Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Synchronisation of quantum systems

Synchronizace kvantových systémů

Research Project

Výzkumný úkol

Author: **Daniel Štěrba**

Supervisor: **Ing. Jaroslav Novotný, Ph.D.**

Consultant: **prof. Ing. Igor Jex, DrSc.**

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Author's declaration:

I hereby declare to have written this work by myself and to have listed all sources used.

Prague, September 17, 2019 Daniel Štěrba

Prohlášení:

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V Praze dne 17. září 2019 Daniel Štěrba

Title:
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Author: Daniel Štěrba

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Abstract: The work addresses the question of synchronisation in quantum systems. It introduces the formalism of quantum Markovian dynamical semigroups and Lindblad dynamics as a suitable model to describe the evolution of open quantum systems and to study their asymptotic behaviour. The concept of synchronisation in the current literature is discussed and a suitable definition of synchronisation for composed systems of several identical subsystems is provided. The main part is focused on a system of two identical qubits for which all possible evolution maps within the studied dynamics that lead to asymptotic phase (anti-)synchronisation are found and classified. Corresponding attractor spaces are described and several other properties of the synchronising maps are further investigated.

Key words: synchronisation, antisynchronisation, asymptotic evolution, open quantum systems, QMDS, Lindblad dynamics, attractor space, two-qubit system

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Abstrakt: Práce se zabývá otázkou synchronizace v kvantových systémech. Shrnuje formalismus kvantových Markovovských dynamickcých semigrup a představuje lindbladovskou dynamiku jako vhodný model pro popis evoluce otevřených kvantových systémů a studium jejich asymptotické dynamiky. Je diskutován koncept synchronizace v současné literatuře a synchronizace je definována pro systém složený z několika identických podsystémů. Stěžejní část se věnuje systému dvou qubitů, nalezení veškerých možných evolucí v rámci studované dynamiky vedoucích k asymptotické fázové (anti-)synchronizaci a jejich následné klasifikaci. Jsou popsány příslušné atraktorové prostory a dále studovány rozličné vlastnosti synchronisujících zobrazení.

Klíčová slova: synchronizace, antisynchronizace, asymptotická evoluce, otevřené kvantové systémy, QMDS, lindbladovská dynamika, atraktorový prostor, systém dvou qubitů

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Introduction

The first to observe synchronisation was reportedly Christiaan Huygens who noticed the tendency of two pendulum clocks to adjust to anti-phase oscillations when mounted on a common support bar, and described the discovery in his letters as early as in February 1665 [1]. Since then synchronisation has been thoroughly explored in a great variety of classical systems [2], yet it was not until very recently that the study of this ubiquitous phenomenon entered the quantum realm.

In both classical and quantum domain, synchronisation is a very broad term. Various viewpoints and hence definitions and measures have been introduced [3], [4], [5], [6], [7], and systems investigated are numerous. The first works on the subject were typically concerned with the case of a forced synchronisation induced by an external field, or entrainment, examples include a driven oscillator [8], an oscillator coupled to a qubit [9] or combinations of van der Pol oscillators [10]. Another noteworthy field of research is represented by synchronisation protocols, proposals of how to exploit system properties such as entanglement to achieve clock synchronisation between two parties [11]. Finally, the main focus in current literature is on spontaneous synchronisation, the situation when two or more individual subsystems tune their local dynamics to a common pace due to the presence of coupling. Prevalent is the study of so called transient synchronisation, the emergence of synchronous behaviour in dissipative systems as a result of time-scale separation of decay rates of single modes [12]. In such a case the system goes through a long-lasting yet temporal phase of sychronised evolution, eventually approaching relaxation in the asymptotics. Among the examples of examined systems are oscillator networks [13], [14], spin systems [15], atomic lattices [16], qubits in bosonic environment [17] or collision models [18]. It was nonetheless demonstrated that synchronisation and quantum correlations can arise temporarily as well as asymptotically, and that such an asymptotic behaviour can be associated with the presence of synchronous modes in the decoherence-free subspaces of the state space [15], [19]. Very recently, a different kind of non-vanishing synchronised evolution was presented in the form of an analogue of the classical phase space limit cycles for a spin system of purely quantum nature [20]. The same authors also discuss the minimal quantum system which can exhibit this type of synchronisation [7], providing a promising baseline for studying limit cycles synchronisation in more complex spin networks. Synchronisation can even occur as a concomitant of other phenomena. In one particular instance it was shown to be an accompanying effect of super- and subradiance [17]. There have been various atempts to establish a link between spontaneous synchronisation and several possible local or global indicators such as entanglement [13], classical or quantum correlations [15] and discord [14]. To give a specific example, in [20] a synchronisation of two spins solely through their mutual interaction was demonstrated to always come with a creation of entanglement, the converse not necessarily true. While the results might be promissing in some specific cases, so far no general connection between the emergence of spontaneous synchronisation and any other phenomena has been found and the plausible mechanisms of quantum synchronisation and its very nature remain to great extend unknown.

An endeavour to understand the phenomenon of synchronisation on the quantum level motivates this work. The main idea is based on Huygens' original observation of two clocks. We investigate the emergence of spontaneous synchronisation between two individual identical systems with their own inner dynamics that are coupled together, with the aim of understanding the underlying synchronising mechanism. Identical systems have identical inner dynamics and natural frequencies, hence when it comes to synchronisation we talk about phase synchronisation. We are not concerned with temporal transient phases of synchronous behaviour preceeding dissipation and relaxation, as is often the case in the current literature, see the brief overview above, rather we look into systems exhibiting sychronised dynamics in the asymptotics.

For the process of synchronisation it is necessary to consider not only the possible mutual interaction of the individual constituents of the composite system in question but also the effects of the environment. Apart from the contact with the environment being practically inevitable, a closed system alone is not enough for the study of the phenomenon since unitary evolution in finite dimensions is at least quasiperiodic [21]. A thrid party is essential for non-trivial occurence of synchronisation. To account for possible environments and their interactions with the system it is best to view it as an open quantum system. One of the simplest and most convenient approaches used to describe the open dynamics and to study asymptotic behaviour is Markovian approximation. Within the Markovian approximation a framework of quantum mechanical dynamical semigroups and Lindblad dynamics represents a suitable tool, and is employed in this work.

We begin with an introduction to the formalism and several key elements of the theory. Discussion of the concept of synchronisation follows together with suitable definitions for our setup. In the main part of the work in chapter 3 we investigate in depth a system of two coupled non-interacting qubits, explore and classify all synchronising maps in the studied model and describe their properties. To do so we make use of a theorem presented in the theoretical part in chapter 1 which links generators of the evolution map in the Lindblad form and the attractor space via commutation relations. Firstly we assume all possible attractors corresponding to non-trivial synchronised asymptotic evolution and find generators of the evolution map which permit the existence of such attractors. From the resulting set of evolution generators we then pick those that enforce synchronisation on the entire attractor space and hence lead to synchronous asymptotic behaviour irrespective of initial conditions. Subsequently, we study their properties and conclude with the obtained results.

Chapter 1

Theoretical background

The evolution of an open quantum system is in general described by an irreversible linear completely positive trace non-increasing map acting on the space of linear operators on a Hilbert space. The open dynamics is often too complex for analytical solutions and certain additional simplifying assumptions need to be applied. A common approach is the use of Markovian approximation to describe dynamics of the system. This is suitable e.g. in the case that the changes in the surrounding environment, if present, arising from the interaction with the system dissipate very quickly and thus can be neglected. Two main classes of quantum Markovian processes are commonly studied, namely discrete quantum Markov chains and continuous quantum Markov dynamical semigroups. With the latter being utilized throughout the rest of the work, this section gives a brief introduction to the necessary theory.

1.1 Preliminary

Assume a quantum system is represented by a finite-dimensional Hilbert space $\mathscr{H}$, let $\mathcal{B}(\mathscr{H})$ be the associated space of all bounded linear operators on $\mathscr{H}$. For $A, B \in \mathcal{B}(\mathscr{H})$, $(A, B) = \operatorname{Tr}\left\{A^\dagger B\right\}$ stands for the corresponding scalar product and $\|A\|$ the induced norm thereof, with $A^\dagger$ being the adjoint operator of A defined via the scalar product $\langle\, , \rangle$ on $\mathscr{H}$. A state of such a system is described by a density operator $\rho \in \mathcal{B}(\mathscr{H})$, a (generally not stricly) positive self-adjoint operator with a unit trace.

1.2 Quantum dynamical semigroups

Among all possible evolutions of a state of an open quantum system special attention is paid to the so called quantum Markovian dynamical semigroups. By the Markov property it is meant that the evolution depends only on the present state and is completely independent of its past. Further, we assume that the process is homogenous, that is the evolution from t_1 to t_2 depends solely on the time difference $\Delta t = t_2 - t_1$ and not on the actual points in time themselves. With these properties combined we arrive at the following definition.

Definiton 1.2.1. A one-parameter family of completely positive (CP) trace non-increasing maps $\mathcal{T}_t : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$, parameterized by $t \in \mathbb{R}_0^+$, satisfying

$$\mathcal{T}_t \mathcal{T}_s = \mathcal{T}_{t+s} \quad \text{and} \quad \mathcal{T}_0 = I \tag{1.1}$$

is called a quantum Markovian dynamical semigroup (QDMS).

1.3 Generators of QMDS and Lindblad operators

In the case of continuous quantum dynamical semigroups we make use of the results of [22], further discussed in [23].

Theorem 1.3.1. *Let $\mathcal{T}_t$ be a continuous quantum dynamical semigroup (continuous in the parameter t above). Then the superoperator $\mathcal{T}_t \equiv \mathcal{T}$ is differentiable in t and is of the form*

$$\mathcal{T}_t = \exp(\mathcal{L}t), \tag{1.2}$$

where $\mathcal{L} : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ is a conditional completely positive[1] *map called the generator. The generator $\mathcal{L}$ can be split into*

$$\mathcal{L}(\rho) = \phi(\rho) - K\rho - \rho K^\dagger, \tag{1.3}$$

where ϕ is completely positive and $K \in \mathcal{B}(\mathscr{H})$.

The master equation governing the evolution of an arbitrary state ρ in this model reads

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \mathcal{L}(\rho(t)). \tag{1.4}$$

A CP map ϕ admits a decomposition into Kraus operators [24], denoted here as $\{L_j\}$, and together with further splitting of K into hermitian and antihermitian (denoted iH) part and the introduction of an optical potential $B = B^\dagger, B \geq 0$ defined via

$$K = iH + \frac{1}{2}\phi^\dagger(I) + B \tag{1.5}$$

we arrive at the final expression of the generator, namely

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_j L_j \rho L_j^\dagger - \frac{1}{2}\left\{L_j^\dagger L_j, \rho\right\} - B\rho - \rho B^\dagger. \tag{1.6}$$

Here the operator H can be identified with the hamiltonian. Indeed in the case of $L_i = B = 0$ equations (1.6) and (1.4) reduce to $\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H,\rho]$, standard expression for the unitary evolution of a closed system. It is to mention that an arbitrary choice of a self-adjoint operator H, positive operator B and operators $\{L_j\}$ satisfying $\sum_j L_j^\dagger L_j \leq I$ gives a valid generator $\mathcal{L}$ leading to a CP trace non-increasing map $\mathcal{T}$ and as such describes a physically possible evolution of an open system.
The relation (1.6) simplifies in case of trace-preserving QMDS. A condition equivalent to the trace-preserving property $\mathcal{T}^\dagger(I) = I$ directly implies $0 = \mathcal{L}^\dagger(I) = \phi^\dagger(I) - K^\dagger - K$ and subsequently $B = 0$. The resulting equation is known as the Lindblad equation and reads

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_j L_j \rho L_j^\dagger - \frac{1}{2}\left\{L_j^\dagger L_j, \rho\right\}. \tag{1.7}$$

[1] Meaning its exponential is a completely positive map. This condition is for example equivalent to $(\mathbb{I} - |\Omega\rangle\langle\Omega|)(\mathcal{L}\otimes\mathbb{I})\,|\Omega\rangle\langle\Omega|\,(\mathbb{I} - |\Omega\rangle\langle\Omega|) \geq 0$ for a maximally entagled state $|\Omega\rangle\,\langle\Omega|$ of arbitrary finite dimension greater than $\dim \mathcal{B}(\mathscr{H})$.

In this context, Kraus operators $\{L_j\}$ are usually reffered to as Lindblad operators. In calculations in chapter 3 we only deal with trace-preserving quantum operations.

Note: So far we have only been working with time-evolving states in Schrödinger picture; the evolution of observables in Heisenberg picture can be described in a similar way, see [23], [25].

1.4 Attractor space and asymptotic dynamics

The dynamics of an open system is typically complicated to analyze compared to a closed system as the generally non-unitary generator of the evolution may not be diagonalizable. However, should we only be concerned with the asymptotic dynamics of the system, there is an algebraic method developed in [26], [25] at hand. For details the reader is advised to study the original papers, for the purpose of this work we only state the most important results.

The asymptotic spectrum $\sigma_{as}(\mathcal{L})$ of a generator $\mathcal{L}$ is the set of all purely imaginary points of its spectrum $\sigma(\mathcal{L})$ and eventually zero, i.e.

$$\sigma_{as}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}), \mathrm{Re}\lambda = 0\}. \tag{1.8}$$

The attractor space $\mathrm{Att}(\mathcal{T})$ of superoperator $\mathcal{T} = \exp(\mathcal{L}t)$ is the subspace spanned by the eigenvectors of its generator corresponding to purely imaginary eigenvalues,

$$\mathrm{Att}(\mathcal{T}) = \bigoplus_{\lambda \in \sigma_{as}(\mathcal{L})} \mathrm{Ker}(\mathcal{L} - \lambda I). \tag{1.9}$$

We commonly refer to an element $X \in \mathrm{Att}(\mathcal{T})$ as attractor. An eigenvector X_λ of $\mathcal{L}$ associated with eigenvalue λ is also an eigenvector of $\mathcal{T}_t = \exp(\mathcal{L}t)$ associated with eigenvalue $\exp(\lambda t)$. It holds $|\exp(\lambda t)| = 1, \forall \lambda \in \sigma_{as}, \forall t \in \mathbb{R}_0^+$. It has been shown in [26] that it is always possible to diagonalize the part of generator $\mathcal{L}$ responsible for the asymptotic dynamics and therefore we can decompose (as a direct sum) the Hilbert space $\mathcal{B}(\mathscr{H})$, which represents a superset of the set of all possible states, into two subspaces $\mathrm{Att}(\mathcal{T})$ and Y. The former accounts for the asymptotic dynamics and the latter represents the part dying out during the evolution, i. e.

$$\mathcal{B}(\mathscr{H}) = \mathrm{Att}(\mathcal{T}) \oplus Y. \tag{1.10}$$

Assume $\{X_{\lambda,i}\}$ to be a basis of the attractor space $\mathrm{Att}(\mathcal{T})$, $\{X_k\}$ to be a basis of $\mathcal{B}(\mathscr{H})$ containing $\{X_{\lambda,i}\}$ and $\{X^k\}$ to be the basis of $\mathcal{B}^*(\mathscr{H})$ dual to $\{X_k\}$, i.e. $\mathrm{Tr}\left\{X_k^\dagger X^{k'}\right\} = \delta_{kk'}$. Denoting $X^{\lambda,i}$ the elements of $\{X^k\}$ which constitute the basis dual to $\{X_{\lambda,i}\}$, $\mathrm{Tr}\left\{X_{\lambda,i}^\dagger X^{\lambda',j}\right\} = \delta_{\lambda\lambda'}\delta_{ij}$, and play the role of determining coefficients in the asymptotic dynamics, we can express the asymptotic dynamics of an initial state $\rho(0)$ as

$$\rho_{as}(t) = \sum_{\lambda \in \sigma_{as}(\mathcal{L}), i} \exp(\lambda t)\, \mathrm{Tr}\left\{\left(X^{\lambda,i}\right)^\dagger \rho(0)\right\} X_{\lambda,i} \tag{1.11}$$

and it holds

$$\lim_{t\to\infty} \|\rho(t) - \rho_{as}(t)\| = 0. \tag{1.12}$$

In general the construction of a suitable basis and its dual remains a challenging task.
To state the final structure theorem revealing a possible way of how to find the attractor space of a given continuous QMDS one more definition is needed.

Definiton 1.4.1. A state σ satisfying $\sigma > 0$ is called a faithful $\mathcal{T}$-state if

$$\mathcal{T}_t(\sigma) \leq \sigma, \forall t > 0. \tag{1.13}$$

For a trace-preserving superoperator of the form (1.2) the condition reduces to $\mathcal{L}(\sigma) = 0$. Finally, due to [25] the following holds.

Theorem 1.4.2. *Let $\mathcal{T}_t : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a QMDS with generator $\mathcal{L}$ of the form* (1.6) *equipped with a faithful $\mathcal{T}$-state σ and let $X \in \mathcal{B}(\mathscr{H})$ be an attractor of $\mathcal{T}_t$ in Schrödinger picture associated with eigenvalue λ. Then the following set of equations holds*

$$\left[L_j, X\sigma^{-1}\right] = \left[L_j, \sigma^{-1}X\right] = \left[L_j^\dagger, X\sigma^{-1}\right] = \left[L_j^\dagger, \sigma^{-1}X\right] = 0, \tag{1.14}$$

$$\left[B, X\sigma^{-1}\right] = \left[B, \sigma^{-1}X\right] = 0, \tag{1.15}$$

$$\left[H, X\sigma^{-1}\right] = i\lambda\sigma^{-1}X, \quad \left[H, X\sigma^{-1}\right] = i\lambda X\sigma^{-1}. \tag{1.16}$$

If $\mathcal{T}_t$ is either trace-preserving or the faithful $\mathcal{T}$-state is stationary the reverse statement applies.

This theorem will be employed as a starting point of our calculations.

1.5 Special case of unitary Lindblad operators

In the application in chapter 3 we use a simplifying assumption that all the Lindblad operators L_j in (1.7) are proportional to some unitary operator U_j, i.e. $L_j = \sqrt{p_j}U_j$ and $\sum_j p_j = 1$. The Lindblad equation takes the form

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_j p_j U_j \rho U_j^\dagger - \rho \tag{1.17}$$

and clearly the identity, proportional to the maximally mixed state, is preserved under evolution and satisfies $I > 0$, $\mathcal{T}(I) = I \leq I$. In this particular case, the choice of a suitable basis and construction of its dual becomes a trivial task. The eigenspaces $\mathrm{Ker}(\mathcal{L} - \lambda I), \lambda \in \sigma_{as}(\mathcal{L})$, forming the attractor space $\mathrm{Att}(\mathcal{T})$ are mutually orthogonal and the same holds for $\mathrm{Att}\mathcal{T}$ and Y (see [26])

$$\mathrm{Ker}(\mathcal{L} - \lambda_i I) \perp \mathrm{Ker}(\mathcal{L} - \lambda_j I) \quad \text{for} \quad \lambda_i, \lambda_j \in \sigma_{as}(\mathcal{L}),\, \lambda_i \neq \lambda_j, \tag{1.18}$$

$$\mathrm{Att}(\mathcal{T}) \perp Y. \tag{1.19}$$

Furthermore, the theorem 1.4.2 reduces to

Theorem 1.5.1. *An element $X \in \mathcal{B}(\mathscr{H})$ is an attractor of $\mathcal{T}_t$ with generator $\mathcal{L}$ of the form* (1.17) *associated with eigenvalue λ if and only if it holds*

$$[U_j, X] = \left[U_j^\dagger, X\right] = 0, \tag{1.20}$$

$$[H, X] = i\lambda X. \tag{1.21}$$

In a case where we deal with just a single operator U_1 with weight $p_1 = 1$ we denote it simply as U. The explicit stating of both commutators in (1.20) is rather redundant as one follows from the other for unitary operators, however, evaluating both of them will help to simplify calculations later on.

Chapter 2

Synchronisation and measures

In order to be able to talk about synchronisation we should first clearly define what we mean by saying a quantum system is synchronised. In this chapter we first introduce our viewpoint of synchronisation and definitions suitable for our work, an itroduction of the most common synchronisation measures and a discussion of several other concepts of synchronisation follows.

2.1 Synchronisation and antisynchronisation

Let us first emphasize that quantum synchronisation does not refer to any newly discovered phenomenon of quantum nature, unwitnessed in the classisal domain. It refers to the synchronisation of quantum systems in the classical understanding. Various concepts and measures of synchronisation applicable in the quantum realm appear in the current literature, see [3], [4], [5] for a brief overview. In the classical domain, the notion and measures of synchronisation are typically built upon comparing systems trajectories in the phase space. In quantum systems, there are two different main approaches to consider. The first one is to look at the dynamics of local observables and their expectation values, the second is to directly compare the local density matrices or other representations of the quantum states using a suitable criterion.
We choose the second approach based on the states themselves rather than observables but before we move to the actual definitions of synchronisation for the purpose of this work, we should motivate them briefly by discussing in layman's terms what it is we want to be synchronous and how to choose a suitable criterion, as the choice needs to be done accordingly to the investigated system.
Imagine a case of two detuned oscillators operating at two distinc frequencies. The intuitive understanding of their synchronisation is an evolution towards oscillations at a single common frequency, possibly with a resulting constant phase shift bethween the two oscillators. However, this understanding brings at least two difficulties. First, the resulting common frequency will be set by and dependent on the outer synchronisation mechanism. Second, once this mechanism is turned off the inner dynamics of the oscillators will tend to desynchronise their frequencies again. On the other hand, in the case of two identical pendulum clocks which from the very beginning oscillate with the same frequency, the natural is synchronisation of their phases. When synchronised, they should move with a given phase difference, typically in-phase or anti-phase, irrespective of the initial shift.

The idea behind this works originates from the Huygens' clock experiment. As a result we are interested in the case of two or potentially more identical systems with their own inner dynamics, same for all of them. An example of such systems is a qubit network. To synchronise is not a specific observable, but rather the actual states of the systems in question. This is an analogue to the phase synchronisation of the classical clocks. With identical inner dynamics, the systems will continue to evolve synchronously after the synchronisation process even if the coupling and mutual interactions are interrupted, resembling two clocks taken apart. For the most part we are interested in the asymptotic dynamics and not in any transient effects, consequently we can make use of a more restrictive absolute understanding of synchronisation rather than a measure that would describe the process leading to synchronised evolution. Hence our choice of setup and following definitions.

Assume a quantum system with an associated Hilbert space $\mathscr{H}$. For an n-component composite system of identical subsystems associated with Hilbert space $\mathscr{H}^{\otimes n} = \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$, $\mathscr{H}_i = \mathscr{H}$, $\forall i \in \{1, \ldots, n\}$, in a state $\rho \in \mathcal{B}(\mathscr{H}^{\otimes n})$ let us denote ρ_k the reduced state of the k$^{\text{th}}$ component, obtained as a partial trace over all the remaining n-1 subsystems,

$$\rho_k = \mathrm{Tr}_{\otimes_{j\neq k}\mathscr{H}_j}\, \rho. \tag{2.1}$$

Definiton 2.1.1. Assume a n-component composite system in a state $\rho(t) \in \mathcal{B}(\mathscr{H}^{\otimes n})$ in time t. We say that the n individual systems in the reduced states $\rho_1(t), \ldots, \rho_n(t)$ are synchronised if $\forall j, k$ there exist stationary states $\rho_{c_{jk}}$, $\frac{\partial \rho_{c_{jk}}}{\partial t} = 0$, such that

$$\rho_j(t) - \rho_k(t) = \rho_{c_{jk}}, \quad \forall t, \tag{2.2}$$

and that they achieve (or that the entire composite system achieves) an asymptotic synchronisation if

$$\lim_{t\to\infty} \left\| \rho_j(t) - \rho_k(t) - \rho_{c_{jk}} \right\| = 0. \tag{2.3}$$

We say that the operation $\mathcal{T}$ and its generating operators $\{L_j\}$ or $\{U_j, p_j\}$ respectively synchronise or lead to synchronisation if all the asymptotic reduced states $\lim_{t\to\infty} \rho_j(t)$ of the evolution are synchronised for an arbitrary initial state $\rho(0)$. We call the operation $\mathcal{T}$ itself and its generating operators synchronising.

According to this definition, the subsystems are synchronised if their non-stationary parts undergo the same evolution. We allow for a constant difference between the synchronised states, imposing constraints only on the dynamical part. To be able to distinguish we introduce a more restrictive second definition.

Definiton 2.1.2. We speak of total synchronisation if the reduced states $\rho_1(t), \ldots, \rho_n(t)$ of all systems in question are the same,

$$\rho_j(t) - \rho_k(t) = 0, \quad \forall j, k, t, \tag{2.4}$$

or respectively of asymptotic total synchronisation if

$$\lim_{t\to\infty} \| \rho_j(t) - \rho_k(t) \| = 0, \quad \forall j, k. \tag{2.5}$$

The rest of the terminology is defined analogously.

With this definition we want not only the dynamical parts of the systems to be the same, but also for the systems to oscillate around the same state.

Assuming only two systems we label them with letters A and B and, again denoting $\rho(t) \in \mathcal{B}(\mathscr{H}^{\otimes 2})$ the global state of the composite system, write the condition of total synchronisation as

$$\operatorname{Tr}_A \rho(t) = \operatorname{Tr}_B \rho(t). \tag{2.6}$$

For two systems it also makes sense to speak of antisynchronisation. We again introduce two definitions, the first one requiring the systems' dynamical parts to evolve in anti-phase, the second imposing an additional restriction of oscillating around the same stationary state.

Assuming two systems A and B in reduced states $\operatorname{Tr}_B \rho(t) = \rho_A(t)$ and $\operatorname{Tr}_A \rho(t) = \rho_B(t)$ of a global state $\rho(t)$ we denote the stationary part of the state of system $X \in \{A, B\}$ as $\rho_{X,st}$ and the dynamical part as $\rho_{X,dyn}$,

$$\rho_A(t) = \rho_{A,st} + \rho_{A,dyn}(t), \tag{2.7}$$

$$\rho_B(t) = \rho_{B,st} + \rho_{B,dyn}(t), \tag{2.8}$$

$$\frac{\partial \rho_{A,st}}{\partial t} = \frac{\partial \rho_{B,st}}{\partial t} = 0. \tag{2.9}$$

Definiton 2.1.3. We say that the systems A and B are antisynchronised if

$$\rho_{A,dyn}(t) = -\rho_{B,dyn}(t), \tag{2.10}$$

and totally antisynchronized if in addition to (2.10) it holds

$$\rho_{A,st} = \rho_{B,st}. \tag{2.11}$$

The remaining terminology is defined in a similar fashion.

Note: To satisfy our definition of a synchronisation the quantum states need not be evolving in time, for example the maximally mixed state proportional to identity cleary satisfies (2.4) and simultaneously is stationary in the studied dynamics (1.4),(1.6). In this work, however, we focus on systems with non-trivial asymptotic evolution, i.e. on situations when the synchronisation mechanism does not kill the inner dynamics.

Contrary to our definitions of (anti-)synchronisation, it is possible to address the matter purely via observables of the system, a way which in fact seems prevalent in the current literature. The dynamics of two or more systems are characterized by the expectation values of chosen local observables and their time evolutions are compared by a classical criterion. The advantage is that this provides not only a definition of synchronisation, but also a measure thereof. It

is also well suited for the study of imperfect transient synchronisation and for the study of synchronisation of nonidentical subsystems.
The problem with taking only observables into account and not the states themselves is that it only makes use of partial information about the systems in question. Synchronisation of one observable does not imply synchronisation of other ones, nor does it guarantee that the systems are not in substantially different states. The two concepts are, nonetheless, interwined.
Again assume a n-component composite system of identical subsystems in a state $\rho \in \mathcal{B}(\mathscr{H}^{\otimes n})$. Synchronisation with respect to a local observable $\sigma \in \mathcal{B}(\mathscr{H})$ can be understood as a situation when the expectation value of σ is the same on all of the individual subsystems

$$\mathrm{Tr}(\sigma\rho_1) = \cdots = \mathrm{Tr}(\sigma\rho_n), \tag{2.12}$$

or if we denote $\sigma^{(l)} = I^{\otimes l-1} \otimes \sigma \otimes I^{\otimes n-l}$ the local operator corresponding to the l^{th} component

$$\left\langle \sigma^{(1)} \right\rangle = \cdots = \left\langle \sigma^{(n)} \right\rangle. \tag{2.13}$$

Clearly the synchronisation of states, specifically the total synchronisation in the sense of definition 2.1.2, implies synchronisation of any local observable $\sigma \in \mathcal{B}(\mathscr{H})$. The converse is not true, unless we extend the requirement on all possible observables $\sigma \in \mathcal{B}(\mathscr{H})$ simultaneously. Formally, if a system in a state $\rho \in \mathcal{B}(\mathscr{H}^{\otimes n})$ has equal expectation values $\mathrm{Tr}\left(\sigma^{(i)}\rho\right)$ on all its components i for all observables $\sigma \in \mathcal{B}(\mathscr{H})$, then the reduced states ρ_i of individual component subsystems are the same. This follows immediatly from the fact that the trace is a scalar product on $\mathcal{B}(\mathscr{H})$.
Therefore, when not restricting ourselves to a small set of predetermined observables we actually require synchronisation of states when requiring synchronisation of observables, and vice versa.

We might even impose additional requirements on synchronisation. Imagine the question is when two clocks are synchronised, knowing they should actually be used in application. It is reasonable to ask not only for the expectation values and probabilities of measuring different possible outcomes to match, but also for the clocks to always provide the same information to both parties, that is for the measurements to be prefectly correlated. This requirement is, however, too restrictive. For a given system and a variable there might not exist states that would guarantee correlated measurement. It is not suitable for a generally applicable definition of synchronisation.

2.2 Synchronisation measures

Synchronisation measures provide a way of quantifying synchronisation and describing the synchronisation process. They can also account for possible errors in synchronisation of the established states. Typically, an observable is chosen and a suitable criterion is applied to its expectation values on the subsystems.
Such a criterion is the Pearson's correlation coefficient defined for two real-valued time-dependent functions f, g via

$$C_{f,g}(t, \Delta t) = \frac{\overline{(f-\bar{f})(g-\bar{g})}}{\sqrt{\overline{(f-\bar{f})^2}\,\overline{(g-\bar{g})^2}}}, \tag{2.14}$$

where $t \in \mathbb{R}$ is used to denote time and $\bar{f} = \frac{1}{\Delta t}\int_t^{t+\Delta t} f(t')dt'$ is the mean value of f at time t calculated over a time window Δt, similarly for g, [4], [2]. The coefficient ranges from -1 to 1 with 1 corresponding to synchronisation of f and g and -1 to antisynchronisation of the two, irrespective of a possible constant difference between them. It was successfully applied for example to position and momentum operator expectation values for two dissipating oscillators [14], in oscillator networks [13] or to spin operators in various spin systems [15], atomic lattices [16] or collision models [18]. It will also be used later in this work in illustrating numerical simulations.

Another criterion of the same kind is the so called synchronisation error typically used for the study of chaotic systems [4]. It is defined for two systems as

$$S_c(t) = \left\langle (q_-^2(t) + p_-^2(t) \right\rangle^{-1}, \tag{2.15}$$

where $q_- = \frac{1}{\sqrt{2}}(q_1 - q_2)$ is the difference in position, the same for momentum p, trajectiories in the classical case and operators in the quantum one. This measure is bounded in the quantum domain by the uncertainity relations. It was employed for example in [3] to compare a pair of coupled optomechanical oscillators.

2.3 Other concepts

Last but not least, there is a very recent insight into the topic of synchronisation of quantum systems to be mentioned. In [7], [20] the Husimi Q representation is presented as a suitable tool for description and detection of synchronisation. Husimi Q representation is related to the arguably better known Wigner-Weyl representation. The authors use this phase-space formulation to discuss emergence of limit cycles in a two-node network of spin one oscillators and they also show it to be a minimal system for the study of the phenomenon.

Chapter 3

Two-qubit system

In this chapter we examine in detail the simplest case possible - a system of two identical two-level subsystems, a system of two identical qubits, with the evolution of its state ρ described by QMDS whereof the generator takes the form (1.17) with just a single unitary operator U with weight 1, i.e.

$$\mathcal{L}(\rho) = -i[H, \rho] + U\rho U^{\dagger} - \rho. \tag{3.1}$$

Let $\mathscr{H}_1$ be the Hilbert space corresponding to a single qubit and

$$H_1 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \tag{3.2}$$

$E_1, E_2 \in \mathbb{R}$, be the Hamiltonian in the basis of its eigenvectors $|1\rangle$ and $|2\rangle$. For a closed system the evolution of an initial state $|\psi\rangle \in \mathscr{H}_1$,

$$|\psi\rangle = a\,|1\rangle + b\,|2\rangle\,, \tag{3.3}$$

$a, b \in \mathbb{C}$, is generated by the Hamiltonian H_1 and given by

$$|\psi(t)\rangle = e^{-iE_2 t}\left(e^{-i(E_1-E_2)t} a\,|1\rangle + b\,|2\rangle\right). \tag{3.4}$$

Here $e^{-iE_2 t}$ represents an overall phase prefactor, irrelevant as far as the qubit alone is concerned, while the intrinsic frequency of the system dynamics is $\omega = E_1 - E_2$, given as the difference of eigenvalues.

Let $(|11\rangle, |12\rangle, |21\rangle, |22\rangle)$ be the basis of the entire system with the Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_1$ and Hamiltonian $H = H_1 + H_2 \equiv H_1 \otimes I + I \otimes H_1$, explicitly

$$H = \begin{pmatrix} 2E_1 & 0 & 0 & 0 \\ 0 & E_1 + E_2 & 0 & 0 \\ 0 & 0 & E_1 + E_2 & 0 \\ 0 & 0 & 0 & 2E_2 \end{pmatrix}. \tag{3.5}$$

Note that we use the standard notation $|ij\rangle = |i\rangle \otimes |j\rangle$. We will stick to this computational basis throughout the entire chapter.

3.1 Two-qubit synchronisation

Applying the theorem 1.5.1 to find all elements of the attractor space for a given evolution map we first make use of the commutation relation (1.21) to separate the space of all states $\mathcal{B}(\mathscr{H})$ into five subspaces $X_{i\lambda}$ based on the corresponding associated eigenvalue λ,

$$X_0 = \text{span}\{|11\rangle\langle 11|, |22\rangle\langle 22|, |12\rangle\langle 12|, |21\rangle\langle 21|, |12\rangle\langle 21|, |21\rangle\langle 12|\}, \tag{3.6}$$

$$X_{2E_1-2E_2} = \text{span}\{|11\rangle\langle 22|\}, \tag{3.7}$$

$$X_{2E_2-2E_1} = \text{span}\{|22\rangle\langle 11|\}, \tag{3.8}$$

$$X_{E_1-E_2} = \text{span}\{|11\rangle\langle 12|, |21\rangle\langle 22|, |11\rangle\langle 21|, |12\rangle\langle 22|\}, \tag{3.9}$$

$$X_{E_2-E_1} = \text{span}\{|21\rangle\langle 11|, |22\rangle\langle 12|, |12\rangle\langle 11|, |22\rangle\langle 21|\}. \tag{3.10}$$

The first one, X_0, corresponds to the stationary part of a possible asymptotic state that does not evolve in time and as such a vector from this subspace automatically satisfies our condition of synchronisation (2.1.1). It plays, however, an important role in the question of total synchronisation discusssed in the next section.

The following two subspaces, namely $X_{2E_1-2E_2}$ and $X_{2E_2-2E_1}$, are trivial from the point of view of synchronisation in the sense that any vectors $X_1 \in X_{2E_1-2E_2}$ and $X_2 \in X_{2E_2-2E_1}$ satisfy $\text{Tr}_A X_1 = Tr_B X_1 = 0$ and $\text{Tr}_A X_2 = Tr_B X_2 = 0$ respectively. They however contribute to the asymptotic evolution of the composite system.

Finally, the last two subspaces $X_{E_2-E_1}$ and $X_{E_1-E_2}$ correspond to the non-trivial evolution of the reduced one-qubit states. The two subspaces are connected by the operation of complex conjugation. Solving the commutation relations (1.20) and (1.21) for one of the subspaces provides the solution for the other one.
It holds that X is an eigenvector of a linear map ϕ with eigenvalue λ iff $X^\dagger$ is an eigenvector of ϕ with eigenvalue $\bar{\lambda}$,

$$\phi(X) = \lambda X \iff \phi(X^\dagger) = \bar{\lambda} X^\dagger. \tag{3.11}$$

It can also be seen from the fact that for the commutation relations it holds

$$[X, U] = \left[X, U^\dagger\right] = 0 \iff \left[X^\dagger, U\right] = \left[X^\dagger, U^\dagger\right] = 0, \tag{3.12}$$

$$[H, X] = i\lambda X \iff \left[H, X^\dagger\right] = i\bar{\lambda} X^\dagger, \tag{3.13}$$

for any matrices X, U, H, where H is self-adjoint, $\lambda \in \mathbb{C}$, and for which the expressions make sense.

Thus we restrict to work only with the space $X_{E_1-E_2}$ and choose to parameterize its arbitrary element $X \in X_{E_1-E_2}$ as

$$X = \alpha |11\rangle\langle 12| + \beta |21\rangle\langle 22| + \gamma |11\rangle\langle 21| + \delta |12\rangle\langle 22|, \tag{3.14}$$

equivalently also

$$X = \begin{pmatrix} 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.15}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. The partial trace condition (2.2) of synchronisation reduces to

$$\alpha + \beta = \gamma + \delta. \tag{3.16}$$

In light of theorem 1.5.1, our goal is to find all possible unitaries U in (3.1) such that the solution to the commutation relations

$$[U, X] = \left[U^{\dagger}, X\right] = 0 \tag{3.17}$$

for $X \in X_{E_1-E_2}$ is non-trivial and satisfies the condition of synchronisation. Stated less formally, we want to find all possible two-qubit couplings that can be represented by a single unitary operator in the Lindblad equation such that the evolution described by the corresponding QMDS leads to an asymptotically synchronized state.

To achieve this we will go through all such possible solutions X and find the unitaries that permit them to subsequently pick out those unitaries that permit exclusively such solutions. We will work in the parameterization given by (3.14) and discuss separatelly all possible attractors $X \in X_{E_1-E_2}$ satisfying the synchronisation condition (3.16), sorted by the number of non-zero coefficients (denoted $\alpha, \beta, \gamma, \delta$) in the parametrization. Evaluating of the commutation relations (3.17) will give us a set of unitaries U for each possible attractor X and from these sets we will extract those operators U that not only commute with the synchronised attractor X, but also enforce the synchronisation condition on the entire associated attractor space. Since by our definition the synchronisation condition is also necessary for the total synchronisation, we will make use of the results in the next section where we further extract those operators U that enforce total synchronisation.
Before we begin note that any two unitary operators that differ only by an arbitrary overall phase factor lead to the same evolution map due to the form of the generator (3.1), we will therefore omit the phase prefactor in the expressions below for simplicity. Now for the actual calculations.

I. One non-zero coefficient:

This situation cannot occur as the synchronisation condition (3.16) requires at least two non-zero coefficients for non-trivial solutions.

II. Two non-zero coefficients:

a) $\alpha = -\beta \neq 0 \wedge \gamma = \delta = 0$ or $\gamma = -\delta \wedge \alpha = \beta = 0$

This corresponds to a stationary asymptotic evolution of the resulting reduced states of individual qubits. Indeed, it can be seen from the parameterization (3.14) that both reduced operators of such attrator X reduce to zero. Consequently, it can only contribute to the evolution of mutual correlations.

b) $\beta = \gamma = 0$ and $\alpha = \delta \neq 0$

The attractor X now reads

$$X = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.18}$$

Denoting u_{ij} the matrix elements of U we can explicitly evaluate the commutation relations (3.17) in the form $XU = UX$ and $XU^\dagger = U^\dagger X$, resulting straightforwardly into a set of equations

$$u_{ij} = 0 \quad \text{for} \quad i \neq j, \tag{3.19}$$

$$u_{11} = u_{22} = u_{44}, \tag{3.20}$$

giving U of the form

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.21}$$

where $a \in \mathbb{C}, |a| = 1$, since we further require $UU^\dagger = U^\dagger U = I$ and factor out an arbitrary phase prefactor.

We have found our first candidate for a synchronising map, yet so far we have only shown that if the attractor X is suppposed to have a certain form satisfying synchronisation condition (3.16) and the commutation relations (3.17) hold, then U has to have the form (3.21). However, for U to lead to synchronisation, the existence of a synchronised attractor X is only a necessary condition, not a sufficient one. There might be other elements of the corresponding attractor space that do not satisfy the synchronisation condition. To show that an operator U generates a synchronising map, we need to prove the opposite relation, that is given an operator U, here by equation (3.21), and a general attractor $X' \in X_{E_1 - E_2}$ satisfying commutation relations $[U, X'] = \left[U^\dagger, X'\right] = 0$, then X' necessarily satisfies the synchronisation condition $\mathrm{Tr}_A X' = \mathrm{Tr}_B X'$.

Given the diagonal form of U close to identity, it is no surprise that this converse statement does not hold and that our candidate does not lead to a synchronising map. We postpone the proof of our claim to the section 3.3 where a counterexample is provided.

c) $\alpha = \delta = 0$ and $\beta = \gamma \neq 0$

Analogously to the previous case we arrive at relations

$$u_{ij} = 0 \quad \text{for} \quad i \neq j, \tag{3.22}$$

$$u_{11} = u_{33} = u_{44}, \tag{3.23}$$

giving U of the form

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.24}$$

for $b \in \mathbb{C}, |b| = 1$.

Similarly to the previous case, this candidate for U does not lead to synchronisation.

d) $\beta = \delta = 0$ and $\alpha = \gamma \neq 0$

Noticing that

$$X = \alpha \left|11\right\rangle \left(\left\langle 12\right| + \left\langle 21\right|\right), \tag{3.25}$$

we introduce a new orthonormal basis (e_1, e_2, e_3, e_4) where

$$e_1 = \left|11\right\rangle, \tag{3.26}$$

$$e_2 = \frac{1}{\sqrt{2}}(\left|12\right\rangle + \left|21\right\rangle), \tag{3.27}$$

$$e_3 = \frac{1}{\sqrt{2}}(\left|21\right\rangle - \left|12\right\rangle), \tag{3.28}$$

$$e_4 = \left|22\right\rangle, \tag{3.29}$$

so that the transition matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.30}$$

is unitary and the attractor X in the new basis reads

$$\tilde{X} = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.31}$$

Using the fact that the commutation relations (3.17) are invariant with respect to the change of basis we evaluate them directly to obtain $\tilde{U}$, the operator U in the new basis, which can then be transformed back into the original computational basis as $U = T\tilde{U}T^\dagger$. (3.17) implies

$$\tilde{u}_{11} = \tilde{u}_{22}, \tag{3.32}$$

$$\tilde{u}_{12} = \tilde{u}_{13} = \tilde{u}_{14} = \tilde{u}_{21} = \tilde{u}_{23} = \tilde{u}_{24} = \tilde{u}_{31} = \tilde{u}_{32} = \tilde{u}_{41} = \tilde{u}_{42} = 0, \tag{3.33}$$

leaving the lower right 2x2 submatrix arbitrary. The unitarity condition $\tilde{U}\tilde{U}^\dagger = \tilde{U}^\dagger\tilde{U} = I$ can be written in blocks implying that the 2x2 submatrix is also unitary and we can factor out a global phase factor such that its determinant is equal to one, enabling us to parameterize it as an element of $SU(2)$. Put together we get

$$\tilde{U} = \begin{pmatrix} c\, I_{2\times 2} & 0 \\ 0 & A \end{pmatrix}, \tag{3.34}$$

where $c \in \mathbb{C}, |c| = 1$ and $A \in SU(2)$. Consequently

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.35}$$

where $a, b, c \in \mathbb{C}, |c| = 1$ and $|a|^2 + |b|^2 = 1$.

For U given by (3.35) to be synchronising two additional conditions need to be imposed, namely that

$$a \neq c, \tag{3.36}$$

$$a \neq \pm 1. \tag{3.37}$$

As the proof is analogous to the one presented below when discussing the case II.e), we skip it here.

e) $\alpha = \gamma = 0$ and $\beta = \delta \neq 0$

This case can be solved similarly to the previous one. We only make a small change switching the second element of the new basis (3.27) for the third one (3.28) and multiplying the latter by minus one, that is taking $(e_1, -e_3, e_2, e_4)$, with e_1, e_2, e_3, e_4 given by equations (3.26) to (3.29), to be the new basis. The result in a familiar form $U = T\tilde{U}T^\dagger$ reads

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.38}$$

where again $a, b, c \in \mathbb{C}, |c| = 1$ and $|a|^2 + |b|^2 = 1$.

To see if our candidate for U leads to a synchronising map, assume a general attractor $X' \in X_{E_1 - E_2}$ parameterized by $\alpha', \beta', \gamma', \delta' \in \mathbb{C}$ as follows

$$X' = \alpha' \,|11\rangle\langle 12| + \beta' \,|21\rangle\langle 22| + \gamma' \,|11\rangle\langle 21| + \delta' \,|12\rangle\langle 22| \,. \tag{3.39}$$

In our new basis it can be expressed using $\tilde{X'} = T^\dagger X T$, resulting in

$$\tilde{X'} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}(\alpha' - \gamma') & \frac{1}{\sqrt{2}}(\alpha' + \gamma') & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}(\delta' - \beta') \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}(\delta' + \beta') \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.40}$$

As $\tilde{X}$ is assumed to be an attractor, $\left[\tilde{U}, \tilde{X'}\right] = \left[\tilde{U}^\dagger, \tilde{X'}\right] = 0$ holds. Written explicitly in the chosen parameterization and comparing matrix elements it gives the following set of equations.

$$c(\delta' - \beta') = \bar{a}(\delta' - \beta') \tag{3.41}$$
$$0 = b(\delta' - \beta') \tag{3.42}$$
$$c(\alpha' + \gamma') = a(\gamma' + \alpha') \tag{3.43}$$
$$\bar{a}(\alpha' - \gamma') = a(\alpha' - \gamma') \tag{3.44}$$
$$0 = \bar{b}(\alpha' - \gamma') \tag{3.45}$$
$$0 = b(\alpha' + \gamma') \tag{3.46}$$

The first two comprise conditions on β' and δ'. It follows directly from (3.42) that $b \neq 0$ implies $\beta' = \delta'$ and from (3.41) that also $c \neq \bar{a}$ implies $\beta' = \delta'$. If on the contrary $c = \bar{a}$ held it would imply $b = 0$ since $|a|^2 + |b|^2 = |c| = 1$, meaning that if $c \neq \bar{a}$ is not satisfied, nor is $b \neq 0$. Furthermore, $b \neq 0$ implies $c \neq \bar{a}$, leaving (3.42) redundant, all the information is contained in (3.41). The remaining equations impose no constraints on parameters β', δ'. Consequently, for $c = \bar{a}$ the operator U does not enforce synchronisation condition on X'. Hence the requirement

$$c \neq \bar{a} \tag{3.47}$$

is necessary for U to be synchronising. It is, however, not sufficient. Assume (3.47) holds. It follows $\beta' = \delta'$ and the synchronisation condition we want to be enforced on X' by the commutation relations $\left[\tilde{U}, \tilde{X'}\right] = \left[\tilde{U}^\dagger, \tilde{X'}\right] = 0$ reduces to $\alpha' = \gamma'$. From (3.45) and (3.46) it follows that $b \neq 0$ implies $\alpha' = \gamma' = 0$ and from the equation (3.44) that $a \neq \bar{a}$ implies $\alpha' = \gamma'$. In these cases the synchronisation condition is enforced. If on the contrary $b = 0$ and $a = \bar{a}$ (if and only if $a = \pm 1$), the equations (3.44), (3.45), (3.46) vanish and (3.43), (3.47) imply $\alpha' = -\gamma'$, contradicting the synchronisation condition. To achieve synchronisation the requirement

$$a \neq \pm 1 \tag{3.48}$$

is necessary and together with (3.47) also sufficient.

To sum up, the operator U of the form (3.38) generates a synchronising map if and only if $c \neq \bar{a}$ and $a \neq \pm 1$.

III. Three non-zero coefficients:

Remarkably, choosing any three of the coefficients non-zero and one equal to zero, the commutation relations (3.17) simply lead to

$$u_{ij} = 0 \quad \text{for} \quad i \neq j, \tag{3.49}$$

$$u_{11} = u_{22} = u_{33} = u_{44}, \tag{3.50}$$

so that the only unitary matrices U such that both U and $U^\dagger$ commute with X are multiples of identity,

$$U = I. \tag{3.51}$$

Since the identity operator commutes with any other operator, the commutation relation $[X', I] = 0$ trivially holds for any $X' \in X_{E_1 - E_2}$ and there are no constraints for X'. This case provides us with no synchronising operators U.

IV. Four non-zero coefficients:

The attractor X takes the form

$$X = \begin{pmatrix} 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.52}$$

with $\alpha, \beta, \gamma, \delta \neq 0$. Looking for submatrices with nonzero determinant in the upper right corner of X we see immediately that $\operatorname{rank} X = 2$. Thus, to simlify evaluation of the commutation relations we introduce a new basis (e_1, e_2, e_3, e_4) such that $e_1, e_2 \in \operatorname{Ker} X$, spanning the two-dimensional kernel, and $e_3, e_4 \in (e_1, e_2)^{\perp}$. Let

$$e_1 = |11\rangle\,, \tag{3.53}$$

$$e_2 = \gamma\,|12\rangle - \alpha\,|21\rangle\,, \tag{3.54}$$

$$e_3 = \bar{\alpha}\,|12\rangle + \bar{\gamma}\,|21\rangle\,, \tag{3.55}$$

$$e_4 = |22\rangle\,, \tag{3.56}$$

and without loss of generality the parameters α, γ are supposed to satisfy a normalization condition $|\alpha|^2 + |\gamma|^2 = 1$. This only impacts rescalling of the attractor X and is thus irrelevant for the result. The reason behind is that at the same time the normalization ensures that the new basis is orthonormal, the transition matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \bar{\alpha} & 0 \\ 0 & -\alpha & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.57}$$

is unitary and it holds $Xe_3 = e_1$, with no additional numerical prefactor, which further simplifies the form of X in the new basis. For the remaining basis element $e_4 = |22\rangle$, which comes from the original computational basis in order not to unnecessarily make the transition matrix T more complicated, we have $Xe_4 = \delta\,|12\rangle + \beta\,|21\rangle$. Clearly $Xe_4 \in \operatorname{span}(e_2, e_3)$, a fact that can be used to define two new parameters $s, r \in \mathbb{C}$ via

$$Xe_4 = se_2 + re_3 \tag{3.58}$$

to take over the role of the parameters $\beta = -s\alpha + r\bar{\gamma}$ and $\delta = s\gamma + r\bar{\alpha}$. This change of parameterization merely helps structurize the disscusion below in simpler terms. The attractor X in the new basis reads

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.59}$$

and the partial trace condition of synchronisation (3.16) takes the form

$$(1-s)\alpha - r\bar{\alpha} = (1+s)\gamma - r\bar{\gamma}. \tag{3.60}$$

This way we only need to examine the dependence on two parameters s and r while the other two, α and γ, keep their role of defining a unitary change of basis (3.57). Again, the result will be of the form $U = T\tilde{U}T^\dagger$. In the folllowing we explore all possible situations one can meet.

a) $s \neq 0$

Rewriting both matrices $\tilde{X}$ and $\tilde{U}$ in a block form

$$\tilde{X} = \begin{pmatrix} 0 & S \\ 0 & R \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.61}$$

introducing 2x2 matrices $A, B, C, D, S, R \in \mathbb{C}^{2x2}$, the commutation relations (3.17) imply $SC = 0 = SB^\dagger$. Hence $C = B = 0$ and the block-diagonal form of $\tilde{U}$ further simplifies the commutaion relations into $SD = AS$ and $TD = DT$. Note the same constraint holds for $A^\dagger, D^\dagger$ as well. Comparing matrix elements of the former, denoting $A = (a_{ij})$, $D = (d_{ij})$, we obtain

$$a_{11} = d_{11}, \tag{3.62}$$

$$a_{22} = d_{22}, \tag{3.63}$$

$$d_{12} = s a_{12}, \tag{3.64}$$

$$d_{21} = \frac{1}{s} a_{21}, \tag{3.65}$$

$$\bar{s} = \frac{1}{s} \implies |s| = 1. \tag{3.66}$$

If furthermore $r \neq 0$, by comparing matrix elements of the latter we get

$$d_{12} = d_{21} = 0 \implies a_{12} = a_{21} = 0, \tag{3.67}$$

$$d_{11} = d_{22}, \tag{3.68}$$

and thus the only solutions for $\tilde{U}$ and consequently for U are multiplies of identity. The consequence heeded we set $r = 0$ and since the matrices A and D are unitary we can parameterize them up to a phase prefactor as elements of $SU(2)$. From (3.60) and (3.66) it follows that

$$\frac{\bar{\alpha}}{\bar{\gamma}} = -\frac{\alpha}{\gamma}, \tag{3.69}$$

consequence of which is that the difference in phase between α and γ is $\pm i$, and further that

$$s = \frac{\alpha - \gamma}{\alpha + \gamma}. \tag{3.70}$$

Note that the seemingly problematic cases $\alpha = \gamma$, implying $s = 0$, and $\alpha = -\gamma$, for which neither the equation (3.70) is defined nor the synchronisation condition (3.60) is satisfied unless $s = 0$, are excluded due to the consequence of (3.69). This reflects the fact that setting $s = r = 0$ is equivalent to $e_4 \in \operatorname{Ker} X$ and $\beta = \delta = 0$, the situation discussed in II.d).

Together with the normalization condition $|\alpha|^2 + |\gamma|^2 = 1$, (3.69) and (3.70) show that the choice of the parameter α determines two non-equivalent pairs (γ, s), the two possibilities stemming from the two possible phase differences between α and γ, non-equivalent in the sense that they correspond each to a different attractor X. Hence there exist two distinct classes of unitaries U,

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \bar{\alpha} & 0 \\ 0 & -\alpha & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & \bar{s}b & 0 & 0 \\ -s\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\gamma} & -\bar{\alpha} & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.71}$$

given by parameters $a, b, \alpha, \gamma, s \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, $0 < |\alpha|^2 < 1$,

$$\gamma = \pm i\alpha \frac{\sqrt{1 - |\alpha|^2}}{|\alpha|}, \tag{3.72}$$

$$s = 2|\alpha|^2 - 1 \mp i \left(2|\alpha| \sqrt{1 - |\alpha|2} \right), \tag{3.73}$$

that commute each with the corresponding non-trivial attractor X of the form

$$X = \begin{pmatrix} 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & s\gamma \\ 0 & 0 & 0 & -s\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.74}$$

Not apparent at first sight, the parameter s runs around the entire unit circle except for the points ± 1 and it holds

$$|\alpha|^2 = \frac{(s+1)^2}{4s}. \tag{3.75}$$

Due to (3.72), (3.73), the attractor X is entirely determined by a single parameter α, taking into account the choice of the class of unitaries U, i. e. the choice of $\pm$ in (3.72), (3.73).

One can multiply the matrices in (3.71) to see that the phase of α can be included in the parameter b, whether $b \neq 0$ or not, without affecting the attractor X, as can be seen e. g. from the fact that the attractor X itself scales by α. We could therefore choose $\alpha \in \mathbb{R}$, $\alpha \in (0, 1)$, removing a small redundancy in our description.

In order to determine whether the operators U of the form (3.71) truly lead to synchronisation we once again parameterize $X' \in X_{E_1 - E_2}$ as in (3.39). In the new basis $\tilde{X}'$ reads

$$\tilde{X}' = \begin{pmatrix} 0 & \gamma\alpha' - \alpha\gamma' & \bar{\alpha}\alpha' + \bar{\gamma}\gamma' & 0 \\ 0 & 0 & 0 & \bar{\gamma}\delta' - \bar{\alpha}\beta' \\ 0 & 0 & 0 & \alpha\delta' + \gamma\beta' \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.76}$$

The commutation relations (3.17), which now take the form $\left[\tilde{X}', \tilde{U}\right] = \left[\tilde{X}', \tilde{U}^\dagger\right] = 0$, give rise to the following set of equations

$$
\begin{aligned}
0 &= -s\bar{b}(\gamma\alpha' - \alpha\gamma'), && (3.77)\\
a(\gamma\alpha' - \alpha\gamma') &= \bar{a}(\gamma\alpha' - \alpha\gamma'), && (3.78)\\
\bar{s}b(\bar{\gamma}\delta' - \bar{\alpha}\beta') &= b(\bar{\alpha}\alpha' + \bar{\gamma}\gamma'), && (3.79)\\
s\bar{b}(\bar{\alpha}\alpha' + \bar{\gamma}\gamma') &= -\bar{b}(\bar{\gamma}\delta - \bar{\alpha}\beta'), && (3.80)\\
0 &= -\bar{b}(\alpha\delta' + \gamma\beta'), && (3.81)\\
a(\alpha\delta' + \gamma\beta') &= \bar{a}(\alpha\delta' + \gamma\beta'). && (3.82)
\end{aligned}
$$

Let us first assume the case $b \neq 0$. It follows from (3.77) that

$$\gamma' = \frac{\gamma}{\alpha}\alpha' \tag{3.83}$$

and from (3.81) that

$$\delta' = -\frac{\gamma}{\alpha}\beta'. \tag{3.84}$$

Inserting these results into (3.79) yields

$$\alpha' = -\bar{s}\beta'. \tag{3.85}$$

The fact that the synchronisation condition $\alpha' + \beta' = \gamma' + \delta'$ holds follows, using the inverse of relation (3.70). This shows that

$$b \neq 0 \tag{3.86}$$

is a sufficient condition for U to be synchronising.

On the other hand, consider the case $b = 0$. If also $a = \bar{a}$, U is a multiple of identity and as such does not enforce synchronisation. Assume therefore $a \neq \bar{a}$. The equations (3.78) and (3.82) are the only non-trivial remaining constraints and they retrieve the results (3.83) and (3.84) respectively. The commutation relations between U and X' as well as between $U^\dagger$ and X' are hereby satisfied. However, equation (3.85) is not enforced in this case and consequently the synchronisation condition does not hold. For example, $\alpha' = \beta' \neq 0$ results in an attractor X' commuting with U and not satisfying the synchronisation condition as

$$\alpha' + \beta' = 2\alpha' \neq 0 \tag{3.87}$$

does not equal

$$\gamma' + \delta' = \frac{\gamma}{\alpha}(\alpha' - \beta') = 0, \tag{3.88}$$

where we used equations (3.83) and (3.84). This proves that U of the form (3.71) is synchronising if and only if $b \neq 0$.

b) $r \neq 0$

We have already demonstrated that if both s and r are non-zero, the only commuting unitary operators U are multiples of identity. Therefore we assume $s = 0$ further on. This case is solved analogously to the previous one. Using the block-matrix form (3.61) one immediately arrives at $C = B^\dagger = 0$ and from the relations $SD = AS$ and $RD = DR$ gets

$$\alpha_{11} = d_{11} = d_{22}, \tag{3.89}$$

$$a_{12} = a_{21} = d_{12} = d_{21} = 0. \tag{3.90}$$

Hence, U simlifies into the form

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \bar{\alpha} & 0 \\ 0 & -\alpha & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\gamma} & -\bar{\alpha} & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.91}$$

where $|a| = 1$. For the parameter r it holds

$$r = \frac{\alpha - \gamma}{\bar{\alpha} - \bar{\gamma}} \quad \text{for} \quad \alpha \neq \gamma \tag{3.92}$$

and r can be arbitrary for $\alpha = \gamma$. Note $\alpha = \gamma$ implies $\beta = \delta$ due to (3.58). In the case $\alpha = \gamma$, r arbitrary, the parameter r does not affect the operator U nor the transition matrix T, it merely parameterizes attractors X which are given by the parameters α and r. We use the plural here as different values of parameter r correspond to different attractors X. In the case $\alpha \neq \gamma$, the attractor is fully determined by the parameters α, γ.

The operator U of the form (3.91), however, does not lead to a synchronising map for $\alpha \neq \gamma$, section 3.3 gives a counterexample, and it reduces to an already discovered form (3.38) for $\alpha = \gamma$.

c) $s = r = 0$

By the definition of s and r, see equation (3.58), this is equivalent to $e_4 \in \operatorname{Ker} X$ and $\beta = \delta = 0$. The case was already discussed.

3.2 Two-qubit total synchronisation

In this part we investigate continuous quantum Markovian dynamical semigroups with a generator of the form (3.1) which enforce asymptotical total synchronisation of two qubits for an arbitrary initial state. As this requirement is stronger than that of synchronisation, it is sufficient to inspect the unitaries U found in section 3.1 to pick out those that lead to total synchronisation. We proceed with assuming a general stationary attractor $X_{st} \in X_0$ given by

$$X_{st} = A\,|11\rangle\langle 11| + B\,|22\rangle\langle 22| + C\,|12\rangle\langle 12| + D\,|21\rangle\langle 21| + E\,|12\rangle\langle 21| + F\,|21\rangle\langle 12| \tag{3.93}$$

parameterized by six variables $A, B, C, D, E, F \in \mathbb{C}$. The condition of total synchronisation (2.6) reduces to

$$C = D. \tag{3.94}$$

We now require that for a given operator U the commutation relations (3.17) impose the condition of total synchronisation (3.94) on X_{st}. As we only consider operators synchronising the dynamical part of asymptotic evolution, the result will be a total synchronisation of the entire system. To discuss are the cases II.d), II.e) and IV.a) from section 3.1. We will stick to the notation used.

The first two can be solved in a similar fashion and therefore only the case II.d) is presented in detail. Using the same change of basis as introduced in the respective part of section 3.1, the attractor X_{st} in the new basis reads

$$\tilde{X}_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \frac{1}{2}(C+D+E+F) & \frac{1}{2}(-C+D+E-F) & 0 \\ 0 & \frac{1}{2}(-C+D-E+F) & \frac{1}{2}(C+D-E-F) & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \tag{3.95}$$

Given operator U of the form (3.35), the commutation relations (3.17) written explicitly in the new basis give rise to a set of equations

$$c(C-D-E+F) = a(C-D-E+F), \tag{3.96}$$
$$0 = b(C-D-E+F), \tag{3.97}$$
$$a(C-D+E-F) = c(C-D+E-F), \tag{3.98}$$
$$2bB = b(C+D-E-F), \tag{3.99}$$
$$\bar{b}(C-D+E-F) = 0, \tag{3.100}$$
$$\bar{b}(C+D-E-F) = 2\bar{b}B. \tag{3.101}$$

Since $a \neq c$ and $c \neq 0$, equations (3.96) and (3.98) imply

$$C-D-E+F = 0, \tag{3.102}$$
$$C-D+E-F = 0, \tag{3.103}$$

which combined together gives $C-D=0$, so that the condition (3.94) is always satisfied. We find out that the whole class of unitary operators U (3.35), originally designed to synchronise the two subsystems, actually enforces total synchronisation.
The same holds for the oprators U given by (3.38), found in II.e). The proof is analogous to the previous one and as such is not presented here.

For the case IVa) we consider U of the form (3.71). The stationary part of attractor X_{st} in the respective basis reads

$$\tilde{X}_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & |\gamma|^2 C + |\alpha|^2 D - \alpha\bar{\gamma}E - \bar{\alpha}\gamma F & \bar{\alpha}\bar{\gamma}C - \bar{\alpha}\bar{\gamma}D + \bar{\gamma}^2 E - \bar{\alpha}^2 F & 0 \\ 0 & \alpha\gamma C - \alpha\gamma D - \alpha^2 E + \gamma^2 F & |\alpha|^2 C + |\gamma|^2 D + \alpha\bar{\gamma}E + \bar{\alpha}\gamma F & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \tag{3.104}$$

and the commutation relations (3.17) yield the following set of equations

$$|\gamma|^2 C + |\alpha|^2 D - \alpha\bar{\gamma}E - \bar{\alpha}\gamma F = A, \tag{3.105}$$
$$\bar{\alpha}\bar{\gamma}C - \bar{\alpha}\bar{\gamma}D + \bar{\gamma}^2 E - \bar{\alpha}^2 F = 0, \tag{3.106}$$
$$\alpha\gamma C - \alpha\gamma D - \alpha^2 E + \gamma^2 F = 0, \tag{3.107}$$
$$|\alpha|^2 C + |\gamma|^2 D + \alpha\bar{\gamma}E + \bar{\alpha}\gamma F = B. \tag{3.108}$$

We can express the difference $C - D$ from (3.106) and (3.107) to obtain

$$C - D = \frac{\bar{\gamma}}{\bar{\alpha}}E - \frac{\bar{\alpha}}{\bar{\gamma}}F, \tag{3.109}$$
$$C - D = \frac{\gamma}{\alpha}F - \frac{\alpha}{\gamma}E, \tag{3.110}$$

equations which in general admit non-trivial solutions (two linear equations for four variables). Summed together after some manipulations using (3.69) and (3.72) yield

$$C - D = \frac{\alpha}{2\gamma}\,\frac{1 - 2|\alpha|^2}{|\alpha|^2}(E - F). \tag{3.111}$$

Consequently, for

$$|\alpha| = \frac{1}{\sqrt{2}} \implies \gamma = \pm i\alpha,\ s = \mp i \tag{3.112}$$

the condition (3.94) is satisfied and U leads to total synchronisation.

To show that this condition is not only sufficient, but also necessary, a simple counterexample can be given. Let $|\alpha| \neq \frac{1}{\sqrt{2}}$ which immediately implies $|\alpha| \neq |\gamma|$ due to imposed normalization, and simplify by choosing $E = -F$. The equations (3.106), (3.107) merge into

$$C - D = -\left(\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}\right)E, \tag{3.113}$$

which can be solved for $E \neq 0$ implying $C - D \neq 0$. Moreover, (3.105) and (3.108) can be seen as only introducing new variables A and B respectively, hence they trivially hold. A non-zero attractor X_{st} not satisfying the condition (3.94) exists. The operator U leads to total synchronisation if and only if $|\alpha| = \frac{1}{\sqrt{2}}$. With this condition we can write

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm i\alpha & \bar{\alpha} & 0 \\ 0 & -\alpha & \mp i\bar{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & \pm ib & 0 & 0 \\ \pm i\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mp i\bar{\alpha} & -\bar{\alpha} & 0 \\ 0 & \alpha & \pm i\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.114}$$

where $|\alpha| = \frac{1}{\sqrt{2}}$, $|a|^2 + |b|^2 = 1$ and $b \neq 0$.

This can be simplified as mentioned before, noticing that the phase of α plays no role in the form of the resulting attractor X and can be included in the parameter b when matrices in (3.114) are multiplied, meaning we can choose $\alpha \in \mathbb{R}$. Thus the result

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \mp i\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & \pm ib & 0 & 0 \\ \pm i\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mp i\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \pm i\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.115}$$

where $|a|^2 + |b|^2 = 1$ and $b \neq 0$.

3.3 Two-qubit antisynchronisation

The derivation in section 3.1 can easily be modified to achieve antisynchronisation in the sense of definition 2.1.3. We impose antisynchronisation condition (2.10), which in our parameterization reads

$$\alpha + \beta + \gamma + \delta = 0. \tag{3.116}$$

The calculation itself can be done in a similar fashion with following minor changes. In II.b) $\delta \to -\delta$, in II.c) $\gamma \to -\gamma$ and in IV.b) $r = \frac{\alpha-\gamma}{\bar{\alpha}-\bar{\gamma}} \to r = \frac{\alpha+\gamma}{-\bar{\alpha}-\bar{\gamma}}$ as a result of $\gamma \to -\gamma$, $\delta \to -\delta$, leave the rest of the calculation and the resulting form of the operator U unchanged, confirming the fact that those candidates for U are not maps that would lead to synchronisation, nor antisynchronisation.

In II.d) $\gamma \to -\gamma$, we can make a slight modification in the definition of new basis (reasoning is the same, we want to simplify $\tilde{X}$) by replacing $e_2 = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle)$, $e_3 = \frac{1}{\sqrt{2}}(|21\rangle - |12\rangle)$ with $e_2 = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle)$, $e_3 = \frac{1}{\sqrt{2}}(|21\rangle + |12\rangle)$, effectively changing $T \to T^\dagger$, to achieve the very same form of the attractor $\tilde{X}$, the rest follows. Same approach works for II.e) with the modification of the result again being merely $T \to T^\dagger$,

$$U_{as}^{II.d} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.117}$$

where $a, b, c \in \mathbb{C}$, $|c| = 1$, $|a|^2 + |b|^2 = 1$, $a \neq c$ and $a \neq \pm 1$. Analogously

$$U_{as}^{II.e} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.118}$$

where $a, b, c \in \mathbb{C}$, $|c| = 1$, $|a|^2 + |b|^2 = 1$, $\bar{a} \neq c$ and $a \neq \pm 1$.

For IV.a) the change becomes $s \to \bar{s}$,

$$U_{as}^{IV.a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & \bar{\alpha} & 0 \\ 0 & -\alpha & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & sb & 0 & 0 \\ -\bar{s}\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\gamma} & -\bar{\alpha} & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.119}$$

where $|a|^2 + |b|^2 = 1$, $b \neq 0$, $0 < |\alpha|^2 < 1$, equations (3.72) and (3.73) hold and we can again choose $\alpha \in \mathbb{R}$ to remove redundancy.

All the other remaining cases are analogous as well, leading only to trivial results.

3.4 Two-qubit total antisynchronisation

Using the results of sections 3.3 and section 3.2, we arrive at the generators of totally antisynchronising maps. Starting from antisynchronising maps obtained in section 3.3, we are once again only concerned with the condition of total antisynchronisation (2.11) applied to the stationary part. This constraint happens to coincide with the condition of total synchronisation (2.6) applied to the stationary part, which is expressed by the equation (3.94) in the parameterization (3.93). Consequently, we can exploit the close similarity between the forms of synchronising and antisynchronising maps and follow the derivation of section 3.2. The reader is welcomed to easily verify that the calculations remain unaffected up to an occasional change of sign if we exchange synchronising for antisynchronising maps.
Hence the antisychronising generating unitaries U given by (3.117) and (3.118) also enforce total antisynchronisation, and the unitary U given by (3.119) generates a totally antisynchronising map if and only if $|\alpha| = \frac{1}{\sqrt{2}}$, implying $\gamma = \pm i\alpha$, $s = \mp i$, or explicitly

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \mp i\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & \mp ib & 0 & 0 \\ \mp i\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mp i\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \pm i\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.120}$$

where $|a|^2 + |b|^2 = 1$ and $b \neq 0$.

3.5 Attractor spaces of synchronising maps on two qubits

Previously, we established three distinct classes of unitary Lindblad operators U that induced total synchronisation of two qubits. We did, however, only partially discussed the corresponding attractors, whereof the role is essential in determining asymptotic dynamics of the system. Having a basis of the attractor space we can write the asymptotic evolution of an arbitrary initial state via (1.11). Revealing the entire structure of the attractor spaces of the respective maps is the subject of this section.

We will go through all the synchronising maps we found and describe all their attractors, consecutively considering elements of the subspaces $X_{E_1-E_2}, X_{E_2-E_1}, X_0, X_{2E_1-2E_2}$ and $X_{2E_2-2E_1}$ of the respective attractor spaces. Reminding that attractors corresponding to different eigenvalues are mutually orthogonal, see (1.18), these subspaces are to be dealt with separately.

Starting with the unitaries U of the form (3.38), earlier referred to as case II.e), we already have some partial results. Let us first assume $a \neq c$, the case $a = c$ will later be discussed seperately. It was shown that the first dynamical part of the attractor $X_{d1} \in X_{E_1-E_2}$ has the form

$$X_{d1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.121}$$

for $\beta \in \mathbb{C}$. Consequently, the second dynamical part of the attractor $X_{d2} \in X_{E_2-E_1}$, related to the previous one by the operation of complex conjugation, has the form

$$X_{d2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \eta & \eta & 0 \end{pmatrix} \tag{3.122}$$

for $\eta \in \mathbb{C}$. The parameters β and η are unrelated, the complex conjugation is merely a connection between the two subspaces $X_{E_1-E_2}$ and $X_{E_2-E_1}$.
The attractors corresponding to the stationary part of the asymptotic state were already discussed for a similar generating operator U given by (3.35), referred to as case II.d). Analogously, having parameterized $X_{st} \in X_0$ again via (3.93), we arrive at the set of constraints

$$2bA = b(C + D - E - F), \tag{3.123}$$
$$-2\bar{b}A = -\bar{b}(C + D - E - F), \tag{3.124}$$
$$0 = b(C - D + E - F), \tag{3.125}$$
$$0 = -\bar{b}(C - D - E + F), \tag{3.126}$$
$$\bar{a}(C - D + E - F) = c(C - D + E - F), \tag{3.127}$$
$$\bar{a}(C - D - E + F) = c(C - D - E + F). \tag{3.128}$$

Combining (3.127) and (3.128) we arrive at

$$C = D, \tag{3.129}$$
$$E = F, \tag{3.130}$$

making equations (3.125) and (3.126) trivial and equations (3.123) and (3.124) for $b \neq 0$ (consequently $a \neq c$) imply

$$A = C - E. \tag{3.131}$$

Hence any stationary element of the attractor space has the form

$$X_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & E & 0 \\ 0 & E & C & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \tag{3.132}$$

for $A, B, C, E \in \mathbb{C}$, and in case $b \neq 0$, $A = C - E$ is determined, otherwise A is a free parameter.

Lastly, assume possible attractors $X_{c1} \in X_{2E_1-2E_2}$ and $X_{c2} \in X_{2E_2-2E_1}$, associated purely with system correlations, with parameterization of the former given by

$$X_{c1} = \sigma \left|11\right\rangle \left\langle 22\right| \tag{3.133}$$

for $\sigma \in \mathbb{C}$. The commutation relations give rise to constraints

$$a\sigma = c\sigma, \tag{3.134}$$

$$-\bar{b}\sigma = 0, \tag{3.135}$$

which result in

$$\sigma = 0 \tag{3.136}$$

for every but the excluded case $a = c$ (which implies $b = 0$). It follows that both X_{c1} and X_{c2} are trivial.

Put together, an element X of the attrator space of the synchronising map generated by U of the form (3.38) has a general form

$$X = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & E & \beta \\ 0 & E & C & \beta \\ 0 & \eta & \eta & B \end{pmatrix}, \tag{3.137}$$

where $\beta, \eta, A, B, C, E \in \mathbb{C}$ and in case $b \neq 0$ for the parameter of U in (3.38) it holds $A = C - E$.

Deliberately, we postpone the discussion of the special case $a = c$ and first state the results for the case of U given by (3.35), previously reffered to as II.d), which can be obtained by the same reasoning as above. Keeping the notation and sticking to the computational basis, assuming $\bar{a} \neq c$, the individual parts of the attractor space read

$$X_{d1} = \begin{pmatrix} 0 & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.138}$$

$$X_{d2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.139}$$

$$X_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & E & 0 \\ 0 & E & C & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \tag{3.140}$$

$$X_{c1} = X_{c2} = 0, \tag{3.141}$$

where $\alpha, \zeta, A, B, C, E \in \mathbb{C}$ and $B = C - E$ if $b \neq 0$ in the parameterization of U (3.35).

Together, an element X of the attractor space of the totally sychronising map generated by U of the form (3.35) has a general form

$$X = \begin{pmatrix} A & \alpha & \alpha & 0 \\ \zeta & C & E & 0 \\ \zeta & E & C & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \tag{3.142}$$

with parameters defined above.

To address the aforementioned special case, the situation $a = c$ in (3.38) is of particular interest. We already noted that U of the form (3.38) with the choice of $a = c$ coincides with U given by (3.91) with the choice of $\alpha = \gamma$, the case excluded from otherwise not synchronising maps (3.91). In addition it coincides with U of the form (3.35) with the choice of $\bar{a} = c$. This can be seen from multiplying the matrices in the expressions mentioned. Such an operator U can be parameterized by a single parameter $c \in \mathbb{C}, |c| = 1, c \neq \pm 1$, as

$$U = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & \mathrm{Re}\, c & i\mathrm{Im}\, c & 0 \\ 0 & i\mathrm{Im}\, c & \mathrm{Re}\, c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \tag{3.143}$$

and marks the overlap of the otherwise distinct classes of totally synchronising maps (3.35) and (3.38). The discussion around (3.47), (3.44) and (3.43) in section 3.1 already revealed yet not pointed out the noticeably more complex structure of the attractor space of the corresponding generated map, needed for consistency and going hand in hand with the overlap. From the equations (3.41), (3.44), (3.127), (3.128) together with the vanishing of (3.134), (3.135) and all the remaining constraints it follows

$$X_{d1} = \begin{pmatrix} 0 & \alpha & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.144}$$

$$X_{d2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & \eta & \eta & 0 \end{pmatrix}, \tag{3.145}$$

$$X_{st} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & E & 0 \\ 0 & E & C & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \tag{3.146}$$

$$X_{c1} = \begin{pmatrix} 0 & 0 & 0 & \sigma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.147}$$

$$X_{c2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0 \end{pmatrix}, \tag{3.148}$$

where $\alpha, \beta, \zeta, \eta, A, B, C, E, \sigma, \tau \in \mathbb{C}$.

Combined, an element X of the attractor space of the totally synchronising map generated by U given by (3.143) has a general form

$$X = \begin{pmatrix} A & \alpha & \alpha & \sigma \\ \zeta & C & E & \beta \\ \zeta & E & C & \beta \\ \tau & \eta & \eta & B \end{pmatrix}, \tag{3.149}$$

parameters defined above.

Once we finish the discussion below regarding the attractor space of the remaining class of synchronising maps it will have been demonstrated that the operators U of the form (3.143) are associated with the highest possible and exclusively achieved dimension of the attractor space among all synchronising maps, as well as that they are the only ones preserving the subspaces $X_{2E_1-2E_2}$ and $X_{2E_2-2E_1}$. In a sense they can preserve the greatest piece of information about the initial state. Indeed, in terms of attractors the resulting asymptotic state is given by (1.11). Furthermore, to each linearly independent attractor one can associate one linearly independent constant of motion, whereof expectation value remains unchanged along all trajectories.

It is worth mentioning that for $c = \pm i$ the operator U in (3.143) reduces to

$$U = \pm i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.150}$$

which up to an irrelevant phase prefactor is the well-known and studied SWAP operator [6], acting simply as the exchange of states on two qubits, i. e. $|i\rangle \otimes |j\rangle \xleftrightarrow{SWAP} |j\rangle \otimes |i\rangle$. In fact, to show the connection to the SWAP operator we can rewrite (3.143) as

$$U = cos(\phi)I + isin(\phi)SWAP, \tag{3.151}$$

where we parameterized $c = cos(\phi) + isin(\phi)$ for $\phi \in \mathbb{R}$. This operator U is sometimes referred to as a partial swap operator in the literature [18].
From the point of view of asymptotic dynamics, (3.143) constitues a certain generalisation of the SWAP operator as it represents the maximal set of unitary Lindblad operators that result in the same asymptotic behaviour as the SWAP operator in the studied type of evolution.

Last but not least we examinate in detail the attractor spaces of synchronising maps generated by operators U of the form (3.71). The dynamical part was already found and is given by (3.72), (3.73), (3.74). The stationary part is, in parameterisation (3.93), constrained by equations (3.105) to (3.108). We distinguish between two cases.

Firstly, for totally synchronising maps it holds $|\alpha| = \frac{1}{\sqrt{2}}$ which implies $\gamma = \pm i\alpha$, $s = \mp i$. This simplifies (3.74) into

$$X_{d1} = \begin{pmatrix} 0 & \alpha & \pm i\alpha & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \pm i\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.152}$$

and the equations (3.105) to (3.108) into

$$C + D \pm iE \mp iF = 2A, \tag{3.153}$$
$$C - D \mp iE \mp iF = 0, \tag{3.154}$$
$$C - D \pm iE \pm iF = 0, \tag{3.155}$$
$$C + D \mp iE \pm iF = 2B, \tag{3.156}$$

implying after summation

$$C = D, \tag{3.157}$$
$$E = -F, \tag{3.158}$$
$$A = C \pm iE, \tag{3.159}$$
$$B = C \mp iE. \tag{3.160}$$

For $X_{c1} \in X_{2E_1-2E_2}$ in parameterisation (3.133) the commutation relations give rise to

$$a\sigma = \bar{a}\sigma, \tag{3.161}$$
$$-s\bar{b}\sigma = 0, \tag{3.162}$$
$$-\bar{b}\sigma = 0, \tag{3.163}$$

resulting in simple

$$\sigma = 0, \tag{3.164}$$

since $b \neq 0$ was required for U to posses synchronising property. Put all together, an element of the attractor space associated with the totally synchronising map generated by U of the form (3.115) can be parameterised as

$$X = \begin{pmatrix} C \pm iE & \alpha & \pm i\alpha & 0 \\ \zeta & C & E & \alpha \\ \mp i\zeta & -E & C & \pm i\alpha \\ 0 & \zeta & \mp i\zeta & C \mp iE \end{pmatrix}, \tag{3.165}$$

where $\alpha, \zeta, C, E \in \mathbb{C}$. We remind that $\pm$ and $\mp$ in the expression above correlate with the two distinct classes of operators U in (3.115).

Secondly, we discuss the synchronising but not totally synchronising maps generated by U of the form (3.71), where $|\alpha| \neq \frac{1}{\sqrt{2}}$. As mentioned before, the dynamical part X_{d1} of an attractor X is up to rescaling given by (3.74), (3.72), (3.73), i. e.

$$X_{d1} = \begin{pmatrix} 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & s\gamma \\ 0 & 0 & 0 & -s\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.166}$$

where α, $0 < |\alpha| < 1$, is a parameter of the operator U in (3.71) and γ, s are determined by α by the expressions (3.72), (3.73). Introducing a new parameter $\beta \in \mathbb{C}$ to account for the possible scaling and to give a full parameterization we can write

$$X_{d1} = \begin{pmatrix} 0 & \beta & \pm i\beta\frac{\sqrt{1-|\alpha|^2}}{|\alpha|} & 0 \\ 0 & 0 & 0 & \pm i\beta\left(2|\alpha|^2 - 1\right)\frac{\sqrt{1-|\alpha|^2}}{|\alpha|} + 2\beta\left(1 - |\alpha|^2\right) \\ 0 & 0 & 0 & \beta\left(1 - 2|\alpha|^2\right) \pm 2i\beta|\alpha|\sqrt{1 - |\alpha|^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.167}$$

For the stationary part X_{st}, the constraints (3.105) to (3.108) can be rewritten in a simpler form, denoting

$$\omega = \frac{\sqrt{1 - |\alpha|^2}}{|\alpha|}. \tag{3.168}$$

This results in

$$\omega^2 C + D \pm i\omega(E - F) = (1 + \omega^2)A, \tag{3.169}$$

$$\mp i\omega(C - D) - \omega^2 E - F = 0, \tag{3.170}$$

$$\pm i\omega(C - D) - E - \omega^2 F = 0, \tag{3.171}$$

$$C + \omega^2 D \mp i\omega(E - F) = (1 + \omega^2)B. \tag{3.172}$$

The equations (3.170) and (3.171) can be summed together to yield

$$E = -F, \tag{3.173}$$

simplifying themselves into

$$E = \frac{\pm i\omega}{1 - \omega^2}(C - D), \tag{3.174}$$

or equivalently

$$E = \mp i|\alpha|\sqrt{1 - |\alpha|^2}(C - D). \tag{3.175}$$

We then insert these results into (3.169) and (3.172) to obtain

$$A = \left(1 - |\alpha|^2 \pm 2|\alpha|^2 \mp 2|\alpha|^4\right) C + \left(|\alpha|^2 \mp 2|\alpha|^2 \pm 2|\alpha|^4\right) D, \tag{3.176}$$

$$B = \left(|\alpha|^2 \mp 2|\alpha|^2 \pm 2|\alpha|^4\right) C + \left(1 - |\alpha|^2 \pm 2|\alpha|^2 \mp 2|\alpha|^4\right) D. \tag{3.177}$$

The remaining parts X_{c1}, X_{c2} are again trivially zero.

Hence, an element X of the attractor space associated with the synchronising map generated by U of the form (3.71) has a general form

$$X = \begin{pmatrix} \begin{matrix}\left(1-|\alpha|^2 \pm 2|\alpha|^2 \mp 2|\alpha|^4\right)C \\ +\left(|\alpha|^2 \mp 2|\alpha|^2 \pm 2|\alpha|^4\right)D\end{matrix} & \beta & \pm i\beta\frac{\sqrt{1-|\alpha|^2}}{|\alpha|} & 0 \\ \zeta & C & \mp i|\alpha|\sqrt{1-|\alpha|^2}(C-D) & \begin{matrix}\pm i\beta\left(2|\alpha|^2-1\right)\frac{\sqrt{1-|\alpha|^2}}{|\alpha|} \\ +2\beta\left(1-|\alpha|^2\right)\end{matrix} \\ \mp i\zeta\frac{\sqrt{1-|\alpha|^2}}{|\alpha|} & \pm i|\alpha|\sqrt{1-|\alpha|^2}(C-D) & D & \begin{matrix}\beta\left(1-2|\alpha|^2\right)\pm \\ 2i\beta|\alpha|\sqrt{1-|\alpha|^2}\end{matrix} \\ 0 & \begin{matrix}\mp i\zeta\left(2|\alpha|^2-1\right)\frac{\sqrt{1-|\alpha|^2}}{|\alpha|} \\ +2\zeta\left(1-|\alpha|^2\right)\end{matrix} & \begin{matrix}\zeta\left(1-2|\alpha|^2\right)\mp \\ 2i\zeta|\alpha|\sqrt{1-|\alpha|^2}\end{matrix} & \begin{matrix}\left(|\alpha|^2 \mp 2|\alpha|^2 \pm 2|\alpha|^4\right)C+ \\ \left(1-|\alpha|^2 \pm 2|\alpha|^2 \mp 2|\alpha|^4\right)D\end{matrix} \end{pmatrix}, \tag{3.178}$$

where $\beta, \zeta, C, D \in \mathbb{C}$ and α, $0 < |\alpha| < 1$, $|\alpha| \neq \frac{1}{\sqrt{2}}$, is a parameter of the operator U.

To sum up, we observe that the attractor space of a synchronising or totally synchronising map generated by unitary Lindblad operators U given by (3.35), (3.38) or (3.71) is in general four- to ten-dimensional, subject to several additional conditions, with these conditions and a precise structure of the attractor space explicitly presented above.

Note: Due to the close similarity of the form of synchronising and antisynchronising maps, one could follow the same scheme of calculations and with only minor changes obtain attractor spaces of antisynchronising and totally antisynchronising maps as well.

Chapter 4

Properties of synchronising maps on two qubits

In chapter 3 we found and described the generators of all possible evolution maps that enforce (anti-)synchronisation or total (anti-)synchronisation of two qubits in the studied dynamics. We further found their respective attractor spaces and provided their full parameterization. This chapter is devoted to the study of their other properties.

4.1 Visibility

Synchronisation is mediated by the interaction of subsystems with their common environment. Such an irreversible process is typically accompanied by decoherence and dephasing, i. e. with information leak into the environment. Thus in this part we address the following question. Once an initial state is synchronised by one of the evolution maps described in the previous chapter, how visible and detectable the resulting time evolution of the individual qubit states is? To what extend does the internal dynamics of qubits survive the process of synchronisation?

For a global two-qubit state $\rho(t) \equiv (\rho_{ij})(t) \in \mathcal{B}(\mathscr{H}_1^{\otimes 2})$ the reduced states $\rho_A(t), \rho_B(t) \in \mathcal{B}(\mathscr{H}_1)$ read

$$\rho_A = \begin{pmatrix} \rho_{11}(t) + \rho_{22}(t) & \rho_{13}(t) + \rho_{24}(t) \\ \rho_{31}(t) + \rho_{42}(t) & \rho_{33}(t) + \rho_{44}(t) \end{pmatrix}, \tag{4.1}$$

$$\rho_B = \begin{pmatrix} \rho_{11}(t) + \rho_{33}(t) & \rho_{12}(t) + \rho_{34}(t) \\ \rho_{21}(t) + \rho_{43}(t) & \rho_{22}(t) + \rho_{44}(t) \end{pmatrix}. \tag{4.2}$$

A general density matrix $\rho(t) \in \mathcal{B}(\mathscr{H}_1)$ describing a state of a qubit has the form

$$\rho(t) = \begin{pmatrix} x & ye^{iEt} \\ \bar{y}e^{-iEt} & 1-x \end{pmatrix}, \tag{4.3}$$

where $y \in \mathbb{C}$, $x, E \in \mathbb{R}$, and from the positivity of $\rho(t)$ it holds $|y| \leq \sqrt{x - x^2}$. In our case $E = E_1 - E_2$ as given by the Hamiltonian (3.2). In the asymptotics, the evolution is driven by (1.11) and hence the coefficents x, y are determined by the projection of the initial state on the generators of the attractor space.

Remark: The asymptotic state and evolution are strongly dependent on initial conditions. Seemingly, for a given synchronising map we could choose such an initial state that when projected onto the attractor space, the part responsible for non-trivial asymptotic time evolution vanishes. In such case the synchronisation mechanism kills the internal dynamics in spite of the presence of an attractor associated with a non-zero eigenvalue in the attractor space. This in inevitable since quantum evolution always has a fixed point. However, this would require a very specific choice of the initial state as it would have to lie in the orthogonal complement of the mentioned attractor in the space of all operators on the system Hilbert space. This orthogonal complement is a set of codimension at least one (there might exist more independent such attractors) and will consequently constitute a set of measure zero in the space of all states. We can conclude that a synchronising map enforces synchronisation with non-trivial asymptotic evolution for almost every initial condition.

The question remains how perceptible this asymptotic evolution will be. To extract the information about the time evolution we can calculate the expectation value $\langle \sigma_1 \rangle (t) = \mathrm{Tr}\left\{\rho^\dagger(t)\sigma_1\right\}$ of the observable

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.4}$$

proportional to the spin operator in the first axis for a spin-$\frac{1}{2}$ particle. After manipulation

$$\langle \sigma_1 \rangle (t) = |y| \cos(Et + \phi), \tag{4.5}$$

where $\phi \in \mathbb{R}$ accounts for the phase of y. Equivalently, we could express the probabilities $p_1 = \mathrm{Tr}\{\rho(t)M_1\}$, $p_2 = \mathrm{Tr}\{\rho(t)M_2\}$ of the corresponding projective measurements $M_1 = \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|)$ and $M_2 = \frac{1}{2}(|1\rangle - |2\rangle)(|1\rangle - |2\rangle)$. It holds

$$p_1(t) = \frac{1}{2}\left\{1 + 2|y|\cos(Et + \phi)\right\}, \tag{4.6}$$

$$p_1(t) = \frac{1}{2}\left\{1 - 2|y|\cos(Et + \phi)\right\}, \tag{4.7}$$

$$(p_1 - p_2)(t) = 2|y|\cos(Et + \phi). \tag{4.8}$$

Therefore, the visibility of the time evolution in the asymptotics is scaled with $|y|$, the absolute value of the off-diagonal parameter y. The greater the $|y|$, the bigger the amplitude in (4.5) and the easier it is to distinguish $\rho(t)$ from a stationary state.

We will now compare the attractor spaces of synchronising maps described in section 3.5, as they limit the possible asymptotic states, to see if the choice of the evolution map in general affects how well the internal dynamics is preserved. Our aim is to study visibility independently of the initial conditions, despite the fact that they play a crucial role in determining the asymptotic state and hence a very few conclusions may be drawn without specifying them. It will be shown that in some cases the evolution map may actually limit or suppress the visibility of the time evolution of the resulting synchronised asymptotic states, irrespective of the initial conditions.

If we keep the parameterizations (3.14), (3.93) for the dynamical and the stationary part of an element X of the attractor space and assume X is a state, then its reduced states are of the form (4.3) with

$$x = A + C \quad \text{or} \quad x = A + D, \tag{4.9}$$

$$y = \alpha + \beta, \tag{4.10}$$

and additional conditions for A, C, D, α, β may apply. Note that the other parameters appearing in the parameterization of attractor spaces in section 3.5 are given by the synchronisation condition and requirement on X to be a state.

Regardless of the particular synchronising map, the constraints on the stationary part yield effectively no limitations on the value of parameter x. It varies dependently on the initial conditions. For all totally synchronising maps, described in section 3.2, the parameters α and β determining y are either independent, as is the case for generating operators U of the form (3.35), (3.38), or linearly dependent for (3.115) where $\beta = \pm i\alpha$. They are simply coefficients specified by the initial condition and its projection onto the attractor space. As such the evolution map does not in general put any limitations on the value of the visibility parameter y.

We conclude that all totally synchronising maps show the same visibility of the asymptotic evolution.

On the other hand, for the synchronising maps generated by U given by (3.71), the form of the attractor (3.167), (3.178) results in the value of y depending not only on the initial conditions, but also on a parameter of U itself. It holds

$$y \propto 2|\alpha|\sqrt{1 - |\alpha|^2}, \tag{4.11}$$

where α is a parameter of U in (3.71). Remember α may be chosen real. At the same time, varying $|\alpha|$ does not change the norm of the attractor (3.167) onto which the initial state is projected. We can plot the factor above as a function of $|\alpha|$.

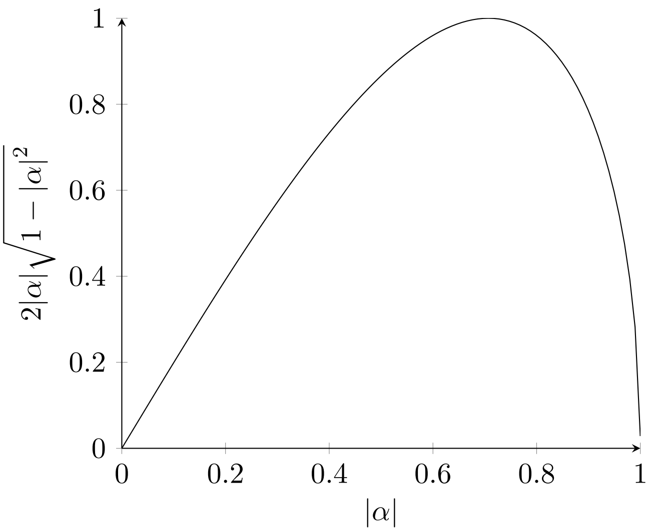

Figure 4.1: Plot of the proportional factor $2|\alpha|\sqrt{1-|\alpha|^2}$ of the off-diagonal parameter y in the parameterization (4.3) of the asymptotic reduced states for synchronising maps generated by U given by (3.71), characterizing the asymptotic states time evolution visibility.

We can see that in case of Lindblad operators U with $|\alpha|$ close to 0 or 1 the asymptotic dynamics will be strongly suppressed. The asymptotic reduced states will resemble stationary ones, irrespective of the initial conditions. On the contrary, the plotted function reaches its maximum for $|\alpha| = \frac{1}{\sqrt{2}}$ when the map generated by U becomes totally synchronising. From this point of view it appears that the total synchronisation results in, perhaps counterintuitively, better visibility than the less restrictive synchronisation.

Expectedly, analogous results can be obtained for antisychronising and totally antisynchronising maps from sections 3.3 and 3.4. Again, in the case of the totally antisynchronising maps visibility is determined solely by the initial conditions. Antisynchronising maps with U given by (3.119) behave exactly as their synchronising counterparts (3.71), with the visibility of the asymptotic time evolution being limited by the proportional factor (4.11) depending on the parameter $|\alpha|$ of U in (3.119), as visualised in the figure 4.1.

4.2 Global symmetry of synchronised states

The total synchronisation mechanism makes both qubits locally indistinguishable. But does it mean that the two become indistinguishable from a global point of view as well? It turns out to depend on the particular synchronising mechanism.

Indistinguishability of a bipartite quantum state means that at any time of its evolution it is described by a permutationally state. Consequently, no measurement can discern the order of the subsystems. By permutation invariance we mean that the global state is invariant with respect to the exchange of the two qubits.

For a map to enforce asymptotic permutation invariance for arbitrary initial conditions, any state lying in its attractor space need be permutation invariant. Formally an attractor $X \in \mathcal{B}(\mathscr{H}_1^{\otimes 2})$ is permutation invariant if it is invariant with respect to conjugation by the SWAP operator, denoted here as $\Pi = \Pi^\dagger = \Pi^{-1} = |11\rangle\langle 11| + |12\rangle\langle 21| + |21\rangle\langle 12| + |22\rangle\langle 22|$,

$$X = \Pi\, X\, \Pi^{-1}. \tag{4.12}$$

Denoting $(X_{ij}) \equiv X$, this condition simplifies into

$$X_{12} = X_{21}, \tag{4.13}$$
$$X_{34} = X_{24}, \tag{4.14}$$
$$X_{22} = X_{33}, \tag{4.15}$$
$$X_{23} = X_{32}, \tag{4.16}$$
$$\tag{4.17}$$

or equivalently, in the parameterization (3.14), (3.93) previously used, into

$$\alpha = \gamma, \tag{4.18}$$
$$\beta = \delta, \tag{4.19}$$
$$C = D, \tag{4.20}$$
$$E = F. \tag{4.21}$$

We do not explicitly state the condition of permutation invariance (4.12) for the dynamical part of the attractor X lying in the subspace $X_{E_2-E_1}$, related to the subspace $X_{E_1-E_2}$ by complex conjugation, as it is satisfied if and only if the conditions (4.13), (4.14), or equivalently (4.18), (4.19), are.
Having fully described the attractor spaces of all synchronising maps in section 3.5, we can directly conjugate a general element X of each of them by Π to see whether (4.12) holds.

Firstly, the equations (4.18), (4.19) together give the synchronisation condition (3.16) which contradicts the antisynchronisation one (3.116), unless the reduced states of the attractor X in question are stationary. Consequently, no antisynchronising map is compatible with permutation invariance of the asymptotic state, as is to be expected.

Moving to the synchronising maps, consider a map generated by U given (3.71), including its special case of a totally synchronising map (3.115). An element X of its attractor space has the form (3.178) or (3.165) respectively, that violates (4.13), (4.14) and (4.16) unless all coefficients involved are zero, which is the case of X stationary with an additional restriction. That would also require very specific initial condition. In general, any non-stationary asymptotic state of an evolution map generated by U of the form (3.71), including (3.115), is guaranteed not to be permutation invariant.

On the other hand, in case of a totally synchronising map generated by U of the form (3.35), (3.38), including the overlap of this two classes represented by the partial swap operator (3.143), an arbitrary element X of the corresponding attractor space has the form (3.142), (3.137) or (3.149) respectively, which always satisfies (4.12). For arbitrary initial conditions the resulting asymptotic state is permutation invariant.

We can sumarize these observations. A non-stationary asymptotic two-qubit state is permutation invariant if and only if the evolution map is generated by U of the form (3.35), (3.38), including the partial swap operator (3.143). The reduced one-qubit states of a global non-stationary permutation invariant states are totally synchronised, however, the global state of totally synchronised one-qubit reduced states does not need to be permutation invariant.

4.3 Algebraic structure of synchronising mechanisms

Each of the classes of the unitary Lindblad operators U generating (anti-)synchronising maps described in chapter 3 resemble a group structure. In a sense the operators U form robust structures preserving the asymptotic dynamics under perturbations and operator composition within the respective class. Furthermore, nearly a group structure offers a possibility to study corresponding generators (algebras), which might provide an additional insight into the mechanism of synchronisation.

Firstly, take a look at U given by (3.35). In this case U has a block diagonal structure (3.34) with a phase multiplied identity $c\, I_{2\times 2}$ and a special unitary matrix $A \in SU(2)$, conjugated by a special unitary matrix $T \in SU(4)$ which accounts for a change of basis between the computational basis and another orthonormal basis given by T. Furthermore, all the operators U correspond to the same attractor space (with the only exception being an overlap (3.143) with the class of operators U given (3.38), a set of measure zero withing U given by (3.35), in which case the resulting attractor space is a union of the two attractor spaces). As such, they generate the same asymptotic dynamics, irrespective of the particular choice of U within the class. If we reintroduce the identity operator, then the unitaries U given by (3.35) form a group structure with respect to map composition (matrix multiplication), up to a set of measure zero given by the additional condition $c \neq a$ in the parameterization. The condition is necessary for U to enforce synchronisation, not however for U to commute with the elements of the attractor space. For a given phase c multiplying the identity operator I in the block form, it excludes only a single element of the $SU(2)$ group. This is roughly a structure of $SU(2) \times U(1)$, or $U(2)$. Note the limiting condition is not necessarily preserved under composition.
Nonetheless, it holds that for a given U of the form (3.35), $U^\dagger$ is also of the form (3.35), and for almost every two U_1,U_2 given by (3.35), $U_1\, U_2$ is also of the form (3.35). As such, U, $U^\dagger$ and U_1U_2, for almost all U_1,U_2 of the same class, lead to the very same asymptotic dynamics.

The same can be said about the unitaries U of the form (3.38). The reasoning is, just as the form, analogous to the previous case. Consequently, it holds for the overlap of the two classes, resulting in the partial SWAP operator (3.143), that such unitaries U roughly form a structure of $U(1)$.

Secondly, consider U given by (3.71). The parameter α, which can be chosen positive real and in such case satisfies $\alpha \in (0, 1) \simeq \mathbb{R}$, determines the special unitary matrix T accounting for a basis change, the parameter s in the parameterization of U and the form of the attractor

X corresponding to the part of the attractor space responsible for non-stationary asymptotic evolution of the reduced states. Recall that there are two distinct classes of operators U, generating maps with two distinct attractor spaces, associated with a single value of α, distinguished only by the $\pm$ sign in (3.71) and corresponding parameterizations of elements of the attractor spaces. With α fixed, there are two classes of Lindblad operators U of the form (3.71), each corresponding to their respective attroctor space. They both have a block diagonal form with two copies of a 2×2 special unitary matrix A, with the off-diagonal elements multiplied by the parameter s in one instance and a condition $b \neq 0$ required to enforce synchronisation. This restriction excludes diagonal matrices and is again not preserved under composition. Still approximately a structure of $SU(2)$.
Just as in the previous case it holds that for U of the form (3.71) with α fixed, $U^\dagger$ is also of the form (3.71) with the same α, and for almost all U_1,U_2 given by (3.71) with α fixed, U_1U_2 is also of the form (3.71) with the same α. U, $U^\dagger$ and U_1U_2, for almost all U_1,U_2 of the same class, result in the very same asymptotic dynamics.
The coefficient $\alpha \in (0, 1)$ parameterizes a family of classes of unitary Lindblad operators with the described properties.

Analogous results hold for antisynchronising maps.

4.4 Synchronisation with more than one Lindblad operator

Up to now we have studied synchronisation mechanisms generated by an evolution generator $\mathcal{L}$ with a single unitary Lindblad operator U with weight 1, that is (3.1). However, the obtained solutions allow us to address a more general problem, namely, which Lindbladians of the type (1.17) with an arbitrary number of unitary Lindblad operators lead to evolution (anti-)synchronising qubits.
It follows from the governing equation (1.11) and the theorem 1.5.1 that the asymptotic evolution is determined by the projection of the initial state on the attractor space and that the attractor space is in turn determined by the commutation relations with the Lindblad operators (1.20). Hence for several Lindblad operators in the generator an attractor of the evolution needs to commute with all of them. In other words, the resulting attractor space is given by the intersection of the attractor spaces of the maps with generators (3.1) with individual operators U apearing in (1.17).
We described several classes of the operators U leading to (anti-)synchronisation and discussed their structure, discovering that the attractor space and consequently the asymptotic dynamics is preserved within each of the classes. As a result, we can arbitrarily combine any number of the operators U within the same class without affecting the asymptotic evolution, as they satisfy the same commutation relations and result in the same attractor space. We can even include any number of unitaries U of the same type which do not satisfy the respective conditions needed to enforce (anti-)synchronisation, as they still commute with all elements of the attractor space. From the construction, where we began with the attractor and found all commuting unitaries to subsequently pick out the ones that enforce (anti-)synchronisation, those are all unitary Lindblad operators we can include in the generator without making changes to the asymptotic dynamics. Weights assigned to the individual operators do not affect the resulting asymptotic evolution either. They determine merely the first stages of evolution and convergence rate towards the asymptotic regime.

If, on the other hand, operators U from two or more distinct classes are combined in the generator, the evolution will always result in a stationary state, irrespective of the initial conditions. This is due to the fact that in the subspace (in intersection with) $X_{E_1-E_2} \subset \mathcal{B}(\mathscr{H})$, responsible for the non-trivial asymptotic evolution of the reduced states, the intersection of the attractor spaces associated with unitaries U from different classes is trivial.

4.5 Entanglement generation and destruction

Synchronisation is a form of correlation between the involved parties. It is natural to ask whether during the evolution towards a synchronised state another form of correlation, such as entanglement, arises. The connection between synchronisation and entanglement formation has been tackled in the literature without much success, so far it remains an open question. This section briefly addresses the relation between two-qubit (anti-)synchronisation mechanisms and entanglement of the asymptotic states.

For quantification of the entanglement of two-qubit states we used concurrence, an explicitly calculable entanglement monotone, monotonously related to the entanglement of formation [27], [28]. The two concepts are defined as follows.

Given a pure state $\rho = |\psi\rangle\langle\psi|$, the entropy of entanglement E is defined as the entropy of either subsystem

$$E(\rho) = -\operatorname{Tr}(\rho_A \log \rho_A) = -\operatorname{Tr}(\rho_B \log \rho_B). \tag{4.22}$$

For a mixed state ρ the entanglement of formation is defined to be the average entropy of entanglement of the pure states in a pure state decomposition, minimized over all possible decompositions

$$E(\rho) = \inf \sum_i p_i E(\rho_i), \tag{4.23}$$

where $\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ stands for the pure state decomposition. It turns out that for two-qubit states this quantity can be explicitly expressed and calculated.

For a pure state $|\psi\rangle$ the concurrence C is defined as

$$C(|\psi\rangle) = \left|\left\langle\psi\middle|\tilde{\psi}\right\rangle\right|, \tag{4.24}$$

where $\tilde{\psi}$ stands for the result of applyting a spin-flop operation $(\sigma_y \otimes \sigma_y)\left|\bar{\psi}\right\rangle$, $\left|\bar{\psi}\right\rangle$ being the complex conjugation of $|\psi\rangle$ in the standard basis and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4.25}$$

The concurrence of a mixed state ρ is defined to be the average concurrence of the pure states in a pure state decomposition, minimized over all possible decompositions

$$C(\rho) = \inf \sum_i p_i C(\rho_i), \tag{4.26}$$

where $\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ is a pure state decomposition of ρ. Remarkably, there exists an explicit formula for the concurrence C. Denote $\tilde{\rho}$ the spin-fliped operator ρ

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\bar{\rho}(\sigma_y \otimes \sigma_y), \tag{4.27}$$

where $\bar{\rho}$ stands for the complex conjugation of ρ in the standard basis. Then

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{4.28}$$

where λ_i are the square roots of eigenvalues of $\rho\tilde{\rho}$ in descending order[1] .

Concurrence is related to the entanglement of formation via

$$E(C) = f\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \tag{4.29}$$

$$f(x) = -x \log x - (1 - x)log(1 - x). \tag{4.30}$$

The entanglement of formation E as a function of concurrence C is monotonous, hence the concurrence is a suitable measure of entanglement and the quantifying of entanglement of two qubits reduces to the problem of finding the eigenvalues of $\rho\tilde{\rho}$. We used this tool to study the mechanisms of synchronisation.

Basically, one of the following options occurs during an evolution

a) An initially separable state remains separable throughout the evolution and in the asymptotics.
b) A separable state temporarily becomes entangled during the evolution yet results in a separable state again.
c) A separable state becomes entangled.
d) An initially entangled state evolves towards a separable asymptotic state.
e) An entangled state remains entangled with the entanglement remaining the same according to the chosen measure.
f) An entangled state remains entangled and the entanglement increases or decreases, possibly nonmonotonously, according to the chosen measure.

Analytically, however, we were unable to obtain any conclusive results regarding for what initial conditions does any of the (anti-)synchronising mechanisms enforce one of the cases described above. The concurrence of the asymptotic state is strongly dependent on initial conditions in a manner theoretically simple yet practically too complicated to draw any conclusions.
In numerical simulations, we witnessed all possible scenarios for each of the classes of (anti-)synchronisation generating Lindblad operators, based on the initial conditions. See appendix A for examples.

Every single (anti-)synchronisation mechanism described in this work is capable of creating, destroying, preserving and both increasing and decreasing entanglement of a pair of qubits. No form of synchronisation studied is in a simple relation to entanglement and neither is entanglement a suitable indicator of synchronisation.

[1] All the eigenvalues are real non-negative as $\rho\tilde{\rho}$ is a product of two positive-semidefinite matrices.

Conclusion

The work addressed the question of synchronisation in quantum systems. We dealt with the concept of synchronisation of quantum systems in the current literature and provided suitable definitions of (anti-)synchronisation and total (anti-)synchronisation for composite systems of identical subsystems with internal dynamics. Within the formalism of quantum Markovian dynamical semigroups and Lindblad dynamics with unitary Lindblad operators we then extensively investigated a system of two qubits.

Using a theorem that connects Lindblad operators and the attractor space of the associated evolution map via commutation relations we found all unitary Lindblad operators that enforce (anti-)synchronisation of the asymptotic reduced states of a pair of qubits, and described their respective attractor spaces. From the (anti-)synchronisation mechanisms we subsequently picked out those which not only (anti-)synchronise the dynamically evolving parts of the asymptotic states, but which also lead to synchronisation of the stationary parts, resulting in the case of synchronisation in identical reduced states of the individual qubits.

From the construction, the Lindblad operators were separated into several classes based on the corresponding asymptotic evolution, which was shown to be preserved within each class. Apart from that, each of the classes was found to roughly have a group structure. Respectivelly, to have a stucture of a group if we include a set of measure zero containing the operators that do not enforce (anti-)synchronisation themselves but do have the same form and do not further affect the asymptotic evolution when added to the Lindbladian. Using the results obtained for a single Lindblad operator in the generator it was possible to generalize the situation to an arbitrary combination of unitary operators.

We also studied some additional properties of the two qubit (anti-)synchronising maps and of the asymptotic states they lead to. It turned out that not all of the synchronisation processes necessarily make the resulting global state permutation invariant, in fact same do the exact opposite. We were able to demonstrate that the visibility of the non-trivial asymptotic evolution of the reduced one-qubit states depends mostly merely on the initial conditions, with an exception of a family of classes of (anti-)synchronising Lindblad operators for which it it suppressed by a factor depending on a parameter of the operator. Last but not least, we tackled the question of connection between synchronisation and entanglement, using concurrence as a measure. However, we were unable to analytically obtain any conclusive results due to the strong and rather complicated dependence on the initial conditions.

A numerical model was created to get a better idea about the (anti-)synchronisation process and to verify the analytical results. It showed, among other things, that each of the (anti-)synchronising mechanisms is capable of both generating and destroying of entanglement, based on the initial state. Hence, to uncover the possible relation between the two phenomena a further research is necessary.

By having thoroughly explored the two qubit system, this work might serve as a starting point for future study of synchronisation in more complex systems and contribute to understanding of the nature of the phenomenon.

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Appendix A

Numerical simulation, illustrating examples

In order to get a better idea about the evolution towards asymptotic (anti-)synchronised states and to verify the analytical results, a numerical simulation in MATLAB was created. Some examples are provided on the following pages.

The evolution map is generated by a Lindbladian (1.17), the particular classes of Lindblad operators U_i and their weights p_i are always specified. Within the classes the operators U_i are chosen randomly. The initial conditions are randomly generated, either from the space of separable states or from the space of entangled states, as indicated.

In each step we calculate the reduced states $\rho_A(t)$, $\rho_B(t)$ from the global state $\rho(t)$ and plot the distance of the reduced states from a fixed randomly generated non-evolving test state ρ_{test} (red and blue), i.e. $\|\rho_A(t) - \rho_{test}\|$ and $\|\rho_B(t) - \rho_{test}\|$, together with the norm of their difference (green) $\|\rho_A(t) - \rho_B(t)\|$ in the first graph. The second plot shows the expectation values $\left\langle \sigma_1^{(A)}(t) \right\rangle$,$\left\langle \sigma_1^{(B)(t)} \right\rangle$ of a local observable σ_1 (4.4) for both reduced states (green and blue). The third plot displays the value of Pearson's correlation coefficient (2.14) for the expectations values of σ_1 taken over a time window Δt of 25 time units (black) and the concurrence C (4.28) of the global state $\rho(t)$ (red). The time axes are aligned and equally scaled.

The first example shows a situation of a single Lindblad operator U_1 given by (3.143) (the partial swap), weight $p_1 = 1$, with an entangled initial state $\rho(0)$. The evolution results in a separable totally synchronised state, entanglement is not preserved.

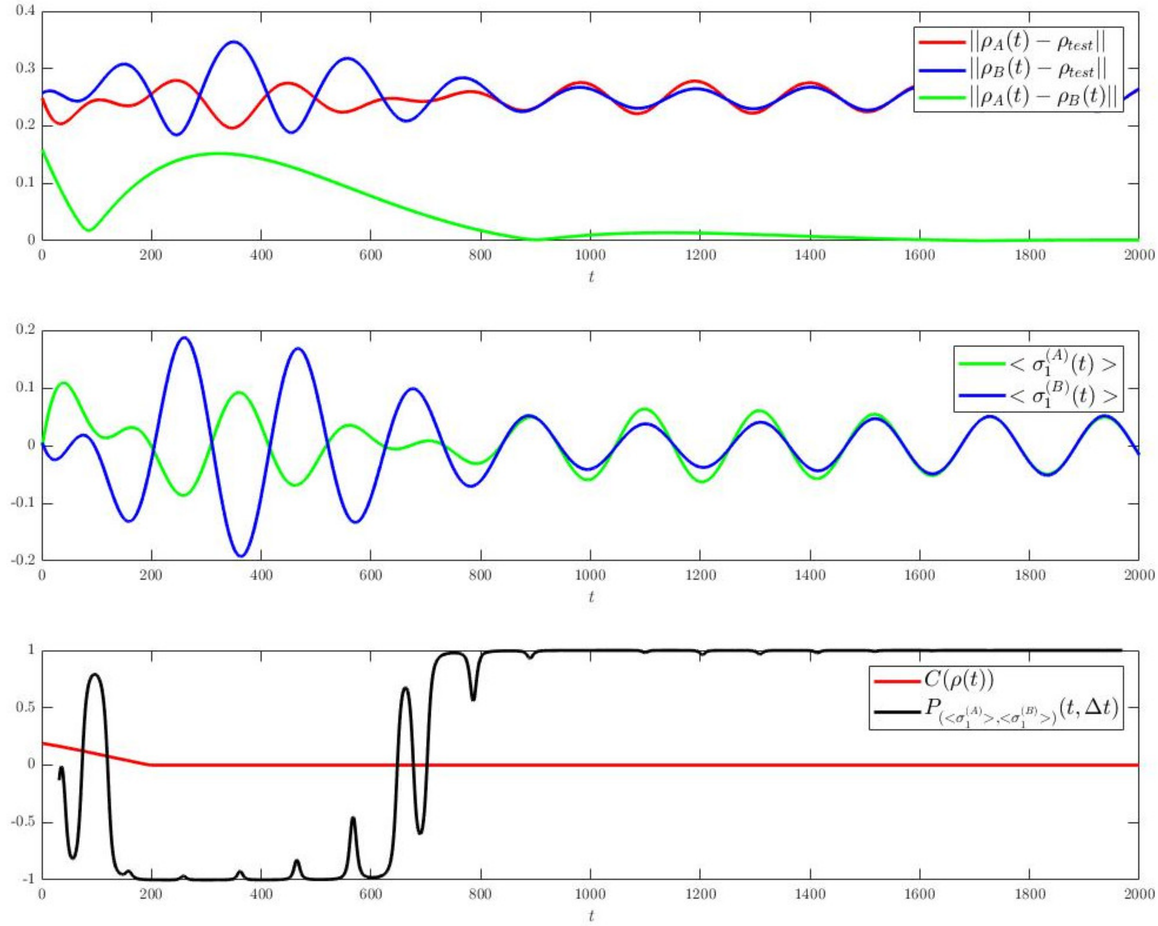

Figure 4.2: Numerical simulation of the evolution of a pair of qubits. A single Lindblad operator U_1 given by (3.143) (the partial swap) with weight $p_1 = 1$, an entangled initial state.

The second example displays evolution of an initially separable state with two Lindblad operators in the generator, U_1 of the form (3.38) with weight $p_1 = 0.7$ and U_2 of the form (3.143) with weight $p_2 = 0.3$. Entanglement is generated during the synchronisation process. Low visibility of the resulting totally synchronised states.

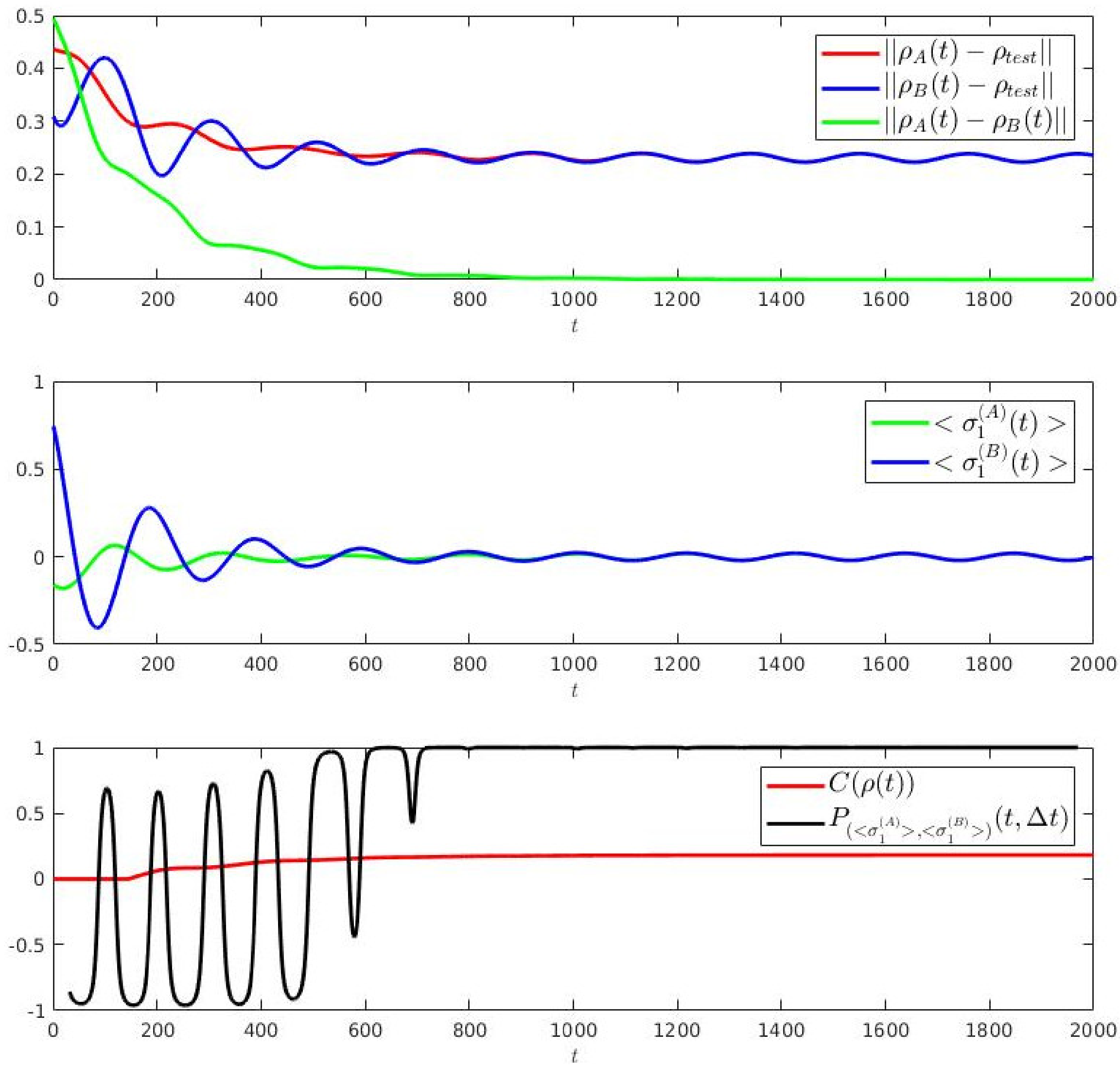

Figure 4.3: Numerical simulation of the evolution of a pair of qubits. Two Lindblad operators, U_1 of the form (3.38) with weight $p_1 = 0.7$ and U_2 of the form (3.143) with weight $p_2 = 0.3$. Separable initial state.

The third example depicts evolution towards an antisynchronised state. A single Lindblad operator U_1 given by (3.119) with weight $p_1 = 1$ is found in the generator. An initially separable state temporarily becomes entangled only to further evolve into a separable state again. The whole process is nearly instantaneous, which is not uncommon.

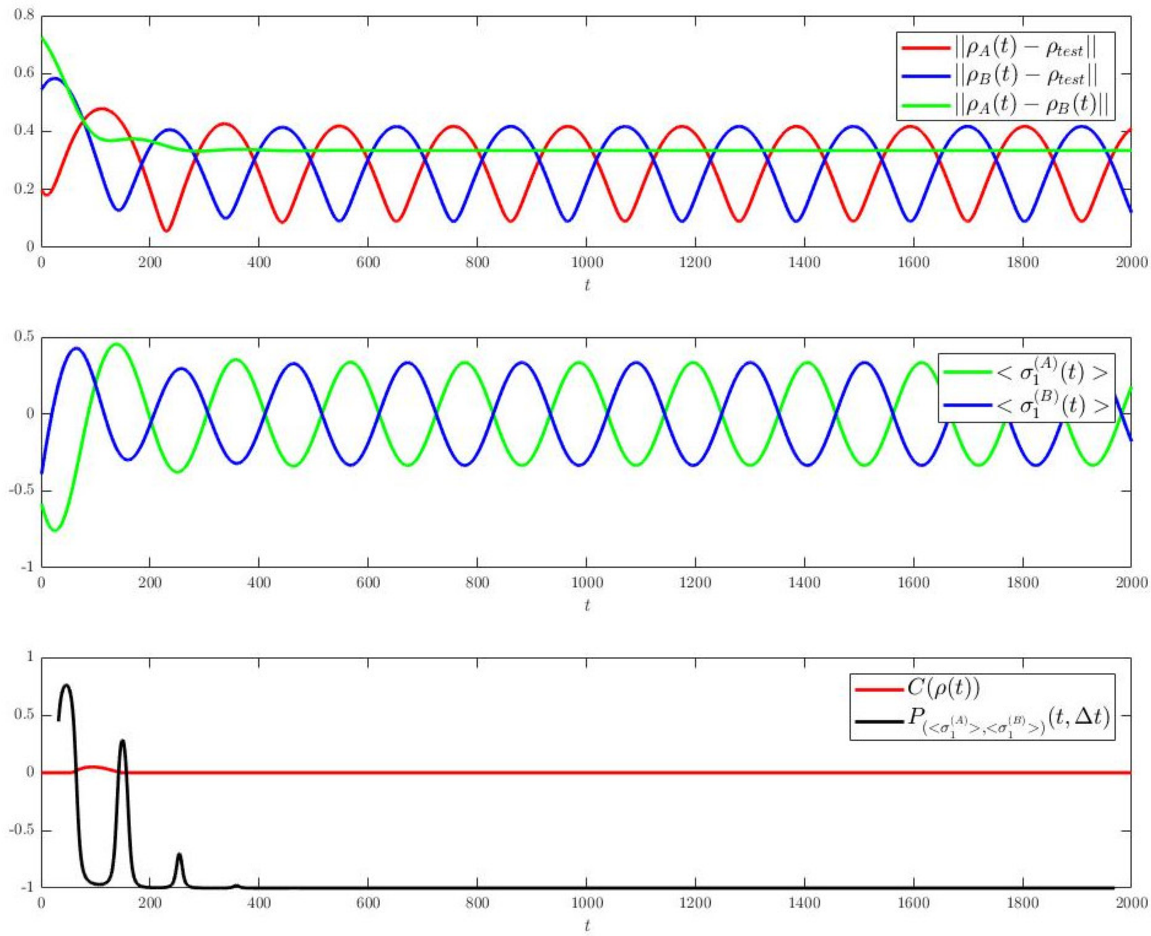

Figure 4.4: Numerical simulation of the evolution of a pair of qubits. A single Lindblad operator U_1 given by (3.119) with weight $p_1 = 1$ and a separable inital state.