



ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE  
Fakulta jaderná a fyzikálně inženýrská



# Optické transformace a manipulace orbitálního momentu hybnosti svazku

## Optical transforms and optical angular momentum manipulation

Výzkumný úkol

Autor: **Bc. Matyáš Staněk**  
Vedoucí práce: **Ing. Václav Potoček, Ph.D.**  
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*Čestné prohlášení:*

Prohlašuji, že jsem tuto práci vypracoval samostatně a uvedl jsem všechnu použitou literaturu.

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Bc. Matyáš Staněk

*Název práce:*

**Optické transformace a manipulace orbitálního momentu hybnosti svazku**

*Autor:* Bc. Matyáš Staněk

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*Vedoucí práce:* Ing. Václav Potoček, Ph.D., Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská ČVUT v Praze

*Abstrakt:* Cílem této práce je představit různé techniky fázové modulace. Nejprve se zkoumají optické transformace souřadnic matematicky, a pak jsou použity na příkladu. Hlavní část textu je soustředěna na manipulaci skalárních Gaussových svazků, jelikož ty představují základní řešení paraxiální rovnice a navíc se za určitých podmínek chovají jako paprsky. Výsledky jsou pak použity na svazky s orbitálním momentem hybnosti a jsou demonstrovány možnosti aritmetických operací s hodnotou OAM.

*Klíčová slova:* fázové modulace, Gaussov svazek, optické transformace souřadnic, orbitální moment hybnosti

*Title:*

**Optical transforms and optical angular momentum manipulation**

*Author:* Bc. Matyáš Staněk

*Abstract:* The goal of this work is to introduce different techniques of phase modulation. Optical coordinate transformations are mathematically examined and used in an example. The main body is focused on manipulation of scalar Gaussian beams, as they represent the most basic solution to the paraxial equation and can behave as rays in geometrical optics. The results are then used on beams carrying the orbital angular momentum and the possibilities of performing arithmetic operations with the value of the OAM are presented.

*Key words:* Gaussian beam, optical coordinate transformations, orbital angular momentum, phase modulation

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# Introduction

The ability to shape light and manipulate beams is desired by scientist in many different fields. Whether it is to modify the displayed image or rearrange data in an optical processor, optical coordinate transformations using the spatial phase modulation offer a simple method in a system with only a few elements.

In the first chapter we investigate the light propagation from two different perspectives, firstly in the scalar paraxial approximation and secondly the geometrical optics. Gaussian beams with special properties are examined, because they can unite these two approaches as is demonstrated on an example.

The introduction of the Fresnel diffraction integral in the first chapter is followed by presenting the Fraunhofer integral in the second chapter. Another example of phase modulation on a tilted displaced Gaussian beam is computed.

The third chapter focuses on optical coordinate transformations. A general transformation is described mathematically and its realizability by optical elements is investigated.

The OAM is presented in the last chapter with a review of an experiment about its measurement. Several techniques for arithmetic operations with the value of the OAM are proposed, including division by a natural number  $n$ , multiplication by 2 and addition of a set amount.

# Chapter 1

## Gaussian beams in geometrical optics

Gaussian beam is very useful representative of the light propagation, not just because of its wide practical applicability. Gaussian beams are introduced as the simplest solution to the paraxial equation in the first section. Then the behaviour of rays and their interaction with optical elements via ABCD-matrices is briefly explained. After limiting the dimensions of a Gaussian beam according to the geometrical optics, it can be treated in similar fashion as a ray and the same methods can be applied. With that, in the last section we compare a Gaussian beam as a wave and as a ray in an example where it propagates through a thin lens.

### 1.1 Gaussian beams in the paraxial approximation

The scalar wave theory describes behaviour of individual components of electro-magnetic field without taking into account vector properties of the field, like polarization. Additionally, we limit our research to coherent monochromatic waves. Any such wave can be written as a complex function of position and time in the form

$$u(x, y, z, t) = U(x, y, z)e^{-i\omega t}, \quad (1.1)$$

where  $\omega$  is the angular frequency of the wave. (In the following text, the time dependency  $e^{-i\omega t}$  will be omitted and only the function of position  $U(x, y, z)$  will be of interest.)

The focus of this paper is on the transformation of optical beams, which are usually studied in the paraxial approximation. Assuming that the beam propagates along  $z$  axis, the dominant factor  $e^{ikz}$  ( $k = 2\pi/\lambda$  is the wave number) can be extracted out of  $U(x, y, z)$

$$U(x, y, z) = \tilde{U}(x, y, z)e^{ikz}. \quad (1.2)$$

Paraxial approximation means that the remainder  $\tilde{U}(x, y, z)$  satisfies (in free space) *the paraxial equation* [1, p. 628]

$$\frac{\partial^2 \tilde{U}}{\partial x^2} + \frac{\partial^2 \tilde{U}}{\partial y^2} + 2ik \frac{\partial \tilde{U}}{\partial z} = 0 \quad (1.3)$$

with the following restrictions. First, the field distribution should vary slowly in the  $z$  direction compared to the transverse variations on the scale of a wave length. Written in mathematical terms

$$\left| \frac{\partial^2 \tilde{U}}{\partial z^2} \right| \ll \left| 2k \frac{\partial \tilde{U}}{\partial z} \right| \quad \text{or} \quad \left| \frac{\partial^2 \tilde{U}}{\partial x^2} \right| \quad \text{or} \quad \left| \frac{\partial^2 \tilde{U}}{\partial y^2} \right|. \quad (1.4)$$

Furthermore, the paraxial approximation deals with such focused beams that all trigonometric functions of small angles, as shown on Fig.1.1, can be expressed only by their first order expansion. In other words, each plane wave making up the optical beam can only diverge from  $z$  axis by angle up to

$$\theta \leq 0,5 \text{ rad.} \quad (1.5)$$

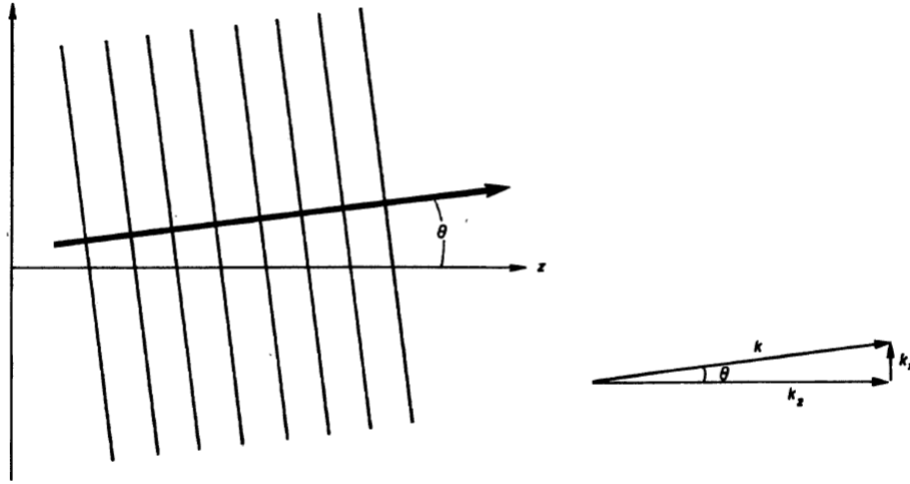


Figure 1.1: A plane wave propagating with a small angle  $\theta$ . Taken from [1, p. 629].

One of the most useful family of solutions of the paraxial equation are *Gaussian beams*. In general, Gaussian beam centred on the  $z$  axis travelling in the positive direction can be written as [2, p. 269]

$$U(x, y, z) = \frac{w_0}{w(z)} \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \right] \exp \left[ i \left( kz + \frac{k(x^2 + y^2)}{2R(z)} + \psi(z) \right) \right], \quad (1.6)$$

where  $w_0$ ,  $w(z)$ ,  $R(z)$  and  $\psi(z)$  are *the waist*, the spot size, the radius of curvature and *the Gouy phase shift*, respectively. All these properties can be expressed by two characteristic parameters of the Gaussian beam, the waist  $w_0$  and *the Rayleigh range*  $z_R$  (related to the wavelength  $\lambda$  in the medium by  $\lambda = \pi w_0^2 / z_R$ )

$$\begin{aligned} w(z) &= w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \\ R(z) &= z + \frac{z_R^2}{z} \\ \psi(z) &= \arctan \left( \frac{z}{z_R} \right). \end{aligned} \quad (1.7)$$

The Fig.1.2 illustrates the practical meaning of some of the parameters, including the angle of divergence  $\theta$  expressed as

$$\theta = \frac{\lambda}{\pi w_0}. \quad (1.8)$$



Assuming the small angle condition (1.5) is satisfied, combining it with (1.8) gives us an important relation of the waist and the wavelength of the beam

$$\lambda \leq \frac{\pi w_0}{2}. \quad (1.9)$$

Additional restrictions arise from geometrical optics presented in the next section.

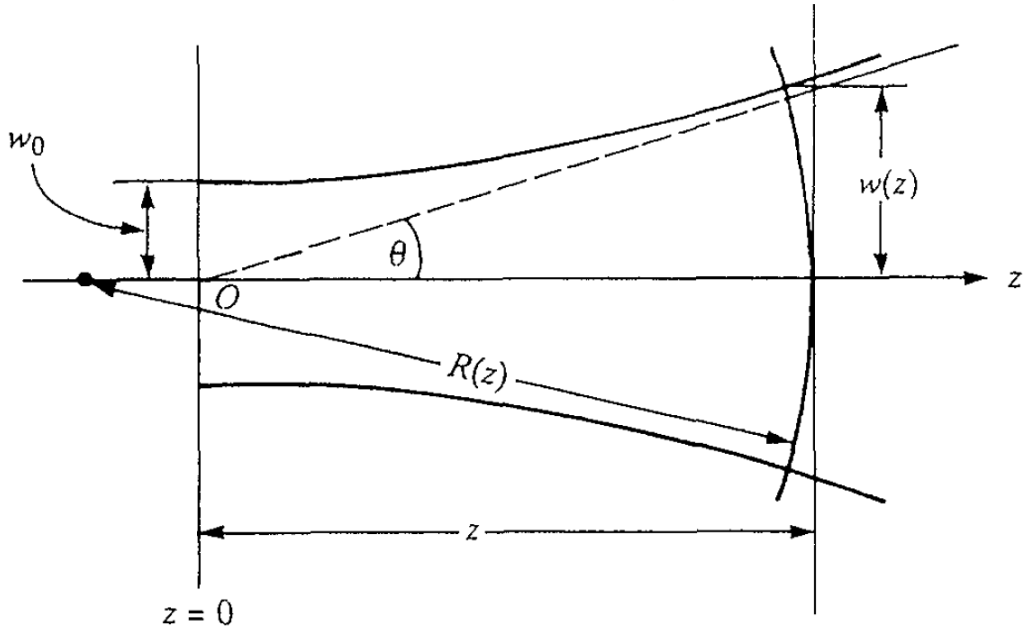


Figure 1.2: Illustration of some important parameters characterizing a Gaussian beam. Taken from [2, p. 271].

A more general form of (1.6) would be a slightly tilted Gaussian beam transversally displaced to the location  $(x_0, y_0)$ . Any beam tilted in the direction of the transversal vector  $(a, b)$  and with the field distribution  $U(x, y, 0)$  is given by

$$U_t(x, y, 0) \approx e^{ik(ax+by)}U(x, y, 0). \quad (1.10)$$

Explanation of this additional phase can be summarized as follows. Suppose a beam  $U(x, y, z)$  is travelling parallel to the  $z$  axis (and satisfies the paraxial equation). If the direction of propagation is slightly changed, the coordinates  $x, y$  will be transformed and the beam gains an additional phase, because it propagates a different distance. Focussing only on one transversal coordinate  $x$ , let  $a$  be the angle between the  $z$  axis and the direction of propagation. According to [3], the tilted beam becomes

$$U_t(x) = e^{ikx \sin a}U(x \cos a). \quad (1.11)$$

Under the condition of the paraxial approximation, that the angle  $a$  is small, all trigonometric functions can be expressed in the first order as

$$\sin a \approx a \quad \text{and} \quad \cos a \approx 1, \quad (1.12)$$

which leads to the expression (1.10).

Returning to our general beam, we can use the formula (1.10) on the transversally shifted  $x \rightarrow x - x_0$ ,  $y \rightarrow y - y_0$  Gaussian beam (1.6) to obtain displaced tilted Gaussian beam at  $z = 0$

$$U_t(x, y, 0) = \exp \left[ -\frac{(x - x_0)^2 + (y - y_0)^2}{w_0^2} \right] \exp[ik(a(x - x_0) + b(y - y_0))], \quad (1.13)$$

that will be used in future calculations.

## 1.2 Geometrical optics and ray matrices

Geometrical optics is a branch of electro-magnetic wave theory studying behaviour of rays. An *optical ray* is a model of light with negligible wavelength, often expressed as a limit  $\lambda \rightarrow 0$ . Geometrical optics describes the trajectory of rays, including media boundary reflection, transmission and changes in polarization [4, p. 116].

A Gaussian beam can, under certain conditions, behave as a ray. The first restriction (1.9) mentioned in the previous section comes from the paraxial approximation and limits the waist size of the beam to be greater than its wavelength. The inequality can be made even more strict ( $\leq \rightarrow \ll$ ) and with the definition of the Rayleigh range  $z_R = \pi w_0^2 / \lambda$  rewritten as

$$w_0 \ll z_R. \quad (1.14)$$

We introduced the quantity  $z_R$ , because the second condition is connected to the distance the Gaussian beam travels before significantly diverging, which is precisely the Rayleigh range. Written mathematically, the Gaussian beam is a valid model of a ray only on the distance  $L$  from the waist satisfying

$$L \leq z_R. \quad (1.15)$$

At last, unlike a ray, the Gaussian beam is not infinitesimally thin. Therefore all optical elements need to be sufficiently large, so that the beam “fits inside” of them. For example, any lens interacting with the Gaussian beam should have a diameter  $d$  much larger than the beam waist  $w_0$ .

To describe behaviour of Gaussian beams representing rays, we first introduce a formalism for working with rays propagating through various optical elements, similarly as in [1, p. 581]. Afterwards we generalize the methods to be compatible with Gaussian beams and compute a simple example.

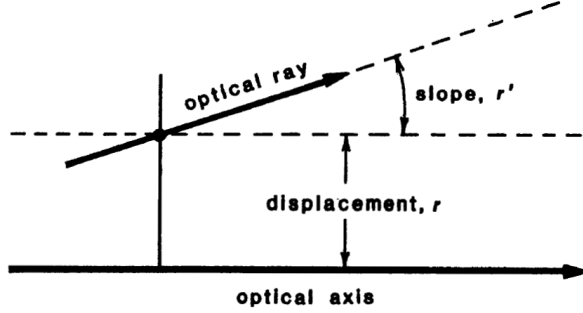
An optical ray travelling approximately in the  $z$  direction can be described by its distance  $r(z)$  from the axis and by its slope  $dr/dz$ , as in Fig.1.3. Considering propagation in free space, the ray travels in a straight line from a plane at  $z_1$  to a plane at  $z_2 = z_1 + L$ , where its coordinates can be derived from (see Fig.1.3)

$$\begin{aligned} r_2 &= r_1 + L \frac{dr_1}{dz} \\ \frac{dr_2}{dz} &= \frac{dr_1}{dz}. \end{aligned} \quad (1.16)$$

For simplicity we work in only 2 dimensions, in 3 dimensional space and rectangular coordinates one would need to calculate  $x$ ,  $dx/dz$  and  $y$ ,  $dy/dz$  separately. Additionally, instead of writing

the slope as a derivative, we define variable  $r'(z)$  (*reduced slope*) as a product of the actual slope and local index of refraction

$$r'(z) = n(z) \frac{dr(z)}{dz}. \quad (1.17)$$



Equation (1.16) gives a linear transformation between the input and output values (displacement and slope) of the rays. Most of the basic optical elements in ray optics (free space being the simplest one) can be described by similar linear transformation in general form

$$\begin{aligned} r_2 &= Ar_1 + Br'_1 \\ r'_2 &= Cr_1 + Dr'_1, \end{aligned} \quad (1.18)$$

where the coefficients A, B, C, D characterize the paraxial focusing properties of this element.

Naturally, equation (1.18) can be rearranged to matrix form

$$\mathbf{r}_2 \equiv \begin{pmatrix} r_2 \\ r'_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r'_1 \end{pmatrix} \equiv \mathbf{M}\mathbf{r}_1, \quad (1.19)$$

where  $\mathbf{M}$  represents *the ray matrix* for the optical element. For example, the ray matrix for propagation in free space with index of refraction  $n_0$  over distance  $L$  is

$$\begin{pmatrix} 1 & L/n_0 \\ 0 & 1 \end{pmatrix}, \quad (1.20)$$

as can be seen from equation (1.16). Other matrices for wide variety of optical elements can be found in literature. However, for a future discussion in the following chapter, we will mention one more example, that is the ray matrix of a thin lens.

Input coordinates  $r_1, r'_1$  of an arbitrary paraxial ray directly in front of a lens of focal length  $f$  are transformed into output coordinates  $r_2, r'_2$  directly behind the lens according to the lens law [1, p. 585]

$$\begin{aligned} r_2 &= r_1 \\ r'_2 &= -(1/f)r_1 + r'_1. \end{aligned} \quad (1.21)$$

Consequently, the ray matrix of a thin lens of focal length  $f$  is

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}. \quad (1.22)$$

Generalization of the ray matrix method for Gaussian beams is a powerful tool capable of handling many different systems. Rigorous explanation and proofs can be found in [1, p. 777], here we only briefly summarize the results and use them on an example.

Working again in only one transversal dimension, we introduce a “two-dimensional” Gaussian beam at  $z = 0$

$$u_1(x_1) = \exp \left[ -\frac{(x_1 - x_0)^2}{w_0^2} \right] \exp [ika(x_1 - x_0)]. \quad (1.23)$$

Effects of a ray matrix on the Gaussian beam (or a general paraxial beam) are computed by the *Huygens-Fresnel integral* [1, p. 783]

$$u_2(x_2) = \sqrt{\frac{1}{iB\lambda}} \int_{\mathbb{R}} u_1(x_1) \exp \left[ \frac{ik}{2B} (Ax_1^2 - 2x_1x_2 + Dx_2^2) \right] dx_1. \quad (1.24)$$

In case of a thin lens, the element  $B = 0$  and a trick is needed to evaluate the integral [1, p. 795]. First, a matrix of propagation in free space over a distance  $\Delta z$  is introduced behind the matrix of an optical element

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & \Delta z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \Delta z \\ -1/f & 1 - \Delta z/f \end{pmatrix}. \quad (1.25)$$

Then the Huygens-Fresnel integration is performed and finally a limit  $\Delta z \rightarrow 0$  is done. The integral (1.24) with the new matrix gives

$$u_2(x_2) = \lim_{\Delta z \rightarrow 0} \sqrt{\frac{1}{i\Delta z \lambda}} \int_{\mathbb{R}} u_1(x_1) \exp \left[ \frac{ik}{2\Delta z} (x_1^2 - 2x_1x_2 + (1 - \Delta z/f)x_2^2) \right] dx_1. \quad (1.26)$$

To calculate the limit a substitution  $x_1 = x_2 + y$  is performed

$$\begin{aligned} u_2(x_2) &= \lim_{\Delta z \rightarrow 0} \sqrt{\frac{1}{i\Delta z \lambda}} \int_{\mathbb{R}} u_1(x_2 + y) \exp \left[ \frac{ik}{2\Delta z} ((x_2 + y)^2 - 2(x_2 + y)x_2 + (1 - \Delta z/f)x_2^2) \right] dy \\ &= \lim_{\Delta z \rightarrow 0} \sqrt{\frac{1}{i\Delta z \lambda}} \int_{\mathbb{R}} u_1(x_2 + y) \exp \left[ \frac{ik}{2\Delta z} (y^2 - x_2^2 \Delta z/f) \right] dy. \end{aligned} \quad (1.27)$$

A formula for a delta function can be seen in the last expression (after we switch the order of limit and integration and manipulate the constants  $\lambda = 2\pi/k$ )

$$\begin{aligned} u_2(x_2) &= \exp \left[ -\frac{ik}{2f} x_2^2 \right] \int_{\mathbb{R}} u_1(x_2 + y) \lim_{\Delta z \rightarrow 0} \sqrt{\frac{k}{2\pi i \Delta z}} \exp \left[ \frac{ik}{2\Delta z} y^2 \right] dy \\ &= \exp \left[ -\frac{ik}{2f} x_2^2 \right] \int_{\mathbb{R}} u_1(x_2 + y) \delta(y) dy \\ &= \exp \left[ -\frac{ik}{2f} x_2^2 \right] u_1(x_2). \end{aligned} \quad (1.28)$$

As a result, the effect of a thin lens on a Gaussian beam (1.23) in geometrical optics at  $z = 0$  is

$$u_2(x_2) = \exp \left[ -\frac{ik}{2f} x_2^2 \right] \exp \left[ -\frac{(x_2 - x_0)^2}{w_0^2} \right] \exp [ika(x_2 - x_0)]. \quad (1.29)$$

To propagate the beam further (say over the distance  $z$ ), the matrix of the optical element would have to be multiplied from the right by the matrix of propagation in free space. However, rest of the calculation is left for the next section, where we compare the results with a phase modulation technique. The field distribution directly behind the lens will be equal in both methods and the free space propagation shows to be the same as well. As a proof, take the propagation in geometrical optics. It can be performed by either multiplying the corresponding ABCD-matrices and then integrating or by calculating the Huygens-Fresnel integral separately for each matrix. In the second case the Huygens-Fresnel integral (1.24) with a matrix of free space propagation  $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$  becomes the Fresnel diffraction integral used in the phase modulation technique.

To summarize, under certain conditions a Gaussian beam can behave and be manipulated as an optical ray. The beam needs to be in the paraxial approximation, cannot diverge too rapidly and its cross section has to be small compared to the elements it is interacting with.

### 1.3 Effects of spatial phase modulation

Now we propagate the same Gaussian beam through the thin lens as in the previous calculation, but with the method of phase modulation.

Consider a monochromatic wave propagating in the  $z$  direction and a diffracting structure located at  $z = 0$ . If the scalar field immediately in front of the structure is  $U_i(x, y)$  and the field immediately behind  $U_o(x, y)$ , then we can define the transmittance function of the aperture as [5, p. 59]

$$t(x, y) = \frac{U_o(x, y)}{U_i(x, y)}. \quad (1.30)$$

In practical cases, a common optical element (for example diffraction grating, hologram or phase plate) is used as the diffraction structure and its representing transmittance function is known.

In the following calculation, we illustrate the effects of the transmittance function approximating a thin lens [5, p. 98]

$$t_l(x, y) = \exp \left[ -\frac{ik}{2f}(x^2 + y^2) \right]. \quad (1.31)$$

Note that the added phase is the same as we obtained from the geometrical optics calculation (1.29) using a ray matrix.

Assume a slightly tilted transversally displaced Gaussian beam (1.13) at  $z = 0$

$$U_1(x_1, y_1) = \exp \left[ -\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{w_0^2} \right] \exp [ik(x'_1(x - x_0) + y'_1(y - y_0))], \quad (1.32)$$

where we changed the notation for the angles  $a \equiv x'_1, b \equiv y'_1$  in order to easily compare the results of our calculations in the next chapters.

At  $z = 0$  the diffracting structure is introduced and the Gaussian beam gains additional phase

$$U_2(x_1, y_1) = U_1(x_1, y_1)t_l(x_1, y_1). \quad (1.33)$$

Propagation in the near field of the aperture is governed by the *Fresnel diffraction integral* (which is a special case of Huygens-Fresnel integral (1.24)) [5, p. 67]

$$U_3(x_2, y_2, z) = \frac{e^{ikz}}{i\lambda z} \int_{\mathbb{R}^2} U_2(x_1, y_1) \exp \left[ \frac{ik}{2z} \left( (x_2 - x_1)^2 + (y_2 - y_1)^2 \right) \right] dx_1 dy_1, \quad (1.34)$$

where  $U_3(x_2, y_2, z)$  is the desired field distribution at location  $(x_2, y_2, z)$  in the positive  $z$  region.

Skipping the details of the calculation (shown in Appendix A), the resulting expression becomes

$$\begin{aligned} U_3(x_2, y_2, z) &= \frac{fk w_0^2}{fk w_0^2 + 2ifz - k w_0^2 z} \\ &\times \exp \left[ \frac{-2kf - ik^2 w_0^2}{2fk w_0^2 + 4ifz - 2k w_0^2 z} \left( (x_2 - \tilde{x}_0(z))^2 + (y_2 - \tilde{y}_0(z))^2 \right) \right] \\ &\times \exp \left[ ik(x'_2 x_2 + y'_2 y_2) - \frac{ik}{2} (x'_2 \tilde{x}_0(z) + y'_2 \tilde{y}_0(z) + x'_1 x_0 + y'_1 y_0) \right], \end{aligned} \quad (1.35)$$

where the new variables are defined as

$$\begin{aligned} \tilde{x}_0(z) &= x'_1 z + x_0(1 - z/f) \\ \tilde{y}_0(z) &= y'_1 z + y_0(1 - z/f) \\ x'_2 &= x'_1 - x_0/f \\ y'_2 &= y'_1 - y_0/f. \end{aligned} \quad (1.36)$$

Even though the calculation in the previous section using ray-matrices was not completed fully, for the reasons explained at the end of the section, the results would be same as (1.35) in this example with the phase modulation.

Under certain conditions (narrow Gaussian beam propagating with small angle over a short distance) both methods are equal and perform the same transformations. That means, instead of computing Fresnel integrals with a phase modulation, a multiplication of ABCD-matrices and the application of the Huygens-Fresnel integral can be done (which may save some computation).

## Chapter 2

# Phase modulation

General spatial phase modulation can be examined from another point of view. In practice, many optical problems rely on knowing the light disturbance very far away from the aperture [4, p. 426]. For that reason we want to determine what effects the phase manipulation has on a wave propagating to infinity. Instead of making measurements of a field on infinitely distant image screen, a converging lens can be used to display the output into the focus plane.

### 2.1 Fraunhofer diffraction

The optical system is simply a phase filter with a transmittance  $t(x_1, y_1)$  placed directly against a converging lens of the focal length  $f$ . In an ideal scenario of an infinitesimally thin lens and a filter directly next to it, the relative position of the filter has no effect. In other words, whether is the filter in front of or behind the lens is irrelevant [5, p. 103].

The incident wave  $U(x_1, y_1)$  (with a cross-section smaller than that of the optical system in order to “fit inside”) is transformed by the lens and filter into

$$U_1(x_1, y_1) = U(x_1, y_1)t(x_1, y_1) \exp\left[-\frac{ik}{2f}(x_1^2 + y_1^2)\right]. \quad (2.1)$$

The desired field distribution in the focal plane ( $z = f$ ) at location  $(x_2, y_2)$  is found using the Fresnel diffraction formula (1.34)

$$U_2(x_2, y_2) = \frac{e^{ikf}}{i\lambda f} \int_{\mathbb{R}^2} U_1(x_1, y_1) \exp\left[\frac{ik}{2f}\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)\right] dx_1 dy_1. \quad (2.2)$$

After substituting (2.1) into (2.2), dropping the constant phase factor  $e^{ikf}$  and manipulating the quadratic phase terms we get

$$U_2(x_2, y_2) = \frac{1}{i\lambda f} \exp\left[\frac{ik}{2f}(x_2^2 + y_2^2)\right] \int_{\mathbb{R}^2} U(x_1, y_1)t(x_1, y_1) \exp\left[-\frac{ik}{f}(x_1x_2 + y_1y_2)\right] dx_1 dy_1. \quad (2.3)$$

The expression (2.3) is called the *Fraunhofer diffraction integral* and is usually derived from the Fresnel integral as a far-field approximation. Note that the formula is basically a Fourier transform of the input wave and transmittance function which is then multiplied by additional quadratic phase.

## 2.2 Phase modulation

We will investigate the effects of the phase modulation on the general Gaussian beam (1.13). The modulation performed by the transmittance function will be given by  $t(x_1, y_1) = \exp[ik\phi(x_1, y_1)]$ . Putting everything into the Fraunhofer integral (2.3) results in

$$U_2(x_2, y_2) = \frac{1}{i\lambda f} \exp\left[\frac{ik}{2f}(x_2^2 + y_2^2)\right] \int_{\mathbb{R}^2} \exp\left[-\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{w_0^2}\right] \times \exp[ik(x'_1(x_1 - x_0) + y'_1(y_1 - y_0))] \exp[ik\phi(x_1, y_1)] \exp\left[-\frac{ik}{f}(x_1x_2 + y_1y_2)\right] dx_1 dy_1. \quad (2.4)$$

The presence of the general function  $\phi(x_1, y_1)$  renders the calculation of the integral to be rather difficult.

To solve it analytically, certain assumptions about the Gaussian beam and  $\phi(x_1, y_1)$  need to be made. The width of the Gaussian beam, already limited by the paraxial and geometrical approximations, is assumed to be so small, that the variance of the phase function  $\phi(x_1, y_1)$  across the beam can be approximated by the first order Taylor expansion

$$\phi(x_1, y_1) \approx \phi(x_0, y_0) + \frac{\partial\phi}{\partial x}(x_0, y_0)(x_1 - x_0) + \frac{\partial\phi}{\partial y}(x_0, y_0)(y_1 - y_0), \quad (2.5)$$

for  $\phi(x_1, y_1)$  with both partial derivatives finite. With (2.5) the integral (2.4) becomes similar to the last calculation in the previous chapter and can be computed following the same steps. The final expression then becomes

$$U_2(x_2, y_2) = \frac{kw_0^2}{2if} \exp\left[\frac{2ifk - k^2w_0^2}{4f^2}((x_2 - \tilde{x}_0)^2 + (y_2 - \tilde{y}_0)^2)\right] \times \exp\left[ik(x'_2x_2 + y'_2y_2) - \frac{ik}{2}\left(\frac{\tilde{x}_0^2 + \tilde{y}_0^2}{f} - 2\phi(x_0, y_0)\right)\right], \quad (2.6)$$

where the new variables are defined as

$$\begin{aligned} \tilde{x}_0 &= f\left(x'_1 + \frac{\partial\phi}{\partial x}(x_0, y_0)\right) \\ \tilde{y}_0 &= f\left(y'_1 + \frac{\partial\phi}{\partial y}(x_0, y_0)\right) \\ x'_2 &= x'_1 + \frac{\partial\phi}{\partial x}(x_0, y_0) - x_0/f \\ y'_2 &= y'_1 + \frac{\partial\phi}{\partial y}(x_0, y_0) - y_0/f. \end{aligned} \quad (2.7)$$

Now we compare the transformations from our two calculations, (2.7) with (1.36). Since the phase modulation case was computed in the focal plane, the transformations (1.36) need to be evaluated for  $z = f$ . The only differences left are in the angle of propagation. Whenever there is  $x'_1$  in (1.36), the transformation (2.7) has  $x'_1 + \frac{\partial\phi}{\partial x}(x_0, y_0)$  and similarly for the other direction,  $y'_1 \rightarrow y'_1 + \frac{\partial\phi}{\partial y}(x_0, y_0)$ . In other words, if we imagine the beam as a bundle of rays, the phase modulation allows us to change the angle of propagation of each ray passing through the filter by the amount  $(\frac{\partial\phi}{\partial x}(x_0, y_0), \frac{\partial\phi}{\partial y}(x_0, y_0))$ .

The conclusion is, the phase modulation in the Fraunhofer diffraction greatly improves the possibilities of manipulations in optical systems compared to the ray-matrices. Instead of using just a quadratic terms in the ABCD-matrices, the general phase modulation with  $\phi(x, y)$  is mostly limited only by the smoothness and differentiability of the function.



## Chapter 3

# Coordinate transformation

In the previous sections we have introduced a method of phase modulation as a powerful tool for manipulation with an optical beam. The same method can be used in imaging technologies for geometrical image modifications such as general coordinate transformations, translation, reflection and local stretching. Following the work of [6],[7] two techniques are introduced.

The first scheme of phase modulation performs coordinate transformation between an object plane  $O$  and a frequency plane  $F$  in a single lens setup as shown in Fig.3.1 using holographic optical elements (HOE's). We will examine this method mathematically, whether all transformations are possible and what are the conditions of realizability.

The second scheme is a modified imaging system consisting of two elements from the first case. As illustrated in Fig.3.2, the telecentric arrangement of two lenses ( $L_1$  and  $L_2$ ) forms an image  $I$  of the object  $O$ . If we think of the object as a collection of different points, then the first lens would display them all into a single point in the focus (in  $F$ ). To avoid this problem, a filter is placed in plane  $O$  to separate the points in the frequency plane. Now individual points can be influenced differently by a second phase filter (placed in  $F$ ) to produce a transformed image in  $I$ .

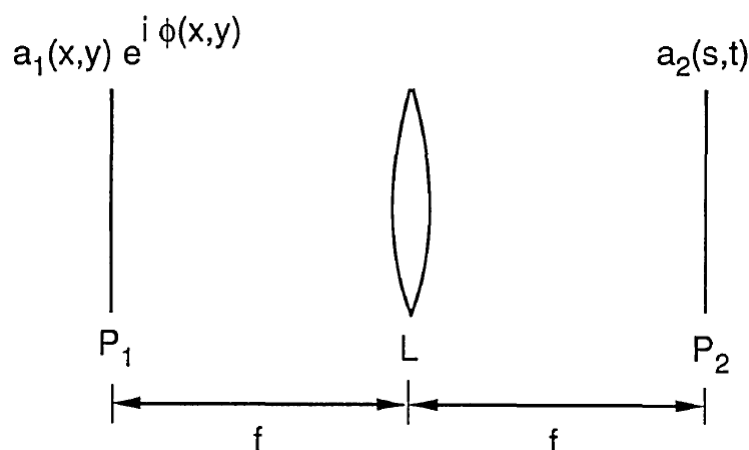


Figure 3.1: Illustration of transformation scheme between object plane  $P_1$  and frequency plane  $P_2$  with a lens  $L$  with an optical element (in  $P_1$ ) introducing a phase  $\phi(x,y)$ . Taken from [8].

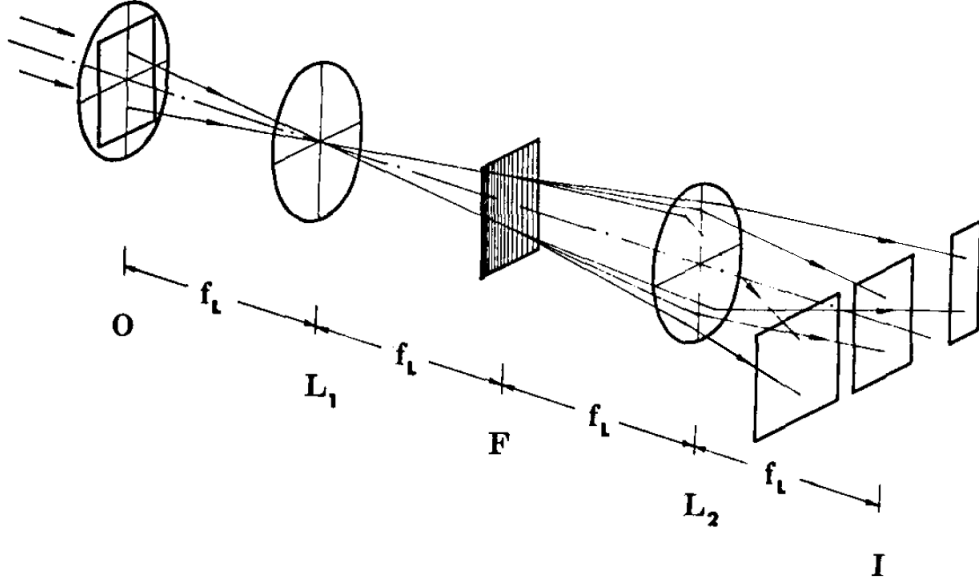


Figure 3.2: A telecentric optical system with two lenses ( $L_1$  and  $L_2$ ) transforming object  $O$  on to the image plane  $I$  using two optical elements, one in  $O$  second in the frequency plane  $F$ . Taken from [7].

### 3.1 Conditions for general transformations

Focusing on the first technique, our task is following. For an input uniphase field with an amplitude distribution  $a_1(x, y)$  and given coordinate transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s(x, y) \\ t(x, y) \end{pmatrix} \quad (3.1)$$

find a phase function  $\phi(x, y)$ , which performs the desired transformation in the first scheme. Are all transformations possible? What are the restrictions on functions  $s(x, y)$  and  $t(x, y)$ ? How many optical elements are necessary to perform the transformation?

The phase modulation is introduced with a holographic optical element in the front focal plane  $P_1$  of the lens  $L$  and the resulting field  $a_2(s, t)$  is found in the back focal plane  $P_2$  as shown in Fig.3.1. The complex amplitude distribution  $a_2(s, t)$  can be calculated with the method of ray matrices from the first chapter. The input beam  $a_1(x, y)$  with the added phase  $\phi(x, y)$  propagates over the distance  $f$ , is transformed by the lens and propagates again over  $f$ , which gives the final matrix

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & f \\ -1/f & 0 \end{pmatrix}. \quad (3.2)$$

After substituting this matrix into the Huygens-Fresnel integral (1.24) (performed for each coordinate separately and the results multiplied) the output distribution becomes

$$a_2(s, t) = -\frac{i}{\lambda f} \int_{\mathbb{R}^2} a_1(x, y) \exp \left[ i\phi(x, y) - \frac{ik}{f}(xs + yt) \right] dx dy. \quad (3.3)$$

Following the steps from [8], the method of stationary phase is used [4, p. 888] to approximate this integral. For large  $k$ , the phase of the expression inside the integral varies rapidly except

for the saddle-points, where the partial derivatives of the exponent are zero. That means the contribution to the integral is insignificant almost everywhere apart from the neighbourhood of the saddle-points

$$\frac{\partial\phi(x,y)}{\partial x} - \frac{k}{f}s = 0 \quad \text{and} \quad \frac{\partial\phi(x,y)}{\partial y} - \frac{k}{f}t = 0. \quad (3.4)$$

If there are multiple saddle-points, each area containing a saddle-point adds to the final value of the integral. However, our prime interest is in the new relative locations of the image points transformed from the original object distribution. The new coordinates can be easily found from the saddle-point equation (3.4)

$$s = \frac{f}{k} \frac{\partial\phi(x,y)}{\partial x} \quad \text{and} \quad t = \frac{f}{k} \frac{\partial\phi(x,y)}{\partial y}. \quad (3.5)$$

By applying our desired transformation (3.1), we obtain a system of partial differential equations for the phase  $\phi(x,y)$ , which we want to find.

Now we will make a few assumptions about the functions (3.1). For simplicity and future calculations we consider only such  $s(x,y)$  and  $t(x,y)$ , which are continuously differentiable and their domain (region in  $x,y$  plane to be transformed) is simply connected. On top of that we assume all transformations are one-to-one, that is no two points  $(x_1, y_1)$ ,  $(x_2, y_2)$  can be mapped to the same point  $(s, t)$ .

With these conditions the system of equations (3.5) has a solutions if and only if

$$\frac{\partial^2\phi(x,y)}{\partial x\partial y} = \frac{\partial^2\phi(x,y)}{\partial y\partial x}, \quad (3.6)$$

which is equivalent to

$$\frac{\partial s(x,y)}{\partial y} = \frac{\partial t(x,y)}{\partial x}. \quad (3.7)$$

To conclude, a coordinate transformation given by (3.1) is said to be realizable with a single HOE in the stationary phase approximation if and only if the equation (3.7) is satisfied.

## 3.2 Two element transformation

Condition (3.7) is very strict and limits our choices of transformation with only one HOE too much. However, it can be shown, that two elements in series may perform most of coordinate transformations of interest.

The desired transformation is again

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s(x,y) \\ t(x,y) \end{pmatrix}, \quad (3.8)$$

for which (3.7) doesn't hold. If there exists decomposition into two transformations

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \quad (3.9)$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} p(u,v) \\ q(u,v) \end{pmatrix}, \quad (3.10)$$

that satisfy

$$\frac{\partial u(x, y)}{\partial y} = \frac{\partial v(x, y)}{\partial x} \quad (3.11)$$

and

$$\frac{\partial p(u, v)}{\partial v} = \frac{\partial q(u, v)}{\partial u}, \quad (3.12)$$

then the whole transformation can be done with two HOE's in series. Of course the second transformation highly depends on the first one and its realizability is not guaranteed. Without going into much detail (found in [8]), the first transformation needs to be one-to-one, continuously differentiable, to satisfy (3.11), to have a differentiable inverse and to be onto a simply connected region. On top of that, for the second transformation to be realizable (i.e. satisfy equation (3.12)), the first one has to satisfy

$$\frac{\partial t}{\partial x} \frac{\partial v}{\partial y} = \frac{\partial t}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial s}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial s}{\partial x} \frac{\partial u}{\partial y}. \quad (3.13)$$

Solution to the system of partial differential equations given by (3.11) and (3.13) exists, if the system is linear hyperbolic. A sufficient condition for the system to be linear hyperbolic is that the total transformation (3.8) has a negative Jacobian determinant. Note that any transformation with positive Jacobian determinant everywhere can be made negative by simply reversing the direction of one of its output coordinates (details in [8]).

In summary, a total coordinate transformation given by (3.8) is realizable by two HOE's in series, if there exists a solution to the system (3.11) and (3.13) and the first transformation (3.9) is one-to-one. Such a solution always exists, if the total transformation has a negative (or positive with mandatory redefining of an output coordinate) Jacobian determinant everywhere in the transformed region.

### 3.3 Example

As a demonstration of the coordinate transformation method via phase modulation an illustrative example is presented [8]. The goal is to perform polar formatting between two orthogonal coordinate sets  $x$ - $y$  and  $s$ - $t$  with the transformation functions

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} s(x, y) \\ t(x, y) \end{pmatrix} = \begin{pmatrix} x \cos y \\ x \sin y \end{pmatrix}. \quad (3.14)$$

In order for the transformation to be one-to-one, the domain needs to be limited to  $x > 0$  or  $x < 0$ .

First we determine, whether the transformation is realizable with a single HOE, i. e. if the equation (3.7) is satisfied

$$\frac{\partial s(x, y)}{\partial y} = -x \sin y \quad , \quad \frac{\partial t(x, y)}{\partial x} = \sin y. \quad (3.15)$$

The partial derivatives are not equal, meaning that two HOE's will be necessary.

The second step is to verify the positivity/negativity of the Jacobian determinant. It is easily shown that the Jacobian is positive and the trick with reversing an output coordinate is needed. One of the possible choices for  $x > 0$  is

$$\begin{aligned} s(x, y) &= x \cos y, \\ t(x, y) &= -x \sin y. \end{aligned} \quad (3.16)$$

In the third step we find the decomposition into two intermediate transformations (3.9) and (3.10). The first transformation is a solution to the system of partial equations (3.11) and (3.13), which for this example (3.16) are

$$\begin{aligned} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial y} + (-\sin y)^{-1} \left[ x \sin y \frac{\partial u}{\partial x} + ((x+1) \cos y) \frac{\partial v}{\partial x} \right] &= 0. \end{aligned} \quad (3.17)$$

Although this system may seem complicated, one solution can be found with the additional assumption that  $u = u(x)$  and  $v = v(y)$ . The equations (3.17) are then simplified to

$$\frac{dv}{dy} - x \frac{du}{dx} = 0. \quad (3.18)$$

The solution gives the first transformation in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_1 \ln x + c_2 \\ c_1 y + c_3 \end{pmatrix}, \quad (3.19)$$

where  $c_{1,2,3}$  are constants. As can be easily verified, the transformation is one-to-one, continuously differentiable with a differentiable inverse and is onto a simply connected region. The second transformation is calculated from (3.16) and (3.19) with the constants  $c_1 = 1$ ,  $c_2 = c_3 = 0$

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} e^u \cos v \\ -e^u \sin v \end{pmatrix}. \quad (3.20)$$

The given transformation (3.16) has been successfully split into two separate transformations each realizable with a single HOE.

The last step is to create the optical elements performing the transformations, or rather determine the input of the HOE's. The phase profile of the transformations is found by solving (3.5) with the proper variables each time. For the first transformation

$$\ln x = \frac{f}{k} \frac{\partial \phi_1}{\partial x} \quad \text{and} \quad y = \frac{f}{k} \frac{\partial \phi_1}{\partial y} \quad (3.21)$$

results in

$$\phi_1(x, y) = \frac{k}{f} \left( \frac{y^2}{2} + x(\ln x - 1) \right). \quad (3.22)$$

The second one is

$$e^u \cos v = \frac{f}{k} \frac{\partial \phi_2}{\partial u}, \quad -e^u \sin v = \frac{f}{k} \frac{\partial \phi_2}{\partial v} \quad (3.23)$$

with the result

$$\phi_2(u, v) = \frac{k}{f} e^u \cos v. \quad (3.24)$$

Plots of both phase profiles are illustrated in Fig.3.3. Note that the assumption of separable variables  $u(x), v(y)$  was only made to solve (3.17) more easily, other solutions to the system of PDE's are possible as well.

Usefulness of this method is proven even more when we realize that to perform the inverse transformation, the direction of propagation needs to be simply reversed [8].

In practice, the optical coordinate transformations and modifications can be mostly found in imaging technologies. The possibilities span from local stretching and imaging onto curved surfaces to correcting distortions and producing optical components like zoom lenses [7]. In the next chapter our focus will be on a completely different application, the manipulation the orbital angular momentum of light, i.e. arithmetic operations with polar coordinates.

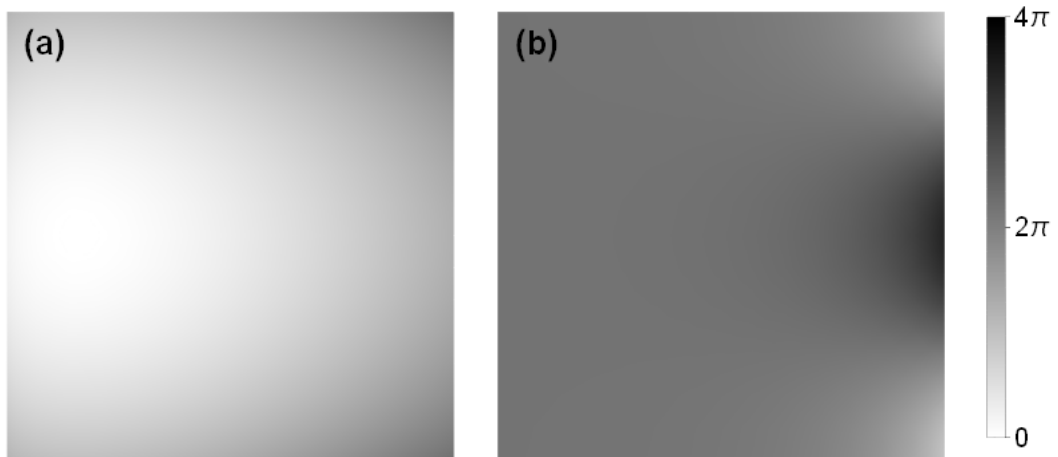


Figure 3.3: (a) phase profile for  $\phi_1$ , (b) phase profile for  $\phi_2$ .

## Chapter 4

# OAM manipulation

There are many applications of optical transformations introduced in the previous chapters, manipulation of the orbital angular momentum is one of them. In general, angular momentum of light has two separate parts: the spin, originating from the circular polarization, and the orbital angular momentum (OAM), the quantity of interest in this text.

The OAM is produced by a helically deformed wavefront. For example a paraxial beam with a planar wavefront has no OAM, but a beam with a phase front deformed into a helix is carrying an orbital angular momentum. The value of the OAM is proportional to the number  $l$  of intertwined helices (as shown in Fig.4.1) and is equal to  $\pm l\hbar$  (the minus corresponds to a helix twisted in the other direction).

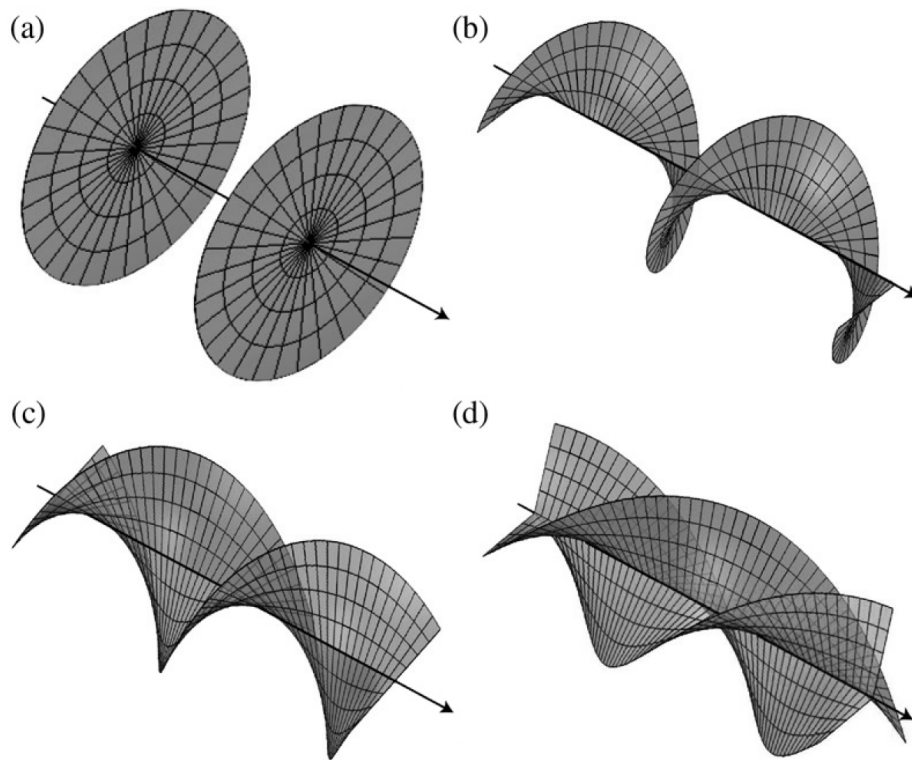


Figure 4.1: An illustration of wavefronts for different values of  $l$ : (a)  $l = 0$ , (b)  $l = 1$ , (c)  $l = 2$ , (d)  $l = 3$ . Taken from [9].

An optical beam with an OAM in the paraxial approximation is most easily described in Laguerre-Gaussian (LG) modes. LG modes form a complete orthogonal basis of solutions to the paraxial equation (1.3) in cylindrical coordinates and are characterized by two numbers  $p \in \mathbb{N}_0$ ,  $l \in \mathbb{Z}$ . They can be written as [9]

$$LG_p^l(r, \varphi, z) = \sqrt{\frac{2p!}{\pi(p+|l|)!} \frac{w_0}{w(z)}} \left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|} L_p^{|l|} \left(\frac{2r^2}{w(z)^2}\right) \exp \left[ \left( \frac{-1}{w(z)^2} + \frac{ik}{2R(z)} \right) r^2 \right] \times \exp(il\varphi) \exp[i(1+|l|+2p)\psi(z)], \quad (4.1)$$

where the only dependence on the azimuthal coordinate  $\varphi$  is in the phase  $\exp(il\varphi)$  responsible for the value of the OAM (and the shape of the phase front).

## 4.1 Arithmetic manipulation with the OAM

With the tools from the previous chapters it is possible to transform the beam and manipulate the OAM it is carrying. Since the value of the orbital momentum is given by  $l$  connected to the coordinate  $\varphi$  in  $\exp(il\varphi)$ , we propose a transformation

$$\varphi \rightarrow n\varphi \quad (4.2)$$

that divides the OAM by  $n$ . For every  $n \in \mathbb{Z}$  and  $l$  divisible by  $n$ , a beam with the OAM  $l\hbar$  is transformed into a beam with  $\frac{l}{n}\hbar$ . In case of non-divisibility the transformed beam has no longer strictly defined value of  $l$  and the OAM cannot be determined.

For the transformation to be one-to-one, the domain of the transformation needs to be limited. Since  $n$  points on a circle would display onto a single point, we restrict the possible input to be only one  $n$ th of the circle, i.e.  $\{(r, \varphi) | \varphi \leq \frac{2\pi}{n}\}$ .

Without a detailed explanation (found in Appendix B), the transformation can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} s(x, y) \\ t(x, y) \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} \\ \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j+1} \binom{n}{2j+1} x^{n-(2j+1)} y^{2j+1} \end{pmatrix}. \quad (4.3)$$

Following the steps from chapter 3, first we compare the partial derivatives (see Appendix B)

$$\frac{\partial s}{\partial y} = \frac{\partial t}{\partial x}. \quad (4.4)$$

They are equal, meaning we can use a single HOE and skip the search for the intermediate transformation.

In the last step, the phase function  $\phi_n(x, y)$  is determined from the formula (3.5) as

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} = \frac{f}{k} \frac{\partial \phi_n}{\partial x}, \quad \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j+1} \binom{n}{2j+1} x^{n-(2j+1)} y^{2j+1} = \frac{f}{k} \frac{\partial \phi_n}{\partial y}. \quad (4.5)$$

These equations can be integrated and with a bit of manipulation of the summation bounds the phase profile becomes

$$\phi_n(x, y) = \frac{k}{f} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j-1} \frac{1}{2j} x^{n-2j+1} y^{2j}. \quad (4.6)$$



A few examples of the phase profiles for small  $n$

$$\begin{aligned}
n=2 &\rightarrow \phi_2(x, y) = \frac{k}{f} \left( \frac{x^3}{3} - xy^2 \right) \\
n=3 &\rightarrow \phi_3(x, y) = \frac{k}{f} \left( \frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} \right) \\
n=4 &\rightarrow \phi_4(x, y) = \frac{k}{f} \left( \frac{x^5}{5} - 2x^3y^2 + xy^4 \right)
\end{aligned} \tag{4.7}$$

are illustrated in Fig.4.2.

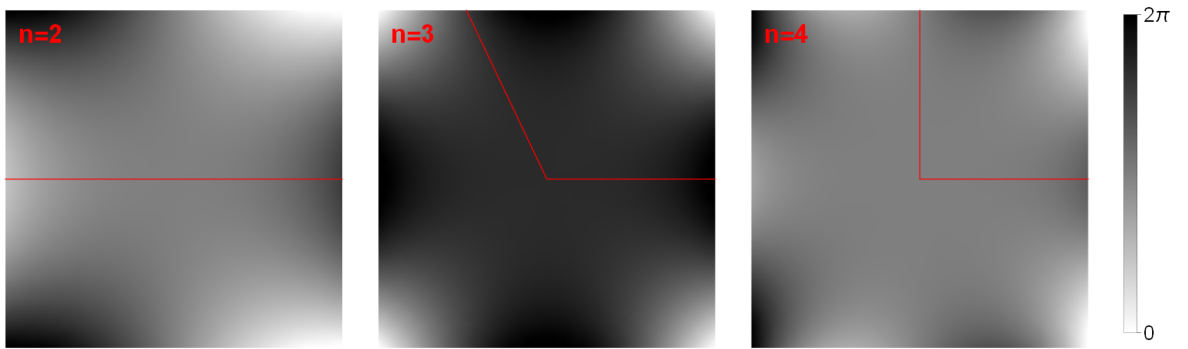


Figure 4.2: Phase profiles for  $\phi_{1,2,3}$ . Red lines represent how the domain is limited.

An alternative to (4.2) would be transformation

$$\varphi \rightarrow \frac{\varphi}{n} \tag{4.8}$$

which multiplies the value of the OAM  $n$  times. In this case there are no restrictions on the numbers  $n$  and  $l$ . However, finding a general transformation for all  $n$  is difficult. Therefore we present only the case with  $n = 2$ .

With help of complex numbers (the same argumentation as in Appendix B) the square root of a complex number  $z = x + iy \in \mathbb{C}$  is given by [12, p. 17]

$$\sqrt{z} = \sqrt{\frac{1}{2} \left( \sqrt{x^2 + y^2} + x \right)} + i \operatorname{sgn} y \sqrt{\frac{1}{2} \left( \sqrt{x^2 + y^2} - x \right)}. \tag{4.9}$$

The proposed transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2} \left( \sqrt{x^2 + y^2} + x \right)} \\ -\sqrt{\frac{1}{2} \left( \sqrt{x^2 + y^2} - x \right)} \end{pmatrix}, \tag{4.10}$$

where we have already changed the sign of the second coordinate for the Jacobian determinant

$$J = \frac{-\operatorname{sgn} y}{4\sqrt{x^2 + y^2}} \tag{4.11}$$

to be negative. To use the methods from chapter 3, we limit the domain to  $\{(x, y) | y > 0\}$  and check the condition (3.7). The partial derivatives

$$\frac{\partial u}{\partial y} = \frac{y}{2\sqrt{2}\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} + x}} = \frac{\partial v}{\partial x} \quad (4.12)$$

are equal and we can use a single optical element. Its phase profile  $\phi(x, y)$  is given by the equations (3.5) in the form of

$$\sqrt{\frac{1}{2}(\sqrt{x^2 + y^2} + x)} = \frac{f}{k} \frac{\partial \phi}{\partial x} \quad , \quad -\sqrt{\frac{1}{2}(\sqrt{x^2 + y^2} - x)} = \frac{f}{k} \frac{\partial \phi}{\partial y}. \quad (4.13)$$

To find  $\phi(x, y)$  we first use the substitution

$$a = x^2 + y^2 \quad , \quad b = x, \quad (4.14)$$

which transforms (4.13) into

$$\sqrt{\frac{1}{2}(\sqrt{a} + b)} = \frac{f}{k} \left( 2b \frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \right) \quad , \quad -\sqrt{\frac{1}{2}(\sqrt{a} - b)} = \frac{f}{k} 2\sqrt{a - b^2} \frac{\partial \phi}{\partial a}. \quad (4.15)$$

These equations can be manipulated to the form

$$-\frac{2\sqrt{2}f}{k} \sqrt{\sqrt{a} + b} \frac{\partial \phi}{\partial a} = 1 \quad , \quad \frac{\sqrt{2}f}{k} \sqrt{\sqrt{a} + b} \frac{\partial \phi}{\partial b} = \sqrt{a} + 2b \quad (4.16)$$

solvable with a computational software (Mathematica). The result becomes

$$\phi(a, b) = -\frac{k\sqrt{2}}{f} \frac{1}{3} (\sqrt{a} - 2b) \sqrt{\sqrt{a} + b}, \quad (4.17)$$

where it is possible to substitute the original variables (4.14) to get the final phase profile

$$\phi(x, y) = -\frac{k\sqrt{2}}{f} \frac{1}{3} \left( \sqrt{x^2 + y^2} - 2x \right) \sqrt{\sqrt{x^2 + y^2} + x}, \quad (4.18)$$

which is illustrated in Fig.4.3.

## 4.2 Measuring the OAM with reverse polar transformation

The article [10] presents a Cartesian to log-polar coordinate transformation to sort beams by their different value of the OAM. The principle of the transformation is to map concentric rings in the input to parallel lines in the output, which translates to LG beams as a transformation from helical phase to transverse phase gradient. A little complication arises from different optical path lengths between the circles and the lines. This introduces an unwanted phase distortion needful of correction by another optical element.

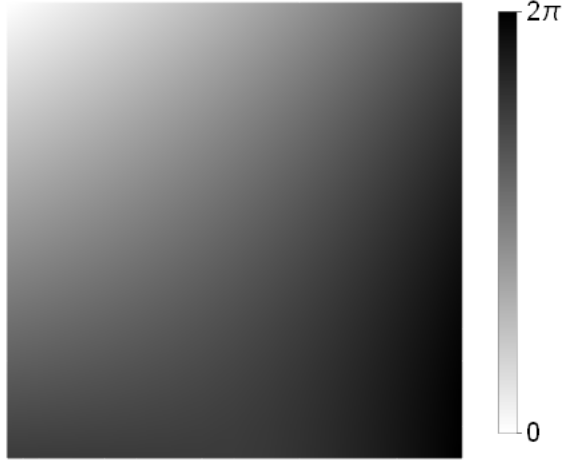


Figure 4.3: Phase profile for multiplying the OAM by 2.

The coordinate transformation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} -a \ln\left(\frac{\sqrt{x^2+y^2}}{b}\right) + x \\ a \arctan\left(\frac{y}{x}\right) \end{pmatrix}, \quad (4.19)$$

where  $a$  is a constant determined by the geometry of the system and  $b$  is an independent value used for scaling the image in the  $u$  direction. The partial derivatives

$$\frac{\partial u}{\partial y} = \frac{-ay}{x^2 + y^2} = \frac{\partial v}{\partial x} \quad (4.20)$$

are equal, meaning only one HOE is necessary. The phase profile of (4.19) is given by

$$\phi_1(x, y) = \frac{ka}{f} \left[ y \arctan\left(\frac{y}{x}\right) - x \ln\left(\frac{\sqrt{x^2 + y^2}}{b}\right) + x \right]. \quad (4.21)$$

The phase correcting element is required to collimate the transformed beam. Each output ray needs to have its angle of propagation corrected by the opposite of what the phase filter and the lens introduced. The corresponding phase profile can be derived from the inverse transformation [11]

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix} = \begin{pmatrix} b \exp\left(-\frac{u}{a}\right) \cos\left(\frac{v}{a}\right) \\ b \exp\left(-\frac{u}{a}\right) \sin\left(\frac{v}{a}\right) \end{pmatrix}, \quad (4.22)$$

which leads to

$$\phi_2(u, v) = -\frac{kab}{f} \exp\left(-\frac{u}{a}\right) \cos\left(\frac{v}{a}\right). \quad (4.23)$$

Note, that this is the same profile as we have already found in (3.24).

The optical setup starts with an iris allowing only the desired light to enter the first HOE, which in this case was (for both transformations) a spatial light modulator. Then the beam travels through a lens and the second transformation is performed in the Fourier plane of that lens. Finally, a second lens focuses the output beam on a display.

In the experiment several LG beams with different values of the OAM were used. Each OAM state was displayed in the output as a distinct line. A beam without angular momentum showed as a line in the center, with the increase of  $l$  the lined moved more to the right to its specific position (and for negative  $l$  to the left). Even an example of a combination of OAM states was performed and from the two bright spots in the output it was possible to correctly identify a superposition of  $l = -1$  and  $l = 2$ . Although some overlap between the different output lines was observed, the method proved to be successful.

### 4.3 Alternative OAM manipulation

The previous experiment gives us an idea for an alternative method to perform arithmetic operations with the OAM. The principle is to transform the beam into orthogonal coordinates as was done in the experiment and display each value of the OAM as a separate line. Then shift the line by a predefined amount to a different position and perform the inverse transformation. Since each position of a line corresponds to a specific value of the OAM, the output beam will carry that OAM. In other words, this method can add (or subtract) the OAM  $l\hbar \rightarrow (n + l)\hbar$ .

The first transformation is going to be again

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\ln(\sqrt{x^2 + y^2}) + x \\ \arctan(\frac{y}{x}) \end{pmatrix}, \quad (4.24)$$

which turns the beam into parallel lines (or a line in case of one value of the OAM). The phase profile is (4.21) with the constants  $a, b = 1$ .

If we want to add  $n$  to the OAM, we need to shift the line by  $n$  positions to the right. (This of course depends on the orientation of the system. Following the experiment, if the negative values were displayed on the left and positive on the right, then shifting to the right means addition and shifting to the left subtraction.) The distance between the lines can be derived from their position [10]

$$t_1 = \frac{\lambda f}{d} l, \quad (4.25)$$

where  $d$  is a parameter of the system. The transformation adding  $n$  to the OAM is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} u + \frac{\lambda f}{d} n \\ v \end{pmatrix}. \quad (4.26)$$

Before making the phase profile for this transformation, we need to remember that the previous one (4.24) required an additional phase correction (4.23). According to [8], such corrections can be combined with the transformation that follows immediately after into a single element giving us phase profile for (4.26) as

$$\phi_2(u, v) = -\frac{k}{f} e^{-u} \cos v + \frac{k}{2f} \left( u^2 + v^2 + u \frac{\lambda f}{d} n \right). \quad (4.27)$$

Last step is to transform the beam back into the polar coordinates. The inverse to (4.24) is given by

$$\begin{pmatrix} s \\ t \end{pmatrix} \rightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} e^s \cos t \\ -e^s \sin t \end{pmatrix}. \quad (4.28)$$

Again, the phase profile will be a combination of the correcting and the transforming part in the form

$$\phi_3(s, t) = \frac{k}{2f} \left( s^2 + t^2 - s \frac{\lambda f}{d} n \right) + \frac{k}{f} e^s \cos t. \quad (4.29)$$

This completes the transformation  $l\hbar \rightarrow (n+l)\hbar$  of the OAM. All the phase profiles are illustrated in Fig.4.4 for  $n = 1$ .

Note that instead of shifting by a set amount (4.26), it would be possible to transform the first coordinate as

$$u \rightarrow s = nu \quad \text{or} \quad u \rightarrow s = \frac{u}{n}, \quad (4.30)$$

which would perform the OAM multiplication and division, respectively. In the second case, if the parameter  $l$  of the input beam was not divisible by  $n$ , the transformed line would be located in an incorrect position not corresponding to any value of  $l$ .

To conclude, each operation described in the sections 4.1 and 4.3 presents a theoretical possibility to manipulate the OAM. From the practical side, their applicability is not certain and even though the methods are simple in design, there are some complications. In the first case, the division of the OAM limits its input domain. This can be achieved by blocking a portion of the input, but that may, at the same time, introduce unwanted diffraction effects. The last method in section 4.3 uses many different optical elements, each reducing the intensity of the output and weakening the precision of the approximations.

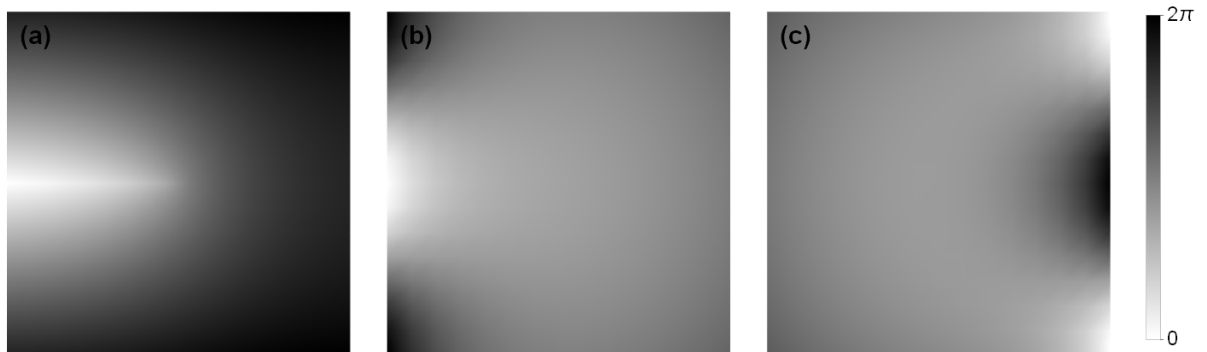


Figure 4.4: Phase profiles for  $\phi_{1,2,3}$ , respectively, in the setup to add a value ( $n = 1$ ) to the OAM.

# Conclusion

The goal of this work was to introduce the spatial phase modulation as a tool to manipulate and shape beams. First, the paraxial approximation and the geometrical optics were presented. The example with the tilted displaced Gaussian beam was meant to show that under certain conditions the beam can behave as a ray and that techniques from both approximations can be used.

Understanding of the effects of the phase modulation lead us to the mathematical description of the optical coordinate transformations. For any given transformation its conditions of realizability were investigated as well as the number of optical elements needed to perform it.

Some of these results were used to manipulate the OAM of a Laguerre-Gaussian beam. We proposed several theoretical possibilities how to perform arithmetic operations with the value of the OAM.

The division of the OAM was fully described for all positive integers  $n$  and phase profiles for  $n = 2, 3, 4$  were designed. However, change in the size of the beam (connected to loosing the strict definition of the parameter  $p$ ) had to be neglected for the transformation to be realizable with a single optical element.

Another presented operation was the OAM multiplication. Since we were unable to find a transformation performing polar angle division in Cartesian coordinates for general  $n$ , only the case of  $n = 2$  (doubling of the OAM) was investigated.

The OAM addition ( $l\hbar \rightarrow (n+l)\hbar$ ) was the last proposed method. In contrast to the previous ones, the addition was achieved by first transforming the beam into the Cartesian coordinates, shifting the value of the OAM and then transforming back into the polar coordinates.

Since this text focused only on the theoretical description of the optical coordinate transformations, the applicability of the presented techniques is left to be discussed by experimenters.

Future works may follow on improving the approximations to find better formulas usable on a broader range of beams or continue the search for optical elements manipulating the value of the OAM. Another option is to verify the current results, preferably in a lab on a real optical system or at least from behind a computer with numerical simulations.

# Appendix A

We want to propagate a Gaussian beam

$$U_1(x_1, y_1) = \exp \left[ -\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{w_0^2} \right] \exp [ik(x'_1(x - x_0) + y'_1(y - y_0))], \quad (4.31)$$

with the additional phase from a thin lens

$$U_2(x_1, y_1) = U_1(x_1, y_1) \exp \left[ -\frac{ik}{2f}(x_1^2 + y_1^2) \right] \quad (4.32)$$

with the Fresnel diffraction integral

$$U_3(x_2, y_2, z) = \frac{e^{ikz}}{i\lambda z} \int_{\mathbb{R}^2} U_2(x_1, y_1) \exp \left[ \frac{ik}{2z} \left( (x_2 - x_1)^2 + (y_2 - y_1)^2 \right) \right] dx_1 dy_1. \quad (4.33)$$

The integral can be, in principle simply, evaluated using the standard formula for computing Gaussian integrals [1, p. 783]

$$\int_{\mathbb{R}} \exp(-ax^2 - 2bx) dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{a}\right), \quad (4.34)$$

where  $a, b \in \mathbb{C}$  and  $\text{Re}(a) > 0$ .

In practice, the help of a computing software (Mathematica) was used to perform the calculations to obtain (after few algebraic manipulations)

$$\begin{aligned} U_3(x_2, y_2, z) &= \frac{e^{ikz}}{i\lambda z} \frac{2\pi}{\frac{ik}{f} + \frac{2}{w_0^2} - \frac{ik}{z}} \exp \left[ \frac{-ik}{2fk w_0^2 + 4ifz - 2k w_0^2 z} \left( k w_0^2 (x_2^2 + y_2^2) \right) \right] \\ &\times \exp \left[ \frac{-ik}{\dots} \left( f k w_0^2 z (x_1'^2 + y_1'^2) + 2iz(x_0^2 + y_0^2) - 2if((x_2 - x_0)^2 + (y_2 - y_0)^2) \right) \right] \\ &\times \exp \left[ \frac{-ik}{\dots} \left( 2k w_0^2 x_1' (f(x_0 - x_2) - x_0 z) + 2k w_0^2 y_1' (f(y_0 - y_2) - y_0 z) \right) \right]. \end{aligned} \quad (4.35)$$

From here we manipulate the expression into a form of a Gaussian beam with easy to read displacement (which will be a function of  $z$ ), angle of propagation and some additional phase

$$U_3(x_2, y_2, z) = c_1 \exp \left[ c_2 \left( (x_2 - \tilde{x}_0(z))^2 + (y_2 - \tilde{y}_0(z))^2 \right) \right] \exp [ik(x'_2 x_2 + y'_2 y_2) + \text{phase}], \quad (4.36)$$

where the variables  $\tilde{x}_0(z), \tilde{y}_0(z), x'_2, y'_2$  are real.

The final expression than becomes

$$\begin{aligned}
U_3(x_2, y_2, z) &= \frac{fk w_0^2}{fk w_0^2 + 2ifz - kw_0^2 z} \\
&\times \exp \left[ \frac{-2kf - ik^2 w_0^2}{2fk w_0^2 + 4ifz - 2kw_0^2 z} \left( (x_2 - \tilde{x}_0(z))^2 + (y_2 - \tilde{y}_0(z))^2 \right) \right] \\
&\times \exp \left[ ik(x'_2 x_2 + y'_2 y_2) - \frac{ik}{2} (x'_2 \tilde{x}_0(z) + y'_2 \tilde{y}_0(z) + x'_1 x_0 + y'_1 y_0) \right],
\end{aligned} \tag{4.37}$$

with the new variables are defined as

$$\begin{aligned}
\tilde{x}_0(z) &= x'_1 z + x_0(1 - z/f) \\
\tilde{y}_0(z) &= y'_1 z + y_0(1 - z/f) \\
x'_2 &= x'_1 - x_0/f \\
y'_2 &= y'_1 - y_0/f.
\end{aligned} \tag{4.38}$$



# Appendix B

Our goal is to perform a transformation  $\varphi \rightarrow n\varphi$  in Cartesian coordinates. A solution can be found in complex numbers, if we treat the transversal  $x, y$  plane of a beam as a complex plane.

Let  $z \in \mathbb{C}$  be written in two forms, a Cartesian  $z = x + iy$  and in a polar form  $z = |z|(\cos(\varphi) + i \sin(\varphi))$ . De Moivre's identity states, that powers of a complex number can be easily computed from its polar form

$$z^n = |z|^n(\cos(n\varphi) + i \sin(n\varphi)), \quad (4.39)$$

where  $n$  can be any real number, but we consider only integers. In this identity, we can see our desired transformation, since taking  $n$ th power of a complex number multiplies its polar angle by  $n$ . To convert the identity into a Cartesian form, we need to compute "the complex binomial expansion" with a positive integer  $n$  [12, p. 16]

$$\begin{aligned} z^n &= (x + iy)^n \\ &= \left[ x^n - \binom{n}{2}x^{n-2}y^2 + \binom{n}{4}x^{n-4}y^4 - \dots \right] + \\ &\quad + i \left[ \binom{n}{1}x^{n-1}y - \binom{n}{3}x^{n-3}y^3 + \binom{n}{5}x^{n-5}y^5 - \dots \right]. \end{aligned} \quad (4.40)$$

First few examples are

$$\begin{aligned} z^2 &= (x^2 - y^2) + i(2xy) \\ z^3 &= (x^3 - 3xy^2) + i(3x^2y - y^3) \\ z^4 &= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3). \end{aligned} \quad (4.41)$$

At last a scaling factor is needed to compensate for the change in magnitude introduced by  $|z|^n$  in (4.39) in order for the transformed points to stay on the same circle.

The final transformation is then

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} s(x, y) \\ t(x, y) \end{pmatrix} = \frac{1}{(x^2 + y^2)^{\frac{n-1}{2}}} \begin{pmatrix} x^n - \binom{n}{2}x^{n-2}y^2 + \binom{n}{4}x^{n-4}y^4 - \dots \\ \binom{n}{1}x^{n-1}y - \binom{n}{3}x^{n-3}y^3 + \binom{n}{5}x^{n-5}y^5 - \dots \end{pmatrix}, \quad (4.42)$$

which multiplies the polar angle by  $n$  ( $\varphi \rightarrow n\varphi$ ).

The partial derivatives are

$$\begin{aligned}
\frac{\partial s}{\partial x} &= \frac{-(n-1)x}{(x^2+y^2)^{\frac{n-3}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} + \\
&\quad + \frac{1}{(x^2+y^2)^{\frac{n-1}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j+1} (2j+1) x^{n-2j-1} y^{2j} \\
\frac{\partial s}{\partial y} &= \frac{-(n-1)y}{(x^2+y^2)^{\frac{n-3}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} + \\
&\quad + \frac{1}{(x^2+y^2)^{\frac{n-1}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j+1} \binom{n}{2j+2} (2j+2) x^{n-2j-2} y^{2j+1} \\
\frac{\partial t}{\partial x} &= \frac{-(n-1)x}{(x^2+y^2)^{\frac{n-3}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1} + \\
&\quad + \frac{1}{(x^2+y^2)^{\frac{n-1}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{j+1} \binom{n}{2j+2} (2j+2) x^{n-2j-2} y^{2j+1} \\
\frac{\partial t}{\partial y} &= \frac{-(n-1)y}{(x^2+y^2)^{\frac{n-3}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1} + \\
&\quad + \frac{1}{(x^2+y^2)^{\frac{n-1}{2}}} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j+1} (2j+1) x^{n-2j-1} y^{2j}.
\end{aligned} \tag{4.43}$$

With a computational software it can be shown that the Jacobian determinant is

$$\begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix} = n, \tag{4.44}$$

i.e. for every  $n$  the determinant is positive and equal to  $n$ . The process of finding the phase profile requires the Jacobian determinant to be negative and therefore one of the coordinates in the final transformation (4.42) needs to change sign. Another step is to compare partial derivatives  $\frac{\partial s}{\partial y}$ ,  $\frac{\partial t}{\partial x}$ . If they were equal, only one HOE would be needed and we could avoid computing complicated system of partial derivatives. Although that is not the case in (4.43), the only problematic term arises from differentiating the scaling pre-factor. Omitting the factor  $\frac{1}{(x^2+y^2)^{\frac{n-1}{2}}}$  in (4.42) would, on one hand, result in the transformation no longer being norm-preserving (the transformation would be  $(r, \varphi) \rightarrow (r^{|n|}, n\varphi)$ ), on the other hand, the calculations would be greatly simplified thanks to  $\frac{\partial s}{\partial y} = \frac{\partial t}{\partial x}$ . Changing the radius of the beam would not effect the value of its OAM as it is determined only by  $\exp(i\ell\varphi)$ . With these modifications the final transformation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} s(x, y) \\ t(x, y) \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} \\ - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j+1} x^{n-(2j+1)} y^{2j+1} \end{pmatrix}. \tag{4.45}$$

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