)} \end{pmatrix},$$

the dual metric and the dilaton for $|\tilde{x}_1| < 1$ are expressed in Brinkmann coordinates as

$$\begin{aligned}
ds^2 &= 2dudv - \frac{(z_3^2 + z_4^2)\left(1 + 2\beta^2 \mathrm{sech}^2(u)\right)}{\beta^2} du^2 + dz_3^2 + dz_4^2, \\
\Phi(u) &= c_1 + c_2\, u + 4\ln(\cosh(u)).
\end{aligned}$$

Using $\coth(u)$ instead of $\tanh(u)$ in the transformation to Brinkmann coordinates, the dual metric and dilaton for $|\tilde{x}_1| > 1$ are

$$\begin{aligned}
ds^2 &= 2dudv - \frac{(z_3^2 + z_4^2)\left(1 - 2\beta^2 \mathrm{csch}^2(u)\right)}{\beta^2} du^2 + dz_3^2 + dz_4^2, \\
\Phi(u) &= c_1 + c_2\, u + 4\ln(\sinh(u)).
\end{aligned}$$

In both cases the torsion is of the form

$$H = -\frac{2}{\beta}\, du \wedge dz_3 \wedge dz_4.$$

To find the solution of equations of the dual sigma model, we also need $\tilde{x}_j, h^k$ expressed in terms of x^j and $\tilde{h}_k$ as

$$
\begin{aligned}
\tilde{x}_1 &= x^2\tilde{h}_2\beta + x^3\tilde{h}_3\beta + x^4\tilde{h}_4\beta + \tilde{h}_1 + x^3\tilde{h}_2 - x^2\tilde{h}_3,\\
\tilde{x}_2 &= e^{-\beta\tilde{x}_1}\left(\tilde{h}_3\sin(x^1) + \tilde{h}_2\cos(x^1)\right),\\
\tilde{x}_3 &= e^{-\beta\tilde{x}_1}\left(-\tilde{h}_2\sin(x^1) + \tilde{h}_3\cos(x^1)\right),\\
\tilde{x}_4 &= \tilde{h}_4 e^{-\beta\tilde{x}_1}.
\end{aligned}
$$

Subalgebra S_{29}

$$S_{29} = Span[\mathcal{K}_1 = L_3 - \beta\, M_3, \mathcal{K}_2 = P_0 - P_3, \mathcal{K}_3 = P_1, \mathcal{K}_4 = P_2], \qquad \beta \neq 0.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = \beta\,\mathcal{K}_2, \qquad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_4, \qquad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_3.$$

The transformation of coordinates in the flat background

$$t = -\frac{1}{2}e^{x^1\beta} + x^2, \qquad x = x^3, \qquad y = x^4, \qquad z = -\frac{1}{2}\left(e^{x^1\beta}\right) - x^2$$

gives the flat metric in the group coordinates:

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & e^{\beta x^1}\beta & 0 & 0\\ e^{\beta x^1}\beta & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background is

$$\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & \frac{1}{\tilde{x}_2\beta+\beta} & 0 & 0\\ \frac{1}{\beta-\beta\tilde{x}_2} & \frac{\tilde{x}_3^2+\tilde{x}_4^2}{\beta^2\left(\tilde{x}_2^2-1\right)} & \frac{\tilde{x}_4}{\beta-\beta\tilde{x}_2} & \frac{\tilde{x}_3}{\beta(\tilde{x}_2-1)}\\ 0 & -\frac{\tilde{x}_4}{\tilde{x}_2\beta+\beta} & 1 & 0\\ 0 & \frac{\tilde{x}_3}{\tilde{x}_2\beta+\beta} & 0 & 1 \end{pmatrix}.$$

The dual metric, dilaton and torsion in Brinkmann coordinates are the same as for the algebra S_{17} in section 9.5:

$$
\begin{aligned}
ds^2 &= 2dudv - \left(z_3^2 + z_4^2\right)du^2 + dz_3^2 + dz_4^2,\\
\Phi(u) &= c_1 + c_2\,u,\\
H &= -2\,du \wedge dz_3 \wedge dz_4.
\end{aligned}
$$

To find the solution of equations of motion of the dual sigma model, we also need $\tilde{x}_j, h^k$ expressed in terms of $x^j, \tilde{h}_k$ as

$$
\begin{aligned}
\tilde{x}_1 &= x^2\tilde{h}_2\beta + \tilde{h}_1 - x^4\tilde{h}_3 + x^3\tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_3\cos(x^1) - \tilde{h}_4\sin(x^1),\\
\tilde{x}_2 &= \tilde{h}_2 e^{-x^1\beta}, & \tilde{x}_4 &= \tilde{h}_3\sin(x^1) + \tilde{h}_4\cos(x^1).
\end{aligned}
$$

Subalgebras S_{31}, S_{33}

$$
\begin{aligned}
S_{31} &= Span[\mathcal{K}_1 = M_3,\ \mathcal{K}_2 = P_1 + \beta\, P_2,\ \mathcal{K}_3 = P_0 - P_3,\ \mathcal{K}_4 = L_2 + M_1],\\
S_{33} &= Span[\mathcal{K}_1 = M_3 + \alpha\, P_2,\ \mathcal{K}_2 = P_1 + \beta\, P_2,\\
&\qquad \mathcal{K}_3 = P_0 - P_3,\ \mathcal{K}_4 = L_2 + M_1], \qquad\qquad \beta \neq 0,
\end{aligned}
$$

The subalgebras differ only in the value of the parameter α: it is positive for S_{33}, while $\alpha = 0$ for S_{31}. In both cases $\beta \neq 0$. The subalgebras are isomorphic even though their Lie groups are not equivalent under conjugacy by proper orthochronous Poincaré transformations. Their commutation relations are

$$
[\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4, \qquad [\mathcal{K}_2, \mathcal{K}_4] = -\mathcal{K}_3.
$$

The transformation of coordinates in the flat background

$$
\begin{aligned}
t &= \ x^3 - x^2 x^4 - \frac{1}{2} e^{-x^1} \left((x^4)^2 + 1\right), &\qquad x &= x^2 + e^{-x^1} x^4,\\
z &= -x^3 + x^2 x^4 + \frac{1}{2} e^{-x^1} \left((x^4)^2 - 1\right), &\qquad y &= x^1 \alpha + x^2 \beta
\end{aligned}
$$

gives the flat metric in the group coordinates as

$$
F_{\mu\nu}(x) = \begin{pmatrix} \alpha^2 & \alpha\beta & -e^{-x^1} & e^{-x^1} x^2 \\ \alpha\beta & \beta^2 + 1 & 0 & e^{-x^1} \\ -e^{-x^1} & 0 & 0 & 0 \\ e^{-x^1} x^2 & e^{-x^1} & 0 & e^{-2x^1} \end{pmatrix}.
$$

For the dual background

$$
\widetilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & -\frac{1}{\tilde{x}_3+1} & 0 \\ 0 & \frac{1}{\beta^2+\tilde{x}_3^2} & \frac{\alpha\beta+(\tilde{x}_3+1)\tilde{x}_4}{(\tilde{x}_3+1)\left(\beta^2+\tilde{x}_3^2\right)} & -\frac{\tilde{x}_3+1}{\beta^2+\tilde{x}_3^2} \\ \frac{1}{\tilde{x}_3-1} & \frac{-\alpha\beta-\tilde{x}_3\tilde{x}_4+\tilde{x}_4}{(\tilde{x}_3-1)\left(\beta^2+\tilde{x}_3^2\right)} & \frac{\alpha^2\tilde{x}_3^2-2\alpha\beta\tilde{x}_4\tilde{x}_3+\left(\beta^2+1\right)\tilde{x}_4^2}{\left(\tilde{x}_3^2-1\right)\left(\beta^2+\tilde{x}_3^2\right)} & \frac{\alpha\beta(\tilde{x}_3+1)-\left(\beta^2+1\right)\tilde{x}_4}{(\tilde{x}_3-1)\left(\beta^2+\tilde{x}_3^2\right)} \\ 0 & \frac{\tilde{x}_3-1}{\beta^2+\tilde{x}_3^2} & \frac{\alpha\beta(\tilde{x}_3-1)-\left(\beta^2+1\right)\tilde{x}_4}{(\tilde{x}_3+1)\left(\beta^2+\tilde{x}_3^2\right)} & \frac{\beta^2+1}{\beta^2+\tilde{x}_3^2} \end{pmatrix}
$$

we can find a rather complicated coordinate transformation that enables us to eliminate the dependence of the background on the parameter α. The dual metric, torsion and dilaton for $|x_3| < 1$ in Brinkmann coordinates are

$$
\begin{aligned}
ds^2 = 2dudv + \Bigg[& z_3^2 \frac{\mathrm{sech}^4(u) \left(2\left(\beta^2+1\right)\sinh^2(u) - \beta^2\right)}{\left(\tanh^2(u) + \beta^2\right)^2} \\
& -z_4^2 \frac{\beta^2 \mathrm{sech}^4(u) \left(2(\beta^2+1)\cosh^2(u) + 1\right)}{\left(\tanh^2(u) + \beta^2\right)^2} \Bigg] du^2 + dz_3^2 + dz_4^2,
\end{aligned}
$$

$$
\begin{aligned}
H &= \left(\frac{2\beta}{\beta^2\cosh^2(u)+\sinh^2(u)}\right) du\wedge dz_3\wedge dz_4,\\
\Phi(u) &= c_1 + c_2\, u + \ln\Big((\beta^2+1)\cosh(2u)+\beta^2-1\Big).
\end{aligned}
$$

The dual metric, torsion and dilaton for $|x_3|>1$ in Brinkmann coordinates are

$$
\begin{aligned}
ds^2 &= 2dudv + \left[z_4^2\frac{\beta^2\mathrm{csch}^4(u)\Big(2(\beta^2+1)\sinh^2(u)-1\Big)}{\Big(\coth^2(u)+\beta^2\Big)^2}\right.\\
&\left.\quad -z_3^2\frac{\mathrm{csch}^4(u)\Big(2(\beta^2+1)\cosh^2(u)+\beta^2\Big)}{\Big(\coth^2(u)+\beta^2\Big)^2}\right] du^2 + dz_3^2 + dz_4^2,\\
H &= -\left(\frac{2\beta}{\cosh^2(u)+\beta^2\sinh^2(u)}\right) du\wedge dz_3\wedge dz_4,\\
\Phi(u) &= c_1 + c_2\, u + \ln\Big((\beta^2+1)\cosh(2u)-\beta^2+1\Big).
\end{aligned}
$$

To find the general solution of field equations of the dual sigma model, it is sufficient to express the coordinates $\tilde{x}_\mu$ in terms of x^ν and $\tilde{h}_k$ for $\alpha=0$, i.e. to tackle the problem for the subalgebra S_{31}. We get

$$
\begin{aligned}
\tilde{x}_1 &= \tilde{h}_1 - x^3\tilde{h}_3 - x^4\tilde{h}_4, & \tilde{x}_2 &= \tilde{h}_2 - x^4\tilde{h}_3,\\
\tilde{x}_3 &= e^{x^1}\tilde{h}_3, & \tilde{x}_4 &= e^{x^1}\Big(x^2\tilde{h}_3+\tilde{h}_4\Big).
\end{aligned}
$$

9.8 Results for other subalgebras II – diagonalizable metrics with nontrivial scalar curvature

Subalgebra S_{11}

$$
S_{11} = Span[\mathcal{K}_1 = M_3 + \alpha\, P_2, \mathcal{K}_2 = P_0, \mathcal{K}_3 = P_3, \mathcal{K}_4 = P_1], \quad \alpha>0.
$$

The commutation relations are

$$
[\mathcal{K}_1,\mathcal{K}_2] = \mathcal{K}_3, \qquad [\mathcal{K}_1,\mathcal{K}_3] = \mathcal{K}_2.
$$

The flat metric in the group coordinates

$$
x^1 = \frac{y}{\alpha}, \qquad x^2 = t, \qquad x^3 = z, \qquad x^4 = x
$$

reads

$$
F_{\mu\nu} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The dual background

$$\widetilde{F}_{\mu\nu} = \begin{pmatrix} \frac{1}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & -\frac{\tilde{x}_3}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & \frac{\tilde{x}_2}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & 0 \\ \frac{\tilde{x}_3}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & -\frac{\alpha^2+\tilde{x}_2^2}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & \frac{\tilde{x}_2\tilde{x}_3}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & 0 \\ -\frac{\tilde{x}_2}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & \frac{\tilde{x}_2\tilde{x}_3}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & \frac{\alpha^2-\tilde{x}_3^2}{\alpha^2+\tilde{x}_2^2-\tilde{x}_3^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gives a metric with non-vanishing scalar curvature

$$\widetilde{R} = \frac{2\left(2\tilde{x}_2^2 - 2\tilde{x}_3^2 - 5\alpha^2\right)}{\left(\tilde{x}_2^2 - \tilde{x}_3^2 + \alpha^2\right)^2},$$

so it cannot be transformed to the plane wave form. On the other hand, the metric following from this background can be diagonalized to the time-dependent form

$$ds^2 = -dy_1^2 + dy_2^2 + \frac{y_1^2\alpha^2}{y_1^2+\alpha^2}\,dy_3^2 + \frac{1}{y_1^2+\alpha^2}\,dy_4^2$$

via

$$\tilde{x}_1 = y_4, \qquad \tilde{x}_2 = y_1\cosh y_3, \qquad \tilde{x}_3 = y_1\sinh y_3, \qquad \tilde{x}_4 = y_2.$$

The torsion then acquires the form

$$H = -\frac{2y_1\alpha^2}{\left(y_1^2+\alpha^2\right)^2}\,dy_1\wedge dy_3\wedge dy_4,$$

and the dilaton satisfying (1.27)–(1.29) is

$$\Phi = \ln(y_1^2+\alpha^2) + c$$

for an arbitrary constant c.

To find the solution of equations of this dual sigma model, we need the above transformation between y_j and $\tilde{x}_j$ and also $\tilde{x}_j$ expressed in terms of $x^j, \tilde{h}_k$. It reads

$$\tilde{x}_1 = \tilde{h}_1 + x^2\tilde{h}_3 + x^3\tilde{h}_2, \qquad \tilde{x}_2 = \tilde{h}_2\cosh x^1 - \tilde{h}_3\sinh x^1,$$
$$\tilde{x}_4 = \tilde{h}_4, \qquad \tilde{x}_3 = \tilde{h}_3\cosh x^1 - \tilde{h}_2\sinh x^1.$$

Subalgebra S_{18}

$$S_{18} = Span[\mathcal{K}_1 = L_3 + \alpha\,P_0,\ \mathcal{K}_2 = P_1,\ \mathcal{K}_3 = P_2,\ \mathcal{K}_4 = P_3], \qquad \alpha > 0.$$

The commutation relations are the same as for S_{17} and S_{19},

$$[\mathcal{K}_1,\mathcal{K}_2] = \mathcal{K}_3, \qquad [\mathcal{K}_1,\mathcal{K}_3] = -\mathcal{K}_2,$$

and these subalgebras are isomorphic. The flat metric in the group coordinates

$$x^1 = \frac{t}{\alpha}, \qquad x^2 = x, \qquad x^3 = y, \qquad x^4 = z$$

reads

$$F_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background

$$\widetilde{F}_{\mu\nu} = \begin{pmatrix} \frac{1}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_3}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & -\frac{\tilde{x}_2}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & 0 \\ -\frac{\tilde{x}_3}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_2^2-\alpha^2}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_2\tilde{x}_3}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & 0 \\ \frac{\tilde{x}_2}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_2\tilde{x}_3}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & \frac{\tilde{x}_3^2-\alpha^2}{-\alpha^2+\tilde{x}_2^2+\tilde{x}_3^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gives a metric with non-vanishing scalar curvature

$$\widetilde{R} = -\frac{10\alpha^2 + 4\left(\tilde{x}_2^2 + \tilde{x}_3^2\right)}{\left(\tilde{x}_2^2 + \tilde{x}_3^2 - \alpha^2\right)^2},$$

so it cannot be transformed to the plane wave form. On the other hand, the metric of this background can be diagonalized to the form

$$ds^2 = \frac{1}{y_3^2 - \alpha^2}\, dy_1^2 + \frac{y_3^2\alpha^2}{\alpha^2 - y_3^2}\, dy_2^2 + dy_3^2 + dy_4^2,$$

via

$$\tilde{x}_1 = y_1, \qquad \tilde{x}_2 = y_3 \cos y_2, \qquad \tilde{x}_3 = y_3 \sin y_2, \qquad \tilde{x}_4 = y_4.$$

Note the singularity on the surfaces $y_3 = \pm\alpha$. For $|y_3| < \alpha$ the time-like direction is given by the vector ∂_{y_1}, whereas for $|y_3| > \alpha$ the time-like vector is ∂_{y_2}.

The torsion acquires the form

$$H = \frac{2y_3\alpha^2}{\left(y_3^2 - \alpha^2\right)^2}\, dy_1 \wedge dy_2 \wedge dy_3,$$

and the dilaton satisfying (1.27)–(1.29) is

$$\Phi = \ln(y_3^2 - \alpha^2) + c.$$

To find the solution of equations of this dual sigma model, we need the above transformation between y_j and $\tilde{x}_j$, and to express $\tilde{x}_j$ in terms of $x^j, \tilde{h}_k$. As the commutation relations are the same as for S_{17} and S_{19}, we get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2\tilde{h}_3 - x^3\tilde{h}_2, & \tilde{x}_2 &= \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1, \\ \tilde{x}_4 &= \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1. \end{aligned}$$

9.9 Conclusions

We have classified all the atomic non-Abelian duals of the four-dimensional flat spacetime with respect to four-dimensional subgroups of the Poincaré group. As a result we have obtained 14 different kinds of exactly solvable sigma models in four-dimensional curved backgrounds. Due to non-Abelian T-duality, one can find general solutions of classical field equations for all of these dual models in terms of d'Alembert solutions of the wave equation. The method of obtaining the solutions is described in Section 9.3 and examples are given in sections 9.4, 9.5, 9.6 and 9.7, 9.8. The one-loop beta equations for all of the dual backgrounds yield simple ordinary differential equations for dilatons. Their solutions are given in sections 9.4, 9.5, 9.6, 9.7 and 9.8.

Eleven of the dual backgrounds are plane-parallel waves whose metrics can be brought to Brinkmann form

$$ds^2 = 2dudv - [K_3(u)z_3^2 + K_4(u)z_4^2]du^2 + dz_3^2 + dz_4^2.$$

The torsion then is

$$H = dB = H(u)\, du \wedge dz_3 \wedge dz_4.$$

Depending on the chosen subgroup, the functions $K_3(u), K_4(u), H(u)$ may acquire the following forms:

$$K_3(u) = 1, \qquad K_4(u) = 1, \qquad H(u) = -2, \tag{9.53}$$

$$K_3(u) = \frac{3}{\left(u^2+1\right)^2}, \qquad K_4(u) = -\frac{\left(2u^2-1\right)}{\left(u^2+1\right)^2}, \qquad H(u) = \pm\frac{2}{u^2+1}, \tag{9.54}$$

$$K_3(u) = 2\,\mathrm{sech}^2(u), \qquad K_4(u) = 2\,\delta\ \mathrm{sech}^2(u), \quad \delta = 0,1, \qquad H(u) = 0, \tag{9.55}$$

$$K_3(u) = -2\,\mathrm{csch}^2(u), \qquad K_4(u) = -2\,\delta\,\mathrm{csch}^2(u), \quad \delta = 0,1, \qquad H(u) = 0, \tag{9.56}$$

$$K_3(u) = K_4(u) = \frac{\left(1+2\beta^2\mathrm{sech}^2(u)\right)}{\beta^2}, \qquad H(u) = -\frac{2}{\beta}, \tag{9.57}$$

$$K_3(u) = K_4(u) = \frac{\left(1-2\beta^2\mathrm{csch}^2(u)\right)}{\beta^2}, \qquad H(u) = -\frac{2}{\beta}, \tag{9.58}$$

$$\begin{aligned}
K_3(u) &= -\frac{\mathrm{sech}^4(u)\left(2\left(\beta^2+1\right)\sinh^2(u)-\beta^2\right)}{\left(\tanh^2(u)+\beta^2\right)^2},\\
K_4(u) &= \frac{\beta^2\mathrm{sech}^4(u)\left(2(\beta^2+1)\cosh^2(u)+1\right)}{\left(\tanh^2(u)+\beta^2\right)^2},\\
H(u) &= \frac{2\beta}{\beta^2\cosh^2(u)+\sinh^2(u)},
\end{aligned} \tag{9.59}$$

$$
\begin{aligned}
K_3(u) &= \frac{\operatorname{csch}^4(u)\left(2(\beta^2+1)\cosh^2(u)+\beta^2\right)}{\left(\coth^2(u)+\beta^2\right)^2}, \\
K_4(u) &= -\frac{\beta^2\operatorname{csch}^4(u)\left(2(\beta^2+1)\sinh^2(u)-1\right)}{\left(\coth^2(u)+\beta^2\right)^2}, \\
H(u) &= -\frac{2\beta}{\cosh^2(u)+\beta^2\sinh^2(u)},
\end{aligned}
\tag{9.60}
$$

where $\beta \in \mathbb{R} \setminus \{0\}$.

Even though the B-fields obtained by T-duality are usually not of the form $B = B_i(u)\,du \wedge dz_i$, they are gauge equivalent to

$$
B' = H(u)\,du \wedge (z_3 dz_4 - z_4 dz_3),
$$

and the corresponding sigma models are exactly conformal [25].

Except for (9.57), (9.58), these pp-wave backgrounds can be transformed to the gauged WZW background forms (9.1) by the standard transformation from Brinkmann to Rosen coordinates, see section 7.2 or Ref. [57]. In most of the transformed backgrounds the function g_1 acquires the form $g_1(u) = 1$ and the function g_2 acquires the form of one of the functions (9.2). Nevertheless, some other combinations of functions (g_1, g_2) also arise, namely $(u^{-2}, \tanh^2 u)$, $(u^{-2}, \coth^2 u)$, $(\tanh^2 u, \tanh^2 u)$ and $(\coth^2 u, \coth^2 u)$.

Consequently, the pp-waves of the form (9.1) are duals of the flat metric not only for $g_1(u) = 1$ and $g_2(u) = u^2$ as mentioned in [63], but also for many other combinations of functions g_1, g_2 from the set (9.2).

It is a remarkable fact that duals with respect to subgroups corresponding to non-isomorphic algebras may lead to the same background (up to a coordinate transformation). These are the cases of the metric (9.53), produced by the subalgebras S_{17} and S_{29}, and of the metrics (9.55), (9.56), obtained from S_7 and S_8 for $\delta = 0$ and S_{26} and S_{27} for $\delta = 1$. The metric (9.53) is a homogeneous exactly solvable model with a nontrivial constant torsion. On the other hand, isomorphic (but not equivalent under proper ortochronous Poincaré transformations) algebras S_{23} and S_{25} give the same metrics, namely (9.54), but opposite torsions. Isomorphic algebras S_{31} and S_{33} give the same metrics and torsions, namely (9.59), (9.60).

Besides the plane waves, we also get the dual metrics with non-vanishing scalar curvature and torsion:

$$
\begin{aligned}
ds^2 &= -dy_1^2 + dy_2^2 + \frac{y_1^2}{y_1^2+\alpha^2}\,dy_3^2 + \frac{1}{y_1^2+\alpha^2}\,dy_4^2, \\
H &= -\frac{2y_1\alpha}{\left(y_1^2+\alpha^2\right)^2}\,dy_1 \wedge dy_3 \wedge dy_4,
\end{aligned}
\tag{9.61}
$$

$$\begin{aligned} ds^2 &= \frac{1}{y_3^2-\alpha^2}\,dy_1^2 + \frac{y_3^2}{\alpha^2-y_3^2}\,dy_2^2 + dy_3^2 + dy_4^2, \\ H &= \frac{2y_3\alpha}{\left(y_3^2-\alpha^2\right)^2}\,dy_1\wedge dy_2\wedge dy_3, \end{aligned} \tag{9.62}$$

$$\begin{aligned} ds^2 &= -dy_1^2 + dy_2^2 + \frac{y_2^2}{y_2^2+\alpha^2}\,dy_3^2 + \frac{1}{y_2+\alpha^2}\,dy_4^2, \\ H &= \frac{2y_2\alpha}{\left(y_2^2+\alpha^2\right)^2}\,dy_2\wedge dy_3\wedge dy_4. \end{aligned} \tag{9.63}$$

They are obtained as non-Abelian duals with respect to groups corresponding to S_{11}, S_{18} and S_{19}. Note that isomorphic (but non-equivalent under proper ortochronous Poincaré transformations) subalgebras S_{17}, S_{18}, S_{19} lead to backgrounds with both vanishing (for S_{17}) and non-vanishing (for S_{18}, S_{19}) curvature scalar. The metrics (9.61)–(9.63) remind us of black hole [64] and cosmological backgrounds [65] rewritten in [28] into diagonal forms depending again on particular functions g_1, g_2. The difference from (9.61) – (9.63) is in these functions.

Chapter 10

Summary

Throughout the whole thesis we have tried to develop the concept of T-duality in string theory in a systematic way, heading from basic notions to more involved concepts, which were later demonstrated using specific examples. In the current chapter we conclude the thesis, emphasize important properties of Poisson–Lie T-duality and summarize some of our results.

In the beginning we have learned that the dynamics of bosonic strings propagating in a general manifold is given by the sigma model (1.15). We have found the classical equations that have to be fulfilled in order to get a plausible theory, and saw that these equations might be quite hard to solve if the spacetime manifold is not flat or if the torsion is not trivial. Nevertheless, T-duality offers a way to solve them once we know the solution of the dual model.

In chapter 2 we summarized the necessary algebraic and geometric background, and introduced the Drinfel'd double as a connected and simply connected Lie group $\mathscr{D}$ whose Lie algebra $\mathfrak{d}$, equipped with a symmetric ad-invariant nondegenerate bilinear form $\langle .,. \rangle_{\mathfrak{d}}$, splits as $\mathfrak{d} = \mathfrak{g} \oplus \widetilde{\mathfrak{g}}$ into a pair of maximally isotropic subalgebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$. Their corresponding connected and simply connected Lie groups $\mathscr{G}$ and $\widetilde{\mathscr{G}}$, which form Poisson–Lie groups, were shown in chapter 4 to accommodate dualizable sigma models.

Following the historical development, we started the study of T-duality in chapter 3 by a careful examination of Abelian and non-Abelian T-duality. We have seen that in order to perform the duality transformation within these frameworks, it is necessary that string backgrounds have symmetries. However, chapters 8 and 9 offer many examples which demonstrate that non-Abelian T-duality does not preserve these symmetries. This is why T-duality should be understood in the broader context of Poisson–Lie T-duality.

The concept of Poisson–Lie T-duality was described in chapter 4. We have seen that the presence of symmetries of the background is in fact not necessary. Instead, there has to be a Lie group $\mathscr{G}$ acting freely on the target manifold $\mathscr{M}$. We have seen that having a particular string background, one can find its dual if and only if the condition (4.13) is satisfied. We have learned that mutually dual sigma models can be build only on subgroups $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ of some Drinfel'd double via the procedure presented in section 4.3, and that the dual backgrounds are specified by the structure of the

Drinfel'd double and some matrix $E(e)$ through (4.36), (4.37), or (4.41) respectively. We also discussed Poisson–Lie T-plurality, which takes into consideration the fact that the particular Drinfel'd double can be decomposed in several ways, thus offering more possibilities to relate apparently different sigma models.

In chapters 5 and 6 we presented the results of the paper [44]. Namely, we focused on the case when the action of $\mathscr{G}$ on the target manifold $\mathscr{M}$ is free but not transitive and we cannot identify $\mathscr{G} \approx \mathscr{M}$. Using a suitable extension of the Drinfel'd double, we reconstructed the previously known formulas for Poisson–Lie T-duality with spectator fields, and derived the formulas for Poisson–Lie T-plurality transformation of sigma models with spectators. Moreover, in chapter 6 we have found formulas for the Poisson–Lie transformation of gluing matrices in the presence of spectators, thus shedding some light into the behavior of boundary conditions of open strings under T-duality.

Having developed Poisson–Lie T-duality/plurality into its full strength, we decided to apply this tool on a particularly interesting class of spacetime metrics called plane-parallel waves. In chapter 7 we summarized some of the most important properties of plane waves, emphasizing their exceptional curvature properties and rich structure of their symmetry group.

The simple structure of plane wave metrics in Brinkmann coordinates allows us to adopt the light-cone gauge (8.6). The field equations for transversal coordinates then simplify to linear equations, and one may try to solve them using the expansion into Fourier modes. This standard technique was used in [24] to find the solution of a sigma model living in the homogeneous isotropic plane wave background. The model can be dualized with respect to subgroups of the group of symmetries, which were previously found for the four-dimensional case in [37]. Continuing in this work, we have found the solutions of the dual models using Poisson–Lie T-duality. Moreover, investigating the symmetries of these duals, we have learned that the dual metrics again represent plane waves. These results, which enlarge the family of exactly solvable sigma models, were published in the paper [45] and summarized in chapter 8.

Finally, in chapter 9 we have summarized the results of the paper [46], and classified the atomic non-Abelian duals of the sigma model living in the four-dimensional flat Minkowski background. The dualization was carried out with respect to 17 four-dimensional subgroups of the Poincaré group that act freely and transitively on the target manifold. Investigating the curvature properties of the dual backgrounds, we realized that three of them again represent sigma models in the flat spacetime with vanishing torsion. Eleven of them, however, give sigma models with various plane wave metrics. The remaining duals have diagonalizable metrics with non-vanishing curvature. Since the string equations of motion can be solved easily in the flat background, we were able to use the Poisson–Lie T-duality transformation to find general solutions of the dual models as well. Such solvable models are still rare, and the results may be valuable for further investigation of string behavior in curved backgrounds.

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