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INVARIANTS OF LIE ALGEBRAS

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This thesis is the result of my own work, except where explicit reference is made to the work of others and has not been submitted for another qualification to this or any other university.

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Introduction

Finite-dimensional complex Lie algebras form extremely useful and frequent part of Lie theory in physics and elsewhere. With an increasing amount of theory and applications concerning Lie algebras of various dimensions, it is becoming necessary to ascertain applicable tools for handling them. Their miscellaneous characteristics constitute such tools and have also found applications: *Casimir operators* [2], *derived, lower central and upper central sequences*, *Lie algebra of derivations, radical, nilradical, ideals, subalgebras* [21, 32] and recently *megaideals* [31]. These characteristics are central when considering possible affinities among Lie algebras.

Physically motivated relations between two Lie algebras, namely *contractions* and *deformations*, were extensively studied for instance in [15, 25]. When investigating these kinds of relations in dimensions higher than five, one can encounter insurmountable difficulties.

Firstly, aside the *semisimple* ones, Lie algebras are completely classified only up to dimension 5 and the *nilpotent* ones up to dimension 6. In higher dimensions, lists of only special types such as *rigid* Lie algebras [17] or Lie algebras with fixed structure of nilradical are determined [33]. For a detailed survey of classification results in lower dimensions see the very recent paper [31] and references therein.

Secondly, known invariant characteristics of Lie algebras are, in some cases, insufficient for their classification. After performing the standard identification procedure in [32], one ends with the set of classical invariants [29]. Even though there has been progress in extending this set of classical invariants, the results still turn out to be insufficient, especially for the description of nilpotent Lie algebras. Alongside the classification of orbit closures of four-dimensional complex Lie algebras [11], so called C_{pq} invariants were introduced; in [3] the invariants χ_i were used for the identification of four-dimensional complex Lie algebras. In [9, 10], the dimensions of cohomology spaces with respect to the adjoint and trivial representations were used as invariants. However, none of the above invariants is able to resolve a nilpotent parametric continuum of Lie algebras. These nilpotent continua

frequently appear as results of the graded contraction procedure.

The graded contraction procedure, originally introduced in [25], was later extended to graded contractions of the representations of Lie algebras [26] and to the Jordan algebras [22]. In [III, 1] all graded contractions of $\mathfrak{sl}(3, \mathbb{C})$ corresponding to the Pauli grading and toroidal grading, respectively, were found. Due to the difficulties mentioned above, the thorough study of resulting parametric continua was omitted in these papers. Later it became evident that the ultimate result – graded contractions corresponding to all four gradings of $\mathfrak{sl}(3, \mathbb{C})$ – involves classifying hundreds of parametric Lie algebras and is, without more appropriate invariants, out of reach. Of course, when one has two identical sets of invariants for two given algebras, one can always try to find explicitly an isomorphism in a given basis – that is, find a regular solution of a very large system of quadratic equations. If the search for an explicit regular solution fails, one has to exclude the existence of any regular solution – this, in some cases, can be done by hand. There has been progress in developing algorithms for these direct calculations: employing modern computational methods and the theory of Gröbner bases, the classification of the three and four-dimensional solvable Lie algebras has been obtained [18]. Application of these types of direct computations seems, however, not surmountable for hundreds of parametric Lie algebras in higher dimensions.

Thus, the *goals* of this thesis are the following:

- add new objects to the existing set of invariants of Lie algebras
- formulate the properties of these invariants and – in view of possible alternative classifications – investigate their behaviour on known lower-dimensional Lie algebras
- demonstrate that these invariants are – in view of their application on graded contractions of $\mathfrak{sl}(3, \mathbb{C})$ – also effective in higher dimensions
- formulate a necessary contraction criterion involving these invariants and apply it to lower-dimensional cases
- investigate possible application of these invariant characteristics to Jordan algebras

One can see from the references that our investigation belongs to a very lively domain of Lie theory. Only Chapter 1. is completely of *review character*. Chapters 2, 3, 4 and Appendices are devoted to thorough description of *new results*.

Original results of this thesis are

- contained in [I]—[IV]: Chapter 2, Section 4.2
- unpublished: Chapter 3, Section 4.1, Appendices

The *most significant original results* are the following:

- Chapter 2: Corollaries 2.1.2, 2.1.5, 2.2.6, Theorems 2.2.1, 2.2.4 and 2.4.2
- Chapter 3: Corollaries 3.2.2, 3.2.3, Theorems 3.1.3 and 3.2.9
- Chapter 4: Theorems 4.1.2, 4.1.7, 4.1.9 and 4.2.1
- Appendix A: classification of (α, β, γ) -derivations of two and three-dimensional Lie algebras
- Appendix B: the invariant functions allowing the classification of four-dimensional Lie algebras

In Chapter 1, the definitions and theorems used in this work are summarized. We define invariant characteristics of algebras and their mutual independence. We review the concept of a Lie algebra and their cohomology. We also state basic facts about linear groups and Jordan algebras.

In Chapter 2, the concept of the derivation of a Lie algebra is generalized; (α, β, γ) -derivations are introduced and their pertinent properties shown. All possible intersections of spaces containing these derivations are investigated. Examples of spaces of (α, β, γ) -derivations for low-dimensional Lie algebras are presented. In special cases, the spaces of (α, β, γ) -derivations form Lie or Jordan operator algebras. These algebras are investigated and the corresponding Lie groups constructed. Invariant functions ψ, ψ^0 are defined and their values estimated. The invariant function ψ is used as the classification tool of three-dimensional Lie algebras.

In Chapter 3, the concept of cocycles of Chevalley cohomology is generalized; κ -twisted cocycles are introduced and shown that for two-dimensional twisted cocycles, analogous properties to the properties of (α, β, γ) -derivations hold. Examples of selected spaces of twisted cocycles are presented. Two invariant functions φ and φ^0 are defined and their behaviour on low-dimensional Lie algebras demonstrated. The invariant functions ψ and φ are used to classify all four-dimensional Lie algebras. New algorithm for the identification of a four-dimensional Lie algebra is also formulated.

In Chapter 4, possible application of the invariant functions ψ, φ, φ^0 to contractions is considered. Necessary criterion for existence of a continuous contraction is formulated. The invariant function ψ is used to classify continuous contractions among three-dimensional Lie algebras. This function is also employed to the classification of two-dimensional Jordan algebras and their continuous contractions. The invariant functions are used to distinguish among results of graded contraction procedure. Application of the invariant functions on nilpotent parametric continua of Lie algebras resulting from contractions of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$ is demonstrated.

In Conclusion, we shortly review other generalizations of derivations, make notes on a computation of (α, β, γ) -derivations and twisted cocycles. We summarize achieved results and make comments concerning further applications.

In Appendix A, explicit matrices of (α, β, γ) -derivations for two and three-dimensional Lie algebras and for two-dimensional Jordan algebras are completely classified.

In Appendix B, the tables of the invariant functions ψ, φ, φ^0 for two, three and four-dimensional Lie algebras and for two-dimensional Jordan algebras are listed.

Chapter 1

Invariants of Lie Algebras

1.1 Invariant Characteristics of Algebras

In order to unify the notation and definitions, we amass in this section basic notions concerning linear algebras. Except for the definition of independent invariant, the content of this section may be found e. g. in [16, 21].

We call a vector space V of finite dimension n over the field of complex numbers \mathbb{C} together with a bilinear map $V \times V \ni (a, b) \mapsto a \cdot b \in V$ a (complex) **algebra** $\mathcal{A} = (V, \cdot)$. We call the map \cdot a **multiplication** of \mathcal{A} . An algebra \mathcal{A} is **associative** if the rule $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds for all $a, b, c \in \mathcal{A}$. Let $\mathcal{X} = (x_1, \dots, x_n)$ be some basis of \mathcal{A} . Then the numbers $c_{ij}^k \in \mathbb{C}$ defined by

$$x_i \cdot x_j = \sum_{k=1}^n c_{ij}^k x_k \quad (1.1)$$

are called **structural constants** with respect to the basis \mathcal{X} . A vector subspace B of \mathcal{A} is called a **subalgebra** of \mathcal{A} if $a \cdot b \in B$ for all $a, b \in B$. For arbitrary subsets B, C of \mathcal{A} the symbol $B \cdot C$ denotes the linear span of all products of elements $b \cdot c$, where $b \in B$ and $c \in C$. A subalgebra B is called an **ideal** in \mathcal{A} if $\mathcal{A} \cdot B \subset B$ and $B \cdot \mathcal{A} \subset B$.

Given an ideal B in an algebra \mathcal{A} then the factor space

$$\mathcal{A}/B = \{[a] = a + B \mid a \in \mathcal{A}\}$$

with a well-defined multiplication $[a] \cdot [b] = [a \cdot b]$, $\forall a, b \in \mathcal{A}$ is called the **factor algebra** and denoted \mathcal{A}/B .

Suppose we have two algebras $\mathcal{A}, \tilde{\mathcal{A}}$ over \mathbb{C} with multiplications \cdot and $*$, respectively. Then a linear map $f : \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ is called a **homomorphism** if the relation

$$f(a \cdot b) = f(a) * f(b) \quad (1.2)$$

holds for all $a, b \in \mathcal{A}$. The kernel $\ker f$ is an ideal in \mathcal{A} . If $\ker f = 0$ then f is called an **isomorphism** and algebras the $\mathcal{A}, \tilde{\mathcal{A}}$ are called **isomorphic**, $\mathcal{A} \cong \tilde{\mathcal{A}}$. Let us denote the group of all regular linear maps on an arbitrary vector space V by $GL(V)$; for an algebra $\mathcal{A} = (V, \cdot)$ we define the symbol $GL(\mathcal{A})$ by $GL(\mathcal{A}) = GL(V)$. An isomorphism $f : \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ is called an **automorphism** of an algebra $\mathcal{A} = (V, \cdot)$; the set of all automorphisms forms a multiplicative group $\text{Aut } \mathcal{A} \subset GL(\mathcal{A})$, i. e.

$$\text{Aut } \mathcal{A} = \{f \in GL(\mathcal{A}) \mid f(a \cdot b) = f(a) \cdot f(b) \quad \forall a, b \in \mathcal{A}\}. \quad (1.3)$$

We denote by $\text{End } V$ a vector space of all linear operators on the vector space V ; if we have an algebra $\mathcal{A} = (V, \cdot)$, we define the symbol $\text{End } \mathcal{A}$ by $\text{End } \mathcal{A} = \text{End } V$. Considering the composition of linear operators as a multiplication, $\text{End } V$ becomes an associative algebra. If $f : \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ is an isomorphism of complex algebras \mathcal{A} and $\tilde{\mathcal{A}}$, then the mapping $\varrho : \text{End } \mathcal{A} \rightarrow \text{End } \tilde{\mathcal{A}}$, defined for all $D \in \text{End } \mathcal{A}$ by

$$\varrho(D) = f D f^{-1} \quad (1.4)$$

is an isomorphism of the associative algebras $\text{End } \mathcal{A}$ and $\text{End } \tilde{\mathcal{A}}$, i. e. $\text{End } \mathcal{A} \cong \text{End } \tilde{\mathcal{A}}$.

A map $D \in \text{End } \mathcal{A}$ which satisfies $D(a \cdot b) = (Da) \cdot b + a \cdot (Db)$ for all $a, b \in \mathcal{A}$ is called a **derivation** of \mathcal{A} ; we denote the set of all derivations by $\text{der } \mathcal{A}$, i. e.

$$\text{der } \mathcal{A} = \{D \in \text{End } \mathcal{A} \mid D(a \cdot b) = (Da) \cdot b + a \cdot (Db) \quad \forall a, b \in \mathcal{A}\}. \quad (1.5)$$

One can easily prove that $\text{der } \mathcal{A}$ forms a linear subspace of the vector space $\text{End } \mathcal{A}$.

The relation of isomorphism \cong between two algebras is an equivalence relation, i. e. it is symmetric, reflexive and transitive. Thus, the set of all algebras is decomposed into **isomorphism classes** – cosets of this equivalence; we denote such coset containing an algebra \mathcal{A} by $[\mathcal{A}]$. Then, indeed, $\mathcal{B} \in [\mathcal{A}]$ holds if and only if $\mathcal{B} \cong \mathcal{A}$. Suppose we have a non-empty set M and some subset Θ of the set of all isomorphism classes of algebras. We call a map

$$\Phi : \Theta \rightarrow M \quad (1.6)$$

an **invariant characteristic** of Θ or shortly an **invariant**. In other words, the mapping Φ assigns to each coset $[\mathcal{A}] \in \Theta$, which contains all mutually isomorphic algebras, some element in M .

For example, we may consider Θ equal to the set of all isomorphism classes of algebras, set $M = \mathbb{N}_0$ and for any $[\mathcal{A}] \in \Theta$ we may define $\Phi[\mathcal{A}] = \dim \mathcal{A} = n$. It is clear that this

mapping Φ is well-defined and indeed has the same value for all isomorphic algebras. We sometimes say that the dimension of \mathcal{A} is an invariant characteristic of algebras. We call such an invariant, for which $M = \mathbb{N}_0$, a **numerical invariant**. As another example of a numerical invariant for an arbitrary algebra may serve $\Phi_{\text{der}}[\mathcal{A}] = \dim \text{der } \mathcal{A}$; indeed, for $\mathcal{A}, \tilde{\mathcal{A}} \in [\mathcal{A}]$ one has

$$\varrho(\text{der } \mathcal{A}) = \text{der } \tilde{\mathcal{A}}, \quad (1.7)$$

i. e. $\dim \text{der } \mathcal{A} = \dim \text{der } \tilde{\mathcal{A}}$. Thus, the mapping Φ_{der} is well-defined.

As a number of known invariants for any given type of algebra grows, there arises natural claim to reflect their independence. We refine the notion of independence in the following way. Suppose we have a subset Θ of the set of all isomorphism classes of algebras and a set of invariants Ω of Θ . Then we call an invariant Φ of Θ **independent** on the set of invariants Ω if there exist two cosets $[\mathcal{A}], [\mathcal{B}] \in \Theta$ such that

$$\begin{aligned} 1. \quad & \Psi[\mathcal{A}] = \Psi[\mathcal{B}], \quad \forall \Psi \in \Omega \\ 2. \quad & \Phi[\mathcal{A}] \neq \Phi[\mathcal{B}]. \end{aligned} \quad (1.8)$$

Note that this definition does not, in general, exclude some possible relation among invariants in Ω and Φ . However, it is pertinent that the invariant Φ does not depend *only* on those in Ω . Since the main aim of the notion of invariant characteristic is to distinguish among different cosets of algebras, the independent invariant Φ thus distinguishes cosets $[\mathcal{A}]$ and $[\mathcal{B}]$.

1.2 Lie Algebras

Basic facts included in this section, concerning Lie algebras, their representations and ideals may be found for instance in [16, 21]. Suppose we have a (complex) algebra \mathcal{L} with the multiplication $[\ , \]$ which for all $x, y, z \in \mathcal{L}$ satisfies

- (1) $[x, x] = 0$ (anti-commutativity)
- (2) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi's identity).

Such an algebra \mathcal{L} is then called a (complex) **Lie algebra**. In terms of structure constants (1.1) anti-commutativity and Jacobi's identity may be written as

$$(1') \quad c_{ij}^m + c_{ji}^m = 0$$

$$(2') \sum_{l=1}^n (c_{jk}^l c_{il}^m + c_{ij}^l c_{kl}^m + c_{ki}^l c_{jl}^m) = 0, \forall i, j, k, m \in \{1, \dots, n\}.$$

Suppose V is an arbitrary vector space. Then we may introduce a new multiplication on the associative algebra $\text{End } V$. For two linear operators $X, Y \in \text{End } V$ we put

$$[X, Y] = XY - YX. \quad (1.9)$$

Then we indeed obtain a Lie algebra which we denote by $\text{gl } V$. For an algebra $\mathcal{A} = (V, \cdot)$ we define $\text{gl } \mathcal{A} = \text{gl } V$. Then we have:

Proposition 1.2.1. The set of all derivations $\text{der } \mathcal{A}$ of an algebra \mathcal{A} is a Lie subalgebra of $\text{gl } \mathcal{A}$.

If $f : \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ is an isomorphism of complex algebras \mathcal{A} and $\tilde{\mathcal{A}}$, then the mapping $\varrho : \text{gl } \mathcal{A} \rightarrow \text{gl } \tilde{\mathcal{A}}$, defined by (1.4) is an isomorphism of the Lie algebras $\text{gl } \mathcal{A}$ and $\text{gl } \tilde{\mathcal{A}}$, i. e. $\text{gl } \mathcal{A} \cong \text{gl } \tilde{\mathcal{A}}$. From this fact and from Proposition 1.2.1 and (1.7) one also obtains that

$$\text{der } \mathcal{A} \cong \text{der } \tilde{\mathcal{A}}. \quad (1.10)$$

holds.

If we choose some basis in \mathcal{A} then to each operator from $\text{gl } \mathcal{A}$ is assigned a matrix; the space of these matrices is denoted by $\text{gl}(n, \mathbb{C})$ and forms also a Lie algebra with respect to the matrix multiplication $[\cdot, \cdot]$. An important matrix algebra of traceless matrices $\text{sl}(n, \mathbb{C})$ is defined by the relation

$$\text{sl}(n, \mathbb{C}) = \{S \in \text{gl}(n, \mathbb{C}) \mid \text{tr } S = 0\}. \quad (1.11)$$

Having two Lie algebras $\mathcal{L}_1 = (V_1, [\cdot, \cdot]_1)$, $\mathcal{L}_2 = (V_2, [\cdot, \cdot]_2)$ one may define on the direct sum of the vector spaces

$$V_1 \oplus V_2 = \{(x, y) \mid x \in V_1, y \in V_2\}$$

a Lie multiplication

$$[(x_1, y_1), (x_2, y_2)]_{\oplus} = ([x_1, x_2]_1, [y_1, y_2]_2)$$

and obtain a Lie algebra $\mathcal{L}_1 \oplus \mathcal{L}_2 = (V_1 \oplus V_2, [\cdot, \cdot]_{\oplus})$ called the **direct sum** of \mathcal{L}_1 and \mathcal{L}_2 .

Let V be a vector space over \mathbb{C} . We call a homomorphism

$$f : \mathcal{L} \rightarrow \text{gl } V$$

a **representation** of a Lie algebra \mathcal{L} over \mathbb{C} on the vector spaces V and we denote it by (V, f) . A map $\text{ad}_{\mathcal{L}} : \mathcal{L} \rightarrow \text{gl } \mathcal{L}$ defined for all $x, y \in \mathcal{L}$ by the relation

$$(\text{ad}_{\mathcal{L}} x)y = [x, y]$$

is a representation $(\text{ad}_{\mathcal{L}}, \mathcal{L})$ and is called the **adjoint representation**.

Proposition 1.2.2. For any Lie algebra \mathcal{L} the set $\text{ad } \mathcal{L} = \{\text{ad}_{\mathcal{L}} x \mid x \in \mathcal{L}\}$ is an ideal in $\text{der } \mathcal{L}$.

Having (V, f) a representation of \mathcal{L} we define a representation on the dual space V^* . For $s \in \text{gl } V$, $\lambda \in V^*$ and $v \in V$ consider the mapping $s^t \in \text{gl } V^*$ defined by

$$(s^t \lambda)v = \lambda(sv).$$

If we put $f^* = -f^t$ then (V^*, f^*) is a representation called the **dual representation** of (V, f) .

We point out some important ideals of a Lie algebra \mathcal{L} . We denote by $C(\mathcal{L})$ a **center** of \mathcal{L} defined by

$$C(\mathcal{L}) = \{x \in \mathcal{L} \mid [x, y] = 0, \forall y \in \mathcal{L}\} \quad (1.12)$$

and a **derived algebra** \mathcal{L}^2 of \mathcal{L}

$$\mathcal{L}^2 = [\mathcal{L}, \mathcal{L}]. \quad (1.13)$$

A **centralizer of the adjoint representation** $C_{\text{ad}}(\mathcal{L}) \subset \text{gl } \mathcal{L}$ is defined as follows:

$$C_{\text{ad}}(\mathcal{L}) = \{A \in \text{gl } \mathcal{L} \mid [A, \text{ad}_{\mathcal{L}}(x)] = 0, \forall x \in \mathcal{L}\}. \quad (1.14)$$

Next, we introduce sequences of ideals. Sequence of ideals $D^0(\mathcal{L}) \supset D^1(\mathcal{L}) \supset \dots$ defined by

$$D^0(\mathcal{L}) = \mathcal{L}, \quad D^{k+1}(\mathcal{L}) = [D^k(\mathcal{L}), D^k(\mathcal{L})], \quad k \in \mathbb{N}_0 \quad (1.15)$$

is called a **derived sequence** of \mathcal{L} . Sequence of ideals $\mathcal{L}^1 \supset \mathcal{L}^2 \supset \dots$ defined by

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \quad k \in \mathbb{N} \quad (1.16)$$

is called a **descending central sequence**. Sequence of ideals $C^0(\mathcal{L}) \subset C^1(\mathcal{L}) \dots$ defined by

$$C^0(\mathcal{L}) = 0, \quad C^{k+1}(\mathcal{L})/C^k(\mathcal{L}) = C(\mathcal{L}/C^k(\mathcal{L})), \quad k \in \mathbb{N}_0 \quad (1.17)$$

is called an **ascending central sequence**. We define three sequences of numerical invariants $d_k, c_k, l_k, k = 0, 1, \dots$ by the relations

$$d_k(\mathcal{L}) = \dim D^k(\mathcal{L}) \quad (1.18)$$

$$l_k(\mathcal{L}) = \dim \mathcal{L}^{k+1} \quad (1.19)$$

$$c_k(\mathcal{L}) = \dim C^{k+1}(\mathcal{L}). \quad (1.20)$$

If, for some k and a Lie algebra \mathcal{L} , $D^k(\mathcal{L}) = 0$ holds then \mathcal{L} is called **solvable**; if $\mathcal{L}^k = 0$ then it is called **nilpotent**. If \mathcal{L} contains no solvable ideal then it is called **semisimple**; if it contains only trivial ideals \mathcal{L} and $\{0\}$ and $\dim(\mathcal{L}) > 1$ then \mathcal{L} is called **simple**.

Theorem 1.2.3 (Engel). Let \mathcal{L} be a Lie algebra. Then \mathcal{L} is nilpotent if and only if $\text{ad}_{\mathcal{L}} x$ is nilpotent for every $x \in \mathcal{L}$.

It is well known that the sum of two nilpotent ideals in a Lie algebra is again a nilpotent ideal. The sum of all nilpotent ideals is a maximal nilpotent ideal called a **nilradical**. It is also true that the sum of two solvable ideals is a solvable ideal. The sum of all solvable ideals is a maximal solvable ideal called a **radical**. It is clear that if radical is zero then such a Lie algebra is semisimple.

The element F of the universal enveloping algebra [16, 21] of \mathcal{L} which satisfies

$$xF - Fx = 0, \quad \forall x \in \mathcal{L}, \quad (1.21)$$

is called a **Casimir operator** [2, 29]. These operators can be calculated in the following way. We take the representation of the elements of the basis (e_1, \dots, e_n) of \mathcal{L} by vector fields

$$e_i \rightarrow \hat{x}_i = \sum_{j,k=1}^n c_{ij}^k x_k \frac{\partial}{\partial x_j}. \quad (1.22)$$

These vector fields act on the space of continuously differentiable functions $F(x_1, \dots, x_n)$. We call a function F **formal invariant** of \mathcal{L} if it is a solution of

$$\hat{x}_i F = 0, \quad i \in \{1, \dots, n\}.$$

The number of algebraically independent formal invariants is

$$\tau(\mathcal{L}) = \dim \mathcal{L} - r(\mathcal{L}), \quad (1.23)$$

where $r(\mathcal{L})$ is the rank of the antisymmetric matrix $M_{\mathcal{L}}$ and $(M_{\mathcal{L}})_{ij} = \sum_k c_{ij}^k e_k$:

$$r(\mathcal{L}) = \sup_{(e_1, \dots, e_n)} \text{rank}(M_{\mathcal{L}}).$$

The map τ defined via (1.23) forms a numerical invariant. In [2], a procedure for obtaining Casimir invariant from a polynomial formal invariant $F(x_1, \dots, x_n)$ is formulated.

Let $p, q \in \mathbb{N}$ be fixed numbers. Suppose there exist $u, v \in \mathcal{L}$ such that $\text{tr}(\text{ad}_{\mathcal{L}} u)^p \neq 0$, $\text{tr}(\text{ad}_{\mathcal{L}} v)^q \neq 0$ and $\text{tr}[(\text{ad}_{\mathcal{L}} u)^p (\text{ad}_{\mathcal{L}} v)^q] \neq 0$. If there exists $C_{pq} \in \mathbb{C}$ such that for all $x, y \in \mathcal{L}$ the equality

$$\text{tr}(\text{ad}_{\mathcal{L}} x)^p \text{tr}(\text{ad}_{\mathcal{L}} y)^q = C_{pq} \text{tr}[(\text{ad}_{\mathcal{L}} x)^p (\text{ad}_{\mathcal{L}} y)^q] \quad (1.24)$$

holds, then C_{pq} is called the C_{pq} -**invariant** of \mathcal{L} [11].

In [3] were introduced the functions:

$$\begin{aligned} p_{111}(x) &= -\operatorname{tr} \operatorname{ad}_{\mathcal{L}} x \\ p_{222}(x) &= \frac{1}{2} ((\operatorname{tr} \operatorname{ad}_{\mathcal{L}} x)^2 - \operatorname{tr}(\operatorname{ad}_{\mathcal{L}} x)^2) \\ p_{333}(x) &= -\frac{1}{6} ((\operatorname{tr} \operatorname{ad}_{\mathcal{L}} x)^3 - 3 \operatorname{tr} \operatorname{ad}_{\mathcal{L}} x \operatorname{tr}(\operatorname{ad}_{\mathcal{L}} x)^2 + 2 \operatorname{tr}(\operatorname{ad}_{\mathcal{L}} x)^3) \end{aligned}$$

If there exists $u \in \mathcal{L}$ such that $p_{222}(u) \neq 0$, $p_{111}^2(u) \neq 0$ and exists $\chi_1 \in \mathbb{C}$ such that for all $x \in \mathcal{L}$ it holds:

$$p_{222}(x) = \chi_1 p_{111}^2(x) \quad (1.25)$$

then we have the **invariant** $\chi_1(\mathcal{L})$. Similarly are defined invariants $\chi_2(\mathcal{L})$ and $\chi_3(\mathcal{L})$, i. e. by relations:

$$p_{333}(x) = \chi_2 p_{111}^3(x), \quad p_{333}^2(x) = \chi_3 p_{222}^3(x). \quad (1.26)$$

1.3 Chevalley Cohomology of Lie Algebras

The content of this section may be found for instance in [16]. Let V be a vector space over \mathbb{C} and let (V, f) be a representation of \mathcal{L} . We call a q -linear map $c : \underbrace{\mathcal{L} \times \mathcal{L} \times \dots \times \mathcal{L}}_{q\text{-times}} \rightarrow V$ a V -**cochain** of dimension q , if for all pairs of indices i, j , ($1 \leq i < j \leq q$) the relation

$$c(x_1, \dots, \underset{i}{x_i}, \dots, \underset{j}{x_j}, \dots, x_q) + c(x_1, \dots, \underset{i}{x_j}, \dots, \underset{j}{x_i}, \dots, x_q) = 0$$

holds. We denote by $C^q(\mathcal{L}, V)$ the vector space of all V -cochains of dimension q for $q \in \mathbb{N}$ and $C^0(\mathcal{L}, V) = V$. We define a map $d : C^q(\mathcal{L}, V) \rightarrow C^{q+1}(\mathcal{L}, V)$ for $q = 0, 1, 2, \dots$ by

$$\begin{aligned} dc(x) &= f(x)c \quad c \in C^0(\mathcal{L}, V) \quad (1.27) \\ dc(x_1, \dots, x_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} f(x_i) c(x_1, \dots, \hat{x}_i, \dots, x_{q+1}) + \\ &\quad + \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \end{aligned}$$

where the symbol \hat{x}_i means that the term x_i is omitted. We summarize the crucial results concerning the map d .

Theorem 1.3.1. For the map $d : C^q(\mathcal{L}, V) \rightarrow C^{q+1}(\mathcal{L}, V)$, defined by (1.27), it holds:

$$dd = 0. \quad (1.28)$$

Such $z \in C^q(\mathcal{L}, V)$ for which $dz = 0$ holds is called **cocycle** of dimension q corresponding to f ; the set of all cocycles of dimension q corresponding to f is denoted by $Z^q(\mathcal{L}, f)$. An element $w \in C^q(\mathcal{L}, V)$ for which such $c \in C^{q-1}(\mathcal{L}, V)$ exists that $dc = w$ is called **coboundary**; the set of all coboundaries of dimension q is denoted by $B^q(\mathcal{L}, f)$, i. e. $B^q(\mathcal{L}, f) = dC^{q-1}(\mathcal{L}, V)$. The spaces $B^q(\mathcal{L}, f)$ and $Z^q(\mathcal{L}, f)$ are vector subspaces of $C^q(\mathcal{L}, V)$ and from (1.28) we have $B^q(\mathcal{L}, f) \subset Z^q(\mathcal{L}, f)$. The factor space $Z^q(\mathcal{L}, f)/B^q(\mathcal{L}, f) = H^q(\mathcal{L}, f)$ is then called a **cohomology space** of dimension q of \mathcal{L} with respect to the representation (V, f) . Directly from the definition, one obtains the following proposition.

Proposition 1.3.2.

$$Z^1(\mathcal{L}, \text{ad}_{\mathcal{L}}) = \text{der } \mathcal{L}, \quad B^1(\mathcal{L}, \text{ad}_{\mathcal{L}}) = \text{ad } \mathcal{L}.$$

1.4 Linear Groups

The content of this section may be found for instance in [16]. We denote the set of $n \times n$ regular matrices by $GL(n, \mathbb{C})$. A closed set which is a subgroup of $GL(n, \mathbb{C})$ is called a **linear group**. If a linear group is \mathbb{C} -holomorphic submanifold of $GL(n, \mathbb{C})$ then we call it **complex**. In general, a group G is called a **complex Lie group** if G is a \mathbb{C} -holomorphic manifold and the map

$$G \times G \ni (a, b) \mapsto ab^{-1} \in G$$

is \mathbb{C} -holomorphic. A complex linear group forms a complex Lie group. The exponential map $\exp : \mathfrak{gl}(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$ for $A \in \mathfrak{gl}(n, \mathbb{C})$ has the form

$$\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

It is well known that if G is a linear group then the set

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \exp(\mathbb{R}X) \subset G\} \tag{1.29}$$

forms a Lie algebra over \mathbb{R} and the following propositions hold:

Proposition 1.4.1. Let G be a linear group in $GL(n, \mathbb{C})$ and \mathfrak{g} its Lie algebra. Then G is complex if and only if \mathfrak{g} is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ over \mathbb{C} .

Proposition 1.4.2. Let G_1 and G_2 be linear groups in $GL(n, \mathbb{C})$ and let \mathfrak{g}_1 and \mathfrak{g}_2 be their Lie algebras. Then the Lie algebra of $G_1 \cap G_2$ is $\mathfrak{g}_1 \cap \mathfrak{g}_2$.

A subgroup G of $GL(n, \mathbb{C})$ is called an **algebraic group** if there exists a set of polynomials $P \subset \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}]$ such that

$$G = \{(a_{ij}) \in GL(n, \mathbb{C}) \mid p(a_{11}, a_{12}, \dots, a_{nn}) = 0 \quad \forall p \in P\}$$

An algebraic group is a linear group. Moreover,

Proposition 1.4.3. An algebraic group in $GL(n, \mathbb{C})$ is a complex linear group.

A subgroup of $GL(\mathcal{A})$, where $\mathcal{A} = (V, \cdot)$, is called an **algebraic group**, if it is represented by an algebraic group in $GL(n, \mathbb{C})$ with respect to some basis of V .

Theorem 1.4.4. Let \mathcal{A} be an algebra over \mathbb{C} . Then the automorphism group $\text{Aut } \mathcal{A}$ is an algebraic group in $GL(\mathcal{A})$ and the Lie algebra of $\text{Aut } \mathcal{A}$ is $\text{der } \mathcal{A}$.

1.5 Jordan Algebras

Basic facts about Jordan algebras included in this section may be found in [7, 14, 23]. Suppose we have a (complex) algebra \mathcal{J} with multiplication \circ which satisfies for all $x, y \in \mathcal{J}$

- (1) $x \circ y = y \circ x$ (commutativity)
- (2) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ (Jordan's identity)

where $x^2 = x \circ x$. Such an algebra \mathcal{J} is called a (complex) **Jordan algebra**. In terms of structure constants (1.1) commutativity and Jordan's identity may be written as

- (1') $c_{ij}^m - c_{ji}^m = 0$
- (2') $\sum_{h,l=1}^n (c_{ik}^h c_{mj}^l c_{hl}^r - c_{ik}^h c_{hj}^l c_{lm}^r + c_{im}^h c_{kj}^l c_{hl}^r - c_{im}^h c_{hj}^l c_{lk}^r + c_{mk}^h c_{ij}^l c_{hl}^r - c_{mk}^h c_{hj}^l c_{li}^r) = 0,$
 $\forall i, j, k, m, r \in \{1, \dots, n\}.$

For an arbitrary vector space V we may introduce a new multiplication on the associative algebra $\text{End } V$. For two linear operators $X, Y \in \text{End } V$ we put

$$X \circ Y = \frac{1}{2}(XY + YX). \tag{1.30}$$

In this way we obtain a Jordan algebra which we denote by $\text{jor } V$.

Note that we have defined two different products on $\text{End } V$ (formulas (1.9) and (1.30)). It was pointed out in [4] that these two products together with the original associative

composition of linear mappings exhaust all products of the type $\lambda XY + \mu YX$ on $\text{End } V$. We refine this statement in the following way. We say that a subspace $U \subset \text{End } V$ is (λ, μ) -**closed** if there exist $\lambda, \mu \in \mathbb{C}$, not both zero, such that for all $X, Y \in U$

$$\lambda XY + \mu YX \in U$$

holds. Then one can easily prove the following result [4]:

Proposition 1.5.1. Let $\lambda, \mu \in \mathbb{C}$ not both zero and U is (λ, μ) -closed set of $\text{End } V$. Then U is some of the following:

- (a) an associative subalgebra of $\text{End } V$
- (b) a Lie subalgebra of $\text{gl } V$
- (c) a Jordan subalgebra of $\text{jor } V$.

Chapter 2

Generalized Derivations

2.1 (α, β, γ) -derivations

We defined a derivation of an arbitrary algebra \mathcal{A} as a linear operator $D \in \text{End } \mathcal{L}$ satisfying relation (1.5). For Lie algebras, several non-equivalent ways generalizing this definition have recently been studied [8, 19, 24]. In this chapter we bring forward another type of generalization introduced in [I, II, IV].

Let $\mathcal{A} = (V, \cdot)$ be an arbitrary algebra. We call a linear operator $D \in \text{End } \mathcal{A}$ an (α, β, γ) -**derivation** of \mathcal{A} if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that for all $x, y \in \mathcal{A}$ the following relation is satisfied

$$\alpha D(x \cdot y) = \beta(Dx) \cdot y + \gamma x \cdot (Dy). \quad (2.1)$$

For given $\alpha, \beta, \gamma \in \mathbb{C}$ we denote the set of all (α, β, γ) -derivations as $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}$, i. e.

$$\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A} = \{D \in \text{End } \mathcal{A} \mid \alpha D(x \cdot y) = \beta(Dx) \cdot y + \gamma x \cdot (Dy), \quad \forall x, y \in \mathcal{A}\}. \quad (2.2)$$

It is clear that $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}$ is a linear subspace of $\text{End } \mathcal{A}$. The advantage of such a generalization of derivations can be seen from the following crucial results.

Theorem 2.1.1. Let $f : \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ be an isomorphism of complex algebras \mathcal{A} and $\tilde{\mathcal{A}}$. Then the mapping $\varrho : \text{End } \mathcal{A} \rightarrow \text{End } \tilde{\mathcal{A}}$, defined by (1.4), is an isomorphism of the vector spaces $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}$ and $\text{der}_{(\alpha, \beta, \gamma)} \tilde{\mathcal{A}}$, i. e. for any $\alpha, \beta, \gamma \in \mathbb{C}$

$$\varrho(\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}) = \text{der}_{(\alpha, \beta, \gamma)} \tilde{\mathcal{A}}. \quad (2.3)$$

Proof. Suppose we have $\mathcal{A} = (V, \cdot)$ and $\tilde{\mathcal{A}} = (\tilde{V}, *)$. The isomorphism relation (1.2) implies that for all $x, y \in \tilde{\mathcal{A}}$

$$x * y = f(f^{-1}(x) \cdot f^{-1}(y)).$$

By rewriting the definition (2.1) we have for $D \in \text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}$

$$\alpha D(f^{-1}(x) \cdot f^{-1}(y)) = \beta(Df^{-1}x) \cdot f^{-1}y + \gamma f^{-1}x \cdot (Df^{-1}y).$$

Applying the mapping f on this equation and taking into account that $\alpha, \beta, \gamma \in \mathbb{C}$ one has

$$\alpha f D f^{-1}(x * y) = \beta(f D f^{-1}x) * y + \gamma x * (f D f^{-1}y), \quad (2.4)$$

i. e. $f D f^{-1} \in \text{der}_{(\alpha, \beta, \gamma)} \tilde{\mathcal{A}}$. □

Corollary 2.1.2. For any $\alpha, \beta, \gamma \in \mathbb{C}$ the dimension of the vector space $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}$ is an invariant characteristic of algebras.

Now we restrict our investigations to commutative or anti-commutative algebras and it follows immediately from (2.1) that for any $\varepsilon \in \mathbb{C} \setminus \{0\}$ it holds:

$$\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A} = \text{der}_{(\varepsilon\alpha, \varepsilon\beta, \varepsilon\gamma)} \mathcal{A} = \text{der}_{(\alpha, \gamma, \beta)} \mathcal{A}. \quad (2.5)$$

Furthermore, we have the following important property.

Lemma 2.1.3. Let \mathcal{A} be a commutative or anti-commutative algebra. Then for any $\alpha, \beta, \gamma \in \mathbb{C}$

$$\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A} = \text{der}_{(0, \beta - \gamma, \gamma - \beta)} \mathcal{A} \cap \text{der}_{(2\alpha, \beta + \gamma, \beta + \gamma)} \mathcal{A} \quad (2.6)$$

holds.

Proof. Suppose any $\alpha, \beta, \gamma \in \mathbb{C}$ are given. We carry out the proof for an anti-commutative algebra \mathcal{A}_- with a multiplication $[\]_-,$ i. e. for all $x, y \in \mathcal{A}_-$ the relation $[x, y]_- = -[y, x]_-$ is satisfied; the proof for a commutative algebra is analogous. Then for $D \in \text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}_-$ and arbitrary $x, y \in \mathcal{A}_-$ we have

$$\begin{aligned} \alpha D[x, y]_- &= \beta[Dx, y]_- + \gamma[x, Dy]_- \\ \alpha D[y, x]_- &= \beta[Dy, x]_- + \gamma[y, Dx]_- \end{aligned} \quad (2.7)$$

By adding and subtracting equations (2.7) we obtain

$$\begin{aligned} 0 &= (\beta - \gamma)([Dx, y]_- - [x, Dy]_-) \\ 2\alpha D[x, y]_- &= (\beta + \gamma)([Dx, y]_- + [x, Dy]_-) \end{aligned} \quad (2.8)$$

and thus $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}_- \subset \text{der}_{(0, \beta - \gamma, \gamma - \beta)} \mathcal{A}_- \cap \text{der}_{(2\alpha, \beta + \gamma, \beta + \gamma)} \mathcal{A}_-$. Similarly, starting with equations (2.8) we obtain equations (2.7) and the remaining inclusion is proven. □

Further, we proceed to formulate the theorem which reveals the structure of the spaces $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A}$; the three original parameters are in fact reduced to only one.

Theorem 2.1.4. Let \mathcal{A} be a commutative or anti-commutative algebra. Then for any $\alpha, \beta, \gamma \in \mathbb{C}$ there exists $\delta \in \mathbb{C}$ such that the subspace $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} \subset \text{End } \mathcal{A}$ is equal to some of the four following subspaces:

1. $\text{der}_{(\delta,0,0)} \mathcal{A}$
2. $\text{der}_{(\delta,1,-1)} \mathcal{A}$
3. $\text{der}_{(\delta,1,0)} \mathcal{A}$
4. $\text{der}_{(\delta,1,1)} \mathcal{A}$.

Proof. 1. Suppose $\beta + \gamma = 0$. Then either $\beta = \gamma = 0$ or $\beta = -\gamma \neq 0$.

(a) For $\beta = \gamma = 0$, we have

$$\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} = \text{der}_{(\alpha,0,0)} \mathcal{A}.$$

(b) For $\beta = -\gamma \neq 0$, it follows from (2.5), (2.6):

$$\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} = \text{der}_{(0,\beta-\gamma,\gamma-\beta)} \mathcal{A} \cap \text{der}_{(2\alpha,0,0)} \mathcal{A} = \text{der}_{(0,1,-1)} \mathcal{A} \cap \text{der}_{(\alpha,0,0)} \mathcal{A}.$$

On the other hand it holds

$$\text{der}_{(\alpha,1,-1)} \mathcal{A} = \text{der}_{(0,2,-2)} \mathcal{A} \cap \text{der}_{(2\alpha,0,0)} \mathcal{A} = \text{der}_{(0,1,-1)} \mathcal{A} \cap \text{der}_{(\alpha,0,0)} \mathcal{A}$$

and therefore

$$\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} = \text{der}_{(\alpha,1,-1)} \mathcal{A}.$$

2. Suppose $\beta + \gamma \neq 0$. Then either $\beta - \gamma \neq 0$ or $\beta = \gamma \neq 0$.

(a) For $\beta - \gamma \neq 0$, we have

$$\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} = \text{der}_{(0,\beta-\gamma,\gamma-\beta)} \mathcal{A} \cap \text{der}_{(2\alpha,\beta+\gamma,\beta+\gamma)} \mathcal{A} = \text{der}_{(0,1,-1)} \mathcal{A} \cap \text{der}_{(\frac{2\alpha}{\beta+\gamma},1,1)} \mathcal{A}$$

and taking into account (2.6), this is equal to $\text{der}_{(\frac{\alpha}{\beta+\gamma},1,0)} \mathcal{A}$, i. e.

$$\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} = \text{der}_{(\frac{\alpha}{\beta+\gamma},1,0)} \mathcal{A}.$$

(b) For $\beta = \gamma \neq 0$ we have

$$\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} = \text{der}_{(\frac{\alpha}{\beta},1,1)} \mathcal{A}.$$

□

We define two complex functions with fundamental property — invariance under isomorphisms. We use the one-parametric sets $\text{der}_{(\alpha,1,0)} \mathcal{A}$ and $\text{der}_{(\alpha,1,1)} \mathcal{A}$ from Theorem 2.1.4 to define invariant functions of an arbitrary algebra \mathcal{A} . Functions $\psi \mathcal{L}, \psi^0 \mathcal{L} : \mathbb{C} \rightarrow \{0, 1, \dots, (\dim \mathcal{A})^2\}$ defined by the formulas

$$(\psi \mathcal{A})(\alpha) = \dim \text{der}_{(\alpha,1,1)} \mathcal{A} \quad (2.9)$$

$$(\psi^0 \mathcal{A})(\alpha) = \dim \text{der}_{(\alpha,1,0)} \mathcal{A} \quad (2.10)$$

are called **invariant functions** corresponding to (α, β, γ) -derivations of an algebra \mathcal{A} .

The following statement follows immediately from Theorem 2.1.1.

Corollary 2.1.5. If two complex algebras $\mathcal{A}, \tilde{\mathcal{A}}$ are isomorphic, $\mathcal{A} \cong \tilde{\mathcal{A}}$, then it holds:

1. $\psi \mathcal{A} = \psi \tilde{\mathcal{A}}$,
2. $\psi^0 \mathcal{A} = \psi^0 \tilde{\mathcal{A}}$.

Note that sometimes in the literature the name 'invariant functions' denotes (formal) Casimir invariants; their form, however, depends on the choice of the basis of \mathcal{L} . Here by invariant functions we rather mean 'basis independent' complex functions, such as ψ and ψ^0 .

2.2 (α, β, γ) -derivations of Lie Algebras

Suppose we have a complex Lie algebra \mathcal{L} and let us discuss in detail possible outcome of Theorem 2.1.4.

Theorem 2.2.1. Let \mathcal{L} be a complex Lie algebra and $\alpha, \beta, \gamma \in \mathbb{C}$ not all zero. Then the space $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ is equal to some of the following:

1. Lie algebra of derivations $\text{der}_{(1,1,1)} \mathcal{L} \subset \text{gl } \mathcal{L}$,
2. Lie algebra $\text{der}_{(0,1,1)} \mathcal{L} \subset \text{gl } \mathcal{L}$,
3. associative algebra $\text{der}_{(1,1,0)} \mathcal{L} = C_{\text{ad}}(\mathcal{L}) \subset \text{gl } \mathcal{L}$,

4. associative algebra $\text{der}_{(1,0,0)} \mathcal{L} \subset \text{End } \mathcal{L}$ of dimension

$$\dim \text{der}_{(1,0,0)} \mathcal{L} = \text{codim } \mathcal{L}^2 \dim \mathcal{L}, \quad (2.11)$$

5. associative algebra $\text{der}_{(0,1,0)} \mathcal{L} \subset \text{End } \mathcal{L}$ of dimension

$$\dim \text{der}_{(0,1,0)} \mathcal{L} = \dim \mathcal{L} \dim C(\mathcal{L}), \quad (2.12)$$

6. Jordan algebra $\text{der}_{(1,1,-1)} \mathcal{L} \subset \text{jor } \mathcal{L}$,

7. Jordan algebra $\text{der}_{(0,1,-1)} \mathcal{L} \subset \text{jor } \mathcal{L}$,

8. subspace $\text{der}_{(\delta,1,0)} \mathcal{L}$, for some $\delta \in \mathbb{C}$, $\delta \neq 0, 1$.

9. subspace $\text{der}_{(\delta,1,1)} \mathcal{L}$, for some $\delta \in \mathbb{C}$, $\delta \neq 0, 1$.

Proof. We list the four cases in Theorem 2.1.4 and discuss all possible values of the parameter $\delta \in \mathbb{C}$:

1. $\text{der}_{(\delta,0,0)} \mathcal{L}$:

(a) Since we assumed some $\alpha, \beta, \gamma \in \mathbb{C}$ non-zero, the case $\delta = 0$ cannot occur.

(b) For $\delta \neq 0$, the space $\text{der}_{(1,0,0)} \mathcal{L}$ is an associative subalgebra of $\text{End } \mathcal{L}$, which maps the derived algebra $\mathcal{L}^2 = [\mathcal{L}, \mathcal{L}]$ to the zero vector:

$$\text{der}_{(1,0,0)} \mathcal{L} = \{A \in \text{End } \mathcal{L} \mid A(\mathcal{L}^2) = 0\},$$

and therefore its dimension is as follows:

$$\dim \text{der}_{(1,0,0)} \mathcal{L} = \text{codim } \mathcal{L}^2 \dim \mathcal{L}.$$

2. $\text{der}_{(\delta,1,-1)} \mathcal{L}$:

(a) For $\delta = 0$, we have a Jordan algebra $\text{der}_{(0,1,-1)} \mathcal{L} \subset \text{jor } \mathcal{L}$,

$$\text{der}_{(0,1,-1)} \mathcal{L} = \{A \in \text{End } \mathcal{L} \mid [Ax, y] = [x, Ay], \quad \forall x, y \in \mathcal{L}\}.$$

The proof of the property $A, B \in \text{der}_{(0,1,-1)} \mathcal{L} \Rightarrow \frac{1}{2}(AB + BA) \in \text{der}_{(0,1,-1)} \mathcal{L}$ is straightforward.

- (b) For $\delta \neq 0$, we obtain the Jordan algebra $\text{der}_{(1,1,-1)} \mathcal{L} \subset \text{jor } \mathcal{L}$ as an intersection of two Jordan algebras:

$$\begin{aligned} \text{der}_{(\delta,1,-1)} \mathcal{L} &= \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(\delta,0,0)} \mathcal{L} = \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(1,0,0)} \mathcal{L} \\ &= \text{der}_{(1,1,-1)} \mathcal{L}. \end{aligned}$$

3. $\text{der}_{(\delta,1,0)} \mathcal{L}$:

- (a) For $\delta = 0$, we get an associative algebra of all linear operators of the vector space \mathcal{L} , which maps the whole \mathcal{L} into its center $C(\mathcal{L})$:

$$\text{der}_{(0,1,0)} \mathcal{L} = \{A \in \text{End } \mathcal{L} \mid A(\mathcal{L}) \subseteq C(\mathcal{L})\},$$

and its dimension is

$$\dim \text{der}_{(0,1,0)} \mathcal{L} = \dim \mathcal{L} \dim C(\mathcal{L}).$$

- (b) For $\delta = 1$, the space $\text{der}_{(1,1,0)} \mathcal{L}$ is the centralizer of the adjoint representation $C_{\text{ad}}(\mathcal{L})$, see (1.14).
(c) For the remaining values of δ , the space $\text{der}_{(\delta,1,0)} \mathcal{L}$ forms, in the general case of a Lie algebra \mathcal{L} , only a vector subspace of $\text{End } \mathcal{L}$. Thus we have the one-parametric set of vector spaces:

$$\text{der}_{(\delta,1,0)} \mathcal{L} = \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L}.$$

4. $\text{der}_{(\delta,1,1)} \mathcal{L}$:

- (a) For $\delta = 0$, we have a Lie algebra

$$\text{der}_{(0,1,1)} \mathcal{L} = \{A \in \text{End } \mathcal{L} \mid [Ax, y] = -[x, Ay], \quad \forall x, y \in \mathcal{L}\}.$$

Verification of the property $A, B \in \text{der}_{(0,1,1)} \mathcal{L} \Rightarrow (AB - BA) \in \text{der}_{(0,1,1)} \mathcal{L}$ is straightforward.

- (b) For $\delta = 1$, we get the algebra of derivations of \mathcal{L} ,

$$\text{der}_{(1,1,1)} \mathcal{L} = \text{der } \mathcal{L}.$$

- (c) For the remaining values of δ , the space $\text{der}_{(\delta,1,1)} \mathcal{L}$ forms, in the general case of a Lie algebra \mathcal{L} , only a vector subspace of $\text{End } \mathcal{L}$.

□

Since the definition of (α, β, γ) -derivations partially overlaps other generalizations, some of the sets from the above theorem naturally appeared already in the literature. For instance, a considerable amount of theory concerning relations between $\text{der}_{(1,1,0)} \mathcal{L}$ and $\text{der}_{(0,1,-1)} \mathcal{L}$ has been developed in [24]. Later on, we are mostly interested in the form of the one-parametric spaces $\text{der}_{(\delta,1,1)} \mathcal{L}$ and $\text{der}_{(\delta,1,0)} \mathcal{L}$.

Example 1. If \mathcal{L} is a simple complex Lie algebra then $\text{der}_{(1,1,1)} \mathcal{L} \cong \mathcal{L}$,

$$\text{der}_{(1,0,0)} \mathcal{L} = \text{der}_{(0,1,0)} \mathcal{L} = \{0\},$$

and $\text{der}_{(1,1,0)} \mathcal{L}$ is the one-dimensional Lie algebra containing multiples of the identity operator.

2.2.1 Intersections of the Spaces $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$

A thorough study of various intersections of two different subspaces $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ turned out to be very valuable. Besides new independent invariants, we also obtain a new operator algebra with a non-trivial structure, as well as restrictions on $\psi \mathcal{L}$, $\psi^0 \mathcal{L}$. We commence with:

Theorem 2.2.2. Let $f : \mathcal{A} \rightarrow \tilde{\mathcal{A}}$ be an isomorphism of complex algebras \mathcal{A} and $\tilde{\mathcal{A}}$. Then the mapping $\varrho : \text{End } \mathcal{A} \rightarrow \text{End } \tilde{\mathcal{A}}$, defined by (1.4), is an isomorphism of the vector spaces $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} \cap \text{der}_{(\alpha',\beta',\gamma')} \mathcal{A}$ and $\text{der}_{(\alpha,\beta,\gamma)} \tilde{\mathcal{A}} \cap \text{der}_{(\alpha',\beta',\gamma')} \tilde{\mathcal{A}}$, i. e. for any $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}$

$$\varrho(\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} \cap \text{der}_{(\alpha',\beta',\gamma')} \mathcal{A}) = \text{der}_{(\alpha,\beta,\gamma)} \tilde{\mathcal{A}} \cap \text{der}_{(\alpha',\beta',\gamma')} \tilde{\mathcal{A}}. \quad (2.13)$$

Proof. Suppose we have $\mathcal{A} = (V, \cdot)$ and $\tilde{\mathcal{A}} = (\tilde{V}, *)$. The isomorphism relation (1.2) implies that for all $x, y \in \tilde{\mathcal{A}}$ the relation $x * y = f(f^{-1}(x) \cdot f^{-1}(y))$ holds. By rewriting the definition (2.1) we obtain $D \in \text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} \cap \text{der}_{(\alpha',\beta',\gamma')} \mathcal{A}$ if and only if both of the equations

$$\begin{aligned} \alpha D(f^{-1}(x) \cdot f^{-1}(y)) &= \beta(Df^{-1}x) \cdot f^{-1}y + \gamma f^{-1}x \cdot (Df^{-1}y) \\ \alpha' D(f^{-1}(x) \cdot f^{-1}(y)) &= \beta'(Df^{-1}x) \cdot f^{-1}y + \gamma' f^{-1}x \cdot (Df^{-1}y) \end{aligned}$$

are satisfied for all $x, y \in \tilde{\mathcal{A}}$. Applying the mapping f on these two equations and taking into account that $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are complex numbers, one has

$$\begin{aligned} \alpha f D f^{-1}(x * y) &= \beta(f D f^{-1}x) * y + \gamma x * (f D f^{-1}y), \\ \alpha' f D f^{-1}(x * y) &= \beta'(f D f^{-1}x) * y + \gamma' x * (f D f^{-1}y), \end{aligned}$$

i. e. $fDf^{-1} \in \text{der}_{(\alpha,\beta,\gamma)} \tilde{\mathcal{A}} \cap \text{der}_{(\alpha',\beta',\gamma')} \tilde{\mathcal{A}}$. □

Corollary 2.2.3. For any $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}$ the number

$$\dim(\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A} \cap \text{der}_{(\alpha',\beta',\gamma')} \mathcal{A})$$

is an invariant characteristic of algebras.

In our search for new invariants of complex Lie algebras, we systematically explored all possible intersections of the spaces $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$. We classify these intersections in the following theorem.

Theorem 2.2.4. Let \mathcal{L} be a complex Lie algebra. Suppose $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero and $\alpha', \beta', \gamma' \in \mathbb{C}$ are not all zero. Then the intersection $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L} \cap \text{der}_{(\alpha',\beta',\gamma')} \mathcal{L}$ is equal to some of the cases 1. – 9. of Theorem 2.2.1 or to some of the following:

1. associative algebra $\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L} \subset \text{End } \mathcal{L}$ of dimension

$$\dim(\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L}) = \text{codim } \mathcal{L}^2 \dim C(\mathcal{L}), \quad (2.14)$$

2. Lie algebra $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L} \subset \text{gl } \mathcal{L}$.

Proof. According to Theorem 2.2.1, for given $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}$ there exists $\delta \in \mathbb{C}$ such that $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ is equal to some of the spaces:

$$\text{der}_{(\delta,1,1)} \mathcal{L}, \text{der}_{(\delta,1,0)} \mathcal{L}, \text{der}_{(1,0,0)} \mathcal{L}, \text{der}_{(1,1,-1)} \mathcal{L}, \text{der}_{(0,1,-1)} \mathcal{L},$$

and $\delta' \in \mathbb{C}$ such that $\text{der}_{(\alpha',\beta',\gamma')} \mathcal{L}$ is equal to some of the spaces:

$$\text{der}_{(\delta',1,1)} \mathcal{L}, \text{der}_{(\delta',1,0)} \mathcal{L}, \text{der}_{(1,0,0)} \mathcal{L}, \text{der}_{(1,1,-1)} \mathcal{L}, \text{der}_{(0,1,-1)} \mathcal{L}.$$

There are 17 possible pairs the above spaces. Five pairs consist of equal spaces and corresponding intersections lead to some of the cases 1. – 9. of Theorem 2.2.1. Then there are 10 obvious pairs with different subspaces, plus two pairs $\text{der}_{(\delta,1,1)} \mathcal{L} \cap \text{der}_{(\delta',1,1)} \mathcal{L}$ and $\text{der}_{(\delta,1,0)} \mathcal{L} \cap \text{der}_{(\delta',1,0)} \mathcal{L}$ with $\delta \neq \delta'$. Assuming $\delta \neq \delta'$, the following equalities among intersections are obvious:

$$\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L} = \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(\delta,1,0)} \mathcal{L} = \text{der}_{(\delta,1,0)} \mathcal{L} \cap \text{der}_{(\delta',1,0)} \mathcal{L} \quad (2.15)$$

$$\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L} = \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(\delta,1,1)} \mathcal{L} = \text{der}_{(\delta,1,1)} \mathcal{L} \cap \text{der}_{(\delta',1,1)} \mathcal{L} \quad (2.16)$$

$$\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,-1)} \mathcal{L} = \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(1,1,-1)} \mathcal{L} = \text{der}_{(1,1,-1)} \mathcal{L} \cap \text{der}_{(0,1,-1)} \mathcal{L} \quad (2.17)$$

Using Lemma 2.6 we obtain:

$$\text{der}_{(1,1,-1)} \mathcal{L} = \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(1,0,0)} \mathcal{L} \quad (2.18)$$

$$\text{der}_{(\delta,1,0)} \mathcal{L} = \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} \quad (2.19)$$

The equality (2.18) implies that all *three* intersections in (2.17) are equal to the case 6. of Theorem 2.2.1. Using successively (2.18), (2.19) and (2.15), we obtain:

$$\begin{aligned} \text{der}_{(1,1,-1)} \mathcal{L} \cap \text{der}_{(\delta,1,1)} \mathcal{L} &= \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(\delta,1,1)} \mathcal{L} \\ &= \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(\frac{\delta}{2},1,0)} \mathcal{L} = \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L}. \end{aligned} \quad (2.20)$$

Using successively (2.19), (2.17) and (2.18), (2.20) we obtain:

$$\begin{aligned} \text{der}_{(1,1,-1)} \mathcal{L} \cap \text{der}_{(\delta,1,0)} \mathcal{L} &= \text{der}_{(1,1,-1)} \mathcal{L} \cap \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} \\ &= \text{der}_{(1,1,-1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} = \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L}. \end{aligned}$$

Using successively (2.19), (2.16), (2.18), (2.20) and assuming firstly $\delta' \neq 2\delta$, we obtain:

$$\begin{aligned} \text{der}_{(\delta,1,0)} \mathcal{L} \cap \text{der}_{(\delta',1,1)} \mathcal{L} &= \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} \cap \text{der}_{(\delta',1,1)} \mathcal{L} \\ &= \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L} \\ &= \text{der}_{(1,1,-1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L} = \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L}. \end{aligned}$$

Secondly, using twice (2.19) we obtain:

$$\begin{aligned} \text{der}_{(\delta,1,0)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} &= \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} \cap \text{der}_{(2\delta,1,1)} \mathcal{L} \\ &= \text{der}_{(\delta,1,0)} \mathcal{L}. \end{aligned}$$

Since from (2.5) follows $\text{der}_{(\delta,1,0)} \mathcal{L} = \text{der}_{(\delta,0,1)} \mathcal{L}$, we have for $A \in \text{der}_{(\delta,1,0)} \mathcal{L}$ and all $x, y \in \mathcal{L}$:

$$\delta A[x, y] = [Ax, y] = [x, Ay].$$

Thus, we have $A \in \text{der}_{(0,1,-1)} \mathcal{L}$ and the inclusion

$$\text{der}_{(\delta,1,0)} \mathcal{L} \subset \text{der}_{(0,1,-1)} \mathcal{L} \quad (2.21)$$

implies

$$\text{der}_{(\delta,1,0)} \mathcal{L} = \text{der}_{(0,1,-1)} \mathcal{L} \cap \text{der}_{(\delta,1,0)} \mathcal{L}.$$

The space $\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L}$, as the intersection of two associative subalgebras of $\text{End } \mathcal{L}$, forms also an associative subalgebra of $\text{End } \mathcal{L}$; the space

$$\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L} = \text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L},$$

as an intersection of two Lie subalgebras of $\text{gl } \mathcal{L}$, forms also a Lie subalgebra of $\text{gl } \mathcal{L}$. \square

Lemma 2.2.5. Let \mathcal{L} be a complex Lie algebra. Then for all $\alpha, \beta, \gamma \in \mathbb{C}$

$$\text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L} \subset \text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}.$$

Proof. Let $A \in \text{der}_{(1,0,0)} \mathcal{L} \cap \text{der}_{(0,1,0)} \mathcal{L}$. Since from (2.5) follows $\text{der}_{(0,1,0)} \mathcal{L} = \text{der}_{(0,0,1)} \mathcal{L}$, we have for all $x, y \in \mathcal{L}$:

$$A[x, y] = 0 \tag{2.22}$$

$$0 = [Ax, y] \tag{2.23}$$

$$0 = [x, Ay] \tag{2.24}$$

Multiplying (2.22), (2.23), (2.24) by $\alpha, \beta, \gamma \in \mathbb{C}$, respectively, and summing these equations one has:

$$\alpha A[x, y] = \beta[Ax, y] + \gamma[x, Ay],$$

i. e. $A \in \text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$. □

Corollary 2.2.6. Let \mathcal{L} be a complex Lie algebra. Then the following inequalities hold:

$$\text{codim } \mathcal{L}^2 \dim C(\mathcal{L}) \leq \psi^0 \mathcal{L} \leq \dim \text{der}_{(0,1,-1)} \mathcal{L} \tag{2.25}$$

$$\psi^0 \mathcal{L}(\alpha) \leq \psi \mathcal{L}(2\alpha), \quad \forall \alpha \in \mathbb{C} \tag{2.26}$$

$$\text{codim } \mathcal{L}^2 \dim C(\mathcal{L}) \leq \psi \mathcal{L}. \tag{2.27}$$

Proof. The inequality (2.27) and the first part of the inequality (2.25) follow directly from Lemma 2.2.5 and Theorem 2.2.4. Since we have from (2.19) the inclusion $\text{der}_{(\delta,1,0)} \mathcal{L} \subset \text{der}_{(2\delta,1,1)} \mathcal{L}$, the inequality (2.26) follows. The second part of the inequality (2.25) follows directly from (2.21). □

Example 2. We demonstrate the non-triviality of the inequalities in Corollary 2.2.6. Consider the four-dimensional Lie algebra $\mathcal{L} = \text{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_1$ with non-zero commutation relations: $[e_1, e_2] = e_1$, $[e_2, e_3] = e_3$, $[e_1, e_3] = 2e_2$. The invariant functions $\psi \mathcal{L}$ and $\psi^0 \mathcal{L}$ have the following form:

α	1	0	-1	2	
$\psi \mathcal{L}(\alpha)$	4	4	6	2	1

α	1	0	
$\psi^0 \mathcal{L}(\alpha)$	2	4	1

A blank space in the table of the function ψ denotes a general complex number, different from all previously listed values, i. e. it holds: $\psi \mathcal{L}(\alpha) = 1$, $\alpha \neq 0, \pm 1, 2$. It is clear that $\dim C(\mathcal{L}) = 1$ and $\dim \mathcal{L}^2 = 3$. Hence we have $\text{codim } \mathcal{L}^2 = 1$. We also calculate

$$\dim \text{der}_{(0,1,-1)} \mathcal{L} = 5$$

and obtain from Corollary 2.2.6 the following inequalities

$$\begin{aligned}
1 &\leq \psi^0 \mathcal{L} \leq 5 \\
4 &= \psi^0 \mathcal{L}(0) \leq \psi \mathcal{L}(0) = 4 \\
2 &= \psi^0 \mathcal{L}(1) \leq \psi \mathcal{L}(2) = 2 \\
1 &= \psi^0 \mathcal{L}(\alpha) \leq \psi \mathcal{L}(2\alpha) = 1, \quad \forall \alpha \in \mathbb{C}, \alpha \neq 0, 1, \pm 1/2 \\
1 &\leq \psi \mathcal{L}.
\end{aligned}$$

2.2.2 (α, β, γ) –derivations of Low–dimensional Lie Algebras

In Section A.1 of Appendix A all (α, β, γ) –derivations of all two and three dimensional non–abelian Lie algebras are listed. Here we present one typical example of (α, β, γ) –derivations for a four–dimensional Lie algebra.

Example 3. The four–dimensional Lie algebra $\mathfrak{g}_{4,2}(a)$ has non–zero commutation relations $[e_1, e_4] = ae_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$, with a complex parameter ($a \neq 0, \pm 1, -2$). We present the complete set of its (α, β, γ) –derivations. Commutation relations of the presented matrix Lie and Jordan algebras are placed in Appendix B. Note especially the form of the one–parametric subspace $\text{der}_{(\delta,1,1)} \mathfrak{g}_{4,2}(a)$. We encounter here for the first time an important phenomenon: the dimensionality of the matrix subspace $\text{der}_{(\delta,1,1)} \mathfrak{g}_{4,2}(a)$ *depends* on the value of the parameter $a \in \mathbb{C}$. In the following formulas, we abbreviate the notation and write $\text{der}_{(\alpha,\beta,\gamma)}$ instead of $\text{der}_{(\alpha,\beta,\gamma)} \mathfrak{g}_{4,2}(a)$.

$$\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\} \cong \mathfrak{g}_{4,5}(1, 1)$$

$$\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_1$$

$$\text{der}_{(1,0,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_{4,5}(1, 1)$$

$$\text{der}_{(0,1,0)} = \{0\}$$

$$\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cong 3 \mathfrak{g}_1$$

$$\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$$

$$\text{der}_{(1,1,-1)} = \{0\}$$

$$\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{j}_1$$

$$\text{der}_{(\delta,1,0)} = \{0\}_{\delta \neq 1}$$

$$\text{der}_{(\delta,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1+\delta \end{pmatrix} \right\}_{\delta \neq 1, a, 1/a}$$

$$\text{der}_{(a,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1+a \end{pmatrix} \right\}$$

$$\text{der}_{(\frac{1}{a},1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1+\frac{1}{a} \end{pmatrix} \right\}.$$

2.3 Associated Lie Algebras

In [III] we have investigated the problem when the subspace $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}$ forms a Lie subalgebra of $\text{gl } \mathcal{L}$, i. e. for which $\alpha, \beta, \gamma \in \mathbb{C}$ is this set closed with respect to the Lie product in $\text{gl } \mathcal{L}$:

$$A, B \in \text{der}_{(\alpha,\beta,\gamma)} \mathcal{L} \Rightarrow AB - BA \in \text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}.$$

We found out the solutions which are now included in Theorem 2.2.1 as the cases 1. – 5. and called them **associated Lie algebras** of the Lie algebra \mathcal{L} . There were two reasons for this investigation. Firstly, from Corollaries 2.1.2 and 2.2.3 we know that for fixed $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}$ the map $d_{(\alpha,\beta,\gamma)}$, defined by

$$d_{(\alpha,\beta,\gamma)} \mathcal{L} = \dim \text{der}_{(\alpha,\beta,\gamma)} \mathcal{L}, \quad (2.28)$$

as well as the map $d_{(\alpha,\beta,\gamma)(\alpha',\beta',\gamma')}$, defined by

$$d_{(\alpha,\beta,\gamma)(\alpha',\beta',\gamma')} \mathcal{L} = \dim(\text{der}_{(\alpha,\beta,\gamma)} \mathcal{L} \cap \text{der}_{(\alpha',\beta',\gamma')} \mathcal{L}), \quad (2.29)$$

constitute invariant characteristics of Lie algebras. Secondly, the following Proposition allows us to consider not only the dimensions but also the Lie structure of the associated Lie algebras.

Proposition 2.3.1. If two complex Lie algebras are isomorphic, $\mathcal{L} \cong \tilde{\mathcal{L}}$, then the associated Lie algebras and their intersections are isomorphic as well, i. e. it holds:

1. $\text{der}_{(1,1,1)} \mathcal{L} \cong \text{der}_{(1,1,1)} \tilde{\mathcal{L}}$
2. $\text{der}_{(0,1,1)} \mathcal{L} \cong \text{der}_{(0,1,1)} \tilde{\mathcal{L}}$
3. $\text{der}_{(1,1,0)} \mathcal{L} \cong \text{der}_{(1,1,0)} \tilde{\mathcal{L}}$
4. $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L} \cong \text{der}_{(1,1,1)} \tilde{\mathcal{L}} \cap \text{der}_{(0,1,1)} \tilde{\mathcal{L}}$.

Proof. Since the map $\varrho : \text{gl } \mathcal{L} \rightarrow \text{gl } \tilde{\mathcal{L}}$, defined by (1.4), is a homomorphism, then its restriction on subalgebras $\text{der}_{(1,1,1)} \mathcal{L} \subset \text{gl } \mathcal{L}$, $\text{der}_{(0,1,1)} \mathcal{L} \subset \text{gl } \mathcal{L}, \dots$ is also a homomorphism. It follows from (2.3) and (2.13) that ϱ is an isomorphism. \square

Let us consider the following set of invariants of Lie algebras:

$$\text{inv} = \{(d_k), (l_k), (c_k), \tau, d_{(1,1,1)}, d_{(0,1,1)}, d_{(1,1,0)}, d_{(1,1,1)(0,1,1)}\},$$

where the sequences d_k, l_k, c_k and the number τ were defined by relations (1.18) – (1.20) and (1.23). We arrange the values of the invariants in the set inv corresponding to some coset of complex Lie algebras $[\mathcal{L}]$ into the following tuple:

$$\begin{aligned} \text{inv } \mathcal{L} = & (d_0(\mathcal{L}), d_1(\mathcal{L}), \dots) (l_0(\mathcal{L}), l_1(\mathcal{L}), \dots) (c_0(\mathcal{L}), c_1(\mathcal{L}), \dots) \tau(\mathcal{L}) \\ & [d_{(1,1,1)}\mathcal{L}, d_{(0,1,1)}\mathcal{L}, d_{(1,1,0)}\mathcal{L}, d_{(1,1,1)(0,1,1)}\mathcal{L}]. \end{aligned} \quad (2.30)$$

In this notation the equations (2.11), (2.12) yield $d_{(1,0,0)}\mathcal{L} = d_0(\mathcal{L})(d_0(\mathcal{L}) - d_1(\mathcal{L}))$ and $d_{(0,1,0)}\mathcal{L} = d_0(\mathcal{L})c_0(\mathcal{L})$. There is a natural question if there exists a similar dependence among invariants in the set inv . Using the notion of independence defined by (1.8), we give the answer in the following proposition.

Proposition 2.3.2. Let \mathcal{L} be a complex Lie algebra. Then the invariant

1. $d_{(1,1,1)}$ is independent on the set $\text{inv} \setminus \{d_{(1,1,1)}\}$
2. $d_{(0,1,1)}$ is independent on the set $\text{inv} \setminus \{d_{(0,1,1)}\}$
3. $d_{(1,1,0)}$ is independent on the set $\text{inv} \setminus \{d_{(1,1,0)}\}$
4. $d_{(1,1,1)(0,1,1)}$ is independent on the set $\text{inv} \setminus \{d_{(1,1,1)(0,1,1)}\}$.

Proof. 1. $d_{(1,1,1)}$: In order to satisfy the definition (1.8), we have to construct two Lie algebras $\mathcal{L}, \tilde{\mathcal{L}}$ whose invariants from the set inv differ only in the values $d_{(1,1,1)}\mathcal{L} \neq d_{(1,1,1)}\tilde{\mathcal{L}}$. We may consider the following two seven-dimensional Lie algebras $\mathcal{L}, \tilde{\mathcal{L}}$:

$$\begin{aligned} \mathcal{L} : & [e_4, e_6] = e_1, [e_4, e_7] = e_2, [e_5, e_6] = e_2, [e_5, e_7] = e_3 \\ \tilde{\mathcal{L}} : & [e_4, e_6] = e_1, [e_4, e_7] = e_2, [e_5, e_7] = e_3 \end{aligned}$$

and find the corresponding values of the invariants from inv:

$$\begin{aligned}\text{inv } \mathcal{L} &= (7, 3, 0)(7, 3, 0)(3, 7) \quad 3 \quad [19, 24, 13, 15] \\ \text{inv } \tilde{\mathcal{L}} &= (7, 3, 0)(7, 3, 0)(3, 7) \quad 3 \quad [20, 24, 13, 15].\end{aligned}$$

We observe that the only different values of $\text{inv } \mathcal{L}$ and $\text{inv } \tilde{\mathcal{L}}$ are

$$19 = d_{(1,1,1)}\mathcal{L} \neq d_{(1,1,1)}\tilde{\mathcal{L}} = 20.$$

The proof of the remaining cases is analogous; we present the pairs $\mathcal{L}, \tilde{\mathcal{L}}$ and the values $\text{inv } \mathcal{L}, \text{inv } \tilde{\mathcal{L}}$ for each case.

2. $d_{(0,1,1)}$:

$$\begin{aligned}\mathcal{L} : \quad & [e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_2, e_6] = -e_7, [e_2, e_8] = e_1, \\ & [e_3, e_7] = e_1, [e_4, e_6] = e_1, [e_6, e_8] = e_5 \\ \tilde{\mathcal{L}} : \quad & [e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_2, e_8] = e_1, [e_3, e_6] = e_8, \\ & [e_3, e_7] = e_1, [e_4, e_6] = e_1, [e_6, e_8] = e_5 \\ \text{inv } \mathcal{L} &= (8, 4, 0)(8, 4, 2, 0)(2, 5, 8) \quad 2 \quad [17, 19, 9, 11] \\ \text{inv } \tilde{\mathcal{L}} &= (8, 4, 0)(8, 4, 2, 0)(2, 5, 8) \quad 2 \quad [17, 20, 9, 11]\end{aligned}$$

3. $d_{(1,1,0)}$:

$$\begin{aligned}\mathcal{L} : \quad & [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_6] = e_1, [e_1, e_7] = e_3, \\ & [e_2, e_6] = -e_2, [e_3, e_6] = e_3, [e_5, e_6] = 2e_5 \\ \tilde{\mathcal{L}} : \quad & [e_1, e_2] = e_4, [e_1, e_4] = e_5, [e_1, e_6] = e_1, [e_1, e_7] = e_3, \\ & [e_2, e_6] = -2e_2, [e_3, e_6] = e_3, [e_4, e_6] = -e_4 \\ \text{inv } \mathcal{L} &= (7, 5, 2, 0)(7, 5)(1) \quad 3 \quad [10, 11, 3, 3] \\ \text{inv } \tilde{\mathcal{L}} &= (7, 5, 2, 0)(7, 5)(1) \quad 3 \quad [10, 11, 4, 3]\end{aligned}$$

4. $d_{(1,1,1)(0,1,1)}$:

$$\begin{aligned}\mathcal{L} : \quad & [e_1, e_3] = -e_3, [e_1, e_4] = e_4, [e_1, e_6] = 2e_6, [e_1, e_7] = -e_7, \\ & [e_1, e_8] = e_8, [e_3, e_6] = e_8, [e_4, e_5] = e_8, [e_4, e_7] = e_2 \\ \tilde{\mathcal{L}} : \quad & [e_1, e_2] = -2e_2, [e_1, e_3] = -e_3, [e_1, e_4] = e_4, [e_1, e_6] = 2e_6, \\ & [e_1, e_8] = e_8, [e_2, e_6] = e_7, [e_3, e_6] = e_8, [e_4, e_5] = e_8 \\ \text{inv } \mathcal{L} &= (8, 6, 2, 0)(8, 6)(1) \quad 2 \quad [12, 13, 4, 3] \\ \text{inv } \tilde{\mathcal{L}} &= (8, 6, 2, 0)(8, 6)(1) \quad 2 \quad [12, 13, 4, 4]\end{aligned}$$

□

As the following example shows, analyzing the Lie structure of the associated Lie algebras can be very useful.

Example 4. Let us present two 8-dimensional nilpotent Lie algebras as a list of their non-zero commutation relations in \mathbb{Z}_3 -labeled basis $(l_{01}, l_{02}, l_{10}, l_{20}, l_{11}, l_{22}, l_{12}, l_{21})$:

$$\mathcal{L}_{17,9} : \quad \begin{aligned} [l_{01}, l_{10}] &= l_{11}, & [l_{01}, l_{20}] &= l_{21}, & [l_{01}, l_{11}] &= l_{12}, & [l_{01}, l_{22}] &= l_{20}, \\ [l_{02}, l_{10}] &= l_{12}, & [l_{10}, l_{11}] &= l_{21}, & [l_{20}, l_{22}] &= l_{12} \end{aligned}$$

$$\mathcal{L}_{17,12} : \quad \begin{aligned} [l_{01}, l_{10}] &= l_{11}, & [l_{01}, l_{20}] &= l_{21}, & [l_{01}, l_{22}] &= l_{20}, & [l_{02}, l_{10}] &= l_{12}, \\ [l_{02}, l_{22}] &= l_{21}, & [l_{10}, l_{11}] &= l_{21}, & [l_{20}, l_{22}] &= l_{12}. \end{aligned}$$

Because we have

$$\text{inv } \mathcal{L}_{17,9} = \text{inv } \mathcal{L}_{17,12} = \quad (8, 4, 0)(8, 4, 2, 0) (2, 5, 8) \quad 2 \quad [16, 19, 9, 11],$$

the algebras $\mathcal{L}_{17,9}$, $\mathcal{L}_{17,12}$ cannot be distinguished using the set of invariants $\text{inv } \mathcal{L}$. We can advance to the higher level by computing:

$$\begin{aligned} \text{inv } \text{der}_{(1,1,1)} \mathcal{L}_{17,9} &= \text{inv } \text{der}_{(1,1,1)} \mathcal{L}_{17,12} & (16, 15, 6, 0)(16, 15)(0) & \quad 6 & [16, 15, 1, 6] \\ \text{inv } \text{der}_{(0,1,1)} \mathcal{L}_{17,9} &= \text{inv } \text{der}_{(0,1,1)} \mathcal{L}_{17,12} & (19, 15)(19, 15)(0) & \quad 5 & [32, 0, 1, 0] \\ \text{inv } \text{der}_{(1,1,0)} \mathcal{L}_{17,9} &= \text{inv } \text{der}_{(1,1,0)} \mathcal{L}_{17,12} & (9, 0)(9, 0)(9) & \quad 9 & [81, 81, 81, 81]. \end{aligned}$$

We see that the algebras $\mathcal{L}_{17,9}$, $\mathcal{L}_{17,12}$ are still not decidedly non-isomorphic. Surprisingly, the algebras $\mathcal{L}_{17,9}$, $\mathcal{L}_{17,12}$ have very different Lie structure of the intersection of the operator algebras $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}$:

$$\begin{aligned} \text{inv } \text{der}_{(1,1,1)} \mathcal{L}_{17,9} \cap \text{der}_{(0,1,1)} \mathcal{L}_{17,9} &= (11, 6, 0) (11, 6, 0) (6, 11) \quad 7 \quad [43, 67, 31, 31] \\ \text{inv } \text{der}_{(1,1,1)} \mathcal{L}_{17,12} \cap \text{der}_{(0,1,1)} \mathcal{L}_{17,12} &= (11, 4, 0) (11, 4, 0) (7, 11) \quad 7 \quad [57, 78, 50, 50]. \end{aligned}$$

The conclusion $\mathcal{L}_{17,9} \not\cong \mathcal{L}_{17,12}$ follows from Proposition 2.3.1.

2.3.1 Associated Lie Groups

The fact that $\text{der } \mathcal{L}$ is a Lie algebra of the group $\text{Aut } \mathcal{L}$ was stated in Theorem 1.4.4. In order to answer the question whether the associated Lie algebras $\text{der}_{(0,1,1)} \mathcal{L}$ and $\text{der}_{(1,1,0)} \mathcal{L}$ and intersection $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}$ are also Lie algebras of some linear groups, we firstly define the sets:

$$\text{Aut}_{(0,1,1)} \mathcal{L} = \{f \in GL(\mathcal{L}) \mid [fx, fy] = [x, y], \quad \forall x, y \in \mathcal{L}\} \quad (2.31)$$

$$\text{Aut}_{(1,1,0)} \mathcal{L} = \{f \in GL(\mathcal{L}) \mid f[x, y] = [fx, y], \quad \forall x, y \in \mathcal{L}\}. \quad (2.32)$$

Proposition 2.3.3. Let \mathcal{L} be a complex Lie algebra. The sets $\text{Aut}_{(0,1,1)} \mathcal{L}$ and $\text{Aut}_{(1,1,0)} \mathcal{L}$ are subgroups of $GL(\mathcal{L})$.

Proof. Both $\text{Aut}_{(0,1,1)} \mathcal{L}$ and $\text{Aut}_{(1,1,0)} \mathcal{L}$ contain the identity operator and therefore are non-empty. Let $f, g \in \text{Aut}_{(0,1,1)} \mathcal{L}$. Substituting $x = g^{-1}z, y = g^{-1}w$ into the equation $[gx, gy] = [x, y]$ one obtains $[g^{-1}z, g^{-1}w] = [z, w]$, i. e. $g^{-1} \in \text{Aut}_{(0,1,1)} \mathcal{L}$. Then for all $x, y \in \mathcal{L}$

$$[fg^{-1}x, fg^{-1}y] = [g^{-1}x, g^{-1}y] = [x, y],$$

i. e. $fg^{-1} \in \text{Aut}_{(0,1,1)} \mathcal{L}$.

Similarly, let $f, g \in \text{Aut}_{(1,1,0)} \mathcal{L}$. Substituting $x = g^{-1}z$ into the equation $g[x, y] = [gx, y]$ one obtains $g[g^{-1}z, y] = [z, y]$, or equivalently $[g^{-1}z, y] = g^{-1}[z, y]$, i. e. $g^{-1} \in \text{Aut}_{(1,1,0)} \mathcal{L}$. Then for all $x, y \in \mathcal{L}$

$$fg^{-1}[x, y] = f[g^{-1}x, y] = [fg^{-1}x, y].$$

□

Proposition 2.3.4. Let \mathcal{L} be a complex Lie algebra. Then $\text{Aut}_{(0,1,1)} \mathcal{L}$ and $\text{Aut}_{(1,1,0)} \mathcal{L}$ are algebraic groups.

Proof. Let c_{ij}^k denote the structural constants of \mathcal{L} , $\dim \mathcal{L} = n$. Then $f = (f_{ij}) \in \text{Aut}_{(0,1,1)} \mathcal{L}$ if and only if

$$\sum_{p,q=1}^n f_{pi} f_{qj} c_{pq}^k = c_{ij}^k, \quad i, j, k \in \{1, \dots, n\}$$

and also $\tilde{f} = (\tilde{f}_{ij}) \in \text{Aut}_{(1,1,0)} \mathcal{L}$ if and only if

$$\sum_{p=1}^n \tilde{f}_{pi} c_{pj}^k - \tilde{f}_{kp} c_{ij}^p = 0, \quad i, j, k \in \{1, \dots, n\}.$$

□

Lemma 2.3.5. Let \mathcal{L} be a complex Lie algebra and $A \in \text{der}_{(0,1,1)} \mathcal{L}$. Then for all $x, y \in \mathcal{L}$ and $m \in \mathbb{N}$

$$\sum_{i=0}^m \binom{m}{i} [A^i x, A^{m-i} y] = 0. \quad (2.33)$$

Proof. We prove the assertion by induction. For $m = 1$ we obtain $[Ax, y] + [x, Ay] = 0$, i. e. the definition of $\text{der}_{(0,1,1)} \mathcal{L}$. Assuming that (2.33) is valid for m , we proceed to $m + 1$

as follows:

$$\begin{aligned}
\sum_{i=0}^{m+1} \binom{m+1}{i} [A^i x, A^{m+1-i} y] &= [x, A^{m+1} y] + \sum_{i=0}^{m-1} \binom{m+1}{i+1} [A^{i+1} x, A^{m-i} y] + [A^{m+1} x, y] \\
&= [x, A^{m+1} y] + \sum_{i=0}^{m-1} \binom{m}{i+1} [A^{i+1} x, A^{m-i} y] + \\
&\quad + \sum_{i=0}^{m-1} \binom{m}{i} [A^{i+1} x, A^{m-i} y] + [A^{m+1} x, y] \\
&= \sum_{i=0}^m \binom{m}{i} [A^i x, A^{m-i}(Ay)] + \sum_{i=0}^m \binom{m}{i} [A^i(Ax), A^{m-i} y].
\end{aligned}$$

According to the assumption, each of the last two sums is equal to zero. \square

Theorem 2.3.6. Let \mathcal{L} be a complex Lie algebra. Then it holds:

1. $\text{der}_{(0,1,1)} \mathcal{L}$ is the Lie algebra of $\text{Aut}_{(0,1,1)} \mathcal{L}$,
2. $\text{der}_{(1,1,0)} \mathcal{L}$ is the Lie algebra of $\text{Aut}_{(1,1,0)} \mathcal{L}$,
3. $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}$ is the Lie algebra of $\text{Aut}_{(0,1,1)} \mathcal{L} \cap \text{Aut} \mathcal{L}$.

Proof. 1. We prove the equality of the sets

$$\text{der}_{(0,1,1)} \mathcal{L} = \{A \in \text{gl } \mathcal{L} \mid \exp(\mathbb{R}A) \subset \text{Aut}_{(0,1,1)} \mathcal{L}\}$$

in two steps.

\supset : Consider any $A \in \text{gl } \mathcal{L}$ such that $\exp(tA) \in \text{Aut}_{(0,1,1)} \mathcal{L}$, i. e. for all $x, y \in \mathcal{L}$ and all $t \in \mathbb{R}$ it holds:

$$[\exp(tA)x, \exp(tA)y] = [x, y]. \quad (2.34)$$

Rewriting equation (2.34) in the form

$$\left[\frac{1}{t}(\exp(tA) - 1)x, \exp(tA)y \right] + \left[x, \frac{1}{t}(\exp(tA) - 1)y \right] = 0$$

and taking the limit $t \rightarrow 0$ we obtain $[Ax, y] + [x, Ay] = 0$, i. e. $A \in \text{der}_{(0,1,1)} \mathcal{L}$.

\subset : Let $A \in \text{der}_{(0,1,1)} \mathcal{L}$. It is sufficient to prove that for all $x, y \in \mathcal{L}$ the relation

$$[\exp(A)x, \exp(A)y] = [x, y]$$

holds. Then, since also $tA \in \text{der}_{(0,1,1)} \mathcal{L}$ for all $t \in \mathbb{R}$, the relation (2.34) follows. Using (2.33), we calculate

$$\begin{aligned} \left[\sum_{i=0}^{\infty} \frac{1}{i!} A^i x, \sum_{i=0}^{\infty} \frac{1}{i!} A^i y \right] &= \sum_{m=0}^{\infty} \sum_{i=0}^m \left[\frac{1}{i!} A^i x, \frac{1}{(m-i)!} A^{m-i} y \right] \\ &= [x, y] + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i=0}^m \binom{m}{i} [A^i x, A^{m-i} y] = [x, y]. \end{aligned}$$

2. \supset : Consider any $A \in \text{gl } \mathcal{L}$ such that $\exp(tA) \in \text{Aut}_{(1,1,0)} \mathcal{L}$, i. e. for all $x, y \in \mathcal{L}$ and all $t \in \mathbb{R}$ it holds:

$$\exp(tA)[x, y] = [\exp(tA)x, y]. \quad (2.35)$$

Rewriting equation (2.35) in the form

$$\frac{1}{t}(\exp(tA) - 1)[x, y] = \left[\frac{1}{t}(\exp(tA) - 1)x, y \right]$$

and taking the limit $t \rightarrow 0$ we obtain $A[x, y] = [Ax, y]$, i. e. $A \in \text{der}_{(1,1,0)} \mathcal{L}$.

\subset : Let $A \in \text{der}_{(1,1,0)} \mathcal{L}$. It is sufficient to prove that for all $x, y \in \mathcal{L}$

$$\exp(A)[x, y] = [\exp(A)x, y].$$

Then, since also $tA \in \text{der}_{(1,1,0)} \mathcal{L}$ for all $t \in \mathbb{R}$, the relation (2.35) follows. Since for all $m \in \mathbb{N}$ is obviously the relation $A^m[x, y] = [A^m x, y]$ satisfied, we have:

$$\sum_{m=0}^{\infty} \frac{1}{m!} A^m [x, y] = \sum_{m=0}^{\infty} \frac{1}{m!} [A^m x, y] = \left[\sum_{m=0}^{\infty} \frac{1}{m!} A^m x, y \right].$$

3. The statement follows directly from case 1., Theorem 1.4.4 and Proposition 1.4.2. \square

2.4 Invariant Function ψ of Low-dimensional Algebras

Since among three-dimensional Lie algebras infinite continuum appears already, it is clear that the finite set $\text{inv } \mathcal{L}$ of certain dimensions, though useful, can never completely characterize Lie algebras of dimension higher or equal to 3. On the contrary, it turns out that the invariant function ψ alone(!) forms a complete set of invariant(s) for 3-dimensional Lie algebras. We use the notation for 3-dimensional Lie algebras as in [29] and begin with the following lemma, which states that the invariant function ψ provides the classification of infinitely many Lie algebras in the continuum $\mathfrak{g}_{3,4}(a)$.

Lemma 2.4.1. Let $\mathfrak{g}_{3,4}(a)$ be a three-dimensional Lie algebra with non-zero brackets $[e_1, e_3] = e_1$, $[e_2, e_3] = ae_2$, where $a \in \mathbb{C}$ and $a \neq 0, \pm 1$. If $\psi_{\mathfrak{g}_{3,4}(a)} = \psi_{\mathfrak{g}_{3,4}(a')}$ then $\mathfrak{g}_{3,4}(a) \cong \mathfrak{g}_{3,4}(a')$.

Proof. From the explicit form of matrices listed in Appendix A we instantly see that the invariant function ψ has the following form

α	1	a	$\frac{1}{a}$	
$[\psi_{\mathfrak{g}_{3,4}(a)}](\alpha)$	4	4	4	3

A blank space in the table of the function ψ denotes a general complex number, different from all previously listed values, i. e. it holds: $[\psi_{\mathfrak{g}_{3,4}(a)}](\alpha) = 3$, $\alpha \neq 1, a, 1/a$. Let us consider some $\mathfrak{g}_{3,4}(a')$, $a' \neq 0, \pm 1$. If $\psi_{\mathfrak{g}_{3,4}(a)} = \psi_{\mathfrak{g}_{3,4}(a')}$ then we need $a = a'$ or $a = 1/a'$ – otherwise $\psi_{\mathfrak{g}_{3,4}(a)} \neq \psi_{\mathfrak{g}_{3,4}(a')}$. The relation

$$\mathfrak{g}_{3,4}(a) \cong \mathfrak{g}_{3,4}(1/a), \quad (2.36)$$

can be verified directly. □

Example 5. The invariant function ψ^0 has for $\mathfrak{g}_{3,1}$ the value $\psi^0_{\mathfrak{g}_{3,1}}(\alpha) = 3$ and for the remaining algebras $\mathfrak{g}_{3,i}$, $i = 2, 3, 4$ and $\mathfrak{sl}(2, \mathbb{C})$ it holds:

$$\psi^0_{\mathfrak{g}_{3,i}}(\alpha) = \begin{cases} 1, & \alpha = 1 \\ 0, & \alpha \neq 1. \end{cases}$$

Theorem 2.4.2 (Classification of three-dimensional complex Lie algebras).

Two three-dimensional complex Lie algebras \mathcal{L} , $\tilde{\mathcal{L}}$ are isomorphic if and only if $\psi_{\mathcal{L}} = \psi_{\tilde{\mathcal{L}}}$.

Proof. \Rightarrow : See Corollary 2.1.5.

\Leftarrow : According to Lemma 2.4.1 and observing the tables in Appendix B.1, we conclude that all tables of the invariant function ψ of non-isomorphic three-dimensional complex Lie algebras are mutually different. □

Example 6. In Example 3 explicit matrices of (α, β, γ) -derivations of four-dimensional one-parametric Lie algebra $\mathfrak{g}_{4,2}(a)$, $a \neq 0, \pm 1, -2$, were presented. Thus we see that the invariant functions ψ and ψ^0 have the following form

α	1	a	$\frac{1}{a}$	
$\psi_{\mathfrak{g}_{4,2}(a)}(\alpha)$	6	5	5	4

α	1	
$\psi^0_{\mathfrak{g}_{4,2}(a)}(\alpha)$	1	0

Let us consider some $\mathfrak{g}_{4,2}(a')$, $a' \neq 0, \pm 1, -2$. If $\psi \mathfrak{g}_{4,2}(a) = \psi \mathfrak{g}_{4,2}(a')$ then we need $a = a'$ or $a = 1/a'$ – otherwise $\psi \mathfrak{g}_{4,2}(a) \neq \psi \mathfrak{g}_{4,2}(a')$. However an isomorphism relation similar to (2.36) does not hold. The invariant function ψ thus does not distinguish between pairs $\mathfrak{g}_{4,2}(a)$ and $\mathfrak{g}_{4,2}(1/a)$ and consequently does not provide a complete classification of Lie algebras in the continuum $\mathfrak{g}_{4,2}(a)$. In the next Chapter, we use additional invariant functions which, together with ψ , allow the classification of $\mathfrak{g}_{4,2}(a)$ as well as all four–dimensional Lie algebras.

Chapter 3

Twisted Cocycles of Lie Algebras

3.1 Two-dimensional Twisted Cocycles of the Adjoint Representation

Recall that an f -cocycle of a Lie algebra \mathcal{L} is defined as a q -linear operator z satisfying the relation $dz = 0$, where the map d is defined by (1.27). We generalize this definition analogously to (α, β, γ) -derivations. The content of this chapter has not been previously published.

Let \mathcal{L} be an arbitrary complex Lie algebra, (V, f) its representation and $\kappa = (\kappa_{ij})$ a $(q+1) \times (q+1)$ complex symmetric matrix. We call $c \in C^q(\mathcal{L}, V)$, $q \in \mathbb{N}$ for which

$$0 = \sum_{i=1}^{q+1} (-1)^{i+1} \kappa_{ii} f(x_i) c(x_1, \dots, \hat{x}_i, \dots, x_{q+1}) + \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} \kappa_{ij} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \quad (3.1)$$

a κ -twisted cocycle or shortly κ -cocycle of dimension q corresponding to (V, f) ; the set of all κ -cocycles of dimension q is denoted by $Z^q(\mathcal{L}, f, \kappa)$. It is clear that $Z^q(\mathcal{L}, f, \kappa)$ is a linear subspace of $C^q(\mathcal{L}, V)$. Similarly to (2.5) we observe that for any $\varepsilon \in \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{N}$ it holds:

$$Z^q(\mathcal{L}, f, \kappa) = Z^q(\mathcal{L}, f, \varepsilon \kappa). \quad (3.2)$$

In Chapter 2 we analyzed in detail (α, β, γ) -derivations; they are now included in the definition of twisted cocycles. Considering the adjoint representation, we immediately have the generalization of Proposition 1.3.2:

$$Z^1(\mathcal{L}, \text{ad}_{\mathcal{L}}, \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix}) = \text{der}_{(\alpha, \beta, \gamma)} \mathcal{L}. \quad (3.3)$$

The most logical next step is to set $q = 2$ and investigate in detail the space $Z^2(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa)$ – we devote this chapter to this goal. For this purpose it may be more convenient to use different notation, analogous to that of derivations, defined by

$$\text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} = Z^2 \left(\mathcal{L}, \text{ad}_{\mathcal{L}}, \begin{pmatrix} \beta_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \beta_3 & \alpha_1 \\ \alpha_3 & \alpha_1 & \beta_2 \end{pmatrix} \right), \quad (3.4)$$

i. e. in the space $\text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$ are such $B \in C^2(\mathcal{L}, \mathcal{L})$ which for all $x, y, z \in \mathcal{L}$ satisfy

$$\begin{aligned} 0 &= \alpha_1 B(x, [y, z]) + \alpha_2 B(z, [x, y]) + \alpha_3 B(y, [z, x]) \\ &+ \beta_1 [x, B(y, z)] + \beta_2 [z, B(x, y)] + \beta_3 [y, B(z, x)]. \end{aligned} \quad (3.5)$$

Six permutations of the variables $x, y, z \in \mathcal{L}$ in the defining equation (3.5) give

Lemma 3.1.1. Let \mathcal{L} be a complex Lie algebra. Then for any $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ are all the following six spaces equal:

1. $\text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$
2. $\text{coc}_{(\alpha_3, \alpha_1, \alpha_2, \beta_3, \beta_1, \beta_2)} \mathcal{L}$
3. $\text{coc}_{(\alpha_2, \alpha_3, \alpha_1, \beta_2, \beta_3, \beta_1)} \mathcal{L}$
4. $\text{coc}_{(\alpha_1, \alpha_3, \alpha_2, \beta_1, \beta_3, \beta_2)} \mathcal{L}$
5. $\text{coc}_{(\alpha_2, \alpha_1, \alpha_3, \beta_2, \beta_1, \beta_3)} \mathcal{L}$
6. $\text{coc}_{(\alpha_3, \alpha_2, \alpha_1, \beta_3, \beta_2, \beta_1)} \mathcal{L}$

Lemma 3.1.2. Let \mathcal{L} be a complex Lie algebra. Then for any $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ is the space $\text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$ equal to all of the following:

1. $\text{coc}_{(\alpha_1 + \alpha_3, \alpha_2 + \alpha_1, \alpha_3 + \alpha_2, \beta_1 + \beta_3, \beta_2 + \beta_1, \beta_3 + \beta_2)} \mathcal{L} \cap \text{coc}_{(\alpha_1 - \alpha_3, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \beta_1 - \beta_3, \beta_2 - \beta_1, \beta_3 - \beta_2)} \mathcal{L}$
2. $\text{coc}_{(0, \alpha_2 - \alpha_3, \alpha_3 - \alpha_2, 0, \beta_2 - \beta_3, \beta_3 - \beta_2)} \mathcal{L} \cap \text{coc}_{(2\alpha_1, \alpha_2 + \alpha_3, \alpha_3 + \alpha_2, 2\beta_1, \beta_2 + \beta_3, \beta_3 + \beta_2)} \mathcal{L}$
3. $\text{coc}_{(0, \alpha_1 - \alpha_2, \alpha_2 - \alpha_1, 0, \beta_1 - \beta_2, \beta_2 - \beta_1)} \mathcal{L} \cap \text{coc}_{(2\alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_1, 2\beta_3, \beta_1 + \beta_2, \beta_2 + \beta_1)} \mathcal{L}$
4. $\text{coc}_{(0, \alpha_3 - \alpha_1, \alpha_1 - \alpha_3, 0, \beta_3 - \beta_1, \beta_1 - \beta_3)} \mathcal{L} \cap \text{coc}_{(2\alpha_2, \alpha_3 + \alpha_1, \alpha_1 + \alpha_3, 2\beta_2, \beta_3 + \beta_1, \beta_1 + \beta_3)} \mathcal{L}$

Proof. Suppose $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ are given. We demonstrate the proof on the case 1. – the proof of the other cases is analogous. Let $B \in \text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$; then for arbitrary $x, y, z \in \mathcal{L}$ we have

$$\begin{aligned} 0 &= \alpha_1 B(x, [y, z]) + \alpha_2 B(z, [x, y]) + \alpha_3 B(y, [z, x]) \\ &\quad + \beta_1 [x, B(y, z)] + \beta_2 [z, B(x, y)] + \beta_3 [y, B(z, x)] \end{aligned} \quad (3.6)$$

$$\begin{aligned} 0 &= \alpha_1 B(z, [x, y]) + \alpha_2 B(y, [z, x]) + \alpha_3 B(x, [y, z]) \\ &\quad + \beta_1 [z, B(x, y)] + \beta_2 [y, B(z, x)] + \beta_3 [x, B(y, z)]. \end{aligned} \quad (3.7)$$

By adding and subtracting equations (3.6) and (3.7) we obtain

$$\begin{aligned} 0 &= (\alpha_1 + \alpha_3) B(x, [y, z]) + (\alpha_2 + \alpha_1) B(z, [x, y]) + (\alpha_3 + \alpha_2) B(y, [z, x]) \\ &\quad + (\beta_1 + \beta_3) [x, B(y, z)] + (\beta_2 + \beta_1) [z, B(x, y)] + (\beta_3 + \beta_2) [y, B(z, x)] \end{aligned} \quad (3.8)$$

$$\begin{aligned} 0 &= (\alpha_1 - \alpha_3) B(x, [y, z]) + (\alpha_2 - \alpha_1) B(z, [x, y]) + (\alpha_3 - \alpha_2) B(y, [z, x]) \\ &\quad + (\beta_1 - \beta_3) [x, B(y, z)] + (\beta_2 - \beta_1) [z, B(x, y)] + (\beta_3 - \beta_2) [y, B(z, x)] \end{aligned} \quad (3.9)$$

and thus $\text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$ is the subset of the intersection 1. Similarly, starting with equations (3.8), (3.9) we obtain equations (3.6), (3.7) and the remaining inclusion is proven. \square

Further, we proceed to formulate an analogous theorem to Theorem 2.1.4; the six original parameters are reduced to four.

Theorem 3.1.3. Let \mathcal{L} be a Lie algebra. Then for any $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that the subspace $\text{coc}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} \subset C^2(\mathcal{L}, \mathcal{L})$ is equal to some of the following sixteen subspaces:

1. $\text{coc}_{(\alpha, 0, 0, \beta, 0, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, 0, 0, \beta, 1, -1)} \mathcal{L}$; $\text{coc}_{(\alpha, 1, -1, \beta, 0, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, \beta, -\beta, \gamma, 1, -1)} \mathcal{L}$
2. $\text{coc}_{(\alpha, 0, 0, \beta, 1, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, 0, 0, \beta, 1, 1)} \mathcal{L}$; $\text{coc}_{(\alpha, \beta, -\beta, \gamma, 1, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, 1, -1, \beta, 1, 1)} \mathcal{L}$
3. $\text{coc}_{(\alpha, 1, 0, \beta, 0, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, 1, 1, \beta, 0, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, 1, 0, \beta, \gamma, -\gamma)} \mathcal{L}$; $\text{coc}_{(\alpha, 1, 1, \beta, 1, -1)} \mathcal{L}$
4. $\text{coc}_{(\alpha, \beta, \gamma, \delta, 1, 0)} \mathcal{L}$; $\text{coc}_{(\alpha, \beta+1, \beta-1, \gamma, 1, 1)} \mathcal{L}$; $\text{coc}_{(\alpha, 1, 1, \beta, \gamma+1, \gamma-1)} \mathcal{L}$; $\text{coc}_{(\alpha, \beta, \beta, \gamma, 1, 1)} \mathcal{L}$

Proof. 1. Suppose $\alpha_2 + \alpha_3 = 0$ and $\beta_2 + \beta_3 = 0$. Then the following four cases are possible:

(a) $\alpha_2 = -\alpha_3 = 0$ and $\beta_2 = -\beta_3 = 0$. In this case we have

$$\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} = \text{COC}_{(\alpha_1, 0, 0, \beta_1, 0, 0)} \mathcal{L}.$$

(b) $\alpha_2 = -\alpha_3 = 0$ and $\beta_2 = -\beta_3 \neq 0$. In this case we have:

$$\begin{aligned} \text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} &= \text{COC}_{(\alpha_1, 0, 0, \beta_1, \beta_2, \beta_3)} \mathcal{L} \\ &= \text{COC}_{(0, 0, 0, 0, \beta_2 - \beta_3, \beta_3 - \beta_2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(0, 0, 0, 0, 2, -2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(\alpha_1, 0, 0, \beta_1, 1, -1)} \mathcal{L} \end{aligned}$$

(c) $\alpha_2 = -\alpha_3 \neq 0$ and $\beta_2 = -\beta_3 = 0$. In this case we have:

$$\begin{aligned} \text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} &= \text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(0, \alpha_2 - \alpha_3, \alpha_3 - \alpha_2, 0, 0, 0)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(0, 2, -2, 0, 0, 0)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(\alpha_1, 1, -1, \beta_1, 0, 0)} \mathcal{L} \end{aligned}$$

(d) $\alpha_2 = -\alpha_3 \neq 0$ and $\beta_2 = -\beta_3 \neq 0$. In this case we have:

$$\begin{aligned} \text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} &= \text{COC}_{(0, \alpha_2 - \alpha_3, \alpha_3 - \alpha_2, 0, \beta_2 - \beta_3, \beta_3 - \beta_2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(0, 2 \frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3}, -2 \frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3}, 0, 2, -2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 0, 0)} \mathcal{L} \\ &= \text{COC}_{(\alpha_1, \frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3}, -\frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3}, \beta_1, 1, -1)} \mathcal{L} \end{aligned}$$

2. Suppose $\alpha_2 + \alpha_3 = 0$ and $\beta_2 + \beta_3 \neq 0$. Then the following four cases are possible:

(a) $\alpha_2 = -\alpha_3 = 0$ and $\beta_2 - \beta_3 \neq 0$. In this case we have:

$$\begin{aligned} \text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} &= \text{COC}_{(\alpha_1, 0, 0, \beta_1, \beta_2, \beta_3)} \mathcal{L} \\ &= \text{COC}_{(0, 0, 0, 0, \beta_2 - \beta_3, \beta_3 - \beta_2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, \beta_2 + \beta_3, \beta_3 + \beta_2)} \mathcal{L} \\ &= \text{COC}_{(0, 0, 0, 0, 1, -1)} \mathcal{L} \cap \text{COC}_{(2 \frac{\alpha_1}{\beta_2 + \beta_3}, 0, 0, 2 \frac{\beta_1}{\beta_2 + \beta_3}, 1, 1)} \mathcal{L} \\ &= \text{COC}_{(\frac{\alpha_1}{\beta_2 + \beta_3}, 0, 0, \frac{\beta_1}{\beta_2 + \beta_3}, 1, 0)} \mathcal{L} \end{aligned}$$

(b) $\alpha_2 = -\alpha_3 = 0$ and $\beta_2 = \beta_3 \neq 0$. In this case we have

$$\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} = \text{COC}_{(\alpha_1, 0, 0, \beta_1, \beta_2, \beta_2)} \mathcal{L} = \text{COC}_{(\frac{\alpha_1}{\beta_2}, 0, 0, \frac{\beta_1}{\beta_2}, 1, 1)} \mathcal{L}.$$

(c) $\alpha_2 = -\alpha_3 \neq 0$ and $\beta_2 - \beta_3 \neq 0$. In this case we have:

$$\begin{aligned}
\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} &= \text{COC}_{(\alpha_1, \alpha_2, -\alpha_2, \beta_1, \beta_2, \beta_3)} \mathcal{L} \\
&= \text{COC}_{(0, 2\alpha_2, -2\alpha_2, 0, \beta_2 - \beta_3, \beta_3 - \beta_2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, \beta_2 + \beta_3, \beta_3 + \beta_2)} \mathcal{L} \\
&= \text{COC}_{(0, 2\frac{\alpha_2}{\beta_2 - \beta_3}, -2\frac{\alpha_2}{\beta_2 - \beta_3}, 0, 1, -1)} \mathcal{L} \cap \text{COC}_{(2\frac{\alpha_1}{\beta_2 + \beta_3}, 0, 0, 2\frac{\beta_1}{\beta_2 + \beta_3}, 1, 1)} \mathcal{L} \\
&= \text{COC}_{(\frac{\alpha_1}{\beta_2 + \beta_3}, \frac{\alpha_2}{\beta_2 - \beta_3}, -\frac{\alpha_2}{\beta_2 - \beta_3}, \frac{\beta_1}{\beta_2 + \beta_3}, 1, 0)} \mathcal{L}
\end{aligned}$$

(d) $\alpha_2 = -\alpha_3 \neq 0$ and $\beta_2 = \beta_3 \neq 0$. In this case we have:

$$\begin{aligned}
\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L} &= \text{COC}_{(\alpha_1, \alpha_2, -\alpha_2, \beta_1, \beta_2, \beta_2)} \mathcal{L} \\
&= \text{COC}_{(0, 2\alpha_2, -2\alpha_2, 0, 0, 0)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, 0, 0, 2\beta_1, 2\beta_2, 2\beta_2)} \mathcal{L} \\
&= \text{COC}_{(0, 2, -2, 0, 0, 0)} \mathcal{L} \cap \text{COC}_{(2\frac{\alpha_1}{\beta_2}, 0, 0, 2\frac{\beta_1}{\beta_2}, 2, 2)} \mathcal{L} \\
&= \text{COC}_{(\frac{\alpha_1}{\beta_2}, 1, -1, \frac{\beta_1}{\beta_2}, 1, 1)} \mathcal{L}
\end{aligned}$$

3. Suppose $\alpha_2 + \alpha_3 \neq 0$ and $\beta_2 + \beta_3 = 0$. This case is a complete analogue of the previous one.

4. Suppose $\alpha_2 + \alpha_3 \neq 0$ and $\beta_2 + \beta_3 \neq 0$. Then the following four cases are possible:

(a) $\alpha_2 - \alpha_3 \neq 0$ and $\beta_2 - \beta_3 \neq 0$. The space $\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$ is in this case equal to:

$$\begin{aligned}
&\text{COC}_{(0, \alpha_2 - \alpha_3, \alpha_3 - \alpha_2, 0, \beta_2 - \beta_3, \beta_3 - \beta_2)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, \alpha_2 + \alpha_3, \alpha_3 + \alpha_2, 2\beta_1, \beta_2 + \beta_3, \beta_3 + \beta_2)} \mathcal{L} \\
&= \text{COC}_{(0, \frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3}, -\frac{\alpha_2 - \alpha_3}{\beta_2 - \beta_3}, 0, 1, -1)} \mathcal{L} \cap \text{COC}_{(2\frac{\alpha_1}{\beta_2 + \beta_3}, \frac{\alpha_2 + \alpha_3}{\beta_2 + \beta_3}, \frac{\alpha_2 + \alpha_3}{\beta_2 + \beta_3}, 2\frac{\beta_1}{\beta_2 + \beta_3}, 1, 1)} \mathcal{L} \\
&= \text{COC}_{(\frac{\alpha_1}{\beta_2 + \beta_3}, \frac{\alpha_2 + \alpha_3}{2(\beta_2 + \beta_3)} + \frac{\alpha_2 - \alpha_3}{2(\beta_2 - \beta_3)}, \frac{\alpha_2 + \alpha_3}{2(\beta_2 + \beta_3)} - \frac{\alpha_2 - \alpha_3}{2(\beta_2 - \beta_3)}, \frac{\beta_1}{\beta_2 + \beta_3}, 1, 0)} \mathcal{L}
\end{aligned}$$

(b) $\alpha_2 - \alpha_3 \neq 0$ and $\beta_2 = \beta_3 \neq 0$. The space $\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$ is in this case equal to:

$$\begin{aligned}
&\text{COC}_{(0, \alpha_2 - \alpha_3, \alpha_3 - \alpha_2, 0, 0, 0)} \mathcal{L} \cap \text{COC}_{(2\alpha_1, \alpha_2 + \alpha_3, \alpha_3 + \alpha_2, 2\beta_1, 2\beta_2, 2\beta_2)} \mathcal{L} \\
&= \text{COC}_{(0, 2, -2, 0, 0, 0)} \mathcal{L} \cap \text{COC}_{(2\frac{\alpha_1}{\beta_2}, \frac{\alpha_2 + \alpha_3}{\beta_2}, \frac{\alpha_2 + \alpha_3}{\beta_2}, 2\frac{\beta_1}{\beta_2}, 2, 2)} \mathcal{L} \\
&= \text{COC}_{(\frac{\alpha_1}{\beta_2}, \frac{\alpha_2 + \alpha_3}{2\beta_2} + 1, \frac{\alpha_2 + \alpha_3}{2\beta_2} - 1, \frac{\beta_1}{\beta_2}, 1, 1)} \mathcal{L}
\end{aligned}$$

(c) $\alpha_2 = \alpha_3 \neq 0$ and $\beta_2 - \beta_3 \neq 0$. The space $\text{COC}_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)} \mathcal{L}$ is in this case

equal to:

$$\begin{aligned}
& \text{coc}_{(0,0,0,0,\beta_2-\beta_3,\beta_3-\beta_2)} \mathcal{L} \cap \text{coc}_{(2\alpha_1,2\alpha_2,2\alpha_2,2\beta_1,\beta_2+\beta_3,\beta_3+\beta_2)} \mathcal{L} \\
&= \text{coc}_{(0,0,0,0,2,-2)} \mathcal{L} \cap \text{coc}_{(2\frac{\alpha_1}{\alpha_2},2,2,2\frac{\beta_1}{\alpha_2},\frac{\beta_2+\beta_3}{\alpha_2},\frac{\beta_2+\beta_3}{\alpha_2})} \mathcal{L} \\
&= \text{coc}_{(\frac{\alpha_1}{\alpha_2},1,1,\frac{\beta_1}{\alpha_2},\frac{\beta_2+\beta_3}{2\alpha_2}+1,\frac{\beta_2+\beta_3}{2\alpha_2}-1)} \mathcal{L}
\end{aligned}$$

(d) $\alpha_2 = \alpha_3 \neq 0$ and $\beta_2 = \beta_3 \neq 0$. In this case we have

$$\text{coc}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)} \mathcal{L} = \text{coc}_{(\frac{\alpha_1}{\beta_2},\frac{\alpha_2}{\beta_2},\frac{\alpha_2}{\beta_2},\frac{\beta_1}{\beta_2},1,1)} \mathcal{L}.$$

□

3.1.1 Twisted Cocycles of Low-dimensional Lie Algebras

It may be more convenient, sometimes, to use different distribution of the cocycle spaces $\text{coc}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)} \mathcal{L}$ than in Theorem 3.1.3. Henceforth, we investigate mainly the space $\text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathcal{L}$ which for $\lambda \neq 0$ fits in the class $\text{coc}_{(\alpha,\beta,\beta,\gamma,1,1)} \mathcal{L}$, with $\alpha = \beta = 1/\lambda$, $\gamma = 1$. For $\lambda = 0$, the space $\text{coc}_{(1,1,1,0,0,0)} \mathcal{L}$ is a special case of the space $\text{coc}_{(\alpha,1,1,\beta,0,0)} \mathcal{L}$, with $\alpha = 1, \beta = 0$. We also put $\alpha = 0, \beta = 1, \gamma = \lambda$ into $\text{coc}_{(\alpha,\beta,\beta,\gamma,1,1)} \mathcal{L}$ and investigate the space $\text{coc}_{(0,1,1,\lambda,1,1)} \mathcal{L}$.

Example 7. In Lemma 2.4.1 the Lie algebra $\mathfrak{g}_{3,4}(a)$, $a \neq 0, \pm 1$ was introduced. We present the explicit form of the spaces $\text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathfrak{g}_{3,4}(a)$ and $\text{coc}_{(0,1,1,\lambda,1,1)} \mathfrak{g}_{3,4}(a)$. Let us define six $\mathfrak{g}_{3,4}(a)$ -cochains $t_1, \dots, t_4, t_5^\lambda, t_6^\lambda$, $\lambda \in \mathbb{C}$ by listing their non-zero commutation relations:

$$t_1 : t_1(e_1, e_3) = e_1$$

$$t_2 : t_2(e_1, e_3) = e_2$$

$$t_3 : t_3(e_2, e_3) = e_1$$

$$t_4 : t_4(e_2, e_3) = e_2$$

$$t_5^\lambda : t_5^\lambda(e_1, e_2) = \lambda a e_2, t_5^\lambda(e_1, e_3) = (-\lambda a + a + 1)e_3$$

$$t_6^\lambda : t_6^\lambda(e_1, e_2) = -\lambda e_1, t_6^\lambda(e_2, e_3) = (-\lambda + a + 1)e_3.$$

Then we have:

$$\cdot \text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathfrak{g}_{3,4}(a) = \text{span}_{\mathbb{C}}\{t_1, \dots, t_4, t_5^\lambda, t_6^\lambda\}$$

- $\text{coc}_{(0,1,1,\lambda,1,1)} \mathfrak{g}_{3,4}(a) = \{0\}_{\lambda \neq 2, 1+a, 1+\frac{1}{a}}$
- $\text{coc}_{(0,1,1,2,1,1)} \mathfrak{g}_{3,4}(a) = \text{span}_{\mathbb{C}}\{t_1, t_4\}$
- $\text{coc}_{(0,1,1,1+a,1,1)} \mathfrak{g}_{3,4}(a) = \text{span}_{\mathbb{C}}\{t_3\}$
- $\text{coc}_{(0,1,1,1+\frac{1}{a},1,1)} \mathfrak{g}_{3,4}(a) = \text{span}_{\mathbb{C}}\{t_2\}$.

Example 8. In Examples 3 and 6 the explicit matrices of (α, β, γ) -derivations and the corresponding invariant functions of $\mathfrak{g}_{4,2}(a)$, $a \neq 0, \pm 1, -2$ were presented. In this example, we calculate the explicit form of the spaces $\text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathfrak{g}_{4,2}(a)$ and $\text{coc}_{(0,1,1,\lambda,1,1)} \mathfrak{g}_{4,2}(a)$. We first define the following nine $\mathfrak{g}_{4,2}(a)$ -cochains b_1, \dots, b_9 by listing their non-zero commutation relations:

$$b_1 : b_1(e_1, e_4) = e_1$$

$$b_2 : b_2(e_1, e_4) = e_2$$

$$b_3 : b_3(e_1, e_4) = e_3$$

$$b_4 : b_4(e_2, e_4) = e_1$$

$$b_5 : b_5(e_2, e_4) = e_2$$

$$b_6 : b_6(e_2, e_4) = e_3$$

$$b_7 : b_7(e_3, e_4) = e_1$$

$$b_8 : b_8(e_3, e_4) = e_2$$

$$b_9 : b_9(e_3, e_4) = e_3,$$

and second three more $\mathfrak{g}_{4,2}(a)$ -cochains b'_1, \dots, b'_3 :

$$b'_1 : b'_1(e_1, e_3) = 2ae_1, b'_1(e_2, e_3) = (-1+a)e_3, b'_1(e_3, e_4) = (-1+a)e_4$$

$$b'_2 : b'_2(e_1, e_2) = (1+a)e_2, b'_2(e_1, e_3) = e_3$$

$$b'_3 : b'_3(e_2, e_3) = e_1.$$

Moreover, for $\lambda \in \mathbb{C}$, we define λ -parametric sets of $\mathfrak{g}_{4,2}(a)$ -cochains:

$$\begin{aligned}
f_1^\lambda : f_1^\lambda(e_1, e_2) &= \lambda a(\lambda a - a - 1)(\lambda - 2)^2 e_1, \quad f_1^\lambda(e_1, e_3) = \lambda a(\lambda - 2)^2 e_1, \\
f_1^\lambda(e_2, e_3) &= 2\lambda(\lambda a - a - 1)^2 e_2, \\
f_1^\lambda(e_2, e_3) &= \lambda(\lambda - 2)(\lambda a - a - 1)^2 e_3, \\
f_1^\lambda(e_2, e_4) &= (\lambda - 2)^2(\lambda a - a - 1)^2 e_3
\end{aligned}$$

$$\begin{aligned}
f_2^\lambda : f_2^\lambda(e_1, e_2) &= \lambda(-\lambda + a + 1)e_2, \quad f_2^\lambda(e_1, e_3) = \lambda a e_2, \\
f_2^\lambda(e_1, e_3) &= \lambda(-\lambda + a + 1)e_3, \quad f_2^\lambda(e_1, e_4) = (-\lambda + a + 1)^2 e_4
\end{aligned}$$

$$\begin{aligned}
f_3^\lambda : f_3^\lambda(e_1, e_2) &= -\lambda a(\lambda - 2)^2 e_1, \quad f_3^\lambda(e_2, e_3) = \lambda(2\lambda a - 2a - \lambda)e_2, \\
f_3^\lambda(e_2, e_3) &= \lambda(\lambda - 2)(-\lambda a + a + 1)e_3, \quad f_3^\lambda(e_2, e_4) = \lambda(\lambda - 2)^2(-\lambda a + a + 1)e_4, \\
f_3^\lambda(e_2, e_3) &= (\lambda - 2)^2 e_4
\end{aligned}$$

Then it holds:

- $\text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_1, \dots, b_9, f_1^\lambda, f_2^\lambda, f_3^\lambda\}_{\lambda \neq 2, a+1, 2/a}$
- $\text{coc}_{(1,1,1,2,2,2)} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_1, \dots, b_9, f_1^2, f_2^2, b'_1\}$
- $\text{coc}_{(1,1,1,a+1,a+1,a+1)} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_1, \dots, b_9, f_1^{a+1}, f_2^{a+1}, f_3^{a+1}, b'_2\}$
- $\text{coc}_{(1,1,1,\frac{2}{a},\frac{2}{a},\frac{2}{a})} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_1, \dots, b_9, f_1^{2/a}, f_2^{2/a}, f_3^{2/a}, b'_3\}$.
- $\text{coc}_{(0,1,1,\lambda,1,1)} \mathfrak{g}_{4,2}(a) = \{0\}_{\lambda \neq 2, 1+a, 1+\frac{1}{a}}$
- $\text{coc}_{(0,1,1,2,1,1)} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_1, b_8, b_5 + b_9\}$
- $\text{coc}_{(0,1,1,1+a,1,1)} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_2\}$
- $\text{coc}_{(0,1,1,1+\frac{1}{a},1,1)} \mathfrak{g}_{4,2}(a) = \text{span}_{\mathbb{C}}\{b_7\}$

3.2 Invariant Functions

We directly generalize Theorem 2.1.1:

Theorem 3.2.1. Let $g : \mathcal{L} \rightarrow \tilde{\mathcal{L}}$ be an isomorphism of Lie algebras \mathcal{L} and $\tilde{\mathcal{L}}$. Then the mapping $\varrho : C^q(\mathcal{L}, \mathcal{L}) \rightarrow C^q(\tilde{\mathcal{L}}, \tilde{\mathcal{L}})$, $q \in \mathbb{N}$ defined for all $c \in C^q(\mathcal{L}, \mathcal{L})$ and all $x_1, \dots, x_q \in \tilde{\mathcal{L}}$ by

$$(\varrho c)(x_1, \dots, x_q) = gc(g^{-1}x_1, \dots, g^{-1}x_q)$$

is an isomorphism of vector spaces $C^q(\mathcal{L}, \mathcal{L})$ and $C^q(\tilde{\mathcal{L}}, \tilde{\mathcal{L}})$. For any complex symmetric $(q+1)$ -square matrix κ

$$\varrho(Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa)) = Z^q(\tilde{\mathcal{L}}, \text{ad}_{\tilde{\mathcal{L}}}, \kappa).$$

Proof. Suppose we have $g : \mathcal{L} \rightarrow \tilde{\mathcal{L}}$ such that for all $x, y \in \tilde{\mathcal{L}}$

$$[x, y]_{\tilde{\mathcal{L}}} = g[g^{-1}x, g^{-1}y]_{\mathcal{L}}$$

holds. It is clear that the map $\varrho : C^q(\mathcal{L}, \mathcal{L}) \rightarrow C^q(\tilde{\mathcal{L}}, \tilde{\mathcal{L}})$, $q \in \mathbb{N}$ is linear and bijective, i. e. it is an isomorphism of these vector spaces. By putting $f = \text{ad}_{\mathcal{L}}$ and rewriting definition (3.1) we have for $c \in Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa)$ and all $x_1, \dots, x_q \in \tilde{\mathcal{L}}$

$$\begin{aligned} 0 &= \sum_{i=1}^{q+1} (-1)^{i+1} \kappa_{ii} [g^{-1}x_i, c(g^{-1}x_1, \dots, \widehat{g^{-1}x_i}, \dots, g^{-1}x_{q+1})]_{\mathcal{L}} + \\ &+ \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} \kappa_{ij} c([g^{-1}x_i, g^{-1}x_j]_{\mathcal{L}}, g^{-1}x_1, \dots, \widehat{g^{-1}x_i}, \dots, \widehat{g^{-1}x_j}, \dots, g^{-1}x_{q+1}) \end{aligned}$$

Applying the mapping g on this equation and taking into account that $\kappa_{ij} \in \mathbb{C}$ one has

$$\begin{aligned} 0 &= \sum_{i=1}^{q+1} (-1)^{i+1} \kappa_{ii} [x_i, (\varrho c)(x_1, \dots, \hat{x}_i, \dots, x_{q+1})]_{\tilde{\mathcal{L}}} + \\ &+ \sum_{\substack{i,j=1 \\ i < j}}^{q+1} (-1)^{i+j} \kappa_{ij} (\varrho c)([x_i, x_j]_{\tilde{\mathcal{L}}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \end{aligned}$$

i. e. $\varrho c \in Z^q(\tilde{\mathcal{L}}, \text{ad}_{\tilde{\mathcal{L}}}, \kappa)$. □

Corollary 3.2.2. For any $q \in \mathbb{N}$ and any complex symmetric $(q+1)$ -square matrix κ is the dimension of the vector space $Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa)$ an invariant characteristic of Lie algebras.

Sixteen parametric spaces in Theorem 3.1.3 allow us to define sixteen invariant functions of up to four variables. However, a complete analysis of possible outcome is beyond the scope of this work. Rather empirically, following calculations in dimension four and eight, we pick up two one-parametric sets of vector spaces to define two new invariant functions of a n -dimensional Lie algebra \mathcal{L} . We call functions $\varphi \mathcal{L}, \varphi^0 \mathcal{L} : \mathbb{C} \rightarrow \{0, 1, \dots, n^2(n-1)/2\}$ defined by the formulas

$$(\varphi \mathcal{L})(\alpha) = \dim \text{coc}_{(1,1,1,\alpha,\alpha)} \mathcal{L} \tag{3.10}$$

$$(\varphi^0 \mathcal{L})(\alpha) = \dim \text{coc}_{(0,1,1\alpha,1,1)} \mathcal{L} \tag{3.11}$$

the **invariant functions** corresponding to two-dimensional twisted cocycles of the adjoint representation of a Lie algebra \mathcal{L} .

From Theorem 3.2.1 follows immediately:

Corollary 3.2.3. If two complex Lie algebras $\mathcal{L}, \tilde{\mathcal{L}}$ are isomorphic, $\mathcal{L} \cong \tilde{\mathcal{L}}$, then it holds:

1. $\varphi \mathcal{L} = \varphi \tilde{\mathcal{L}}$,
2. $\varphi^0 \mathcal{L} = \varphi^0 \tilde{\mathcal{L}}$.

3.2.1 Invariant Functions φ and φ^0 of Low-dimensional Lie Algebras

We now investigate the behaviour of the functions φ, φ^0 in dimensions three and four.

Example 9. In Example 7 we found the explicit form of the spaces $\text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathfrak{g}_{3,4}(a)$ and $\text{coc}_{(0,1,1,\lambda,1,1)} \mathfrak{g}_{3,4}(a)$, where $a \neq 0, \pm 1$. Observing these spaces we immediately have:

$$\begin{array}{|c|c|} \hline \alpha & \\ \hline \varphi \mathfrak{g}_{3,4}(a)(\alpha) & 6 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|c|c|} \hline \alpha & 2 & 1+a & 1+\frac{1}{a} & \\ \hline \varphi^0 \mathfrak{g}_{3,4}(a)(\alpha) & 2 & 1 & 1 & 0 \\ \hline \end{array}
 \quad (3.12)$$

Note that the function φ^0 behaves like the function ψ on $\mathfrak{g}_{3,4}(a)$. This fact allows us to derive a quite interesting fact – the function φ^0 also classifies three-dimensional Lie algebras:

Lemma 3.2.4. For Lie algebra $\mathfrak{g}_{3,4}(a)$, $a \neq 0, \pm 1$ from Lemma 2.4.1 it holds: if $\varphi^0 \mathfrak{g}_{3,4}(a) = \varphi^0 \mathfrak{g}_{3,4}(a')$ then $\mathfrak{g}_{3,4}(a) \cong \mathfrak{g}_{3,4}(a')$.

Proof. Observing (3.12), we conclude that the proof is completely analogous to the proof of Lemma 2.4.1. □

Theorem 3.2.5 (Classification of three-dimensional complex Lie algebras II).

Two three-dimensional complex Lie algebras \mathcal{L} and $\tilde{\mathcal{L}}$ are isomorphic if and only if $\varphi^0 \mathcal{L} = \varphi^0 \tilde{\mathcal{L}}$.

Proof. \Rightarrow : See Corollary 3.2.3.

\Leftarrow : According to Lemma 3.2.4 and observing the tables of φ^0 in Appendix B.1, we conclude that all tables of the invariant function φ^0 of non-isomorphic three-dimensional complex Lie algebras are mutually different. □

Example 10. In Example 8 we found the explicit form of the spaces $\text{coc}_{(1,1,1,\lambda,\lambda,\lambda)} \mathfrak{g}_{4,2}(a)$ and $\text{coc}_{(0,1,1,\lambda,1,1)} \mathfrak{g}_{4,2}(a)$, where $a \neq 0 \pm 1, -2$. Thus, the invariant function φ of the algebra $\mathfrak{g}_{4,2}(a)$ has the following form:

$$\begin{array}{|c|c|c|c|} \hline \alpha & 1+a & \frac{2}{a} & \\ \hline \varphi \mathfrak{g}_{4,2}(a)(\alpha) & 13 & 13 & 12 \\ \hline \end{array}$$

and the function φ^0 has the form:

α	2	$1 + a$	$1 + \frac{1}{a}$	
$\varphi^0 \mathfrak{g}_{4,2}(a)(\alpha)$	3	1	1	0

Similar calculations as in the above examples lead us to the tables of the functions φ and φ^0 of all three and four-dimensional Lie algebras. These tables are placed in Appendix B.1.

3.2.2 Classification of Four-dimensional Complex Lie Algebras

Theorem 2.4.2 (or 3.2.5) provided complete classification of three-dimensional complex Lie algebras. We show in this section that combined power of the functions ψ and φ distinguishes among all complex four-dimensional Lie algebras. The most challenging problem in the identification process in any dimension is to describe the parametric continua. The first natural question is, whether the table of an invariant function of some parametric continuum cannot 'degenerate' for some special value of the parameter. For example, examining the table of $\psi \mathfrak{g}_{4,5}(a, -1 - a)$ labeled by (g-18) in Appendix B.1, we see that generally the value 5 appears six times there. Is it possible that for some special value of parameter a this table has different shape? We devote to this problem the following part of this section. In order to be more precise, let us firstly define the **number of occurrences** of $j \in \mathbb{C}$ in a complex function f . Let j be in the range of values of f . If there exist only finitely many mutually distinct numbers $x_1, \dots, x_m \in \mathbb{C}$ for which $f(x_1) = \dots = f(x_m) = j$ holds then we write

$$f : j_m$$

and say that j **occurs** in f m -times; otherwise we write $f : j$.

Example 11. Consider the continuum $\mathfrak{g}_{4,5}(a, -1 - a)$, where $a \neq 0, \pm 1, -2, -\frac{1}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, labeled by (g-18) in Appendix B.1. In order to verify that

$$\psi \mathfrak{g}_{4,5}(a, -1 - a) : 6_1, 5_6, 4$$

$$\varphi \mathfrak{g}_{4,5}(a, -1 - a) : 15_1, 12$$

we have to check for solutions each of 15 possible equalities

$$\begin{aligned}
a &= \frac{1}{a} \\
a &= -1 - a \\
a &= -\frac{a}{a+1} \\
&\vdots \\
-\frac{a}{1+a} &= -\frac{a+1}{a}.
\end{aligned}$$

These equations have all solutions in the set

$$\left\{ 0, \pm 1, -2, -\frac{1}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right\}$$

– these values we excluded from the beginning.

Analogous calculations allow us to conclude:

Lemma 3.2.6. For the following complex four–dimensional Lie algebras defined in Appendix B.1 it holds:

$$(g-8) \quad \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1, \quad a \neq 0, \pm 1$$

$$\psi \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 : 7_1, 6_3, 5$$

$$\varphi \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 : 13_3, 12$$

$$(g-11) \quad \mathfrak{g}_{4,2}(a), \quad a \neq 0, \pm 1, -2$$

$$\psi \mathfrak{g}_{4,2}(a) : 6_1, 5_2, 4$$

$$\varphi \mathfrak{g}_{4,2}(a) : 13_2, 12$$

$$(g-17) \quad \mathfrak{g}_{4,5}(a, b), \quad a \neq 0, \pm 1, \pm b, \frac{1}{b}, b^2, -1 - b, \quad b \neq 0, \pm 1, \pm a, \frac{1}{a}, a^2, -1 - a$$

$$\psi \mathfrak{g}_{4,5}(a, b) : 6_1, 5_6, 4$$

$$\varphi \mathfrak{g}_{4,5}(a, b) : 13_3, 12$$

$$(g-18) \quad \mathfrak{g}_{4,5}(a, -1 - a), \quad a \neq 0, \pm 1, -2, -\frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\psi \mathfrak{g}_{4,5}(a, -1 - a) : 6_1, 5_6, 4$$

$$\varphi \mathfrak{g}_{4,5}(a, -1 - a) : 15_1, 12$$

$$(g-19) \quad \mathfrak{g}_{4,5}(a, a^2), \quad a \neq 0, \pm 1, \pm i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\psi \mathfrak{g}_{4,5}(a, a^2) : 6_3, 5_2, 4$$

$$\varphi \mathfrak{g}_{4,5}(a, a^2) : 13_3, 12$$

$$(g-20) \quad \mathfrak{g}_{4,5}(a, 1), \quad a \neq 0, \pm 1, -2$$

$$\psi \mathfrak{g}_{4,5}(a, 1) : 8_1, 6_2, 4$$

$$\varphi \mathfrak{g}_{4,5}(a, 1) : 15_1, 13_1, 12$$

$$(g-21) \quad \mathfrak{g}_{4,5}(a, -1), \quad a \neq 0, \pm 1, \pm i$$

$$\psi \mathfrak{g}_{4,5}(a, -1) : 6_2, 5_4, 4$$

$$\varphi \mathfrak{g}_{4,5}(a, -1) : 16_1, 13_2, 12$$

$$(g-28) \quad \mathfrak{g}_{4,8}(a), \quad a \neq 0, \pm 1, \pm 2, \pm \frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\psi \mathfrak{g}_{4,8}(a) : 5_1, 4_3, 3$$

$$\varphi \mathfrak{g}_{4,8}(a) : 12_5, 11$$

Example 12. Suppose we have two-parametric family of Lie algebras $\mathfrak{g}_{4,5}(a, b)$ and the conditions $a \neq 0, \pm 1, \pm b, \frac{1}{b}, b^2, -1 - b, b \neq 0, \pm 1, \pm a, \frac{1}{a}, a^2, -1 - a$ are satisfied. Then the function φ has the form:

α	$a + b$	$\frac{1+a}{b}$	$\frac{1+b}{a}$	
$\varphi \mathfrak{g}_{4,5}(a, b)(\alpha)$	13	13	13	12

(3.13)

We discuss now the following question: Is it possible to recover the exact values of parameters a, b if only the function φ of some algebra in $\mathfrak{g}_{4,5}(a, b)$ is known? If we have some member of the family $\mathfrak{g}_{4,5}(a', b')$, its function φ is of the general form

α	z_1	z_2	z_3	
$\varphi \mathfrak{g}_{4,5}(a', b')(\alpha)$	13	13	13	12

(3.14)

where $z_k \in \mathbb{C}$ and $z_k \neq -1, k = 1, 2, 3$. There are $3! = 6$ possibilities of correspondence (permutations) between (3.13) and (3.14). One of them imply the system of equations

$$\frac{1 + a'}{b'} = z_1, \quad \frac{1 + b'}{a'} = z_2, \quad a' + b' = z_3. \quad (3.15)$$

with the solution

$$a' = \frac{z_3 + 1}{z_2 + 1}, \quad b' = \frac{z_2 z_3 - 1}{z_2 + 1}. \quad (3.16)$$

We prove later that the other five permutations lead, in fact, to the isomorphic realizations of the algebra $\mathfrak{g}_{4,5}(a', b')$.

Lemma 3.2.7. For the four-dimensional Lie algebras from Lemma 3.2.6 it holds:

(g-8) If $\psi \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 = \psi \mathfrak{g}_{3,4}(a') \oplus \mathfrak{g}_1$ then $a' = a, \frac{1}{a}$.

(g-11) If $\varphi \mathfrak{g}_{4,2}(a) = \varphi \mathfrak{g}_{4,2}(a')$ then $a' = a$.

(g-17) If $\varphi \mathfrak{g}_{4,5}(a, b) = \varphi \mathfrak{g}_{4,5}(a', b')$ then

$$(a', b') = (a, b), (b, a), \left(\frac{1}{a}, \frac{b}{a}\right), \left(\frac{b}{a}, \frac{1}{a}\right), \left(\frac{1}{b}, \frac{a}{b}\right), \left(\frac{a}{b}, \frac{1}{b}\right).$$

(g-18) If $\psi \mathfrak{g}_{4,5}(a, -1 - a) = \psi \mathfrak{g}_{4,5}(a', -1 - a')$ then

$$a' = a, \frac{1}{a}, -\frac{a}{1+a}, -1 - \frac{1}{a}, -1 - a, -\frac{1}{1+a}.$$

(g-19) If $\psi \mathfrak{g}_{4,5}(a, a^2) = \psi \mathfrak{g}_{4,5}(a', a'^2)$ then $a' = a, \frac{1}{a}$.

(g-20) If $\varphi \mathfrak{g}_{4,5}(a, 1) = \varphi \mathfrak{g}_{4,5}(a', 1)$ then $a' = a$.

(g-21) If $\varphi \mathfrak{g}_{4,5}(a, -1) = \varphi \mathfrak{g}_{4,5}(a', -1)$ then $a' = a, -a$.

(g-28) If $\psi \mathfrak{g}_{4,8}(a) = \psi \mathfrak{g}_{4,8}(a')$ then $a' = a, \frac{1}{a}$.

Proof. Cases (g-8), (g-19), (g-20), (g-21) and (g-28). In these cases is the proof completely analogous to the proof of Lemma 2.4.1.

Case (g-11). The function φ of $\mathfrak{g}_{4,2}(a')$ has the form

α	$1 + a'$	$\frac{2}{a'}$	
$\varphi \mathfrak{g}_{4,2}(a')(\alpha)$	13	13	12

(3.17)

and there are two possibilities:

(12) If $a + 1 = a' + 1, \frac{2}{a} = \frac{2}{a'}$ then $a' = a$.

(21) If $a + 1 = \frac{2}{a'}, a' + 1 = \frac{2}{a}$ then $a = a' = 1, -2$, which is not possible.

Case (g-17). The function φ of $\mathfrak{g}_{4,5}(a', b')$ has the form

α	$a' + b'$	$\frac{1+a'}{b'}$	$\frac{1+b'}{a'}$	
$\varphi \mathfrak{g}_{4,5}(a', b')(\alpha)$	13	13	13	12

(3.18)

and there are six possible correspondences between this table and (3.13). Thus, substituting corresponding z_k 's into (3.16) we obtain:

$$(123) \text{ If } z_2 = \frac{1+b}{a}, z_3 = a + b \text{ then } a' = a, b' = b.$$

$$(213) \text{ If } z_2 = \frac{1+a}{b}, z_3 = a + b \text{ then } a' = b, b' = a.$$

$$(132) \text{ If } z_2 = a + b, z_3 = \frac{1+b}{a} \text{ then } a' = \frac{1}{a}, b' = \frac{b}{a}.$$

$$(312) \text{ If } z_2 = \frac{1+a}{b}, z_3 = \frac{1+b}{a} \text{ then } a' = \frac{b}{a}, b' = \frac{1}{a}.$$

$$(231) \text{ If } z_2 = a + 1, z_3 = \frac{1+a}{b} \text{ then } a' = \frac{1}{b}, b' = \frac{a}{b}.$$

$$(312) \text{ If } z_2 = \frac{1+b}{a}, z_3 = \frac{1+a}{b} \text{ then } a' = \frac{a}{b}, b' = \frac{1}{b}.$$

Case 4. It is convenient to note that six values in the table (g-18) can be arranged in the triple of pairs $\{a, \frac{1}{a}\}$, $\{-1 - a, -\frac{1}{1+a}\}$ and $\{-\frac{1+a}{a}, -\frac{a}{1+a}\}$. Then one checks directly only $6 \cdot 2^3 = 48$ permutations and obtains the solutions like in the previous case. \square

Corollary 3.2.8. For the four-dimensional Lie algebras from Lemma 3.2.6 it holds:

$$(g-8) \text{ If } \psi \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 = \psi \mathfrak{g}_{3,4}(a') \oplus \mathfrak{g}_1 \text{ then } \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 \cong \mathfrak{g}_{3,4}(a') \oplus \mathfrak{g}_1.$$

$$(g-17) \text{ If } \varphi \mathfrak{g}_{4,5}(a, b) = \varphi \mathfrak{g}_{4,5}(a', b') \text{ then } \mathfrak{g}_{4,5}(a, b) \cong \mathfrak{g}_{4,5}(a', b').$$

$$(g-18) \text{ If } \psi \mathfrak{g}_{4,5}(a, -1 - a) = \psi \mathfrak{g}_{4,5}(a', -1 - a') \text{ then } \mathfrak{g}_{4,5}(a, -1 - a) \cong \mathfrak{g}_{4,5}(a', -1 - a').$$

$$(g-19) \text{ If } \psi \mathfrak{g}_{4,5}(a, a^2) = \psi \mathfrak{g}_{4,5}(a', a'^2) \text{ then } \mathfrak{g}_{4,5}(a, a^2) \cong \mathfrak{g}_{4,5}(a', a'^2).$$

$$(g-21) \text{ If } \varphi \mathfrak{g}_{4,5}(a, -1) = \varphi \mathfrak{g}_{4,5}(a', -1) \text{ then } \mathfrak{g}_{4,5}(a, -1) \cong \mathfrak{g}_{4,5}(a', -1).$$

$$(g-28) \text{ If } \psi \mathfrak{g}_{4,8}(a) = \psi \mathfrak{g}_{4,8}(a') \text{ then } \mathfrak{g}_{4,8}(a) \cong \mathfrak{g}_{4,8}(a').$$

Proof. The statement follows from Lemma 3.2.7 and from the relations

$$\mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 \cong \mathfrak{g}_{3,4}(1/a) \oplus \mathfrak{g}_1 \tag{3.19}$$

$$\mathfrak{g}_{4,5}(a, b) \cong \mathfrak{g}_{4,5}(b, a) \cong \mathfrak{g}_{4,5}\left(\frac{1}{a}, \frac{b}{a}\right) \cong \mathfrak{g}_{4,5}\left(\frac{b}{a}, \frac{1}{a}\right) \cong \mathfrak{g}_{4,5}\left(\frac{1}{b}, \frac{a}{b}\right) \cong \mathfrak{g}_{4,5}\left(\frac{a}{b}, \frac{1}{b}\right) \tag{3.20}$$

$$\mathfrak{g}_{4,8}(a) \cong \mathfrak{g}_{4,8}(1/a), \tag{3.21}$$

which hold for all $a, b \neq 0$ and can be directly verified. \square

Theorem 3.2.9 (Classification of four-dimensional complex Lie algebras). Two four-dimensional complex Lie algebras \mathcal{L} and $\tilde{\mathcal{L}}$ are isomorphic if and only if $\psi \mathcal{L} = \psi \tilde{\mathcal{L}}$ and $\varphi \mathcal{L} = \varphi \tilde{\mathcal{L}}$.

Proof. \Rightarrow : See Corollaries 2.1.5 and 3.2.3.

\Leftarrow : According to Lemmas 3.2.6, 3.2.7, Corollary 3.2.8 and observing the tables in Appendix B.1, we conclude that all non-isomorphic four-dimensional complex Lie algebras differ at least in one of the functions ψ or φ . \square

3.2.3 Identification of Four-dimensional Complex Lie Algebras

An efficient algorithm for the identification of four-dimensional Lie algebras was quite recently published in [3]. Using Lemmas 3.2.6, 3.2.7, Corollary 3.2.8 and Theorem 3.2.9, we may now formulate an alternative *algorithm*: take a four-dimensional complex Lie algebra \mathcal{L} and

1. Calculate $\psi \mathcal{L}$ and $\varphi \mathcal{L}$.
2. The range of values of the functions ψ and φ and the number of their occurrences determines the label (g- k), $k = 1, \dots, 34$ in Appendix B.1.
3. The algebra is now identified up to the exact value of parameter(s) of the parametric continuum. These parameters are determined in the following cases:
 - (g-8) Pick any of the two values $z \in \mathbb{C}, z \neq 1$, which satisfy $\psi \mathcal{L}(z) = 6$, and put $a = z$. Then $\mathcal{L} \cong \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1$ holds.
 - (g-11) There are two different complex numbers $z_1, z_2 \neq 0$ which satisfy $\varphi \mathcal{L}(z_1) = \varphi \mathcal{L}(z_2) = 13$. If $z_1 - 1 = 2/z_2$ holds then put $a = z_1 - 1$, otherwise put $a = z_2 - 1$. Then $\mathcal{L} \cong \mathfrak{g}_{4,2}(a)$ holds.
 - (g-17) There are three mutually different complex numbers $z_1, z_2, z_3 \neq 0, -1$ which satisfy $\varphi \mathcal{L}(z_1) = \varphi \mathcal{L}(z_2) = \varphi \mathcal{L}(z_3) = 13$. Put $a = \frac{z_3+1}{z_2+1}$, $b = \frac{z_2 z_3 - 1}{z_2 + 1}$. Then $\mathcal{L} \cong \mathfrak{g}_{4,5}(a, b)$ holds.
 - (g-18) Pick any of the six values $z \in \mathbb{C}$, which satisfy $\psi \mathcal{L}(z) = 5$, and put $a = z$. Then $\mathcal{L} \cong \mathfrak{g}_{4,5}(a, -1 - a)$ holds.
 - (g-19) Pick any of the two values $z \in \mathbb{C}, z \neq 1$, which satisfy $\psi \mathcal{L}(z) = 6$, and put $a = z$. Then $\mathcal{L} \cong \mathfrak{g}_{4,5}(a, a^2)$ holds.

(g-20) Take the value $z \in \mathbb{C}$, which satisfies $\varphi \mathcal{L}(z) = 15$, and put $a = z - 1$. Then $\mathcal{L} \cong \mathfrak{g}_{4,5}(a, 1)$ holds.

(g-21) Pick any of the two values $z \in \mathbb{C}$, which satisfy $\varphi \mathcal{L}(z) = 13$, and put $a = z + 1$. Then $\mathcal{L} \cong \mathfrak{g}_{4,5}(a, -1)$ holds.

(g-28) Pick any of the two values $z \in \mathbb{C}$, $z \neq 2$ which satisfy $\psi \mathcal{L}(z) = 4$, and put $a = z$. Then $\mathcal{L} \cong \mathfrak{g}_{4,8}(a)$ holds.

We demonstrate the above algorithm of identification on the following two examples.

Example 13. In [3], a four-dimensional algebra \mathcal{L}_1 was introduced:

$$\mathcal{L}_1 : \begin{aligned} [e_1, e_2] &= -e_1 - e_2 + e_3, & [e_1, e_3] &= -6e_2 + 4e_3, & [e_1, e_4] &= 2e_1 - e_2 + e_4, \\ [e_2, e_3] &= 3e_1 - 9e_2 + 5e_3, & [e_2, e_4] &= 4e_1 - 2e_2 + 2e_4, & [e_3, e_4] &= 6e_1 - 3e_2 + 3e_4. \end{aligned}$$

1. Computing the functions $\psi \mathcal{L}_1$ and $\varphi \mathcal{L}_1$ one obtains:

α	1	2	$\frac{1}{2}$	
$\psi \mathcal{L}_1(\alpha)$	6	5	5	4

α	3	1	
$\varphi \mathcal{L}_1(\alpha)$	13	13	12

2. The combination of occurrences $\psi \mathcal{L}_1 : 6_1, 5_2, 4$ and $\varphi \mathcal{L}_1 : 13_2, 12$ is unique for the case (g-11).

3. Since for $z_1 = 3$, $z_2 = 1$ the equality $z_1 - 1 = 2/z_2$ holds, one has $a = z_1 - 1 = 2$ and $\mathcal{L}_1 \cong \mathfrak{g}_{4,2}(2)$.

Example 14. In [3], a four-dimensional algebra \mathcal{L}_2 was also introduced:

$$\mathcal{L}_2 : \begin{aligned} [e_1, e_2] &= 4e_1 + 3e_2 - 6e_3 + 2e_4, & [e_1, e_3] &= 15e_1 + 5e_2 - 15e_3 + 5e_4, \\ [e_1, e_4] &= 50e_1 + 15e_2 - 48e_3 + 16e_4, & [e_2, e_3] &= 21e_1 + 2e_2 - 15e_3 + 5e_4, \\ [e_2, e_4] &= 93e_1 + 21e_2 - 81e_3 + 27e_4, & [e_3, e_4] &= 90e_1 + 25e_2 - 84e_3 + 28e_4. \end{aligned}$$

1. Computing the functions $\psi \mathcal{L}_2$ and $\varphi \mathcal{L}_2$ one obtains:

α	0	
$\psi \mathcal{L}_2(\alpha)$	6	4

α	0	1	
$\varphi \mathcal{L}_2(\alpha)$	12	12	10

2. The combination of occurrences $\psi \mathcal{L}_2 : 6_1, 4$ and $\varphi \mathcal{L}_2 : 12_2, 10$ is unique for the case (g-3) and one has $\mathcal{L}_2 \cong \mathfrak{g}_{2,1} \oplus \mathfrak{g}_{2,1}$.

Chapter 4

Contractions of Algebras

4.1 Continuous Contractions of Algebras

Except where explicitly stated, Section 4.1 of this chapter contains original and unpublished results. The content of Section 4.2 is based on [II, III, IV].

Suppose we have an arbitrary algebra $\mathcal{A} = (V, \cdot)$ and a continuous mapping $U : (0, 1) \rightarrow GL(V)$, i. e. $U(\varepsilon) \in GL(V)$, $0 < \varepsilon \leq 1$. If the limit

$$x \cdot_0 y = \lim_{\varepsilon \rightarrow 0^+} U(\varepsilon)^{-1}(U(\varepsilon)x \cdot U(\varepsilon)y) \quad (4.1)$$

exists for all $x, y \in V$ then we call the algebra $\mathcal{A}_0 = (V, \cdot_0)$ a **one-parametric continuous contraction** (or simply a **contraction**) of the algebra \mathcal{A} and write $\mathcal{A} \rightarrow \mathcal{A}_0$. We call the contraction $\mathcal{A} \rightarrow \mathcal{A}_0$ **proper** if $\mathcal{A} \not\cong \mathcal{A}_0$. Contraction to the Abelian algebra \mathcal{A}_0 ,

$$x \cdot_0 y = 0, \forall x, y \in \mathcal{A}_0$$

is always possible via $U(\varepsilon) = \varepsilon 1$. We call all improper contractions and contractions to the Abelian algebra **trivial**.

It is well known that if $\mathcal{L} \rightarrow \mathcal{L}_0$ is any one-parametric continuous contraction of a Lie algebra \mathcal{L} then \mathcal{L}_0 is also a Lie algebra. Invariant characteristics of Lie algebras change after a contraction. The relation among these characteristics before and after a contraction form useful necessary contraction criteria. For example, such a set of these criteria, which provided the complete classification of contractions of three and four-dimensional Lie algebras, has been found in [27]. Our aim is to state new necessary contraction criteria using (α, β, γ) -derivations and twisted cocycles. We first give a criterion from [9, 13]:

Theorem 4.1.1. Let \mathcal{L} be a complex Lie algebra and $\mathcal{L} \rightarrow \mathcal{L}_0$, and $q \in \mathbb{N}$. Then it holds:

$$\dim Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}) \leq \dim Z^q(\mathcal{L}_0, \text{ad}_{\mathcal{L}_0}). \quad (4.2)$$

There is a straightforward generalization of the above theorem:

Theorem 4.1.2. Let \mathcal{L} be a complex Lie algebra, $\mathcal{L} \rightarrow \mathcal{L}_0$ and $q \in \mathbb{N}$. Then for any $(q+1) \times (q+1)$ complex symmetric matrix κ

$$\dim Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa) \leq \dim Z^q(\mathcal{L}_0, \text{ad}_{\mathcal{L}_0}, \kappa) \quad (4.3)$$

holds.

Proof. Suppose that the contraction $\mathcal{L} \rightarrow \mathcal{L}_0$ is performed by the mapping U , i. e. $[x, y]_0 = \lim_{\varepsilon \rightarrow 0^+} [x, y]_\varepsilon$, where

$$[x, y]_\varepsilon = U(\varepsilon)^{-1}[U(\varepsilon)x, U(\varepsilon)y], \quad \forall x, y \in \mathcal{L}.$$

Suppose $\mathcal{L} = (V, [\cdot, \cdot])$ and let us fix a basis $\{x_1, \dots, x_n\}$ of V . We denote the structural constants of the algebra \mathcal{L} by c_{ij}^k and the structural constants of the algebras $\mathcal{L}_\varepsilon = (V, [\cdot, \cdot]_\varepsilon)$ by $c_{ij}^k(\varepsilon)$. Then it holds

$$c_{ij}^k(0) = \lim_{\varepsilon \rightarrow 0^+} c_{ij}^k(\varepsilon), \quad (4.4)$$

where $c_{ij}^k(0)$ are the structural constants of \mathcal{L}_0 . The dimension of the space $Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa)$ is determined via the relation

$$\dim Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa) = \dim C^q(\mathcal{L}, \mathcal{L}) - \text{rank } S^q(\mathcal{L}, \kappa), \quad (4.5)$$

where $S^q(\mathcal{L}, \kappa)$ is the matrix corresponding to the linear system of equations generated from (3.1). We write the explicit form of this system for $q = 1$. Then we obtain from (3.1) that $D = (D_{ij}) \in Z^1(\mathcal{L}, \text{ad}_{\mathcal{L}}, \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix})$ if and only if the linear system with the matrix $S^1(\mathcal{L}, \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix})$ is satisfied

$$S^1(\mathcal{L}, \begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix}) : \sum_{r=1}^n -\alpha c_{ij}^r D_{sr} + \beta c_{rj}^s D_{ri} + \gamma c_{ir}^s D_{rj} = 0, \quad \forall i, j, s \in \{1, \dots, n\}, \quad (4.6)$$

and similarly for $q > 1$. Since $\mathcal{L}_\varepsilon \cong \mathcal{L}$ holds for all $0 < \varepsilon \leq 1$, we see from Corollary 3.2.2 that

$$\dim Z^q(\mathcal{L}, \text{ad}_{\mathcal{L}}, \kappa) = \dim Z^q(\mathcal{L}_\varepsilon, \text{ad}_{\mathcal{L}_\varepsilon}, \kappa), \quad 0 < \varepsilon \leq 1, q \in \mathbb{N}. \quad (4.7)$$

Since the relation

$$\dim C^q(\mathcal{L}, \mathcal{L}) = \dim C^q(\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon) = \dim C^q(\mathcal{L}_0, \mathcal{L}_0), \quad 0 < \varepsilon \leq 1, q \in \mathbb{N} \quad (4.8)$$

holds, the relations (4.5), (4.7) then imply that

$$\text{rank } S^q(\mathcal{L}, \kappa) = \text{rank } S^q(\mathcal{L}_\varepsilon, \kappa), \quad 0 < \varepsilon \leq 1, q \in \mathbb{N}. \quad (4.9)$$

The rank of the matrix $S^q(\mathcal{L}, \kappa)$ is equal to r if and only if there exists a non-zero minor of the order r and every minor of order higher than r is zero. It follows from (4.9) that all minors of the orders higher than r of the matrices $S^q(\mathcal{L}_\varepsilon, \kappa)$ are zeros. Since the equality (4.4) holds, all minors of the matrices $S^q(\mathcal{L}_\varepsilon, \kappa)$ converge to the minors of the matrix $S^q(\mathcal{L}_0, \kappa)$. Thus, as the limits of zero functions, all minors of order higher than r of the matrix $S^q(\mathcal{L}_0, \kappa)$ are also zero. Therefore $\text{rank } S^q(\mathcal{L}_0, \kappa) \leq r$ and the statement of the theorem follows from (4.5) and (4.8). \square

There exist other necessary contraction criteria, similar to (4.2), (4.3) – certain inequalities between invariants. However, one very powerful criterion is quite unique. This highly non-trivial theorem, very useful in [9, 11, 27], was originally proved in [6].

Theorem 4.1.3. If \mathcal{L}_0 is a proper contraction of a complex Lie algebra \mathcal{L} then it holds:

$$\dim \text{der } \mathcal{L} < \dim \text{der } \mathcal{L}_0. \quad (4.10)$$

Corollary 4.1.4. If \mathcal{L}_0 is a proper contraction of a complex Lie algebra \mathcal{L} then it holds:

1. $\psi \mathcal{L} \leq \psi \mathcal{L}_0$
2. $\psi \mathcal{L}(1) < \psi \mathcal{L}_0(1)$.

Proof. Since $\psi \mathcal{L}(\alpha) = \dim Z^1(\mathcal{L}, \text{ad}_{\mathcal{L}}, \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix})$ the first inequality follows from (4.3) and the second from $\psi \mathcal{L}(1) = \dim \text{der } \mathcal{L}$ and (4.10). \square

Corollary 4.1.5. If \mathcal{L}_0 is a contraction of a complex Lie algebra \mathcal{L} then it holds:

1. $\varphi \mathcal{L} \leq \varphi \mathcal{L}_0$
2. $\varphi^0 \mathcal{L} \leq \varphi^0 \mathcal{L}_0$.

Proof. Since $\varphi \mathcal{L}(\alpha) = \dim Z^2(\mathcal{L}, \text{ad}_{\mathcal{L}}, \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix})$ the first inequality follows from (4.3); the proof of the second condition is analogous. \square

Example 15. Consider the contraction $\mathfrak{g}_{3,2} \rightarrow \mathfrak{g}_{3,3}$. In Table 4.1 the dimensions (2.28) of the associated Lie and Jordan algebras of $\mathfrak{g}_{3,2}$ and $\mathfrak{g}_{3,3}$ are listed. Note that except for the dimensions of the algebras of derivations

$$4 = \dim \text{der } \mathfrak{g}_{3,2} < \dim \text{der } \mathfrak{g}_{3,3} = 6$$

none of these dimensions grows after the contraction.

Table 4.1: *Dimensions of the associated Lie and Jordan algebras of $\mathfrak{g}_{3,2}$ and $\mathfrak{g}_{3,3}$*

	$d_{(1,1,1)}$	$d_{(0,1,1)}$	$d_{(1,1,0)}$	$d_{(1,1,1)(0,1,1)}$	$d_{(1,1,-1)}$	$d_{(0,1,-1)}$
$\mathfrak{g}_{3,2}$	4	3	1	2	0	1
$\mathfrak{g}_{3,3}$	6	3	1	2	0	1

4.1.1 Contractions of Low-dimensional Lie Algebras

In Section 2.4 we have used the invariant function ψ to classify all three-dimensional Lie algebras. We now employ the necessary contraction criterion of Corollary 4.1.4 to describe all possible contractions among these algebras. The behaviour of the function ψ determines the classification and contractions of three-dimensional Lie algebras. Contractions of three-dimensional algebras were the most recently classified in [27]:

Theorem 4.1.6. Only the following non-trivial contractions among three-dimensional Lie algebras exist:

1. $\mathfrak{g}_{3,4}(-1)$ is a contraction of $\mathfrak{sl}(2, \mathbb{C})$,
2. $\mathfrak{g}_{3,3}$ is a contraction of $\mathfrak{g}_{3,2}$,
3. $\mathfrak{g}_{3,1}$ is a contraction of $\mathfrak{g}_{3,2}$, $\mathfrak{g}_{3,4}(a)$, $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$ and $\mathfrak{sl}(2, \mathbb{C})$.

We present two examples which show that both items in Corollary 4.1.4 are effective for the description of contractions among three-dimensional Lie algebras.

Example 16. Observing the corresponding tables of the functions ψ , we see that the contraction of $\mathfrak{g}_{3,2}$ to $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$ is not possible due to $\psi \mathfrak{g}_{3,2}(1) = \psi \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1(1) = 4$ – a contradiction to the item 2. of the Corollary 4.1.4. The necessary condition $\psi \mathfrak{g}_{3,2} \leq \psi \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$ is satisfied and, thus, does not exclude the existence of a contraction.

Example 17. On the contrary to the previous example, observing the corresponding tables of the functions ψ , we see that the contraction of $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$ is not possible due to $5 = \psi \mathfrak{sl}(2, \mathbb{C})(-1) > \psi \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1(-1) = 4$ – a contradiction to the item 1. of the Corollary 4.1.4. The necessary condition $3 = \psi \mathfrak{sl}(2, \mathbb{C})(1) < \psi \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1(1) = 4$ is satisfied and, thus, does not exclude the existence of a contraction.

A similar analysis of all possible pairs of three-dimensional Lie algebras leads us to the following theorem.

Theorem 4.1.7 (Contractions of three-dimensional complex Lie algebras).

Let $\mathcal{L}, \mathcal{L}_0$ be two three-dimensional complex Lie algebras. Then there exists a proper one-parametric continuous contraction $\mathcal{L} \rightarrow \mathcal{L}_0$ if and only if

$$\psi \mathcal{L} \leq \psi \mathcal{L}_0 \quad \text{and} \quad \psi \mathcal{L}(1) < \psi \mathcal{L}_0(1).$$

Proof. \Rightarrow : This implication is, in fact, Corollary 4.1.4.

\Leftarrow : This implication follows from a direct comparison of the tables of the invariant functions ψ of three-dimensional Lie algebras in Appendix B.1 and Theorem 4.1.6. \square

In Section 3.2.2 we used the invariant functions ψ and φ to classify all four-dimensional Lie algebras. In order to obtain stronger contraction criteria, we also defined the function φ^0 – a supplement to the functions ψ and φ (see Example 20). The combined forces of the Corollaries 4.1.4 and 4.1.5, though strong, do not provide us with a complete classification of contractions of four-dimensional Lie algebras. So far, we were unable to find more suitable definitions of invariant functions, which offers the concept of two-dimensional twisted cocycles, allowing such classification. The complete description of the spaces of two-dimensional twisted cocycles for four-dimensional Lie algebras, similar to the classification of (α, β, γ) -derivations in Appendix A.1, would solve the existence of such functions explicitly. However, such a complete description seems, at the moment, out of reach. We discuss the application of the criteria of the Corollaries 4.1.4 and 4.1.5 to the four-dimensional Lie algebras in the following examples.

Example 18. To demonstrate behaviour of the functions ψ, φ and φ^0 in dimension four, we consider the following sequence of contractions [11, 27] :

$$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_1 \rightarrow \mathfrak{g}_{4,8}(-1) \rightarrow \mathfrak{g}_{3,4}(-1) \oplus \mathfrak{g}_1 \rightarrow \mathfrak{g}_{4,1} \rightarrow \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \rightarrow 4\mathfrak{g}_1.$$

Note in Table 4.2, how the value of each invariant function is greater or equal than the value in the previous row. As expected, the strict inequality for the values $\psi(1)$ holds – in this case the sequence of dimensions: 4, 5, 6, 7, 10, 16. The strict increase of values is also identified in the following cases: $\psi(2)$, 'generic values' of ψ , $\varphi(1/2)$ and 'generic values' of φ . These conjectures of strict inequalities are, however, not valid for the general case of a contraction in dimension four.

Example 19. Consider the pair of four-dimensional Lie algebras $\mathfrak{g}_{4,2}(a)$, $a \neq 0, \pm 1, -2$ and $\mathfrak{g}_{4,5}(a', 1)$, $a' \neq 0, \pm 1, -2$. There are two possibilities, how the corresponding tables of the invariant functions φ , in Appendix B.1 cases (g-11) and (g-20), can satisfy $\varphi \mathfrak{g}_{4,2}(a) \leq$

Table 4.2: *Invariant functions ψ , φ and φ^0 of the contraction sequence: $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_1 \rightarrow \mathfrak{g}_{4,8}(-1) \rightarrow \mathfrak{g}_{3,4}(-1) \oplus \mathfrak{g}_1 \rightarrow \mathfrak{g}_{4,1} \rightarrow \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \rightarrow 4\mathfrak{g}_1$.*

α	$\psi(\alpha)$					$\varphi(\alpha)$					$\varphi^0(\alpha)$			
	-1	0	1	2		-1	0	1	$\frac{1}{2}$		0	1	2	
$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_1$	6	4	4	2	1	14	12	12	10	9	0	0	1	0
$\mathfrak{g}_{4,8}(-1)$	6	4	5	4	4	14	12	13	12	12	0	0	1	0
$\mathfrak{g}_{3,4}(-1) \oplus \mathfrak{g}_1$	7	7	6	5	5	16	16	15	14	14	3	3	3	1
$\mathfrak{g}_{4,1}$	7	7	7	7	7	16	16	15	15	15	3	3	3	3
$\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$	10	11	10	10	10	19	20	19	19	19	8	11	8	8
$4\mathfrak{g}_1$	16	16	16	16	16	24	24	24	24	24	24	24	24	24

$\varphi \mathfrak{g}_{4,5}(a', 1)$. The first possibility leads to conditions $a' + 1 = 2/a$ and $a + 1 = 2/a'$ – these have solutions $a = a' = 1, -2$ and we excluded them. The second possibility implies $a = a'$. The necessary condition 1. of Corollary 4.1.5 therefore admits only the contraction $\mathfrak{g}_{4,2}(a) \rightarrow \mathfrak{g}_{4,5}(a, 1)$. This contraction indeed exists [27]. In Table 4.3 we summarize the behaviour of the functions ψ , φ and φ^0 . Note that the function ψ grows only at the points $1, a, \frac{1}{a}$ and the function φ only at one(!) point $1 + a$.

Table 4.3: *Invariant functions ψ , φ and φ^0 of the contraction: $\mathfrak{g}_{4,2}(a) \rightarrow \mathfrak{g}_{4,5}(a, 1)$*

α	$\psi(\alpha)$				$\varphi(\alpha)$			$\varphi^0(\alpha)$			
	1	a	$\frac{1}{a}$		$1 + a$	$\frac{2}{a}$		2	$1 + a$	$1 + \frac{1}{a}$	
$\mathfrak{g}_{4,2}(a)$	6	5	5	4	13	13	12	3	1	1	0
$\mathfrak{g}_{4,5}(a, 1)$	8	6	6	4	15	13	12	7	2	2	0

Example 20. Consider the pair of four-dimensional Lie algebras $\mathfrak{g}_{4,7}$ and $\mathfrak{g}_{4,2}(1)$. The necessary conditions $\psi \mathfrak{g}_{4,7} \leq \psi \mathfrak{g}_{4,2}(1)$, $[\psi \mathfrak{g}_{4,7}](1) < [\psi \mathfrak{g}_{4,2}(1)](1)$ and $\varphi \mathfrak{g}_{4,7} \leq \varphi \mathfrak{g}_{4,2}(1)$ are satisfied. But since it holds

$$1 = [\varphi^0 \mathfrak{g}_{4,7}] \left(\frac{3}{2} \right) > [\varphi^0 \mathfrak{g}_{4,2}(1)] \left(\frac{3}{2} \right) = 0,$$

a contraction is not possible.

4.1.2 Contractions of Two-dimensional Jordan Algebras

The classification of Jordan algebras is a far more challenging problem than the classification of Lie algebras. Indeed, even the classification in dimension 2 involves the solution of a system of 13 cubic equations from Section 1.5 with 6 variables. This problem was solved recently [5]. The resulting five two-dimensional complex Jordan algebras, together with the explicit form of the spaces $\text{der}_{(\alpha,\beta,\gamma)} \mathcal{A}$ and the function ψ , are listed in Appendices A.2 and B.2. We have:

Theorem 4.1.8 (Classification of two-dimensional complex Jordan algebras).

Two two-dimensional complex Jordan algebras \mathcal{J} and $\tilde{\mathcal{J}}$ are isomorphic if and only if $\psi \mathcal{J} = \psi \tilde{\mathcal{J}}$.

Proof. The result follows from the direct comparison of the classification of two-dimensional complex Jordan algebras and the corresponding values of the invariant function ψ – see Appendix B.2. \square

It was also pointed out [5] that the necessary contraction criterion (4.10) holds for two-dimensional Jordan algebras. The modification of the proof of Theorem 4.1.2 for Jordan algebras and $q = 1$ is straightforward. Thus, the necessary criterion in Corollary 4.1.4 is also valid for Jordan algebras. The behaviour of the function ψ determines the classification and contractions of two-dimensional complex Jordan algebras. The contractions of two-dimensional complex Jordan algebras were recently also classified [5]:

Theorem 4.1.9. Only the following non-trivial contractions among two-dimensional complex Jordan algebras exist:

1. $\mathfrak{j}_{2,1}$ is a contraction of $\mathfrak{j}_{2,5}$,
2. $\mathfrak{j}_{2,2}$ is a contraction of $\mathfrak{j}_{2,5}$,
3. $\mathfrak{j}_{2,3}$ is a contraction of $\mathfrak{j}_{2,1}$, $\mathfrak{j}_{2,2}$ and $\mathfrak{j}_{2,5}$.

Comparing the tables of the invariant functions ψ of two-dimensional complex Jordan algebras in Appendix B.2 and Theorem 4.1.9 we formulate an analogous theorem to Theorem 4.1.7.

Theorem 4.1.10 (Contractions of two-dimensional complex Jordan algebras).

Let \mathcal{J} and \mathcal{J}_0 be two two-dimensional complex Jordan algebras. Then there exists a proper one-parametric continuous contraction $\mathcal{J} \rightarrow \mathcal{J}_0$ if and only if

$$\psi \mathcal{J} \leq \psi \mathcal{J}_0 \quad \text{and} \quad \psi \mathcal{J}(1) < \psi \mathcal{J}_0(1).$$

Similarly to Examples 16, 17, we show that both conditions in Theorem 4.1.10 are necessary for the description of contractions of two-dimensional Jordan algebras.

Example 21. Observing the corresponding tables of the functions ψ , we see that the contraction of $\mathfrak{j}_{2,1}$ to $\mathfrak{j}_{2,2}$ is not possible due to $\psi_{\mathfrak{j}_{2,1}}(1) = \psi_{\mathfrak{j}_{2,2}}(1) = 1$. The necessary condition $\psi_{\mathfrak{j}_{2,1}} \leq \psi_{\mathfrak{j}_{2,2}}$ is satisfied and, thus, does not exclude the existence of a contraction.

Example 22. On the contrary to the previous example, observing the corresponding tables of the functions ψ , we see that the contraction of $\mathfrak{j}_{2,1}$ to $\mathfrak{j}_{2,4}$ is not possible due to $2 = \psi_{\mathfrak{j}_{2,1}}(2) > \psi_{\mathfrak{j}_{2,4}}(2) = 1$. The necessary condition $1 = \psi_{\mathfrak{j}_{2,1}}(1) < \psi_{\mathfrak{j}_{2,4}}(1) = 2$ is satisfied and does not exclude the existence of a contraction.

4.2 Graded Contractions of Lie Algebras

Suppose we have some Abelian group G . We say that a complex Lie algebra $\mathcal{L} = (V, [\cdot, \cdot])$ is G -**graded**, if there is a decomposition into subspaces L_i , $i \in G$ – called a **grading** Γ –

$$\Gamma : \mathcal{L} = \bigoplus_{i \in G} L_i \quad (4.11)$$

and the relation

$$[L_i, L_j] \subseteq L_{i+j}, \quad \forall i, j \in G \quad (4.12)$$

holds. The decomposition $\mathcal{L} = \bigoplus_{i \in G} gL_i$, where $g \in \text{Aut } \mathcal{L}$, is also a grading of \mathcal{L} and is **equivalent** to Γ .

We define for all $x \in L_i$, $y \in L_j$, $i, j \in G$ and $\varepsilon_{ij} \in \mathbb{C}$ a new bilinear mapping on V by the formula

$$[x, y]_\varepsilon = \varepsilon_{ij}[x, y]. \quad (4.13)$$

Since we claim the bilinearity of $[\cdot, \cdot]_\varepsilon$, the condition (4.13) determines this mapping on the whole V . If $\mathcal{L}^\varepsilon := (V, [\cdot, \cdot]_\varepsilon)$ is a Lie algebra, then it is called a **graded contraction** of the Lie algebra \mathcal{L} . Note that the contraction preserves a grading because it is also true that

$$\mathcal{L}^\varepsilon = \bigoplus_{i \in G} L_i \quad (4.14)$$

is a grading of \mathcal{L}^ε .

There are two conditions which the parameters ε_{ij} must fulfill. Antisymmetry of $[\cdot, \cdot]_\varepsilon$ immediately gives

$$\varepsilon_{ij} = \varepsilon_{ji}. \quad (4.15)$$

The validity of the Jacobi identity requires: for all (unordered) triples $i, j, k \in G$

$$e(i j k) : [x, [y, z]_\varepsilon]_\varepsilon + [z, [x, y]_\varepsilon]_\varepsilon + [y, [z, x]_\varepsilon]_\varepsilon = 0 \quad (\forall x \in L_i)(\forall y \in L_j)(\forall z \in L_k). \quad (4.16)$$

is satisfied. Each set of ε_{ij} 's which satisfies the above conditions – determines a Lie algebra – can be written in the form of a symmetric matrix $\varepsilon = (\varepsilon_{ij})$ which is called a **contraction matrix**.

4.2.1 Invariant Functions and Graded Contractions of $\mathfrak{sl}(3, \mathbb{C})$

The Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ has four non-equivalent gradings [20]. One of them is a $\mathbb{Z}_3 \times \mathbb{Z}_3$ -grading, called the **Pauli grading** [30], and has the following explicit form:

$$\begin{aligned} \mathfrak{sl}(3, \mathbb{C}) = & \mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \\ & \oplus \mathbb{C} \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \end{aligned}$$

where $\omega = \exp(2\pi i/3)$.

In [III], all graded contractions corresponding to the Pauli grading of $\mathfrak{sl}(3, \mathbb{C})$ have been found. Classifying the outcome of the contraction, it turned out that the set of invariant characteristics of Lie algebras, basically the set $\text{inv } \mathcal{L}$ without the dimension of associated Lie algebras, is insufficient to distinguish among the results. This difficulty was dealt with by the explicit calculation of the isomorphism problem: complex Lie algebras \mathcal{L} and $\tilde{\mathcal{L}}$ of the same dimension n determined by structure constants c_{ij}^k and \tilde{c}_{ij}^k in are isomorphic if and only if there exists a regular matrix $A = (A_{ij}) \in GL(n, \mathbb{C})$, whose elements satisfy the following system of quadratic equations

$$\sum_{r=1}^n c_{ij}^r A_{kr} = \sum_{s,t=1}^n A_{si} A_{tj} \tilde{c}_{st}^k, \quad \forall i, j, k \in \{1, \dots, n\}. \quad (4.17)$$

This direct computation was used in [III] to show that the two graded contractions of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$ denoted by $\mathcal{L}_{17,9}$, $\mathcal{L}_{17,12}$ and defined already in Example 4 are non-isomorphic. We have proved this fact in Example 4 by analyzing the structure of associated Lie algebra $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}$. It was pointed out in [II] that the set of invariants $\text{inv } \mathcal{L}$ enriched by the sets $\text{inv } \text{der}_{(1,1,1)} \mathcal{L}$ and $\text{inv}[\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}]$ is sufficient for the description of all results of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$ – parametric continua excluded.

Employing the invariant function ψ , one may also distinguish between $\mathcal{L}_{17,9}$ and $\mathcal{L}_{17,12}$.

Example 23. Consider the graded contractions of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$ denoted by $\mathcal{L}_{17,9}$ and $\mathcal{L}_{17,12}$ and defined in Example 4. We have seen that calculating the set of invariants $\text{inv } \mathcal{L}$ one has:

$$\text{inv } \mathcal{L}_{17,9} = \text{inv } \mathcal{L}_{17,12} \quad (8, 4, 0)(8, 4, 2, 0)(2, 5, 8) \quad 2 \quad [16, 19, 9, 11]$$

Calculating the dimensions of the associated Jordan algebras we obtain:

$$\begin{aligned} \dim \text{der}_{(1,1,-1)} \mathcal{L}_{17,9} &= \dim \text{der}_{(1,1,-1)} \mathcal{L}_{17,12} = 8 \\ \dim \text{der}_{(0,1,-1)} \mathcal{L}_{17,9} &= \dim \text{der}_{(0,1,-1)} \mathcal{L}_{17,12} = 17 \end{aligned}$$

We may add the values of the invariant function ψ^0

α	0	1	
$\psi^0 \mathcal{L}_{17,9}(\alpha)$	16	9	8
$\psi^0 \mathcal{L}_{17,12}(\alpha)$	16	9	8

and we see that the unique characterization is not attained. However, calculating the invariant function ψ yield:

α	0	-2		α	0	$-\frac{1}{2}$	
$\psi \mathcal{L}_{17,9}(\alpha)$	19	17	16	$\psi \mathcal{L}_{17,12}(\alpha)$	19	17	16

and since $\psi \mathcal{L}_{17,9} \neq \psi \mathcal{L}_{17,12}$, we see that the conclusion $\mathcal{L}_{17,9} \not\cong \mathcal{L}_{17,12}$ provides the function ψ , as well as the set of invariants $\text{inv}[\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}]$; laborious direct computation of the isomorphism problem is now unnecessary.

When solving the isomorphism problem for parametric continua of Lie algebras, the situation is far more challenging.

Example 24. Consider the following graded contraction of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$:

$$\varepsilon_{18,25}(a) = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.18)$$

We determine this graded contraction by listing its non-zero commutation relations in \mathbb{Z}_3 -labeled basis $(l_{01}, l_{02}, l_{10}, l_{20}, l_{11}, l_{22}, l_{12}, l_{21})$:

$$\begin{aligned} \mathcal{L}_{18,25}(a) \quad [l_{01}, l_{10}] &= l_{11}, \quad [l_{01}, l_{20}] = -al_{21}, \quad [l_{02}, l_{10}] = l_{12}, \quad [l_{02}, l_{20}] = l_{22}, \\ [l_{10}, l_{11}] &= l_{21}, \quad [l_{10}, l_{12}] = l_{22}, \quad a \in \mathbb{C} \end{aligned}$$

Infinitely many Lie algebras $\mathcal{L}_{18,25}(a)$ – the parametric continuum – are all indecomposable and nilpotent. It is clear, that the set $\text{inv } \mathcal{L}$, or a similar finite set of certain dimensions, can never completely characterize an infinite number of algebras in $\mathcal{L}_{18,25}(a)$. Moreover, all invariants based on the trace of the adjoint representation, such as C_{pq} (1.24) and χ_i (1.25) are, due to Theorem 1.2.3, worthless in the case of the nilpotent parametric continuum. The behaviour of the function ψ turns out to be quite valuable, otherwise we would be forced to try to solve the isomorphism problem explicitly.

First, we achieve partial characterization by isolating two points of $\mathcal{L}_{18,25}(a)$, namely $a = 0, -1$, and obtain

$\text{inv } \mathcal{L}_{18,25}(0)$	$(8, 4, 0)(8, 4, 2, 0)(2, 5, 8)$	4	[21, 23, 10, 14]
$\text{inv } \mathcal{L}_{18,25}(-1)$	$(8, 4, 0)(8, 4, 2, 0)(2, 5, 8)$	4	[22, 22, 10, 13]
$\text{inv } \mathcal{L}_{18,25}(a) \quad a \neq 0, -1$	$(8, 4, 0)(8, 4, 2, 0)(2, 5, 8)$	4	[20, 22, 10, 13]

Calculating the dimensions of the associated Jordan algebras we obtain for all $a \in \mathbb{C}$:

$$\dim \text{der}_{(1,1,-1)} \mathcal{L}_{18,25}(a) = 9$$

$$\dim \text{der}_{(0,1,-1)} \mathcal{L}_{18,25}(a) = 18.$$

The invariant function ψ^0 is not again of much use:

α	0	1	
$\psi^0 \mathcal{L}_{18,25}(a)(\alpha)$	16	10	9

However, the calculation of the invariant function ψ yield:

$a = 0$			$a = 1$				
α	0		α	0	-1	1	
$\psi \mathcal{L}_{18,25}(0)(\alpha)$	23	21	$\psi \mathcal{L}_{18,25}(1)(\alpha)$	22	21	20	19

$a = -1$			
α	0	1	
$\psi \mathcal{L}_{18,25}(-1)(\alpha)$	22	22	19

$a \neq 0, \pm 1$					
α	0	1	$-a$	$-\frac{1}{a}$	
$\psi \mathcal{L}_{18,25}(a)(\alpha)$	22	20	20	20	19

In that last table ($a \neq 0, \pm 1$) we observe that the function ψ has the same form for pairs $\mathcal{L}_{18,25}(a)$ and $\mathcal{L}_{18,25}(1/a)$. This indicates possible isomorphism between these pairs and,

indeed, we verify directly that

$$\mathcal{L}_{18,25}(a) \cong \mathcal{L}_{18,25}(1/a), \quad a \neq 0, \pm 1.$$

Finally since for $a, a' \neq 0, \pm 1$, $a \neq a'$, $a \neq 1/a'$ the inequality

$$\psi \mathcal{L}_{18,25}(a)(-a) = 20 \neq \psi \mathcal{L}_{18,25}(a')(-a) = 19$$

holds, the relation $\mathcal{L}(a) \not\cong \mathcal{L}(a')$ is thus guaranteed. The invariant function ψ provides us therefore with a complete characterization of the continuum $\mathcal{L}_{18,25}(a)$.

Example 25. Similarly to the previous example, consider the following graded contraction of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$:

$$\varepsilon^{17,13}(a) = \begin{pmatrix} 0 & 0 & a & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.19)$$

We determine this graded contraction by listing its non-zero commutation relations in \mathbb{Z}_3 -labeled basis $(l_{01}, l_{02}, l_{10}, l_{20}, l_{11}, l_{22}, l_{12}, l_{21})$:

$$\begin{aligned} \mathcal{L}_{17,13}(a) \quad [l_{01}, l_{10}] &= al_{11}, \quad [l_{01}, l_{20}] = l_{21}, \quad [l_{01}, l_{22}] = l_{20}, \quad [l_{02}, l_{10}] = l_{12}, \\ [l_{10}, l_{11}] &= l_{21}, \quad [l_{10}, l_{22}] = l_{02}, \quad [l_{20}, l_{22}] = l_{12}, \quad a \neq 0 \end{aligned}$$

Lie algebras $\mathcal{L}_{17,13}(a)$ are all indecomposable and nilpotent. Isolating one point, $a = -1$ we obtain

$$\begin{array}{ll} \text{inv } \mathcal{L}_{17,13}(-1) & (8, 5, 0)(8, 5, 2, 0)(2, 5, 8) \quad 4 \quad [19, 19, 8, 9] \\ \text{inv } \mathcal{L}_{17,13}(a) \quad a \neq 0, -1 & (8, 5, 0)(8, 5, 2, 0)(2, 5, 8) \quad 4 \quad [17, 19, 8, 9] \end{array}$$

Calculating the dimensions of the associated Jordan algebras we obtain for all $a \in \mathbb{C}$:

$$\begin{aligned} \dim \text{der}_{(1,1,-1)} \mathcal{L}_{17,13}(a) &= 7 \\ \dim \text{der}_{(0,1,-1)} \mathcal{L}_{17,13}(a) &= 18. \end{aligned}$$

The invariant function ψ^0 has the form

α	0	1	
$\psi^0 \mathcal{L}_{17,13}(a)(\alpha)$	16	8	7

and the invariant function ψ has the form

$a = 1$				
α	-1	0	1	
$\psi \mathcal{L}_{17,13}(1)(\alpha)$	19	19	17	16

$a = -1$				
α	0	1	-1	
$\psi \mathcal{L}_{17,13}(-1)(\alpha)$	19	19	17	16

$a \neq \pm 1$						
α	0	1	-1	$-a$	$-\frac{1}{a}$	
$\psi \mathcal{L}_{17,13}(a)(\alpha)$	19	17	17	17	17	16

After verifying $\mathcal{L}_{17,13}(a) \cong \mathcal{L}_{17,13}(1/a)$, $a \neq 0, \pm 1$, we conclude that the function ψ again represents a priceless instrument providing complete description of presented parametric continuum of Lie algebras.

Example 26. Consider the following graded contraction of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$:

$$\varepsilon^{17,7}(a) = \begin{pmatrix} 0 & 0 & a & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.20)$$

where $a \neq 0$. We may also determine this graded contraction by listing its non-zero commutation relations in \mathbb{Z}_3 -labeled basis $(l_{01}, l_{02}, l_{10}, l_{20}, l_{11}, l_{22}, l_{12}, l_{21})$:

$$\mathcal{L}_{17,7}(a) : \quad [l_{01}, l_{10}] = -al_{11}, \quad [l_{01}, l_{20}] = l_{21}, \quad [l_{01}, l_{11}] = l_{12}, \quad [l_{01}, l_{22}] = l_{20}, \\ [l_{02}, l_{10}] = l_{12}, \quad [l_{02}, l_{22}] = l_{21}, \quad [l_{10}, l_{11}] = l_{21}, \quad a \neq 0.$$

Lie algebras $\mathcal{L}_{17,7}(a)$ are all indecomposable and nilpotent. We obtain for all $a \in \mathbb{C}$, $a \neq 0$:

$$\text{inv } \mathcal{L}_{17,7}(a) \quad (8, 4, 0)(8, 4, 2, 0)(2, 5, 8) \quad 2 \quad [19, 20, 9, 12]$$

Computing the dimensions of the associated Jordan algebras we obtain for all $a \in \mathbb{C}$:

$$\dim \text{der}_{(1,1,-1)} \mathcal{L}_{17,7}(a) = 8 \\ \dim \text{der}_{(0,1,-1)} \mathcal{L}_{17,7}(a) = 17.$$

The invariant functions ψ^0 and ψ have the following form:

α	0	1	
$\psi^0 \mathcal{L}_{17,7}(a)(\alpha)$	16	9	8

α	0	1	
$\psi \mathcal{L}_{17,7}(a)(\alpha)$	20	19	18

In this case, the function ψ completely fails – does not depend on $a \neq 0$. We are able, however, to advance by calculation of the function φ :

$$a = 1$$

α	0	1	-1	
$\varphi \mathcal{L}_{17,7}(1)(\alpha)$	112	83	81	80

$$a = -1$$

α	0	1	-1	
$\varphi \mathcal{L}_{17,7}(-1)(\alpha)$	104	83	81	80

$$a = \frac{1}{4} + \frac{\sqrt{7}}{4}i$$

α	0	1	$-\frac{1}{4} - \frac{\sqrt{7}}{4}i$	
$\varphi \mathcal{L}_{17,7}\left(\frac{1}{4} + \frac{\sqrt{7}}{4}i\right)(\alpha)$	104	82	82	80

$$a = \frac{1}{4} - \frac{\sqrt{7}}{4}i$$

α	0	1	$-\frac{1}{4} + \frac{\sqrt{7}}{4}i$	
$\varphi \mathcal{L}_{17,7}\left(\frac{1}{4} - \frac{\sqrt{7}}{4}i\right)(\alpha)$	104	82	82	80

$$a = \frac{1}{3}$$

α	0	1	$-\frac{1}{3}$	
$\varphi \mathcal{L}_{17,7}\left(\frac{1}{3}\right)(\alpha)$	104	83	81	80

$$a \neq 0, \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$$

α	0	1	$-a$	$-\frac{1}{2} + \frac{1}{2a}$	
$\varphi \mathcal{L}_{17,7}(a)(\alpha)$	104	82	81	81	80

In order to verify that

$$\varphi \mathcal{L}_{17,7}(a) : 104_1, 82_1, 81_2, 80, \quad a \neq 0, \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4}i \quad (4.21)$$

we have to check the equality

$$-a = -\frac{1}{2} + \frac{1}{2a}$$

which has the solutions $\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$. Thus, (4.21) is verified.

We proceed to solve the relation

$$\varphi \mathcal{L}_{17,7}(a) = \varphi \mathcal{L}_{17,7}(a'), \quad a, a' \neq 0, \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$$

and we obtain

1. If $-a = -a'$, $-\frac{1}{2} + \frac{1}{2a} = \frac{1}{2} + \frac{1}{2a'}$ then $a = a'$.

2. If $-a = -\frac{1}{2} + \frac{1}{2a'}$, $-a' = -\frac{1}{2} + \frac{1}{2a}$ then $a = a' = \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$.

The second case is not possible. Observing that all other tables of the function

$$\varphi \mathcal{L}_{17,7}(a), a = \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$$

are mutually different, we have: if $\varphi \mathcal{L}_{17,7}(a) = \varphi \mathcal{L}_{17,7}(a')$, $a, a' \neq 0$ then $a = a'$. We conclude that even though the function ψ did not distinguish the algebras in nilpotent parametric continuum, the function φ provided their complete description.

A further analysis of the parametric continua resulting from the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$ as well as the analysis of all graded contractions of $\mathfrak{sl}(3, \mathbb{C})$ will be done elsewhere [28]. We summarize the results from Examples 24, 25 and 26 into the following theorem.

Theorem 4.2.1. Let $\mathcal{L}_{18,25}(a)$, $\mathcal{L}_{17,13}(a)$ and $\mathcal{L}_{17,7}(a)$, $a \neq 0$ be graded contractions of the Pauli graded $\mathfrak{sl}(3, \mathbb{C})$, defined by the contraction matrices (4.18), (4.19) and (4.20), respectively. Then it holds:

1. $\mathcal{L}_{18,25}(a) \cong \mathcal{L}_{18,25}(a')$ if and only if $a' = a, 1/a$.
2. $\mathcal{L}_{17,13}(a) \cong \mathcal{L}_{17,13}(a')$ if and only if $a' = a, 1/a$.
3. $\mathcal{L}_{17,7}(a) \cong \mathcal{L}_{17,7}(a')$ if and only if $a' = a$.

Conclusion

In Chapter 2 we have introduced a new concept – the (α, β, γ) –derivations and presented results related to their structure and significance as invariants. There exist, however, several non-equivalent ways of generalizing the derivation of a Lie algebra. For example in [19], a linear operator $A \in \text{End } \mathcal{L}$ is called a (σ, τ) –derivation of \mathcal{L} if for some $\sigma, \tau \in \text{End } \mathcal{L}$ the property $A[x, y] = [Ax, \tau y] + [\sigma x, Ay]$ holds for all $x, y \in \mathcal{L}$. This generalization for σ, τ homomorphisms appears already in [21]. If there exists $B \in \text{der } \mathcal{L}$ such that for all $x, y \in \mathcal{L}$ the condition $A[x, y] = [Ax, y] + [x, By]$ holds, then the operator A forms another generalization [8]. A more general definition emerged in [24] and runs as follows: $A \in \text{End } \mathcal{L}$ is called a *generalized derivation* of \mathcal{L} if there exist $B, C \in \text{End } \mathcal{L}$ such that for all $x, y \in \mathcal{L}$ the property $C[x, y] = [Ax, y] + [x, By]$ holds.

We see that various definitions of generalized derivations were formulated with the aid of some operators inserted into the equation $D[x, y] = [Dx, y] + [x, Dy]$. But for the ‘invariance’ of the definition (2.1), i. e. the validity of the equation (2.3), it is essential that these operators commute with an arbitrary isomorphism of the Lie algebra. Thus, we have chosen in (2.1), from this point of view, a ‘maximal’ set of operators, i. e. multiples of the identity operator.

Among the results of the concept of (α, β, γ) –derivations are ‘associated’ Lie and Jordan algebras. Jordan algebra $\text{der}_{(1,1,-1)} \mathcal{L}$ and Lie algebras $\text{der}_{(0,1,1)} \mathcal{L}$, $\text{der}_{(1,1,1)} \mathcal{L} \cap \text{der}_{(0,1,1)} \mathcal{L}$ seem to be unnoticed in the existing literature. We demonstrated in Proposition 2.3.2 and Example 4 that their dimension and structure are a really useful addition to the knowledge about given Lie algebra. The sets $\text{der}_{(1,1,0)} \mathcal{L}$ and $\text{der}_{(0,1,-1)} \mathcal{L}$ appeared in [24] and are called *centroid* and *quasicentroid*, respectively. The inquiry under which conditions these sets coincide has also been discussed in [24].

In Chapter 3 we have defined the set of κ –twisted cocycles $Z^q(\mathcal{L}, f, \kappa)$ for an arbitrary representation. Then we investigated in detail the case $f = \text{ad}_{\mathcal{L}}$. This study provided a complete description of four–dimensional complex Lie algebras and the algorithm for

their identification. There are, however, two other obvious choices for the representation f : either the representation $\text{ad}_{\mathcal{L}}^*$, or the trivial representation $f : \mathcal{L} \rightarrow \mathbb{C}$. Another option is to define generalized prederivations (see e. g. [12]) – i. e. insert four parameters into the equation

$$P[x, [y, z]] = [Px, [y, z]] + [x, [Py, z]] + [x, [y, Pz]], \quad \forall x, y, z \in \mathcal{L}.$$

None of these possibilities turned out to be very fruitful, therefore we did not investigate such generalizations in detail. Note that according to Proposition 1.5.1 it is pointless to modify by parameters the operator Lie multiplication $XY - YX$, where $X, Y \in \text{End } \mathcal{L}$.

In Chapter 4 we have applied the concept of twisted cocycles to formulate a necessary criterion for the existence of continuous contraction. We demonstrated that Corollary 4.1.4 decides about the existence of a continuous contraction for three–dimensional Lie and two–dimensional Jordan algebras. In Examples 24, 25 and 26, we have successfully applied the invariant functions to the eight–dimensional graded contractions of the Pauli graded $\text{sl}(3, \mathbb{C})$.

Resolving parametric continua of Lie algebras belongs to the most challenging parts of their classification. Except for the explicit calculation, all known tools for such resolving were based on the adjoint representation. 'Trace' invariants χ_i and C_{pq} , defined by relations (1.25), (1.24) and able to resolve a parametric continuum, have been successfully used in [3, 11, 27]. Advantages of these invariants are easy calculation and invariance under continuous contractions. One may also consider action of the adjoint representation on the nilradical of a Lie algebra – considering eigenvalues of this action one may also resolve a parametric continuum. All these approaches, however, fail in the case of a nilpotent continuum – due to Theorem 1.2.3 none of the invariants χ_i nor C_{pq} exists, eigenvalues of the adjoint representation are all zeros. We have seen in Examples 24, 25 and 26 that such cases may resolve our new invariant functions ψ, φ – their form is non–trivial and depends on the parameter of a nilpotent continuum. The main idea behind the present work – the classification of all graded contractions of $\text{sl}(3, \mathbb{C})$ – can now be easier to achieve.

It is computationally advantageous that determining associated Lie algebras is a linear problem: in order to determine the space $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{L}$ for fixed $\alpha, \beta, \gamma \in \mathbb{C}$, one has to solve homogeneous system of linear equations (4.6). The investigation how the dimension of the vector space $\text{der}_{(\delta, 1, 1)} \mathcal{L}$ depends on δ – computing the function ψ – is more challenging. One has to analyze the rank of the δ –dependent matrix corresponding to the linear system (4.6). Determining the rank of parametric matrices is called a *specialization problem*.

There has been some progress concerning the application of various computational algorithms to parametric matrices. The results, however, are unsatisfactory and computation of the rank of parametric matrices is not implemented in standard symbolic mathematical tools – MAPLE, Mathematica. It was therefore necessary to develop a new algorithm. This new algorithm is based on the standard Gaussian elimination with row pivoting. The given column is at first searched for a non-zero complex number. If this number is not found, then the δ -dependent element with minimal length is assumed to be non-zero and this assumption is added to some set – the set of assumptions. The resulting set of assumptions is analyzed and solved for the equality of its elements to zero. For these solutions, special values of δ , the standard Gaussian elimination is performed again. This algorithm, implemented in MAPLE VIII, turned out to be sufficient for our purpose. However, it is in some cases quite computationally demanding. Namely, computing Examples 13, 14 from [3] took a significant amount of computational time. Even more demanding is the computing the functions φ, φ^0 for eight-dimensional Lie algebras – the δ -dependent matrices have 224 columns and contain approximately two thousand rows.

Thus, allowing a compact formulation in Theorem 3.2.9, the significance of the functions ψ, φ in dimension four is more theoretical than computationally advantageous. The invariants χ_i provide sufficient description of parametric four-dimensional Lie algebras. On the other hand, the knowledge about the dimensions of highly non-trivial structures, interlaced with given Lie algebra, may also be very valuable. Considering parametric continua of nilpotent algebras the invariant functions ψ, φ, φ^0 seem to be even more important. In these cases, the behaviour of the invariant functions ψ, φ, φ^0 is quite unique and irreplaceable.

Appendix A

Classification of (α, β, γ) -derivations of two and three-dimensional complex Lie and Jordan Algebras

Appendix A is divided into two sections. Section A.1 contains matrices of (α, β, γ) -derivations of two and three-dimensional non-abelian complex Lie algebras. Section A.2 contains matrices of (α, β, γ) -derivations of two-dimensional non-abelian complex Jordan algebras. Instead of symbols $\text{der}_{(\alpha, \beta, \gamma)} \mathcal{A}$, abbreviated symbols $\text{der}_{(\alpha, \beta, \gamma)}$ are used.

A.1 Lie Algebras

Two-dimensional Complex Lie Algebras

$$\mathfrak{g}_{2,1} : \quad [e_1, e_2] = e_2$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong \text{sl}(2, \mathbb{C})$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$

- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,-1)} = \{0\}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$
- $\text{der}_{(\delta,1,0)} = \{0\}_{\delta \neq 1}$
- $\text{der}_{(\delta,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\delta \end{pmatrix} \right\}_{\delta \neq \pm 1}$
- $\text{der}_{(-1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$

$$\mathfrak{g}_{3,4}(a) : \quad [e_1, e_3] = e_1, [e_2, e_3] = ae_2, a \neq 0, \pm 1$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1} \oplus \mathfrak{g}_{2,1}$
- $\text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong \mathfrak{g}_{3,3}$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_{3,3}$
- $\text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cong 2\mathfrak{g}_1$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,-1)} = \{0\}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$
- $\text{der}_{(\delta,1,0)} = \{0\}_{\delta \neq 1}$
- $\text{der}_{(\delta,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\delta \end{pmatrix} \right\}_{\delta \neq 1, a, 1/a}$
- $\text{der}_{(a,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+a \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$
- $\text{der}_{(\frac{1}{a},1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\frac{1}{a} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$

$$\mathfrak{sl}(2, \mathbb{C}) : \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = 2e_2$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong \mathfrak{sl}(2, \mathbb{C})$
- $\text{der}_{(0,1,1)} = \{0\}$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \{0\}$
- $\text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \{0\}$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,-1)} = \{0\}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}_{j_{1,1}}$
- $\text{der}_{(\delta,1,0)} = \{0\}_{\delta \neq 1}$
- $\text{der}_{(\delta,1,1)} = \{0\}_{\delta \neq \pm 1, 2}$
- $\text{der}_{(2,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$
- $\text{der}_{(-1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$

A.2 Jordan Algebras

Two-dimensional Complex Jordan Algebras

$$j_{2,1} : \quad e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(0,1,1)} = \{0\}$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong 2\mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \{0\}$
- $\text{der}_{(0,1,0)} = \{0\}$

- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \{0\}$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,-1)} = \{0\}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{j}_{2,1}$
- $\text{der}_{(\delta,1,0)} = \{0\}_{\delta \neq 1}$
- $\text{der}_{(\delta,1,1)} = \{0\}_{\delta \neq 1,2}$
- $\text{der}_{(2,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$\mathfrak{j}_{2,2} : \quad e_2 \circ e_2 = e_2$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong 2\mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(0,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{j}_{1,1}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- $\text{der}_{(\delta,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}_{\delta \neq 0,1}$
- $\text{der}_{(\delta,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}_{\delta \neq 0,2}$
- $\text{der}_{(2,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$\mathfrak{j}_{2,3} : \quad e_1 \circ e_1 = e_2$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$

- $\text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong 2\mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(0,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathfrak{j}_{2,4}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- $\text{der}_{(\delta,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- $\text{der}_{(\delta,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \delta & 0 \\ 0 & 2 \end{pmatrix} \right\}$

$$\mathfrak{j}_{2,4} : \quad e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = \frac{1}{2}e_2$$

- $\text{der}_{(1,1,1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_{2,1}$
- $\text{der}_{(0,1,1)} = \{0\}$
- $\text{der}_{(1,1,0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{g}_1$
- $\text{der}_{(1,0,0)} = \{0\}$
- $\text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,1)} \cap \text{der}_{(0,1,1)} = \{0\}$
- $\text{der}_{(1,0,0)} \cap \text{der}_{(0,1,0)} = \{0\}$
- $\text{der}_{(1,1,-1)} = \{0\}$
- $\text{der}_{(0,1,-1)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathfrak{j}_{2,1}$
- $\text{der}_{(\delta,1,0)} = \{0\}_{\delta \neq \frac{1}{2}, 1}$
- $\text{der}_{(\frac{1}{2}, 1, 0)} = \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

$$\cdot \operatorname{der}_{(\delta,1,1)} = \{0\}_{\delta \neq 1,2}$$

$$\cdot \operatorname{der}_{(2,1,1)} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$j_{2,5} : \quad e_1 \circ e_1 = e_1, \quad e_2 \circ e_2 = -e_1, \quad e_1 \circ e_2 = e_2$$

$$\cdot \operatorname{der}_{(1,1,1)} = \{0\}$$

$$\cdot \operatorname{der}_{(0,1,1)} = \{0\}$$

$$\cdot \operatorname{der}_{(1,1,0)} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong 2g_1$$

$$\cdot \operatorname{der}_{(1,0,0)} = \{0\}$$

$$\cdot \operatorname{der}_{(0,1,0)} = \{0\}$$

$$\cdot \operatorname{der}_{(1,1,1)} \cap \operatorname{der}_{(0,1,1)} = \{0\}$$

$$\cdot \operatorname{der}_{(1,0,0)} \cap \operatorname{der}_{(0,1,0)} = \{0\}$$

$$\cdot \operatorname{der}_{(1,1,-1)} = \{0\}$$

$$\cdot \operatorname{der}_{(0,1,-1)} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong j_{2,1}$$

$$\cdot \operatorname{der}_{(\delta,1,0)} = \{0\}_{\delta \neq 1}$$

$$\cdot \operatorname{der}_{(\delta,1,1)} = \{0\}_{\delta \neq 2}$$

$$\cdot \operatorname{der}_{(2,1,1)} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Appendix B

Invariant Functions of complex Lie and Jordan Algebras of dimensions 2, 3 and 4

Appendix B is divided into two sections. Section B.1 contains the classification of complex Lie algebras up to dimension four and the invariant functions ψ, φ, φ^0 . We basically follow the notation of [27] and present a list connecting it to other notations. Section B.2 contains the classification of one and two-dimensional complex Jordan algebras [5] and the tables of the invariant function ψ . Instead of the symbols $\psi \mathcal{A}, \varphi \mathcal{A}, \varphi^0 \mathcal{A}$, abbreviated symbols ψ, φ, φ^0 are used. Blank spaces in the tables of the functions ψ, φ, φ^0 denote general complex numbers, different from all previously listed values, e. g. it holds:

$$\psi_{\mathfrak{g}_{3,4}(-1)}(\alpha) = 3, \quad \alpha \in \mathbb{C}, \alpha \neq \pm 1.$$

B.1 Lie Algebras

Two-dimensional Complex Lie Algebras

$2\mathfrak{g}_1$: Abelian

α	1	
$\psi(\alpha)$	4	4

α	
$\varphi(\alpha)$	2

α	
$\varphi^0(\alpha)$	2

$\mathfrak{g}_{2,1}$: $[e_1, e_2] = e_1$

α	1	0
$\psi(\alpha)$	2	3

α	
$\varphi(\alpha)$	2

α	2
$\varphi^0(\alpha)$	1

Three-dimensional Complex Lie Algebras

$3g_1$: Abelian

α	1	
$\psi(\alpha)$	9	9

α		
$\varphi(\alpha)$	9	

α		
$\varphi^0(\alpha)$	9	

$g_{2,1} \oplus g_1$: $[e_1, e_2] = e_2$

α	1	0	
$\psi(\alpha)$	4	6	4

α		
$\varphi(\alpha)$	6	

α	1	2	
$\varphi^0(\alpha)$	2	2	1

$g_{3,1}$: $[e_2, e_3] = e_1$

α	1	
$\psi(\alpha)$	6	6

α	0	
$\varphi(\alpha)$	9	8

α		
$\varphi^0(\alpha)$	3	

$g_{3,2}$: $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$

α	1	
$\psi(\alpha)$	4	3

α		
$\varphi(\alpha)$	6	

α	2	
$\varphi^0(\alpha)$	2	0

$g_{3,3}$: $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$

α	1	
$\psi(\alpha)$	6	3

α		
$\varphi(\alpha)$	6	

α	2	
$\varphi^0(\alpha)$	6	0

$g_{3,4}(-1)$: $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$

α	1	-1	
$\psi(\alpha)$	4	5	3

α	0	
$\varphi(\alpha)$	9	7

α	0	2	
$\varphi^0(\alpha)$	2	2	0

$g_{3,4}(a)$: $[e_1, e_3] = e_1$, $[e_2, e_3] = ae_2$, $a \neq 0, \pm 1$

α	1	a	$\frac{1}{a}$	
$\psi(\alpha)$	4	4	4	3

α		
$\varphi(\alpha)$	6	

α	2	$1+a$	$1+\frac{1}{a}$	
$\varphi^0(\alpha)$	2	1	1	0

$$\mathfrak{sl}(2, \mathbb{C}) : \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = 2e_2$$

α	1	-1	2	
$\psi(\alpha)$	3	5	1	0

α	0	
$\varphi(\alpha)$	9	6

α	2	
$\varphi^0(\alpha)$	1	0

Four-dimensional Complex Lie Algebras

(g-1) $4\mathfrak{g}_1$: Abelian

α	1	
$\psi(\alpha)$	16	16

α	
$\varphi(\alpha)$	24

α	
$\varphi^0(\alpha)$	24

(g-2) $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$: $[e_1, e_2] = e_1$

α	1	0	
$\psi(\alpha)$	8	11	8

α	0	
$\varphi(\alpha)$	16	14

α	1	2	
$\varphi^0(\alpha)$	8	7	6

(g-3) $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_{2,1}$: $[e_1, e_2] = e_1, [e_3, e_4] = e_3$

α	1	0	
$\psi(\alpha)$	4	6	4

α	0	1	
$\varphi(\alpha)$	12	12	10

α	1	2	
$\varphi^0(\alpha)$	2	2	0

(g-4) $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$: $[e_2, e_3] = e_1$

α	1	0	
$\psi(\alpha)$	10	11	10

α	0	
$\varphi(\alpha)$	20	19

α	1	
$\varphi^0(\alpha)$	11	8

(g-5) $\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$: $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$

α	1	0	
$\psi(\alpha)$	6	7	5

α	1	
$\varphi(\alpha)$	13	12

α	1	2	
$\varphi^0(\alpha)$	3	3	1

(g-6) $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$: $[e_1, e_3] = e_1, [e_2, e_3] = e_2$

α	1	0	
$\psi(\alpha)$	8	7	5

α	1	
$\varphi(\alpha)$	15	12

α	1	2	
$\varphi^0(\alpha)$	3	7	1

(g-7) $\mathfrak{g}_{3,4}(-1) \oplus \mathfrak{g}_1$: $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$

α	1	0	-1	
$\psi(\alpha)$	6	7	7	5

α	1	0	-1	
$\varphi(\alpha)$	15	16	16	14

α	1	2	0	
$\varphi^0(\alpha)$	3	3	3	1

(g-8) $\mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1$: $[e_1, e_3] = e_1, [e_2, e_3] = ae_2, a \neq 0, \pm 1$

α	1	0	a	$\frac{1}{a}$	
$\psi(\alpha)$	6	7	6	6	5

α	1	a	$\frac{1}{a}$	
$\varphi(\alpha)$	13	13	13	12

α	1	2	$1+a$	$1+\frac{1}{a}$	
$\varphi^0(\alpha)$	3	3	2	2	1

(g-9) $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_1$: $[e_1, e_2] = e_1, [e_2, e_3] = e_3, [e_1, e_3] = 2e_2$

α	1	0	-1	2	
$\psi(\alpha)$	4	4	6	2	1

α	1	0	-1	$\frac{1}{2}$	
$\varphi(\alpha)$	12	12	14	10	9

α	2	
$\varphi^0(\alpha)$	1	0

(g-10) $\mathfrak{g}_{4,1}$: $[e_2, e_4] = e_1, [e_3, e_4] = e_2$

α	1	
$\psi(\alpha)$	7	7

α	-1	0	
$\varphi(\alpha)$	16	16	15

α		
$\varphi^0(\alpha)$	3	

(g-11) $\mathfrak{g}_{4,2}(a)$: $[e_1, e_4] = ae_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3, a \neq 0, \pm 1, -2$

α	1	a	$\frac{1}{a}$	
$\psi(\alpha)$	6	5	5	4

α	$1+a$	$\frac{2}{a}$	
$\varphi(\alpha)$	13	13	12

α	2	$1+a$	$1+\frac{1}{a}$	
$\varphi^0(\alpha)$	3	1	1	0

(g-12) $\mathfrak{g}_{4,2}(1)$: $[e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$

α	1	
$\psi(\alpha)$	8	4

α	2	
$\varphi(\alpha)$	15	12

α	2	
$\varphi^0(\alpha)$	7	0

(g-13) $g_{4,2}(-2)$: $[e_1, e_4] = -2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$

α	1	-2	$-\frac{1}{2}$	
$\psi(\alpha)$	6	5	5	4

α	-1	
$\varphi(\alpha)$	15	12

α	2	-1	$\frac{1}{2}$
$\varphi^0(\alpha)$	3	1	1
			0

(g-14) $g_{4,2}(-1)$: $[e_1, e_4] = -e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$

α	1	-1	
$\psi(\alpha)$	6	6	4

α	-2	0
$\varphi(\alpha)$	13	16
		12

α	0	2
$\varphi^0(\alpha)$	2	3
		0

(g-15) $g_{4,3}$: $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$

α	1	0
$\psi(\alpha)$	6	7
		6

α	0
$\varphi(\alpha)$	16
	13

α	1	2
$\varphi^0(\alpha)$	3	3
		2

(g-16) $g_{4,4}$: $[e_1, e_4] = e_1$, $[e_2, e_4] = e_1 + e_2$, $[e_3, e_4] = e_2 + e_3$

α	1
$\psi(\alpha)$	6
	4

α	2
$\varphi(\alpha)$	13
	12

α	1	2
$\varphi^0(\alpha)$	0	3
		0

(g-17) $g_{4,5}(a, b)$: $[e_1, e_4] = ae_1$, $[e_2, e_4] = be_2$, $[e_3, e_4] = e_3$,
 $a \neq 0, \pm 1, \pm b, 1/b, b^2, -1 - b$, $b \neq 0, \pm 1, \pm a, 1/a, a^2, -1 - a$

α	1	a	$\frac{1}{a}$	b	$\frac{1}{b}$	$\frac{a}{b}$	$\frac{b}{a}$	
$\psi(\alpha)$	6	5	5	5	5	5	5	4

α	$a + b$	$\frac{1+a}{b}$	$\frac{1+b}{a}$
$\varphi(\alpha)$	13	13	13
			12

α	2	$1 + a$	$1 + b$	$1 + \frac{1}{a}$	$1 + \frac{1}{b}$	$1 + \frac{a}{b}$	$1 + \frac{b}{a}$	
$\varphi^0(\alpha)$	3	1	1	1	1	1	1	0

(g-18) $g_{4,5}(a, -1 - a)$: $[e_1, e_4] = ae_1$, $[e_2, e_4] = (-1 - a)e_2$, $[e_3, e_4] = e_3$,
 $a \neq 0, \pm 1, -2, -1/2, -1/2 \pm i\sqrt{3}/2$

α	1	a	$\frac{1}{a}$	$-1 - a$	$-\frac{1}{1+a}$	$-\frac{a}{a+1}$	$-\frac{a+1}{a}$	
$\psi(\alpha)$	6	5	5	5	5	5	5	4

α	-1
$\varphi(\alpha)$	15
	12

α	2	$1 + a$	$-a$	$1 + \frac{1}{a}$	$\frac{a}{a+1}$	$\frac{1}{a+1}$	$-\frac{1}{a}$	
$\varphi^0(\alpha)$	3	1	1	1	1	1	1	0

(g-19) $g_{4,5}(a, a^2)$: $[e_1, e_4] = ae_1$, $[e_2, e_4] = a^2e_2$, $[e_3, e_4] = e_3$,
 $a \neq 0, \pm 1, \pm i, -1/2 \pm i\sqrt{3}/2$

α	1	a	$\frac{1}{a}$	a^2	$\frac{1}{a^2}$	
$\psi(\alpha)$	6	6	6	5	5	4

α	$a + a^2$	$\frac{a+1}{a^2}$	$\frac{a^2+1}{a}$	
$\varphi(\alpha)$	13	13	13	12

α	2	$1 + a$	$1 + a^2$	$1 + \frac{1}{a}$	$1 + \frac{1}{a^2}$	
$\varphi^0(\alpha)$	3	2	1	2	1	0

(g-20) $g_{4,5}(a, 1)$: $[e_1, e_4] = ae_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$,
 $a \neq 0, \pm 1, -2$

α	1	a	$\frac{1}{a}$	
$\psi(\alpha)$	8	6	6	4

α	$1 + a$	$\frac{2}{a}$	
$\varphi(\alpha)$	15	13	12

α	2	$1 + a$	$1 + \frac{1}{a}$	
$\varphi^0(\alpha)$	7	2	2	0

(g-21) $g_{4,5}(a, -1)$: $[e_1, e_4] = ae_1$, $[e_2, e_4] = -e_2$, $[e_3, e_4] = e_3$,
 $a \neq 0, \pm 1, \pm i$

α	1	a	$\frac{1}{a}$	-1	-a	$-\frac{1}{a}$	
$\psi(\alpha)$	6	5	5	6	5	5	4

α	$-1 + a$	$-1 - a$	0	
$\varphi(\alpha)$	13	13	16	12

α	2	$1 + a$	0	$1 + \frac{1}{a}$	$1 - a$	$1 - \frac{1}{a}$	
$\varphi^0(\alpha)$	3	1	2	1	1	1	0

(g-22) $g_{4,5}(1, 1)$: $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$

α	1	
$\psi(\alpha)$	12	4

α	2	
$\varphi(\alpha)$	18	12

α	2	
$\varphi^0(\alpha)$	18	0

(g-23) $g_{4,5}(-1, 1)$: $[e_1, e_4] = -e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$

α	1	-1	
$\psi(\alpha)$	8	8	4

α	0	-2	
$\varphi(\alpha)$	20	13	12

α	2	0	
$\varphi^0(\alpha)$	7	4	0

(g-24) $g_{4,5}(-2, 1) : [e_1, e_4] = -2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$

α	1	-2	$-\frac{1}{2}$	
$\psi(\alpha)$	8	6	6	4

α	-1	
$\varphi(\alpha)$	16	12

α	2	-1	$\frac{1}{2}$	
$\varphi^0(\alpha)$	7	2	2	0

(g-25) $g_{4,5}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i) : [e_1, e_4] = (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)e_1, [e_2, e_4] = (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)e_2, [e_3, e_4] = e_3$

α	1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}i$	
$\psi(\alpha)$	6	7	7	4

α	-1	
$\varphi(\alpha)$	15	12

α	2	$\frac{1}{2} - \frac{\sqrt{3}}{2}i$	$\frac{1}{2} + \frac{\sqrt{3}}{2}i$	
$\varphi^0(\alpha)$	3	3	3	0

(g-26) $g_{4,5}(i, -1) : [e_1, e_4] = ie_1, [e_2, e_4] = -e_2, [e_3, e_4] = e_3$

α	1	i	$-i$	-1	
$\psi(\alpha)$	6	6	6	6	4

α	$-1 + i$	$-1 - i$	0	
$\varphi(\alpha)$	13	13	16	12

α	2	$1 + i$	0	$1 - i$	
$\varphi^0(\alpha)$	3	2	2	2	0

(g-27) $g_{4,7} : [e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$

α	1	2	
$\psi(\alpha)$	5	4	3

α	0	1	3	
$\varphi^0(\alpha)$	12	12	12	11

α	$\frac{3}{2}$	2	
$\varphi^0(\alpha)$	1	1	0

(g-28) $g_{4,8}(a) : [e_2, e_3] = e_1, [e_1, e_4] = (1 + a)e_1, [e_2, e_4] = e_2, [e_3, e_4] = ae_3$
 $a \neq 0, \pm 1, \pm 2, \pm 1/2, -1/2 \pm \sqrt{3}i/2$

α	1	2	a	$\frac{1}{a}$	
$\psi(\alpha)$	5	4	4	4	3

α	0	1	$1 + 2a$	$1 + \frac{2}{a}$	$\frac{5}{3} + \frac{2}{3}(a + \frac{1}{a})$	
$\varphi(\alpha)$	12	12	12	12	12	11

α	2	$\frac{1+2a}{1+a}$	$\frac{2+a}{1+a}$	
$\varphi^0(\alpha)$	1	1	1	0

(g-29) $g_{4,8}(1)$: $[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$

α	1	2	
$\psi(\alpha)$	7	4	3

α	0	1	3	
$\varphi(\alpha)$	12	12	14	11

α	2	$\frac{3}{2}$	
$\varphi^0(\alpha)$	1	2	0

(g-30) $g_{4,8}(2)$: $[e_2, e_3] = e_1$, $[e_1, e_4] = 3e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = 2e_3$

α	1	2	$\frac{1}{2}$	
$\psi(\alpha)$	5	5	4	3

α	0	1	5	2	$\frac{10}{3}$	
$\varphi(\alpha)$	12	12	12	12	12	11

α	2	$\frac{5}{3}$	$\frac{4}{3}$	
$\varphi^0(\alpha)$	1	1	1	0

(g-31) $g_{4,8}(0)$: $[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$

α	1	0	
$\psi(\alpha)$	5	6	4

α	0	1	
$\varphi(\alpha)$	12	13	11

α	1	2	
$\varphi^0(\alpha)$	2	2	0

(g-32) $g_{4,8}(-1)$: $[e_2, e_3] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -e_3$

α	1	-1	
$\psi(\alpha)$	5	6	4

α	1	-1	
$\varphi(\alpha)$	13	14	12

α	2	
$\varphi^0(\alpha)$	1	0

(g-33) $g_{4,8}(-2)$: $[e_2, e_3] = e_1$, $[e_1, e_4] = -e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -2e_3$

α	1	2	-2	$-\frac{1}{2}$	
$\psi(\alpha)$	5	4	4	4	3

α	0	1	-3	
$\varphi(\alpha)$	16	12	12	11

α	2	3	0	
$\varphi^0(\alpha)$	1	1	1	0

(g-34) $g_{4,8}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$: $[e_2, e_3] = e_1$, $[e_1, e_4] = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)e_1$, $[e_2, e_4] = e_2$,
 $[e_3, e_4] = (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)e_3$

α	1	2	$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}i$	
$\psi(\alpha)$	5	4	4	4	3

α	0	1	$\sqrt{3}i$	$-\sqrt{3}i$	
$\varphi(\alpha)$	12	12	12	12	11

α	2	$\frac{3}{2} + \frac{\sqrt{3}}{2}i$	$\frac{3}{2} - \frac{\sqrt{3}}{2}i$	
$\varphi^0(\alpha)$	1	1	1	0

Comparison of Notations of Lie Algebras

We reproduce the list from [27] of isomorphisms of Lie algebras which compares notation based on [27] to those in [11] and [3]. The ordering of items is the same as above. We label four-dimensional Lie algebras by corresponding labels (g-1), ..., (g-34).

$$\begin{array}{l}
 2 \mathfrak{g}_1 \cong \mathbb{C}^2 \\
 \mathfrak{g}_{2,1} \cong \mathfrak{t}_2(\mathbb{C}) \\
 \\
 3 \mathfrak{g}_1 \cong \mathbb{C}^3 \\
 \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1 \cong \mathfrak{t}_2(\mathbb{C}) \oplus \mathbb{C} \\
 \mathfrak{g}_{3,1} \cong \mathfrak{n}_3(\mathbb{C}) \\
 \mathfrak{g}_{3,2} \cong \mathfrak{t}_3(\mathbb{C}) \\
 \mathfrak{g}_{3,3} \cong \mathfrak{t}_{3,1}(\mathbb{C}) \\
 \mathfrak{g}_{3,4}(-1) \cong \mathfrak{t}_{3,-1}(\mathbb{C}) \\
 \mathfrak{g}_{3,4}(a) \cong \mathfrak{t}_{3,a}(\mathbb{C}), \quad a \neq 0, \pm 1 \\
 \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \\
 \\
 \begin{array}{llll}
 \text{(g-1)} & 4 \mathfrak{g}_1 \cong & \mathbb{C}^4 & \cong L_0 \\
 \text{(g-2)} & \mathfrak{g}_{2,1} \oplus 2 \mathfrak{g}_1 \cong & \mathfrak{t}_2(\mathbb{C}) \oplus \mathbb{C}^2 & \cong L_4(\infty) \\
 \text{(g-3)} & \mathfrak{g}_{2,1} \oplus \mathfrak{g}_{2,1} \cong & \mathfrak{t}_2(\mathbb{C}) \oplus \mathfrak{t}_2(\mathbb{C}) & \cong L_9 \\
 \text{(g-4)} & \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \cong & \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C} & \cong L_1 \\
 \text{(g-5)} & \mathfrak{g}_{3,2} \oplus \mathfrak{g}_1 \cong & \mathfrak{t}_3(\mathbb{C}) \oplus \mathbb{C} & \cong L_7(1, 0) \\
 \text{(g-6)} & \mathfrak{g}_{3,3} \oplus \mathfrak{g}_1 \cong & \mathfrak{t}_{3,1}(\mathbb{C}) \oplus \mathbb{C} & \cong L_4(0) \\
 \text{(g-7)} & \mathfrak{g}_{3,4}(-1) \oplus \mathfrak{g}_1 \cong & \mathfrak{t}_{3,-1}(\mathbb{C}) \oplus \mathbb{C} & \cong L_7(-1, 0) \\
 \text{(g-8)} & \mathfrak{g}_{3,4}(a) \oplus \mathfrak{g}_1 \cong & \mathfrak{t}_{3,a}(\mathbb{C}) \oplus \mathbb{C}, & \cong L_7(a, 0) \quad a \neq 0, \pm 1 \\
 \text{(g-9)} & \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_1 \cong & \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} & \cong L_6
 \end{array}
 \end{array}$$

$$\begin{aligned}
\text{(g-10)} \quad & \mathfrak{g}_{4,1} \cong \mathfrak{n}_4 \cong L_2 \\
\text{(g-11)} \quad & \mathfrak{g}_{4,2}(a) \cong \mathfrak{g}_2 \left(\frac{a}{(a+2)^3}, \frac{2a+1}{(a+2)^2} \right) \cong L_7(a, 1) \quad a \neq 0, \pm 1, -2 \\
\text{(g-12)} \quad & \mathfrak{g}_{4,2}(1) \cong \mathfrak{g}_5 \cong L_4(1) \\
\text{(g-13)} \quad & \mathfrak{g}_{4,2}(-2) \cong \mathfrak{g}_3 \left(\frac{27}{4} \right) \cong L_7(-2, 1) \\
\text{(g-14)} \quad & \mathfrak{g}_{4,2}(-1) \cong \mathfrak{g}_2(-1, -1) \cong L_7(-1, 1) \\
\text{(g-15)} \quad & \mathfrak{g}_{4,3} \cong \mathfrak{g}_2(0, 0) \cong L_7(0, 0) \\
\text{(g-16)} \quad & \mathfrak{g}_{4,4} \cong \mathfrak{g}_2 \left(\frac{1}{27}, \frac{1}{3} \right) \cong L_7(1, 1) \\
\text{(g-17)} \quad & \mathfrak{g}_{4,5}(a, b) \cong \mathfrak{g}_2 \left(\frac{ab}{(a+b+1)^3}, \frac{ab+a+b}{(a+b+1)^2} \right) \cong L_7(a, b) \quad (*) \\
\text{(g-18)} \quad & \mathfrak{g}_{4,5}(a, -1-a) \cong \mathfrak{g}_3 \left(\frac{a^2+a+1}{a^2(a+1)^2} \right) \cong L_7(a, -1-a) \quad (**) \\
\text{(g-19)} \quad & \mathfrak{g}_{4,5}(a, a^2) \cong \mathfrak{g}_2 \left(\frac{a^3}{(a+a^2+1)^3}, \frac{a^3+a+a^2}{(a+a^2+1)^2} \right) \cong L_7(a, a^2) \quad (***) \\
\text{(g-20)} \quad & \mathfrak{g}_{4,5}(a, 1) \cong \mathfrak{g}_1(a) \cong L_4(a) \quad a \neq 0, \pm 1, -2 \\
\text{(g-21)} \quad & \mathfrak{g}_{4,5}(a, -1) \cong \mathfrak{g}_2 \left(-\frac{1}{a^2}, -\frac{1}{a^2} \right) \cong L_7(a, -1) \quad a \neq 0, \pm 1, \pm i \\
\text{(g-22)} \quad & \mathfrak{g}_{4,5}(1, 1) \cong \mathfrak{g}_1(1) \cong L_3 \\
\text{(g-23)} \quad & \mathfrak{g}_{4,5}(-1, 1) \cong \mathfrak{g}_1(-1) \cong L_4(-1) \\
\text{(g-24)} \quad & \mathfrak{g}_{4,5}(-2, 1) \cong \mathfrak{g}_1(-2) \cong L_4(-2) \\
\text{(g-25)} \quad & \mathfrak{g}_{4,5} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \cong \mathfrak{g}_4 \cong L_7 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \\
\text{(g-26)} \quad & \mathfrak{g}_{4,5}(i, -1) \cong \mathfrak{g}_2(1, 1) \cong L_7(i, -1) \\
\text{(g-27)} \quad & \mathfrak{g}_{4,7} \cong \mathfrak{g}_{4,8} \left(\frac{1}{4} \right) \cong L_8(1) \\
\text{(g-28)} \quad & \mathfrak{g}_{4,8}(a) \cong \mathfrak{g}_8 \left(\frac{a}{(a+1)^2} \right) \cong L_8(a) \quad a \neq 0, \pm 1, \pm 2, \pm 1/2 \\
\text{(g-29)} \quad & \mathfrak{g}_{4,8}(1) \cong \mathfrak{g}_6 \cong L_5 \\
\text{(g-30)} \quad & \mathfrak{g}_{4,8}(2) \cong \mathfrak{g}_8 \left(\frac{2}{9} \right) \cong L_8(2) \\
\text{(g-31)} \quad & \mathfrak{g}_{4,8}(0) \cong \mathfrak{g}_8(0) \cong L_8(0) \\
\text{(g-32)} \quad & \mathfrak{g}_{4,8}(-1) \cong \mathfrak{g}_7 \cong L_8(-1) \\
\text{(g-33)} \quad & \mathfrak{g}_{4,8}(-2) \cong \mathfrak{g}_8(-2) \cong L_8(-2) \\
\text{(g-34)} \quad & \mathfrak{g}_{4,8} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \cong \mathfrak{g}_8 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \cong L_8 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)
\end{aligned}$$

(*) $a \neq 0, \pm 1, \pm b, 1/b, b^2, -1-b, b \neq 0, \pm 1, \pm a, 1/a, a^2, -1-a$

(**) $a \neq 0, \pm 1, -2, -1/2, -1/2 \pm i\sqrt{3}/2$

(***) $a \neq 0, \pm 1, \pm i, -1/2 \pm i\sqrt{3}/2$

B.2 Jordan Algebras

One-dimensional Complex Jordan Algebras

j_1 : Abelian

α	1	
$\psi(\alpha)$	1	1

$j_{1,1}$: $e_1 \circ e_1 = e_1$

α	1	2	
$\psi(\alpha)$	0	1	0

Two-dimensional Complex Jordan Algebras

$2j_1$: Abelian

α	1	
$\psi(\alpha)$	4	4

$j_{2,1}$: $e_1 \circ e_1 = e_1, e_1 \circ e_2 = e_2$

α	1	2	
$\psi(\alpha)$	1	2	0

$j_{2,2}$: $e_2 \circ e_2 = e_2$

α	1	0	2	
$\psi(\alpha)$	1	2	2	1

$j_{2,3}$: $e_1 \circ e_1 = e_2$

α	1	
$\psi(\alpha)$	2	2

$$j_{2,4} : \quad e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = \frac{1}{2}e_2$$

α	1	2	
$\psi(\alpha)$	2	1	0

$$j_{2,5} : \quad e_1 \circ e_1 = e_1, \quad e_2 \circ e_2 = -e_1, \quad e_1 \circ e_2 = e_2$$

α	1	2	
$\psi(\alpha)$	0	2	0

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