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Simulations of Agents on Social Network

Computer Simulations of Multi-Agent Systems placed on Complex Networks
Dissertation

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Contents

Chapter 1

Foreword

In the dark ages, enhanced and more efficient language in combination with biological versatility caused victory of *Homo Sapiens Sapiens* in biological battle over Europe resources with *Homo Sapiens Neanderthalensis*, according to one possible evolution theory. Specially, the language allows to exchange ideas and coordination of the groups of people and coordination of the individuals. Better coordination can facilitate creation of bigger organizationals. The society evolved from small communities to small villages to the cities, where civilization has been born and the cities allow to form first city states and after then empires. All that was allowed by forming of the formal organizations that are based on communication.

Scientific understanding of natural processes since late the 18th century in combination with its practical usage allowed extraordinary progress of mankind ability including exploration of space but it brings new problems - energy consumption, pollution and overcrowding of cities - as it is mentioned for example by Meadows in [1], Mesarovic et al. in [2], Fromm in [3] and Schweitzer in [10]. Although, the first alarming messages were published in early 70s of the 20th century, almost nothing has been done since its publication. Many authors f.e. Fromm in [3] or Keller in [5] showed global irresponsibility of mankind against foretold crisis that could burst out before 2050 and they proposed system change of moral values and imperatives. They also mentioned that such shift is improbable due to the radical personal requirements. Nevertheless, it can be reasonable to try to change external parameters of the society what would prepare society to fix the problems. The very first point to fix problems is to understand what are the processes in the society and Game Theory shows to be excellent tool [8], [9] and [10].

Mathematician von Neumann and economist Morgenstern published the first book about the Game Theory [8], where they focused on understanding of the economical behavior. Their original imagine later inspired sociologists, biologist, psychologist and philosophers to use and to understand processes their field of study through prism of the Game Theory what can be illustrated by famous book by R. Dawkins [9].

The author thinks that this extraordinary success of the Game Theory in understanding of wide spectrum of problems can be the possible tool, in combination with the other tools and the faith, to solve the global problems ahead. There is one visible large-scale application (**ABM** treaty and **détente** policy). Briefly, the author hopes that the thesis will contribute to formulation of more accurate models that can be used to formulate a policy that would help preventing the problems on the threshold of the 21st century.

1.1 Organization of the thesis

The thesis is divided into four parts. The first part deals with complex networks which are the base bricks of models that being simulated. The next part has three chapters which are focused on models of human behavior based on the networks. The first chapter deals with opinion formation and voting process. In the following chapter, we study a model that can explain structure and evolution of organizations. Finally, wealth distribution within society is investigated in the last chapter. The third part of the thesis describes the simulation platform within which all the results presented in this thesis were obtained. In the final part we deal with conclusions of the thesis.

Chapter 2

Contemporary state of the research

In the last few years, there has been huge development in the mathematical understanding of society where certain subsystem of society was investigated and mapped to a graph. It allows to view society as a set of networks, f.e., contacts of the actors and some of the networks is the substrate for processes running within the society[1]. Thus, it seems to be natural that multi-agent systems (MAS) is used in the investigation the processes. Due to complicated structure of the networks we are forced to use models of complex networks and the dynamic of the process can be specified using the following mental processes:

- Inductive understanding of the process from microlevel to macro-level
- Application of a model to a system without detailed understanding

[1]That view completely neglects spatial behavior of actors which nowadays uses long distance communication devices like a telephone or Internet, where no direct contact is possible.

of interactions at microlevel but giving qualitatively the same results.

Inductive understanding is based on knowledge of internal behavior of the key elements (agents). It will be used for the opinion formation process that can mainly influence votes and Minority Game with imitation.

Application of a new model and a data comparison of simulation and reality with final reasoning is used when internal properties of the key elements are not known. It is used for the analysis of wealth distribution and this involved the use of scattering models, where a qualitative comparison with the data is done. Generally, we are trying to use the tools taken from physics and apply them on society.

We are attempting to imagine society as an evolving complex system, where some certain subsystems can be modeled on a computer. However, this point of view does not take into account the psycho-social base of human behavior and this field of study are called Sociophysics or Econophysics respectively depending on application of the model. The main aim is to answer a question how socio-economic and the political organization of system works by using statistical physics methods.

2.1 General state of the research

Computers that form computer networks simplified communication and allowed to access huge storages of data of physical experiments. It allowed as well to access the data from the other fields of study, e.g., economy, sociology or psychology, and we can use analyze them in a similar way like data from physical experiments. Moreover, the experiment can bring new results that attacking on mainstream theoretical explanations and it allows to formulate alternative theories that mainly ignore psycho-social base of behavior.

The thesis deals with three agent-based models that are implemented on complex networks. The investigated models are

- Sznajd model as model of opinion formation
- Minority Game with imitation
- Scattering model as model of wealth distribution.

Briefly, the complex networks were described in review article [16] by Albert et al., in books like [15] by Dorogovtsev et al. or at Wikipedie[14] [2]. We can find broad area of different systems that are mapped to a graph going far behind main focus of the thesis.

Sznajd model is a model that is allowed to explain opinion formation. It and its modifications was reviewed by Stauffer in [106] and comparisons with real experiments (votes) are discussed. Success of the model to explain experiments allowed to present articled in public journals and newspapers.

Minority Game is formalization of bar attendance that have many variants and modifications. The model was formulated in 1997 by Zhang and Challet in [127]. The model was widely investigated and modified and the summary of articles by Challet et al. can be found in [126]. Easy-to-understand introduction to Minority Game is in [14][3].

Wealth distribution was investigated during last century in, e.g., [150] by Pareto, [152] by Gibrat, [159] by Dragulescu et al. and explained in, e.g., [166] by Solomon or [171] by Bouchaud et al. using different base mechanisms. So, the models are able to explain the main characteristics

[2] using key-phrases Scale-free network, Small-world network
[3] using key-phrase Minority Game

of wealth distribution but they are unable to form a policy due to internal incompatibility.

Part I

Networks

Chapter 3

Complex networks

3.1 Introduction

The first investigation of social contacts was performed by Stanley Milgram in [95], which motivated scientists to question the structure of network of social contacts. Moreover, they started to question human behavior that lay on the substrate of social contacts, e.g., a sexual contact network, a network of co-authorship in different fields of studies, network of contacts made by long telephone calls and network of collaboration of movie actors. Then, power grid structure, Internet structure (WWW pages, routers or domains) and structure of word cooccurence and word synonyms were investigated. Finally, network structure is not privileged only for human behavior but nature gives us many examples of networks, e.g., metabolic network in an organism, protein network in a cell or food chain networks.

The first paragraph deals with empirical networks and their properties, where the mapping of the system is to a graph. Using such mapping we completely ignore some qualities of the system like interaction with neighborhood (other systems) or other level of interaction. The mapping is usually maps the objects to vertices and interaction (collaboration, cooperation, connection or communication) of the objects is an edge between two vertices. This simplification of the system transforms the problem to investigation of a graph. Thus, it is natural to question to investigate properties of graphs like average of the shortest distance (number of edges between initial and final vertex), the connectivity distribution or the clustering coefficient.

The results of the experiments were challenge for Graph Theory to explain such properties. One of the first questions to arise is the randomness of the networks. The networks are products of a complex behavior of complex system and so the topology of the network must display some organization principles. Due to fact that the system is interconnected with neighborhood the graph must also contain certain level of disorderliness. So it means that randomness plays one of the key roles in the construction. So modeling of the graph forces us to use of stochastically generated graphs.

The following paragraphs begin to deal with empirical results. The question is how to get qualitative correspondence with empirical graphs. The simplest model of stochastic graphs is Erdös-Renyi random network. Properties of the model differs from empirical results and so it forces us to search for the other models that are introduced in the next paragraphs.

3.2 Empirical results

Empirical results are the base of models that allows to build theories [14][1]. In particular, measurement of properties of a society allows us to understand principles of organization of society. To proceed we define a graph, then measured variables are introduced and finally, results of measurements of wide spectrum of examples are discussed.

3.2.1 Graph theory

A *graph* $G = (V, E, \epsilon)$ with oriented edges is defined in book by Demel [13], where V is a set of vertices, E is a set of the edges and ϵ is mapping $\epsilon : E \to V \times V$. $V_G(i)$ is a set of nodes that are neighbors (in sense of directed connection by a single edge) of node i. Finally, $E_G(i)$ is a set of edges connecting node i with its neighbors.

The original definition involves consideration of the oriented edges. However, the orientation of the edges can be redundant information in some models and in such a situation orientation can be neglected. Such a structure is called a *graph* with non-oriented edges.

Both definitions of the graphs include multiple edges[2] but effectively single edges[3] are sufficient. This property does not influence functions that are defined on graphs. Finally, the graph can be generalized to carry extra information. These graphs can be defined as $G = (V, E, \epsilon, \nu_V)$, $G = (V, E, \epsilon, \nu_E)$ and $G = (V, E, \epsilon, \nu_V, \nu_E)$, where $\nu_V : V \to O_V$, $\nu_E \to O_E$ and O_V, O_E are sets of the extra object which can be a real or natural number[4]

[1] Using key-phrase Theory
[2] A pair of vertices is connected by many edges.
[3] A pair of vertices can be connected once at most.
[4] It is *weighted graph*.

or, more generally, an extra object can be a set of agents playing a game[5].

The crucial variable of the thesis is connectivity k_i of a vertex i what number of elements of $E_G(i)$. Lattices that are used in physics for modeling materials, have usually $k_i = c$ where c is constant and the lattice is periodical without irregularities. On the other hand, we can observe graph-like structures in society but the connectivities k_i are random variables with certain fixed probability distribution in contrast to the lattices in physics. Thus we are forced use tools of probability and mathematical statistics to Graph Theory.

3.2.2 Measured variables

The networks are represented by graphs and analyzed variables must be defined on graphs where 3 main properties of graphs are investigated:

- small world property (average distance between vertices of graph)
- clustering
- degree distribution (probability distribution of connectivities).

Small worlds The small-world effect shows that although, the number of nodes can be hundreds of millions but the average of the distance of two nodes, which is defined as a number of edges along the shortest path connecting them, is relatively short. The first manifestation of the property on the networks was shown by psychologist Stanley Milgram in paper [95]. However, the small-world property is not an indicator of organization in the system, because the property is held by random networks.

[5]It is *generalized weighted graph*.

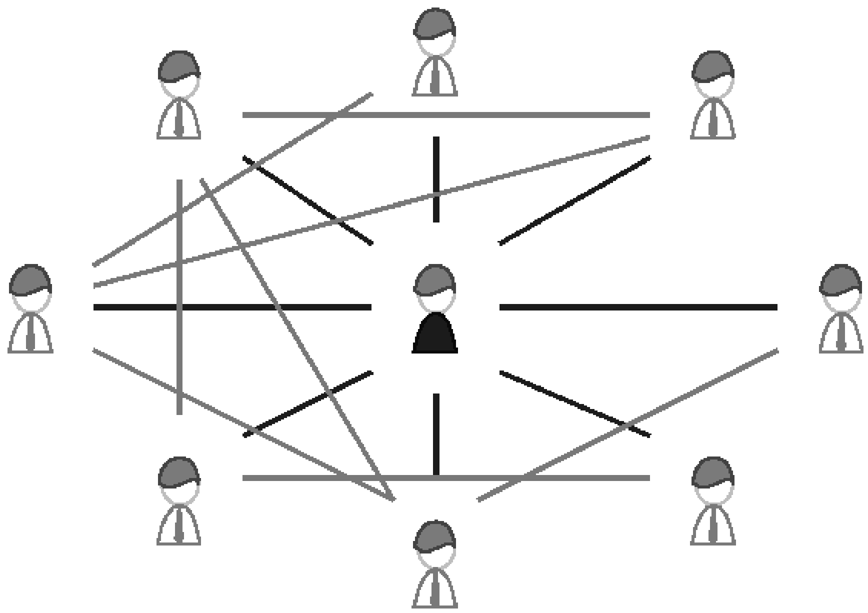

Figure 3.1: Effect of clusterization. The blue edges are connections of the actual person and the red edges are counted as connection (acquaintance) between friends.

Clustering All social networks hold a property that it is very probable to find a common friend of a pair of friends, as it can be seen from figure 3.1. The property is measured by a clustering coefficient C, which is defined as the average of clustering coefficients C_i of node i.

$$C = \frac{1}{|V|} \sum_{i \in V} C_i \tag{3.1}$$

The definition of C_i is

$$C_i = \frac{\sum_{j \in V_G(i)} \sum_{k \in V_G(i) \setminus j} \omega(j,k)}{|V_G(i)| \left(|V_G(i)| - 1\right)} \tag{3.2}$$

where

$$\omega(j,k) = |\bigcup_{l \in E_G(j) \bigcap E_G(k)} \{l\}|. \tag{3.3}$$

$|A|$ means a number of members in the set A. The denominator of the equation 3.2 number of possible connections between neighbors of node i and the numerator is real number of connections (Non-oriented edges are counted twice.).

Watts and Strogatz published in [22] results which showed that for a real network, that clustering is typically larger than in random networks of an equal number nodes and edges.

Degree distribution The number of edges can differ from a node to the another node, thus it seems reasonable to ask for the distribution of connectivities $P(k)$. In the case of random graphs with maximal value and the mean value at $P(\langle k \rangle)$ it is Poisson distribution.

However, a huge real network has a distribution of connectivities significantly different and this is characterized with a "fat-tail." It means that the higher end of the distribution $P(k)$ has property

$$P(k) \sim k^{-\alpha}, \tag{3.4}$$

where α is a power-law exponent. Some networks display an exponential tail which still deviates from the Poisson distribution of the random network model.

3.2.3 Examples of networks

Networks were observed in many field of studies but the most significant network is the Internet. It is described as a symbol of Globalization. However, humans are not the only ones who can produce complex networks as the production of the networks can also be by nature.

Internet

The hardware of the Internet is formed by the physical links between the computers and the other communication devices. There are two different level of observation of the Internet. At the router level, routers are mentioned as nodes and the edges are the physical connections between a pair of routers. At the inter-domain (or autonomous system) level, each node represents a domain, which includes many routers and computers, and the edge is constructed if two domains have at least one physical connection.

The power-law exponent around year 1998 fluctuates between $\alpha_I^{as} = 2.15$ and $\alpha_I^{as} = 2.2$ for the inter-domain level but the exponent for the router level is $\alpha_I^r = 2.48$, which was recorded by Faloutsos et al. in [23]. However, another set of routers give a power-law exponent of $\alpha_I^r \simeq 2.3$.

A question of emergence of small average path length and the clustering coefficient C was investigated by Yook et al. in [25] and by Pastor-Satorras in [26] between 1997 and 1999 and these authors found that the clustering coefficient C is between 0.18 and 0.3. It can be compared with $C_{rand} \simeq 0.001$ for random graph with similar parameters.

The average path length was between 3.7 and 3.77 at the domain level in [25] and [26] and it was around 9 at the router level in [25].

World-Wide Web

The World Wide Web is the largest network for which topological information is available. The number of nodes was close to 10^9 as reported by Lawrence et al. in [27], [28] but at present it is now close to 10^{10}. The web pages are represented by nodes and the edges are the hyper-links (URLs[6]) which point from one page to the another.

The hyper-links are oriented such that it can be plotted as 2 degree distributions - for outgoing pages $P_{out}(k)$ and incoming pages $P_{in}(k)$. Both of them show a power-law degree distribution over several magnitudes - $P_{out}(k) \sim k^{-\alpha_{out}^{WWW}}$ and $P_{in}(k) \sim k^{-\alpha_{in}^{WWW}}$. It was found that $\alpha_{out}^{WWW} = 2.45$ and $\alpha_{in}^{WWW} = 2.1$ in [29] by Albert et al. However, another group found that $\alpha_{out}^{WWW} = 2.38$ and $\alpha_{in}^{WWW} = 2.1$ in [30] by Kumar et al. Broder et al. in [31] used Altavista and they observed that $\alpha_{out}^{WWW} = 2.72$ and $\alpha_{in}^{WWW} = 2.1$.

Another definition of the graph was reported by Adamic et al. in [33], where the nodes are the domains and the edges are present if there is a web page with the URL to a web page at the second domain. The incoming degree distribution is the power-law with exponent $\alpha_{in}^{dom} = 1.94$, which is stable. However, the distribution of outgoing edges follows the power-law but the exponent α_{out}^{dom} grows up during couple years between measurements.

The WWW networks have the small world property and the first measurement of the WWW is reported by Albert et al. in [29] and their data (approximately $3 \cdot 10^5$ nodes) show an average path length at 11.2 and the extrapolation for whole Internet (assumed size of $8 \cdot 10^8$ nodes) in 1999, is 19. Broder et al. reported in [31] that they acquired different data (around $2 \cdot 10^8$ nodes) with an average path length 16, which is in a good agree-

[6]Universal Resource Locator

ment with previous measurements. The domain level network displays an average path length of 3.1 as reported by Adamic et al. in [32].

URLs are directed objects so is not possible to use formula 3.2. Adamic et al. of [33] transformed the directed edges to the undirected edges and they dropped web pages with only one edge and the final network consisted of approximately $1.5 \cdot 10^5$ nodes. The clustering coefficient with these modifications of the network was shown $C_{WWW} = 0.1078$. This value is larger than $C_{rand} = 0.00023$ for the random graph with the same size and the average degree.

Movie actor collaboration network

The Internet Movie Database shows a interesting network of movie-actor collaboration. The network is still growing from approximately $2 \cdot 10^5$ [22] in 1998 reported by Watts et al. to $4.5 \cdot 10^5$ [34] by Newman et al. by May 2000. These networks have a small world property with average path length 3.65 whereas a random network with the same parameters has value of 2.9. However, the clustering coefficient is more than 100 times larger than a random graph. The degree distribution $P_{actor}(k)$ follows the power-law with exponent around $\alpha_{actor} \simeq 2.3$. It was reported by Barabási et al. in [35], [36] and Amaral et al. [37].

Science collaboration network

The science collaboration network is very similar to the movie collaboration network. The scientists are the nodes and the edge is participation in a paper. A community of physicists, biomedicine researchers, high-energy physicists, computer scientists , neuroscientists and mathematicians were investigated in papers by Newman et al.[38], [39], [40] and by Barabási et al. [41] during the 5 years between 1995 and 1999. The small world property was observed in all cases. The higher end of the degree distribution follows the power-law and, finally, the clustering coefficient is significantly higher than for a random network with similar parameters.

Network of human sexual contacts

Human sexuality is a delicate theme, which was investigated and reported in papers by Liljeros et al. [42] and [43]. A network was constructed from a survey carried out in Sweden in 1996 with a population of 2810 individuals. The duration of the survey was relatively short but one year seemed to be enough to support the conclusion that the degree distribution follows the power law where $P_{female}(k)$ and $P_{male}(k)$ for males as well as females with exponents around $\alpha_{female} = 3.5$ and $\alpha_{male} = 3.3$.

Cellular networks

The metabolism of 43 organisms was studied in paper by Jeong et al. [44]. A network was constructed from chemical substances and an edge represent when two chemical substances reacting. All the networks follow the power law with exponents between 2.0 and 2.4. The average path length was almost the same for all organisms giving a value of 3.3.

The complex networks properties were investigated in papers by Fell et al.[45] and by Wagner et al. [46], where the authors focused on energy and biosynthesis of the *Escherichia Coli* bacterium. A short average path length, a higher clustering and the power-law behavior were all observed.

Another important network can be extracted from protein-protein interactions. The nodes are proteins produced from DNA[7] and the edges are

[7]Deoxyribose Nucleic Acid

present when two proteins bind together. The degree distribution follows the power-law with an exponential cutoff

$$P(k) \sim (k + k_0)^{-\alpha} \exp\left(-\frac{k + k_0}{k_c}\right) \tag{3.5}$$

with $k_0 = 1$, $k_c = 20$ and $\alpha = 2.4$ as was reported by Jeong et al. in [47] .

Ecological networks

Ecologists use food webs to present interaction between the species as reported by Pimm et al. in [11]. A food web has nodes, which mean different species, and the edges represent predator-prey relationship. The topology of seven of the largest and the most documented food webs were studied by Williams et al. in [48]. The small-world property was observed.

Food webs are highly clustered if the edges are taken as non-oriented [49] by Montoya et al., [50] by Camacho et al. The first research on degree distribution was reported by Montoya et al. in [49]. However, the networks were relatively small - the largest having $N = 186$ nodes. The authors fit the food web by the power law with exponent $\alpha_{food} \simeq 1.1$ but an exponential fit was also reported as being used [50], [51] by Camacho et al.

Phone-call networks

Telephone numbers as the nodes and long-distance telephone calls during one day between the numbers can produce a huge network which was investigated by Abello et al. in [52] and Aiello et al. in [53]. Degree distributions $P^{in}_{phone}(k)$ and $P^{out}_{phone}(k)$ follow the power law with exponents $\alpha_{out} = \alpha_{in} = 2.1$

Citation networks

An article published in a journal, could be taken as a node in a citation network and an oriented edge is the citation of an article by another one. Orientation is from the citing article to the cited article. Redner in [54] examined papers cataloged by the Institute of Scientific Information (almost $8 \cdot 10^5$ papers) and by Physical Review D (almost $2.5 \cdot 10^4$) between 1975 and 1994. The data shows a power-law degree distribution $P_{cite}^{inco}(k)$ with the power-law exponent $\alpha_{cite}^{inco} = 3$.

Vázquez reported in [55] investigation of a similar network where an edge is present in the network if an article cite another one but the edge is oriented from citing article to the cited article. The degree distribution P_{cite}^{outgo} has an exponential tail.

Networks in linguistics

Human languages provide a very interesting system due to the complexity and there are many possibilities to define and study such complex networks. English was investigated by Ferrer et al. in [56], where the British National Corpus was taken as source of a network. A word is a node in the network and an edge connects two words if they were observed in the sentences next to or one word apart. This network size is around $4.5 \cdot 10^5$ nodes. The network has the small-world property with average path length 2.67 and a large clustering coefficient $C_{ling} = 0.437$ and, finally, the degree distribution $P_{ling}^{cooc}(k)$ follows the power law in two regimes. The first relates to low connected words $k \leq 10^3$ with the power-law exponent $\alpha_{ling}^{cooc\,<} = 1.5$ and the word with $10^3 \leq k \leq 10^5$ follows the power law with the exponents $\alpha_{ling}^{cooc\,>} \simeq 2.7$.

In another study by Yook et al. in [57], the authors reported the construction of a network from Merriam-Webster Dictionary. Here, a word is represented as a node and an edge is present if a pair of word are synonyms. The size of the network was $2.4 \cdot 10^4$ but the giant cluster was formed by $2.2 \cdot 10^3$ nodes. The average path length was $l = 4.5$, the clustering coefficient was $C^{syn}_{ling} = 0.7$ and the degree distribution $P^{sym}_{ling}(k)$ has the power law with the exponent $\alpha^{syn}_{ling} = 2.8$.

Power and neural networks

The power grid of the USA forms a complex network, where generators, substations and transformers are nodes in the network and an edge is made when a connection between a pair exists. The networks were studied by Watts et al. in [22] where its size was shown as $N \simeq 5000$, within average connectivity of $\overline{k} = 2.67$. The average path length is a longer than that for a random network, but a higher clustering coefficient was measured. However, the degree distribution follows exponential decay.

Another example of a network is a neural network of the nematode *Caenorhabditis elegans* reported by Watts et al. in [22]. This network consisted of nodes which represent neurons and an edge is present if two neurons are connected by a synapse or a gap junction. The average path length was relatively small and the clustering coefficient was larger than that found for a random network of similar parameters. The degree distribution also follows an exponential decay.

Protein folding networks

The folding of proteins can also give an interesting network, where the nodes are conformations of a protein and an edge is made between two conformations if one can be transformed to the second by an elementary move. The network was investigated for a $2D$ lattice polymer in paper by Scala et al. [58] and the small-world property was found with a higher clustering coefficient. However, the degree distribution was reported by Amaral et al. to be Gaussian in [36].

3.3 Models of Complex networks

Graph Theory was based as the work of Leonhard Euler and early investigation was focused to a small graph with a high degree of regularity as reported by Albert et al. in [16] and such graphs are useful in physics. On the other hand, previous section showed networks in various fields of studies and its mapping to a graph produce irregular graphs contrasting with regular lattices on physics. Modeling of these networks requests to use stochastic graphs where the edges of the graph are placed with certain level of randomness.

3.3.1 Random networks

The theory of random graphs was introduced by Paul Erdös and Alfréd Rényi in a series of papers [59], [60], [61]. Afterthen, probabilistic approach to Graph Theory was introduced.

Construction

Erdös and Rényi in the first article on random graphs define it as a graph with N nodes and the edges are taken randomly from $\frac{N(N-1)}{2}$ possible edges [59]. In a set of such graphs, there are $C_n^{\frac{N(N-1)}{2}} = \binom{\frac{N(N-1)}{2}}{n}$graphs. Where n is number of edges.

An alternative definition of a random graph is called a binomial model. We start with N nodes and a node is connected to the other node with probability p. Total number of edges E is a random variable with expectation value $\langle E \rangle = p\frac{N(N-1)}{2}$. Lets have a graph G_0 with N nodes and n edges then probability obtaining the graph G_0 using the random construction is $P(G_0) = p^n(1-p)^{\frac{N(N-1)}{2}-n}$.

The random graph theory study graphs with N nodes but its properties are questioned in limit $N \to +\infty$. Such ensemble of graphs can be investigated having property Q. The ensemble have property Q if for almost all elements of the ensemble have property Q in limit $N \to +\infty$.

Properties

Small-world Fronczak et al. in [62] derive the general formula for average path length on a graph

$$l = \frac{\ln\left(\langle k^2 \rangle - \langle k \rangle\right) - 2\langle \ln k \rangle + \ln N - \gamma}{\ln\left(\frac{\langle k^2 \rangle}{\langle k \rangle} - 1\right)} + \frac{1}{2}, \tag{3.6}$$

where γ is Euler's constant. However, an exact value of the average path length for a random graph cannot be acquired from 3.6 and simplification must be used to give the following

$$l_{rand} = \frac{\ln N - \gamma}{\ln(pN)} + \frac{1}{2}, \tag{3.7}$$

which scales the same way like the diameter of a random graph, which means the maximal distance between a pair of nodes. The theoretical value of the diameter of a random graph is shown by Albert et al. in [63] and by Dorogovtsev et al. in [64] to be

$$d_{rand} = \frac{\ln N}{\ln (pN)}. \tag{3.8}$$

Clustering coefficient Using equation 3.2, the clustering coefficient can be computed and in [16], they obtain

$$C_{rand} = p = \frac{\langle k \rangle}{N}. \tag{3.9}$$

Simple comparison of real networks and a random network show a large deviance for the clustering coefficient [16].

Degree distribution The founders of random graph theory, Erdös and Rényi studied the degree distribution [59] but the whole distribution was derived by Bollobás in [65]. He found that the distribution approximately followed the Poisson distribution

$$P(k) \simeq \exp(-pN) \frac{(pN)^k}{k!} = \exp(-\langle k \rangle) \frac{\langle k \rangle^k}{k!}, \tag{3.10}$$

where $\langle k \rangle$ is constant and it is parameter of Poisson distribution.

3.3.2 Watts-Strogatz model

In the previous section, there were many examples of complex networks and all of them had the small-world property in combination with the clustering coefficient higher than in a random network of the same parameters.

Watts and Strogatz in [22] suggest the following algorithm. The algorithm starts with an ordered network, which had a high clustering coeffi-

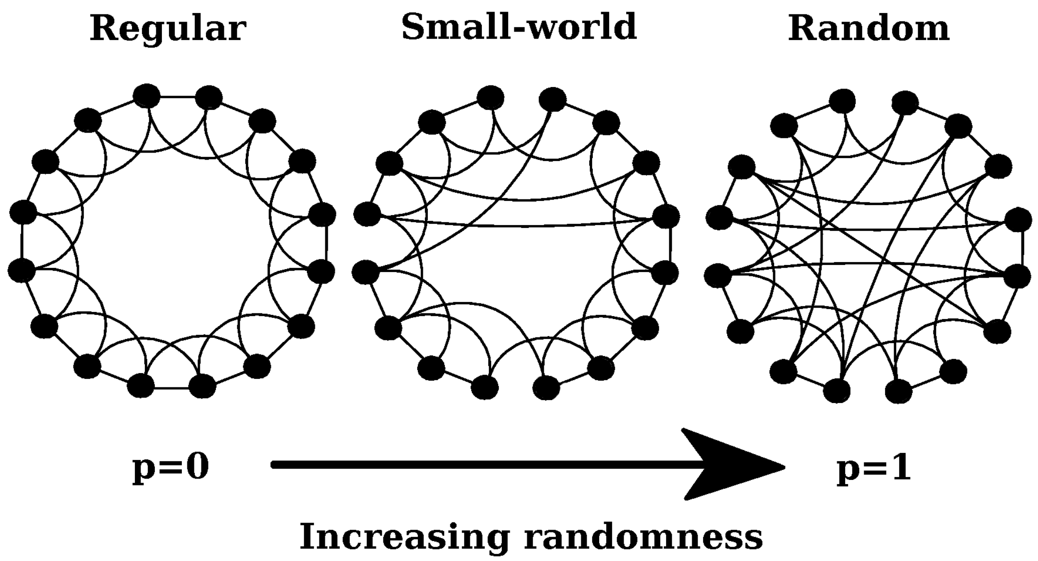

Figure 3.2: Algorithm of Watts-Strogatz model, which can be tuned by parameter $p \in [0, 1]$. Increasing of the parameter p produce randomness in the network.

cient and a high average path length. Finally, the network is randomized, which produces shortcuts in the structure.

Construction

Watts and Strogatz [22] introduced an one-parameter model which interpolates between an ordered lattice and a random graph and its root is in social systems where most people have friends, which are common in a community, e.g., in a house, in a street, in a workplace or an interest group.

Start with order The algorithm starts with an ordered network, which can be a one-dimensional lattice with periodic boundary conditions where node has connection to the K closest nodes (see figure 3.2). The assumed initial network has the clustering coefficient

$$C_{init} = \frac{3}{4}\frac{K-2}{K-1}. \tag{3.11}$$

However, low-dimensional regular lattices do not have the short path length - for a d-dimensional hyper-cubic lattice, the average path length scales as $N^{\frac{1}{d}}$, which grows faster for increasing N than logarithmic increase, which was observed in a random network or real networks.

Randomization In this case the edges of a graph are rewired with probability $p \in [0, 1]$, but self-connections and duplicate edge are disabled. The process creates $\frac{pNK}{2}$ long range connection (shortcuts), which significantly decrease the average path length.

Properties

The research reported in [22] by Watts et al. discusses the influence of research on the properties of small-world networks and the Watts-Strogatz model. The next investigated variant is only with adding more edges than simple rewiring [66], [67] by Newman et al. This model is easier to analyze than the original Watts-Strogatz model because it does not lead to the formation of isolated clusters. However, for high N and small p, the models are equivalent.

Small-world As it was mentioned above, in the Watts-Strogatz model, a change of scaling take place for the average path length l. For a small p, l linearly increases but a large p tends to logarithmic scaling. Appearance of shortcuts influences of behavior of l [68] by Watts et al., [69] by Pandit et al., where a shortcut connects widely separated communities. Thus, the shortcut has significant impact on the length of path between persons in the communities.

We may ask about dependence of the small-world property on the system size as in statistical physics. The first remark about that was reported by Watts [68], where it was noticed that l cannot decrease until boundary $p_0 = \frac{2}{NK}$ is reached. The nonexistence of a shortcut in the system in region $p \in [0, p_0]$ leads to the permanence of the average path length. The property implies that transition p_0 depends on system size, or conversely there exists N^*, which is depended on p which was introduced by Barthélémy et al. in [70]. $l \sim N$ if N is the region $N < N^*$ and $l \sim \ln N$ if $N > N^*$. Barthélémy and Amaral reported in [70] that the average path length should scale as

$$l(N, p) \sim N^* F(\frac{N}{N^*}), \tag{3.12}$$

where

$$F(u) = \begin{cases} u & \text{if } u \ll 1; \\ \ln u & \text{if } u \gg 1. \end{cases}$$

From investigations reported by Newman et al. in [67], by Barthélémy et al. [70], by Barrat et al. [71], by Argollo et al. [73], by Barrat et al. [74], it can be concluded that the crossover length N^* scales with p as $N^* \sim p^{-\tau}$, where $\tau = \frac{1}{d}$. d means dimension of starting lattice where random rewiring is used. In the simplest case $d = 1$, it can be got $p_0 \sim \frac{1}{N}$.

Now, it is accepted that l has the general scaling form as

$$l(N, p) \sim \frac{N}{K} f(pKN^d), \tag{3.13}$$

where $f(u)$ is universal scaling function with property

$$f(u) = \begin{cases} constant & \text{if } u \ll 1; \\ \frac{\ln u}{u} & \text{if } u \gg 1. \end{cases}$$

Further comments are reported by Albert et al. in [16].

Clustering coefficient If an initial lattice of the model has a large clustering coefficient $C(0)$ then it still can be high after the rewiring process. Thus, cooccurence of the small-world property and the high clustering coefficient can be found for region of parameter p.

Calculation of $C(p)$ needs to keep in mind that if $p > 0$ then a triplet of nodes which were connected decay as $(1-p)^3$. So, the clustering coefficient dependence is

$$C(p) \simeq C(0)\,(1-p)^3\,. \tag{3.14}$$

Verification of the deviance from 3.14 vanish in limit $N \to \infty$ and it was done in [74] by Barrat et al.

Degree distribution The degree distribution is a delta function for $p = 0$ because every node has the same degree. Non-zero p causes disorder in the network and it influences the degree distribution. Only the single end of an edge is changed. Thus, a node can minimally have $\frac{K}{2}$ edges after rewiring process. For $K > 1$, there are no isolated nodes and the network is usually connected.

The following computation was reported by Barrat and Weigt in [74]. For $p > 0$, the degree k_i of vertex i can be written as $k_i = \frac{K}{2} + c_i$, where c_i can be decomposed into two parts c_i^{left}, $c_i^{rewired}$. c_i^{left} is connected with edges, which were not rewired with probability $1-p$ and $c_i^{rewired}$ is connected with incoming connections during rewiring towards i (with weight $\frac{1}{N}$). The probability of distributions for c_i^{left}, $c_i^{rewired}$ are

$$P_1^{left}(c) = C_{\frac{K}{2}}^{c}(1-p)^c p^{\frac{K}{2}-c} \tag{3.15}$$

$$\begin{aligned} P_2^{rewired}(c) &= C_{\frac{pNK}{2}} \left(\frac{1}{N}\right)^c \left(1-\frac{1}{N}\right)^{\frac{pNK}{2}-c} \\ &\simeq \frac{\left(\frac{pK}{2}\right)^c}{c!} e^{\frac{-pK}{2}} \end{aligned} \tag{3.16}$$

A combination of the factors led to

$$P(k) = \sum_{n=2}^{f(k,K)} C_{\frac{K}{2}}^{n} \left(1-p\right)^n p^{\frac{K}{2}-n} \frac{\left(\frac{pK}{2}\right)^{k-\frac{K}{2}-n}}{\left(k-\frac{K}{2}-n\right)!} e^{-\frac{pK}{2}}, \tag{3.17}$$

for $k \in \mathcal{N}$ with condition $k \geq \frac{K}{2}$, where $f(k,K) = \min\left(k-\frac{K}{2}, \frac{K}{2}\right)$.

The topology of the network is relatively homogeneous because $\langle k \rangle = K$ and it exponentially decays. The connectivity distribution was simulated by the author of the thesis and the results are in the figure 3.3.

3.3.3 Barabási-Albert model

Some empirical studies of the real networks show an exponential decay for the degree distribution but mainly, the decay follows the power-law with exponent α. The random network model and Watts-Strogatz model did not allow the power-law degree distribution

$$P(k) \sim k^{-\alpha} \tag{3.18}$$

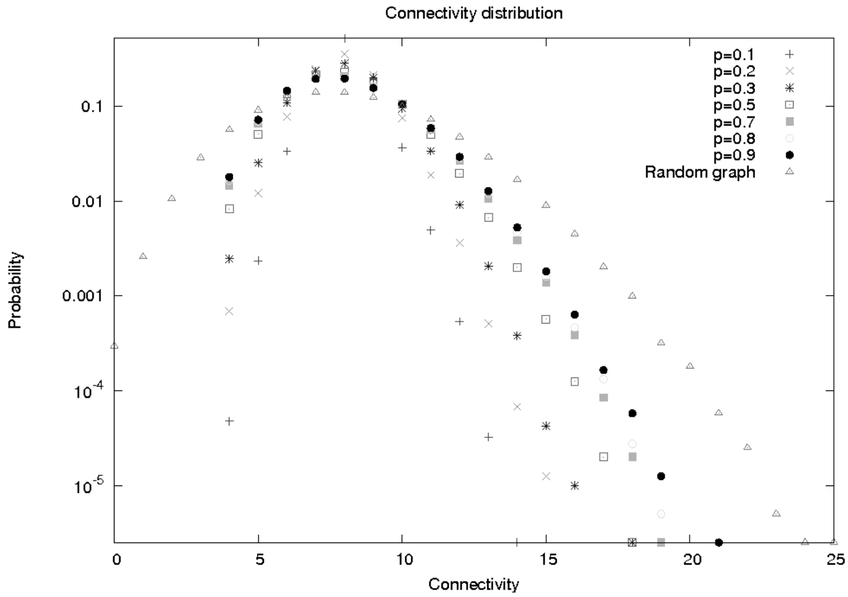

Figure 3.3: Comparison of the connectivity distributions for the Watts-Strogatz model of networks with various rewiring probabilities and the Erdös-Renyi random graph.

What is the mechanism of the emergence of the power-law degree distribution. This section should give an answer that shifts from modeling topology to modeling the network assembly and evolution. Generally, these aspects are not counter to each other.

There is the fundamental difference between the modeling approach used for random graphs and the Watts-Strogatz model and an algorithm which can reproduce the power-law degree distribution. The previous models try to fit correct topological features but the following model is described by the dynamic of the model.

Construction

Barabási and Albert defined an algorithm in [75] which has two common properties with the real networks. The first one is *growth*, continuous addition of new nodes and new edges to the network and starting with a small kernel network. An example of such mechanism can be found in the WWW network, where the number of pages grows exponentially. The second property of most networks is *preferential attachment*. It means that a new node is more likely to be connected to a node with more connections than less connections. For example, a web page often includes hyper-links to popular documents.

The two properties mentioned above inspired the introduction of the Barabási-Albert model in following formulation.

Growth The algorithm starts with small number of N_0 of nodes. At the every time-step, a new node is added until final size of network N is reached (We must fulfill $N_0 \leq N$). The initial network can be generated as a random graph or a fully-connected graph, because if $N_0 \ll N$ then it does not depend on initial network.

Preferential attachment When a node is added to a network, it is assumed in [75] that the probability $\Pi(k_i)$, which means that the probability to be connected to a node i, depends on k_i and k_i is degree of node i. The form of $\Pi(k_i)$ initially suggested linearly which depended on k_i and its full form is

$$\Pi(k_i) = \frac{k_i}{\sum_{j \in \widehat{N'}} k_j}, \tag{3.19}$$

where $\widehat{N'} = \{x \in \mathcal{N} | x < N'\}$ and $N' = N_0 + N_{time}$, where N_{time} is the number of nodes that were added to the kernel network of the size N_0. Every time-step, it is connected by m new edges to the kernel network and every connection influences $\Pi(k_i)$. However, the parameter m does not influence the power-law exponent α_{BA} so it is an internal parameter of the model.

Properties

Paper by Watts et al. [22] influenced research of small-world networks and paper by Barabási [75] stimulated research of scale-free networks and it inspired the investigation of their properties in papers by Dorogovtsev et al. [76], by Krapivsky et al. [77] and by Barabási [78].

Small-world Simulations of the Barabási-Albert model showed in paper [16] that the average path length is smaller than for a random graph and this property is due to the heterogeneous structure of the scale-free networks. Barabási and Albert in [16] showed in simulations that the average path length follows a generalized logarithmic form

$$l_{BA} = A \log(N - B) + C, \tag{3.20}$$

where A, B and C are constants. It can be concluded that a scale-free structure of the network decreases the average path length but the behavioral dependence remain the same as in the random networks - $l_{BA} \sim l_{rand}$. So far there is no theoretical explanation relating to small-world property for the Barabási-Albert model. However some comments on it can be found in paper [16].

Clustering coefficient There is also no theoretical prediction for the clustering coefficient but simulations in [16] showed that the clustering coefficient is higher than in a random network and it behaves as $C_{rand} \sim N^{-1}$. The actual model behaves as $C_{BA} \sim N^{-\frac{3}{4}}$, which leads to a slower decay

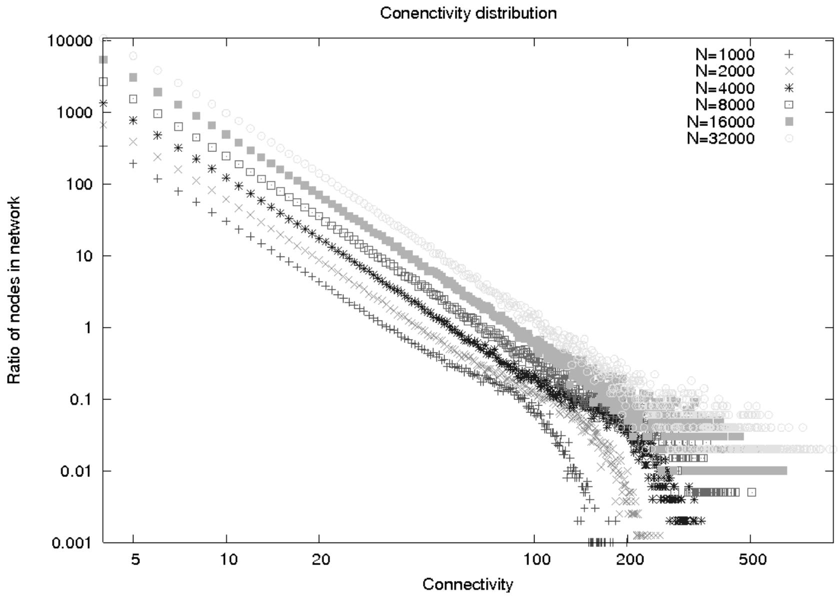

Figure 3.4: Connectivity distribution for the Barabási-Albert model of networks for different sizes of system.

than in random graphs. However, it still decreases with the network size.

Degree distribution It was shown in the simulations that the degree distribution $P_{BA}(k)$ follows the power law $P_{BA}(k) \sim k^{-\alpha_{BA}}$, with the power-law exponent $\alpha_{BA} = 3$. These results were simulated by the author of the thesis in the figure 3.4.

Several theoretical studies by Barabási et al. [75], [78], by Dorogovtsev et al. [76] and by Krapivsky et al. [77] agree with the size of the power-law exponent obtained in the simulations. The papers used three different approaches using *continuum theory*, *master equation* and *rate equation*. A summary of the computations is shown in paper [16].

3.4 Conclusions

This chapter introduces elementary information about complex networks. Real networks have special properties which are simulated using stochastic graphs. These graphs will be used as substrates of the agent-based models that were investigated, where the agents are bound to the vertices of a graph. In the special case of binding to the vertices, the edges are used for collaboration, cooperation, connection or communication between the agents.

The author of the thesis shows comparable character variables like the other authors in figures 3.3 and 3.4 that we obtained using **Zarja** simulation library.

Part II

Socio-economic models

Chapter 4

Sznajd model

4.1 Introduction

Opinion formation process is responsible for creating consensus in society. In particular, it can be found as a result of the sharing of personal philosophy (religion), public relation (market propaganda) or political propaganda due to Internet, transportation, urbanization, economical competition, meetings etc. Every of these processes could have some distinguished specific properties but the key point is change of persuasion due to interaction.

In this chapter we deal with the opinion formation from the perspective of convincing, where some dispositions (psychological and social dispositions of the participant) are neglected. We also forbid external interventions of neighborhood in the model.

4.2 Path to the Sznajd model

Political changes in east Europe during year 1989 stimulated some of physicists to search parallels of socio-political behavior and physics, e.g., Galam in [97]. Physics was used to investigate systems which consisted of small elements that constitute global behavior. The metaphors of socio-political behavior would be like: a man - an agent, manipulation - interaction. On every step we have to keep in mind that such a metaphor has limits - psycho-social base of a man, only qualitative description of a phenomenon and non-predictability of reality.

Galam in [97] discussed a model of a state (organization) with votes in many levels. He investigated a state where citizens were organized into groups, 3 independent persons, and where each group could vote for one of two options using majority rule at level n. Similar groups are constructed at level $n + 1$ from voted options at level n. The question was how big support to option A has to be at level 1 to become the ruling option at infinite level. In [97] the author showed, using renormalization transformation, that it is necessary to be 50% of the A option. Moreover, it was demonstrated, that in a real organization (finite number of levels) the minority option lost its power in the organization very rapidly.

The next interesting question which was answered in [97] was the problem of emergence of new ideas and how large its support has to be to become the leading idea, when groups are consisted of 4 persons and 3 persons are necessary for majority (in reality, there could be a prefect to avoid fast changes of power). It was found that 77% support is needed to get full power. After that the formula for majority in bigger groups is derived and its stable points are investigated. There are only three points fixed in infinite level structure 0%, 100%, when are stable, and 50% that is unstable. So, the dynamics of the system lead to totalitarianism. In real systems, there is a constitution, which determines the number of levels to n. Moreover, we are interested in "democratic" regime, where the probability to get majority at nth level is not 100%. The existence of the regime is proved in [97] and a lower and upper threshold is derived. Below the lower and above the upper threshold are "totalitarian" regimes.

In the end, a model, where there are three groups in three cells voting rooms, is introduced in paper [97]. When parity occurs then the third group is elected (this is known by polimetrician as model of 2.5 parties and this really happened in Germany in the 80s of 20th century). Galam obtained in infinite level model such, that when one of two opinions becomes more than 50%, then the opinion wins, otherwise the third opinion gets the majority.

Having knowledge of the analysis of impacts of the voting processes for the political system, makes it possible investigate different strategies for the manipulation of the agents and its impact on the dynamics of the voting process.

The model of Sznajd-Weron and Sznajd reported in [98] was designed to explain certain features of opinion dynamics. Well known slogan "United we stand, divided we fall" or in Arabic culture "Union is force and division is weakness" leads to simple dynamics, in which individuals placed on a string can choose between two different opinions (political parties, products etc.) and in each update step a pair of neighbors sharing a common opinion can persuade their neighbors to join their opinion. This kind of majority game can be compared to Ising model which was formulated in the 1920s and also simulated during the last 40 years. In contrast to the Ising model, information does not flow from the neighborhood to the selected spin, but conversely, it flows out from the selected cluster to its neighbors.

In the original article [98], the opinions of the agents are numbers $+1$ and -1 and the agents are placed on string and Monte Carlo step is characterized by choosing two neighbors and if both opinions agree then their neighbors are (ferromagnetic) convinced to share the same opinion and if the pair of agents don't agree then (anti-ferromagnetic) disagreement is spread. There are three limiting points

- all agents share the same opinion $+1$,
- all agents share the same opinion -1,
- both opinions share 50% of agents.

The results of the simulations of dependence of probabilities of the limiting cases on the initial concentration is shown and no phase transition is found in [98]. Next, correspondence of the model and reality using autocorrelation function is presented. After that, a question of distribution of time intervals needed to change opinion of random chosen agent were submitted and it was shown that the time intervals are distributed as power law with exponent close to $\frac{3}{2}$ and it is independent of the initial concentration of $+1$s and -1s. Finally, the problem, where the agents are not able to follow dynamic rules and make random choices with probability p, was simulated. The simulations suggests that there is some critical value p^*, where if $p < p^*$ then the system reaches one of the three steady states whereas if $p > p^*$ it is completely disordered. After that, the dependence of the distribution of time intervals needed to swap opinion on the noise probability p, was then investigated. In the case where $p < p^*$, the distribution follows the power law and, in case $p > p^*$, the distribution follows the exponential law.

Using this as our starting point the model can be enhanced in many ways as follows;

- different social structure of the agents,
- generalization of the interactions and
- allowing more possible states of the agents.

The model was simulated in 2 dimensions in [99] by Stauffer et al., [100] by Moreira et al., [101] by Bernardes et al., [103] by Chang et al., [108] and [109] by Stauffer et al., [113], [114] by Stauffer, in 3 dimensions, in [107] by Bernardes et al. and [109] by Stauffer et al. and in higher dimensions, in [109] by Stauffer et al. Generalization of the Sznajd model on small-world networks was made in [110] by Elgazzar and simulations on Barabási-Albert network were done in [107] by Bernardes et al. and [120] by Bonnekoh et al. Finally, the fully-connected network was described by Slanina in [123].

When we move from a one-dimensional case of the underlying network to a higher dimension, then there are various possible definitions of interactions. If these changes are ignored then the following cases of interaction can be considered: the Ochrombel case in [102]; the advertising effects in [113] by Schultze and in [116] by Sznajd; the long-range interaction in [115] by Schultze; the simultaneous updating in [114] by Stauffer and [119] by Sabatelli et al. and the simple model of elections shown by Bernardes et al. in [101] and [107].

The standard definition of the Sznajd model supposes two possible opinions. Generalization of the rule is discussed in a range of papers by

Bernardes et al. [107], by Stauffer et al. [108], [112] and by Schultze et al. [115].

As has been discussed, the model was simulated in various variants. Analytical investigation of the Sznajd model variants was shown in papers by Krapivsky et al. [118] and by Slanina et al. [123].

The model is general and application of its modification to explain elections is shown by Bernardes et al. in [107] with discussion of the results process by Lyra et al. in [105]. Its application to the market is investigated by Sznajd et al. in [104] and comparison with the results of the real sociological investigation was shown by Sznajd et al. in [98].

Generalization of the model on a square lattice was discussed with six different rules by Stauffer et al. in [99] and fixed points of dynamics and relaxation times were investigated and a phase transition was observed for an initial concentration of 50% of $+1$s and -1s.

The Sznajd model on an incipient cluster of the square lattice at percolation threshold was simulated by Moreira et al. in [100] and it was shown that the phase transition is fixed to equal concentration of the opinions and it is robust against changes in geometry.

The Ising model with Glauber kinetics is compared to the Sznajd model using cluster size of s in [101] by Bernardes et al. and the differences were found in time evolution. In the Sznajd model, the cluster sizes permanently decrease with time. Distribution of sizes of clusters corresponds each other as well as distribution of time-scaled distribution of sizes of clusters. Next, the simple model of elections was introduced and its agreement with experimental data described in papers by Costa Filho et al. [96] and by Bernardes et al. [107] was shown.

In paper by Chang et al. [103], two models of the spreading of opinions on a triangular lattice are discussed. The first one, in where spreading mixed opinions and the second one is the spreading anti-ferromagnetic opinions. Mixed spreading always leads to a fixed point for all values of the initial concentrations and the spreading of disagreement is irrelevant. The second model has always fixed points except for special cases of the lattice. Relaxation times fit to the same curve, so the initial conditions do not affect the model. No phase transition was found in either models.

A cubic lattice and the Barabási-Albert network were used in [107]. The model was a 2-step process and the agents have a finite set of possible opinions. In the first step, the agents are manipulated to accept a candidate and, in the cubic lattice case, the candidates have different skills to convince the others. The second step means to run the Monte Carlo simulation in the system. Both cases give a good approximation to the results from elections, but the case of Barabási-Albert network is more realistic, because there is no more assumptions of skills of the candidates as in the cubic model.

The Sznajd model with limited persuasion was described in [108]. Limited persuasion means ability to change an opinion of an agent with q possible states only for ± 1. In case of $q = 3$, the dynamics nearly always lead to a fixed point where all opinions were a central opinion. In rare cases, the dynamics lead to the inhomogeneous fixed configuration. In the case $q = 4$, almost always the dynamics lead to an inhomogeneous fixed point and rarely lead to never-ending dynamics or an parallel fixed point. Next, cyclicity of the opinions was discussed. Case $q = 3$ is trivial, but for the case $q = 4$ the dynamics lead to a inhomogeneous fixed point.

A ratio of never changed opinions and its comparison with the Ising model were studied in [109]. Generally, the ratio decays as $t^{-\theta}$, where t

means time. In one dimension, the exact result for the Ising model compares well with $\theta = 0.2$, which was obtained from simulations, but in higher dimensions these two models differ very much when we compare the θs.

A modification of the Sznajd model on small-world networks was studied in [110]. First of all, simple analytical results of the small-world network are derived. This showed two fixed points and the time needed to change the opinion of an agent decays as a power-law with exponent of 1.36. Thereafter, a model with leaders is introduced. In this case no fixed points are observed and this result is independent of the initial concentration of the opinions and the leaders don't influence the exponent of time needed to change opinion. Finally, an analysis of the time series of magnetization is made and in [110] it was found that it decays as a power-law with exponent 0.934 and this means a Hurst exponent close to 0.5. Thus, the results are very close to the normal distribution, which is widely observed in social phenomena.

In [111], the Sznajd model is generalized to look similar to the Glauber dynamics of the Ising model and this model is described as a two-component model. Hamiltonian of the model for one dimension is

$$H = -J_1 \sum_i S_i S_{i+1} - J_2 \sum_i S_i S_{i+2}. \tag{4.1}$$

The dynamics lead to several fixed points which depends on the constants J_1 and J_2, which could be;

- ferromagnetic and anti-ferromagnetic fixed points,
- ferromagnetic fixed points,

- anti-ferromagnetic fixed points,
- (2,2) anti-phase.

In [111] the author provides a detailed phase diagram and relaxation toward the fixed points are discussed. For the first pair of the cases relaxation is very slow or sometimes very fast. For the second pair, the system fluctuates around the final state and the fluctuations decrease in time.

A simple effect of advertising and feedback of advertising on society and diffusion of the agents on the square lattice are discussed in [113]. An advertising effect is significant when advertising is sufficiently strong. A master of submission of an advertising campaign must start its campaign fast with a small time lag, otherwise the market will be lost for the minority company.

The Sznajd model with simultaneous updating which is traditional in cellular automata is investigated in [114]. The agents, which are influenced by different opinions, do nothing. Simultaneous updating prevents reaching complete dominance one of the opinions in sufficient long time. When the dominance was reached, initial majority leads to its total dominance.

Long range interaction with multiple options in the Sznajd model was studied in [115]. A fraction of the agents follow the Sznajd rule and the next part follows the Ochrombel rule and one of the options is chosen and its ratio in the system is varied. A phase transition was observed for different strength of long range interaction when the author of [115] increases bias probability. In strictly Ochrombel case, no phase transition is observed.

Theoretical explanation in the mean-field limit of the Majority Rule model, which is strongly related to the Sznajd model in [98], is done in

[118]. Two types of networks were used, the fully-connected network and a string and two questions were solved:

- Function probability reaching total dominance one of the opinions from a given initial state of initial concentrations of the opinions.
- Function, that returns time needed to reach consensus, of system size and initial ratio of the opinions.

The authors in [118] derive Master equations of the processes. On the fully-connected network (mean-field approximation), an exponential decay of the probability reaching total dominance was observed in the finite system for an initial concentration of less than $\frac{1}{2}$. The maximal time to reach consensus scale as $2 \ln N$, where N is size of the system and the leading behavior is $\ln N$. On a string, equations of motion for the mean spin and for the number of domains are derived and the number of the domain decreases as $t^{-\frac{1}{2}}$ in time t. Computer simulations showed the power-law behavior of the most probable consensus time in different dimension of the hyper-cubic lattice.

In [119], the Sznajd model was investigated with simultaneous updating and the agents have a memory. The system ends up at a fixed point when there is asymmetry and the concentrations of the agents are higher than $\triangle p_c$, which in turn depends on the memory length and the size of lattice. Phase transition could be observed from simulations for a fixed size, various initial concentrations and memory lengths.

Simulation of the Ising model and the Sznajd model on a growing Barabási-Albert network is developed in [120]. Simulations were completed for these models using:

- one randomly chosen node

- two randomly chosen neighbors
- four randomly chosen neighbors

The previous pattern of agent then is able to convince all their neighbors. Significant differences in magnetization were found between the first model and the last two models, where consensus was always found and where an area of coexistence of both opinions is present. When limited persuasion is present in the model and a prior distribution of opinions of new agents is taken, we can see that even opinions almost lose supporters.

Reformulation of the Sznajd model on a string as a linear voting model and new simulations of the Sznajd model are shown in [122]

4.3 Sznajd model on the fully-connected network

A mean-field solution of Sznajd model is used as the starting point to understand the dynamics of opinion formation in a society.

4.3.1 Definition of the Sznajd model on a network

[1] Consider a system with N agents placed on the nodes of graph Λ (network of social contacts) , which is defined by $G = (V, E, \epsilon, \nu_V)$. V is a set of nodes and E is a set that is mapped using ϵ to unordered pairs (i, j), where $i, j \in V$, of nodes (edges). ν_V is a mapping to an opinion of an agent.

An opinion of an agent can be, in sense of the paper [123], can be denoted as σ_i, $i \in V$ and in case of only 2 options it is $\sigma_i \in \{-, +\} = S$. The state of whole system at time t is described by $\Sigma(t) = [\sigma_1(t), \sigma_2(t), \ldots, \sigma_N(t)] \in$

[1] Theoretical solution were derived by coauthor in [123]

S^V. $\Sigma(t)$ performs Markov process, where the dynamics could be evolved in three ways:

- two against one,
- Ochrombel simplification,
- edge initiated dynamics.

Two against one

First, $i \in V$ are chosen with a uniform distribution. Next, $j \in E_G(i)$ are selected with the uniform distribution. If $\sigma_i(t) = \sigma_j(t)$ then an agent $k \in E_G(i) \cap E_G(j) \setminus \{i, j\}$ is manipulated by players i and j $\sigma_k(t+1) = \sigma_i(t) = \sigma_j(t)$, otherwise opinions remain unchanged.

Ochrombel simplification

First, an agent $i \in V$ is chosen with the uniform distribution. Then, an agent $j \in E_G(i)$ with the uniform distribution is used and his (her) opinion is set to $\sigma_j(t+1) = \sigma_i(t)$, otherwise the others do not change opinion.

Edge initiated dynamics

An edge (i, j) is randomly chosen with the uniform distribution from E. If $\sigma_i(t) = \sigma_j(t)$ then an agent $k \in E_G(i) \cap E_G(j) \setminus \{i, j\}$ is set to $\sigma_k(t+1) = \sigma_i(t) = \sigma_j(t)$, the others remain unchanged.

The models two against one and the edge initiated model could lead in many cases of networks to the same dynamics, especially for the fully-connected network, the string and the hyper-cubic lattices.

4.3.2 Theoretical solution

When a graph is random and densely connected, it can be approximated by the fully-connected graph. In fact, it is a kind of mean-field approximation.

The social network, where the agents are placed, is the fully-connected network. The state of the system is described by the occupation numbers $N_\sigma = \sum_{i=1}^{N} \delta_{\sigma_i \sigma}$ or by densities $n_\sigma = \frac{N_\sigma}{N}$, for the opinion σ and its dynamic fully describes the evolution of the model, where the total number of the agents is conserved. Thus it is in the system $q-1$ independent dynamical variables.

Two against one

Lets start only with $q = 2$ and so $S = \{-, +\}$. My system will be described by only one dynamical variable

$$m = \frac{N_+ - N_-}{N}. \tag{4.2}$$

There are only three possible events at each turn and their probabilities are

$$\begin{aligned} P\left(m \to m + \frac{2}{N}\right) &= \frac{1-m^2}{8}\left(1 + m + \frac{1+3m}{N}\right) \\ P\left(m \to m - \frac{2}{N}\right) &= \frac{1-m^2}{8}\left(1 - m + \frac{1+3m}{N}\right) \\ P\left(m \to m\right) = 1 &- \frac{1-m^2}{4}\left(1 + \frac{1}{N}\right), \end{aligned} \tag{4.3}$$

where any term of a higher order than $\frac{1}{N}$ can be neglected. The probabilities are not symmetric and this has significant consequences (similar a phase transition).

The discrete master equation for the process for probability $P(m,t)$ retrieval the system at time t with variable m is

$$\begin{aligned} &\tfrac{P(m,t+\delta t)-P(m,t)}{\delta t} = \\ &-P(m,t)\left(P\left(m \to m+\tfrac{2}{N}\right)+P\left(m \to m-\tfrac{2}{N}\right)\right) \\ &+P(m-\tfrac{2}{N},t)P\left(m-\tfrac{2}{N} \to m\right)+P(m+\tfrac{2}{N},t)P\left(m+\tfrac{2}{N} \to m\right) \end{aligned} \quad (4.4)$$

and if $N \to \infty$, $\delta t \to 0$ and $\delta t = 2N\delta\tau$ then it leads to the Fokker-Planck equation up to diffusion term

$$\frac{\partial}{\partial \tau}P(m,\tau) = -\frac{\partial}{\partial m}\left(\left(1-m^2\right)mP(m,\tau)\right)+\frac{\partial^2}{\partial m^2}\left(\left(1-m^2\right)P(m,\tau)\right). \quad (4.5)$$

Next, when I start to investigate $q \neq 2$ and $q \gg 1$. The distribution of occupation numbers are defined by

$$D(n) = \frac{N}{q}\sum_{\sigma=1}^{q}\delta(n-n_\sigma), \quad (4.6)$$

where $\delta(x) = 1$ for $x = 0$ and otherwise zero. It would be difficult to write a dynamic equation for all possible $D(n)$, so we start to investigate $P_n(n) = \langle D(n)\rangle$. Assuming limit $N \to \infty$ and $q \to \infty$ and substitution $x = 2n-1$, I get

$$\frac{\partial}{\partial \tau}P_n(x,\tau) = \frac{\partial^2}{\partial x^2}\left(\left(1-x^2\right)P_n(x,\tau)\right). \quad (4.7)$$

Ochrombel simplification

Probabilities of the three possible events are

$$
\begin{aligned}
P\left(m \to m + \frac{2}{N}\right) &= \frac{1-m^2}{4}\left(1 + \frac{1}{N-1}\right)\\
P\left(m \to m - \frac{2}{N}\right) &= \frac{1-m^2}{4}\left(1 + \frac{1}{N-1}\right)\\
P\left(m \to m\right) = 1 &- \frac{1-m^2}{2}\left(1 + \frac{1}{N-1}\right) \quad .
\end{aligned}
\tag{4.8}
$$

So, the master equation is

$$
\begin{aligned}
&\tfrac{P(m,t+\delta t)-P(m,t)}{\delta t} = \\
&-P(m,t)\left(P\left(m \to m + \tfrac{2}{N}\right) + P\left(m \to m - \tfrac{2}{N}\right)\right)\\
&+P(m-\tfrac{2}{N},t)P\left(m - \tfrac{2}{N} \to m\right) + P(m+\tfrac{2}{N},t)P\left(m + \tfrac{2}{N} \to m\right),
\end{aligned}
\tag{4.9}
$$

in limit $N \to \infty$ and assuming $\delta t = N^2 \delta\tau$, it could be rewritten to

$$
\frac{\partial}{\partial\tau}P(m,\tau) = \frac{\partial^2}{\partial m^2}\left(\left(1-m^2\right)P(m,\tau)\right). \tag{4.10}
$$

Thus in 4.10 we describe the dynamics of the Sznajd model with Ochrombel simplification on the fully-connected network. The dynamic is written in form of the Fokker-Plank equation with position-depended diffusion constant.

4.3.3 Solution of the dynamics

Two against one

The equation which describes the dynamics in this case is 4.7, but I only take the equation without diffusion term

$$\frac{\partial}{\partial \tau} P(m, \tau) = -\frac{\partial}{\partial m} \left(\left(1 - m^2\right) m P(m, \tau) \right). \tag{4.11}$$

General solution of equation 4.11 is

$$P(m, \tau) = \frac{1}{m \left(1 - m^2\right)} f \left(\exp\left(-\tau\right) \frac{m}{\sqrt{1 - m^2}} \right), \tag{4.12}$$

where f is arbitrary function of one variable and it is influenced by initial conditions. The equation 4.12 provides the following estimation for the average time $\langle \tau_{st} \rangle$ to reach the stationary state

$$\langle \tau_{st} \rangle \simeq -\ln \left(\frac{|2p - 1|}{\sqrt{p\left(1 - p\right)}} \frac{1}{\sqrt{N}} \right) \tag{4.13}$$

The relaxation time towards the uniform state is $\tau_{relax} \simeq N$ and when I use scaling $t = 2N\tau$, I get the tail of the distribution

$$P\left(\tau_{st}\right) \sim \exp\left(-\frac{\tau_{st}}{\tau_{relax}} \right), \tau_{st} \to \infty, \tag{4.14}$$

$$\tau_{relax} \simeq \frac{1}{2}. \tag{4.15}$$

The most interesting result of the model is the presence of the dynamic phase transition similar to that found in numerical simulations. When I start with any fixed positive magnetization, the dynamic lead to the state with uniform opinion of the society with meaning $+1$. When the initial magnetization is negative then the stationary state is uniform with opinion -1. The possible deviations close to $p = \frac{1}{2}$ are neglected in 4.11. The presence of the dynamical phase transition can be seen in the average time to reach the stationary state 4.13 for $p \to \frac{1}{2}$.

Ochrombel simplification

The Ochrombel simplification of the Sznajd model and the Sznajd model with $q \gg 1$ are governed by

$$\frac{\partial}{\partial \tau} P(m, \tau) = \frac{\partial^2}{\partial m^2} \left(\left(1 - m^2 \right) P(m, \tau) \right). \tag{4.16}$$

The solution of the equation 4.16 is achieved using the expansion in eigenvectors. The eigenvectors corresponding to eigenvalue $-c$ will be denoted by $\Phi_c(m)$ and they are governed by equation

$$(1 - m^2)\Phi_c''(m) - 4m\Phi_c'(m) + (c - 2)\Phi_c(m) = 0. \tag{4.17}$$

So, the full solution of the equation 4.16 is

$$P(m, \tau) = \sum_c A_c \exp\left(-c\tau \right) \Phi_c(x) \tag{4.18}$$

where the coefficients A_c are determined by initial conditions.

What is the space where equations 4.12 and 4.17 are solved? These functions are the probability densities, so they must be normalizable $\int \Phi(m) dm < \infty$ and only relevant interval is $[-1, 1]$ and finally, it is useful to use δ-function as an initial condition. All these conditions lead to the space of distributions with restricted support to the interval $[-1, 1]$.

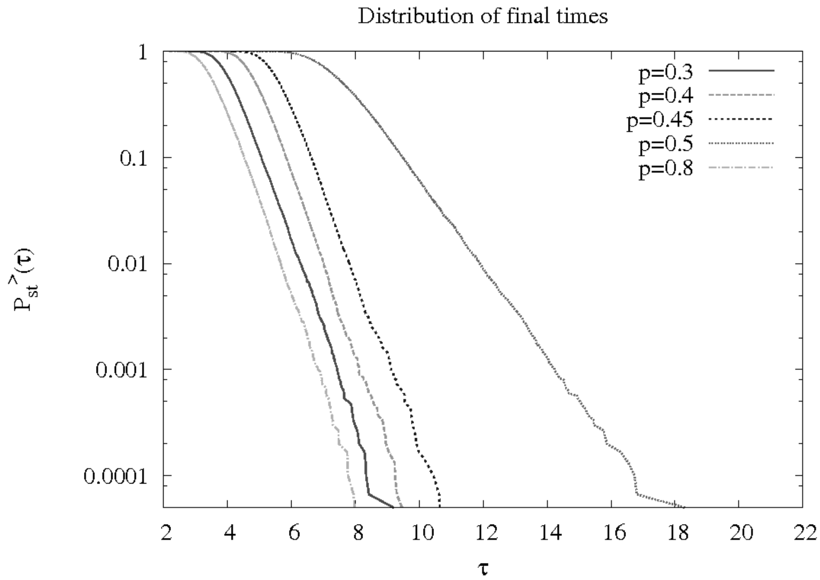

Figure 4.1: Probability of reaching the stationary state in times larger that τ in case of the two against one for $q = 2$ and $N = 2000$.

4.3.4 Results from simulations

The author of the thesis performed the simulation of the Sznajd model using two variants of updating and with algorithms described in the section 2.4.1. The simulation program provided the distribution of times needed to reach the fixed point of the dynamics $P_{st}^{>}(\tau)$. It can be found in the figures 4.1 and 4.2 and it can be easily seen that the distribution decays exponentially.

Two against one

Taking into account the analytical results shown at 4.14 then the author was able to fit an exponential tail of the distribution obtained from simulations by

$$P_{st}^{>}(\tau) \simeq \exp\left(-\frac{\tau - \langle \tau_{st} \rangle}{\tau_{relax}}\right), \tau \to \infty. \tag{4.19}$$

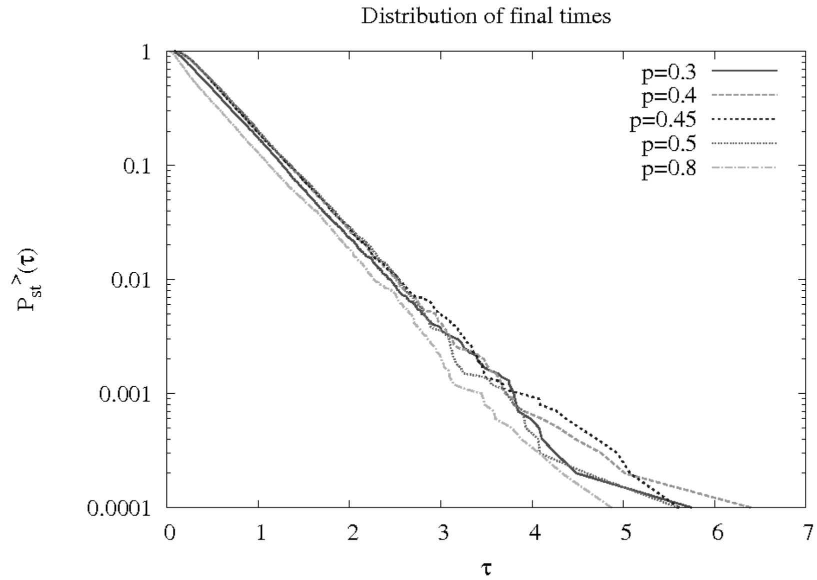

Figure 4.2: Probability of reaching the stationary state in times larger that τ for various initial concentrations in case of Ochrombel simplification for $q = 2$ and $N = 2000$.

Both, the results from simulations for $\langle \tau_{st} \rangle$ and the analytical prediction can be found in the figure 4.3 and good agreement is observed between the theory and the simulations as well as the results from simulations and the theory 4.15 for τ_{relax} in the figure 4.4 except such initial ratio p close to the point of the phase transition $p_0 = \frac{1}{2}$. The deviations are caused by finite size effects.

Ochrombel simplification

The analytical predictions, which were shown in [123], could provide the leading term of the tail of the distribution $P_{st}^{>}(\tau)$. The expected behavior is

$$P_{st}^{>}(\tau) \simeq \exp\left(-\frac{\tau - \tau_0}{\tau_{r0}}\right), \tau \to \infty, \tag{4.20}$$

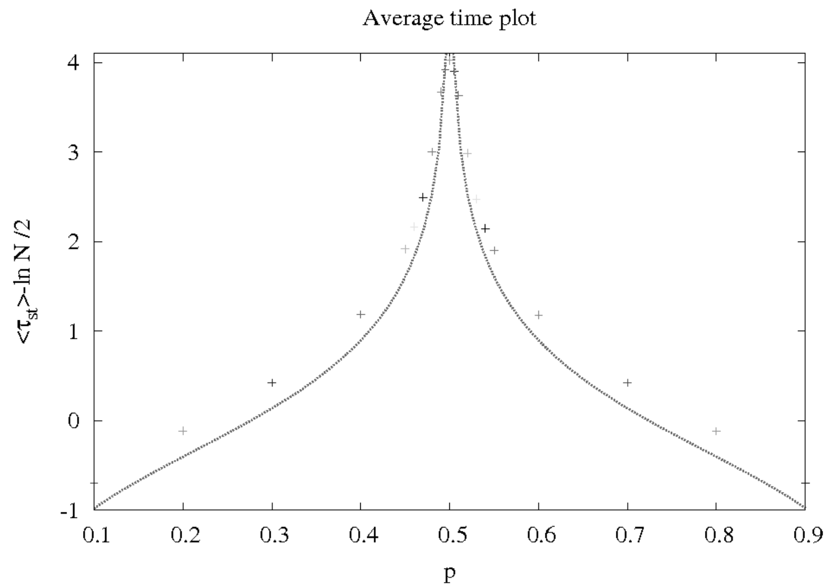

Figure 4.3: The average time to reach the stationary state for two against one in case $q = 2$ and $N = 2000$. The solid line is the analytic prediction 4.13.

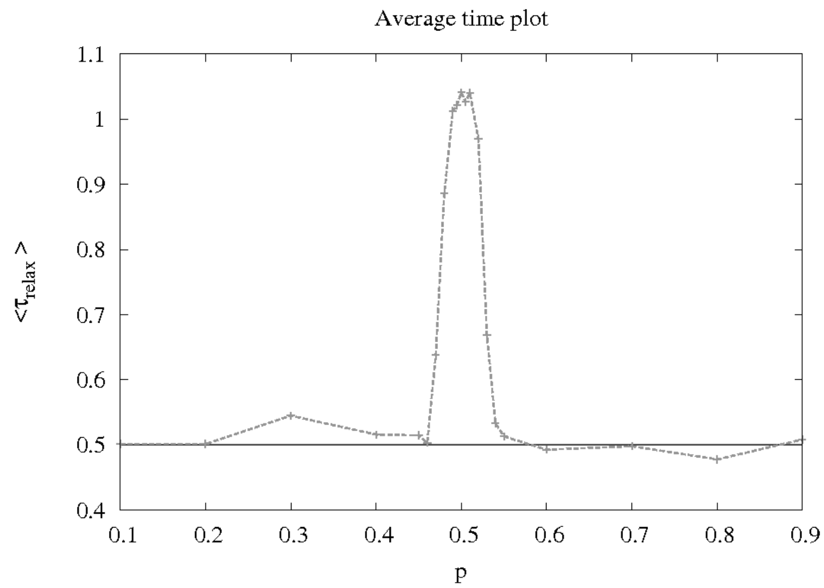

Figure 4.4: The relaxation time towards the stationary state in case $q = 2$ and $N = 2000$. The solid line is the analytic prediction 4.15

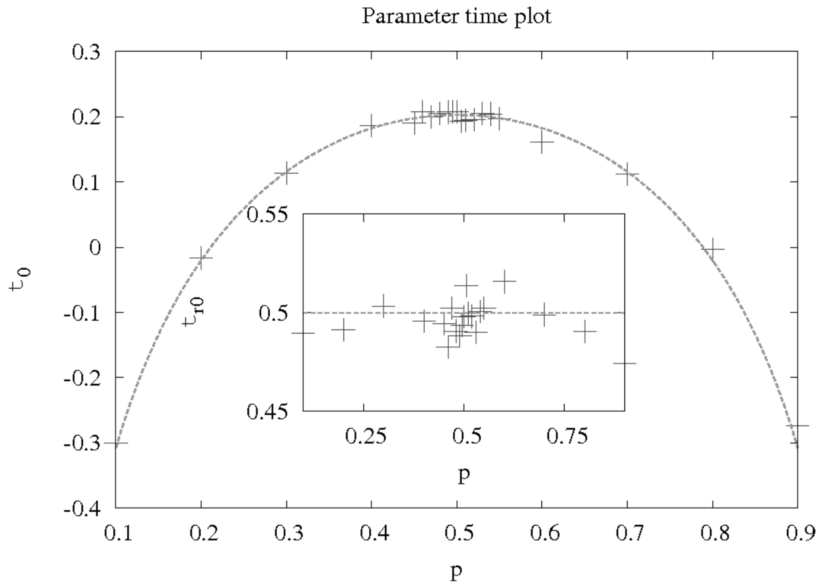

Figure 4.5: Average time to reach the stationary state for Ochrombel simplification in case $q = 2$ and $N = 2000$. The solid line is the analytic prediction 4.21.

where the theoretical expectation for τ_{r0} is $\tau_{r0} = \frac{1}{2}$ and the expectation for τ_0 could be derived from equation 30 in [123] and this leads to

$$\tau_0 \simeq \ln \sqrt{6p(1-p)}. \tag{4.21}$$

In the figure 4.5, there could be found a good agreement between the theory and the results from simulations.

4.4 Conclusions

The theoretical mean-field solution of two modifications of the Sznajd model were introduced in this chapter. It was shown that there is no phase transition in the Ochrombel simplification, in contrast to model two against one, where the phase transition is present. In figures 4.1, 4.2, 4.34.4 and 4.5

we can see results of simulations. The simulation was made by the author using the **Zarja** simulation library and it verifies the theoretical solutions of mean-field model.

Chapter 5

Minority Game

5.1 Introduction

Let us now imagine and simulate a world, where players try to choose the minority option. This situation can be motivated by simple scenario where a person needs to reach a certain point. There are only two possible ways to reach the point for example by car or by bicycle. The person does not play the game alone but with the others. The right option is one which was chosen by minority of the players because the majority freeze in crowd. The person tries to edify herself and he/she is equipped with memory of last success votes.

The scenario can be extended by imitation which is very common strategy by psychologist, e.g., Fromm [3]. Thus, the person is able to follow one of its acquaintances. The players, when they are not successful in choosing the right option, follow the most valuable source of information which can be the source of advantages and wealth. The source is chosen on social network of human contacts. At the top of the pyramid are the leaders who make orders which are executed by the followers and the followers pay a fee to their own advisers as counter-value.

Such a society was studied in papers by Slanina [145] and [146] on ring and in [147] by Anghel et al. for the Erdös-Renyi network and the tree structures of leader-follower on networks which have a structure similar to the structure of human contacts.

5.2 Review of the Minority Games

The Minority Game is an evolutionary game, whose states depend on previous states. It was introduced in paper by Challet et al. [127] where the agents use *inductive reasoning*. The origin of the model was found in the *El Farol* bar problem introduced by economist Arthur in [125].

Minority Game involves an odd number N of players. The agents selfishly try to choose one of two options. The agents, which chose an option that was chosen by minority of all agents, can be rewarded. Every agent keep at disposition S strategies which maps previous history of winning votes to the next possible chosen vote. Official used vote of the agent is chosen under the highest internal scores of strategies.

The Minority Game is a game with a non-zero sum. Challet et al. in [127], [128] searched two extreme cases of attendance at one of sides because it is correlated with number of winners. If one of agents chooses one option and the others the second one then there is only one winner and the waste of the system is huge in contrast to the case with $\lfloor \frac{N}{2} \rfloor$ at one side and with $\lceil \frac{N}{2} \rceil$ at the other side. So the agents can cooperate through memory of winning votes and more effectively exploit the banker of the game. In this sense, the collaboration decreases global waste.

When agents with different memory sizes are placed at the same battlefield, the more intelligent agents are more successful and the success of big-brain agents is saturated.

On the other hand, when the agents are equipped with a bigger idea bag S then they behave worse in contrast to the population of the agents with a smaller "idea bag" and the players tend to switch strategies which causes their failure.

A different payoff function which estimates less winners more than more winners could cause a two-peak distribution of attendance. In [128] by Challet et al., there is a general discussion of payoff functions when the system can manage global payoff which can cause two peaks in the distribution of the attendance. In general, when $\rho = \frac{N}{2^M} > \rho_c$ then the histogram can have only one peak and when $\rho < \rho_c$ then the could be more peaks that are symmetric with respect to $\frac{N}{2}$.

In [127], Challet et al. measured the average success of individual strategies and they found that the strategies are equivalent (none of the strategies have extra ability to win) in the $t \to \infty$. An individual strategy that play good in competition with the other strategies could be last in next set of strategies. There is in the strategy space no risk-free strategy.

When Darwinism is introduced into the system and when cloning of the most successful agents is not perfect with probability p then the total waste of the system is decreased but when cloning is perfect $p = 0$ then the behavior is worse than in the original system without Darwinism. In the case of Darwinism of memory sizes when the length of memory of an agent could rise up or fall down with small probability then average memory of the agents saturates and the saturation values are not universal. In [128], the authors compare systems with various values of ρ with Darwinism and in the original game without Darwinism and they found

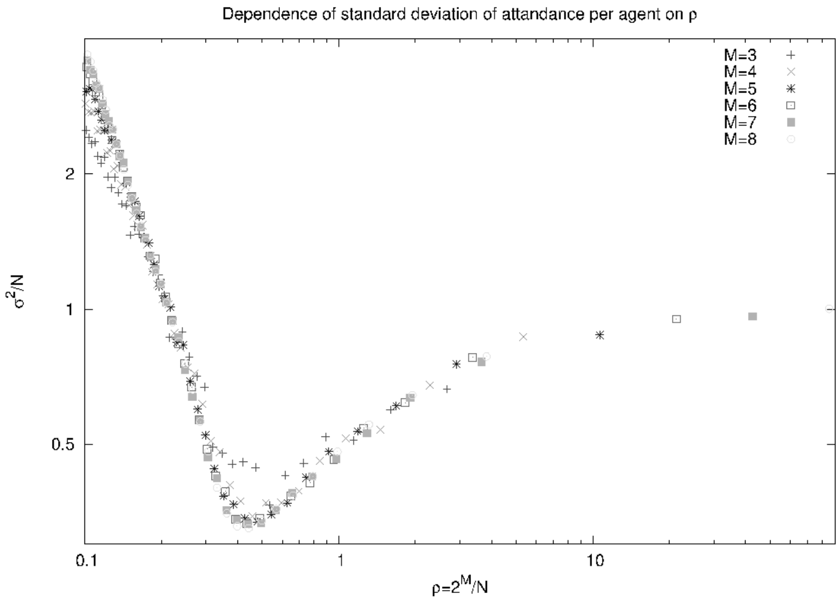

Figure 5.1: Volatility of attendance as function of the control parameter $\rho = \frac{2^M}{N}$ for $s = 2$ in original Minority Game. Random players would generate signal with volatility 1.

that the performance of the system is better and the figure of dependence of volatility per agent against ρ was introduced and they obtain a similar picture to that shown in 5.1. Moreover, species lifetime (where species means the agents with the same strategies) follow the power-law with an exponent that is very close to real life evolution and the law does not depend on p. Finally, observation of the power-law for species against their rank was made for $p < 1$.

A phase transition was observed in [128] and it is illustrated by 5.1. There are two phases which are separated by minimal volatility of attendance per agent for ρ_c but there are two different regions in the first phase. The first phase is for $\rho < \rho_c$, where are 2 regions, for $\rho \ll \rho_c$, $\frac{\sigma^2}{N} \sim \frac{1}{\rho}$ and so $\frac{\sigma^2}{N^2} \sim \frac{1}{2^M}$, when M is constant then different N causes dilatation of the system. The next region is $\rho < \rho_c$, close to ρ_c. In this phase in general, increasing number N of agents causes increase of global waste of payoff points. The second phase is for $\rho > \rho_c$, where increase of agents causes decrease of global waste of payoff points. In case $\frac{\sigma^2}{N} \rightarrow +\infty$, the players are not coordinated and they play as random players (players which randomly votes one of the options). The explanation of the classification based on investigation of strategy space is made in [128], and a similar discussion is seen in [130] by Johnson et al. and [133] by Savit et al.

A more accurate discussion of phase transition was described by Challet et al. in [129] where the Minority Game is described as a spin system and when $\rho = \frac{2^M}{N}$ varies then a dynamical phase transition with symmetry breaking can be observed. There are two phases, the symmetric one ($\rho < \rho_c$) where both options are taken with the same frequency and the asymmetric phase ($\rho > \rho_c$), where a minority of agents provide one action more frequently for every of 2^M possibilities of memory outcome and thus there is opportunity to exploit information from the game and get profit. It is called an *arbitrage* in economics. It means that a fraction of the agents use only one strategy while the others switch between strategies. Approximative computation of variance of attendance lead to a term that is similar to neural network models.

Later, spin glass theory was used in the Minority Game by Challet et al. in [132], namely the replica method by Mezard et al. [124] combined with the saddle point method. The results obtained using previous methods, differ from Nash equilibrium of the game where agents maximize their utility function. Challet et al. in [132] successfully computed the minimal position of minimal variance of attendance as $\rho_c \doteq 0.34$. In [131] by Challet et al., more general analysis of the Minority Game was shown with

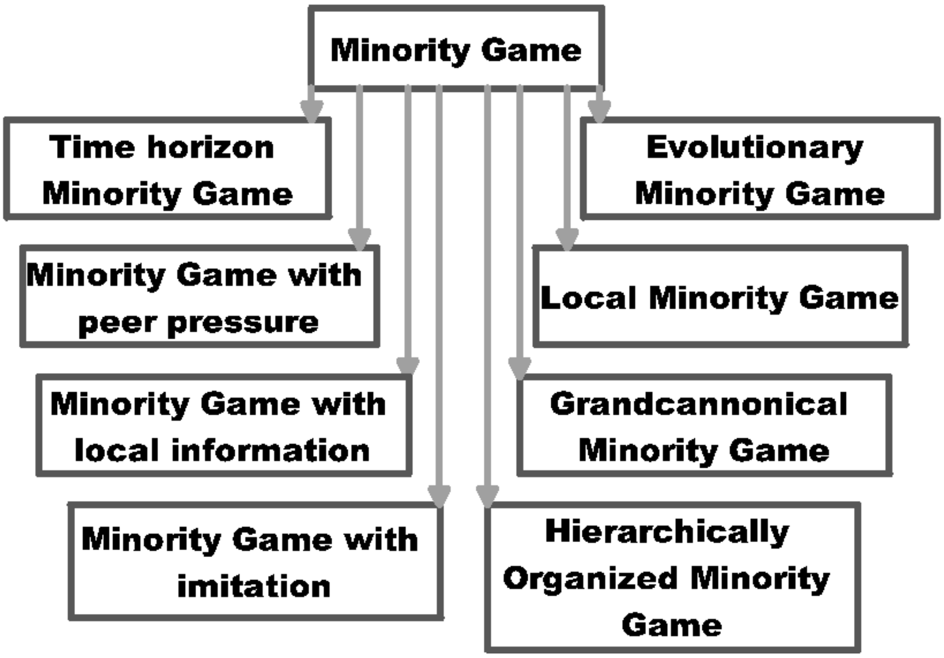

Figure 5.2: Structure of the types of the Minority Games.

detailed computations.

5.3 Review of modifications of Minority Game

In the previous section, there are fundamental properties of the Minority Game, where the agents are placed in a complex world where no agent has complete information for his/her next action. These properties stimulated many authors to make modifications of to the original rules. The evolution of Minority Game can be imagined as in figure 5.2 where different payoff functions in the original Minority Game were omitted.

5.3.1 Minority Game with peer pressure

This model is based on an idea based on the fact, that in the original Minority Game, the agents use global information - memory but when the agents are placed on a network then they can have local information at disposal - memory of the local minority. The model was defined and simulated by Chau et al. in [134]. The agents have no overall strategy space at their disposal with only reduced strategy space and the network is equivalent to a 1-dimensional torus (ring). In general, the model exploits gain better in then the Minority Game for small ρ so local information dilutes the overcrowding effect but when ρ increases then the overcrowding effect of both games will be suppressed but cooperation in the Minority Game with peer pressure is problematic due to two sources of information.

For case of non-zero global memory size M_g and when $M_l > M_g$ then performance become better for a bigger memory size M_l irrespective of value ρ. If $M_l \sim M_g$ then the performance is a similar like system of coin-tossing players.

Case with memory size $M_g = 0$, it allows only the use of local information and the global payoff is bigger than in the original Minority Game as well as in case $M_g \neq 0$ for small ρ. Chau et al. in [134] revealed that the players are frozen and it causes depression of global waste.

5.3.2 Evolutionary Minority Game

This model was defined in paper [135] by Johnson et al. as a system of N agents. Every agent is equipped with one strategy but the strategy is used with probability p and the opposite site with probability $1-p$. Darwinism is implemented such that when an agent's points decrease below value d

then he/she is eliminated and his/her place is occupied with a new agent with the same strategy but different p. The new value of p is chosen with uniform distribution centered in old p with radius of R.

Johnson et al. in [135] measured the distribution of values p distributed among the agents. Most of probability is clustered close to 0 and 1 as well as average life length and that behavior was successfully theoretically explained and classified.

5.3.3 Time Horizon Minority Game

The definition of Time Horizon Minority Game was made by Hart et al. in paper [140] and the model is based on fact that every trader knows only limited time history of successes and failures. Thus, there is defined time of τ when previous successes are kept in mind. Volatility is periodic in 2.2^M time-steps and the authors concluded that this value corresponds to the number of different paths in a De Bruijn graph linking 2^M histories and this is connected with the similarity of dynamic for $t > \tau$ when τ is increased by multiples of 2.2^M. The dynamic of Time Horizon Minority Game for $t < \tau$ is similar to the original Minority Game but the time evolution of the actual modification of Minority Game for $t > \tau$ is strictly deterministic and it differs from case when random histories are generated. Finally, the theoretical explanation of previous mentioned features was published in [140].

5.3.4 Local Minority Game

The Local Minority Game was defined by Moelbert et al. in [136] as a game of N agents that try to be in minority group. Every agent is placed on a network and in [136] there a ring and a two-dimensional lattice was taken. In contrast to the original Minority Game, the agents play in the local area of his/her neighbors and so every agent has available memory of local winning sides. Moelbert et al. in [136] provide theoretical computation of the problem with temperature T similar to [131] and they revealed a phase transition from a phase of frozen agents that use only one of the strategies but there are some frustrated agents on the boundaries of the frozen-agent clusters which vanish in thermodynamic limit and a phase of random-strategy-choosing agents. The organization of the agents in the frozen phase is like antiferromagnetism when we look at actions taken and the results is that everybody can win in contrast to the original Minority Game where at most there are $\lfloor \frac{N}{2} \rfloor$ winners.

5.3.5 Minority Game with local information

Kalinowski et al. in paper [138] defined the Minority Game with local information where they were influenced by the results reported in paper [137] by Paczuski et al. The agents in the model are placed on a ring, where every agent uses memory which contains information on previous actions of his neighbors. It was found that left-right asymmetry of memory plays no role .

First of all, Kalinowski et al. in [138] analyzed the dependence of the performance of the game on the number of strategies and they found that in case of $S > 1$, the performance is worse for constant M and N and increasing S. For case $S = 1$ it behave randomly because the agents have no possibility to adapt to the reality.

The standard deviation of attendance follows the power law with exponent $\frac{1}{2}$ for increasing number N of agents in the system for memory

$M \in \{1, 2\}$ and it follows the power-law with exponent 1 for memory $M \geq 3$. Kalinowski et al. in [138] revealed phase transition between $M = 2$ and $M = 3$ and the cooperation is based on connections between the agents who provide the real previous option taken. Finally, Darwinism is a positive factor for the decrease of global waste only if local evolution has an effect. Local evolution is based on the copying of abilities of neighbors.

5.3.6 Hierarchically Organized Minority Game

The definition of the Hierarchically Organized Minority Game was introduced by Foldy et al. in [141] by Foldy et al. and the central idea is hierarchical organization of society which should be reflected in a model. Implementation of the main idea is done by playing the Minority Game in two levels. The first level is where are the agents attended and there are N' such playgrounds and it can be imagined as in the figure 5.3. The second level is the Minority Game where the first level Minority Games are attended and they use the option which was in minority at the first level. The strength of the second level interaction is controlled by a payoff parameter C and if $C = 0$ then the model simplifies to the original Minority Game. The similar hierarchical structure can be found in Galam's paper [97].

Research of the interaction between the agents and the first level Minority Games leads us to measuring of global waste which shows deviation from the original game but the character of the curve is the same whereas the order parameter in the original Minority Game from [129] shows no more phase transition in the hierarchical structured Minority Game and thus there exists optimal strategy.

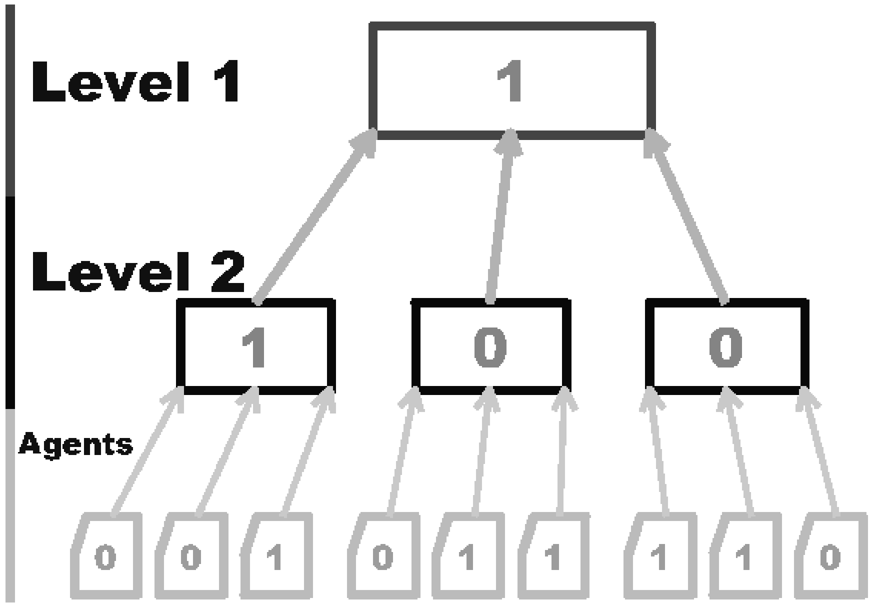

Figure 5.3: Organization of Hierarchically Organized Minority Game.

Correlation of the first level and the second level shows for waste of the system random behavior for weak global interaction but if the global interaction is stronger then global coordination can be observed.

Finally, coordination of the agents and the second level Minority Game show similar behavior for all values of parameter C.

5.3.7 Grandcannonical Minority Game

The Grandcannonical extension of the Minority Game was defined in papers by Slanina et al. in [142], Challet et al. in [143], Jefferies et al. in [144] as models of a real market mechanism where the agents are able to participate in a market that is introduced as a modified Minority Game and there are 2 types of agents - producers and speculators. The above mentioned papers differ in properties of the agents as well as exact market mechanism but the main conclusion, that is relevant to the original Minority Game, is occurrence of phase transition from the asymmetric phase to the symmetric phase (with different meaning from the original Minority Game) for speculators and producers [143].

5.3.8 Minority Game with imitation

Imitation was introduced in the Minority Game by Slanina in [145], [146] as a local information transmission when the agents could imitate each other where they have a disposition for imitation. There are a fixed average ratio p of agents with disposition for imitation and probability p_1 of transmission from a state with disposition to a state without disposition. The substrate network, where the agents are connected to the nodes, is a one-dimensional torus with one-way oriented edges.

The number of imitated agents have an asymptotic value that is monotonously reached for $p_1 = 0$ and with maximum for $p_1 \neq 0$. Imitation seems to be beneficial for global performance of the model in the symmetric phase and harmful factor in the asymmetric phase where neighbors of rich agents are abused paying fees for imitation. Evolution of social tension monotonously reaches asymptotic value for $p = 0$ and a minimum value is asymtotically reached for $p \neq 0$.

Research of imitation structures using computer simulations was completed by Anghel et al. in [147] where the distribution of the average number of leaders with k followers undergo the power-law with sharp cut-off due to using the random graph as a substrate network of the model.

5.3.9 Other important modifications of Minority Game

Lo et al. in [148] studied a networked Minority Game with agents who can use the best of owned strategies and strategies of neighboring agents and random network using crowd-anticrowd theory.

A special kind of Darwinism using a genetic algorithm was studied by Sysi-aho et al. in [149] and the process seems to be useful for global performance of the model.

5.4 Minority Game with imitation on complex networks

The author defines the behavior of the agents in the game as follows: the player use its own strategies if there is no richer friend otherwise the player follows a vote of a richer individual. This kind of behavior is widely spread in society because it is very effective and low cost social strategy.

5.4.1 Definition of the model

Consider a graph with an extra information $G = (V, E, \epsilon, \nu_V)$ that is bound to the vertices V using scheme one-to-one (in our case one agent per one vertex). ϵ is mapping to unordered pairs (i, j), where $i, j \in V$. Every agent is bound to a vertex and every vertex carries only one agent. Every agent carries an identifier that is natural number and it can be used to access the data what is carried on him/her. We have an odd number N of agents and each player has $S = 2$ strategies (setting up the variable to 2 due to simplification), denoted $s_j \in \{-1, 1\}$. A player can take two possible actions. Agents which choose the option 1, win if the most of players take 0 and vice versa. The winning agents receive 1 point but the loser agents will not improve their wealth. The players share the history of the the winning sides of length M, and this can be written as $\mu \in \{-1, 1\}^M$. The strategies are maps which transforms 2^M possible strings of actual memory μ to some possible option $-1, +1$. If the player j takes strategy s at time t then the action will be denoted as $a^{\mu}_{j,s}$. The virtual points of strategy s of agent j will change in the following way

$$U_{j,s}(t+1) = U_{j,s}(t) + 1 - \delta\left(a^{\mu(t)}_{j,s} - \theta\left(\sum_{i\in\Psi} a_i(t)\right)\right), \tag{5.1}$$

where $a_j(t)$ is the action the player j which has been taken at time t and Ψ is a set of all agents. These virtual points are used by appropriate agents for making decision - the agent use such vote as it is offered by the strategy with the highest virtual points.

Two conditions are necessary in order that the agent can imitate his/her neighbors. The player should have a disposition for being an imitator. There are two possible states of imitation. Every player will have a state $\tilde{l} \in \{0, 1\}$ indicating whether the player is a potential natural born leader ($\tilde{l} = 0$) or potential imitator ($\tilde{l} = 1$). Initially, every agent will have probability p to be an imitator and probability $1-p$ to be a natural born leader at the initial time. But the players should swap between imitators and natural born leaders and vice versa p_1 is probability swapping from imitator to natural born leader; p_2 is probability swapping from natural born leader to imitator which must fit $p = \frac{p_2}{p_1+p_2}$ which comes from a condition where it is desired to have the average number of imitators and natural born leaders fixed. The second condition to be an imitator is - there is a neighbor in the social network that has more points than the actual agent. The actual agent chooses the most successful agent - in sense of number of points. The player with $\tilde{l} = 0$ never imitates (he/she is natural born leader).

In brief, an agent uses their own strategy in two cases. Either the agent has no disposition to imitate or the agent has a disposition to imitate others but there is no richer agent among her neighbors. Otherwise, the agent is an imitator.

The potential imitator copies action from his/her most successful neighbor via the edges in the network at time t. If there are more such agents, the leader of the actual agent is randomly chosen with uniform distribution. W_j is the wealth (number of points) of jth player, and the variables $\tilde{l}_j$ describing the potential state of imitation. It can be written as $l_j = \tilde{l}_j \theta(\max_{i \in E_G(j)} W_i - W_j)$, with $\theta(x) = 1$ for $x > 0$ and 0 otherwise, where $E_G(j)$ is a set of agents that are neighbors on the social network (represented by graph) G. l_j is actual state of imitation at certain time. The actions of the players are

$$a_j = l_j f\left(j, a_k\right) + \left(1 - l_j\right) a_{j,s_M}, \tag{5.2}$$

where the function $f(j, a_k)$ returns an action for the wealthiest neighbor of the agent j with wealth $\max_{i \in E_G(j)} W_i$. The imitation is not free of charge. The imitator loses a small fraction ε of his income to the imitated agent (income fee) at time t. So, the rules account for allowing imitation (direct interaction) to the imitators. So, we update the wealth of the players in each time step is

$$Y_j(t) = \varepsilon \sum_{i \in \Omega(j)} Y_i\left(t\right) + 1 - \delta\left(a_j - \theta\left(\sum_{i \in \Psi} a_i(t) - \frac{N}{2}\right)\right), \tag{5.3}$$

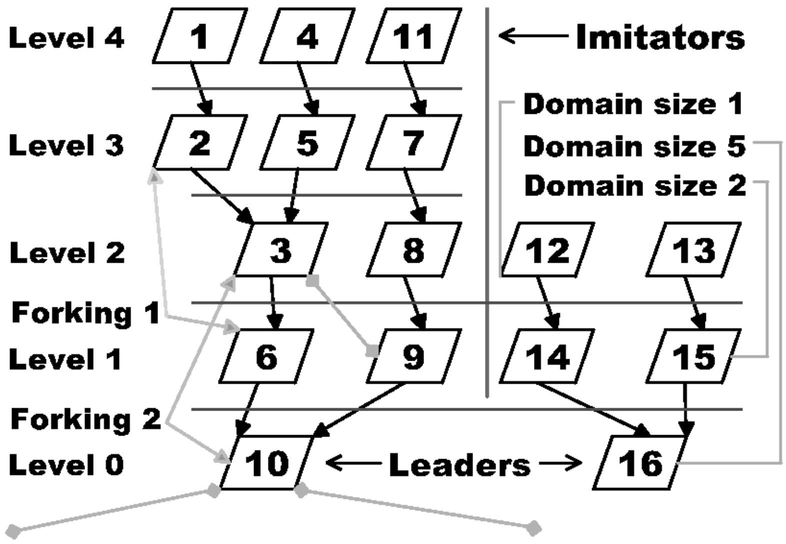

Figure 5.4: An example of two domains. The leaders dictate the next option and their followers point to them using black arrows. The followers are aside the vertical line in the center. The grey lines show other possibilities for imitation - social contacts. The levels of imitation are separated by thin horizontal lines. The grey arrows on left specify different forking values and finally, grey arrows on right specify different sizes of domains.

$$\triangle W_j\,(t) = (1 - \varepsilon l_j)\, Y_j\,(t)\,, \tag{5.4}$$

where $Y_j(t)$ is an auxiliary variable that stores the change of wealth before the agent pays for information if she imitates. $\Omega(j)$ is the set of all agents that imitate the agent j and $\triangle W_j(t) = W_j(t) - W_j(t-1)$, so it is a time difference of wealth.

5.4.2 Interesting variables

All these rules support the formation of domains of agents which follow the option of a leader of the domain. An example of two domains can be seen in the figure 5.4. There are many variables that can be measured:

- Social tension evolution,
- Domain size distribution,
- Leader domain size distribution,
- Imitation depth (level),
- Forking distribution,
- Leader forking distribution,
- Correlation of points and connectivity,
- Correlation of number (direct) imitators and connectivity,
- Correlation of number (direct) imitators and connectivity for leaders,
- Efficiency of habit exploitation.

The first interesting variable is social tension, which measures differences of number of points among society as follows

$$T_\sigma(t) = \frac{1}{\overline{W(t)}} \left(\frac{1}{|V|} \sum_{i \in V} \frac{1}{|V_i|} \sum_{j \in V_i} |W_i(t) - W_j(t)|^\sigma \right)^{\frac{1}{\sigma}}, \tag{5.5}$$

where $\overline{W} = \frac{1}{N} \sum_{i \in V} W_i$ so it is the average[1]. In general, the author has assumed that the social tension of a pair of poor agents with an appropriate difference of wealth is higher than the social tension of rich ones with the same difference of wealth. Such assumption is fulfilled for $\sigma \in (0, 1)$. This is a parameter set up to $\sigma = \frac{1}{2}$ in our simulations.

[1] V_i means nodes in a graph but there is one-to-one correspondence between agents and nodes. Thus, W_i means corresponding agent.

Domain sizes, leader domain sizes, imitation depths, forking and leader forking are explained in figure 5.4 and the statistics of the variables are marked as $D_A(s)$, $D_L(s)$, $I(d)$, $F_A(f)$, $F_L(f)$ respectively. Correlation of points and connectivity means

$$P(c) = \frac{\sum_{i \in V} W_i \delta(k_i - c)}{\sum_{i \in V} \delta(k_i - c)}, \tag{5.6}$$

where k_i is connectivity of i-th agent and V are vertices of a graph. The correlation of number (direct) imitators and connectivity, correlation of number (direct) imitators and connectivity for leaders are defined as

$$I_A(c) = \frac{\sum_{i \in V} f_i \delta(k_i - c)}{\sum_{i \in V} \delta(k_i - c)}, \tag{5.7}$$

$$I_L(c) = \frac{\sum_{i \in V, l_i = 0} f_i \delta(k_i - c)}{\sum_{i \in V} \delta(k_i - c)}, \tag{5.8}$$

respectively. Where f_i means number of direct followers (imitators) and $l_i = 0$ means that i-th agent is leader.

Finally, efficiency of habit exploitation $H(c)$ is defined as

$$H_X(c) = \frac{I_X(c)}{c}, \tag{5.9}$$

where X is A or L.

5.4.3 Results of simulations

The author of the thesis made simulations of the model for two kinds of substrate social networks - Watts-Strogatz network and Barabási-Albert network. The Watts-Strogatz network has one internal parameter which can tune the number of shortcuts in the network and the Barabási-Albert network has no parameter.

After investigation of many variables of the model, the author found that the most interesting parameter is income fee ε which can cause huge changes in the imitation structures. The simulations of the system have been done for case $N = 1001$ agents with memory $M = 5$ and probability of initiation $p = 0.995$.

Results for Watts-Strogatz network

The evolution of social tension $T_{\frac{1}{2}}(t)$ is shown in figure 5.5 where it can be seen that it decreases without asymptotic value for low values of ε for all rewiring probabilities p or it finds its stable asymptotic value for nonzero ε for all rewiring probabilities p. The shortcuts can speed down the decrease of social tension for low values of ε.

The correlation of points and connectivity $P(c)$ in the figure 5.6 is usable for higher rewiring probabilities p due to variability of connectivity in society of agents. A flat function of connectivity c can be observed for $\varepsilon \leq 0.001$ and it is increasing function of connectivity c for $\varepsilon \geq 0.01$.

The imitation depths distribution $I(d)$ in the figure 5.7 behaves as a Poisson distribution except in the region $0 < \varepsilon < 0.001$ and $0 < p < 0.001$ where the distribution is exponential and emergence of large imitation depth are frequent.

The domain size distribution $D_A(s)$ in the figure 5.8 has a power-law tail with exponent $\alpha_{domains} = 2.58$ with exponential finite-size cutoff for $\varepsilon < 0.001$ and $p \sim 0.1$. However, where $\varepsilon > 0.01$ it causes exponential decay for all rewiring probabilities p and the swap of tail is in an interval $0.001 < \varepsilon < 0.01$ for all rewiring probabilities p.

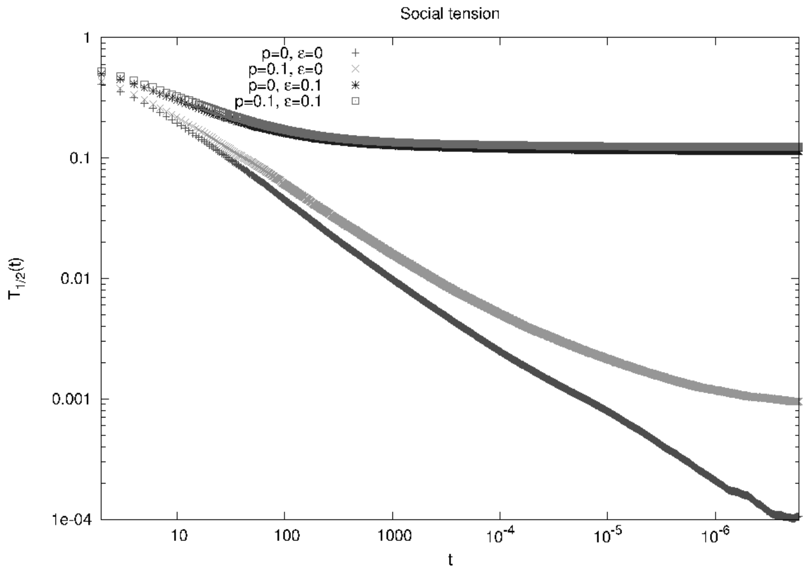

Figure 5.5: Evolution of social tension for different values income fee ε for Watts-Strogatz network.

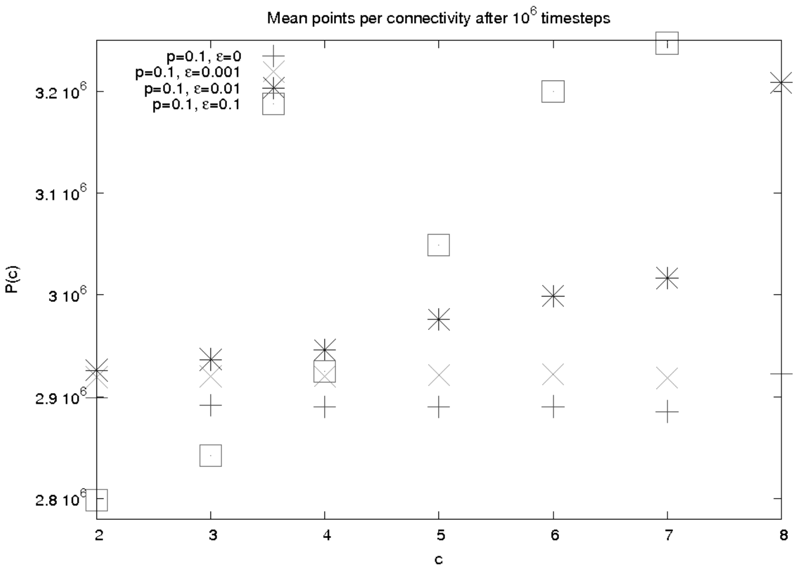

Figure 5.6: Correlation of points and connectivity for different values income fee ε for Watts-Strogatz network after $6 \cdot 10^6$ time-steps.

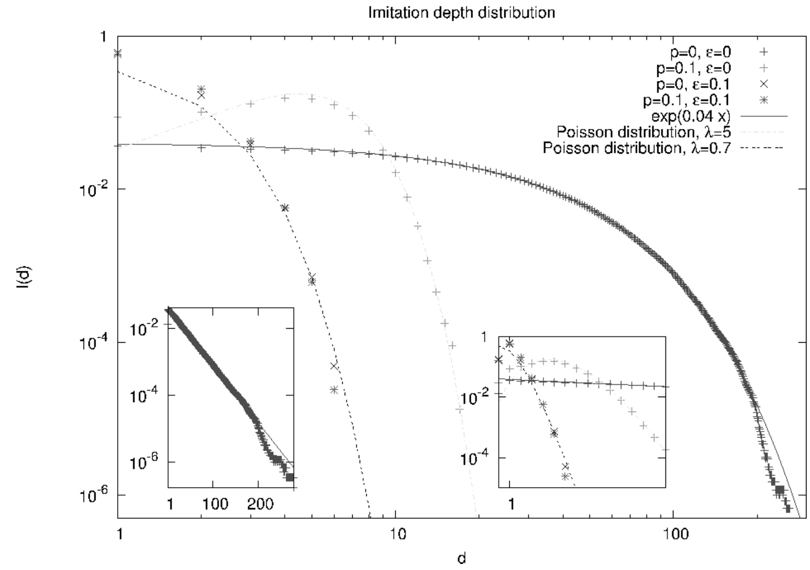

Figure 5.7: Distribution of imitation depths for different values income fee ε for Watts-Strogatz network with rewiring probability p.

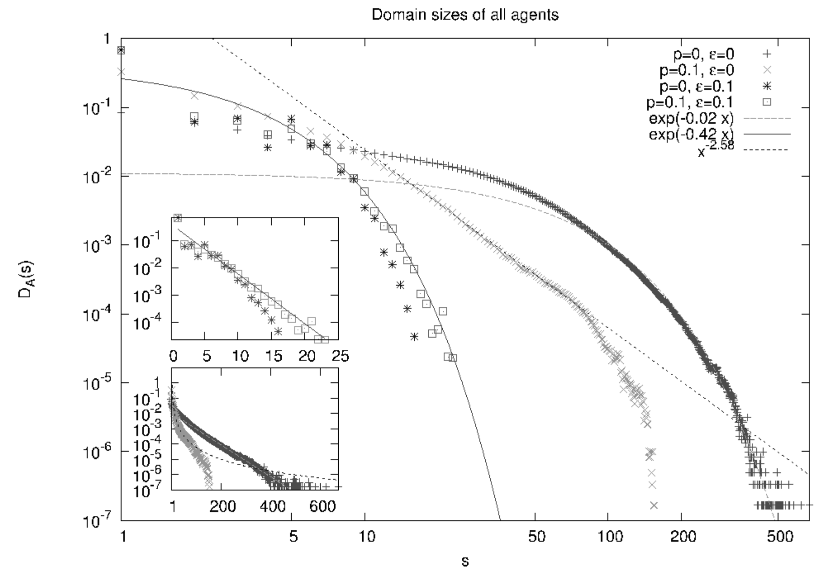

Figure 5.8: Distribution of domain sizes for different values income fee ε for Watts-Strogatz network with rewiring probability p.

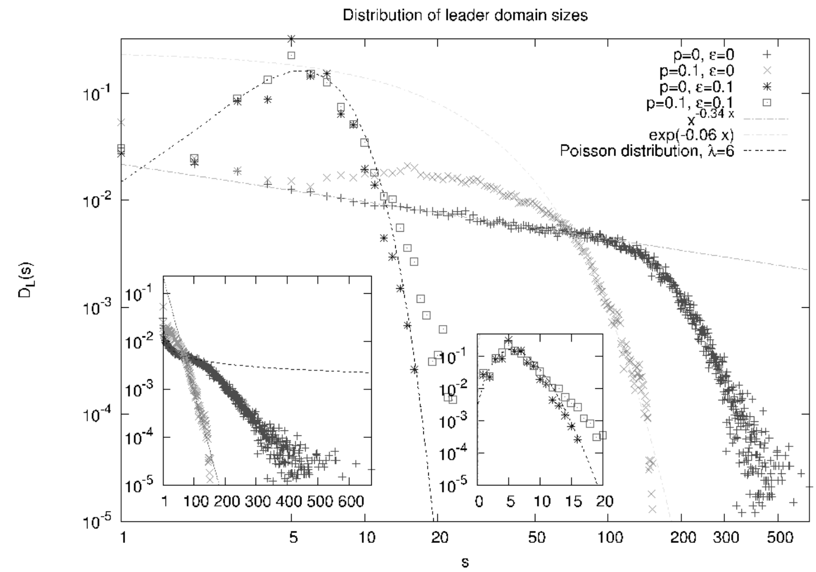

Figure 5.9: Distribution of leader domain sizes for different values income fee ε for Watts-Strogatz network with rewiring probability p.

The leader domain size distribution in the figure 5.9 follows exponential decay for income fee $\varepsilon \geq 0.01$ but the behavior is destroyed in region $0.001 < \varepsilon < 0.01$ for all rewiring probabilities p. Exponential decay becomes a finite size effect for $\varepsilon < 0.001$ and the power-law emerges with exponent $\alpha_{leader\ domains} = 0.34$ for $p < 0.001$ and Poisson distribution emerges for $p > 0.01$.

Finally, correlation of number imitators and connectivity $I_A(c)$, correlation of number imitators and connectivity for leaders $I_L(c)$, and efficiency of habit exploitation $H_A(c)$, $H_L(c)$ is described in the figure 5.10. Leaders have more imitators than common agent in terms of absolute and relative numbers except for the case of $\varepsilon < 0.001$, where high connected leaders have, in absolute and relative numbers, the same number of imitators as a common agent.

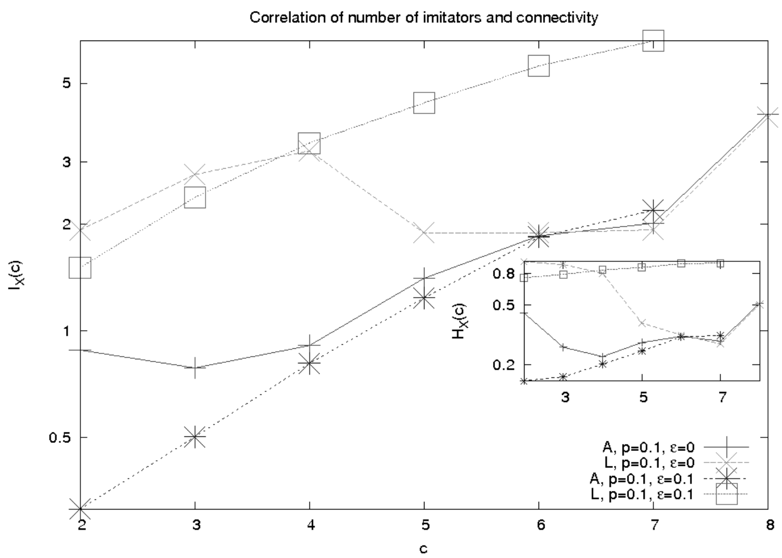

Figure 5.10: Correlation of number imitators and connectivity with combination with correlation of number imitators and connectivity for different values income fee ε for leaders for Watts-Strogatz network with rewiring probability p. There is efficiency of habit exploitation in the inset.

The forking distribution $F_A(f)$ and leader forking distribution $F_L(f)$ are too noisy to provide sensible conclusions.

Results for Barabási-Albert network

Evolution of social tension $T_{\frac{1}{2}}(t)$ is shown in the figure 5.11 and it can be seen that it decreases without asymptotic value for low values of ε or it finds its stable asymptotic value for nonzero ε.

The correlation of points and connectivity $P(c)$ in the figure 5.12 is significant for higher rewiring probabilities p. A flat function can be observed for $\varepsilon < 0.02$ and it is increasing function of connectivity c for $\varepsilon > 0.1$.

Imitation depths distribution $I(d)$ in the figure 5.13 behave as exponential for $\varepsilon \geq 0.02$. However, low imitation fees $\varepsilon < 0.02$ decays faster than exponential.

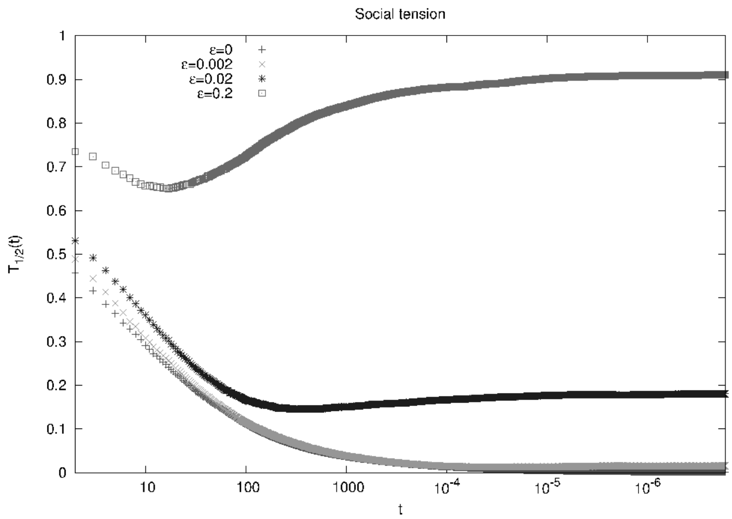

Figure 5.11: Evolution of social tension for different values income fee ε for Barabási-Albert network.

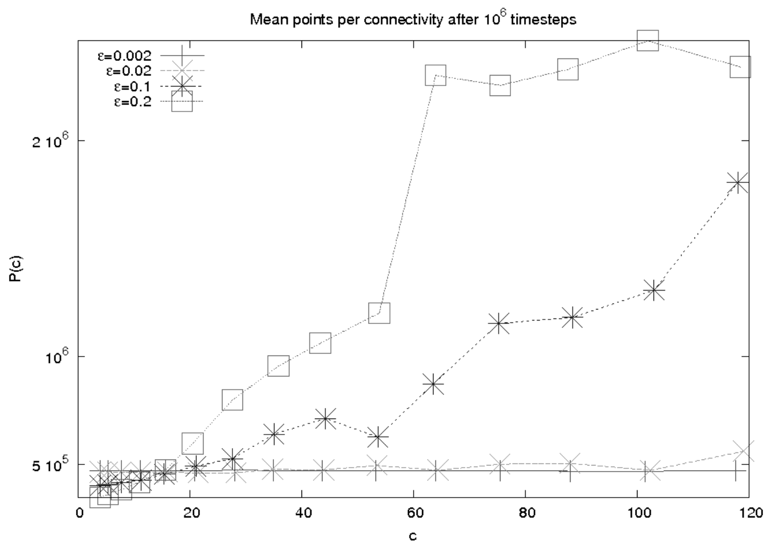

Figure 5.12: Correlation of points and connectivity for different values income fee ε for Barabási-Albert network after 10^6 time-steps.

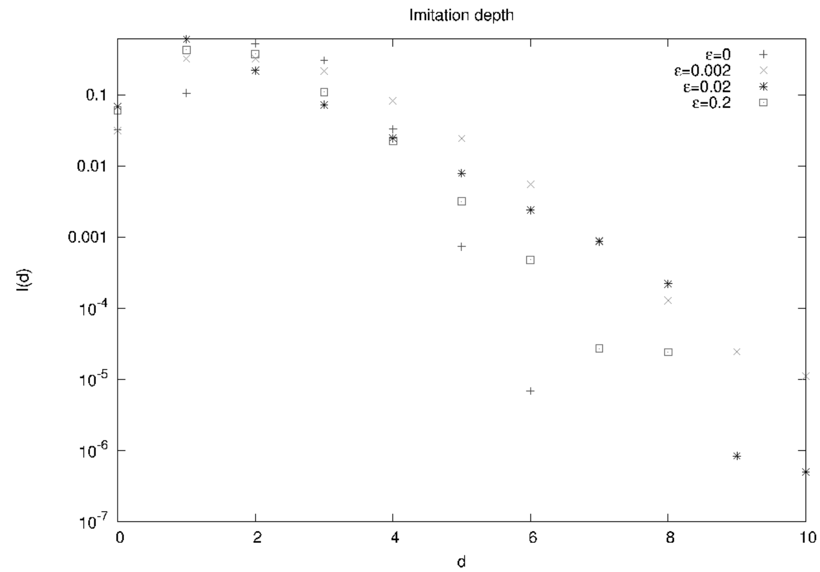

Figure 5.13: Distribution of imitation depths for different values income fee ε for Barabási-Albert network.

Domain size distribution $D_A(s)$ in the figure 5.14 has a power-law tail with exponent $\alpha_{domains} = 1.97$ and exponential finite-size cutoff

The leader domain size distribution in the figure 5.15 follows the power-law with exponent $\alpha_{leader\ domains} = 1.9$ decay for income fee $\varepsilon \geq 0.02$ but the behavior is destroyed for $\varepsilon < 0.02$ where the decay is faster than exponential.

Forking distribution $F_A(f)$ is shown in the figure 5.16 and it has a power-law tail with exponent $\alpha_{forking} = 1.96$ for values $\varepsilon \geq 0.001$ and exponential tail for ε around 0.

The leader forking distribution $F_L(f)$ in the figure 5.17 has a power-law tail with exponent $\alpha_{leader\ forking} = 1.74$ for $\varepsilon > 0.01$ and the behavior is changed to have exponential tail for $\varepsilon < 0.002$.

Finally, the correlation of number imitators and connectivity $I_A(c)$, cor-

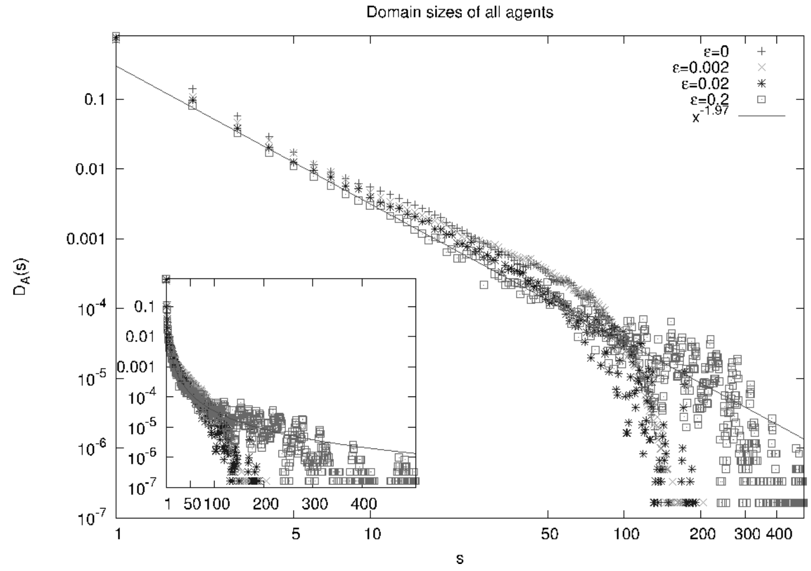

Figure 5.14: Distribution of domain sizes for different values income fee ε for Barabási-Albert network.

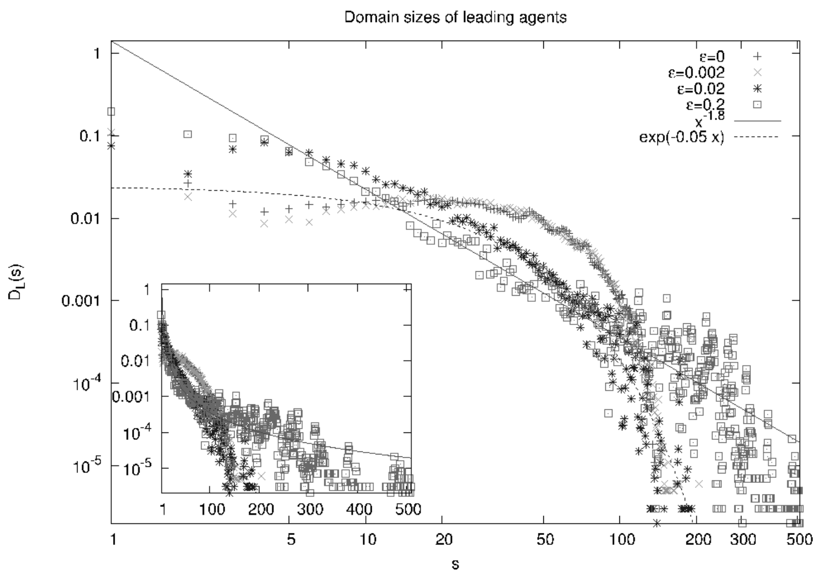

Figure 5.15: Distribution of leader domain sizes for different values income fee ε for Barabási-Albert network.

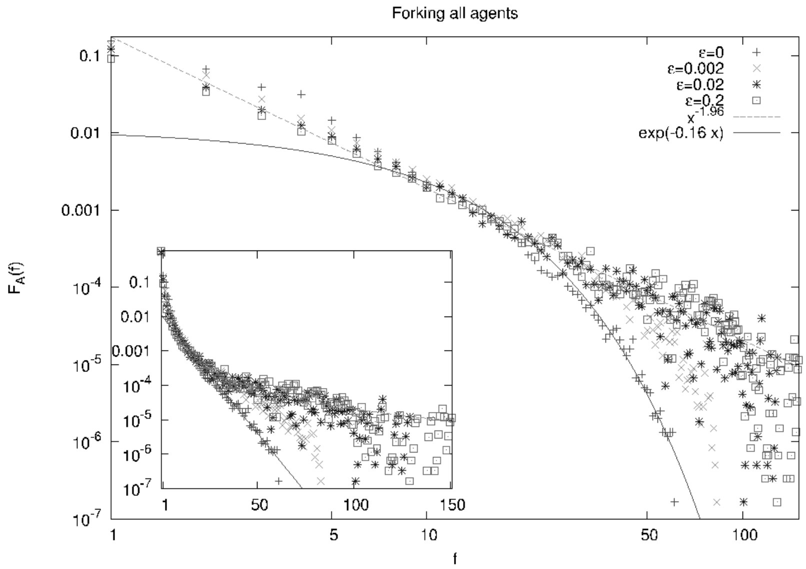

Figure 5.16: Forking distribution for different values income fee ε for Barabási-Albert network.

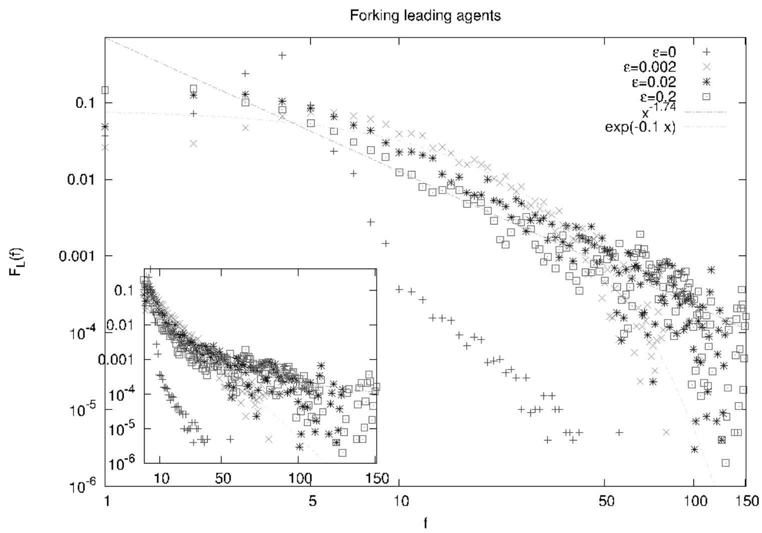

Figure 5.17: Leader forking distribution for different values income fee ε for Barabási-Albert network.

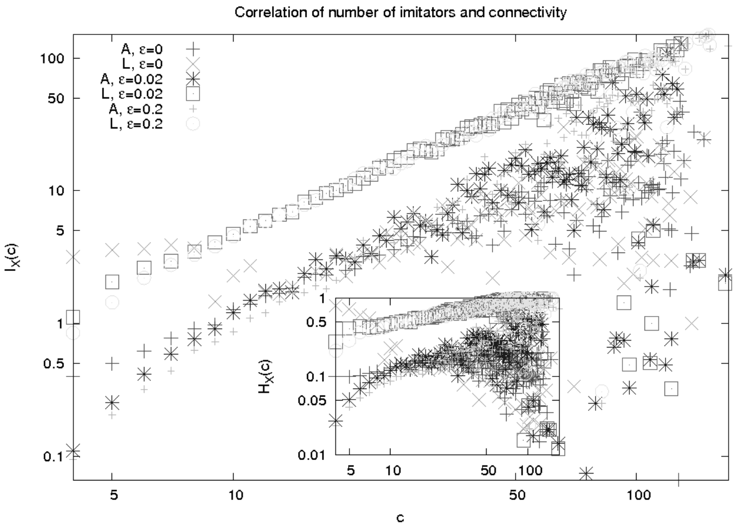

Figure 5.18: Correlation of number imitators and connectivity with combination with correlation of number imitators and connectivity for different values of income fee ε for leaders for Barabási-Albert network. There is efficiency of habit exploitation in the inset.

relation of number imitators and connectivity for leaders $I_L(c)$, and efficiency of habit exploitation $H_A(c)$, $H_L(c)$ is shown in figure 5.18. Low connected leaders play significantly better for $\varepsilon < 0.001$ but high connected leaders are comparable with the effectivity of all agents in absolute as well as relative numbers. However, leaders have significantly more imitators in absolute as well as relative numbers than a common agent with the same connectivity for $\varepsilon \geq 0.001$.

5.5 Conclusions

Simulations of the Minority Game with imitation were made. Payment for imitation is affecting imitation structures for both substrate networks. Correlation of points and connectivity $P(c)$ become a increasing function for fee $\varepsilon > 0.01$. Thus, more connected agents benefit for high fees. Imitation depth has a very different behavior in the substrate networks. In the case of Watts-Strogatz network, the exponential tail for high probability of rewiring and high fee is transformed into a Poissonian distribution but increasing fee transforms faster decay than exponential to exponential for the Barabási-Albert network. The domain sizes distribution follows the power law for the Barabási-Albert network but the Watts-Strogatz network causes complicated behavior which can be the power-law for high probability of rewiring and small fee or exponential for other cases. The correlation of number imitators and connectivity, correlation of number imitators and connectivity for leaders, and the efficiency of habit exploitation behave similarly in all cases. Small payments for imitation does not seem to give an advantage leaders of domains in compared to all other agents. High connectivity of leaders is an advantage only when high payments are present otherwise the is no significant difference between leaders and imitators.

Chapter 6

Scattering model of wealth in society

6.1 Introduction

The mass media reduce actual information on modern capitalist economies[1] to various measures of individual or corporate wealth. Social tensions are caused by disproportion in wealth which in turn is governed by the market activity of people. That activity can cause changes in social structure and, in revolutionary times, social catastrophes.

The origins of the investigation of wealth distribution can be found by looking back for the 19th century when philosophers and scientists started to investigate many natural as well as social and economical phenomena. In fact, the 19th century was not only a revolutionary era for natural sciences which were used in industry but this period led to the development of the industrial revolution which changed society in the euro-

[1]The reader should strictly contra-distinguish between modern, classical and early capitalist economy.

american region from agricultural medieval structures to modern society with new classes and new relations. The same process is expected in the fast-developing and fast-growing regions.

The next generation of thinkers on the edge of the 20th century tried to argue using more exact methods and the first "natural law" of economics (Pareto's law) was formulated. The Italian plutonomist[2] Vilfredo Pareto in [150] observed that the higher end of wealth distribution follow power-law. He thought that the power-law exponent for income distribution (probability distribution) was $\alpha = \frac{3}{2}$ what was a universal constant for all regions and all times. However, Shirras in [153] concluded that there is no Pareto law. On the other hand, Mandelbrot [154] formulated a "weak Pareto law" applicable asymptotically to the higher end of incomes. Thus, the majority of population is outside of the Pareto law.

6.2 Facts of distribution of wealth

The first measurement of wealth distribution was made by Pareto in a book [150] and it later influenced the investigation of wealth distribution in the late 20th century and the beginning of the 21st century by Dragulescu et al. [159]. They analyzed low income data (below $120k\$$) for single persons from USA from different sources and two-earners families from USA. Individual annual income for single person follows the exponential law

$$P_1(r) = \frac{1}{R} \exp\left(-\frac{r}{R}\right), \tag{6.1}$$

for two sources of data where R is mean income $R = \int_0^{+\infty} r P_1(r) dr$.

The next step is focused on families with 2 earners. The total income of

[2]Political economist

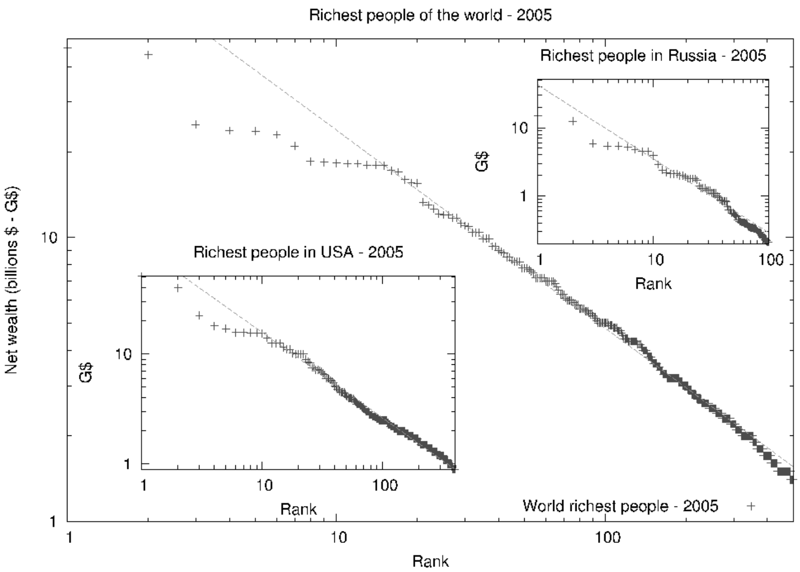

Figure 6.1: Graph of net wealth on rank of the richest people in the world with power-law fit with exponent $\alpha = 1.43$. There are also the richest people with its own appropriate power-law fits in USA with exponent $\alpha = 1.29$ and in Russia with exponent $\alpha = 0.92$ in insets. The data was taken from Forbes magazine in November 2005.

family is $r = r_1 + r_2$ and two person wealth distribution is convolution of individual probability distributions, what lead to

$$P_2(r) = \int_0^r P_1(r')P_1(r - r')dr' = \frac{r}{R^2} \exp\left(-\frac{r}{R}\right), \tag{6.2}$$

what fits experimental data for families with 2 children under 18 year old very well. The next step was a discussion of all families and it was suggested to have superposition of distributions 6.1 and 6.2 which fits data well. Finally, the last step of the article was computation of the Gini coefficient measure of the inequality of wealth distribution. The computed Gini coefficient fits theoretical values, but time evolution shows a small but visible increase of the coefficient from beginning of the 70s. It can correspond with changes in society accepted by sociologists, e.g., Keller in [4], [5] and [6] or Wallerstein [7].

M.Levy et al. in [151] analyzed data from Forbes magazine of the 400 richest people with the power-law exponent $\alpha = 1.36$ and this is in good agreement with figures 6.1 and 6.2 as well as the power-law exponents. However, the power-law exponent of richest people in USA has changed from $\alpha_{1996} = 1.36$ in [151] to $\alpha_{2005} = 1.29$ in 6.1, which can be only a small fluctuation or sign of deep structural changes of wealth distribution.

Wealth and income data from Great Britain and USA was investigated by Dragulescu [160]. Wealth distribution (probability distribution) in Great Britain was reconstructed from Inland Revenue data and the exponential fit is valid below $100k£$ but the power-law fit is valid for wealth more than $100k£$ with exponent $\alpha = 1.9$. The Gini coefficient rose from 64% in 1984 to 68% in 1996. Income distribution in Great Britain from 1994 to 1999 can be approximated by exponential function below $40k£$, temperature of the data rises with one exception to $R_{UK} = 11.7k£$ between 1998 and 1999, and the power-law function upon $40k£$ with exponent $\alpha = 2.0 - 2.3$, which is valid only for 5% of all population.

The higher end of income distribution in the states forming the USA follows the power law with exponent $\alpha = 1.7 \pm 0.1$, where 70% of all states differs less than ± 0.1. The lower end of income distribution (95% of all population) follows the exponential law with US temperature $R_{US} = 36.4k\$$. Temperatures of the states fluctuates $\pm 25\%$ from average US temperature. If the exchange rate between dollars and pounds are used then temperature in the USA is higher than in Great Britain and there is possibility to construct profitable thermal machine. This could be origin of high

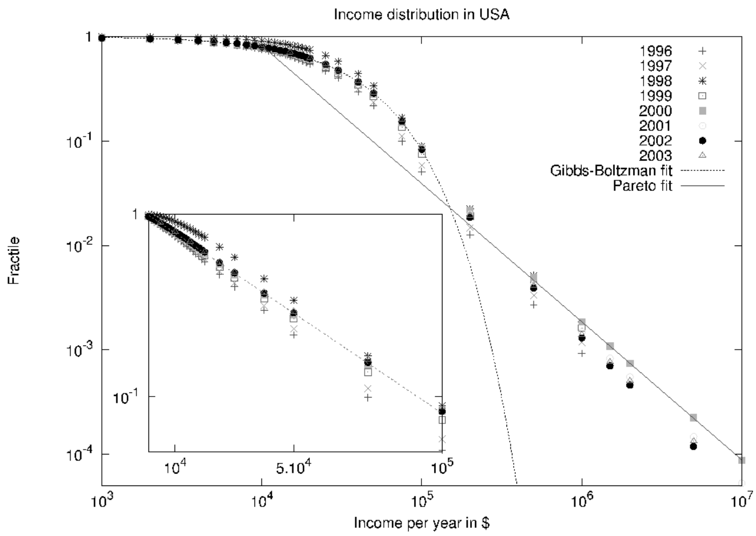

Figure 6.2: Graph of income distribution in USA from 1996 to 2003. Gibbs-Boltzmann fit is done with temperature $39.6k\$$ and Pareto fit with $\alpha = 1.32$.

trade deficit of USA with other countries with smaller temperatures.

Egyptian society in the 14th century BC which is investigated by Abul Magd in paper [164], when Pharaoh Akhenaten founded a new monotheistic religion of god Aten. A new capital of Egypt was found and called Akhenaten. Instability and permanent war related to previous religion resulted in the new capital being populated only for 20-30 years and then it has moved back to Thebes. Akhenaten can be considered as a representative of an accident urban society and modern excavations revealed the distribution of house areas. It is assumed that area of a house is a measure of the wealth. This revealed the power-law exponent is $\alpha = 3.76$ for the ancient society and the area of the houses was nearly proportional to the wealth.

Aoyama et al. in [163] investigated the individual income and debts of the bankrupt companies distributions. The Pareto law with exponent close to $\alpha = 2$ was revealed for income distribution for 2 sources of data. Debts of bankrupt companies follows Zipf law with the power-law exponent close to $\alpha = 1$.

Paper [165] by Sinha et al. is focused on wealth and income distributions in India. Net wealth distribution showed the Pareto exponent close to $\alpha = 0.8$ for the richest Indians but the worlds richest people have the exponent close to $\alpha = 0.9$. The next measurement was a correlation of net worth over an interval of 6 months. The graph did not show significant changes but the impact of stock exchange was discussed and wealth gained or lost by agents was found to be proportional to their overall wealth. The income of top Indian company executives was investigated and the Pareto exponent $\alpha = 1.51$ was revealed. However, lower-income Indian households and individuals with 10 or more years experience of working in IT industry showed log-normal distribution for the low- to middle-income range.

Two sets of high quality income data, one from UK and the second from the USA, were investigated in [185]. Log-normal and Gibbs-Boltzmann distribution can be a good fit of real data and authors prefer Gibbs-Boltzmann distribution.

The author investigated the richest people (by the net worth) in the world, in USA and in Russia and the results are shown in figure 6.1. It was found that there is a power law in all data but the exponents are $\alpha_{world} = 1.43$, $\alpha_{USA} = 1.29$ and $\alpha_{Russia} = 0.92$ respectively. It seems that the highest end of accumulated personal wealth is the power-law but the exponents differ from one region to another.

The next data was investigated from the Income Revenue Service in USA which is shown in figure 6.2. There is the power-law with $\alpha_{USA} =$

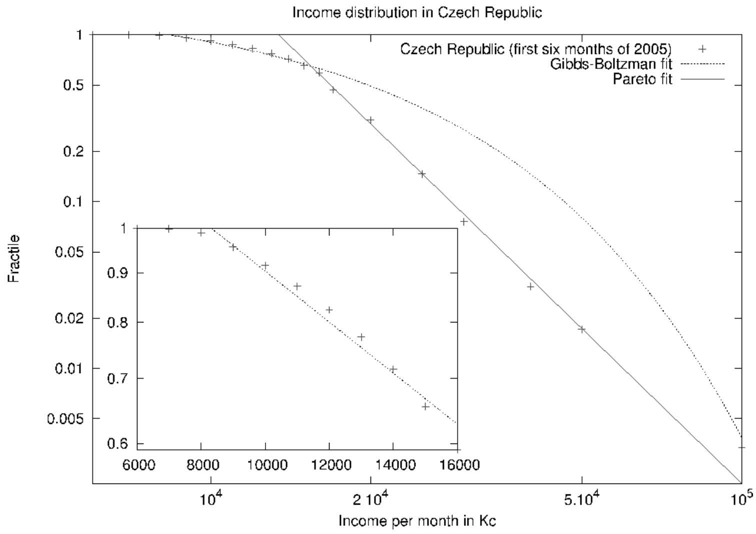

Figure 6.3: Graph of income distribution in Czech Republic in first six months of 2005. Gibbs-Boltzmann fit is done with temperature of 16.5kKč and Pareto fit with $\alpha = 3.08$. Source ČSÚ.

1.32 for top 10% of population and Gibbs-Boltzmann with temperature $T_{USA} = 39.6k\$$/year for low 90% of population. Finally, the Czech Republic income data was obtained from ČSÚ[3] and the data is shown in figure 6.3. The power-law with exponent $\alpha_{CR} = 3.08$ is applicable for 60% of population and possible Gibbs-Boltzmann with temperature $T_{CR} = 16.5kCZK$/month fits the lowest 40% of population. So, temperature in Czech Republic is only 23% of temperature in USA using actual exchange rate. These facts lead to conclusion that Gibbs-Boltzmann in combination with the power-law tail shows the universal behavior of income distribution. However, the position of the transition point between the regions is different as well as parameters of behavior.

[3]Český statistický úřad

6.3 Models of distribution of wealth

A theoretical understanding of the results, which were obtained by measurements of wealth distribution, form two schools:

- socio-economical,
- statistical.

The socio-economic school try to find a theoretical explanation of the effects of economic, political and demographic factors in, e.g., Levy et al. in [155]. Later, it was suggested that stochastic process was able to explain wealth distribution as well. Gibrat proposed in [152] in 1931 that income is governed by a multiplicative random process what causes log-normal wealth distribution (see also Montroll et al. in [156]). This idea suggests that the width of the distribution is not stationary, but it is growing in time (see Kalecki in [157]). This problem was fixed by Levy et al. in [158] where a cut-off at lower incomes was proposed and it stabilizes the distribution to the power law.

Following division into groups under origin of the model is only for orientation. It is due to the fact that most of the models are stochastic processes that are derived using intuitive understanding of microscopic processes from different points of view in society which influence wealth distribution.

6.3.1 Socio-economic models

Two toy stochastic models of capital exchange were investigated theoretically and simulated by Ispolatov et al. in [169]. The first model is the additive stochastic process which has 3 modifications - random exchange, when one point is exchanged, ″greedy″ exchange, when one point is exchanged but rich is favored, and very ″greedy″ exchange, when capital equal to difference of capitals of interacting agents is transfered to a richer agent. Mean-field solutions wealth distributions which were made, ″greedy″ models leads to finite-temperature Fermi distributions, with an effective temperature that goes to zero in long-time limit. It is the case where a small fraction of population is holding most of wealth and it can be reflected as existence of monopolies and oligopolies.

Liquidation of the agents with no wealth lead to generalization, when only a fraction of capital is exchanged. Two modifications were introduced - random exchange with no flavoring rich individuals, ″greedy″ exchange with favoritism of rich individuals. The random exchange model and ″greedy″ exchange model exhibit in different regions of the parameters of the exponential and power-law tail respectively.

A stochastic model, where total wealth is conserved, and interaction transfers a small fraction of wealth from one to a second agent, where interaction can be tuned by one parameter and global agent behavior is regulated by saving propensity was investigated by Chakraborti et al. in [173]. The authors argue that the saving propensity is dictated by an individuals self-interest, which makes the money dynamics cooperative and the global ordering is achieved. A kind of self-organization coming out of pure self-interest of each agent is reminiscent of the ″invisible hand″ effect in the ″free market″ suggested originally by Adam Smith in 1776. Two regimes emerge in the model, Gibbs-Boltzmann distribution and asymmetric and phase transition in scaling of most probable money earned by an agent.

The Conservative Exchange Market Model (CEMM) was investigated by Pianegonda et al. in [177], [178] and by Iglesias et al. in [179]. CEMM is zero-sum model of the economy game which is placed on one-dimensional lattice with periodic boundary conditions which produces exponential wealth distribution. Finally, comparison with real world data was completed.

The risk aversion factor of the agents was taken into account by Iglesias et al. in [180] with two modifications of dynamics and with two extra modifications which benefit poorer agents. In the first case when all agents have the same risk aversion and benefit for poorer agents is the same during the game. 5 regions were observed with its appropriate results - utopian socialism, liberal socialism, moderated capitalism, ruthless capitalism and few rich land. The second modification is disordered risk aversion and benefit for poorer agents, the power-law of wealth distribution was observed with $\alpha = 1$ as well as the exponential distribution.

The money-based model that is an extension of the Equíluz-Zimmermann model, was presented by Xie in [181]. This model is based on two simple rules. The first one is minimalization of costs and maximalization of profits which lead to merging two corporations. The second is disassociation of big corporations into small ones. The power-law with various exponents was observed and theoretically computed.

The nonlinear stochastic trade-investment model (NSTIM) was investigated by Scafetta et al. in [182]. This model benefits the "non-equilibrium" base of the actual model according to neoclassic economy, which assumes that all trades are done in "equilibrium", where price and value are equal (law of one price). The model is based on three assumptions. First, transfers of wealth from one agent to the another because the price paid fluctuates around an equilibrium price (value) and the price can differ from commodity transfered. The second assumption is that transactions are limited to total wealth of agents. And finally, poorer agents give preferential treatment according to rich agents that is necessary to avoid *gambler's ruin*. NSTIM have 3 independent parameters that can tune the strength of the interactions between agents and this separates two investigated models - trade-investment model and trade-alone model. The power law was revealed for the higher end of wealth distribution and the authors used Gamma distribution for fitting the probability density of wealth distributions.

A family based model of wealth distribution was investigated by Coelho et al. in [183]. The model allows aging of agents including dead and born of a new agent. The final wealth distribution has the power-law tail and the network of social contacts has an exponential tail. However, no clusterization was observed.

6.3.2 General stochastic models

A model of wealth distribution with Gaussian noise that causes non-conservation of total wealth and exchange matrix that controls strength of interaction between agents was defined in [171]. The simplest model involved when everybody is in interaction with everybody else with the same strength lead to the power-law tail of wealth distribution. An extension of this model to include income taxes showed wealth distribution with reduced inequalities of wealth but capital taxes with no redistribution led to higher inequalities of wealth. A simulation of this model on a fixed random network of possible interactions shows the power law as well. The emergence of a phase transition was revealed in the model with a randomly chosen random network at every step between the condensed state and the power-law regime of wealth distribution. In the next step, the d dimensional hyper-cubic lattice was investigated and $d \leq 2$ gives $\alpha \to 0$

but $d > 2$ gives a phase transition between a ”social” economy and a rich dominated regime. Finally, a non-symmetric matrix was investigated and an exponential (Boltzmann-Gibbs) distribution was revealed.

A stochastic model of wealth distribution was presented and discussed by Bose et al. in [184] where the agents were in 2 possible states called inactive (E) and active (E^*) and switching between them was allowed. These two states have a different inflow of wealth but the outflow of wealth has the same rate. The wealth distribution exhibited the beta distribution in the steady state.

A different stochastic model of money distribution was introduced by Dragulescu in [159]. The agents exchange small fixed amounts of money and the random fraction of interacting agents of average money. The Gibbs-Boltzmann distribution was observed for the money distribution. The next step involved defining the ”thermodynamics” of a firm and thermodynamical view of the economy was presented. In the next step, a model with possible debts was analyzed and a higher temperature in comparison with the same parameters was derived. The author believes that the actual boom of bank loans for consumers causes ”heating” up of society. Finally, a model of government which collects a value addition tax for every transaction in the system causes the exponential tail of money distribution.

A nonlinear stochastic model of wealth distribution with 3 parameters was presented by Scafetta et al. in [175]. The model is based on article of Bouchaud and Mezard [171] but Scafetta et al. introduced individual investment index and nonlinear stochastic variable, that describe actual exchange of wealth between individuals, whose standard deviation depends on wealth of interacting individuals. Scafetta et al. simulated and obtained Pareto law.

Di Matteo et al. in [176] introduced an additive stochastic process of wealth distribution on networks. The use of the Glauber-Kawasaki dynamics led to the average stationary wealth on each node being the Gaussian distribution. The power-law wealth distribution with the exponential cut-off was observed for networks with scale-free degree distribution.

6.3.3 Stock exchange models

The LLS (Levy-Levy-Solomon) stock exchange market model was defined by Levy et al. in [172] and the Pareto law of wealth distribution was observed by Solomon et al. in [170].

6.3.4 Game theory models

Two models of IFS (iterated function system) were investigated by Sinha et al. in [174] - the first uses the ″Theft-and-Fraud″ where the winner gets a random fraction of the looser′s wealth and the second uses the ″Yard-Sale″ where the winner gets a random fraction of poorer agent′s wealth. The strength of the interaction is controlled by one parameter. The 2-agent, 3-agent and N-agent models were also investigated and again the Pareto law were observed.

6.3.5 Deterministic models

Solomon in [166] introduced the stochastic Lotka-Volterra system of competing auto-catalytic agents with a focus on the stock exchange. The main idea was taken from the late 20s of the 20th century when Lotka and Volterra in [167] and [168] tried to explain fluctuations in the volume of

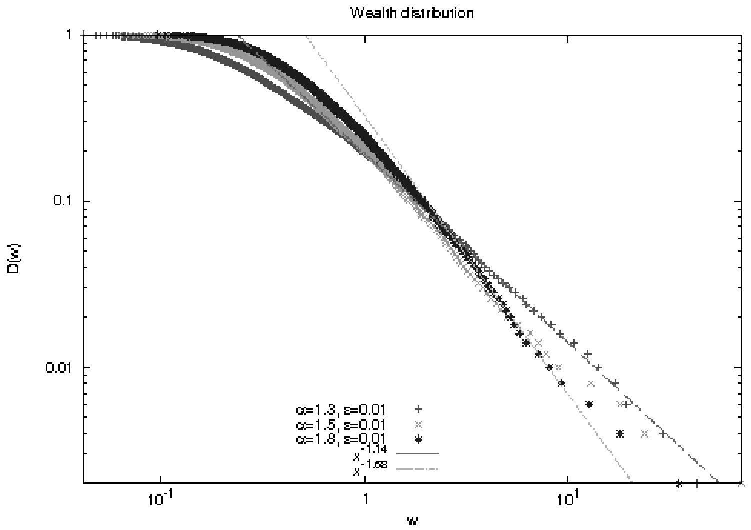

Figure 6.4: Distribution of wealth among agents in the mean-field model.

the fish population in the Adriatic Sea. The generalization of the main idea led to the power-law wealth distribution.

6.3.6 Physical models

A physical based interpretation of wealth distribution was completed in [186] where a model of wealth exchange using inelastic scattering interaction with positive balance of energy (wealth) was introduced. A theoretical mean-field solution of the model was investigated and the power-law tail was derived for a subset of the 2-parameter space of the model. The exponents are in good agreement with simulations made by the author in figure 6.5 with social tension in figure 6.4.

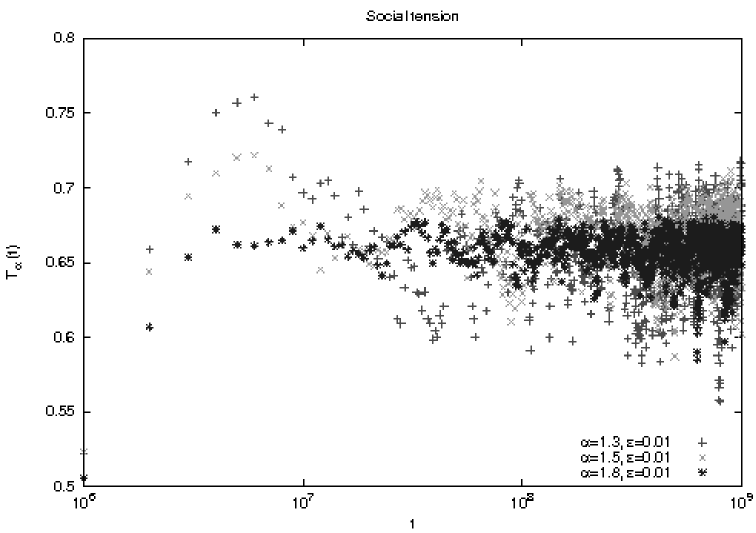

Figure 6.5: Evolution of social tension for the mean-field model. Parameters α and ϵ are as in [186] or below in the text.

6.4 Inelastic scattering model on network

The main idea of the model is an assumption that society can be viewed as "gas-like" particles, which interact using an interaction which has been known for 400 years - inelastic scattering interaction. But the main difference is that there is an inflow of energy (wealth) in contrast to the old physical models.

6.4.1 Definition of the model

Let us imagine a society of N agents which are bound one-to-one to vertices of a graph $G = (V, E, \epsilon, \nu_V)$ with non-oriented edges using ν_V. Each agent has an identificator and one variable which signs his/her wealth $\tilde{w}_i$, $i \in \{1, 2, \ldots, N\}$. Thus, the state of the system in a time-step t is described by $W(t) = \{\tilde{w}_1(t), \tilde{w}_2(t), \ldots, \tilde{w}_N(t)\}$. The agents are able to interact and the interaction is *essentially instantaneous*. Of course, in a real society it is more complicated and many economic interactions can take place at the same time, *pairwise*, although some economic interactions can be taken as multilateral rather than bilateral in a real society. Moreover, it is *positive*, the interaction has a positive effect on the total wealth of the society of the agents. Thus, the interacting agents become, in sum, more wealthy after the interaction than at the beginning of the interaction.

When two agents i and j are chosen to interact, the dynamics of the wealth of agents i and j is governed by interactions which can be formalized as follows

$$\begin{pmatrix} \tilde{w}_i(t+1) \\ \tilde{w}_j(t+1) \end{pmatrix} = \begin{pmatrix} 1+\varepsilon-\beta & \beta \\ \beta & 1+\varepsilon-\beta \end{pmatrix} \begin{pmatrix} \tilde{w}_i(t) \\ \tilde{w}_j(t) \end{pmatrix} \tag{6.3}$$

and all other agents remain unchanged, so $\tilde{w}_k(t+1) = \tilde{w}_k(t)$ for $k \neq i$ and $k \neq j$ where ε and β are parameters of the model. $\beta \in (0,1)$ measures the strength of the exchange and $\varepsilon > 0$ measures the one-step inflow of wealth.

The network is generated by the Watts-Strogatz algorithm [22], which supports the network with basic features which has been found to describe human networks. A rewiring algorithm is applied to a totally ordered network, which means that each edge is rewired to a randomly chosen agent with probability p. The resulting network can be imagined as in the figure 3.2.

There are two possible ways to execute one Monte Carlo step, as follows

- an agent initiated model

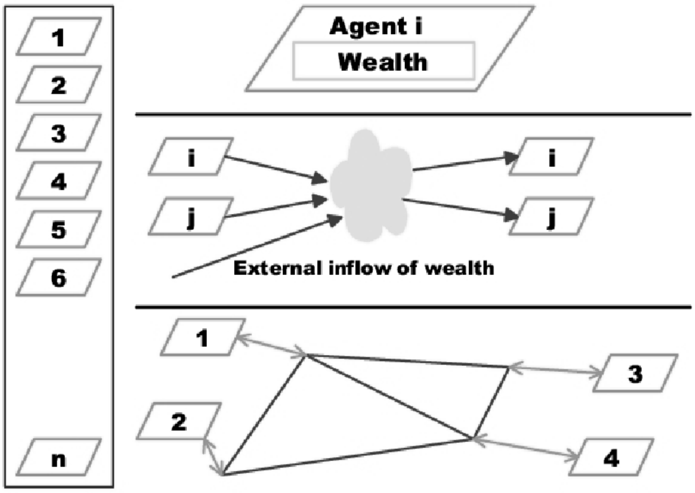

Figure 6.6: Scattering model with inelastic interaction. On the left, there is set of all agents. On top, there is schematic scructure of an agent. In the middle, schematic picture of interaction between 2 agents. Finally, on the bottom we can find agents bound to a graph.

- an edge initiated model

6.4.2 Agent initiated model

The updating mechanism for the Monte Carlo step is based on the choice of agents i.e., agent $i \in V$ is chosen with a uniform distribution and a second agent j is chosen with uniform distribution from his/her neighbors $E_G(i)$. It can be argued that the edges of the graph are only dispositions that can be used by agents. A pair of agents that interact is interested in collaboration. Finally, the collaboration is useful for them.

6.4.3 Edge initiated model

This model is based on the choice of an edge $\epsilon(e) = (i, j)$, $e \in E$ with the uniform distribution. The interacting agents are signed i and j, the rule of interaction is symmetric to the exchange of i for j, so there is no ambiguity. It can be argued that every connection in society is used with the same probability, and highly connected agents will interact very frequently.

6.5 Interesting variables

Measured wealth was normalized $w_i = \frac{\tilde{w}_i}{\overline{w}}$. This means that there are N units of wealth in the society after normalization, and they are distributed among the agents.

The first interesting variable is social tension, which measures differences in wealth as

$$T_\sigma = \frac{1}{\overline{w}} \left(\frac{1}{|E|} \sum_{i \in E} \frac{1}{|\Gamma_i|} \sum_{j \in \Gamma_i} |w_i - w_j|^\sigma \right)^{\frac{1}{\sigma}}, \tag{6.4}$$

where $\overline{w} = \frac{1}{N} \sum_{i \in E} w_i$ so it is the average. $\sigma \in (0,1)$ is a parameter set up to $\sigma = \frac{1}{2}$ in our simulations.

The second interesting variable is distribution of wealth

$$D(w) = P\left(w' > w\right). \tag{6.5}$$

where P means the probability that a randomly chosen agent's wealth is greater than w.

The following variable is the correlation between wealth and connectivity

$$H(c) = \overline{w}_{P(w|c)}. \tag{6.6}$$

Value $H(c)$ is computed as

$$H(c) = \frac{\sum_{i \in V} w_i \delta(k_i - c)}{\sum_{i \in V} \delta(k_i - c)}, \tag{6.7}$$

where k_i is connectivity of individual agent i and c is an integer value.

6.6 Results of simulations

The model was investigated with fixed interaction parameters which were set up to fulfill equation *(10)* from [186]

$$2\beta = (\alpha - 1)\,\varepsilon^2. \tag{6.8}$$

with $\alpha = \frac{3}{2}$, i.e., the same interaction where the power-law exponent of wealth distribution of the model on the fully-connected network was $\alpha = \frac{3}{2}$. Now there is only one freedom, which is fixed by setting up $\varepsilon = 0.01$.

The simulations were performed with the following parameters:

General parameters of the Monte Carlo method

- Number of agents $N = 10000$
- Final time of the simulations $T = 1.5\ 10^9$
- Number of Monte Carlo runs $R = 10$

Parameters of the interaction

- $\beta = 2.5\ 10^{-5}$
- $\varepsilon = 0.01$

Parameters of the network

There are two parameters involved in the construction of the small-world network using the Watts-Strogatz algorithm [22]:

- Initial number of edges from agent $m = 4$ (mean connectivity)
- Probability $p \in [0, 1]$ of rewiring of the edge .

The initial wealth of the agents was set at 1, so the initial wealth dispersed in the society of N agents is N.

6.6.1 Agent initiated model

The model is based on a random choice of agents which will interact using motion equation 6.3. The time evolution of social tension (figure 6.7) for all p values rises and then decreases, but in the case of parameter $p < p_a$, $0.00007 < p_a < 0.0001$, the process is slower and for the subset of cases with $p \neq 0$ there is a trough or plateau in the time evolution after the peak of social tension. For case with $p > p_a$ behaves differently: there is one peak and then a rapid decrease in social tension.

The distribution of wealth (figure 6.8) has the power-law tail for $p > p_a$ (the same symbol is used as a consequence of the power-law behavior and the different social tension evolution) with exponent $\alpha_{agent} = 0.96$, which is stable for the interval of p at the thermodynamic limit $N \to +\infty$, $T \to +\infty$ and $\frac{N}{T}$ constant. The power-law is valid for approximately 1 - 5% of the population, which is in quite good agreement with the measurements recorded in [160] by Dragulescu et al. The deviation of the data from the power-law for the higher end of the distribution is a finite-size effect. If $p > p_a$, the behavior of wealth distribution is no longer following the power-law, and the initial power-law tail is spread by the dynamics of the model.

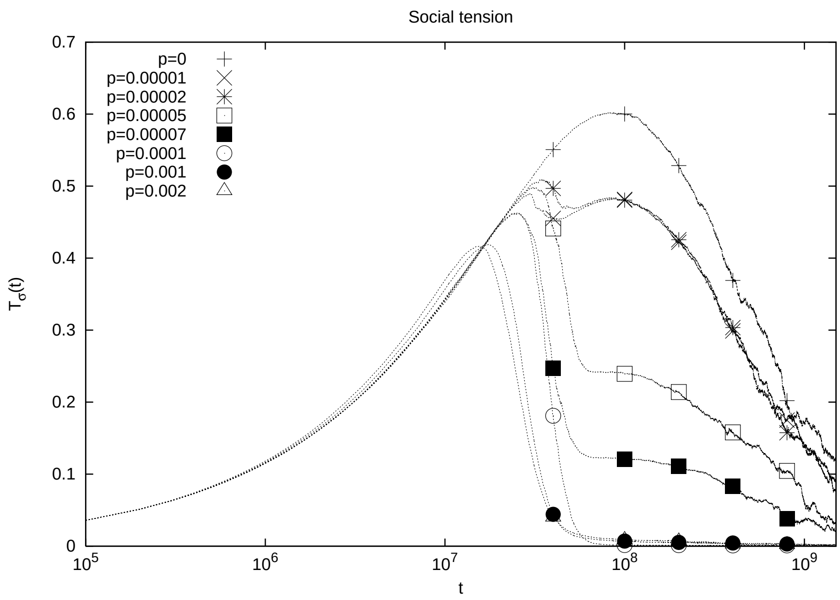

Figure 6.7: Time evolution of social tension in the agent initiated model.

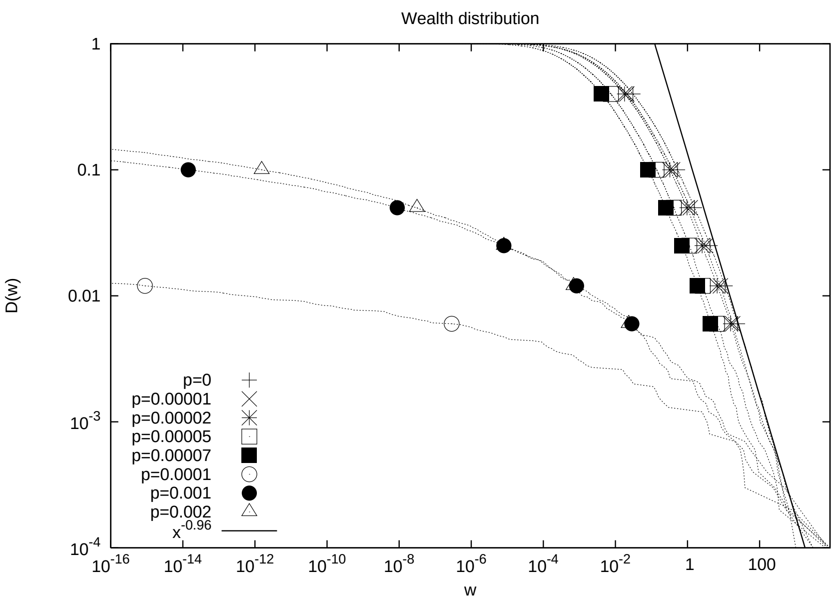

Figure 6.8: Distribution of wealth among agents in the agent initiated model.

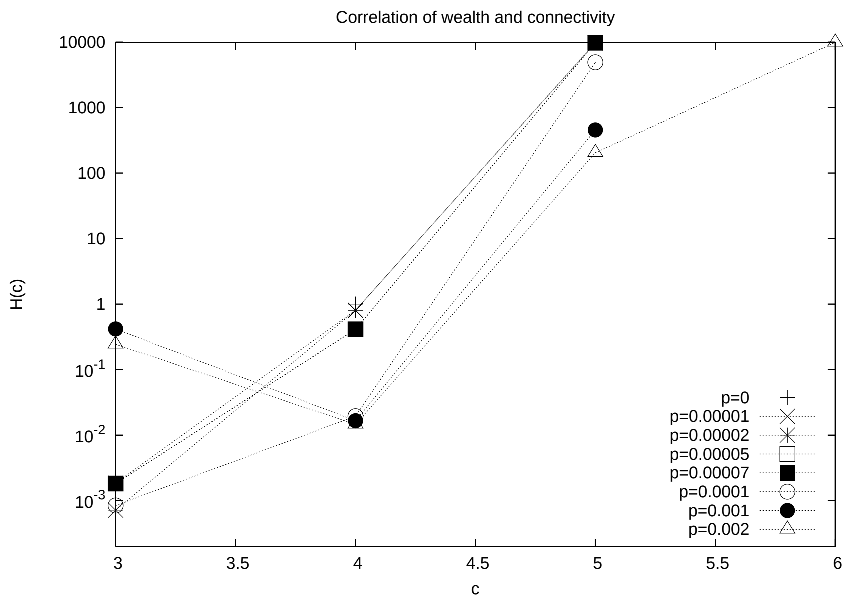

Figure 6.9: Correlation of wealth and connectivity in the agent initiated model.

In figure 6.9, the average connectivity of the network was 4, and the connectivity is dispersed around this value during the rewiring process. In the case $p < p_c$, there is a strong correlation between the average wealth and connectivity, but for the case $p > p_a$ this enables less connected agents to outperform agents with average connectivity.

6.6.2 Edge initiated model

This model is based on a random choice of an edge where the agents connected to the vertices of the edge will interact using the motion equation 6.3. The time evolution of social tension (figure 6.10) seems very similar to the previous case. In the case of $p < p_e$, $0.00007 < p_e < 0.0001$, the dynamic is slower than in the following case, and for $p > 0$ there is a peak and a plateau, or a twin peak. This is in contrast to the case $p > p_e$, where there

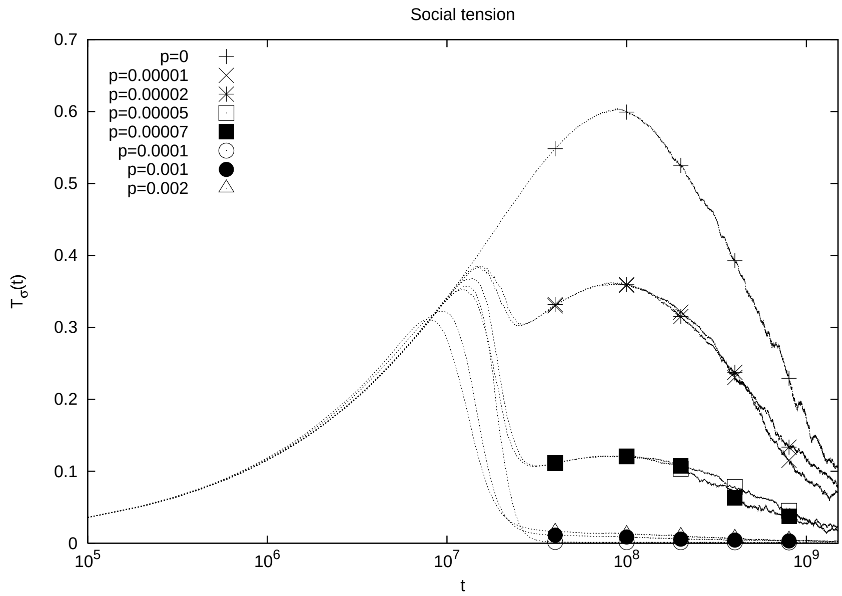

Figure 6.10: Time evolution of social tension in the edge initiated model.

is only one peak followed by a rapid decrease.

The distribution of wealth (figure 6.11) allows the power-law behavior with exponent $\alpha_{edge} = 0.95$ for the case $p < p_e$ and the power-law tail is stable at the thermodynamic limit. As in the previous case it is valid for 1 - 5% of the population and the deviation from the power-law for the higher end of the distribution is a finite-size effect. However, there is no power-law for $p > p_e$.

The correlation of wealth (figure 6.12) shows that the average wealth is a strictly growing function of connectivity c. The average wealth of a player with average connectivity (4) is better for $p < p_e$, which is similar to the previous case.

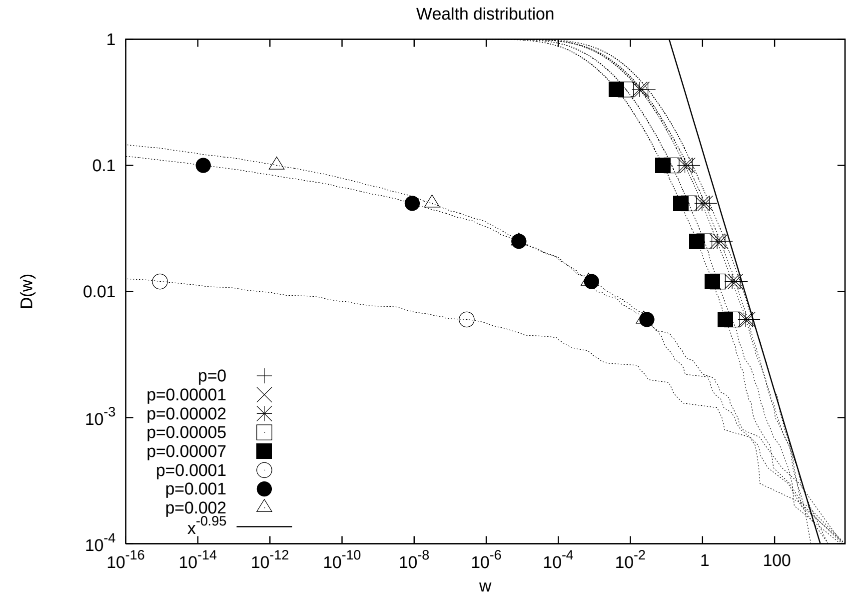

Figure 6.11: Distribution of wealth among agents in the edge initiated model.

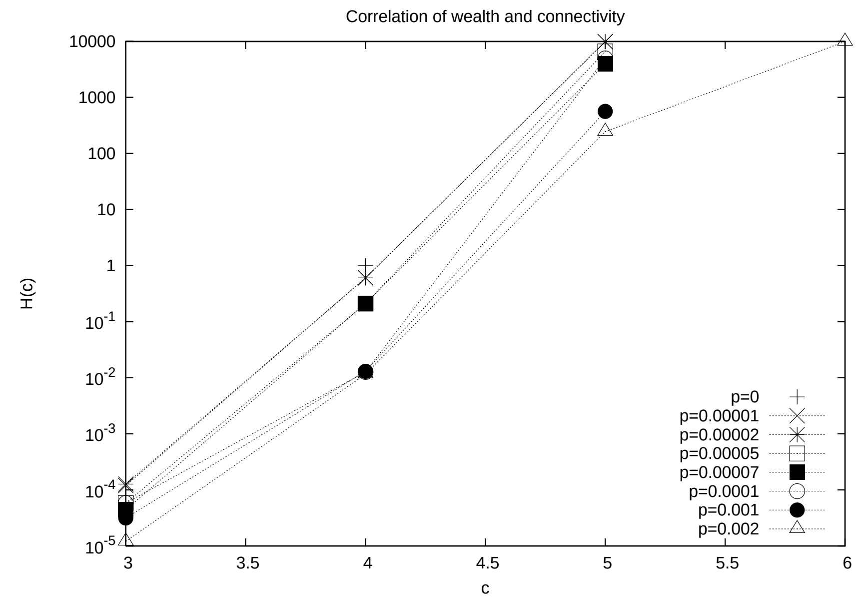

Figure 6.12: Correlation of wealth and connectivity in the edge initiated model.

6.7 Conclusions

The models of distribution of wealth and the experimental measurements of wealth distribution were discussed in this chapter. The author focused on the inelastic scattering model with an inflow of energy (wealth), where agent are placed on a complex network, which can be a good approximation to the structure of social contacts in a society. The model exhibits the power-law tail for a subset of parameter space rewiring probability $p < p_e$, $7\ 10^{-5} < p_e < 10^{-4}$ in both the cases. However, the author thinks that this model cannot be used in a real policy, but on the other hand it is a good demonstration of using of physics to economics.

Part III

Simulation library

Chapter 7

Computer simulation program

7.1 Introduction

During last quarter of the 20th century, computers became a popular and universal tool which were used in science, industry and entertainment. Now in society, there are many activities where computers now considered essential to society's progress. This situation derives from the exceptional commercial success of computers. The main scientific and engineering use of computers relates to simulations (but it is not only one) in contrast to commercial storing and processing of information.

Simulations are a valuable tool in many engineering applications that could fit possible states, cases of evolution of a problem or solutions. Typically, the problem is formulated by a stochastic process. This is caused by our inability to get full information of the system and (or) processes inside the system due to internal properties of the system or external influence on the system. The way how to deal with the stochastic process modeling the system is Monte Carlo method which was used for first time in statistical physics.

Especially, we intends to simulate socio-physical or econo-physical systems then our model is based on Game Theory. The model exhibits the main aspect of a conflict situations in the system. Since than our central object is a game that describes studied social or economical phenomenon. Moreover, the systems are usually consisted of many peer objects that are in the conflict. Thus, the model must also exhibit the fact what forces us to focus on multi-agent systems. Specifically, humans are introduced as the agents in the model and the conflict between them in society is interaction in the model. Many kinds of conflict are allowed only between limited number of objects rather than everybody can be in conflict with everybody else. Mathematical structures that we can be used to exhibit this situation are graphs.

7.1.1 Information in the models

From perspective of information, every object in the model can carry information which can be used in the interactions. Thus we have this kinds of information:

- Internal information
- External information
- Social information.

Internal information is information that is carried by agents. External information is available to every agent and it can be stored in the game as public information. Social information is information which pair of agents can interact.

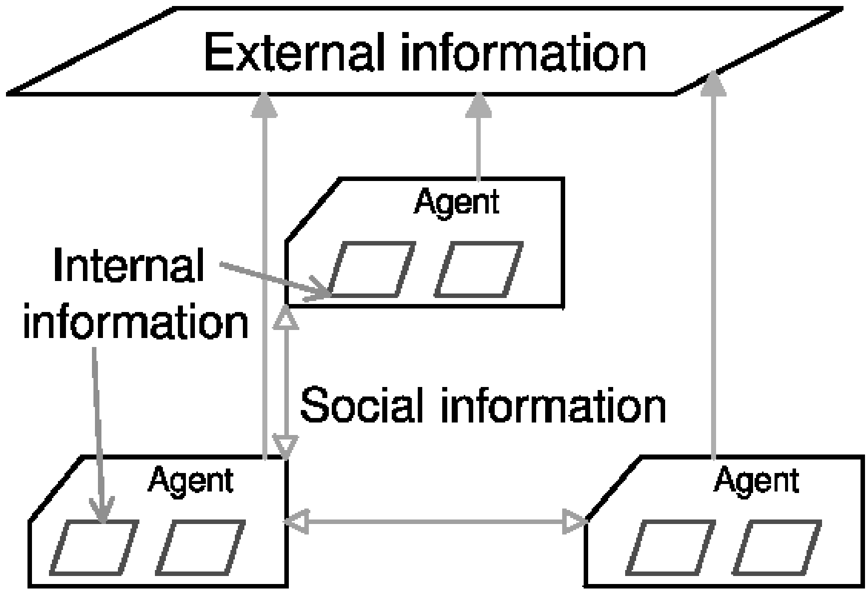

Figure 7.1: General structure of model of socio-physical or econo-physical system with information sources.

7.2 Implementation

The main aim of good programming is the development of working, easy-to-understand and extensible program. In particular for the thesis, the author developed the simulation program of the models introduced in the previous chapters. The main aim of development of the code was that it must be able to be used and reused in future applications in science and industry. The best solution to do so is to prepare a library[1].

[1]Piece of code that is linked to the main program, but it is not executable and its functionality lie in supporting of the main program with prefabricated functions and classes. In general, there are two forms of linking. The first one is a statically-linked library where the code of the supporting library is incorporated in the executable program. The second one is dynamically-linked library where the code of the library is in a operating system placed only once and the program have only links to the library in the code. There are two types of loading the library when a dynamically-linked library is used. The code of library is immediately loaded when the program starts in statically loaded dynamically-linked library. Programmer could load needed library in program using system calls in dynamically loaded dynamically-linked library. All of them are supported in the simula-

The library then consisted of general tools which forms environment where the simulations are embedded. This idea is not new and we can find on market many tools that are accompanied with GUI that are called CAS[2] (*Mathematica*, *Maple*, *Matlab*) or specialized simulation programs (*Swarm*, *Anylogic*). The main aim of this library is to focus simulation multi-agent systems on supercomputers rather than desktops.

Since start of development of the library author prepared simulations of this problem showing range of its abilities:

- Complex networks
- Sznajd model
- Minority Game
- Scattering models
- Percolation
- Quantum random walks
- Numerical solving of systems of ordinary differential equations
- General stochastic models.

7.2.1 Technical prerequisites

Data structure from figure 7.1 can be implemented in many different programming languages or CASs. The author's previous experience with different programming languages like Pascal, Basic, C, C++, Java or Python

tion program. However, the author prefers static linking.

[2]Computed Algebra System

and with previous scientific work on simulations of charged particles in plasma led him choose the C++ because of:

- Low level programming
- Object oriented programming
- Polymorphism
- Operator overloading
- Multi-fold inheritance
- Namespaces
- Templates
- Exceptions.

Low level programming is important for *fast programs*. Language C++ allows to compile fast programs that is because it is based on language C. **Object oriented programming** allows to logically encapsulate data structures with its own methods. **Polymorphism** allows to reuse code and modify certain methods using *virtual methods*. **Namespaces** allows name-structural programming that lead to better understanding of the code. **Operator overloading** allows the reuse of *standard operators* in user-defined operations. **Multi-fold inheritance** supports objects with *multiple ancestors* (in literature it is rarely mentioned as questionable ability). **Templates** allows to writing of type-general and value-general code. **Exceptions** can handle exceptional states of a program and libraries,. It allows to use different data structures and hardware enhancements (kernel threads, MPI, XML, STL).

These requirements are successfully fulfilled in other programming languages like Python and Java but C++ was chosen due to **standardization**, general **UNIX support** and mainly due to **fast programs**. Recent developments of Python[3] show that it can be interesting platform in near future.

The main platform which was chosen for support was IA32[4] due to its extensibility but UNIX like operation system with its own standardization in combination with C++ allow to use it on the different platforms, which are provided now or which will be developed in the future. Finally, it allows the use of the simulation platform headers in different programming languages like Python and write code in high level programming languages including scripting languages.

7.2.2 General structure of simulation program

One of the first problems which has to be solved before writing some source code involved finding out if the program will be a terminal program or if it will be graphical application for the X-Window system. Keeping in mind maximal simplicity of the program, the author chose the terminal application but the source codes have a structure that is similar to the structure of the source code of graphical applications which are generated from graphical *RAD tools*[5] like *C++ Builder* or *Delphi*. This simulation library could be accompanied with its own graphical *RAD tool* like *Simulink* in *Matlab*) or *Anylogic* that fast helps to develop an appropriate application. Moreover, the author believes that the *RAD tool* based on

[3]Please, check actual development at http://python.org/
[4]32-bit Intel architecture
[5]Rapid Application Development

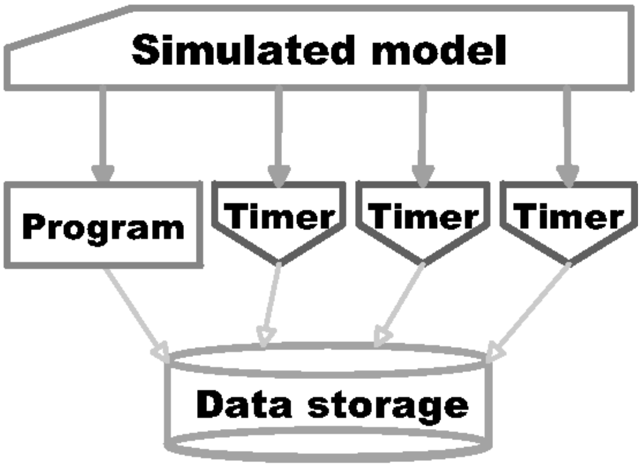

Figure 7.2: General functionality of the simulation

UML[6] is the right way for wider use of the platform with a combination of rapid development.

All the pevious remarks on development and structure of code were materialized in a computer library called **Zarja**[7]. The name was chosen because of its extendibility and modularity which is the same as its preimage in space.

7.2.3 Programming models of multi-agent systems

Implementation of the previous remarks into the code has changed during the scientific work of the author during development. It led to two different styles of programming of the main code.

The first model was implemented the Minority Game and scattering model and it's structure is called classical[8]. The second structure is called modern because it structure is similar to functionality as modern program-

[6]Unified Modeling Language
[7]Zarja is the first module of ISS (International Space Station).
[8]Classical, because of structure of the code that is similar to C programming.

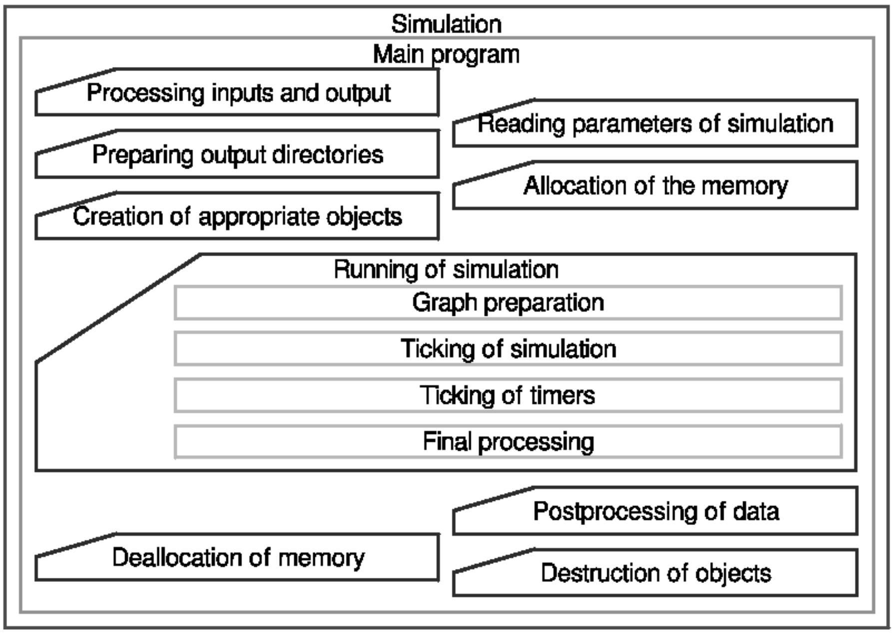

Figure 7.3: Structure of simulation of Minority Game and scattering models

ming libraries[9].

Classical simulation structure

The main program `int main (int argc,char * argv[])` executes which governs the whole simulation calling appropriate functions. The Minority Game is based on deriving from *class agency*. However, scattering models are derived from *class network* and it is connected with the investigation of the possibility of the usability of the **Zarja** simulation library. The main aim of a different base class of the models lay in different views of the incorporation of the network. The Minority Game has the network incorporated as a variable but the scattering models are directly based on *class network*.

[9]Modern, because using of the library is C++-like

The general structure of simulations of the Minority Game and the scattering models can be found in 7.3. It can be seen that the whole functionality is done by the main program which is called interface of class simulation. First of all, standard inputs and outputs[10] must be processed, then the output directory must be set up and prepared for data storage and a general log of the simulation is created. After then, class of the simulation is created with many supporting classes and parameters of the simulation are set up. The next step is the allocation of the memory is needed during simulation. The main simulation is executed and the whole simulation can be broken down into 4 sub-steps. The first one is generation of graphs. The second is the main execution of simulation calling function `int Tick()`, that evolves the simulation by one step. Measurements are done by timers and these must be evolved synchronously with simulation. Finally, the simulation has to process the stored data. After finishing the main body of simulation, the data stored in the main program, then processed and the memory deallocated. Finally, the objects are allocated (destructed).

Modern simulation structure

The author describes this style as modern because the style is thought as standard in C++ programming, although C++ is more than 20 years old. So it is not surprising that the author found an amazing similarity of the structure of the simulation library and graphical applications written in MFC[11], Borland Delphi[12], QT library[13] and Clanlib[14]. A general structure

[10]standard input - `stdin`, standard output - `stdout` and standard error output - `stderr`

[11]Microsoft Foundation Classes

[12]and it's clones like Kylix and Borland C++ Builder

[13]www.trolltech.org

[14]www.clanlib.org

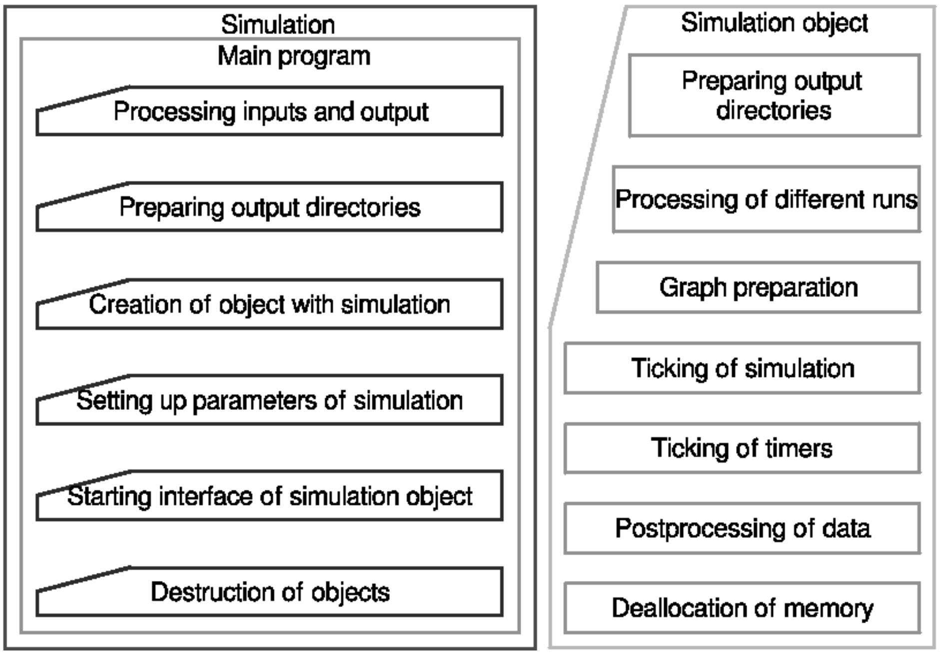

Figure 7.4: Structure of simulation of Quantum games and Sznajd model

of a simulation is shown in figure 7.4.

At beginning of the program, standard input, standard output and standard error output need to be processed and parameters of a simulation are then extracted. Next the output directory is prepared and a general log of a simulation is created followed by the object of a simulation being created and its parameters are set up. Finally comes the start of the simulation using `int Run()` or `int RunMPI()` or `int RunThread()`[15] followed by the object being destructed.

When an object with the simulation is executed using `int Run()` or, in future, `int RunMPI()` and `int RunThread()` then timers must be created and memory must be allocated. The next step is preparation of the graphs. Following this the simulation can be evolved and the timer

[15]`int RunMPI()` and `int RunThread()` will be allowed when the simulation classes and timer classes will be MPI and thread ready.

can store data. Finally the data obtained is postprocessed, the memory deallocated, the timers destructed followed at the end of simulation with the object being deallocated (destructed).

7.2.4 Components of the simulation library

A typical single-purpose library has many components which helps developers to successfully develop an application[16]. This library will be similar but the components will be oriented for scientific work with a multi-agent system on graphs. The structure of the library has to reflect this fact. Language *C++* supports encapsulation of the data into the objects which are called **classes** in *C++*. The **classes** support *inheritance* which is a useful tool for deriving more complex objects which can handle more complex systems in the future. This can be useful for understanding social and economic phenomena.

The whole work can be split into two parts; the first one is the library and the second one consists of examples of using the library. Although there are two parts it is regarded as one project and it can thus be described as one compact tool.

The library should be consisted of the following components:

- Random generator
- Node
- Edge
- Network
- Agent

[16] e.g. libxml - xmlsoft.org, clanlib - www.clanlib.org, qt - www.trolltech.org

- Game
- Timer
- Data storages
- Computing supporters
- Other tools
  - Special game structures
  - Directory tools
  - Memory tools
  - Processing tools
  - Mathematical tools

The structure of collaboration of the main components of the **Zarja** library is shown in the figure 7.5.

Random generator

Random generators form the core of all Monte Carlo simulations. The structure should provide random numbers with different distributions and its object encapsulation permits its substitution for a different kind of source of coincidence. It means that robust results are obtained from the simulations.

Every random generator is located in a *namespace* random_generator and it is derived from an **abstract class** general_random_generator. The class provides the basic interface of a random generator and supports of a different kind of output random numbers (*int, long , long long, double, long double*) with different distributions are implemented. Recently the

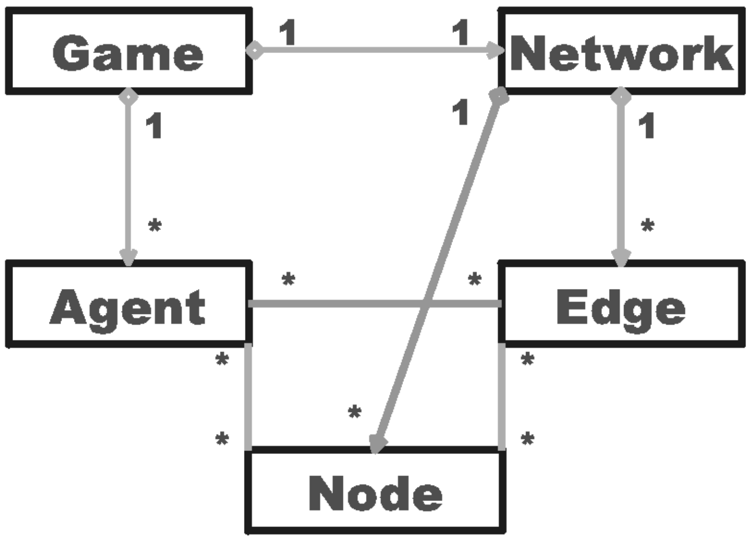

Figure 7.5: General **UML** scheme of structure of **Zarja**. The light arrows are usual connections between components. The dark arrows are extra connections that can be added to the model.

general_random_generator and its derived *classes* support various distributions discreet or continuous.

Node

The node forms the building brick of the all networks. Its interpretation in Graph Theory is a vertex in a graph. Inheritance supports the creation of derived objects which have an extra structure which can support developing new agent-based models, where the agents will be connected to the nodes but where the agents are only one of many applications of the library. Included in the library is a model of percolation which is one of the cases of different applications.

Each node is derived from *class node_general* in *math_network::nodes namespace* as well as all its offsprings in the library **Zarja**. The class supports only features which facilitate the analogy of a general vertex in a graph. Each class have its identifier which is used for identifier and it is unique in the graph.

Class node_general was one of the first objects which were developed in the library. It has two offsprings *node* and *node_weighted_network*. *Node* is simple vertex of a graph which facilitates accessing an edge without any information. Later, *node_with_general_link* was derived and it have a feature to connect to the agents in a game. Thereafter, more complex nodes in a graph are implemented: *class node_with_general_link_and-_dye_infrastructure* which encapsulates previous features with the ability to find clusters in a graph and finally, *class node_with_caching_infrastructure*, which is used by the Minority game to resolve the most successful agents.

The next branch which is derived forms *node_weighted_network* and it is oriented to allow graphs with extra information carried by the edges. It has one offspring *node_weighted_network_with_general_link* which allows extra data on nodes to be carried.

Edge

An edge is a connection between vertices in sense of **Graph Theory**. The vertices usually carry no extra information but recent papers by Barrat et al. in [17], [18], [19], [21] and by Barthelemy et al. in [20] show that weighted graphs are becoming more popular. **Weighted networks** are structures where the edges can carry an extra information [17].

There are 2 **abstract classes** namely *edge_object* and *oriented_edge* in the library. The first one is *edge_object* and it is only a simple edge with no future development. It is used in graphs with a list of edges without

[17]There is contained a real number in an edge in the cited papers

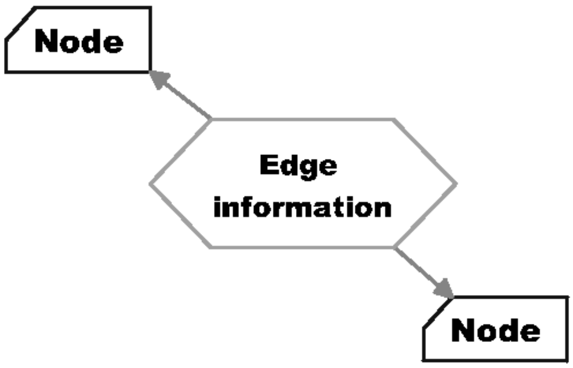

Figure 7.6: An edge with an extra information

extra information. The next base *class* is *oriented_edge*. It and its offspring *nonoriented_edge* plays the role of an edge in Graph Theory which can carry any information as can be seen in the figure 7.6.

Up to now, there is only few simulations with the **weighted networks** or more generally with networks with extra information on the edges. None of the models, used in this thesis use the **weighted networks**. Future work is going to be pointed to the **weighted networks** as it seems to be a good base of modeling processes on complex networks.

Network

The *class network* encapsulates a graph as an *object oriented programming* and it is an **abstract class** of all networks in *namespace math_network*. Its direct offsprings *oriented_network* and *weighted_oriented_network* and their offsprings can carry an extra information. A simple graph have structure as can be imagined from figure 7.7 is constructed from nodes which are connected with edges. In general, programming language C++ supports one of the modern tools of *object oriented programming* - **multiple in-**

heritance[18] which allows the creation of classes with many base *classes*.

The first direct offspring of *network* is *oriented_network* which supports the functionality of a graph with oriented edges without any extra information. Its offspring is *oriented_network_with_edge_list* which has an extra list of edges in a network, *oriented_network_with_links* which is able to make connections between agents and nodes, and finally *nonoriented_network*. The latter is a base *class* for all networks with non-oriented edges. In the next generation, there are *nonoriented_network_with_edge_list* with ability to make connections between the agents in a game and the nodes in a network and with one offspring *nonoriented_network_with_links_and_cache* that is used by Minority Game with possibility use cache, and *nonoriented_network_with_edge_list*. It is the parent of many useful *classes* like *network_percolation* which is used by percolation problems, *nonoriented_network_with_dye_infrastructure* which facilitates the finding of *connected clusters*[19]. Finally, a structure called *nonoriented_network_with_edge_list_and_links* which implements a list of all links and ability to have nodes connected with agents of a game. At the end of the branch is *nonoriented_network_for_quantum_game* used by quantum games and subbranch of scattering models based on *scattering_model*. Its offsprings are *scattering_model_agent_initiated* which governs agent-initiated simulations, *scattering_model_bond_initiated* which governs edge-initiated simulations, *scattering_model_increasing_energy* which is base class of all scattering models with increasing energy. The agent-initiated simulations of the scattering model with increasing energy is supported by *scattering_model_increasing_energy_agent_initiated* and the edge-initiated

[18] There is argued that **multiple inheritance** is questionable functionality in [82]. Notwithstanding, many modern programming languages have **multiple inheritance** implemented, f.e., standard stream in *C++* use the tool.

[19] A cluster is a subset of all vertices that are connected.

simulations are governed by *scattering_model_increasing_energy_bond_-initiated*.

The next offspring of the *class network* is called *weighted_oriented-_network* what is the base *class* of all network with extra information carried by the edges. It has two offsprings namely *weighted_oriented_network-_with_edge_list* what has a list of all edges, with one offspring *weighted-_oriented_network_with_edge_list_with_link* and the next offspring is *weighted_nonoriented_network* which is a base class of a network with nonoriented edges. The last class has one direct derived *class weighted_nonoriented_network_with_edge_list* having a list of all edges and one derived *class weighted_nonoriented_network_with_edge_list_with_link* with possibility to connected nodes and agents. The is only one follower of the *class weighted_nonoriented_network_for_quantum_game* used by quantum games.

The branch of the *class network* has all the necessary functionality for future simulations although some more functionality could be necessary in the future.

Agent

Every structure which can be handled as member of a game that is playing the main role in **multi-agent systems** is derived from the base structure *agent_general*. This base structure is inherited from *class agent_general* which is the **abstract class** having an identifier that is unique in a game. There are five offsprings of the base *class*. The first one is *agent_with_wealth* with two offsprings *agent_with_scattering_interaction*, which is used for all scattering models, and *agent_minority_game* is used by the Minority game simulations without network and its offspring *agent_minority_game-*

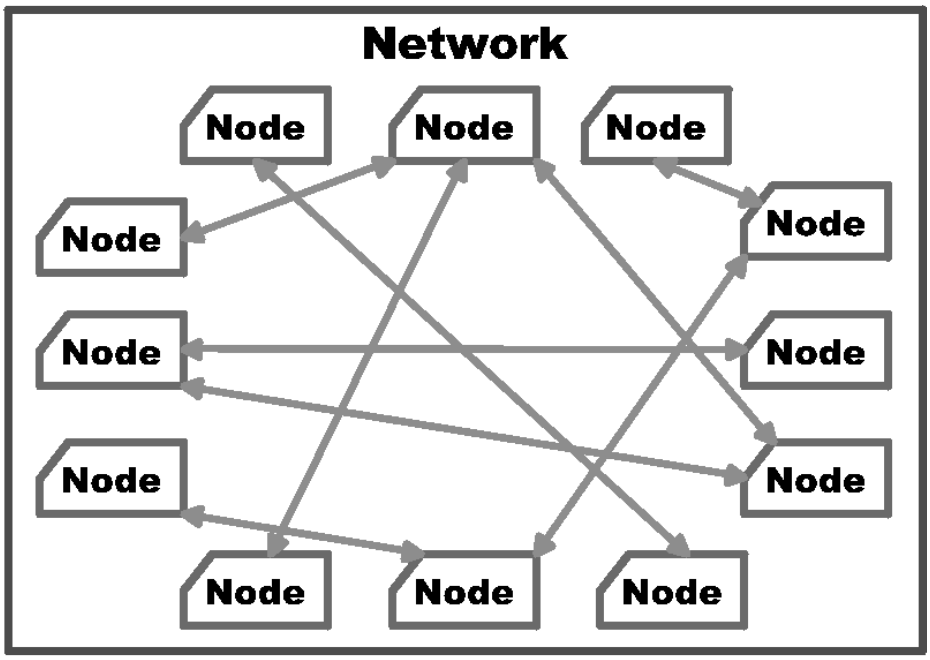

Figure 7.7: Structure of *class* network

_with_link is used by the Minority Game with substrate network. The second offspring is called *agent_general_with_link* with one offspring *agent-_sznajd_model* used by Sznajd model. The third offspring is called *agent-_weighted_network_with_link* with many offsprings used by quantum simulations. The fourth derivative is called *agent_general_with_multiple_links* and the fifth one is called *agent_weighted_network_with_multiple_links,* these allow multiple links from agents to vertices.

A branch of *class agent_general* has an abstract functionality and there are no plans for future enhancements except the case of developing new agents for new simulations.

Game

A computer model of a game in Game theory is discussed here. The *class* must provide the whole functionality that is expected by a model in Game theory. Although, the author tried to write a code as reusable as it was possible, there are more *abstract classes* that could be taken as a base of a general game.

The first developed **abstract class** is derived from *agency* and it is the base of all Minority game based games. In the next generation is *minority_game* which is followed by *minority_game_with_general_network* and next there are two derived of *classes* *minority_game_with_network* and *minority_game_with_oriented_network* and that, finally, are followed by *minority_game_with_network_random_imitation* and *minority_game-_with_oriented_network_random_imitation*.

The second possible way to derive *classes* of a game from offstrings of *network*. An example what has been derived is *scattering_model* and its offsprings *scattering_model_agent_initiated*, *scattering_model_bond_initiated*, *scattering_model_increasing_energy*, *scattering_model_increasing_energy_agent_initiated* and *scattering_model_increasing_energy_bond_initiated*.

The third and the most general **abstract class** which encapsulates multi-agent systems is *general_model*. It has two offsprings but only one of them is related to multi-agent simulations - *agent_based_model*. It has one offspring *network_based_agent_model* which is followed by implementation of quantum games *quantum_simulation*, *quantum_simulation_hexagonal-_lattice*, implementation of Sznajd model *sznajd_model_simulation* and *sznajd_model_with_Ochrombel_simplification_simulation*.

Future multi-agent simulations could be derived from the last mentioned *classes*. Every user of the library could develop his(her) own basis for the simulations. There is one more possibility to develop more advanced simulations of agent-based models on networks which has not been investigated. This is **multiple inheritance** where agent-based simulation *class* and network *class* are used to derive a more complex *class* with 2 parents[20].

Timer

Timers are controlling, recording and measuring tools, see 7.2, which can be plugged in a game which in turn is controlled and measured with the results been stored on a memory media. Control of a game means starting processes which can influence the state of the game. Recording is simple storing interesting variables. Measuring is more advanced measurements of the system using the data storages.

Every timer in the library is derived from an **abstract class** *timer* which is followed by *timer_with_return* and general inheritance tree is followed by *timer_with_return_and_post-processing*. These three timers form a skeleton covered by timers that are specific for different models. The Sznajd model is processed by one timer *timer_sznajd_model*. The Minority game is served by *timer_network_log*, *timer_repeat_domain_imitation_log*, *timer-_repeat_network_log* and *timer_repeat_log_network_log*. The scattering models are supported by *timer_scattering_model_log*, *timer_scattering_model-_wealth_log*, *timer_scattering_normalizer*, *timer_scattering_model_agent_-wealth_log*, *timer_scattering_model_agent_short_time_wealth_log*, and, finally, quantum simulations are recorded by *timer_quantum_simulation*.

The main structure of the inheritance tree of timers for one-process

[20]Similar process was used by scattering models but one one parent is a network with list of edges and the second parent is network with possibility to dye clusters but the difference is in using **multiple virtual inheritance** [82].

(one-processor) simulations is completely done but multi-process (multi-processor) simulation[21] which needs more complex infrastructure which has not been developed yet because of the number of weeks requested for testing.

Data storages

Every simulation program needs to collect and store data what can be visualized. The data storages are constructed as flow-analyzers. This idea is taken from general UNIX idea of programs[22] illustrated by Raymond in [81] . The library has one input, what is served by different functions which in turn can process different kinds of input sources of data and two outputs. The first output, is equivalent to `stdout` and this is used for the output of processed data from the data storage. The second is redirected to the standard error output `stderr` and every error in the storage is immediately shown there.

The **abstract parent** of all data storages is called *data_logger*. This *class* has two offsprings. The first is *mean_data_logger* which provides computing averages and dispersions and the second is *general_histogram_builder* which is an abstract parent of all the histograms. The last one has 3 derivatives. The first is called *average_histogram_builder* and it is responsible for making histograms of the averages of the data. The second one is called *weighted_histogram_builder* that can it allows the addition of weights to the events. The third one is called *histogram_builder* which is a histogram with a fixed box-size, which is not growing under data distribu-

[21]Two biggest personal computer processor producers (Intel, AMD) announced processors with multiple cores on a chip.

[22]General UNIX program have 3 input-output streams that are accessed in *language C* by `stdin`, `stdout`, `stderr`.

Figure 7.8: General scheme of cumulative histogram and its fitting the data.

tion. The last one has three offsprings *log_histogram_builder, log_normal-_histogram_builder* has a fixed domain but different box sizes and *histogram_builder_cumulative* is the base for all histograms (see the figure 7.8) which have a domain undefined at the initial position but the domain resize under incoming data[23] and its followers *log_histogram_builder_cumulative* and *log_quadratic_histogram_builder_cumulative* have different box sizes.

A base girder of future histograms has been laid down and it is robust in so far as it can handle every kind of numeric data and it can be developed as a *class* with appropriate box sizes. Outputs of the *classes* are variable from simple frequencies over probabilities to probability distributions.

[23]When box sizes are chosen very small and data are over several magnitudes then it can be observed **memory-flooding** which depends on memory size. The objects are partially **memory-flooding** proofed - the data are deallocated and no more data are accepted by the object.

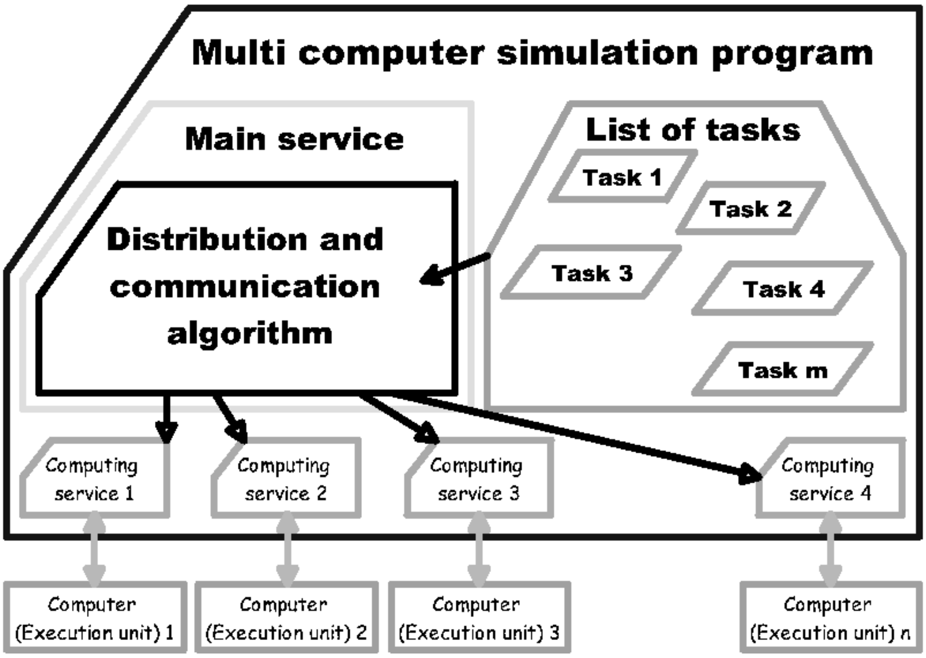

Figure 7.9: General scheme of parallelization of simulation program

Computing supporters

In the last 20 years it can be observed that there has been a concerted endeavor to increase the power of computers using multiprocessors and the clustering of computers.

The *classes*, which would be associated in the section, would allow the exploitation of such types of computers which are based on *threads* or *MPI*. Parallelization of the simulation program would be based on Monte Carlo simulations - a run of simulation = different process and structure of the parallelized program will look as in figure 7.9.

This feature is planned for version 2.0 and it will need a lot of time for writing and, especially, testing, which will be the most time consuming process.

Other tools

The following is a collection of all other functionality which supports functionality of the library in many ways.

Special game structures Functionality that is supporting to build up simulation is associated to the set. The scattering models and the Sznajd model need not have extra functionality. The Minority Games are using *strategy* and its offspring *strategy_nonrandom*. The quantum games are using *classes quantum_state* and *quantum_transformation*. Future simulations will need more extra *classes* which are necessary for functionality.

Directory tools Data is product from the simulation program, which has to be preferably stored in a place which can be a directory in a UNIX system or in a directory and its sub-directories. A couple of functions which set up output directory and functions which prepares output filenames are in `auxiliary_function.h`. There is one option to enhance functionality - dynamical allocation of string in separate data structure.

Memory tools Functionality that allocates, loads and deallocates memory that is used by processing tools. The function are obsolete and they will not be supported in future. The function are declared in `auxiliary-_functions.h`. No more functionality will be developed here.

Processing tools Functionality which is necessary to process out-coming data from simulations. Nowadays, these function are obsolete and will now be supported in future. The functions are implemented in `auxiliary-_functions.h`,`derivation.h` and `statistical_functions.h`. No more functionality is planned.

Mathematical tools Functionality that is related to mathematics. Pack of functions in `math_functions.h`. No more functionality is planned but there is possibility to find similar functions in other libraries and substitute them with a library.

7.2.5 Structure of library

Every program should be decomposable to multi-purpose part and one-purpose part. The multi-purpose part consists of tools which are general and usable in similar simulations. Such tools are packed in `libzarja.a (.so)`[24] and it is accessible via `zarja.h` header. The header is a meta-header which allows access to every random generator, general nodes, general edges, general networks, general agents, general game infrastructure based on *agent_based_model*, general timers, data storages and other tools which were described in previous section.

The one-purpose part differs from simulation to simulation but, in general, it consists of parts which are specific for a simulation. Thus there could be modifications of nodes, edges, networks, games and timers. There could be testing of the tools which would be put inside the multi-purpose library **Zarja**.

7.2.6 General future plans

The author believes that the most interesting plan for the future is the support of multi-computer simulation from the position of user of library. From position of internal structure it must prevent problems with allocation-deallocation of memory due to copying of pointers and use method de-

[24]Suffix of the file depend on type of linking of the library.

scribed by Koenig in [83] and in Clanlib [84] - sharing pointer[25], possessive pointer[26] and copy-on-write pointer.[27] From a mathematical point of view, it would be interesting to implement algorithms from [13], especially the Dijkster algorithm.

7.2.7 Legal notes

The simulation library and its additions are distributed under GPL[28] which allows modification and redistribution the package but the author does not provide any warranty. In general, the author thinks that the GPL contributes to a combination of juristic protection with ability to maximize scientific progress.

7.2.8 Compatibility and release notes

The library is written in *C++* such that it would pass all general rules to be portable to other platforms where GCC[29] is accessible. It was tested on *IA32* architecture because of its spreading. Especially for the architecture, there is supported creation of different optimalization builds[30] which are usable for making the most optimal packages that would be run on different computers with slightly different structure of a processor than the actual computer[31]. So, it exploits the power of the computer to the maximum.

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[25]Sharing pointer is pointer that counts references to the object and deallocation is made when counter is 0.

[26]Possessive pointer is pointer that copies the object on demand of cloning of pointers.

[27]Copy-on-write pointer is pointer that share data between of cloned pointers until request of writing to the object is present.

[28]General Public License

[29]*GNU* C compiler - gcc.gnu.org

[30]f.e. for Pentium 4 - Prescott, Pentium 4, Athlon, i686, ...

[31]It is usable for static linking of the **Zarja** library.

7.3 Conclusions

The simulation library which was developed by the author was introduced component by component. It appears to be a very general and multi-purpose tool which can be very helpful for making structurally difficult computer simulations based on graphs. Plans for the future development are discussed focusing on MPI and thread support for Monte Carlo simulation.

Part IV

Conclusions

Chapter 8

General conclusions

The thesis examines the ideas of Sociophysics and Econophysics which relate to the use of physical models in Sociology and Economics. This idea was initiated by Vilfredo Pareto analyzing the wealth distribution. Breakthroughs in the observation and the formalization of known conflict situations in the last century has facilitated ways to formulate the Game Theory models. The models are mainly stochastic and they possible qualitatively explain human behavior in various aspects: the structure of social networks, opinion formation, the economics.

8.1 Comments of the models

This thesis should provide a basic understanding of processes in modern society using the paradigm of network structure of society in combination with analysis of simple games from Game Theory evolving on it. In particular we focused on Sznajd model of opinion formation, Minority Game and inelastic scattering model of wealth.

Sznajd model is a model of opinion formation in society and it was simulated on wide spectra of graphs including small-world network in [110] and growing Barabasi-Albert network in [120] but author's simulations on fixed Barabasi-Albert network does not follow power law opinion distribution. Thus, the author of the thesis does not think that Sznajd model is simply the general mechanism of opinion formation but it is a good approximation for a community in society. Placing Sznajd model on fully-connected network give certain similarities with measured data [123].

Simulation of Minority Game provide interesting statistical properties of imitation trees. If we let it to be a generalized model of attendance at a bar then the imitation trees are only result. However, the authors of [142], [143] and [144] provide interesting application of Minority Game on a real market and the author of the thesis thinks that imitation trees can have the same structure like word-of-mouth advertisement or the ownership structure of companies.

The last model was inelastic scattering of wealth which was simulated on Watts-Strogatz network and fully-connected graph and both networks allow Pareto law. Authors simulation of the model on Barabasi-Albert network does not lead to Pareto law the same way like simulation on Watts-Strogatz network for high rewiring probability. Thus, it seems that the model can be understand only like metaphor of the real functionality of economy. Moreover, this model can be accepted only like toy model rather test model for policy making. The next comment is pointed to all models of wealth distribution because none of them takes into account existence of formal organizations (corporations, institutions and offices).

The author provide few comments of actual economic situation related to political economy. The comments are in agreement with sociological observations by Keller [4], [5], [6] or Wallerstein [7].

The author thinks that the thesis provides the necessary background of real networks, their models, models of processes in society based on physical models and its simulations. Finally, the author provides a simulation library **Zarja** with documentation which can be used by other authors. At this point, the models are not able to support practical predictions or conclusions to prevent problems of the 21th century but they are impressive metaphors of the important processes in the modern society.

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