Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

# Invariant solutions to and qualitative analysis of reaction-diffusion problems on evolving domains

# Invariantní řešení a kvalitativní analýza reakčně-difuzních rovnic na rostoucích oblastech

Master's Thesis

Author: **Bc. Michal Tichý**

Supervisor: **doc. Ing. Václav Klika, PhD.**

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- Zadání práce -

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*Název práce:*

**Invariantní řešení a kvalitativní analýza reakčně-difuzních rovnic na rostoucích oblastech**

*Autor:* Bc. Michal Tichý

*Obor:* Matematická fyzika

*Druh práce:* Diplomová práce

*Vedoucí práce:* doc. Ing. Václav Klika, PhD., Katedra matematiky, Fakulta jaderná a fyzikálně inženýrská, ČVUT v Praze

*Abstrakt:* Reakčně difuzní rovnice jsou hojně užívaný a studovaný model se širokým uplatněním. V této práci se budeme zabývat dvěma důležitými případy: oblast rostoucí pouze z kraje a stejnoměrně rostoucí oblast s nekonstantní difuzí. Nejprve se seznámíme se symetriemi parciálních diferenciálních rovnic. Dále představíme reakčně-difuzní systémy na rostoucích oblastech. Použijeme symetrie parciálních diferenciálních rovnic k nalezení takzvaných invariantních rešení pro oba dva zmiňované případy. Nakonec nalezneme, která z těchto invariantních řešení splňují Dirichletovy, Neumannovy, Robinovy a periodické okrajové podmínky.

*Klíčová slova:* exaktní řešení, invariantní řešení, Lieovy grupy, neautonomní diferenciální rovnice, oblast rostoucí pouze z kraje, reakčně-difuzní rovnice, rostoucí oblasti, stejnoměrně rostoucí oblast, symetrie parciálních diferenciálních rovnic

*Title:*

**Invariant solutions to and qualitative analysis of reaction-diffusion problems on evolving domains**

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*Abstract:* The reaction-diffusion equations represent a very used and studied model with many applications. In this work we focus on the effect of growth in two important cases: apical growth and uniform growth with non-constant diffusion. We introduce the concept of symmetries of partial differential equations. Next we formulate the reaction-diffusion system on growing domains. We use the symmetries of partial differential equations to find the so-called invariant solutions for the apical growth and the uniform growth with non-constant diffusion. In both cases we consider the scalar case and the vector case for two components. Finally we find invariant solutions which also satisfy Dirichlet, Neumann, Robin a periodic boundary conditions.

*Key words:* apical growth, evolving domains, exact solutions, invariant solutions, Lie groups, non-autonomous differential equations, reaction-diffusion equations, symmetries of partial differential equations, uniform growth

# Contents

# Introduction

The reaction-diffusion equations represent a very used and studied model with many applications, see [1], [2] and [3]. However, the effect of the growing domain is up to exceptions based on static domain behaviour whose size is a parameter. In this work we will focus on the effect of growth in two important cases: apical growth and uniform growth with non-constant diffusion. There is no stability analysis in either case, in the literature and only very recently there were some progress in the most simple case of growth, the uniform expansion [7], [8], [10] and [11]. In our opinion the reason is that classical spectral analysis fails due to the explicit temporal and spatial dependence. Therefore we choose a different approach, namely studying the symmetries of given differential equations, to find a rich family of special explicit solutions. With their help we will try to gain insight into instability (long time behaviour) taking into account relevant boundary conditions.

In the first chapter we will introduce the concept of symmetries of partial differential equations. We will show how to find the infinitesimal generators of symmetries, how to use them to find the so-called invariant solutions and how to find new solutions from known ones using them.

In the second chapter we will formulate a reaction-diffusion system on growing domains. We will express this model in natural Eulerian coordinates and transform it into Lagrangian coordinates, which are defined on constant domain. We will describe two cases of growth: apical growth with constant diffusion coefficient and uniform growth with non-constant diffusion, because the case of uniform growth with constant diffusion coefficient was studied recently to a reasonable degree and generality [7], [10] and [11]. In both cases we will consider the scalar case and the vector case for two components. At the end of this chapter we carefully formulate the relevant boundary conditions, namely Dirichlet, Neumann, Robin and periodic boundary conditions.

In the third chapter we focus on the scalar case of apical growth. We identify its symmetries and use them to find the corresponding invariant solutions. Finally we would like to find invariant solutions which also satisfy the boundary conditions.

In the fourth chapter we shall repeat the analysis on a two component system of apical growth and again we will focus on the symmetries, corresponding invariant solutions and identify invariant solutions satisfying the boundary conditions.

The same procedures is then used in the fifth chapter for a scalar case of the uniform growth with a non-constant diffusion and in the sixth chapter for a two component system undergoing the uniform growth with a non-constant diffusion.

# Chapter 1

# Symmetries of partial differential equations

In this chapter we introduce the concept of symmetries of the partial differential equations and the methods how to use them to find the exact solutions. This approach is inspired mainly by the books [4] and [5].

## 1.1 Symmetries of scalar PDE with two independent variables

First for simplicity we are interested in a scalar partial differential equation (PDE) with two independent variables. Let $x, t$ be the independent variables and $u = u(x,t)$ be the dependent variable. Next $F(x,t,u,u_x,u_t,u_{xx},\dots) = 0$ denotes a partial differential equation, where partial derivatives of $u$ are denoted with a subscript. Let $\Gamma$ be a diffeomorphism:

$$\Gamma : (x,t,u) \mapsto \left(\hat{x}(x,t,u), \hat{t}(x,t,u), \hat{u}(x,t,u)\right). \tag{1.1}$$

Then we can define a symmetry of the partial differential equation:

**Definition 1.1.1.** *Let $u = u(x,t)$ be a function of two independent variables, $F(x,t,u,u_x,u_t,\dots) = 0$ be the partial differential equation and $\Gamma$ be the diffeomorphism of the form* (1.1)*. Then $\Gamma$ is the symmetry if and only if this transformation preserves the set of all solutions of the equation $F = 0$.*

We are interested in the infinite set of the symmetries $\Gamma_\varepsilon$, where $\varepsilon$ is a real parameter, which satisfies conditions:

1. $\Gamma_0$ is the trivial symmetry, which means $\hat{x} = x, \hat{t} = t, \hat{u} = u$, when $\varepsilon = 0$,

2. $\exists \varepsilon > 0 : \Gamma_\delta$ is the symmetry $\forall \delta \in (-\varepsilon, \varepsilon)$,

3. $\Gamma_\varepsilon \Gamma_\delta = \Gamma_{\varepsilon+\delta}$,

4. $$\hat{x} = x + \varepsilon \xi(x,t,u) + \mathcal{O}(\varepsilon^2), \tag{1.2a}$$
$$\hat{t} = t + \varepsilon \tau(x,t,u) + \mathcal{O}(\varepsilon^2), \tag{1.2b}$$
$$\hat{u} = u + \varepsilon \eta(x,t,u) + \mathcal{O}(\varepsilon^2). \tag{1.2c}$$

This set of the symmetries also forms a one-parameter local Lie group. We denote the corresponding infinitesimal generator

$$X = \eta\partial_u + \xi\partial_x + \tau\partial_t,$$

where $\partial_x$ denotes the partial derivative with respect to $x$. The Lie group of transformations can be obtained by exponentiating this generator, with the mapping exp: $\varepsilon X \mapsto \varphi(\varepsilon) \equiv e^{\varepsilon X}$, where $\varphi(\varepsilon) = (\hat{x}(\varepsilon), \hat{t}(\varepsilon), \hat{u}(\varepsilon))$ is the integral curve of the infinitesimal generator $X$, which satisfies

$$\dot{\varphi}(\varepsilon) = X(\varphi(\varepsilon)) \qquad \text{and} \qquad \varphi(0) = (x, t, u).$$

The expressions (1.2) are the Taylor series of the integral curve, see [4]. This definition is more understandable in the differential geometry formalism, cf [6].

Using the expressions (1.2) we can obtain $(\hat{u}, \hat{x}, \hat{t})$ by solving

$$\begin{aligned}
\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} &= \eta(\hat{x}, \hat{t}, \hat{u}),\\
\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} &= \xi(\hat{x}, \hat{t}, \hat{u}),\\
\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} &= \tau(\hat{x}, \hat{t}, \hat{u}),
\end{aligned}$$

subject to the initial conditions

$$(\hat{u}, \hat{x}, \hat{t})\big|_{\varepsilon=0} = (u, x, t).$$

Next we would like to know how to find the derivatives of $\hat{u}$ with respect to $\hat{x}$ and $\hat{t}$. We denote the so-called operators of the total derivative:

$$\begin{aligned}
D_x &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \dots,\\
D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \dots.
\end{aligned}$$

Equations

$$\begin{aligned}
\hat{x} &= \hat{x}\left(x, t, u(x,t)\right),\\
\hat{t} &= \hat{t}\left(x, t, u(x,t)\right)
\end{aligned}$$

are locally invertible if the Jacobian $\mathcal{J}$ is nonzero, which means

$$\mathcal{J} = \det\begin{bmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t}\end{bmatrix} \neq 0.$$

In this case we can locally obtain $x, t$ in terms of $\hat{x}, \hat{t}$ and thus

$$\hat{u} = \hat{u}(\hat{x}, \hat{t}).$$

Next we obtain the formula for the total derivatives of $\hat{u}$ :

$$\begin{bmatrix} D_x\hat{u} \\ D_t\hat{u}\end{bmatrix} = \begin{bmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t}\end{bmatrix}\begin{bmatrix} \hat{u}_{\hat{x}} \\ \hat{u}_{\hat{t}}\end{bmatrix},$$

and using the Cramer's rule we get

$$\hat{u}_{\hat{x}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_x\hat{u} & D_x\hat{t} \\ D_t\hat{u} & D_t\hat{t}\end{vmatrix}, \tag{1.3}$$

$$\hat{u}_{\hat{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u} \\ D_t\hat{x} & D_t\hat{u} \end{vmatrix}. \tag{1.4}$$

The high-order derivatives can be obtained recursively using the same procedure:

$$\hat{u}_{J\hat{x}} = \partial_{\hat{x}}\hat{u}_J = \frac{1}{\mathcal{J}} \begin{vmatrix} D_x\hat{u}_J & D_x\hat{t} \\ D_t\hat{u}_J & D_t\hat{t} \end{vmatrix}, \tag{1.5}$$

$$\hat{u}_{J\hat{t}} = \partial_{\hat{t}}\hat{u}_J = \frac{1}{\mathcal{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_J \\ D_t\hat{x} & D_t\hat{u}_J \end{vmatrix}, \tag{1.6}$$

where $\hat{u}_J$ denotes some derivative of $\hat{u}$ with respect to $\hat{x}$ and $\hat{t}$, i.e. $\hat{u}_J = \hat{u}_{\hat{x}\hat{x}\ldots\hat{x}\hat{t}\hat{t}\ldots\hat{t}}$.

Now we would like to express $\hat{u}_{\hat{x}}$ and $\hat{u}_{\hat{t}}$ as

$$\hat{u}_{\hat{x}} = u_x + \varepsilon\eta^x(x,t,u,u_x,u_t) + \mathcal{O}(\varepsilon^2),$$
$$\hat{u}_{\hat{t}} = u_t + \varepsilon\eta^t(x,t,u,u_x,u_t) + \mathcal{O}(\varepsilon^2)$$

by substituting (1.2) into (1.3) and (1.4):

$$\begin{aligned}\hat{u}_{\hat{x}} &= \frac{1}{\mathcal{J}} \begin{vmatrix} D_x\hat{u} & D_x\hat{t} \\ D_t\hat{u} & D_t\hat{t} \end{vmatrix} = \frac{D_x\hat{u}D_t\hat{t} - D_x\hat{t}D_t\hat{u}}{D_x\hat{x}D_t\hat{t} - D_x\hat{t}D_t\hat{x}} = \\ &= \frac{(u_x + \varepsilon D_x\eta)(1+\varepsilon D_t\tau) - \varepsilon D_x\tau(u_t + \varepsilon D_t\eta) + \mathcal{O}(\varepsilon^2)}{(1+\varepsilon D_x\xi)(1+\varepsilon D_t\tau) + \mathcal{O}(\varepsilon^2)} \\ &= \frac{u_x(1+\varepsilon D_t\tau) + \varepsilon D_x\eta - \varepsilon u_t D_x\tau + \mathcal{O}(\varepsilon^2)}{1 + \varepsilon(D_x\xi + D_t\tau) + \mathcal{O}(\varepsilon^2)}.\end{aligned}$$

Next we use Taylor expansion $\frac{1}{1+x} = 1 - x + \mathcal{O}(x^2), \quad \text{as} \quad x \to 0:$

$$\begin{aligned}\hat{u}_{\hat{x}} &= \left(u_x(1+\varepsilon D_t\tau) + \varepsilon D_x\eta - \varepsilon u_t D_x\tau\right)\left(1 - \varepsilon(D_x\xi + D_t\tau)\right) + \mathcal{O}(\varepsilon^2) = \\ &= u_x + \varepsilon\left(D_x\eta - u_x D_x\xi - u_t D_x\tau\right) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Using the same steps we also obtain

$$\hat{u}_{\hat{t}} = u_t + \varepsilon(D_t\eta - u_t D_t\tau - u_x D_t\xi) + \mathcal{O}(\varepsilon^2),$$

and thus

$$\eta^x(x,t,u,u_x,u_t) = D_x\eta - u_x D_x\xi - u_t D_x\tau,$$
$$\eta^t(x,t,u,u_x,u_t) = D_t\eta - u_t D_t\tau - u_x D_t\xi.$$

Expressions for high-order derivatives can be obtained recursively using (1.5) and (1.6). Finally, suppose that

$$\hat{u}_J = u_J + \varepsilon\eta^J + \mathcal{O}(\varepsilon^2),$$

where again $\hat{u}_J$ is some derivative of $\hat{u}$ with respect to $\hat{x}$ and $\hat{t}$. Then

$$\hat{u}_{J\hat{x}} = u_{Jx} + \varepsilon\left(D_x\eta^J - u_{Jx}D_x\xi - u_{Jt}D_x\tau\right) + \mathcal{O}(\varepsilon^2),$$
$$\hat{u}_{J\hat{t}} = u_{Jt} + \varepsilon\left(D_t\eta^J - u_{Jx}D_t\xi - u_{Jt}D_t\tau\right) + \mathcal{O}(\varepsilon^2),$$

and thus

$$\eta^{Jx} = D_x\eta^J - u_{Jx}D_x\xi - u_{Jt}D_x\tau,$$

$$\eta^{Jt} = D_t\eta^J - u_{Jx}D_t\xi - u_{Jt}D_t\tau.$$

Now the infinitesimal operator $X$ can be prolonged to derivatives by adding all the terms of the form $\eta^J\partial_{u_j}$ and this so-called prolonged infinitesimal generator is denoted by $X^{(n)}$, where $n$ is the highest order of derivatives to which $X$ is prolonged:

$$\begin{aligned}
X^{(1)} &= \xi\partial_x + \tau\partial_t + \eta\partial_u + \eta^x\partial_{u_x} + \eta^t\partial_{u_t},\\
X^{(2)} &= \xi\partial_x + \tau\partial_t + \eta\partial_u + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} + \eta^{xx}\partial_{u_{xx}} + \eta^{xt}\partial_{u_{xt}} + \eta^{tt}\partial_{u_{tt}}.
\end{aligned}$$

In this work we are interested in second order partial differential equations hence we need the second prolonged generator and thus we need explicit expression for these terms:

$$\begin{aligned}
\eta^x =& \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t,\\
\eta^t =& \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2,\\
\eta^{xx} =& \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 -\\
&- \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt},\\
\eta^{tt} =& \eta_{tt} + (2\eta_{tu} - \tau_{tt})u_t - \xi_{tt}u_x + (\eta_{uu} - 2\tau_{tu})u_t^2 - 2\xi_{tu}u_x u_t - \tau_{uu}u_t^3 -\\
&- \xi_{uu}u_x u_t^2 + (\eta_u - 2\tau_t)u_{tt} - 2\xi_t u_{xt} - 3\tau_u u_t u_{tt} - \xi_u u_x u_{tt} - 2\xi_u u_t u_{xt},\\
\eta^{xt} =& \eta_{xt} + (\eta_{tu} - \xi_{xt})u_x + (\eta_{xu} - \tau_{xt})u_t + (\eta_{uu} - \xi_{xu} - \tau_{tu})u_x u_t - \xi_{tu}u_x^2 -\\
&- \tau_{xu}u_t^2 - \xi_{uu}u_x^2 u_t - \tau_{uu}u_x u_t^2 + (\eta_u - \xi_x - \tau_t)u_{xt} - 2\xi_u u_x u_{xt} - 2\tau_u u_t u_{xt} -\\
&- \xi_t u_{xx} - \tau_x u_{tt} - \xi_u u_t u_{xx} - \tau_u u_x u_{tt}.
\end{aligned}$$

Finally, differentiating the symmetry condition

$$F(\hat{x}, \hat{t}, \hat{u}, \hat{u}_{\hat{x}}, \hat{u}_{\hat{t}}, \dots) = 0$$

with respect to $\varepsilon$ at $\varepsilon = 0$ we obtain the linearized symmetry condition in terms of the prolonged infinitesimal generator.

$$X^{(n)}F = 0,$$

when $F = 0$ and $n$ is the order of the partial differential equation as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(\hat{x}, \hat{t}, \hat{u}, \hat{u}_{\hat{x}}, \hat{u}_{\hat{t}}, \dots)\Big|_{\varepsilon=0} = \partial_{\hat{x}}F\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} + \partial_{\hat{t}}F\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} + \partial_{\hat{u}}F\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} + \partial_{\hat{u}_{\hat{x}}}F\frac{\hat{u}_{\hat{x}}}{\mathrm{d}\varepsilon} + \partial_{\hat{u}_{\hat{t}}}F\frac{\hat{u}_{\hat{t}}}{\mathrm{d}\varepsilon} + \dots\Big|_{\varepsilon=0} =\\
&= \partial_x F\xi + \partial_t F\tau + \partial_u F\eta + \partial_{u_x}F\eta^x + \partial_{u_t}F\eta^t + \dots = X^{(n)}F.
\end{aligned}$$

Note that if the PDE is linear in the highest derivative

$$F(x, t, u, u_x, u_t, \dots, u_J) = u_J + f(x, t, u, u_x, u_t, \dots) = 0,$$

we can eliminate the term $u_J$ and the so-called determining equations for $\xi, \tau, \eta$ can be obtained by comparing the prefactors of derivatives of $u$.

**Remark.** *The vector space of all symmetry generators of a partial differential equation forms the Lie algebra.*

## 1.2 How to find symmetries of general PDEs

In this section we are interested in the partial differential equations with $n$ dependent and $m$ independent variables ($n, m \in \mathbb{N}$). We denote the dependent variables as $u = (u_1, u_2, \ldots, u_n)$ and the independent variables as $x = (x_1, x_2, \ldots, x_m)$. For clarity we denote the set of all dependent variables and their partial derivatives of order $p$ or less as $u^{(p)}$.

Next we would like to derive the linearized symmetry condition for PDEs $F_\alpha(u^{(k)}, x) = 0$, where $\alpha = 1, \ldots, n$. This procedure is similar to the one showed in the previous section. Suppose that $X$ is the infinitesimal generator of the one-parameter Lie symmetries:

$$X = \xi_i(x,u)\partial_{x_i} + \eta_j(x,u)\partial_{u_j},$$

where we use the Einstein summation convention. Further we denote the operators of the total derivative:

$$D_{x_i} = \partial_{x_i} + u_{j,x_i}\partial_{u_j} + u_{j,x_i x_k}\partial_{u_{j,x_k}} + \ldots.$$

**Remark.** *In this part we use the notation:*

$$u_{j,x_i} = \frac{\partial u_j}{\partial x_i},$$
$$u_{j,x_i x_k} = \frac{\partial^2 u_j}{\partial x_i \partial x_k}.$$

The first prolongation has the form [4]

$$X^{(1)} = \xi_i(x,u)\partial_{x_i} + \eta_j(x,u)\partial_{u_j} + \eta_j^l(x, u^{(1)})\partial_{u_{j,x_l}},$$

where

$$\eta_j^l(x, u^{(1)}) = D_{x_l}\eta_j - u_{j,x_i} D_{x_l}\xi_i.$$

Next we denote

$$D_M = D_{x_1} D_{x_1} \ldots D_{x_1} D_{x_2} D_{x_2} \ldots D_{x_m}$$

and

$$u_{j,M} = D_M u_j,$$

where $M$ is a multi index. Thus for a partial differential equation of the order $k$ the prolonged infinitesimal generator is

$$X^{(k)} = \xi_i(x,u)\partial_{x_i} + \eta_j(x,u)\partial_{u_j} + \eta_j^M \partial_{u_{j,M}},$$

where

$$\eta_j^M = D_M \eta_j - u_{j,x_i} D_M \xi_i$$

and the multi index $M$ includes all the partial derivatives of order $k$ or less, see [4]. Finally the linearized symmetry condition is in the form

$$X^{(k)} F_\alpha = 0, \qquad \text{when} \qquad F_\alpha = 0, \qquad \alpha = 1, \ldots, n.$$

If the PDEs are linear in the highest derivative

$$F_\alpha = u_{K_\alpha} + f_\alpha(x, u^{(p)}),$$

where $f_\alpha(x, u^{(p)})$ does not contain any term of $u_{K_\alpha}$ or any of its derivatives, the $u_{K_\alpha}$ term can be eliminated and thus the determining equations can be obtained by equating the terms of the remaining derivatives as in the previous section.

## 1.3 Invariant solutions

In this section we are interested in the methods how to find the exact solutions of partial differential equations using their symmetries. For most PDEs, we cannot find the general solution but we can try to use some ansatz. In this part we would like to find the solutions which are invariant under a particular group of symmetries, the so-called invariant solutions. For simplicity we assume a scalar partial differential equation with two independent variables. Suppose that

$$F(x, t, u, u_x, u_t, \dots) = 0 \tag{1.7}$$

is the PDE with the infinitesimal generator of Lie symmetries

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u.$$

The solution $u = u(x, t)$ is invariant under this symmetry if the surface $u = u(x, t)$ is mapped to itself by the group of symmetries generated by $X$, equivalently

$$X(u - u(x, t)) = 0, \qquad \text{when} \qquad u = u(x, t).$$

Next we denote the so-called characteristic

$$\mathcal{Q} = \eta - \xi u_x - \tau u_t.$$

Thus the solution $u = u(x, t)$ is invariant under the symmetry if and only if [4]

$$\mathcal{Q} = \eta - \xi u_x - \tau u_t = 0. \tag{1.8}$$

This partial differential equation is usually much easier to solve than the original one. If we solve this "characteristic" $\mathcal{Q}$ (1.8) we can find out, which solutions also satisfy the equation (1.7).

Suppose that $\xi$ and $\tau$ are both nonzero. Then the characteristic equations of the characteristic $\mathcal{Q}$ are [4]

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\eta}. \tag{1.9}$$

Further we need to find two functionally independent first integrals of (1.9) which we denote $p = p(x, t, u)$ and $v = v(x, t, u)$. Then every invariant solution is a function of $p$ and $v$. If $v$ depends on $u$, the invariant solution is in the form

$$v = f(p).$$

If it is possible we express $u$ from this equation and substitute it into the original PDE to find the function $f$.

**Remark.** *In the case when both first integrals depend on* $u$ *the equation has all invariant solutions of the form* $v = F(p)$ *except a constant solution*

$$p = c,$$

*where* $c$ *is a constant [4]. Hence it is necessary to determine whether the original PDE has a constant solution.*

Now we mention two simple examples:

**Example.** *Suppose that the partial differential equation* (1.7) *has translation symmetries in both independent variables corresponding to generators*

$$X_1 = \partial_x, \qquad X_2 = \partial_t,$$

*where* $X_1$ *corresponds to the symmetry* $(x,t,u) \mapsto (x+\varepsilon, t, u)$ *and* $X_2$ *corresponds to the symmetry* $(x,t,u) \mapsto (x, t+\varepsilon, u), \varepsilon \in \mathbb{R}$. *We would like to find the solution which is invariant under the group of symmetries generated by*

$$X = cX_1 + X_2 = c\partial_x + \partial_t,$$

*which is a generator of the general translation symmetry in both independent variables.*

**Remark.** *Since the sum of symmetry transformations is also a symmetry and the derivative is linear, the sum of generators is also a generator of symmetries.*

*The invariant solution* $u$ *must satisfy*

$$-cu_x - u_t = 0,$$

*which implies that*

$$u = f(x - ct),$$

*where* $f$ *is an arbitrary function of the variable* $x - ct$. *This solution invariant under the general translation symmetry in both independent variables is called a travelling wave.*

**Example.** *Let* $F(x,t,u,u_x,u_t,\dots) = 0$ *be a PDE with scaling symmetries*

$$X_1 = x\partial_x + t\partial_t, \qquad X_2 = u\partial_u,$$

*where* $X_1$ *corresponds to the symmetry* $(x,t,u) \mapsto (e^{\varepsilon}x, e^{\varepsilon}t, u)$ *and* $X_2$ *corresponds to the symmetry* $(x,t,u) \mapsto (x,t,e^{\varepsilon}u), \varepsilon \in \mathbb{R}$. *Every generator of the one-parameter Lie symmetry of scaling has form*

$$X = X_1 + kX_2 = x\partial_x + t\partial_t + ku\partial_u.$$

*The characteristic equations of the invariant solution are*

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{ku}.$$

*Two independent first integrals are, for example*

$$r = \frac{x}{t}, \qquad v = \frac{u}{t^k},$$

*and thus the invariant solution has form*

$$u = t^k f(xt^{-1}).$$

*Substituting this expression into the original equation the function* $f(r)$ *can be identified. Every scale-invariant solution is called a similarity solution.*

Now we assume the system of partial differential equations with $n$ dependent variables $(u_1,\dots,u_n)$ and two independent variables $x,t$. Then there are $n$ characteristics which are zero on any invariant solution, which means

$$\mathcal{Q}_\alpha = X[u_\alpha - u_\alpha(x,t)] = 0, \quad \alpha = 1,2,\dots,n, \tag{1.10}$$

when $u = u(x,t)$ holds. As in the scalar case we have to find new variables which enable us to reduce the original problem into a system of ODEs.

## 1.4 How to find new solutions from known ones

In this part we assume that some solution of the partial differential equation is already known. Using the symmetries we can find new solutions from the known one. Suppose that the solution $u = f(x,t)$ satisfies the equation (1.7) and the equation has symmetry

$$\begin{aligned} \hat{x} &= \hat{x}(x,t,u), \\ \hat{t} &= \hat{t}(x,t,u), \\ \hat{u} &= \hat{u}(x,t,u). \end{aligned}$$

By definition the symmetry maps the original solution into the new one. Suppose that the original variables $x, t, u$ can be written using the new ones. By substituting into the known solution

$$u(\hat{x}, \hat{t}, \hat{u}) = f(\hat{x}(x,t,u), \hat{t}(x,t,u))$$

we obtain the new solution $\hat{u}$. We demonstrate this procedure on an example.

**Example.** *The partial differential equation*

$$u_t = \frac{u_{xx}}{1+u_x^2} = 0$$

*has the solution*

$$u = \sqrt{c - 2t - x^2},$$

*which is invariant under the symmetry generated by*

$$X = u\partial_x - x\partial_u.$$

*This equation has also the translation symmetry*

$$(\hat{x}, \hat{t}, \hat{u}) = (x + a, t, u),$$

*and thus the new solution has form*

$$u = \sqrt{c - 2t - (x-a)^2}.$$

*Using another symmetry*

$$(\hat{x}, \hat{t}, \hat{u}) = (x, t, u + b)$$

*we obtain the solution*

$$u = \sqrt{c - 2t - (x-a)^2} + b.$$

### 1.4.1 Homogeneous linear PDE

Suppose that $F(x, t, u, u_x, u_t, \dots) = 0$ is a homogeneous linear partial differential equation. Every infinitesimal generator of the symmetries of this equation has form [4]

$$X = \xi(x,t)\partial_x + \tau(x,t)\partial_t + g(x,t)u\partial_u + U(x,t)\partial_u,$$

where $U(x,t)$ is an arbitrary solution of the original equation $F = 0$. The Lie algebra of these symmetries consists of the finite-dimensional subalgebra corresponding to generators in form

$$X_f = \xi(x,t)\partial_x + \tau(x,t)\partial_t + g(x,t)u\partial_u$$

and the infinite-dimensional subalgebra spanned by

$$X_i = U(x,t)\partial_u.$$

The commutator of $X_f$ and $X_i$ has form

$$[X_i, X_f] = (U(x,t)g(x,t) - \xi(x,t)\partial_x U(x,t) - \tau(x,t)\partial_t U(x,t))\,\partial_u =: \tilde{U}(x,t)\partial_u,$$

where the commutator is defined as

$$[X_1, X_2] = X_1X_2 - X_2X_1.$$

The generator $\tilde{U}(x,t)\partial_u$ belongs to the infinite-dimensional subalgebra and thus the function $u = \tilde{U}(x,t) = U(x,t)g(x,t) - \xi(x,t)\partial_x U(x,t) - \tau(x,t)\partial_t U(x,t)$ is also a solution of the original PDE. This new function can be used in place of $U$ to obtain another solution. So if we know a solution of the homogeneous linear PDE we can generate new solutions using this procedure.

# Chapter 2

# Reaction-diffusion system on growing domains

In this chapter we will introduce the subject of our study, a reaction-diffusion model of two components on a growing domain and we will use a transformation of coordinates to rewrite the problem on a constant domain. This chapter is inspired by the articles [7] and [8]:

**Definition 2.0.1.** *Let $\Omega \in \mathbb{R}^m, m = 1, 2, 3$ be a simply connected bounded continuously deforming domain at time $t \in [0, t_F], t_F > 0$ and $\partial\Omega_t$ be its boundary. Next let $\boldsymbol{u} = (u(\boldsymbol{x}(t), t), v(\boldsymbol{x}(t), t))^T$ be a vector of two chemical concentrations, where $\boldsymbol{x} = (x(t), y(t), z(t)) \in \Omega_t$. The growth of the domain generates a flow velocity $\boldsymbol{a}(\boldsymbol{x}, t)$. The governing equations for reaction-diffusion on a growing domain follow from balance of mass and have form*

$$u_t + \nabla \cdot (\boldsymbol{a}u) = \nabla \cdot (D_u(u)\nabla u) + f(u, v), \tag{2.1}$$

$$v_t + \nabla \cdot (\boldsymbol{a}v) = \nabla \cdot (D_v(v)\nabla v) + g(u, v), \tag{2.2}$$

*where functions $f(u, v), g(u, v)$ represent nonlinear reaction kinetics and $D_u(u), D_v(v)$ are the diffusion coefficients.*

The system (2.1) – (2.2) can be rewritten in vector form

$$\boldsymbol{u}_t + \nabla \cdot (\boldsymbol{a} : \boldsymbol{u} - \mathbb{D}\nabla u) = \boldsymbol{F}(\boldsymbol{u}), \tag{2.3}$$

where $\boldsymbol{u} = (u, v)^T, \boldsymbol{F} = (f(u, v), g(u, v))^T, \boldsymbol{a} : \boldsymbol{u} = (\boldsymbol{a}u, \boldsymbol{a}v)^T, \mathbb{D} = diag(D_u, D_v)$.

## 2.1 Change of variables

Let $\boldsymbol{\xi}$ be a new spatial coordinate system defined by the bijective mapping

$$\xi_i = \xi_i(\boldsymbol{x}, t), \quad x_j = x_j(\boldsymbol{\xi}, t),$$

where $\boldsymbol{x} \in \Omega_t$. From the chain rule we obtain conditions [7]

$$\frac{\partial \xi_i}{\partial_t} + \frac{\xi_i}{\partial x_j}\frac{\partial x_j}{\partial t} = 0, \quad \frac{\partial \xi_i}{\partial x_j}\frac{\partial x_j}{\partial \xi_i} = \delta_{ij}.$$

Next we transform the equation (2.3) into the new variables:

$$\left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{\xi}} + \frac{\partial \xi_i}{\partial x_j}\frac{\partial x_j}{\partial t} + \frac{\partial \xi_i}{\partial x_j}\frac{\partial}{\partial \xi_i}(a_j \boldsymbol{u}) = \frac{\partial \xi_i}{\partial x_j}\frac{\partial}{\partial \xi_i}\left(\mathbb{D}\frac{\partial \xi_k}{\partial x_j}\frac{\partial \boldsymbol{u}}{\partial \xi_k}\right) + \boldsymbol{F}(\boldsymbol{u}) \implies \tag{2.4}$$

$$\left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{\xi}} + \left[\left(\frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_i}{\partial x_j} a_j\right) I - \frac{\partial^2 \xi_i}{\partial x_j^2}\mathbb{D}\right] \frac{\partial \boldsymbol{u}}{\partial \xi_i} + \frac{\partial \xi_k}{\partial x_j}\frac{\partial a_j}{\partial \xi_k}\boldsymbol{u} = \frac{\partial \xi_i}{\partial x_j}\frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial \xi_i}\left(\mathbb{D}\frac{\partial \boldsymbol{u}}{\partial \xi_k}\right) + \boldsymbol{F}(\boldsymbol{u}). \quad (2.5)$$

## 2.2 Determining the flow

The transformation $(\boldsymbol{x}, t) \to (\boldsymbol{\xi}, t)$ was general so far. Now we restrict ourselves to $\boldsymbol{\xi}$ being the Lagrange coordinates $\boldsymbol{X}$ transforming $\Omega_t$ into $\Omega_{t=0}$ the initial position of the growing domain being independent on time. The movement of the point $\boldsymbol{x}$ in domain $\Omega_t$ can be described in terms of the trajectories $\Gamma$ :

$$\boldsymbol{x} = \Gamma(\boldsymbol{X}, t) = (\Gamma_1(\boldsymbol{X}, t), \Gamma_2(\boldsymbol{X}, t), \Gamma_3(\boldsymbol{X}, t)),$$

with the initial condition

$$\Gamma(\boldsymbol{X}, 0) = \boldsymbol{X}.$$

We disregard solid body translation which implies the boundary condition

$$\Gamma(0, t) = 0.$$

The deformation of the domain due to the growth in $N$-dimensions is described by the $N \times N$ rate of deformation tensor $L_{ij}$ (velocity gradient) which has components determined by

$$\frac{\partial^2 \Gamma_i}{\partial t \partial X_k} = \sum_j L_{ij} \frac{\partial \Gamma_j}{\partial X_k}. \quad (2.6)$$

We will assume dilatational growth, hence the rate of deformation tensor is symmetric (neither rotation nor shearing of the domain). The diagonal elements represent the rate of extension along each axis and hence the trace gives the rate of volume expansion [9]:

$$\sum_i L_{ii} = \nabla \cdot \boldsymbol{a} =: S(\boldsymbol{X}, t), \quad (2.7)$$

where $S(\boldsymbol{X}, t) > 0$.

## 2.3 One-dimensional growth

Due to the complexity we focus on the simplest case, growth in one spatial dimension, but yet having many open problems. In this case the velocity gradient (2.7) is a scalar equal to $S$. From equation (2.6) and initial and boundary condition we obtain

$$\Gamma(X, t) = \int_0^X \left[\exp \int_0^t S(z, \tau) \mathrm{d}\tau\right] \mathrm{d}z, \quad (2.8)$$

$$a(X, t) = \frac{\partial \Gamma}{\partial t}. \quad (2.9)$$

To isolate the effects of transport inside the domain and to enable comparison of results, we transform the coordinate $x$ into the domain $\Omega_0 = [0, L]$ using the uniform spatial scaling

$$X = \frac{x}{r(t)} \in [0, L],$$

where

$$Lr(t) = \Gamma(L,t)$$

denotes the domain length. We use expression (2.5) to transform the original problem (2.1) – (2.2) in the language of new coordinate $X$ :

$$u_t + \left(-\frac{\dot{r}(t)}{r(t)}X + \frac{a}{r(t)}\right)u_X + S(X,t)u = \frac{1}{r^2(t)}\frac{\partial}{\partial X}(D_u u_X) + f(u,v), \tag{2.10}$$

$$v_t + \left(-\frac{\dot{r}(t)}{r(t)}X + \frac{a}{r(t)}\right)v_X + S(X,t)v = \frac{1}{r^2(t)}\frac{\partial}{\partial X}(D_v v_X) + g(u,v), \tag{2.11}$$

where $\dot{r}(t) := \frac{\mathrm{d}r(t)}{\mathrm{d}t}$. Now we discuss two cases which we will study in the rest of the thesis.

## 2.4 Uniform domain growth

First we are interested in the case when the strain rate $S(X,t) = S(t)$ is uniform across the domain, hence the name uniform domain growth. For trajectories we obtain from (2.8) the expression

$$\Gamma(X,t) = X \exp \int_0^t S(\tau)\mathrm{d}\tau = Xr(t).$$

We denote $\xi = X/L$ and from (2.9) we obtain $a = X\dot{r} = \xi L\dot{r}$. The system of reaction-diffusion equations (2.10) – (2.11) has form

$$u_t + S(t)u = \frac{1}{r^2(t)}\frac{\partial}{\partial \xi}(\tilde{D}_u u_\xi) + f(u,v),$$

$$v_t = +S(t)v = \frac{1}{r^2(t)}\frac{\partial}{\partial \xi}(\tilde{D}_v v_\xi) + g(u,v),$$

where $\xi \in [0,1]$ and we absorbed $1/L^2$ into the diffusion coefficients, $\tilde{D}_j = D_j/L^2$ but will immediate drop the tilde below. We can express the strain rate $S(t)$ in a language of $r(t)$ and its derivative:

$$\exp\left(\int_0^t S(\tau)\mathrm{d}\tau\right) = r(t) \implies \int_0^t S(\tau)\mathrm{d}\tau = \ln r(t) \implies S(t) = \frac{\dot{r}(t)}{r(t)}.$$

Because the case of uniform growth for constant diffusion coefficients is well analyzed, see [7], [10] and [11], we will be interested in the linear dependence of diffusion coefficients, which means

$$D_u(u) = d_u u, \quad D_v(v) = d_v v,$$

where $d_u, d_v$ are constants. Furthermore we assume from the onset linearized reaction kinetics:

$$f(u,v) = J_1 u + J_2 v,$$

$$g(u,v) = J_3 u + J_4 v.$$

The reaction-diffusion system now has the form

$$u_t + S(t)u = \frac{D_u}{r^2(t)}uu_{\xi\xi} + \frac{D_u}{r^2(t)}u_\xi^2 + J_1 u + J_2 v,$$

$$v_t + S(t)v = \frac{D_v}{r^2(t)}vv_{\xi\xi} + \frac{D_v}{r^2(t)}v_\xi^2 + J_3 u + J_4 v.$$

Finally we can simplify this system by rescaling $r(t)$ :

$$u_t + S(t)u = \frac{1}{r^2(t)} u u_{\xi\xi} + \frac{1}{r^2(t)} u_\xi^2 + J_1 u + J_2 v, \tag{2.12}$$

$$v_t + S(t)v = \frac{D}{r^2(t)} v v_{\xi\xi} + \frac{D}{r^2(t)} v_\xi^2 + J_3 u + J_4 v, \tag{2.13}$$

where $D$ is a ratio of diffusion constants. We denote

$$\mathbb{J} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}.$$

If the matrix $\mathbb{J}$ is diagonalizable we can separate this system in two scalar equations in form

$$u_t = S(t)u = \frac{D u u_{\xi\xi}}{r^2(t)} + \frac{D u_\xi^2}{r^2(t)} + Ju, \tag{2.14}$$

where $J$ is an eigenvalue of the matrix $\mathbb{J}$.

## 2.5 Apical growth

Next we are interested in the so-called apical growth, which is modelled by assuming that the strain rate is nonzero only in the proliferating region, maintained at the width $\delta$ [8]:

$$S(x,t) = \begin{cases} 0, & 0 \le x < Lr(t) - \delta, \\ S(t), & Lr(t) - \delta \le x \le Lr(t). \end{cases}$$

From (2.8) and (2.9) we obtain

$$a(X,t) = \begin{cases} 0, & 0 \le X < L - \frac{\delta}{r(t)}, \\ (Xr(t) - Lr(t) + \delta)S(t), & L - \frac{\delta}{r(t)} \le X \le L, \end{cases}$$

where $r(t) = 1 + \delta \int_0^t S(\tau)\mathrm{d}\tau$, see [8]. If we assume that the proliferating region is much smaller that the rest of domain, the system can be reduced into a moving boundary problem in the limit. Formally it can be shown by choosing $\|S\| \sim \frac{1}{\delta}$ as $\delta \to 0$. In this case the flow is zero everywhere, while the domain length, $r(t) \sim \mathcal{O}(1)$, increase with time, see [8]. If we denote $\xi = X/L$ the system of reaction-diffusion equations has form:

$$u_t = \frac{\dot{r}(t)}{r(t)} \xi u_\xi + \frac{1}{r^2(t)} \frac{\partial}{\partial \xi}(\tilde{D}_u u_\xi) + f(u,v),$$

$$v_t = \frac{\dot{r}(t)}{r(t)} \xi v_\xi + \frac{1}{r^2(t)} \frac{\partial}{\partial \xi}(\tilde{D}_v v_\xi) + g(u,v),$$

where $\xi \in [0,1]$. In this case we assume, that both diffusion coefficients are constants. As in the previous section we assume the reaction terms in form

$$f(u,v) = J_1 u + J_2 v,$$

$$g(u,v) = J_3 u + J_4 v.$$

After the rescaling $r(t)$ the reaction-diffusion equations for apical growth have form

$$u_t = \frac{\dot{r}(t)}{r(t)}\xi u_\xi + \frac{u_{\xi\xi}}{r^2(t)} J_1 u + J_2 v, \tag{2.15}$$

$$v_t = \frac{\dot{r}(t)}{r(t)}\xi v_\xi + D\frac{v_{\xi\xi}}{r^2(t)} + J_3 u + J_4 v, \tag{2.16}$$

where $D$ is a ratio of diffusion coefficients. As in the previous section, when the matrix $\mathbb{J}$ is diagonalizable we can separate the system into two scalar equations in form

$$u_t = \frac{\dot{r}(t)}{r(t)}\xi u_\xi + D\frac{u_{\xi\xi}}{r^2(t)} + J_u, \tag{2.17}$$

where $J$ is an eigenvalue of the matrix $\mathbb{J}$.

## 2.6 Boundary conditions

In this section we will introduce the relevant Dirichlet, Neumann, Robin and periodic boundary conditions because usually the reaction-diffusion system is equipped by one of these boundary conditions. We list them in the original coordinates $x \in [0, Lr(t)]$ and also in the new coordinates $\xi \in [0, 1]$. For simplicity we assume the scalar case on a growing domain:

### 2.6.1 Dirichlet boundary conditions

$$\begin{aligned} x: \quad & u_D(x,t)\big|_{x=0} = u_D(x,t)\big|_{x=Lr(t)} \stackrel{!}{=} 0, \\ \xi: \quad & u_D(\xi,t)\big|_{\xi=0} = u_D(\xi,t)\big|_{\xi=1} \stackrel{!}{=} 0. \end{aligned}$$

### 2.6.2 Neumann boundary conditions

$$\begin{aligned} x: \quad & \partial_x u_N(x,t)\big|_{x=0} = \partial_x u_N(x,t)\big|_{x=Lr(t)} \stackrel{!}{=} 0, \\ \xi: \quad & \partial_\xi u_N(\xi,t)\big|_{\xi=0} = \partial_\xi u_N(\xi,t)\big|_{\xi=1} \stackrel{!}{=} 0. \end{aligned}$$

### 2.6.3 Robin boundary conditions

$$\begin{aligned} x: \quad & \left(\partial_x u_R(x,t) - \alpha(t) u_R(x,t)\right)\big|_{x=0} = \left(\partial_x u_R(x,t) + \alpha(t) u_R(x,t)\right)\big|_{x=Lr(t)} \stackrel{!}{=} 0, \\ \xi: \quad & \left(\frac{1}{Lr(t)}\partial_\xi u_R(\xi,t) - \alpha(t) u_R(\xi,t)\right)\Bigg|_{\xi=0} = \left(\frac{1}{Lr(t)}\partial_\xi u_R(\xi,t) + \alpha(t) u_R(\xi,t)\right)\Bigg|_{\xi=1} \stackrel{!}{=} 0, \end{aligned}$$

where $\alpha(t)$ is some function of $t$.

### 2.6.4 Periodic boundary conditions

$$
\begin{aligned}
x: \quad & u_P(x,t)\Big|_{x=0} = u_P(x,t)\Big|_{x=Lr(t)}, \quad \partial_x u_P(x,t)\Big|_{x=0} = \partial_x u_P(x,t)\Big|_{x=Lr(t)}, \\
\xi: \quad & u_P(\xi,t)\Big|_{\xi=0} = u_P(\xi,t)\Big|_{\xi=1}, \quad \partial_\xi u_P(\xi,t)\Big|_{\xi=0} = \partial_\xi u_P(\xi,t)\Big|_{\xi=1}.
\end{aligned}
$$

# Chapter 3

# Scalar case of apical growth

In this chapter we are interested in the scalar case of the apical growth. This problem is described by the equation

$$F_S = \frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju - u_t = 0, \tag{3.1}$$

where $\varphi(t) = \frac{\dot{r}(t)}{r(t)}$, see (2.17).

## 3.1 Symmetries and their infinitesimal generators

First of all we would like to find the symmetries of this equation. The prolonged infinitesimal generator acting on this equation is

$$X^{(2)}(F_S) = -2\frac{D\varphi(t)}{r^2(t)}\tau u_{xx} + \frac{D}{r^2(t)}\eta^{xx} + \varphi(t)\xi u_x + x\dot{\varphi}(t)\tau u_x + x\varphi(t)\eta^x + J\eta - \eta^t. \tag{3.2}$$

Using the linearized symmetry condition we obtain the conditions for unknown functions describing the linear part of symmetries:

$$\begin{aligned}
X^{(2)}(F_S) = & -2D\frac{\varphi}{r^2}\tau u_{xx} + \frac{D}{r^2}\eta_{xx} + \frac{D}{r^2}\left(2\eta_{xu} - \xi_{xx}\right)u_x - \frac{D}{r^2}\tau_{xx}\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) + \\
& + \frac{D}{r^2}(\eta_{uu} - 2\xi_{xu})u_x^2 - 2\frac{D}{r^2}\tau_{xu}u_x\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) - \frac{D}{r^2}\xi_{uu}u_x^3 - \\
& - \frac{D}{r^2}\tau_{uu}u_x^2\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) + \frac{D}{r^2}(\eta_u - 2\xi_x)u_{xx} - 2\frac{D}{r^2}\tau_x u_{xt} - 3\frac{D}{r^2}\xi_u u_x u_{xx} - \\
& - \frac{D}{r^2}\tau_u u_{xx}\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) - 2\frac{D}{r^2}\tau_u u_x u_{xt} + \varphi\xi u_x + x\dot{\varphi}\tau u_x + x\varphi\eta_x + \\
& + x\varphi(\eta_u - \xi_x)u_x - x\varphi\tau_x\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) - x\varphi\xi_u u_x^2 - \\
& - x\varphi\tau_u u_x\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) + J\eta - \eta_t + \xi_t u_x - \\
& - (\eta_u - \tau_t)\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) + \xi_u u_x\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right) + \\
& + \tau_u\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + Ju\right)^2 = 0,
\end{aligned}$$

where we used formulas from Section 1.1 and the condition $F_S = 0$. Next we compare the terms multiplied by derivatives of $u$ and their products independently:

$$\begin{aligned} u_{xt}: &\quad \tau_x = 0, \\ u_x u_{xt}: &\quad \tau_u = 0. \end{aligned}$$

These two conditions much simplify the determining equations for $\xi, \tau, \eta$ and also imply $\tau = \tau(t)$ :

$$u_x u_{xx}: \quad -2\frac{D}{r^2}\xi_u = 0 \implies \xi = \xi(x,t), \tag{3.3}$$

$$u_{xx}: \quad \dot{\tau} - 2\varphi\tau - 2\xi_x = 0, \tag{3.4}$$

$$u_x^2: \quad \frac{D}{r^2}\eta_{uu} = 0 \implies \eta = A(x,t)u + B(x,t), \tag{3.5}$$

$$u_x: \quad x\varphi\dot{\tau} + \xi_t - x\varphi\xi_x + x\dot{\varphi}\tau + \xi\varphi + \frac{D}{r^2}(2\eta_{xu} - \xi_{xx}) = 0, \tag{3.6}$$

$$1: \quad \dot{\tau}Ju - \eta_u Ju - \eta_t + J\eta + x\varphi\eta_x + \frac{D}{r^2}\eta_{xx} = 0, \tag{3.7}$$

where $A(x,t)$ and $B(x,t)$ are arbitrary functions. Now we solve these remaining equations. From the condition (3.3) and equation (3.4) we obtain

$$\xi = \left(\frac{\dot{\tau}}{2} - \varphi\tau\right)x + E(t),$$

where $E(t)$ is an arbitrary function. Next we substitute the expression for $\eta$ into equation (3.7):

$$\dot{\tau}Ju - A_t u - B_t + JB + x\varphi A_x u + x\varphi B_x + \frac{D}{r^2}A_{xx}u + \frac{D}{r^2}B_{xx} = 0.$$

By collecting the terms depending on $u$ we obtain

$$u: \quad \dot{\tau}J - A_t + x\varphi A_x + \frac{D}{r^2}A_{xx} = 0, \tag{3.8}$$

$$1: \quad B_t = JB + x\varphi B_x + \frac{D}{r^2}B_{xx}, \tag{3.9}$$

which implies that the function $B(x,t)$ is any function satisfying the original equation (3.1). As a next step we substitute the expressions for $\eta$ and $\xi$ into condition (3.6):

$$x\varphi\dot{\tau} + \left(\frac{\ddot{\tau}}{2} - \dot{(\varphi\tau)}\right)x + \dot{E} - x\varphi\frac{\dot{\tau}}{2} + x\varphi^2\tau + \left(\frac{\dot{\tau}}{2}\varphi - \varphi^2\tau\right)x + \varphi E + 2\frac{D}{r^2}A_x + x\dot{\varphi}\tau = \tag{3.10}$$

$$= \frac{\ddot{\tau}}{2}x + \dot{E} + \varphi E + 2\frac{D}{r^2}A_x = 0, \tag{3.11}$$

which implies

$$\partial_x A(x,t) = v(t)x + w(t),$$

and thus

$$A(x,t) = \frac{1}{2}v(t)x^2 + w(t)x + z(t),$$

where $v(t), w(t)$ and $z(t)$ are arbitrary functions. By substituting this expression into condition (3.8) we obtain

$$\dot{\tau}J - \frac{1}{2}\dot{v}x^2 - \dot{w}x - \dot{z} + \varphi v x^2 + \varphi w x + \frac{D}{r^2}v = 0.$$

By comparing the terms with different powers of $x$ we get

$$\begin{aligned} x^2: &\quad \dot{v} = 2\varphi v \implies v = ae^{2\int\varphi} = ae^{2\int\frac{\dot{r}}{r}} = ae^{2\ln r} = ar^2, \\ x: &\quad \dot{w} = \varphi w \implies w = be^{\int\varphi} = br, \\ 1: &\quad \dot{\tau}J - \dot{z} + \frac{D}{r^2}v = 0 \implies J\dot{\tau} = \dot{z} - aD, \end{aligned}$$

where $a, b$ are arbitrary constants. As a next step we use equation (3.11) and do the same procedure as above:

$$\begin{aligned} &\frac{\ddot{\tau}}{2}x + \dot{E} + \varphi E + 2\frac{D}{r^2}A_x = \frac{\ddot{\tau}}{2}x + \dot{E} + \varphi E + 2Dax + \frac{2Db}{r}, \\ x: &- 4Da = \ddot{\tau} = \frac{\ddot{z}}{J} \implies z(t) = -2DJat^2 + dt + f, \\ &\text{and also } \tau(t) = -2Dat^2 + \frac{d - aD}{J}t + c, \\ 1: &\dot{E} + \varphi E = -\frac{2Db}{r} \implies (\dot{Er}) = -2Db \implies E(t) = -\frac{2Dbt}{r(t)} + \frac{g}{r(t)}, \end{aligned}$$

where $c, d, f, g$ are arbitrary constants. Using all these results we finally obtain

$$\begin{aligned} \eta &= \frac{a}{2}r^2(t)x^2u + br(t)xu - 2DJat^2u + dtu + fu + B(x,t), \\ \xi &= 2Da\varphi(t)t^2x - \frac{d - aD}{J}t\varphi(t)x - 2Datx - c\varphi(t)x + \frac{d - aD}{2J}x - \frac{2Dbt}{r(t)} + \frac{g}{r(t)}, \\ \tau &= -2Dat^2 + \frac{d - aD}{J}t + c, \end{aligned}$$

and thus the infinitesimal generators of the symmetries of equation (3.1) are spanned by

$$\begin{aligned} X_1 &= \left(\frac{r^2(t)}{2}x^2 - 2DJt^2\right)u\partial_u + \left(2D\varphi(t)t^2 + \frac{D}{J}\varphi(t)t - 2Dt - \frac{D}{2J}\right)x\partial_x - \\ &\quad - \left(2Dt^2 + \frac{D}{J}t\right)\partial_t, \\ X_2 &= r(t)xu\partial_u - \frac{2Dt}{r(t)}\partial_x, \\ X_3 &= -\varphi(t)x\partial_x + \partial_t, \\ X_4 &= tu\partial_u + \frac{1}{2J}x\partial_x + \frac{t}{J}\partial_t - \frac{1}{J}\varphi(t)tx\partial_x, \\ X_5 &= u\partial_u, \\ X_6 &= \frac{1}{r(t)}\partial_x, \\ X_B &= B(x,t)\partial_u, \end{aligned}$$

where $B(x, t)$ satisfies the original equation.

Now we verify that these infinitesimal generators form Lie algebra which means that they are closed with respect to the commutator:

$$[X_1, X_2] = \left(\frac{r^2(t)}{2}x^2 - 2DJt^2\right)ur(t)x\partial_u + \left(2D\varphi(t)t^2 + \frac{D}{J}\varphi(t)t - 2Dt - \frac{D}{2J}\right)xr(t)u\partial_u -$$

$$
\begin{aligned}
&-\left(2Dt^2+\frac{D}{J}t\right)\left(\dot r(t)xu\partial_u-\frac{2D}{r(t)}\partial_x+\frac{2Dt\dot r(t)}{r^2(t)}\partial_x\right)-\\
&-\left[r(t)xu\left(\frac{r^2(t)}{2}x^2-2DJt^2\right)\partial_u-\right.\\
&\left.-\frac{2Dt}{r(t)}\left(r^2(t)xu\partial_u+\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)\partial_x\right)\right]\\
&=-\frac{D}{2J}xr(t)u\partial_u+\frac{D^2}{Jr(t)}t\partial_x=-\frac{D}{2J}\left(xr(t)u\partial_u-\frac{2Dt}{r(t)}\partial_x\right)=-\frac{D}{2J}X_2,\\
[X_1,X_3]&=-\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)x\varphi(t)\partial_x+\left(2Dt^2+\frac{D}{J}t\right)\dot\varphi(t)x\partial_x+\\
&+\varphi(t)x^2r^2(t)u\partial_u+\varphi(t)x\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)\partial_x-\\
&-\left(\dot r(t)r(t)x^2-4DJt\right)u\partial_u+\left(4Dt+\frac{D}{J}\right)\partial_t-\\
&-\left(2D\dot\varphi(t)t^2+4D\varphi(t)t+\frac{D}{J}\dot\varphi(t)t+\frac{D}{J}\varphi(t)-2D\right)x\partial_x=\\
&=4DJtu\partial_u-\left(4D\varphi(t)t+\frac{D}{J}\varphi(t)-2D\right)x\partial_x+\left(4Dt+\frac{D}{J}\right)\partial_t=\\
&=4DJ\left(tu\partial_u-\frac{1}{J}\varphi(t)tx\partial_x+\frac{1}{2J}x\partial_x+\frac{1}{J}t\partial_t\right)+\frac{D}{J}(-\varphi(t)x+\partial_t)=\\
&=4DJX_4+\frac{D}{J}X_3,\\
[X_1,X_4]&=\left(\frac{r^2(t)}{2}x^2-2DJt^2\right)ut\partial_u+\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)\times\\
&\times x\left(\frac{1}{2J}-\frac{1}{J}\varphi(t)t\right)\partial_x-\left(2Dt^2+\frac{D}{J}t\right)\left(u\partial_u+\frac{1}{J}\partial_t-\frac{1}{J}\dot\varphi(t)tx\partial_x-\frac{1}{J}\varphi(t)x\partial_x\right)-\\
&-tu\left(\frac{r^2(t)}{2}x^2-2DJt^2\right)\partial_u-\left(\frac{1}{2J}x-\frac{1}{J}\varphi(t)tx\right)\times\\
&\times\left[r^2(t)xu\partial_u+\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)\partial_x\right]-\\
&-\frac{t}{J}\left[(\dot r(t)r(t)x^2-4DJt)u\partial_u+\left(2D\dot\varphi(t)t^2+4D\varphi(t)t+\frac{D}{J}\dot\varphi(t)t+\frac{D}{J}\varphi(t)-2D\right)x\partial_x\right]\\
&+\frac{t}{J}\left(4Dt+\frac{D}{J}\right)\partial_t=\\
&=\left[-\frac{D}{J}tu-\frac{1}{2J}x^2r^2(t)u+2Dt^2u\right]\partial_u+\left[-\frac{2D}{J}\varphi(t)t^2x+\frac{2D}{J}tx\right]\partial_x+\left[\frac{2D}{J}t^2\right]\partial_t=\\
&=-\frac{1}{J}\left[\left(\frac{1}{2}r^2(t)x^2-2Dt^2J\right)\partial_u+\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)x\partial_x+\right.\\
&\left.+\left(-2Dt^2-\frac{D}{J}t\right)\right]-\frac{D}{J}\left[tu\partial_u-\frac{1}{J}\varphi(t)tx\partial_x+\frac{1}{2J}x\partial_x+\frac{1}{J}t\partial_t\right]=\\
&=-\frac{1}{J}X_1-\frac{D}{J}X_4,
\end{aligned}
$$

$$
\begin{aligned}
[X_1, X_5] &= \left(\frac{r^2(t)}{2}x^2 - 2DJt^2\right) u\partial_u - u\left(\frac{r^2(t)}{2}x^2 - 2DJt^2\right)\partial_u = 0,\\
[X_1, X_6] &= \left(2Dt^2 + \frac{D}{J}t\right)\frac{\dot r(t)}{r^2(t)}\partial_x - \frac{1}{r(t)}\Bigg[r^2(t)xu\partial_u +\\
&\quad + \left(2D\varphi(t)t^2 + \frac{D}{J}\varphi(t)t - 2Dt - \frac{D}{2J}\right)\partial_x\Bigg] =\\
&= -\left(r(t)xu\partial_u - \frac{2Dt}{r(t)}\partial_x\right) + \frac{D}{2J}\frac{1}{r(t)}\partial_x = -X_2 + \frac{D}{2J}X_6,\\
[X_2, X_3] &= 2Dt\frac{\dot r(t)}{r^2(t)}\partial_x + \dot r(t)xu\partial_u - \dot r(t)xu\partial_u + \frac{2D}{r(t)}\partial_x - \frac{2Dt\dot r(t)}{r^2(t)}\partial_x = 2DX_6,\\
[X_2, X_4] &= r(t)xut\partial_u - \frac{2Dt}{2Jr(t)}\partial_x + \frac{2Dt^2\varphi(t)}{Jr(t)}\partial_x - tur(t)x\partial_u -\\
&\quad - \left(\frac{x}{2J} - \frac{t\varphi(t)x}{J}\right)r(t)u\partial_u - \frac{t}{J}\dot r(t)xu\partial_u + \frac{2Dt}{Jr(t)}\partial_x - \frac{2Dt^2\dot r(t)}{Jr^2(t)}\partial_x =\\
&= -\frac{1}{2J}xr(t)u\partial_u + \frac{Dt}{Jr(t)}\partial_x = -\frac{1}{2J}X_2,\\
[X_2, X_5] &= r(t)xu\partial_u - ur(t)x\partial_u = 0,\\
[X_2, X_6] &= -\frac{1}{r(t)}r(t)u\partial_u = -X_5,\\
[X_3, X_4] &= -\varphi(t)x\left(\frac{1}{2J} - \frac{1}{J}t\varphi(t)\right)\partial_x + u\partial_u + \frac{1}{J}\partial_t - \frac{1}{J}\varphi(t)x\partial_x - \frac{1}{J}\dot\varphi(t)tx\partial_x +\\
&\quad + \left(\frac{1}{2J}x - \frac{1}{J}t\varphi(t)x\right)\varphi(t)\partial_x + \frac{t}{J}\dot\varphi(t)x\partial_x = u\partial_u + \frac{1}{J}(-\varphi(t)x\partial_x + \partial_t) =\\
&= X_5 + \frac{1}{J}X_3,\\
[X_3, X_5] &= 0,\\
[X_3, X_6] &= -\frac{\dot r(t)}{r^2(t)} + \frac{\varphi(t)}{r(t)} = 0,\\
[X_4, X_5] &= tu\partial_u - ut\partial_u = 0,\\
[X_4, X_6] &= -\frac{t\dot r(t)}{r^2(t)J}\partial_x - \frac{1}{2Jr(t)}\partial_x + \frac{t\varphi(t)}{J}\partial_x = \frac{1}{2J}X_6,\\
[X_5, X_6] &= 0.
\end{aligned}
$$

We can summarize these results in Table 3.1.

Finally we show that the commutators $[X_i, X_B]$ have the form $[X_i, X_B] = B_i(x,t)\partial_u$, and it holds $F_S(B_i) = 0$ when $F_S(B) = 0, \forall i \in \{1, 2, \dots, 6\}$ :

$$
\begin{aligned}
[X_1, X_B] = \Bigg[&\left(2D\varphi(t)t^2 + \frac{D}{J}\varphi(t)t - 2Dt - \frac{D}{2J}\right)xB_x - \left(2Dt^2 + \frac{D}{J}t\right)B_t - \frac{r^(t)x^2}{2}B +\\
&+ 2DJt^2B\Bigg]\partial_u =: B_1(x,t)\partial_u,
\end{aligned}
$$

$$
F_S(B_1) = \frac{D}{r^2(t)}\Bigg[4D\varphi(t)t^2B_{xx} + \frac{2D}{J}\varphi(t)tB_{xx} - 4DtB_{xx} - \frac{D}{J}B_{xx} + 2D\varphi(t)t^2xB_{xxx} +
$$

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ |
|---|---|---|---|---|---|---|
| $X_1$ | $0$ | $-\frac{D}{2J}X_2$ | $4DJX_4+\frac{D}{J}X_3$ | $-\frac{1}{J}X_1-\frac{D}{J}X_4$ | $0$ | $-X_2+\frac{D}{2J}X_6$ |
| $X_2$ | $\frac{D}{2J}X_2$ | $0$ | $2DX_6$ | $-\frac{1}{2J}X_2$ | $0$ | $-X_5$ |
| $X_3$ | $-4DJX_4-\frac{D}{J}X_3$ | $-2DX_6$ | $0$ | $X_5+\frac{1}{J}X_3$ | $0$ | $0$ |
| $X_4$ | $\frac{1}{J}X_1+\frac{D}{J}X_4$ | $\frac{1}{2J}X_2$ | $-X_5-\frac{1}{J}X_3$ | $0$ | $0$ | $\frac{1}{2J}X_6$ |
| $X_5$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $X_6$ | $X_2-\frac{D}{2J}X_6$ | $X_5$ | $0$ | $-\frac{1}{2J}X_6$ | $0$ | $0$ |

Table 3.1: Commutators

$$+\frac{D}{J}\varphi(t)txB_{xxx}-2DtxB_{xxx}-\frac{D}{2J}xB_{xxx}-\left(2dt^2+\frac{D}{J}t\right)B_{xxt}-\frac{r^2(t)x^2}{2}B_{xx}-2r^2(t)xB_x-$$

$$-r^2(t)B+2DJt^2B_{xx}\Bigg]+x\varphi(t)\Bigg[2D\varphi t^2B_x+\frac{D}{J}\varphi(t)tB_x-2DtB_x-\frac{D}{2J}B_x+2D\varphi(t)t^2xB_{xx}+$$

$$+\frac{D}{J}\varphi(t)txB_{xx}-2DtxB_{xx}-\frac{D}{2J}xB_{xx}-\left(2Dt^2+\frac{D}{J}t\right)B_{xt}-\frac{r^2(t)x^2}{2}B_x-r^(t)xB+$$

$$+2DJt^2B_x\Bigg]+J\Bigg[\left(2D\varphi(t)t^2+\frac{D}{J}\varphi(t)t-2Dt-\frac{D}{2J}\right)xB_x-\left(2Dt^2+\frac{D}{J}t\right)B_t-\frac{r^(t)x^2}{2}B+$$

$$+2DJt^2B\Bigg]-\Bigg[4D\varphi(t)txB_x+2D\dot\varphi(t)t^2xB_x+2D\varphi(t)t^2xB_{xt}+\frac{D}{J}\dot\varphi(t)txB_x+\frac{D}{J}\varphi(t)xB_x+$$

$$+\frac{D}{J}\varphi(t)txB_{xt}-2DxB_x-2DtxB_{xt}-\frac{D}{2J}xB_{xt}-\left(4Dt+\frac{D}{J}\right)B_t-\left(2Dt^2+\frac{D}{J}t\right)B_{tt}-$$

$$-\dot r(t)r(t)x^2B-\frac{r^2(t)x^2}{2}B_t+4DJtB+2DJt^2B_t\Bigg]=$$

$$=2D\varphi(t)t^2x\left[\frac{D}{r^2(t)}B_{xxx}+\varphi(t)B_x+x\varphi(t)B_{xx}+JB_x-B_{tx}\right]+$$

$$+\frac{D}{J}\varphi(t)tx\left[\frac{D}{r^2(t)}B_{xxx}+\varphi(t)B_x+x\varphi(t)B_{xx}+JB_x-B_{tx}\right]-$$

$$-2Dtx\left[\frac{D}{r^2(t)}B_{xxx}+\varphi(t)B_x+x\varphi(t)B_{xx}+JB_x-B_{tx}\right]-$$

$$-\frac{D}{2J}x\left[\frac{D}{r^2(t)}B_{xxx}+\varphi(t)B_x+x\varphi(t)B_{xx}+JB_x-B_{tx}\right]-$$

$$-\left(2Dt^2+\frac{D}{J}t\right)\left[\frac{D}{r^2(t)}B_{xxt}-\frac{2\dot r(t)D}{r^3(t)}B_{xx}+x\varphi(t)B_{xt}+\dot\varphi(t)xB_x+JB_t-B_{tt}\right]-$$

$$-\frac{r^2(t)x^2}{2}\left[\frac{D}{r^2(t)}B_{xx}+x\varphi(t)B_x+JB-B_t\right]+2DJt^2\left[\frac{D}{r^2(t)}B_{xx}+x\varphi(t)B_x+JB-B_t\right]-$$

$$-4Dt\left[\frac{D}{r^2(t)}B_{xx}+x\varphi(t)B_x+JB-B_t\right]-\frac{D}{J}\left[\frac{D}{r^2(t)}B_{xx}+x\varphi(t)B_x+JB-B_t\right]=$$

$$
\begin{aligned}
&= 2D\varphi(t)t^2 x\partial_x\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] + \\
&+ \frac{D}{J}\varphi(t)tx\partial_x\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] - \\
&- 2Dtx\partial_x\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] - \frac{D}{2J}x\partial_x\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] - \\
&- \left(2Dt^2 + \frac{D}{J}t\right)\partial_t\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] - \\
&- \frac{r^2(t)x^2}{2}\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] + \\
&+ 2DJt^2\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] - 4Dt\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] - \\
&- \frac{D}{J}\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] = 0,
\end{aligned}
$$

$$
\begin{aligned}
&[X_2, X_B] = \left[-\frac{2Dt}{r(t)}B_x - Br(t)x\right]\partial_u =: B_2(x,t)\partial_u, \\
&F_S(B_2) = \frac{D}{r^2(t)}\left[\frac{-2Dt}{r(t)}B_{xxx} - r(t)xB_{xx} - 2r(t)B_x\right] + x\varphi(t)\left[\frac{-2Dt}{r(t)}B_{xx} - r(t)xB_x - r(t)B\right] \\
&+ J\left[-\frac{2Dt}{r(t)}B_x - Br(t)x\right] - \left[\frac{2D}{r(t)}B_x + \frac{2Dt\dot{r}(t)}{r^2(t)}B_x - \frac{2Dt}{r(t)}B_{xt} - r(t)xB_t - \dot{r}(t)xB\right] = \\
&= -\frac{2Dt}{r(t)}\left[\frac{D}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + JB_x - B_{tx}\right] - \\
&- r(t)x\left[\frac{D}{r^2(t)}B_{xx} + x\varphi(t)B_x + JB - B_t\right] = 0,
\end{aligned}
$$

$$
\begin{aligned}
&[X_3, X_B] = [-\varphi(t)xB_x + B_t]\partial_u =: B_3(x,t)\partial_u, \\
&F_S(B_3) = \frac{D}{r^2(t)}[-2\varphi(t)B_{xx} - \varphi(t)xB_{xxx} + B_{xxt}] + x\varphi(t)[-\varphi(t)B_x - \varphi(t)xB_{xx} + B_{xt}] + \\
&+ J[-\varphi(t)xB_x + B_t] - \left[-\dot{\varphi}(t)xB_x - \varphi(t)xB_{xt} + B_{tt}\right] = \\
&= -\varphi(t)x\left[\frac{D}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + JB_x - B_{tx}\right] + \\
&+ \left[\frac{D}{r^2(t)}B_{xxt} - \frac{2\dot{r}(t)D}{r^3(t)}B_{xx} + x\varphi(t)B_{xt} + \dot{\varphi}(t)xB_x + JB_t - B_{tt}\right] = 0,
\end{aligned}
$$

$$
[X_4, X_B] = \left[\frac{1}{2J}xB_x - \frac{1}{J}\varphi(t)txB_x + \frac{t}{J}B_t - tB\right]\partial_u =: B_4(x,t)\partial_u,
$$

$$
\begin{aligned}
&F_S(B_4) = \frac{D}{r^2(t)}\left[\frac{1}{J}B_{xx} + \frac{1}{2J}xB_{xxx} - \frac{2}{J}\varphi(t)tB_{xx} - \frac{1}{J}\varphi(t)txB_{xxx} + \frac{t}{J}B_{xxx} - tB_{xx}\right]+ \\
&+ x\varphi(t)\left[\frac{1}{2J}B_x + \frac{1}{2J}xB_{xx} - \frac{1}{J}\varphi(t)tB_x - \frac{1}{J}\varphi(t)txB_{xx} + \frac{t}{J}B_{tx} - tB_x\right]+ \\
&+ J\left[\frac{1}{2J}xB_x - \frac{1}{J}\varphi(t)txB_x + \frac{t}{J}B_t - tB\right]- \\
&- \left[\frac{1}{2J}xB_{xt} - \frac{1}{J}\dot\varphi(t)txB_x - \frac{1}{J}\varphi(t)xB_x - \frac{1}{J}\varphi(t)txB_{xt} + \frac{1}{J}B_t + \frac{t}{J}B_{tt} - tB_t - B\right] = \\
&= \frac{x}{2J}\left[\frac{D}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + JB_x - B_{tx}\right]- \\
&- \frac{1}{J}\varphi(t)tx\left[\frac{D}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + JB_x - B_{tx}\right]+ \\
&+ \frac{t}{J}\left[\frac{D}{r^2(t)}B_{xxt} - \frac{2\dot r(t)D}{r^3(t)}B_{xx} + x\varphi(t)B_{xt} + \dot\varphi(t)xB_x + JB_t - B_{tt}\right]+ \\
&t\left[\frac{1}{2J}xB_x - \frac{1}{J}\varphi(t)txB_x + \frac{t}{J}B_t - tB\right] + \frac{1}{J}\left[\frac{1}{2J}xB_x - \frac{1}{J}\varphi(t)txB_x + \frac{t}{J}B_t - tB\right] = 0, \\
&[X_5, X_B] = -B\partial_u =: B_5(x,t)\partial_u, \\
&F_S(B_5) = 0, \\
&[X_6, X_B] = \frac{1}{r(t)}B_x\partial_u =: B_6(x,t)\partial_u, \\
&F_S(B_6) = \frac{D}{r^3(t)}B_{xxx} + x\varphi(t)\frac{1}{r(t)}B_{xx} + J\frac{1}{r(t)}B_x - -\left[-\frac{\dot r(t)}{r^2(t)}B_x + \frac{1}{r(t)}B_{xt}\right] = \\
&= \frac{1}{r(t)}\left[\frac{D}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + JB_x - B_{tx}\right] = 0.
\end{aligned}
$$

**Remark.** *The fact that the infinitesimal generators form Lie algebra guarantees that we found at least some subalgebra of all symmetries.*

**Remark.** *Due to the complexity of the infinitesimal generator $X_1$ we will not use the corresponding symmetry in the analysis below.*

As a next step we find the symmetries $(\hat u, \hat x, \hat t)$ corresponding to the infinitesimal generators:

1. $X_2 = r(t)xu\partial_u - \frac{2Dt}{r(t)}\partial_x$

$$
\begin{aligned}
&\frac{\mathrm{d}\hat t}{\mathrm{d}\varepsilon} = 0 \implies \hat t = t, \\
&\frac{\mathrm{d}\hat x}{\mathrm{d}\varepsilon} = -\frac{2D\hat t}{r(\hat t)} = -\frac{2Dt}{r(t)} \\
&\implies \hat x = x - \frac{2Dt}{r(t)}\varepsilon,
\end{aligned}
$$

$$\frac{\hat{u}}{\mathrm{d}\varepsilon} = r(\hat{t})\hat{x}\hat{u} = r(t)\left(x - \frac{2Dt}{r(t)}\varepsilon\right)\hat{u} \implies \ln\hat{u} = -Dt\varepsilon^2 + r(t)x\varepsilon + c_2$$
$$\implies \hat{u} = ue^{-Dt\varepsilon^2 + r(t)x\varepsilon}.$$

2. $X_3 = -\varphi(t)x\partial_x + \partial_t$

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} = 1 \implies \hat{t} = t + \varepsilon,$$
$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = -\varphi(\hat{t})\hat{x} \implies \ln\hat{x} = \int \varphi(t+\varepsilon)\mathrm{d}\varepsilon = -\ln r(t+\varepsilon) + c_3$$
$$\implies \hat{x} = \frac{c_3}{r(t+\varepsilon)}, \hat{x}(0) = \frac{c_3}{r(t)} \stackrel{!}{=} x \implies c_3 = xr(t)$$
$$\implies \hat{x} = x\frac{r(t)}{r(t+\varepsilon)},$$
$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} = 0$$
$$\implies \hat{u} = u.$$

3. $X_4 = tu\partial_u + \frac{1}{2J}x\partial_x + \frac{t}{J}\partial_t - \frac{1}{J}\varphi(t)tx\partial_x$

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} = \frac{\hat{t}}{J} \implies \ln\hat{t} = \frac{\varepsilon}{J} + c_4$$
$$\implies \hat{t} = te^{\frac{\varepsilon}{J}},$$
$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = \hat{x}\left(\frac{1}{2J} - \frac{\hat{t}}{J}\varphi(\hat{t})\right) = \hat{x}\left(\frac{1}{2J} - \frac{te^{\frac{\varepsilon}{J}}}{J}\varphi\left(te^{\frac{\varepsilon}{J}}\right)\right) \implies \ln\hat{x} = \frac{\varepsilon}{2J} - \int \frac{te^{\frac{\varepsilon}{J}}\dot{r}\left(te^{\frac{\varepsilon}{J}}\right)}{Jr\left(te^{\frac{\varepsilon}{J}}\right)}$$
$$= \begin{bmatrix} te^{\frac{\varepsilon}{J}} = p \\ \frac{t}{J}e^{\frac{\varepsilon}{J}}\mathrm{d}\varepsilon = \mathrm{d}p \end{bmatrix} = \frac{\varepsilon}{2J} - \int \frac{\dot{r}(p)}{r(p)} = \frac{\varepsilon}{2J} - \ln\left(te^{\frac{\varepsilon}{J}}\right) + c_4' \implies \hat{x} = \tilde{c'}_4\frac{e^{\frac{\varepsilon}{2J}}}{r\left(te^{\frac{\varepsilon}{J}}\right)},$$
$$\hat{x}(0) = \frac{\tilde{c'}_4}{r(t)} \stackrel{!}{=} x \implies \tilde{c'}_4 = xr(t)$$
$$\implies \hat{x} = xe^{\frac{\varepsilon}{2J}}\frac{r(t)}{r\left(te^{\frac{\varepsilon}{J}}\right)},$$
$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} = \hat{t}\hat{u} = te^{\frac{\varepsilon}{J}}\hat{u} \implies \ln\hat{u} = Jte^{\frac{\varepsilon}{J}} + c_4'' \implies \hat{u} = \tilde{c''}_4 e^{Jte^{\frac{\varepsilon}{J}}},$$
$$\hat{u}(0) = \tilde{c''}_4 e^{Jt} \stackrel{!}{=} u \implies \tilde{c''}_4 = ue^{-Jt}$$
$$\hat{u} = u\exp\left[Jt\left(e^{\frac{\varepsilon}{J}} - 1\right)\right].$$

4. $X_5 = u\partial_u$

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} = 0 \implies \hat{t} = t,$$
$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = 0 \implies \hat{x} = x,$$
$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} = \hat{u} \implies \hat{u} = ue^{\varepsilon}.$$

5. $X_6 = \frac{1}{r(t)}\partial_x$

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} = 0 \implies \hat{t} = t,$$
$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = \frac{1}{r(t)} \implies \hat{x} = x + \frac{\varepsilon}{r(t)},$$
$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} = 0 \implies \hat{u} = u.$$

6. $X_B = B(x,t)\partial_u$

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}\varepsilon} = 0 \implies \hat{t} = t,$$
$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}\varepsilon} = 0 \implies \hat{x} = x,$$
$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\varepsilon} = B(\hat{x},\hat{t}) = B(x,t) \implies \hat{u} = u + B(x,t)\varepsilon.$$

## 3.2 Invariant solutions

### 3.2.1 Solution invariant under the symmetry generated by $X_3 + a_1X_5$

First we seek the solution invariant under the infinitesimal generator

$$X_3 + a_1X_5,$$

where $a_1$ is an arbitrary constant. We can write down the characteristic equations of the characteristic $\mathcal{Q} = a_1u + \frac{\dot{r}(t)}{r(t)}xu_x - u_t$ :

$$\frac{\mathrm{d}u}{a_1u} = \frac{\mathrm{d}x}{-\varphi(t)x} = \frac{\mathrm{d}t}{1}.$$

Using them we can transform the original partial differential equation into the ordinary differential equation:

$$\begin{aligned}
&\frac{\mathrm{d}x}{-\varphi(t)x} = \frac{\mathrm{d}t}{1}\\
&\implies \ln x = -\int \varphi(t)\mathrm{d}t = -\ln r(t)\\
&\implies p = r(t)x,
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}u}{a_1u} = \frac{\mathrm{d}t}{1}\\
&\implies \ln u = a_1t\\
&\implies v = e^{-a_1t}u\\
&\implies u(x,t) = e^{a_1t}F(r(t)x).
\end{aligned}$$

We substitute it into the original PDE:

$$a_1e^{a_1t}F(r(t)x) + \dot{r}(t)xe^{a_1t}F'(r(t)x) = De^{a_1t}F''(r(t)x) + x\dot{r}(t)e^{a_1t}F'(r(t)x) + Je^{a_1t}F(r(t)x)$$

$$\implies F''(p) = R_1 F(p),$$

where $R_1$ denotes a ratio of constant parameters, $R_1 = \frac{a_1 - J}{D}$. Solution of this ODE has form

$$F(p) = K_1 e^{\sqrt{R_1} p}$$

and thus the invariant solution is

$$u_1(x,t) = K_1 e^{\sqrt{R_1} x r(t) + a_1 t}. \tag{3.12}$$

Now we can use another symmetries to find the so-called new solutions from know one, see Section 1.4:

1. $X_2$ :

   From the symmetry corresponding to the generator $X_2$ we obtain:

$$u = \hat{u} e^{D\hat{t}\varepsilon^2 - r(\hat{t})x\varepsilon} = \hat{u} e^{D\hat{t}\varepsilon^2 - r(\hat{t})\hat{x}\varepsilon - 2Dt\varepsilon^2} = \hat{u} e^{-Dt\varepsilon^2 - r(\hat{t})\hat{x}\varepsilon} \stackrel{!}{=} K_1 e^{\sqrt{R_1} r(\hat{t}) \left(\hat{x} + \frac{2D\hat{t}\varepsilon}{r(\hat{t})}\right) + a_1 \hat{t}}$$
$$\implies \tilde{u}_1(x,t) = K_1 e^{(\sqrt{R_1} + b_1) x r(t) + (Db_1^2 + 2\sqrt{R_1} D b_1 + a_1) t},$$

   where we denoted symmetry parameter $\varepsilon$ as $b_1$.

2. $X_4$ :

$$u = \hat{u} e^{J\hat{t} e^{\frac{-\varepsilon}{J}} (1 - e^{\frac{\varepsilon}{J}})} = \hat{u} e^{J\hat{t} e^{-\frac{\varepsilon}{J}} (e^{-\frac{\varepsilon}{J}} - 1)} \stackrel{!}{=} K_1 e^{e^{-\frac{\varepsilon}{2J}} (\sqrt{R_1} + b_1) \hat{x} r(\hat{t}) + e^{-\frac{\varepsilon}{J}} (Db_1^2 + 2\sqrt{R_1} D b_1 + a_1) \hat{t}}$$

   and hence

$$\tilde{\tilde{u}}_1(x,t) = K_1 e^{\sqrt{c_1} (\sqrt{R_1} + b_1) x r(t) + c_1 (Db_1^2 + 2\sqrt{R_1} D b_1 + a_1 - J) t + Jt}, \tag{3.13}$$

   where $e^{-\frac{\varepsilon}{J}} = c_1 \geq 0$.

3. $X_6$ :

   Using the same procedure we find out that this symmetry only changes the parameter $K_1$ :

$$u = K_1 e^{(\sqrt{R_1} + b_1) \left(\hat{x} - \frac{\varepsilon}{r(\hat{t})}\right) r(\hat{t}) + (Db_1^2 + 2\sqrt{R_1} D b_1 + a_1) \hat{t}} = \tilde{K}_1 e^{(\sqrt{R_1} + b_1) x r(t) + (Db_1^2 + 2\sqrt{R_1} D b_1 + a_1) t}.$$

### 3.2.2 Solution invariant under the symmetry generated by $X_5 + a_2 X_6$

Next we are interested in the solution invariant under the symmetry generated by

$$X_5 + a_2 X_6 = u\partial_u + \frac{a_2}{r(t)} \partial_x,$$

where $a_2$ is an arbitrary parameter. The characteristic corresponding to the symmetry has form

$$\mathcal{Q} = u - \frac{a_2}{r(t)} u_x = 0.$$

This PDE contains the derivative with respect to only one of independent variables which enables us to solve it quite easy. Hence we have

$$u = K_2 e^{\frac{1}{a_2} r(t) x + f_2(t)},$$

where $f_2(t)$ is an arbitrary function. Substituting this solution into the original differential equation we obtain:

$$\begin{aligned}
&\left(\frac{1}{a_2}\dot{r}(t)x + \dot{f}_2(t)\right)u = D\frac{1}{a_2^2}u + \frac{1}{a_2}\dot{r}(t)xu + Ju \\
&\implies \dot{f}_2(t) = \frac{D}{a_2^2} + J \\
&\implies f_2(t) = \left(\frac{D}{a_2^2} + J\right)t.
\end{aligned}$$

The invariant solution has form

$$u_2(x,t) = K_2 e^{\frac{1}{a_2}r(t)x + \left(\frac{D}{a_2^2}+J\right)t}.$$

Using the substitution $d_2 := \left(\frac{D}{a_2^2} + J\right)$ we obtain

$$u_2(x,t) = K_2 e^{\sqrt{\frac{d_2-J}{D}}r(t)x + d_2 t},$$

which is the same solution as (3.12).

### 3.2.3 Solution invariant under the symmetry generated by $X_3 + a_3X_6$

Now we would like to find the solution invariant under the symmetry generated by

$$X_3 + a_3X_6 = \left(-\frac{\dot{r}(t)}{r(t)}x + \frac{a_3}{r(t)}\right)\partial_x + \partial_t,$$

where $a_3$ is an arbitrary constant. The corresponding characteristic has form

$$\mathcal{Q} = \left(\frac{\dot{r}(t)}{r(t)}x - \frac{a_3}{r(t)}\right)u_x - u_t = 0,$$

which is equivalent to the condition

$$(a_3 - \dot{r}(t)x)u_x + r(t)u_t = 0.$$

Each function

$$u = f_3\left(a_3 t - r(t)x\right)$$

is a solution of this characteristic for arbitrary $f_3$. By substituting into the original PDE we obtain

$$\begin{aligned}
&(a_3 - \dot{r}(t)x)f_3' = Df_3'' - \dot{r}(t)xf_3' + Jf_3 \\
&\implies Df_3'' - a_3f_3' + Jf_3 = 0 \\
&\implies f_3(z) = K_3 \exp\left[\left(\frac{a3 - \sqrt{a_3^2 - 4DJ}}{2D}\right)z\right] + L_3 \exp\left[\left(\frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\right)z\right],
\end{aligned}$$

where

$$z = a_3 t - r(t)x.$$

The invariant solution has form

$$u_3(x,t) = K_3 \exp\left[\left(\frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D}\right)(a_3 t - r(t)x)\right] + \\ + L_3 \exp\left[\left(\frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\right)(a_3 t - r(t)x)\right].$$

Now we use another symmetries to generalize this solution:

1. $X_2$ :

$$u = K_3 \exp\left[\left(\frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D}\right)\left(a_3 \hat{t} - r(\hat{t})\left(\hat{x} + \frac{2D\hat{t}\varepsilon}{t(\hat{t})}\right)\right] + \\ + L_3 \exp\left[\left(\frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\right)\left(a_3 \hat{t} - r(\hat{t})\left(\hat{x} + \frac{2D\hat{t}\varepsilon}{t(\hat{t})}\right)\right)\right] = \\ = K_3 \exp\left[\frac{a_3 - 2D\varepsilon}{2D}\left(a_3 - \sqrt{a_3^2 - 4DJ}\right)\hat{t} - \frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D}\hat{x}r(\hat{t})\right] + \\ + L_3 \exp\left[\frac{a_3 - 2D\varepsilon}{2D}\left(a_3 + \sqrt{a_3^2 - 4DJ}\right)\hat{t} - \frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\hat{x}r(\hat{t})\right] = \\ = K_3 \exp\left[\frac{a_3 - 2D\varepsilon}{2D}\left(a_3 - \sqrt{a_3^2 - 4DJ}\right)\hat{t} + D\varepsilon^2\hat{t} + \left(\varepsilon - \frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D}\right)\hat{x}r(\hat{t})\right] + \\ + L_3 \exp\left[\frac{a_3 - 2D\varepsilon}{2D}\left(a_3 + \sqrt{a_3^2 - 4DJ}\right)\hat{t} + D\varepsilon^2\hat{t} + \left(\varepsilon - \frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\right)\hat{x}r(\hat{t})\right].$$

Finally we denote $\varepsilon = b_3$ and obtain the solution

$$\tilde{u}_3(x,t) = K_3 \exp\left[\frac{a_3 - 2Db_3}{2D}\left(a_3 - \sqrt{a_3^2 - 4DJ}\right)t + Db_3^2 t + \right. \\ \left. + \left(b_3 - \frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D}\right)xr(t)\right] + L_3 \exp\left[\frac{a_3 - 2Db_3}{2D}\left(a_3 + \sqrt{a_3^2 - 4DJ}\right)t + \right. \\ \left. + Db_3^2 t + \left(b_3 - \frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\right)xr(t)\right].$$

2. $X_4$ :

Similarly as in Section 3.2.3 we obtain

$$\tilde{\tilde{u}}_3(x,t) = K_3 \exp\left[c_3\left[\frac{a_3 - 2Db_3}{2D}\left(a_3 - \sqrt{a_3^2 - 4DJ}\right) + Db_3^2 - J\right]t + Jt + \right. \\ \left. + \sqrt{c_3}\left(b_3 - \frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D}\right)xr(t)\right] + L_3 \exp\left[c_3\left[\frac{a_3 - 2Db_3}{2D}\left(a_3 + \sqrt{a_3^2 - 4DJ}\right) + \right.\right. \\ \left.\left. + Db_3^2 - J\right]t + Jt + \sqrt{c_3}\left(b_3 - \frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D}\right)xr(t)\right], \tag{3.14}$$

where $e^{-\frac{\varepsilon}{J}} = c_3 \geq 0$.

### 3.2.4 Solution invariant under the symmetry generated by $X_2 + a_4X_5 + b_4X_6$

We are interested in the solution which is invariant under the symmetry generated by

$$X_2 + a_4X_5 + b_4X_6,$$

where $a_4$ and $b_4$ are arbitrary parameters. The corresponding characteristic reads

$$\mathcal{Q} = (r(t)x + a_4)\,u + \frac{2Dt + b_4}{r(t)}u_x = 0.$$

Again there is only a derivative with respect to one variable and thus we can easy solve this equation:

$$\begin{aligned} u_x &= \left(-\frac{r^2(t)x}{2Dt + b_4} - \frac{a_4r(t)}{2Dt + b_4}\right) u \implies \\ u(x,t) &= Ke^{\left(-\frac{r^2(t)x^2}{2(2Dt+b_4)} - \frac{a_4r(t)x}{2Dt+b_4} + f_4(t)\right)}, \end{aligned}$$

where $f_4(t)$ is some function of $t$ which we identify by substituting this solution into original equation (3.1):

$$\begin{aligned} &\left(-\frac{r(t)\dot{r}(t)x^2}{2Dt + b_4} + \frac{Dr(t)^2x^2}{(2Dt + b_4)^2} - \frac{a_4\dot{r}(t)x}{2Dt + b_4} + \frac{2Dar(t)x}{(2Dt + b_4)^2} + \dot{f}_4(t)\right) u = \\ =&\left(-\frac{D}{2Dt + b_4} + \frac{Dr^2(t)x^2}{(2Dt + b_4)^2} + \frac{2Da_4r(t)x}{(2Dt + b_4)^2} + \frac{Da_4^2}{(2Dt + b_4)^2}\right) u + \\ &+\left(-\frac{\varphi(t)r^2(t)x^2}{2Dt + b_4} - \frac{a_4phi(t)r(t)x}{2Dt + b_4} + J\right) u \\ &\implies \dot{f}_4(t) = -\frac{D}{2Dt + b_4} + \frac{Da_4^2}{(2Dt + b_4)^2} + J \\ &\implies f_4(t) = -\frac{1}{2}\ln(2Dt + b_4) - \frac{a_4^2}{2(2Dt + b_4)} + Jt. \end{aligned}$$

The invariant solution is

$$u_4(x,t) = \frac{K_4}{\sqrt{2Dt + b_4}}e^{\left(-\frac{r^2(t)x^2}{2(2Dt+b_4)} - \frac{a_4r(t)x}{2Dt+b_4} - \frac{a_4^2}{2(2Dt+b_4)} + Jt\right)},$$

where $K_4, a_4, b_4$ are arbitrary constants.

Again we use other symmetries:

1. $X_3$ :

$$\tilde{u}_4(x,t) = \frac{\tilde{K}_4}{\sqrt{2Dt + \tilde{b}_4}}e^{\left(-\frac{r^2(t)x^2}{2(2Dt+\tilde{b}_4)} - \frac{a_4r(t)x}{2Dt+\tilde{b}_4} - \frac{a_4^2}{2(2Dt+\tilde{b}_4)} + Jt\right)},$$

where $\tilde{b}_4 = b_4 - 2D\varepsilon$ and $\tilde{K}_4 = K_4e^{J\varepsilon}$.

2. $X_4$ :

We use the generator $X_4$ in the same way as above and obtain:

$$\tilde{\tilde{u}}_4(x,t) = \frac{\tilde{K}_4}{\sqrt{2Dc_4t + \tilde{b}_4}}e^{\left(-\frac{c_4r^2(t)x^2}{2(2Dc_4t+\tilde{b}_4)} - \frac{a_4\sqrt{c_4}r(t)x}{2Dc_4t+\tilde{b}_4} - \frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)} + Jt\right)}. \tag{3.15}$$

### 3.2.5 Solution invariant under the symmetry generated by $a_5X_3 + X_4 + b_5X_5$

In this subsection we would like to find the solution which is invariant under the symmetry generated by

$$a_5X_3 + X_4 + b_5X_5,$$

where $a_5$ and $b_5$ are arbitrary constants. The corresponding characteristic has form

$$\mathcal{Q} = (t+b_5)u - \left(\frac{1}{2J} - \frac{f\varphi(t)}{J} - a_5\varphi(t)\right) xu_x - \left(\frac{t}{J} + a_5\right) u_t = 0.$$

As in the first case we can use the characteristic equations to transform original problem into ordinary differential equation:

$$\frac{\mathrm{d}u}{(t+b_5)u} = \frac{\mathrm{d}x}{\left(\frac{1}{2J} - \frac{f\varphi(t)}{J} - a_5\varphi(t)\right) x} = \frac{\mathrm{d}t}{\left(\frac{t}{J} + a_5\right)},$$

$$\frac{\mathrm{d}x}{\left(\frac{1}{2J} - \frac{f\varphi(t)}{J} - a_5\varphi(t)\right) x} = \frac{\mathrm{d}t}{\left(\frac{t}{J} + a_5\right)}$$

$$\implies \ln x = \int \left(\frac{\frac{1}{2} - t\varphi(t) - Ja_5\varphi(t)}{t+Ja_5}\right) \mathrm{d}t = \frac{1}{2}\ln t + Ja_5 - \ln r(t)$$

$$\implies p = \frac{r(t)}{\sqrt{t+Ja_5}} x,$$

$$\frac{\mathrm{d}u}{(t+b_5)u} = \frac{\mathrm{d}t}{\left(\frac{t}{J} + a_5\right)}$$

$$\implies \ln u = J\int \left(\frac{t+b_5}{t+Ja_5}\right) \mathrm{d}t = Jt + (Jb_5 - J^2a_5)\ln t + Ja_5$$

$$\implies v = (t+Ja_5)^{(J^2a_5 - Jb_5)} e^{-Jt} u$$

$$\implies u(x,t) = (t+Ja_5)^{(Jb_5 - J^2a_5)} e^{Jt} F\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right).$$

Next we substitute this expression into original equation (3.1):

$$(Jb_5 - J^2a_5)(t+Ja_5)^{(Jb_5 - J^2a_5 - 1)} e^{Jt} F\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right) +$$
$$+ J(t+Ja_5)^{(Jb_5 - J^2a_5)} e^{Jt} F\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right) +$$
$$+ \left(\frac{\dot r(t)x}{\sqrt{t+Ja_5}} - \frac{1}{2}\frac{r(t)x}{(t+Ja_5)^{3/2}}\right)(t+Ja_5)^{(Jb_5 - J^2a_5)} e^{Jt} F'\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right) =$$
$$= \frac{D}{t+Ja_5}(t+Ja_5)^{(Jb_5 - J^2a_5)} e^{Jt} F''\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right) +$$
$$+ \frac{\dot r(t)x}{\sqrt{t+Ja_5}}(t+Ja_5)^{(Jb_5 - J^2a_5)} e^{Jt} F'\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right) + J(t+Ja_5)^{(Jb_5 - J^2a_5)} e^{Jt} F\left(\frac{r(t)}{\sqrt{t+Ja_5}} x\right)$$
$$\implies F''(p) + \frac{p}{2D}F'(p) + \frac{(J^2a_5 - Jb_5)}{D}F(p) = 0.$$

This second order ordinary differential equation can be transformed into the so-called Kummer's differential equation, see [12]:

**Theorem 3.2.1** (Kummer's differential equation)**.** *The Kummer's differential equation*

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + (b-z)\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0 \tag{3.16}$$

*has general solution in form*

$$w(z) = C_1 M(a,b,z) + C_2 U(a,b,z),$$

*where*

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n} n!$$

*and*

$$U(a,b,z) = \frac{\pi}{\sin \pi b}\left(\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right),$$

*with*

$$\begin{aligned}
(a)_n &= a(a+1)(a+2)\dots(a+n-1), (a)_0 = 1,\\
\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t, \quad (\Re z > 0),\\
\frac{1}{\Gamma(z)} &= ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right] \quad (|z|<\infty),\\
\gamma &= \lim_{m\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{m} - \ln m\right).
\end{aligned}$$

*We use convention* $0! = 1$.

Substitution

$$F(p) = e^{-\frac{p^2}{4D}} w\left(\frac{p^2}{4D}\right), \frac{p^2}{4D} = z$$

entails

$$\begin{aligned}
F'(p) &= \frac{\mathrm{d}(e^{-z}w(z))}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}p} = \left(-e^{-z}w(z) + e^{z}w'(z)\right)\frac{p}{2D}\\
F''(p) &= \frac{\mathrm{d}}{\mathrm{d}p}\left((-e^{-z}w(z)+e^{z}w'(z))\frac{p}{2D}\right) =\\
&= \frac{1}{2D}(-e^{-z}w(z)+e^{z}w'(z)) + \frac{p}{2D}\frac{\mathrm{d}z}{\mathrm{d}p}\frac{\mathrm{d}(-e^{-z}w(z)+e^{z}w'(z))}{\mathrm{d}z} =\\
&= \frac{1}{2D}(-e^{-z}w(z)+e^{z}w'(z)) + \frac{p^2}{4D}(e^{-z}w(z) - 2e^{-z}w'(z) + e^{-z}w''(z))
\end{aligned}$$

and hence

$$\begin{aligned}
F''(p) &+ \frac{p}{2D}F'(p) + \frac{(J^2a_5 - Jb_5)}{D}F(p) = \frac{1}{2D}\left(-e^{-z}w(z)+e^{z}w'(z)\right) +\\
&+ z(e^{-z}w(z) - 2e^{-z}w'(z) + e^{-z}w''(z)) + \frac{z}{D}\left(-e^{-z}w(z)+e^{z}w'(z)\right) +\\
&+ \frac{(J^2a_5 - Jb_5)}{D}e^{-z}w(z) \stackrel{!}{=} 0
\end{aligned}$$

$$\Longleftrightarrow zw''(z) + \left(\frac{1}{2} - z\right) w'(z) - \left(\frac{1}{2} + Jb_5 - J^2 a_5\right) w(z) = 0.$$

Therefore we obtain Kummer's differential equation with parameters $a = \frac{1}{2} + Jb_5 - J^2 a_5$ and $b = \frac{1}{2}$ and thus

$$F(p) = K_5 e^{-\frac{p^2}{4D}} M\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{p^2}{4D}\right) + L_5 e^{-\frac{p^2}{4D}} U\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{p^2}{4D}\right).$$

The invariant solution of original equation (3.1) reads

$$u_5(x,t) = (t + Ja_5)^{(Jb_5 - J^2 a_5)} e^{-\frac{r(t)^2 x^2}{4D(t+Ja_5)} + Jt} \left[ K_5 M\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{r(t)^2 x^2}{4D(t + Ja_5)}\right) + \right.$$
$$\left. + L_5 U\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{r(t)^2 x^2}{4D(t + Ja_5)}\right)\right].$$

Again we use other symmetries to generate new solutions as above:

1. $X_2$ :

$$\tilde{u}_5(x,t) = (t + Ja_5)^{(Jb_5 - J^2 a_5)} e^{-\frac{(xr(t)+2c_5 Dt)^2}{4D(t+Ja_5)} + c_5 r(t) x + (Dc_5^2 + J)t} \times$$
$$\times \left[ K_5 M\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{(xr(t) + 2c_5 Dt)^2}{4D(t + Ja_5)}\right) + \right.$$
$$\left. + L_5 U\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{(xr(t) + 2c_5 Dt)^2}{4D(t + Ja_5)}\right)\right],$$

where $c_5$ is an arbitrary constant.

2. $X_6$ :

$$\tilde{\tilde{u}}_5(x,t) = (t + Ja_5)^{(Jb_5 - J^2 a_5)} e^{-\frac{(xr(t)+2c_5 Dt+d_5)^2}{4D(t+Ja_5)} + c_5 r(t) x + (Dc_5^2 + J)t} \times$$
$$\times \left[ \tilde{K}_5 M\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{(xr(t) + 2c_5 Dt + d_5)^2}{4D(t + Ja_5)}\right) + \right.$$
$$\left. + \tilde{L}_5 U\left(\frac{1}{2} + Jb_5 - J^2 a_5, \frac{1}{2}, \frac{(xr(t) + 2c_5 Dt + d_5)^2}{4D(t + Ja_5)}\right)\right].$$

where $\tilde{K}_5 = K_5 e^{c_5 d_5}, \tilde{L}_5 = L_5 e^{c_5 d_5}$ and $d_5$ is an arbitrary parameter. Note that $u_5, \tilde{u}_5$ are special cases of $\tilde{\tilde{u}}_5$.

#### 3.2.5.1 Asymptotics of the solution $u_5(x,t)$

Due to the complexity of the solution we expand $u_5(x,t)$, resp. $\tilde{\tilde{u}}_5(x,t)$, in the asymptotic regime $t \to +\infty$. We assume two cases:

1. $r(t) = at^k, k \in \mathbb{Z}, k \geq \frac{1}{2}, a = \text{const}$.

   In this case
$$z = \frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)} \xrightarrow{t\to+\infty} +\infty.$$
   Using known results about behaviour of Kummer's functions for $z \in \mathbb{R}, z \to +\infty$ [12]:
$$\frac{M(a,b,z)}{\Gamma(b)} = \frac{e^{i\pi a}z^{-a}}{\Gamma(b-a)}\left[\sum_{n=0}^{R-1}\frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + \mathcal{O}(z^{-R})\right] + \\ + \frac{e^z z^{a-b}}{\Gamma(a)}\left[\sum_{n=0}^{S-1}\frac{(b-a)_n(1-a)_n}{n!}z^{-n} + \mathcal{O}(z^{-S})\right],$$
$$U(a,b,z) = z^{-a}\left[\sum_{n=0}^{R-1}\frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + \mathcal{O}(z^{-R})\right],$$
   we get in our case for $t \to +\infty$
$$M\left(\frac{1}{2} + Jb_5 - J^2a_5, \frac{1}{2}, \frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right) \sim \frac{\Gamma(1/2)e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5 + J^2a_5)}\times \\ \times\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)} + \\ + \frac{\Gamma(1/2)}{\Gamma(1/2 + Jb_5 - J^2a_5)}\exp\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)^{(Jb_5-J^2a_5)},$$
$$U\left(\frac{1}{2} + Jb_5 - J^2a_5, \frac{1}{2}, \frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right) \sim \left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)},$$
   and thus
$$\tilde{\tilde{u}}_5(x,t) \sim (t + Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)} + c_5r(t)x + (Dc_5^2+J)t}\times \\ \times\left[\tilde{K}_5\frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5 + J^2a_5)}\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)} + \right. \\ + \tilde{K}_5\frac{\sqrt{\pi}}{\Gamma(1/2 + Jb_5 - J^2a_5)}\exp\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)\times \\ \left.\times\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)^{(Jb_5-J^2a_5)} + \tilde{L}_5\left(\frac{(xr(t) + 2c_5Dt + d_5)^2}{4D(t + Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right] =: \\ =: u_{A\infty}(x,t).$$

2. $r(t) = at^k, k \in \mathbb{Z}, k < \frac{1}{2}, a = \text{const}$.

   Now in this case we have instead
$$z = \frac{r(t)^2x^2}{4D(t + Ja_5)} \xrightarrow{t\to+\infty} 0.$$
   We again employ the knowledge of asymptotic behaviour for $z \in \mathbb{R}, z \to 0$ while $b = \frac{1}{2}$ [12]:
$$M(a,b,z) = 1,$$

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \mathcal{O}(|z|^{1/2}),$$

which implies for $t \to +\infty$

$$u_5(x,t) \sim (t+Ja_5)^{(Jb_5 - J^2 a_5)} e^{-\frac{r(t)^2 x^2}{4D(t+Ja_5)} + Jt} \left[ \tilde{K}_5 + \tilde{L}_5 \frac{\sqrt{\pi}}{\Gamma(1+Jb_5 - J^2 a_5)} \right] =: u_{A0}(x,t).$$

## 3.3 Boundary conditions

In this section we would like to find invariant solutions which satisfy the boundary conditions, in particular Dirichlet, Neumann, Robin and periodic boundary conditions, see Section 2.6. The aim is to gain some qualitative insight into the problem's behaviour via invariant solutions.

### 3.3.1 Dirichlet boundary conditions

First we are interested in the solutions $u_D(x,t)$ which satisfy Dirichlet boundary conditions

$$u_D(0,t) = u_D(1,t) = 0.$$

We use this condition for a general invariant solution, i.e. a linear combination of all the identified above:

$$\begin{aligned}
u(x,t) = {} & K_1 e^{\sqrt{c_1}(\sqrt{R_1}+b_1)xr(t) + c_1(Db_1^2 + 2\sqrt{R_1}Db_1 + a_1 - J)t + Jt} + \\
& + K_3 \exp\left[ c_3 \left[ \frac{a_3 - 2Db_3}{2D} \left( a_3 - \sqrt{a_3^2 - 4DJ} \right) + Db_3^2 - J \right] t + Jt + \right. \\
& \left. + \sqrt{c_3} \left( b_3 - \frac{a_3 - \sqrt{a_3^2 - 4DJ}}{2D} \right) xr(t) \right] + \\
& + L_3 \exp\left[ c_3 \left[ \frac{a_3 - 2Db_3}{2D} \left( a_3 + \sqrt{a_3^2 - 4DJ} \right) + Db_3^2 - J \right] t + Jt + \right. \\
& \left. + \sqrt{c_3} \left( b_3 - \frac{a_3 + \sqrt{a_3^2 - 4DJ}}{2D} \right) xr(t) \right] + \\
& + \frac{\tilde{K}_4}{\sqrt{2Dc_4 t + \tilde{b}_4}} e^{\left( -\frac{c_4 r^2(t) x^2}{2(2Dc_4 t + \tilde{b}_4)} - \frac{a_4 \sqrt{c_4} r(t) x}{2Dc_4 t + \tilde{b}_4} - \frac{a_4^2}{2(2Dc_4 t + \tilde{b}_4)} + Jt \right)}, \\
u(0,t) = {} & K_1 e^{c_1(Db_1^2 + 2\sqrt{R_1}Db_1 + a_1 - J)t + Jt} + \\
& + K_3 \exp\left[ c_3 \left[ \frac{a_3 - 2Db_3}{2D} \left( a_3 - \sqrt{a_3^2 - 4DJ} \right) + Db_3^2 - J \right] t + Jt \right] + \\
& + L_3 \exp\left[ c_3 \left[ \frac{a_3 - 2Db_3}{2D} \left( a_3 + \sqrt{a_3^2 - 4DJ} \right) + Db_3^2 - J \right] t + Jt \right] + \\
& + \frac{\tilde{K}_4}{\sqrt{2Dc_4 t + \tilde{b}_4}} e^{\left( -\frac{a_4^2}{2(2Dc_4 t + \tilde{b}_4)} + Jt \right)} + \frac{\tilde{K}_4}{\sqrt{2Dc_4 t + \tilde{b}_4}} e^{\left( -\frac{a_4^2}{2(2Dc_4 t + \tilde{b}_4)} + Jt \right)} \stackrel{!}{=} 0 \\
u(1,t) = {} & K_1 e^{\sqrt{c_1}(\sqrt{R_1}+b_1)r(t) + c_1(Db_1^2 + 2\sqrt{R_1}Db_1 + a_1 - J)t + Jt} + \\
& + K_3 \exp\left[ c_3 \left[ \frac{a_3 - 2Db_3}{2D} \left( a_3 - \sqrt{a_3^2 - 4DJ} \right) + Db_3^2 - J \right] t + Jt + \right.
\end{aligned}$$

$$+\sqrt{c_3}\left(b_3-\frac{a_3-\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\Bigg]+$$
$$+L_3\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3+\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt+\right.$$
$$\left.+\sqrt{c_3}\left(b_3-\frac{a_3+\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\right]\stackrel{!}{=}0.$$

Because this conditions must be satisfied for all $t$ first of them implies $\tilde{K}_4=0$. In the case, when two or three of the remaining solutions have the same coefficients before $r(t)$ and $t$ in the exponential part, sum of coefficients before them must be zero again as the boundary conditions have to be met, which implies zero solution. Hence we can reduce this problem and require boundary conditions for every invariant solution separately. Its easy to see that only zero solution satisfies Dirichlet boundary conditions, and thus

$$u_D(x,t)=0.$$

As the last step we inspect the possibility of Kummer's functions satisfying the boundary conditions via the asymptotic regime for large $t$:

$$u_{A\infty}(x,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}+c_5r(t)x+(Dc_5^2+J)t}\times$$
$$\times\left[\tilde{K}_5\frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}+\right.$$
$$+\tilde{K}_5\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)}\exp\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)}+$$
$$\left.+\tilde{L}_5\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right],$$

and hence at the boundary

$$u_{A\infty}(0,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}+(Dc_5^2+J)t}\times$$
$$\times\left[\tilde{K}_5\frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}+\right.$$
$$+\tilde{K}_5\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)}\exp\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)}+$$
$$\left.+\tilde{L}_5\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right]\stackrel{!}{=}0,$$

$$u_{A\infty}(1,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}+c_5r(t)+(Dc_5^2+J)t}\times$$
$$\times\left[\tilde{K}_5\frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}+\right.$$
$$+\tilde{K}_5\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)}\exp\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\times$$

$$\times\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)}+$$
$$\left.+\tilde{L}_5\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right]\stackrel{!}{=}0,$$

which again implies

$$u_{DA\infty}(x,t)=0.$$

The second case when growth is slower the square root of $t$

$$u_{A0}(x,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{r(t)^2x^2}{4D(t+Ja_5)}+Jt}\left[\tilde{K}_5+\tilde{L}_5\frac{\sqrt{\pi}}{\Gamma(1+Jb_5-J^2a_5)}\right].$$

We have

$$u_{A0}(0,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{Jt}\left[\tilde{K}_5+\tilde{L}_5\frac{\sqrt{\pi}}{\Gamma(1+Jb_5-J^2a_5)}\right]\stackrel{!}{=}0,$$
$$u_{A0}(1,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{r(t)^2x^2}{4D(t+Ja_5)}+Jt}\left[\tilde{K}_5+\tilde{L}_5\frac{\sqrt{\pi}}{\Gamma(1+Jb_5-J^2a_5)}\right]\stackrel{!}{=}0,$$

and again

$$u_{DA0}(x,t)=0.$$

### 3.3.2 Neumann boundary conditions

Now we are interested in solutions $u_N(x,t)$ which satisfy Neumann boundary conditions

$$\partial_x u_N(0,t)=\partial_x u_N(1,t)=0. \tag{3.17}$$

We seek the invariant solutions that satisfy these boundary conditions:

$$\partial_x u(x,t)=K_1\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)e^{\sqrt{c_1}(\sqrt{R_1}+b_1)xr(t)+c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt}+$$
$$+K_3\sqrt{c_3}\left(b_3-\frac{a_3-\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\times$$
$$\times\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3-\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+\right.$$
$$\left.+Jt+\sqrt{c_3}\left(b_3-\frac{a_3-\sqrt{a_3^2-4DJ}}{2D}\right)xr(t)\right]+$$
$$+L_3\sqrt{c_3}\left(b_3-\frac{a_3+\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3+\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+\right.$$
$$\left.+Jt+\sqrt{c_3}\left(b_3-\frac{a_3+\sqrt{a_3^2-4DJ}}{2D}\right)xr(t)\right]+$$
$$+\frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}(-c_4r^2(t)x-a_4\sqrt{c_4}r(t))e^{\left(-\frac{c_4r^2(t)x^2}{2(2Dc_4t+\tilde{b}_4)}-\frac{a_4\sqrt{c_4}r(t)x}{2Dc_4t+\tilde{b}_4}-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)}$$

and hence

$$\begin{aligned}
\partial_x u(0,t) &= K_1\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)e^{\sqrt{c_1}(c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt}+\\
&+ K_3\sqrt{c_3}\left(b_3-\frac{a_3-\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\times\\
&\times \exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3-\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt\right]+\\
&+ L_3\sqrt{c_3}\left(b_3-\frac{a_3+\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\times\\
&\times \exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3+\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt\right]+\\
&+ \frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}(-a_4\sqrt{c_4}r(t))e^{\left(-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)} \stackrel{!}{=} 0,\\
\partial_x u(1,t) &= K_1\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)e^{\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)+c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt}+\\
&+ K_3\sqrt{c_3}\left(b_3-\frac{a_3-\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\times\\
&\times \exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3-\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt+\right.\\
&\left.+\sqrt{c_3}\left(b_3-\frac{a_3-\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\right]+L_3\sqrt{c_3}\left(b_3-\frac{a_3+\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\times\\
&\times \exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3+\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt+\right.\\
&\left.+\sqrt{c_3}\left(b_3-\frac{a_3+\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\right]+\\
&+ \frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}(-c_4r^2(t)-a_4\sqrt{c_4}r(t))e^{\left(-\frac{c_4r^2(t)}{2(2Dc_4t+\tilde{b}_4)}-\frac{a_4\sqrt{c_4}r(t)}{2Dc_4t+\tilde{b}_4}-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)} \stackrel{!}{=} 0,
\end{aligned}$$

The factor before $r^2(t)$ in the second condition is $-\frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}c_4$ and thus $\tilde{K}_4 \stackrel{!}{=} 0$ or $c_4 \stackrel{!}{=} 0$. The second one implies non-trivial solution

$$u(x,t) = \frac{\tilde{K}_4}{\tilde{b}_4^{3/2}}e^{-\frac{a_4^2}{2\tilde{b}_4}+Jt} = K_4' e^{Jt},$$

where $K_4'$ is a new arbitrary constant.

Other solutions can be considered separately as discussed above, because it is easy to see that the equality of exponential functions implies the equality of their prefactors up to the arbitrary parameters $K_1, K_3, L_3$ and it is the same situation as for Dirichlet boundary conditions. For the individual solutions we obtain:

$$K_1\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)e^{\sqrt{c_1}(c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt} \stackrel{!}{=} 0 \implies \sqrt{c_1}(\sqrt{R_1}+b_1) = 0$$

$$
\begin{aligned}
& c_1 = 0 \implies u(x,t) = K_1 e^{Jt}, \\
& b_1 = -\sqrt{R_1} \implies (Db_1^2 + 2\sqrt{R_1} Db_1 + a_1 - J) = 0 \implies u(x,t) = K_1 e^{Jt}, \\
& \sqrt{c_3}\left(b_3 - \frac{a_3 \pm \sqrt{a_3^2 - 4DJ}}{2D}\right) r(t) e^{c_3\left[\frac{a_3 - 2Db_3}{2D}\left(a_3 \pm \sqrt{a_3^2-4DJ}\right) + Db_3^2 - J\right]t + Jt} \overset{!}{=} 0 \\
& \quad \implies \sqrt{c_3}\left(b_3 - \frac{a_3 \pm \sqrt{a_3^2 - 4DJ}}{2D}\right) = 0 \\
& c_3 = 0 \implies u(x,t) = K_3 e^{Jt}, \text{ resp. } L_3 e^{Jt}, \\
& b_3 = \frac{a_3 \pm \sqrt{a_3^2 - 4DJ}}{2D} \implies \frac{a_3 - 2Db_3}{2D}\left(a_3 \pm \sqrt{a_3^2 - 4DJ}\right) + Db_3^2 - J = 0, \\
& \quad \implies u(x,t) = K_3 e^{Jt}, \text{ resp. } L_3 e^{Jt},
\end{aligned}
$$

and thus the invariant solution which satisfies Neumann boundary conditions has form

$$
u_N(x,t) = Ce^{Jt},
$$

where $C$ is an arbitrary constant. Again as the last step we consider Kummer's as the invariant solutions in the large time asymptotics:

$$
\begin{aligned}
& \partial_x u_{A\infty}(x,t) = \left(-r(t)\frac{(xr(t) + 2c_5Dt + d_5)}{2D(t+Ja_5)} + c_5 r(t)\right)(t+Ja_5)^{(Jb_5 - J^2a_5)} \times \\
& \times e^{-\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)} + c_5 r(t) x + (Dc_5^2 + J)t} \times \\
& \times \Bigg[\tilde{K}_5 \frac{\sqrt{\pi} e^{i\pi(1/2 + Jb_5 - J^2a_5)}}{\Gamma(-Jb_5 + J^2a_5)}\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2 - Jb_5 + J^2a_5)} + \\
& + \tilde{K}_5 \frac{\sqrt{\pi}}{\Gamma(1/2 + Jb_5 - J^2a_5)} \exp\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5 - J^2a_5)} + \\
& + \tilde{L}_5 \left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2 - Jb_5 + J^2a_5)}\Bigg] + \\
& (t+Ja_5)^{(Jb_5 - J^2a_5)} e^{-\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)} + c_5 r(t) x + (Dc_5^2 + J)t} \times \\
& \times \Bigg[\tilde{K}_5 \frac{\sqrt{\pi} e^{i\pi(1/2 + Jb_5 - J^2a_5)}}{\Gamma(-Jb_5 + J^2a_5)}\left(r(t)(-1 - 2Jb_5 + 2J^2a_5)\frac{(xr(t)+2c_5Dt+d_5)^{(-2-2Jb_5+2J^2a_5)}}{4D(t+Ja_5)^{(-1/2-Jb_5+J^2a_5)}}\right) + \\
& + \tilde{K}_5 r(t) \frac{(xr(t)+2c_5Dt+d_5)}{2D(t+Ja_5)} \frac{\sqrt{\pi}}{\Gamma(1/2 + Jb_5 - J^2a_5)} \exp\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right) \times \\
& \times \left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5 - J^2a_5)} + \\
& + \tilde{K}_5 \frac{\sqrt{\pi}}{\Gamma(1/2 + Jb_5 - J^2a_5)} \exp\left(\frac{(xr(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right) \times \\
& \times \left(2(Jb_5 - J^2a_5) r(t) \frac{(xr(t)+2c_5Dt+d_5)^{(2Jb_5 - 2J^2a_5 - 1)}}{4D(t+Ja_5)^{(Jb_5 - J^2a_5)}}\right) + \\
& + \tilde{L}_5 \left((-1 - 2Jb_5 + 2J^2a_5) r(t) \frac{(xr(t)+2c_5Dt+d_5)^{(-2-2Jb_5+2J^2a_5)}}{4D(t+Ja_5)^{(-1/2-Jb_5+J^2a_5)}}\right)\Bigg].
\end{aligned}
$$

The left boundary terms then reads

$$\partial_x u_{A\infty}(0,t) = \left(-r(t)\frac{(2c_5Dt+d_5)}{2D(t+Ja_5)} + c_5 r(t)\right)(t+Ja_5)^{(Jb_5-J^2a_5)} e^{-\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}+(Dc_5^2+J)t} \times$$
$$\times \left[\tilde{K}_5 \frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)} + \right.$$
$$+ \tilde{K}_5 \frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)} \exp\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)} +$$
$$\left. + \tilde{L}_5\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right] +$$
$$(t+Ja_5)^{(Jb_5-J^2a_5)} e^{-\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}+(Dc_5^2+J)t} \times$$
$$\times \left[\tilde{K}_5 \frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(r(t)(-1-2Jb_5+2J^2a_5)\frac{(2c_5Dt+d_5)^{(-2-2Jb_5+2J^2a_5)}}{4D(t+Ja_5)^{(-1/2-Jb_5+J^2a_5)}}\right) + \right.$$
$$+ \tilde{K}_5 r(t)\frac{(2c_5Dt+d_5)}{2D(t+Ja_5)}\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)} \exp\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)}$$
$$+ \tilde{K}_5 \frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)} \exp\left(\frac{(2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right) \times$$
$$\times \left(2(Jb_5-J^2a_5)r(t)\frac{(2c_5Dt+d_5)^{(2Jb_5-2J^2a_5-1)}}{4D(t+Ja_5)^{(Jb_5-J^2a_5)}}\right) +$$
$$\left. + \tilde{L}_5\left((-1-2Jb_5+2J^2a_5)r(t)\frac{(2c_5Dt+d_5)^{(-2-2Jb_5+2J^2a_5)}}{4D(t+Ja_5)^{(-1/2-Jb_5+J^2a_5)}}\right)\right] \stackrel{!}{=} 0,$$

and the condition on the right is

$$\partial_x u_{A\infty}(1,t) = \left(-r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)} + c_5 r(t)\right)(t+Ja_5)^{(Jb_5-J^2a_5)} \times$$
$$\times e^{-\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}+c_5r(t)+(Dc_5^2+J)t} \times$$
$$\times \left[\tilde{K}_5 \frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)} + \right.$$
$$+ \tilde{K}_5 \frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)} \exp\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)} +$$
$$\left. + \tilde{L}_5\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right] +$$
$$(t+Ja_5)^{(Jb_5-J^2a_5)} e^{-\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}+c_5r(t)+(Dc_5^2+J)t} \times$$
$$\times \left[\tilde{K}_5 \frac{\sqrt{\pi}e^{i\pi(1/2+Jb_5-J^2a_5)}}{\Gamma(-Jb_5+J^2a_5)}\left(r(t)(-1-2Jb_5+2J^2a_5)\frac{(r(t)+2c_5Dt+d_5)^{(-2-2Jb_5+2J^2a_5)}}{4D(t+Ja_5)^{(-1/2-Jb_5+J^2a_5)}}\right) + \right.$$
$$+ \tilde{K}_5 r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)}\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)} \exp\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right) \times$$

$$\times\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)}+$$
$$+\tilde{K}_5\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)}\exp\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\times$$
$$\times\left(2(Jb_5-J^2a_5)r(t)\frac{(r(t)+2c_5Dt+d_5)^{(2Jb_5-2J^2a_5-1)}}{4D(t+Ja_5)^{(Jb_5-J^2a_5)}}\right)+$$
$$+\tilde{L}_5\left((-1-2Jb_5+2J^2a_5)r(t)\frac{(r(t)+2c_5Dt+d_5)^{(-2-2Jb_5+2J^2a_5)}}{4D(t+Ja_5)^{(-1/2-Jb_5+J^2a_5)}}\right)\Bigg]\stackrel{!}{=}0.$$

Now we focus on the terms which include $\exp\left(\frac{(r(t)+2c_5Dt)^2}{4D(t+Ja_5)}\right)$ :

$$(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}+c_5r(t)+(Dc_5^2+J)t}\tilde{K}_5\frac{\sqrt{\pi}}{\Gamma(1/2+Jb_5-J^2a_5)}\times$$
$$\times\exp\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(Jb_5-J^2a_5)}\times$$
$$\times\left[\left(-r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)}+c_5r(t)\right)+r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)}+\frac{2(Jb_5-J^2a_5)r(t)}{(r(t)+2c_5Dt+d_5)}\right]$$
$$\stackrel{!}{=}0$$

and hence $\tilde{K}_5\stackrel{!}{=}0$ or

$$\left[\left(-r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)}+c_5r(t)\right)+r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)}+\frac{2(Jb_5-J^2a_5)r(t)}{(r(t)+2c_5Dt+d_5)}\right]=$$
$$=r(t)\frac{[-d_5+c_52D(t+Ja_5)](r(t)+2c_5Dt+d_5)+2(Jb_5-J^2a_5)2D(t+Ja_5)}{2D(t+Ja_5)(r(t)+2c_5Dt+d_5)}\stackrel{!}{=}0,$$

which requires $c_5=0\wedge(Jb_5-J^2a_5)=0\wedge d_5=0$. As all these terms have to vanish which implies

$$\Longrightarrow\ M(a,b,z)=M(1/2,1/2,z)=e^z.$$

The first case, $\tilde{K}_5$, gives

$$\partial_x u_{A\infty}(1,t)=(t+Ja_5)^{(Jb_5-J^2a_5)}e^{-\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}+c_5r(t)+(Dc_5^2+J)t}\times$$
$$\times\left[\tilde{L}_5\left(\frac{(r(t)+2c_5Dt+d_5)^2}{4D(t+Ja_5)}\right)^{(-1/2-Jb_5+J^2a_5)}\right]\times$$
$$\times\left(-r(t)\frac{(r(t)+2c_5Dt+d_5)}{2D(t+Ja_5)}+c_5r(t)-\frac{(1+2Jb_5-2J^2a_5)r(t)}{(r(t)+2c_5Dt+d_5)}\right)\neq0.$$

while the second case, $c_5=0\wedge(Jb_5-J^2a_5)=0\wedge d_5=0$, yields

$$u_{A\infty}(x,t)=e^{-\frac{(xr(t))^2}{4D(t+Ja_5)}+Jt}\left[\tilde{K}_5e^{\frac{(xr(t))^2}{4D(t+Ja_5)}}+\tilde{L}_5\left(\frac{(xr(t))^2}{4D(t+Ja_5)}\right)^{-1/2}\right]=$$
$$=\tilde{K}_5e^{Jt}+\tilde{L}_5e^{-\frac{(xr(t))^2}{4D(t+Ja_5)}+Jt}\left(\frac{(xr(t))}{(4D(t+Ja_5))^{-1/2}}\right),$$

$$
\begin{aligned}
&\partial_x u_{A\infty}(x,t) = -\tilde{L}_5 r(t)\frac{(xr(t))}{2D(t+Ja_5)} e^{-\frac{(xr(t))^2}{4D(t+Ja_5)}+Jt}\left(\frac{(xr(t))}{(4D(t+Ja_5))^{-1/2}}\right)+\\
&+\tilde{L}_5 e^{-\frac{(xr(t))^2}{4D(t+Ja_5)}+Jt}\left(\frac{r(t)}{(4D(t+Ja_5))^{-1/2}}\right),\\
&\partial_x u_{A\infty}(0,t) = \tilde{L}_5\left(\frac{r(t)}{(4D(t+Ja_5))^{-1/2}}\right) \overset{!}{=} 0,\\
&\partial_x u_{A\infty}(1,t) = -\tilde{L}_5 r(t)\frac{(r(t))}{2D(t+Ja_5)} e^{-\frac{r^2(t)}{4D(t+Ja_5)}+Jt}\left(\frac{r(t)}{(4D(t+Ja_5))^{-1/2}}\right)+\\
&+\tilde{L}_5 e^{-\frac{r^2(t)}{4D(t+Ja_5)}+Jt}\left(\frac{r(t)}{(4D(t+Ja_5))^{-1/2}}\right) \overset{!}{=} 0
\end{aligned}
$$

and hence

$$u_{NA\infty}(x,t) = \tilde{K}_5 e^{Jt}.$$

Finally for slow growth (slower than $\sqrt{t}$) we have

$$
\begin{aligned}
&\partial_x u_{A0}(x,t) = -\frac{2r(t)^2 x}{4D}(t+Ja_5)^{(Jb_5-J^2a_5)-1} e^{-\frac{r(t)^2x^2}{4D(t+Ja_5)}+Jt}\times\\
&\times\left[\tilde{K}_5+\tilde{L}_5\frac{\sqrt{\pi}}{\Gamma(1+Jb_5-J^2a_5)}\right],\\
&\partial_x u_{A0}(0,t) = 0,\\
&\partial_x u_{A0}(1,t) = -\frac{2r(t)^2}{4D}(t+Ja_5)^{(Jb_5-J^2a_5)-1} e^{-\frac{r(t)^2}{4D(t+Ja_5)}+Jt}\times\\
&\times\left[\tilde{K}_5+\tilde{L}_5\frac{\sqrt{\pi}}{\Gamma(1+Jb_5-J^2a_5)}\right] \overset{!}{=} 0
\end{aligned}
$$

and thus

$$u_{NA0}(x,t) = 0$$

is the only viable invariant solution satisfying Neumann boundary conditions.

### 3.3.3 Robin boundary conditions

We would like to find solutions $u_R(x,t)$ which satisfy Robin boundary conditions

$$\frac{1}{\tilde{L}r(t)}\partial_x u_R(0,t) - \alpha(t)u_R(0,t) = \frac{1}{\tilde{L}r(t)}\partial_x u_R(1,t) + \alpha(t)u_R(1,t) = 0. \tag{3.18}$$

For the same reasons as above we can consider the invariant solutions separately. The first invariant solution (3.13) reads

$$
\begin{aligned}
&K_1\left(\frac{\sqrt{c_1}}{\tilde{L}}(\sqrt{R_1}+b_1)-\alpha\right)e^{\sqrt{c_1}(c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt} \overset{!}{=} 0,\\
&K_1\left(\frac{\sqrt{c_1}}{\tilde{L}}(\sqrt{R_1}+b_1)+\alpha\right)e^{\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)+c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt} \overset{!}{=} 0\\
&\implies u_{R1}(x,t) = 0.
\end{aligned}
$$

The second invariant solution (3.14) gives

$$C\left[\frac{\sqrt{c_3}}{\tilde{L}}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)-\alpha\right]\times$$
$$\times\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3\pm\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt\right]\overset{!}{=}0,$$
$$C\left[\frac{\sqrt{c_3}}{\tilde{L}}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)+\alpha\right]\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3\pm\sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+\right.$$
$$\left.+Jt+\sqrt{c_3}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\right]\overset{!}{=}0,$$
$$\implies u_{R3}(x,t)=0.$$

And finally the third solution (3.15) yields:

$$\frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}\left(-\frac{a_4\sqrt{c_4}}{\tilde{L}}-\alpha\right)e^{\left(-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)}\overset{!}{=}0,$$
$$\frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}\left(-\frac{c_4}{\tilde{L}}r(t)-\frac{a_4\sqrt{c_4}}{\tilde{L}}+\alpha\right)e^{\left(-\frac{c_4r^2(t)}{2(2Dc_4t+\tilde{b}_4)}-\frac{a_4\sqrt{c_4}r(t)}{2Dc_4t+\tilde{b}_4}-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)}\overset{!}{=}0,$$
$$\implies u_{R4}(x,t)=0.$$

### 3.3.4 Periodic boundary conditions

Finally we would like to find the solutions $u_P(x,t)$ which satisfy periodic boundary conditions

$$u_P(0,t)=u_P(1,t),\text{ and }\partial_x u_P(0,t)=\partial_x u_P(1,t).$$

As above we can analyze the solutions separately:

- $u_1(x,t)$ from (3.13):

$$u_1(0,t)=u_1(1,t)\iff$$
$$K_1e^{c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt}=K_1e^{\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)+c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt},$$
$$\partial_x u_1(0,t)=\partial_x u_1(1,t)\iff$$
$$K_1\sqrt{c_1}(\sqrt{R_1}+b_1)xr(t)e^{c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt}=$$
$$=K_1\sqrt{c_1}(\sqrt{R_1}+b_1)xr(t)e^{\sqrt{c_1}(\sqrt{R_1}+b_1)r(t)+c_1(Db_1^2+2\sqrt{R_1}Db_1+a_1-J)t+Jt}$$

yielding

$$u_P(x,t)=K_1e^{Jt}$$

for $c_1=0$ or

$$u_P(x,t)=K_1e^{Jt}$$

for $b_1=-\sqrt{R_1}(\implies Db_1^2+2\sqrt{R_1}Db_1+a_1-J=0)$.

- $u_2(x,t)$ from (3.14):

$$u_2(0,t) = u_2(1,t) \iff$$
$$C\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3 \pm \sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt\right] =$$
$$= C\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3 \pm \sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt+\right.$$
$$\left.+\sqrt{c_3}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\right],$$
$$\partial_x u_2(0,t) = \partial_x u_2(1,t) \iff$$
$$C\sqrt{c_3}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)\times$$
$$\times\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3 \pm \sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt\right] =$$
$$= C\sqrt{c_3}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)\times$$
$$\times\exp\left[c_3\left[\frac{a_3-2Db_3}{2D}\left(a_3 \pm \sqrt{a_3^2-4DJ}\right)+Db_3^2-J\right]t+Jt+\right.$$
$$\left.+\sqrt{c_3}\left(b_3-\frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}\right)r(t)\right]$$

implying
$$u_P(x,t) = K_3e^{Jt},\ \text{resp. } L_3e^{Jt},$$
for $c_3 = 0$ or
$$u_P(x,t) = K_3e^{Jt},\ \text{resp. } L_3e^{Jt},$$
for $b_3 = \frac{a_3\pm\sqrt{a_3^2-4DJ}}{2D}$ $\left(\implies \frac{a_3-2Db_3}{2D}\left(a_3\pm\sqrt{a_3^2-4DJ}\right)+Db_3^2-J=0\right)$.

- $u_3(x,t)$ from (3.15):

$$u_3(0,t) = u_3(1,t) \iff$$
$$\frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}e^{\left(-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)} = \frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}e^{\left(-\frac{c_4r^2(t)}{2(2Dc_4t+\tilde{b}_4)}-\frac{a_4\sqrt{c_4}r(t)}{2Dc_4t+\tilde{b}_4}-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)}$$
$$\partial_x u_3(0,t) = \partial_x u_3(1,t) \iff$$
$$\frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}(-a_4\sqrt{c_4})e^{\left(-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)} =$$
$$= \frac{\tilde{K}_4}{(2Dc_4t+\tilde{b}_4)^{3/2}}(-c_4r(t)-a_4\sqrt{c_4})e^{\left(-\frac{c_4r^2(t)}{2(2Dc_4t+\tilde{b}_4)}-\frac{a_4\sqrt{c_4}r(t)}{2Dc_4t+\tilde{b}_4}-\frac{a_4^2}{2(2Dc_4t+\tilde{b}_4)}+Jt\right)}$$

yielding
$$u_P(x,t) = K_4'e^{Jt}$$
for $c_4 = 0$, where $K_4' = \frac{\tilde{K}_4}{\tilde{b}_4^{3/2}}e^{-\frac{a_4^2}{2\tilde{b}_4}}$.

# Chapter 4

# Two component system of apical growth

In this section we are interested in the two component system of the apical growth. This problem is described by the system of two equations

$$F_u = \frac{1}{r^2(t)} u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v - u_t, \tag{4.1}$$

$$F_v = \frac{D}{r^2(t)} v_{xx} + x\varphi(t)v_x + J_3 u + J_4 v - v_t, \tag{4.2}$$

where $\varphi(t) = \frac{\dot{r}(t)}{r(t)}$, see (2.15).

## 4.1 Symmetries and their infinitesimal generators

As in the scalar case first we would like to find the symmetries of this system of equations. The prolonged infinitesimal generators acting on them are

$$X^{(2)}(F_u) = -2\frac{\varphi(t)}{r^2(t)}\tau u_{xx} + \frac{1}{r^2(t)}\eta^{xx} + \varphi(t)\xi u_x + x\dot{\varphi}(t)\tau u_x + x\varphi(t)\eta^x + J_1\eta + J_2\vartheta - \eta^t,$$

$$X^{(2)}(F_v) = -2\frac{D\varphi(t)}{r^2(t)}\tau v_{xx} + \frac{D}{r^2(t)}\vartheta^{xx} + \varphi(t)\xi v_x + x\dot{\varphi}(t)\tau v_x + x\varphi(t)\vartheta^x + J_3\eta + J_4\vartheta - \vartheta^t.$$

Now we use the linearized symmetry condition to obtain the conditions for unknown functions describing the infinitesimal generators of symmetries:

$$X^{(2)}(F_u) = -2\frac{\varphi}{r^2}\tau u_{xx} + \frac{1}{r^2}\eta_{xx} + \frac{1}{r^2}\left(2\eta_{xu} - \xi_{xx}\right)u_x - \frac{1}{r^2}\tau_{xx}\left(\frac{D}{r^2(t)}u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v\right)$$

$$+ \frac{1}{r^2}(\eta_{uu} - 2\xi_{xu})u_x^2 - 2\frac{1}{r^2}\tau_{xu}u_x\left(\frac{1}{r^2(t)}u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v\right) - \frac{1}{r^2}\xi_{uu}u_x^3 -$$

$$- \frac{1}{r^2}\tau_{uu}u_x^2\left(\frac{1}{r^2(t)}u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v\right) + \frac{1}{r^2}(\eta_u - 2\xi_x)u_{xx} - 2\frac{1}{r^2}\tau_x u_{xt} - 3\frac{1}{r^2}\xi_u u_x u_{xx} -$$

$$- \frac{1}{r^2}\tau_u u_{xx}\left(\frac{1}{r^2(t)}u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v\right) - 2\frac{1}{r^2}\tau_u u_x u_{xt} + \varphi\xi u_x + x\dot{\varphi}\tau u_x + x\varphi\eta_x +$$

$$+ x\varphi(\eta_u - \xi_x)u_x - x\varphi\tau_x\left(\frac{1}{r^2(t)}u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v\right) - x\varphi\xi_u u_x^2 -$$

$$x\varphi\tau_u u_x\left(\frac{1}{r^2(t)}u_{xx} + x\varphi(t)u_x + J_1 u + J_2 v\right) + J_1\eta + J_2\vartheta - \eta_t + \xi_t u_x -$$

$$- (\eta_u - \tau_t) \left( \frac{1}{r^2(t)} u_{xx} + x\varphi(t) u_x + J_1 u + J_2 v \right) +$$
$$+ \xi_u u_x \left( \frac{1}{r^2(t)} u_{xx} + x\varphi(t) u_x + J_1 u + J_2 v \right) + \tau_u \left( \frac{1}{r^2(t)} u_{xx} + x\varphi(t) u_x + J_1 u + J_2 v \right)^2 = 0,$$

$$X^{(2)}(F_v) = -2D\frac{\varphi}{r^2}\tau v_{xx} + \frac{D}{r^2}\vartheta_{xx} + \frac{D}{r^2}(2\vartheta_{xv} - \xi_{xx}) v_x - \frac{D}{r^2}\tau_{xx} \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right)$$
$$+ \frac{D}{r^2}(\vartheta_{vv} - 2\xi_{xv}) v_x^2 - 2\frac{D}{r^2}\tau_{xv} v_x \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) - \frac{D}{r^2}\xi_{vv} v_x^3 -$$
$$- \frac{D}{r^2}\tau_{vv} v_x^2 \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) + \frac{D}{r^2}(\vartheta_v - 2\xi_x) v_{xx} - 2\frac{D}{r^2}\tau_x v_{xt} - 3\frac{D}{r^2}\xi_v v_x v_{xx} -$$
$$- \frac{D}{r^2}\tau_v v_{xx} \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) - 2\frac{D}{r^2}\tau_v v_x v_{xt} + \varphi\xi v_x + x\dot{\varphi}\tau v_x + x\varphi\vartheta_x +$$
$$+ x\varphi(\vartheta_v - \xi_x) v_x - x\varphi\tau_x \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) - x\varphi\xi_v v_x^2 -$$
$$- x\varphi\tau_v v_x \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) + J_3\eta + J_4\vartheta - \vartheta_t + \xi_t v_x -$$
$$- (\vartheta_v - \tau_t) \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) + \xi_v v_x \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right) +$$
$$+ \tau_v \left( \frac{D}{r^2(t)} v_{xx} + x\varphi(t) v_x + J_3 u + J_4 v \right)^2 = 0,$$

where we used formulas from Section 1.2 and conditions $F_u = 0, F_v = 0$. Next we compare the terms multiplied by derivatives of $u$ and $v$ and their products independently:

$$u_{xt}, v_{xt}: \quad \tau_x = 0,$$
$$u_x u_{xt}: \quad \tau_u = 0,$$
$$v_x v_{xt}: \quad \tau_v = 0.$$

These conditions simplify the determining equations for $\xi, \tau, \eta, \vartheta$ and also imply $\tau = \tau(t)$ :

$$u_x u_{xx}: \quad -2\frac{1}{r^2}\xi_u = 0, \tag{4.3}$$
$$v_x v_{xx}: \quad -2\frac{D}{r^2}\xi_v = 0 \tag{4.4}$$
$$\implies \xi = \xi(x,t), \tag{4.5}$$
$$u_{xx}, v_{xx}: \quad \dot{\tau} - 2\varphi\tau - 2\xi_x = 0, \tag{4.6}$$
$$u_x^2: \quad \frac{1}{r^2}\eta_{uu} = 0 \implies \eta = A(x,t,v)u + B(x,t,v), \tag{4.7}$$
$$v_x^2: \quad \frac{D}{r^2}\vartheta_{vv} = 0 \implies \vartheta = \tilde{A}(x,t,u)v + \tilde{B}(x,t,u), \tag{4.8}$$
$$u_x: \quad x\varphi\dot{\tau} + \xi_t - x\varphi\xi_x + x\dot{\varphi}\tau + \xi\varphi + \frac{1}{r^2}(2\eta_{xu} - \xi_{xx}) = 0, \tag{4.9}$$
$$v_x: \quad x\varphi\dot{\tau} + \xi_t - x\varphi\xi_x + x\dot{\varphi}\tau + \xi\varphi + \frac{D}{r^2}(2\vartheta_{xv} - \xi_{xx}) = 0, \tag{4.10}$$
$$1(F_u): \quad (\dot{\tau} - \eta_u)(J_1 u + J_2 v) - \eta_t + J_1\eta + J_2\vartheta + x\varphi\eta_x + \frac{1}{r^2}\eta_{xx} = 0, \tag{4.11}$$

$$1(F_v): \quad (\dot{\tau} - \vartheta_v)(J_3 u + J_4 v) - \vartheta_t + J_3\eta + J_4\vartheta + x\varphi\eta_x + \frac{D}{r^2}\vartheta_{xx} = 0, \tag{4.12}$$

where $A(x,t,v), \tilde{A}(x,t,u)$ and $B(x,t,v), \tilde{B}(x,t,u)$ are arbitrary functions. Now we can solve these equations. From condition (4.5) and equation (4.6) we obtain

$$\xi = \left(\frac{\dot{\tau}}{2} - \varphi\tau\right)x + E(t),$$

where $E(t)$ is an arbitrary function. Next we substitute the expressions for $\eta$ and $\vartheta$ into equations (4.11) and (4.12):

$$\begin{aligned}
&\dot{\tau}(J_1 u + J_2 v) - AJ_2 v - A_t u - B_t + J_1 B + J_2\tilde{A}v + J_2\tilde{B} + \\
&+ x\varphi A_x u + x\varphi B_x + \frac{1}{r^2}A_{xx}u + \frac{1}{r^2}B_{xx} = 0, \\
&\dot{\tau}(J_3 u + J_4 v) - \tilde{A}J_3 u - \tilde{A}_t v - \tilde{B}_t + J_4\tilde{B} + J_3 A u + J_3 B + \\
&+ x\varphi\tilde{A}_x v + x\varphi\tilde{B}_x + \frac{D}{r^2}\tilde{A}_{xx}v + \frac{D}{r^2}\tilde{B}_{xx} = 0.
\end{aligned}$$

Because the conditions are very complicated, we follow a simplified ansatz (and hence restricting the set of symmetries we will find).

$$\begin{aligned}
A(x,t,v) &= A(x,t), \qquad & B(x,t,v) &= B(x,t), \\
\tilde{A}(x,t,u) &= \tilde{A}(x,t), \qquad & \tilde{B}(x,t,u) &= \tilde{B}(x,t).
\end{aligned}$$

By collecting the terms depending on $u$ and $v$ we obtain

$$\dot{\tau}J_1 - A_t + x\varphi A_x + \frac{1}{r^2}A_{xx} = 0, \tag{4.13}$$

$$J_2(\dot{\tau} - A + \tilde{A}) = 0, \tag{4.14}$$

$$\dot{\tau}J_4 - \tilde{A}_t + x\varphi\tilde{A}_x + \frac{1}{r^2}\tilde{A}_{xx} = 0, \tag{4.15}$$

$$J_3(\dot{\tau} - \tilde{A} + A) = 0, \tag{4.16}$$

$$B_t = J_1 B + J_2\tilde{B} + x\varphi B_x + \frac{1}{r^2}B_{xx}, \tag{4.17}$$

$$\tilde{B}_t = J_3 B + J_4\tilde{B} + x\varphi\tilde{B}_x + \frac{D}{r^2}\tilde{B}_{xx}. \tag{4.18}$$

Combining conditions (4.14) and (4.16) we obtain

$$\dot{\tau} = 0 \implies \tau = c,$$

where $c$ is an arbitrary constant. Equations (4.17) and (4.18) imply that the functions $B(x,t)$ and $\tilde{B}(x,t)$ satisfy the original system of equations (4.1). Next we substitute the expressions for $\eta, \vartheta, \tau$ and $\xi$ into conditions (4.9) and (4.10):

$$-c\dot{\varphi}x + \dot{E} + cx\varphi^2 + -c\varphi^2 x + \varphi E + \frac{2}{r^2(t)}A_x = \tag{4.19}$$

$$= \dot{E}(t) + \varphi(t)E(t) + \frac{2}{r^2(t)}A_x = 0 \implies A(x,t) = w(t)x + z(t), \tag{4.20}$$

$$-c\dot{\varphi}x+\dot{E}+cx\varphi^2+-c\varphi^2x+\varphi E+\frac{2D}{r^2(t)}\tilde{A}_x= \tag{4.21}$$

$$=\dot{E}(t)+\varphi(t)E(t)+\frac{2D}{r^2(t)}\tilde{A}_x=0 \implies \tilde{A}(x,t)=\tilde{w}(t)x+\tilde{z}(t), \tag{4.22}$$

where $w(t), \tilde{w}(t)$ and $z(t), \tilde{z}(t)$ are arbitrary functions. Now we substitute these expressions into equations (4.13) and (4.15):

$$\begin{aligned}&-\dot{w}x-\dot{z}+\varphi wx=0,\\&-\dot{\tilde{w}}x-\dot{\tilde{z}}+\varphi\tilde{w}x=0.\end{aligned}$$

By comparing the terms multiplied by different powers of $x$ we obtain

$$\begin{aligned}x:\quad &\dot{w}=\varphi w \implies w=ae^{\int\varphi}=ar(t),\\&\dot{\tilde{w}}=\varphi\tilde{w} \implies \tilde{w}=\tilde{a}e^{\int\varphi}=\tilde{a}r(t),\\1:\quad &-\dot{z}=0 \implies z=b,\\&-\dot{\tilde{z}}=0 \implies \tilde{z}=\tilde{b},\end{aligned}$$

where $a, \tilde{a}, b, \tilde{b}$ are arbitrary constants. As a next step we use equations (4.20) and (4.22):

$$\begin{aligned}&\dot{E}+\varphi E+2\frac{1}{r^2}A_x=\dot{E}+\varphi E+\frac{2a}{r} \implies (\dot{Er})=-2a \implies E(t)=-\frac{2at}{r(t)}+\frac{f}{r(t)},\\&\dot{E}+\varphi E+2\frac{D}{r^2}\tilde{A}_x=\dot{E}+\varphi E+\frac{2D\tilde{a}}{r} \implies (\dot{Er})=-2D\tilde{a} \implies E(t)=-\frac{2D\tilde{a}t}{r(t)}+\frac{\tilde{f}}{r(t)}\\&\implies \tilde{f}=f\wedge a=D\tilde{a},\end{aligned}$$

where $f$ and $\tilde{f}$ are arbitrary constants. As the last step we have to use condition (4.14), resp. (4.16):

$$\begin{aligned}&A=\tilde{A}\iff D\tilde{a}r(t)x+b=\tilde{a}r(t)x+\tilde{b}\\&\iff b=\tilde{b}\wedge D\tilde{a}=\tilde{a} \implies \tilde{a}=0.\end{aligned}$$

Finally we obtain

$$\begin{aligned}\eta&=bu+B(x,t),\\\vartheta&=bv+\tilde{B}(x,t),\\\xi&=-c\varphi(t)x+\frac{f}{r(t)},\\\tau&=c,\end{aligned}$$

where $b, c, f$ are arbitrary constants and $\left(B(x,t),\tilde{B}(x,t)\right)$ satisfies the original problem.The infinitesimal generators of the symmetries of system (3.1) are spanned by

$$\begin{aligned}X_3&=-\varphi(t)x\partial_x+\partial_t,\\X_5&=u\partial_u+v\partial_v,\\X_6&=\frac{1}{r(t)}\partial_x,\end{aligned}$$

$$X_B = B(x,t)\partial_u + \tilde{B}(x,t)\partial_v,$$

where $\Big(B(x,t), \tilde{B}(x,t)\Big)$ satisfies the original equation.

As in the scalar case we verify that these infinitesimal generators form Lie algebra. We use the results from the scalar case:

$$[X_3, X_5] = 0,$$
$$[X_3, X_6] = 0,$$
$$[X_5.X_6] = 0.$$

As in the previous chapter we show that $F_u(B_i(x,t), \tilde{B}_i(x,t)) = 0, F_v(\tilde{B}_i(x,t), B_i(x,t)) = 0$ where $[X_i, X_B] = B_i(x,t)\partial_u + \tilde{B}_i(x,t)\partial_v$ for $i = \{3,5,6\}$ :

$$\begin{aligned}
&[X_3, X_B] = [-\varphi(t)xB_x(x,t) + B_t(x,t)]\partial_u + [-\varphi(t)x\tilde{B}_x(x,t) + \tilde{B}_t(x,t)]\partial_v =: \\
&=: B_3(x,t)\partial_u + \tilde{B}_3(x,t)\partial_v, \\
&F_u(B_3(x,t), \tilde{B}_3(x,t)) = \frac{1}{r^2(t)}[-2\varphi(t)B_{xx} - \varphi(t)xB_{xxx} + B_{xxt}] + x\varphi(t)[-\varphi(t)B_x - \\
&- \varphi(t)xB_{xx} + B_{xt}] + J_1[-\varphi(t)xB_x + B_t] + J_2[-\varphi(t)x\tilde{B}_x(x,t) + \tilde{B}_t(x,t)] - \\
&- [-\dot{\varphi}(t)xB_x - \varphi(t)xB_{xt} + B_{tt}] = \\
&= -\varphi(t)x\left[\frac{1}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + J_1B_x + J_2\tilde{B}_x - B_{tx}\right] + \\
&+ \left[\frac{1}{r^2(t)}B_{xxt} - \frac{2\dot{r}(t)}{r^3(t)}B_{xx} + x\varphi(t)B_{xt} + \dot{\varphi}(t)xB_x + J_1B_t + J_2\tilde{B}_t - B_{tt}\right] = 0, \\
&F_v(\tilde{B}_3(x,t), B_3(x,t)) = \frac{D}{r^2(t)}[-2\varphi(t)\tilde{B}_{xx} - \varphi(t)x\tilde{B}_{xxx} + \tilde{B}_{xxt}] + x\varphi(t)[-\varphi(t)\tilde{B}_x - \\
&- \varphi(t)x\tilde{B}_{xx} + \tilde{B}_{xt}] + J_3[-\varphi(t)xB_x + B_t] + J_4[-\varphi(t)x\tilde{B}_x(x,t) + \tilde{B}_t(x,t)] - \\
&- [-\dot{\varphi}(t)x\tilde{B}_x - \varphi(t)x\tilde{B}_{xt} + \tilde{B}_{tt}] = \\
&= -\varphi(t)x\left[\frac{D}{r^2(t)}\tilde{B}_{xxx} + \varphi(t)\tilde{B}_x + x\varphi(t)\tilde{B}_{xx} + J_3B_x + J_4\tilde{B}_x - \tilde{B}_{tx}\right] + \\
&+ \left[\frac{D}{r^2(t)}\tilde{B}_{xxt} - \frac{2\dot{r}(t)D}{r^3(t)}\tilde{B}_{xx} + x\varphi(t)\tilde{B}_{xt} + \dot{\varphi}(t)x\tilde{B}_x + J_3B_t + J_4\tilde{B}_t - \tilde{B}_{tt}\right] = 0,
\end{aligned}$$

$$\begin{aligned}
&[X_5, X_B] = -B\partial_u - \tilde{B}\partial_v =: B_5(x,t)\partial_u + \tilde{B}(x,t)\partial_v, \\
&F_u(B_5, \tilde{B}_5) = 0, \\
&F_v(\tilde{B}_5, B_5) = 0, \\
&[X_6, X_B] = \frac{1}{r(t)}B_x\partial_u + \frac{1}{r(t)}\tilde{B}_x\partial_v =: B_6(x,t)\partial_u + \tilde{B}_6(x,t)\partial_v, \\
&F_u(B_6, \tilde{B}_6) = \frac{1}{r^3(t)}B_{xxx} + x\varphi(t)\frac{1}{r(t)}B_{xx} + J_1\frac{1}{r(t)}B_x + J_2\frac{1}{r(t)}\tilde{B}_x - \left[-\frac{\dot{r}(t)}{r^2(t)}B_x + \frac{1}{r(t)}B_{xt}\right] = \\
&= \frac{1}{r(t)}\left[\frac{1}{r^2(t)}B_{xxx} + \varphi(t)B_x + x\varphi(t)B_{xx} + J_1B_x + J_2\tilde{B}_x - B_{tx}\right] = 0,
\end{aligned}$$

$$F_v(\tilde{B}_6, B_6) = \frac{D}{r^3(t)}\tilde{B}_{xxx} + x\varphi(t)\frac{1}{r(t)}\tilde{B}_{xx} + J_3\frac{1}{r(t)}B_x + J_4\frac{1}{r(t)}\tilde{B}_x - \left[-\frac{\dot{r}(t)}{r^2(t)}\tilde{B}_x + \frac{1}{r(t)}\tilde{B}_{xt}\right] =$$
$$= \frac{1}{r(t)}\left[\frac{D}{r^2(t)}\tilde{B}_{xxx} + \varphi(t)\tilde{B}_x + x\varphi(t)\tilde{B}_{xx} + J_3B_x + J_4\tilde{B}_x - \tilde{B}_{tx}\right] = 0.$$

Next we identify symmetries corresponding to these infinitesimal generators. We use the results from Section 3.1:

1. $X_3 = -\varphi(t)x\partial_x + \partial_t$ :
$$(\hat{u}, \hat{v}, \hat{x}, \hat{t}) = (ue^{\varepsilon}, ve^{\varepsilon}, x, t),$$

2. $X_5 = u\partial_u + v\partial_v$ :
$$(\hat{u}, \hat{v}, \hat{x}, \hat{t}) = \left(u, v, x + \frac{\varepsilon}{r(t)}, t\right),$$

3. $X_6 = \frac{1}{r(t)}\partial_x$ :
$$(\hat{u}, \hat{v}, \hat{x}, \hat{t}) = \left(u, v, x\frac{r(t)}{r(t+\varepsilon)}, t + \varepsilon\right),$$

   where $\varepsilon$ is a symmetry parameter.

## 4.2 Invariant solutions

### 4.2.1 Solution invariant under the symmetry generated by$X_3 + a_1X_5$

First we would like to find the solution invariant under the infinitesimal generator

$$X_3 + a_1X_5 = a_1u\partial_u + a_1v\partial_v - \frac{\dot{r}(t)}{r(t)}x\partial_x + \partial_t,$$

where $a_1$ is an arbitrary constant. The corresponding characteristics have form:

$$\mathcal{Q}_u = a_1u + \frac{\dot{r}(t)}{r(t)}xu_x - u_t = 0,$$
$$\mathcal{Q}_v = a_1v + \frac{\dot{r}(t)}{r(t)}xv_x - v_t = 0.$$

From the results listed in Subsection 3.2.1 we obtain the substitution formulas:

$$p = xr(t),$$
$$u = e^{a_1t}F(p) = e^{a_1t}F(xr(t)),$$
$$v = e^{a_1t}G(p) = e^{a_1t}G(xr(t)).$$

Using them we transform system (4.1) into the system of ordinary differential equations:

$$a_1e^{a_1t}F(r(t)x) + \dot{r}(t)xe^{a_1t}F'(r(t)x) = e^{a_1t}F''(r(t)x) + x\dot{r}(t)e^{a_1t}F'(r(t)x)+$$
$$+ J_1^{a_1t}F(r(t)x) + J_2^{a_1t}G(r(t)x),$$
$$a_1e^{a_1t}G(r(t)x) + \dot{r}(t)xe^{a_1t}G'(r(t)x) = De^{a_1t}G''(r(t)x) + x\dot{r}(t)e^{a_1t}G'(r(t)x)+$$
$$+ J_3^{a_1t}F(r(t)x) + J_4^{a_1t}G(r(t)x)$$

and hence

$$F'' + (J_1 - a_1)\, F + J_2 G = 0, \tag{4.23a}$$

$$DG'' + (J_4 - a_1)\, G + J_3 F = 0. \tag{4.23b}$$

With the notation

$$\begin{aligned} H :=& F', \\ I :=& G' \end{aligned}$$

system of two linear second order ODEs (4.23) can be transformed into the system of four first order linear ODEs which can be written in the vector form

$$y' = \mathbb{A} y,$$

where

$$y = \begin{bmatrix} F \\ G \\ H \\ I \end{bmatrix}, \qquad \mathbb{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\tilde{J}_1 & -\tilde{J}_2 & 0 & 0 \\ -\tilde{J}_3 & -\tilde{J}_4 & 0 & 0 \end{bmatrix}, \tag{4.24}$$

with

$$\begin{aligned} \tilde{J}_1 &= (J_1 - a_1), \\ \tilde{J}_2 &= J_2, \\ \tilde{J}_3 &= J_3/D, \\ \tilde{J}_4 &= (J_4 - a_1)/D. \end{aligned}$$

Now we would like to find eigenvalues of the matrix $\mathbb{A}$ and corresponding eigenvectors:

$$\det(\mathbb{A} - \lambda \mathbb{I}) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -\tilde{J}_1 & -\tilde{J}_2 & -\lambda & 0 \\ -\tilde{J}_3 & -\tilde{J}_4 & 0 & -\lambda \end{vmatrix} =$$

$$= -\lambda(-\lambda^3 - \tilde{J}_4\lambda) + (\tilde{J}_1\lambda^2 + \tilde{J}_1\tilde{J}_4 - \tilde{J}_2\tilde{J}_3) = \lambda^4 + (\tilde{J}_1 + \tilde{J}_4)\lambda^2 + \tilde{J}_1\tilde{J}_4 - \tilde{J}_2\tilde{J}_3 \overset{!}{=} 0$$

and hence

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{-(\tilde{J}_1 + \tilde{J}_4) \pm \sqrt{(\tilde{J}_1 + \tilde{J}_4)^2 - 4(\tilde{J}_1\tilde{J}_4 - \tilde{J}_2\tilde{J}_3)}}{2}}.$$

Corresponding eigenvectors are:

$$\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -\tilde{J}_1 & -\tilde{J}_2 & -\lambda & 0 \\ -\tilde{J}_3 & -\tilde{J}_4 & 0 & -\lambda \end{bmatrix} \sim \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & \tilde{J}_2\lambda & \lambda^2 + \tilde{J}_1 & 0 \\ 0 & \tilde{J}_4\lambda & \tilde{J}_3 & \lambda^2 \end{bmatrix} \sim \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & \lambda^2 + \tilde{J}_1 & \tilde{J}_2 \\ 0 & 0 & \tilde{J}_3 & \lambda^2 + \tilde{J}_4 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & \lambda^2 + \tilde{J}_1 & \tilde{J}_2 \\ 0 & 0 & 0 & X \end{bmatrix},$$

where $X = (\lambda^2 + \tilde{J}_1)(\lambda^2 + \tilde{J}_4) - \tilde{J}_2\tilde{J}_3 = 0$ and thus

$$v(\lambda) = \begin{bmatrix} -\tilde{J}_2/\lambda \\ (\lambda^2 + \tilde{J}_1)/\lambda \\ -\tilde{J}_2 \\ (\lambda^2 + \tilde{J}_1) \end{bmatrix}.$$

Several cases may occur:

1. All eigenvalues are distinct:

   From the knowledge of the eigenvalues and the corresponding eigenvectors we obtain:

$$F(p) = \sum_{i=1}^{4} K_i \frac{(-\tilde{J}_2)}{\lambda_i} e^{\lambda_i p},$$
$$G(p) = \sum_{i=1}^{4} K_i \frac{\lambda_i^2 + \tilde{J}_1}{\lambda_i} e^{\lambda_i p},$$

   and thus

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{\lambda_i x r(t) + a_1 t}, \\ v(x,t) &= \sum_{i=1}^{4} K_i \frac{\lambda_i^2 + J_1 - a_1}{\lambda_i} e^{\lambda_i x r(t) + a_1 t}, \end{aligned} \tag{4.25}$$

   where $K_i$ is a constant $\forall i \in \{1, 2, 3, 4\}$.

2. $\tilde{J}_1\tilde{J}_4 - \tilde{J}_2\tilde{J}_3 = 0$ :

   We have three distinct eigenvalues where one of them, $\lambda_0$, is degenerate:

$$\lambda_\pm = \pm\sqrt{-(\tilde{J}_1 + \tilde{J}_4)} = \pm\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} =: \pm\lambda,$$
$$\lambda_0 = 0.$$

   The first two components of the solutions of system (4.24) have form

$$F(p) = K_+ e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}p} + K_- e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}p} + Lp + M, \tag{4.26a}$$

$$G(p) = \tilde{K}_+ e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}p} + \tilde{K}_- e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}p} + \tilde{L}p + \tilde{M}, \tag{4.26b}$$

   where $K_\pm, L, M$ and $\tilde{K}_\pm, \tilde{L}, \tilde{M}$ are arbitrary constants. Now we substitute it into system (4.23):

$$\lambda^2 K_+ e^{\lambda p} + \lambda^2 K_- e^{-\lambda p} + (J_1 - a_1)\left(K_+ e^{\lambda p} + K_- e^{-\lambda p} + Lp + M\right) +$$
$$+ J_2\left(\tilde{K}_+ e^{\lambda p} + \tilde{K}_- e^{-\lambda p} + \tilde{L}p + \tilde{M}\right) \stackrel{!}{=} 0$$

   and thus

$$\tilde{M} = \frac{a_1 - J_1}{J_2} M,$$

$$\tilde{L} = \frac{a_1 - J_1}{J_2} L,$$
$$\tilde{K}_\pm = \frac{J_4 - a_1}{J_2 D} K_\pm,$$
$$\lambda^2 K_+ \frac{J_4 - a_1}{J_2} e^{\lambda p} + \lambda^2 K_- \frac{J_4 - a_1}{J_2} e^{-\lambda p} + J_3 \left( K_+ e^{\lambda p} + K_- e^{-\lambda p} + Lp + M \right) +$$
$$+ (J_4 - a_1) \left( K_+ \frac{J_4 - a_1}{J_2 D} e^{\lambda p} + K_- \frac{J_4 - a_1}{J_2 D} e^{-\lambda p} + L \frac{a_1 - J_1}{J_2} p + M \frac{a_1 - J_1}{J_2} \right) = 0.$$

The corresponding solution reads

$$\begin{aligned} u(x,t) &= K_+ e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} x r(t) + a_1 t} + K_- e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} x r(t) + a_1 t} + \\ &\quad + L x r(t) e^{a_1 t} + M e^{a_1 t}, \\ v(x,t) &= K_+ \frac{J_4 - a_1}{J_2 D} e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} x r(t) + a_1 t} + K_- \frac{J_4 - a_1}{J_2 D} e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} x r(t) + a_1 t} + \\ &\quad + L \frac{a_1 - J_1}{J_2} x r(t) e^{a_1 t} + M \frac{a_1 - J_1}{J_2} e^{a_1 t}, \end{aligned} \tag{4.27}$$

where $K_\pm, L, M$ are arbitrary constants.

3. $(\tilde{J}_1 + \tilde{J}_4)^2 - 4(\tilde{J}_1 \tilde{J}_4 - \tilde{J}_2 \tilde{J}_3) = 0:$

We have two degenerate eigenvalues:

$$\lambda_\pm = \pm \sqrt{-\frac{\tilde{J}_1 + \tilde{J}_4}{2}} = \pm \sqrt{\frac{a_1 - J_1 + \frac{a_1 - J_4}{D}}{2}} =: \pm \lambda.$$

The first two components have form:

$$F(p) = (Kp + L)\, e^{\lambda p} + (Mp + N)\, e^{-\lambda p},$$
$$G(p) = \left( \tilde{K} p + \tilde{L} \right) e^{\lambda p} + \left( \tilde{M} p + \tilde{N} \right) e^{-\lambda p},$$

where $K, L, M, N$ and $\tilde{K}, \tilde{L}, \tilde{M}, \tilde{N}$ are constants. Next we substitute them into original system of ODEs (4.23):

$$\lambda^2 (Kp + L) e^{\lambda p} + 2\lambda K e^{\lambda p} + \lambda^2 (Mp + N) e^{-\lambda p} - 2\lambda M e^{-\lambda p} +$$
$$+ (J_1 - a_1) \left( (Kp + L)\, e^{\lambda p} + (Mp + N)\, e^{-\lambda p} \right) + J_2 \left( \left( \tilde{K} p + \tilde{L} \right) e^{\lambda p} + \left( \tilde{M} p + \tilde{N} \right) e^{-\lambda p} \right) \stackrel{!}{=} 0$$

and thus

$$\tilde{K} = -\frac{\lambda^2 + (J_1 - a_1)}{J_2} K,$$
$$\tilde{L} = -\frac{\lambda^2 L + (J_1 - a_1) L + 2\lambda K}{J_2},$$
$$\tilde{M} = -\frac{\lambda^2 + (J_1 - a_1)}{J_2} M,$$
$$\tilde{N} = -\frac{\lambda^2 N + (J_1 - a_1) L - 2\lambda M}{J_2},$$

$$\lambda^2 D\left(-\frac{\lambda^2+(J_1-a_1)}{J_2}Kp-\frac{\lambda^2 L+(J_1-a_1)L+2\lambda K}{J_2}\right)e^{\lambda p}-2\lambda D\frac{\lambda^2+(J_1-a_1)}{J_2}Ke^{\lambda p}+$$
$$+\lambda^2 D\left(-\frac{\lambda^2+(J_1-a_1)}{J_2}Mp-\frac{\lambda^2 N+(J_1-a_1)L-2\lambda M}{J_2}\right)e^{-\lambda p}+$$
$$+2\lambda D\frac{\lambda^2+(J_1-a_1)}{J_2}Me^{-\lambda p}+$$
$$+(J_4-a_1)\left(-\frac{\lambda^2+(J_1-a_1)}{J_2}Kp-\frac{\lambda^2 L+(J_1-a_1)L+2\lambda K}{J_2}\right)e^{\lambda p}+$$
$$+(J_4-a_1)\left(-\frac{\lambda^2+(J_1-a_1)}{J_2}Mp-\frac{\lambda^2 N+(J_1-a_1)L-2\lambda M}{J_2}\right)e^{-\lambda p}+$$
$$+J_3\left((Kp+L)\,e^{\lambda p}+(Mp+N)\,e^{-\lambda p}\right)=0.$$

The invariant solution is then

$$\begin{aligned}u(x,t)&=(Kxr(t)+L)\,e^{\lambda xr(t)+a_1t}+(Mxr(t)+N)\,e^{-\lambda xr(t)+a_1t},\\ v(x,t)&=\left(-\frac{\lambda^2+(J_1-a_1)}{J_2}Kxr(t)-\frac{\lambda^2 L+(J_1-a_1)L+2\lambda K}{J_2}\right)e^{\lambda xr(t)+a_1t}-\\ &-\left(\frac{\lambda^2+(J_1-a_1)}{J_2}Mxr(t)+\frac{\lambda^2 N+(J_1-a_1)N-2\lambda M}{J_2}\right)e^{-\lambda xr(t)+a_1t}.\end{aligned} \tag{4.28}$$

where $K, L, M, N$ are arbitrary constants.

### 4.2.2 Solution invariant under the symmetry generated by $a_2X_2 + X_3$

Next we would like to find the solution invariant under the symmetry generated by

$$a_2X_2+X_3=\frac{a_2}{r(t)}\partial_x+u\partial_u,$$

where $a_2$ is an arbitrary constant. The characteristics corresponding to this symmetry are

$$\mathcal{Q}_u=u-\frac{a_2}{r(t)}u_x=0,$$

$$\mathcal{Q}_v=v-\frac{a_2}{r(t)}v_x=0.$$

These equations were already solved in Subsection 3.2.2 and thus the invariant solutions are

$$u=K_2e^{\frac{r(t)x}{a_2}+f_2(t)}=A_2(t)e^{\frac{r(t)x}{a_2}},$$

$$u=L_2e^{\frac{r(t)x}{a_2}+g_2(t)}=B_2(t)e^{\frac{r(t)x}{a_2}},$$

where $f(t), g(t)$, resp. $A_2(t), B_2(t)$ are arbitrary functions and $K_2, L_2$ are arbitrary constants. Substituting them into original system of differential equations (4.1) we obtain

$$\left(\dot{A}_2(t)+\frac{A_2(t)\dot{r}(t)x}{a_2}\right)e^{\frac{r(t)x}{a_2}}\overset{!}{=}\frac{A_2(t)}{a_2^2}e^{\frac{r(t)x}{a_2}}+\left(\frac{A_2(t)\dot{r}(t)x}{a_2}\right)e^{\frac{A_2(t)r(t)x}{a_2}}+$$
$$+J_1A_2(t)e^{\frac{r(t)x}{a_2}}+J_2B_2(t)e^{\frac{r(t)x}{a_2}}$$

$$\implies \dot{A}_2(t) = \left(\frac{1}{a_2^2} + J_1\right) A_2(t) + J_2 B_2(t),$$

$$\left(\dot{B}_2(t) + \frac{B_2(t)\dot{r}(t)x}{a_2}\right) e^{\frac{r(t)x}{a_2}} \stackrel{!}{=} \frac{DB_2(t)}{a_2^2} e^{\frac{r(t)x}{a_2}} + \left(\frac{B_2(t)\dot{r}(t)x}{a_2}\right) e^{\frac{r(t)x}{a_2}} +$$
$$+ J_3 A_2(t) e^{\frac{r(t)x}{a_2}} + J_4 B_2(t) e^{\frac{r(t)x}{a_2}}$$
$$\implies \dot{B}_2(t) = \left(\frac{D}{a_2^2} + J_4\right) B_2(t) + J_3 A_2(t).$$

Now we separate this system of ODEs:

$$A_2(t) = \frac{\dot{B}_2(t)}{J_3} - \left(\frac{D}{J_3 a_2^2} + \frac{J_4}{J_3}\right) B_2(t),$$
$$\dot{A}_2(t) = \frac{\ddot{B}_2(t)}{J_3} - \left(\frac{D}{J_3 a_2^2} + \frac{J_4}{J_3}\right) \dot{B}_2(t)$$
$$\implies \dot{A}_2(t) - \left(\frac{1}{a_2^2} + J_1\right) A_2(t) - J_2 B_2(t) =$$
$$= \frac{\ddot{B}_2(t)}{J_3} - \left(\frac{D}{J_3 a_2^2} + \frac{J_4}{J_3}\right) \dot{B}_2(t) - \left(\frac{1}{a_2^2} + J_1\right) \left[\frac{\dot{B}_2(t)}{J_3} - \left(\frac{D}{J_3 a_2^2} + \frac{J_4}{J_3}\right) B_2(t)\right] - J_2 B_2(t)$$
$$\implies \ddot{B}_2 - k\dot{B}_2 + lB_2 = 0,$$
$$B_2(t) = \frac{\dot{A}_2(t)}{J_2} - \left(\frac{1}{J_2 a_2^2} + \frac{J_1}{J_2}\right) A_2(t),$$
$$\dot{B}_2(t) = \frac{\ddot{A}_2(t)}{J_2} - \left(\frac{1}{J_2 a_2^2} + \frac{J_1}{J_2}\right) \dot{A}_2(t)$$
$$\implies \dot{B}_2(t) - \left(\frac{D}{a_2^2} + J_4\right) B_2(t) - J_3 A_2(t) =$$
$$= \frac{\ddot{A}_2(t)}{J_2} - \left(\frac{1}{J_2 a_2^2} + \frac{J_1}{J_2}\right) \dot{A}_2(t) - \left(\frac{D}{a_2^2} + J_4\right) \left[\frac{\dot{A}_2(t)}{J_2} - \left(\frac{1}{J_2 a_2^2} + \frac{J_1}{J_2}\right) A_2(t)\right] - J_3 A_2(t)$$
$$\implies \ddot{A}_2 - k\dot{A}_2 + lA_2 = 0,$$

where $k = (J_1 + \frac{1}{a_2^2}) + (J_4 + \frac{D}{a_2^2})$ and $l = (J_1 + \frac{1}{a_2^2})(J_4 + \frac{D}{a_2^2}) - J_2 J_3$. The solutions of these equations have form

$$A(t) = C_1 e^{\frac{1}{2}(k - \sqrt{k^2 - 4l})t} + C_2 e^{\frac{1}{2}(k + \sqrt{k^2 - 4l})t},$$
$$B(t) = C_3 e^{\frac{1}{2}(k - \sqrt{k^2 - 4l})t} + C_4 e^{\frac{1}{2}(k + \sqrt{k^2 - 4l})t},$$

where $C_1, C_2, C_3, C_4$ are arbitrary constants, which implies the following invariant solution:

$$\begin{aligned} u(x,t) &= C_1 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k - \sqrt{k^2 - 4l})t} + C_2 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k + \sqrt{k^2 - 4l})t}, \\ v(x,t) &= C_3 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k - \sqrt{k^2 - 4l})t} + C_4 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k + \sqrt{k^2 - 4l})t}. \end{aligned} \tag{4.29}$$

There are certain restrictions on the constant values which we obtain by substituting these functions into original system (4.1):

$$C_1 \left(\frac{\dot{r}(t)x}{a_2} + \frac{1}{2}(k - \sqrt{k^2 - 4l}\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k - \sqrt{k^2 - 4l})t} +$$

$$+ C_2\left(\frac{\dot{r}(t)x}{a_2} + \frac{1}{2}(k + \sqrt{k^2 - 4l})\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=}$$

$$\stackrel{!}{=} C_1 \frac{1}{a_2^2} e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 \frac{1}{a_2^2} e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} + C_1\left(\frac{\dot{r}(t)x}{a_2}\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} +$$

$$+ C_2\left(\frac{\dot{r}(t)x}{a_2}\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} + J_1 C_1 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + J_1 C_2 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} +$$

$$+ J_2 C_3 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + J_2 C_4 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t}$$

$$\implies C_3 = \frac{\left(J_4 + \frac{D}{a_2^2}\right) - \left(J_1 + \frac{1}{a_2^2}\right) - \sqrt{k^2 - 4l}}{2J_2} C_1, \tag{4.30}$$

$$C_3\left(\frac{\dot{r}(t)x}{a_2} + \frac{1}{2}(k - \sqrt{k^2 - 4l})\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} +$$

$$+ C_4\left(\frac{\dot{r}(t)x}{a_2} + \frac{1}{2}(k + \sqrt{k^2 - 4l}\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=}$$

$$\stackrel{!}{=} C_3 \frac{D}{a_2^2} e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + C_4 \frac{D}{a_2^2} e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} + C_3\left(\frac{\dot{r}(t)x}{a_2}\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} +$$

$$+ C_4\left(\frac{\dot{r}(t)x}{a_2}\right) e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} + J_3 C_1 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + J_3 C_2 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} +$$

$$+ J_4 C_3 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + J_4 C_4 e^{\frac{r(t)x}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t}$$

$$\implies C_4 = \frac{\left(J_4 + \frac{D}{a_2^2}\right) - \left(J_1 + \frac{1}{a_2^2}\right) + \sqrt{k^2 - 4l}}{2J_2} C_2. \tag{4.31}$$

We obtained these formulas only from the first condition. It is easy to verify that for these values the second condition holds as well.

### 4.2.3 Solution invariant under the symmetry generated by $X_1 + a_3 X_2$

Finally we would like to find the solution invariant under the symmetry generated by

$$X_1 + a_3 X_2 = -\frac{\dot{r}(t)}{r(t)} x\partial_x + \partial_t + \frac{a_3}{r(t)}\partial_x,$$

where $a_3$ is an arbitrary constant. The characteristics are

$$\mathcal{Q}_u = (a_3 - \dot{r}(t)x)u_x + r(t)u_t = 0,$$

$$\mathcal{Q}_v = (a_3 - \dot{r}(t)x)v_x + r(t)v_t = 0.$$

As in Subsection 3.2.3 each function in the form

$$u(x,t) = F(a_3 t - r(t)x),$$

$$v(x,t) = G(a_3 t - r(t)x)$$

satisfy the characteristics conditions. By substituting into the original system of PDEs we obtain

$$(a_3 - \dot{r}(t)x)F' = F'' - \dot{r}(t)xF' + J_1 F + J_2 G$$

$$\Longrightarrow F'' - a_3F' + J_1F + J_2G = 0,$$
$$(a_3 - \dot{r}(t)x)G' = DG'' - \dot{r}(t)xG' + J_3F + J_4G$$
$$\Longrightarrow G'' - \frac{a_3}{D}F' + \frac{J_3}{D}F + \frac{J_4}{F}G = 0,$$

As in Subsection 4.2.1 we use the substitution

$$H := F',$$
$$I := G'$$

which transforms these two linear second order ODEs into four linear first order ODEs which can be written as

$$y' = \mathbb{A}y,$$

where

$$y = \begin{bmatrix} F \\ G \\ H \\ I \end{bmatrix}, \qquad \mathbb{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -J_1 & -J_2 & a_3 & 0 \\ -\frac{J_3}{D} & -\frac{J_4}{D} & 0 & \frac{a_3}{D} \end{bmatrix}.$$

Next we need to find out the eigenvalues and corresponding eigenvectors of the matrix $\mathbb{A}$ :

$$\det(\mathbb{A} - \lambda\mathbb{I}) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -J_1 & -J_2 & a_3 - \lambda & 0 \\ -\frac{J_3}{D} & -\frac{J_4}{D} & 0 & \frac{a_3}{D} - \lambda. \end{vmatrix} =$$
$$= -\lambda\left[-\lambda(a_3 - \lambda)\left(\frac{a_3}{D} - \lambda\right) + \frac{J_4(a_3 - \lambda)}{D}\right] + \left[\lambda\left(J_1\lambda - \frac{J_1a_3}{D}\right) + \frac{J_1J_4 - J_2J_3}{D}\right] =$$
$$= \lambda^4 - a_3\left(1 + \frac{1}{D}\right)\lambda^3 + \left(\frac{a_3^2 + J_4}{D} + J_1\right)\lambda^2 - a_3\left(\frac{J_1 + J_4}{D}\right)\lambda + \frac{J_1J_4 - J_2J_3}{D} \overset{!}{=} 0.$$

The corresponding roots of this equation can be identified but they are very complex and thus we do not write them down explicitly. The solution has the same form as in Subsection 4.2.1.

## 4.3 Boundary conditions

Finally we identify the invariant solutions satisfying Dirichlet, Neumann, Robin and periodic boundary conditions, see Section 2.6. All invariant solutions we found are of the form $Ce^{axr(t)+bt}$ or $Dxr(t)e^{cxr(t)+dt}$, where $C, D, a, b, c, d$ are some constants a thus

$$Ce^{axr(t)+bt}\Big|_{x=0} = Ce^{bt},$$
$$Ce^{axr(t)+bt}\Big|_{x=1} = Ce^{ar(t)+bt},$$
$$Dxr(t)e^{cxr(t)+dt}\Big|_{x=0} = 0,$$
$$Dxr(t)e^{cxr(t)+dt}\Big|_{x=1} = Dr(t)e^{cr(t)+dt},$$

$$\left.\frac{\partial Ce^{axr(t)+bt}}{\partial x}\right|_{x=0} = Car(t)e^{bt},$$
$$\left.\frac{\partial Ce^{axr(t)+bt}}{\partial x}\right|_{x=1} = Car(t)e^{ar(t)+bt},$$
$$\left.\frac{\partial Dxr(t)e^{cxr(t)+dt}}{\partial x}\right|_{x=0} = Dr(t)e^{dt},$$
$$\left.\frac{\partial Dxr(t)e^{cxr(t)+dt}}{\partial x}\right|_{x=1} = Dr(t)e^{cr(t)+dt} + Dcr^2(t)e^{cr(t)+dt},$$

which implies that we can inspect these solutions separately as above in Section 3.3.

### 4.3.1 Dirichlet boundary conditions

We would like to find invariant solutions $(u_D(x,t), v_D(x,t))$ which satisfy

$$u_D(0,t) = u_D(1,t) = 0,$$
$$v_D(0,t) = v_D(1,t) = 0.$$

The first invariant solution (4.25) gives

$$u_1(0,t) = \sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{a_1 t} \stackrel{!}{=} 0,$$
$$u_1(1,t) = \sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{\lambda_i r(t)+a_1 t} \stackrel{!}{=} 0$$
$$\implies K_i = 0, \forall i \in \{1,2,3,4\}$$
$$\implies u_{D1}(x,t) = 0.$$

The second invariant solution (4.28) yields

$$u_2(0,t) = K_+ e^{a_1 t} + K_- e^{a_1 t} + Me^{a_1 t} \stackrel{!}{=} 0,$$
$$u_2(1,t) = K_+ e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} r(t) + a_1 t} + K_- e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} r(t) + a_1 t} +$$
$$+ Lr(t)e^{a_1 t} + Me^{a_1 t} \stackrel{!}{=} 0$$
$$\implies u_{D2}(x,t) = 0.$$

The third invariant solution (4.28) reads

$$u_3(0,t) = Le^{a_1 t} + Ne^{a_1 t} \stackrel{!}{=} 0,$$
$$u_3(1,t) = (Kr(t) + L)\, e^{\lambda r(t) + a_1 t} + (Mr(t) + N)\, e^{-\lambda r(t) + a_1 t} \stackrel{!}{=} 0$$
$$\implies u_{D3}(x,t) = 0.$$

And finally the fourth solution (4.29) gives

$$u_4(0,t) = C_1 e^{\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 e^{\frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=} 0,$$
$$u_4(1,t) = C_1 e^{\frac{r(t)}{a_2} + \frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 e^{\frac{r(t)}{a_2} + \frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=} 0$$

$$\implies u_{D4}(x,t) = 0.$$

The invariant solution which satisfies Dirichlet boundary conditions is

$$u_D(x,t) = 0$$

and for the same reason

$$v_D(x,t) = 0.$$

### 4.3.2 Neumann boundary conditions

Next we would like to find the invariant solutions $(u_N(x,t), v_N(x,t))$ which satisfy

$$\begin{aligned}
\partial_x u_N(0,t) &= \partial_x u_N(1,t) = 0,\\
\partial_x v_N(0,t) &= \partial_x v_N(1,t) = 0.
\end{aligned}$$

The first invariant solution (4.25) reads

$$\begin{aligned}
\partial_x u_1(0,t) &= -\sum_{i=1}^{4} K_i r(t) J_2 e^{a_1 t} \stackrel{!}{=} 0,\\
\partial_x u_1(1,t) &= -\sum_{i=1}^{4} K_i r(t) J_2 e^{\lambda_i r(t) + a_1 t} \stackrel{!}{=} 0\\
&\implies K_i = 0, \forall i \in \{1,2,3,4\}\\
&\implies u_{N1}(x,t) = 0.
\end{aligned}$$

The second invariant solution (4.27) gives

$$\begin{aligned}
&\partial_x u_2(0,t) = K_+\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}r(t)e^{a_1 t} - K_-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}r(t)e^{a_1 t} +\\
&+ Lr(t)e^{a_1 t} \stackrel{!}{=} 0,\\
&\partial_x u_2(1,t) = K_+\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}r(t)e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}r(t) + a_1 t} -\\
&- K_-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}r(t)e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}}r(t) + a_1 t} +\\
&+ Lr(t)e^{a_1 t} \stackrel{!}{=} 0\\
&\implies u_{N2}(x,t) = Me^{a_1 t},
\end{aligned}$$

where $a_1$ is an eigenvalue of $\mathbb{J}$.

The third invariant solution (4.28) yields

$$\begin{aligned}
&\partial_x u_3(0,t) = \left(Kr(t) + L\lambda r(t)\right)e^{a_1 t} + \left(Mr(t) - N\lambda r(t)\right)e^{a_1 t} \stackrel{!}{=} 0,\\
&\partial_x u_3(1,t) = \left(Kr(t) + K\lambda r^2(t) + L\lambda r(t)\right)e^{\lambda r(t) + a_1 t} +\\
&+ \left(Mr(t) - M\lambda r^2(t) - N\lambda r(t)\right)e^{-\lambda r(t) + a_1 t} \stackrel{!}{=} 0,\\
&\lambda \neq 0 \implies u_{N3}(x,t) = 0,
\end{aligned}$$

$$\lambda = 0 \implies u_{N3}(x,t) = Le^{a_1 t}$$
$$\lambda = 0 \iff a_1 - J_1 + \frac{a_1 - J_4}{D} = 0 \implies \left(J_1 - a_1 - \frac{J_4 - a_1}{D}\right)^2 - 4\frac{J_2 J_3}{D} =$$
$$= -4\frac{(J_1 - a_1)(J_4 - a_1) - J_2 J_3}{D} = 0 \iff (J_1 - a_1)(J_4 - a_1) - J_2 J_3 = 0.$$

And finally the fourth invariant solution (4.29) gives

$$\partial_x u_4(0,t) = C_1 \frac{r(t)}{a_2} e^{\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 \frac{r(t)}{a_2} e^{\frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=} 0,$$
$$\partial_x u_4(1,t) = C_1 \frac{r(t)}{a_2} e^{\frac{r(t)}{a_2}+\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 \frac{r(t)}{a_2} e^{\frac{r(t)}{a_2}+\frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=} 0$$
$$\implies u_{N4}(x,t) = 0.$$

The invariant solution which satisfies Neumann boundary conditions is

$$u_N(x,t) = Me^{a_1 t}$$

and for the same reason

$$v_N(x,t) = M\frac{a_1 - J_1}{J_2} e^{a_1 t},$$

where $a_1$ is an eigenvalue of $\mathbb{J}$.

### 4.3.3 Robin boundary conditions

We would like to find solutions $(u_N(x,t), v_N(x,t))$, which satisfy

$$\frac{1}{\tilde{L}r(t)} \partial_x u_R(0,t) - \alpha u_R(0,t) = \frac{1}{\tilde{L}r(t)} \partial_x u_R(1,t) + \alpha u_R(1,t) = 0,$$
$$\frac{1}{\tilde{L}r(t)} \partial_x v_R(0,t) - \alpha v_R(0,t) = \frac{1}{\tilde{L}r(t)} \partial_x v_R(1,t) + \alpha v_R(1,t) = 0.$$

The first invariant solution (4.25) yields

$$\frac{1}{\tilde{L}r(t)} \partial_x u_1(0,t) - \alpha u_1(0,t) = -\frac{1}{\tilde{L}} \sum_{i=1}^{4} K_i J_2 e^{a_1 t} - \alpha \sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{a_1 t} \stackrel{!}{=} 0,$$
$$\frac{1}{\tilde{L}r(t)} \partial_x u_1(1,t) + \alpha u_1(1,t) = -\frac{1}{\tilde{L}} \sum_{i=1}^{4} K_i J_2 e^{\lambda_i r(t) + a_1 t} + \alpha \sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{\lambda_i r(t) + a_1 t} \stackrel{!}{=} 0$$
$$\implies u_{R1}(x,t) = 0.$$

The second invariant solution (4.27) gives

$$\frac{1}{\tilde{L}r(t)} \partial_x u_2(0,t) - \alpha u_2(0,t) = \frac{K_+}{\tilde{L}} \sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} e^{a_1 t} -$$
$$- \frac{K_-}{\tilde{L}} \sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} e^{a_1 t} + \frac{L}{\tilde{L}} e^{a_1 t} - \alpha K_+ e^{a_1 t} +$$
$$- \alpha K_- e^{a_1 t} - \alpha M e^{a_1 t} \stackrel{!}{=} 0,$$

$$
\begin{aligned}
&\frac{1}{\tilde{L}r(t)}\partial_x u_2(1,t)+\alpha u_2(1,t)=\frac{K_+}{\tilde{L}}\sqrt{a_1-J_1+\frac{a_1-J_4}{D}}e^{\sqrt{a_1-J_1+\frac{a_1-J_4}{D}}r(t)+a_1t}-\\
&-\frac{K_-}{\tilde{L}}\sqrt{a_1-J_1+\frac{a_1-J_4}{D}}e^{-\sqrt{a_1-J_1+\frac{a_1-J_4}{D}}r(t)+a_1t}+\\
&+\frac{L}{\tilde{L}}e^{a_1t}+\alpha K_+e^{\sqrt{a_1-J_1+\frac{a_1-J_4}{D}}r(t)+a_1t}+\\
&+\alpha K_-e^{-\sqrt{a_1-J_1+\frac{a_1-J_4}{D}}r(t)+a_1t}+\alpha Lr(t)e^{a_1t}+\alpha Me^{a_1t}\overset{!}{=}0\\
&\implies u_{R2}(x,t)=0.
\end{aligned}
$$

The third invariant solution (4.29) reads

$$
\begin{aligned}
&\frac{1}{\tilde{L}r(t)}\partial_x u_3(0,t)-\alpha u_3(0,t)=\frac{1}{\tilde{L}}\left(K+L\lambda\right)e^{a_1t}+\\
&+\frac{1}{\tilde{L}}\left(M-N\lambda\right)e^{a_1t}+\\
&-\alpha Le^{a_1t}-\alpha Ne^{a_1t}\overset{!}{=}0,\\
&\frac{1}{\tilde{L}r(t)}\partial_x u_3(1,t)+\alpha u_3(1,t)=\frac{1}{\tilde{L}}\left(K+K\lambda r(t)+L\lambda\right)e^{\lambda r(t)+a_1t}+\\
&+\frac{1}{\tilde{L}}\left(M-M\lambda r(t)-N\lambda\right)e^{-\lambda r(t)+a_1t}+\\
&+\alpha\left(Kr(t)+L\right)e^{\lambda r(t)+a_1t}+\alpha\left(Mr(t)+N\right)e^{-\lambda r(t)+a_1t}\overset{!}{=}0\\
&\implies u_{R3}(x,t)=0.
\end{aligned}
$$

And finally the fourth invariant solution (4.29) yields

$$
\begin{aligned}
&\frac{1}{\tilde{L}r(t)}\partial_x u_4(0,t)-\alpha u_4(0,t)=C_1\frac{1}{\tilde{L}a_2}e^{\frac{1}{2}(k-\sqrt{k^2-4l})t}+C_2\frac{1}{\tilde{L}a_2}e^{\frac{1}{2}(k+\sqrt{k^2-4l})t}-\\
&-\alpha C_1e^{\frac{1}{2}(k-\sqrt{k^2-4l})t}-\alpha C_2e^{\frac{1}{2}(k+\sqrt{k^2-4l})t}\overset{!}{=}0,\\
&\frac{1}{\tilde{L}r(t)}\partial_x u_4(1,t)+\alpha u_4(1,t)=C_1\frac{1}{\tilde{L}a_2}e^{\frac{r(t)}{a_2}+\frac{1}{2}(k-\sqrt{k^2-4l})t}+C_2\frac{1}{\tilde{L}a_2}e^{\frac{r(t)}{a_2}+\frac{1}{2}(k+\sqrt{k^2-4l})t}+\\
&+\alpha C_1e^{\frac{r(t)}{a_2}+\frac{1}{2}(k-\sqrt{k^2-4l})t}+\alpha C_2e^{\frac{r(t)}{a_2}+\frac{1}{2}(k+\sqrt{k^2-4l})t}\overset{!}{=}0\\
&\implies u_{R4}(x,t)=0.
\end{aligned}
$$

The invariant solution which satisfies Robin boundary conditions is

$$u_R(x,t)=0$$

and for the same reason

$$v_R(x,t)=0.$$

### 4.3.4 Periodic boundary conditions

Finally we would like to find the solutions which satisfy periodic boundary conditions

$$u_P(0,t)=u_P(1,t)\quad\text{and}\quad\partial_x u_P(0,t)=\partial_x u_P(1,t),$$

$$v_P(0,t) = v_P(1,t) \quad \text{and} \quad \partial_x v_P(0,t) = \partial_x v_P(1,t).$$

The first invariant solution (4.25) gives

$$\begin{aligned}
&u_1(0,t) = u_1(1,t) \Longleftrightarrow \\
&\sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{a_1 t} \stackrel{!}{=} \sum_{i=1}^{4} K_i \frac{(-J_2)}{\lambda_i} e^{\lambda_i r(t) + a_1 t}, \\
&\partial_x u_1(0,t) = \partial_x u_1(1,t) \Longleftrightarrow \\
&-\sum_{i=1}^{4} K_i r(t) J_2 e^{a_1 t} \stackrel{!}{=} -\sum_{i=1}^{4} K_i r(t) J_2 e^{\lambda_i r(t) + a_1 t},
\end{aligned}$$

which yields

$$u_{P1}(x,t) = 0$$

and for the same reason

$$v_{P1}(x,t) = 0.$$

The second invariant solution (4.25) reads

$$\begin{aligned}
&u_2(0,t) = u_2(1,t) \Longleftrightarrow \\
&K_+ e^{a_1 t} + K_- e^{a_1 t} + M e^{a_1 t} \stackrel{!}{=} \\
&\stackrel{!}{=} K_+ e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} r(t) + a_1 t} + K_- e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} r(t) + a_1 t} + L r(t) e^{a_1 t} + M e^{a_1 t}, \\
&\partial_x u_2(0,t) = \partial_x u_2(1,t) \Longleftrightarrow \\
&K_+ r(t) \sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} e^{a_1 t} - K_- r(t) \sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} e^{a_1 t} + L r(t) e^{a_1 t} \stackrel{!}{=} \\
&\stackrel{!}{=} K_+ r(t) \sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} e^{\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} r(t) + a_1 t} - \\
&- K_- r(t) \sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} e^{-\sqrt{a_1 - J_1 + \frac{a_1 - J_4}{D}} r(t) + a_1 t} + L r(t) e^{a_1 t},
\end{aligned}$$

which yields

$$u_{P2}(x,t) = M e^{a_1 t}$$

and for the same reason

$$v_{P2}(x,t) = M \frac{a_1 - J_1}{J_2} e^{a_1 t},$$

where $a_1$ is an eigenvalue of $\mathbb{J}$.

The third invariant solution (4.28) gives

$$\begin{aligned}
&u_3(0,t) = u_3(1,t) \Longleftrightarrow \\
&L e^{a_1 t} + N e^{a_1 t} \stackrel{!}{=} (K r(t) + L) e^{\lambda r(t) + a_1 t} + (M r(t) + N) e^{-\lambda r(t) + a_1 t}, \\
&\partial_x u_3(0,t) = \partial_x u_3(1,t) \Longleftrightarrow \\
&(K r(t) + L \lambda r(t)) e^{a_1 t} + (M r(t) - N \lambda r(t)) e^{a_1 t} \stackrel{!}{=} \\
&\stackrel{!}{=} \left(K r(t) + K \lambda r^2(t) + L \lambda r(t)\right) e^{\lambda r(t) + a_1 t} + \left(M r(t) - M \lambda r^2(t) - N \lambda r(t)\right) e^{-\lambda r(t) + a_1 t},
\end{aligned}$$

which yields

$$u_{P3}(x,t) = Le^{a_1 t},$$
$$v_{P3}(x,t) = L\frac{a_1 - J_1}{J_2}e^{a_1 t},$$

for $\lambda = 0$, where $a_1$ is an eigenvalue of $\mathbb{J}$ or

$$u_{P3}(x,t) = 0,$$
$$v_{P3}(x,t) = 0,$$

for $\lambda \neq 0$.

And finally the fourth invariant solution (4.29) reads

$$u_4(0,t) = u_4(1,t) \Longleftrightarrow$$
$$C_1 e^{\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 e^{\frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=} C_1 e^{\frac{r(t)}{a_2}+\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 e^{\frac{r(t)}{a_2}+\frac{1}{2}(k+\sqrt{k^2-4l})t},$$
$$\partial_x u_4(0,t) = \partial_x u_4(1,t) \Longleftrightarrow$$
$$C_1 \frac{r(t)}{a_2} e^{\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 \frac{r(t)}{a_2} e^{\frac{1}{2}(k+\sqrt{k^2-4l})t} \stackrel{!}{=}$$
$$\stackrel{!}{=} C_1 \frac{r(t)}{a_2} e^{\frac{r(t)}{a_2}+\frac{1}{2}(k-\sqrt{k^2-4l})t} + C_2 \frac{r(t)}{a_2} e^{\frac{r(t)}{a_2}+\frac{1}{2}(k+\sqrt{k^2-4l})t},$$

which yields

$$u_{P4}(x,t) = 0$$

and for the same reason

$$v_{P4}(x,t) = 0.$$

# Chapter 5

# Scalar case of uniform growth with non-fickian diffusion

In this chapter we consider the scalar case of the uniform growth with non-fickian diffusion, which is described by the equation

$$F_S = u_t + h(t)u - \frac{Du_x^2}{\varphi^2(t)} - \frac{Duu_{xx}}{\varphi^2(t)} - Ju = 0, \tag{5.1}$$

where $h(t) = \frac{\dot{\varphi}(t)}{\varphi(t)}$, see (2.14).

## 5.1 Symmetries and their infinitesimal generators

First we find the symmetries and the corresponding infinitesimal generators of equation (5.1). The linearized condition has form

$$X^{(2)}(F_S) = \eta^t + h(t)\eta + \dot{h}(t)u\tau - \frac{2Du_x}{\varphi^2(t)}\eta^x + \frac{2D\dot{\varphi}(t)}{\varphi^3(t)}u_x^2\tau - \frac{Du_{xx}}{\varphi^2(t)}\eta -$$
$$- \frac{Du}{\varphi^2(t)}\eta^{xx} + \frac{2D\dot{\varphi}(t)}{\varphi^3(t)}uu_{xx}\tau - J\eta =$$
$$= \eta_t - \xi_t u_x + \eta_u\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right) - \tau_t\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right) -$$
$$- \xi_u u_x\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right) - \tau_u\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right)^2 + \eta h + \tau\dot{h}u -$$
$$- \frac{2D\eta_x}{\varphi^2}u_x(\xi_x - \eta_u)\frac{2Du_x^2}{\varphi^2} + 2\tau_x\frac{Du_x}{\varphi^2}\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right) + \xi_u\frac{2Du_x^3}{\varphi^2} +$$
$$+ \tau_u\frac{2Du_x^2}{\varphi^2}\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right) + \tau\frac{2D\dot{\varphi}}{\varphi^3}u_x^2 - \frac{D\eta}{\varphi^2}u_{xx} + \frac{2D\tau\dot{\varphi}}{\varphi^3}uu_{xx} - \eta_{xx}\frac{Du}{\varphi^2} -$$
$$- (2\eta_{xu} - \xi_{xx})\frac{Du}{\varphi^2}u_x + \tau_{xx}\frac{Du}{\varphi^2}\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right) - (\eta_{uu} - 2\xi_{xu})\frac{Du}{\varphi^2}u_x^2 +$$
$$+ \tau_{xu}\frac{2Du}{\varphi^2}\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right)u_x + \xi_{uu}\frac{Du}{\varphi^2}u_x^3 +$$
$$+ \tau_{uu}\frac{Du}{\varphi^2}\left(\frac{Du_x^2}{\varphi^2} + \frac{Duu_{xx}}{\varphi^2} + Ju - hu\right)u_x^2 - (\eta_u - 2\xi_x)\frac{Du}{\varphi^2}u_{xx} + 2\tau_x\frac{Du}{\varphi^2}u_{xt} + 3\xi_u\frac{Du}{\varphi^2}u_x u_{xx} +$$

$$+\tau_u \frac{Du}{\varphi^2}\left(\frac{Du_x^2}{\varphi^2}+\frac{Duu_{xx}}{\varphi^2}+Ju-hu\right)u_{xx}+2\tau_u\frac{Du}{\varphi^2}u_xu_{xt}-J\eta=0,$$

where we used the formulas from Section 1.1 and the condition $F_S = 0$. As a next step we compare the terms multiplied by derivatives of $u$ and their products independently:

$$\begin{aligned}
u_xu_{xt}: &\quad \frac{2Du}{\varphi^2}\tau_u=0,\\
u_{xt}: &\quad \frac{2Du}{\varphi^2}\tau_x=0,
\end{aligned}$$

which implies $\tau = \tau(t)$. Further

$$\begin{aligned}
u_xu_{xx}: &\quad 2\frac{Du}{\varphi^2}\xi_u=0 \implies \xi=\xi(x,t),\\
u_x^2: &\quad -\eta_u\frac{D}{\varphi^2}-\dot\tau\frac{D}{\varphi^2}+2\xi_x\frac{D}{\varphi^2}+2\frac{D\dot\varphi}{\varphi^3}\tau-\frac{Du}{\varphi^2}\eta_{uu}=0,\\
u_{xx}: &\quad -\dot\tau\frac{Du}{\varphi^2}-\frac{D\eta}{\varphi^2}+2\frac{D\dot\varphi}{\varphi^3}\tau u+2\frac{Du}{\varphi^2}\xi_x=0,\\
u_x: &\quad -\xi_t-2\frac{D\eta_x}{\varphi^2}-2\frac{D\eta_{xu}}{\varphi^2}u+\frac{D\xi_{xx}}{\varphi^2}u=0,\\
1: &\quad \eta_t-\eta_u hu+\eta_u Ju+\eta h+\dot\tau hu-\dot\tau Ju+\tau\dot hu-\frac{Du}{\varphi^2}\eta_{xx}-\eta J=0.
\end{aligned}$$

We proceed to solution of the remaining set of equations:

$$\eta_u+u\eta_{uu}+\dot\tau-2h\tau-2\xi_x=0, \tag{5.2}$$
$$2u\xi_x-\dot\tau u-\eta+2h\tau u=0, \tag{5.3}$$
$$\xi_t\varphi^2+2D\eta_x+2D\eta_{xu}u-Du\xi_{xx}=0, \tag{5.4}$$
$$\eta_t-\eta_u hu+\eta_u Ju+\eta h+\dot\tau hu-\dot\tau Ju+\tau\dot hu-\frac{Du}{\varphi^2}\eta_{xx}-\eta J=0. \tag{5.5}$$

From equation (5.3) we have

$$\eta=(2\xi_x-\dot\tau+2h\tau)u,$$

which we then substitute into equation (5.4):

$$\xi_t\varphi^2+4D\xi_{xx}u+4D\xi_{xx}u-D\xi_{xx}u=0.$$

By collecting prefactors of powers of $u$ we obtain

$$\begin{aligned}
u: &\quad 7D\xi_{xx}=0,\\
1: &\quad \xi_t\varphi^2=0
\end{aligned}$$

and hence

$$\begin{aligned}
\xi&=ax+b,\\
\eta&=(2a-\dot\tau+2h\tau)u,
\end{aligned}$$

where $a, b$ are arbitrary constants. Now we substitute these results into equations (5.2) and (5.5):

$$2a-\dot\tau+2h\tau+\dot\tau-2h\tau-2a=0,$$

$$
\begin{aligned}
&-\ddot{\tau}u + 2\dot{h}\tau u + 2h\dot{\tau}u - (2a - \dot{\tau} + 2h\tau)hu + (2a - \dot{\tau} + 2h\tau)Ju + (2a - \dot{\tau} + 2h\tau)hu+\\
&+\dot{\tau}hu - \dot{\tau}Ju + \tau\dot{h}u - (2a - \dot{\tau} + 2h\tau)Ju = 0,
\end{aligned}
$$

to get

$$
\begin{aligned}
&\ddot{\tau} - \frac{\mathrm{d}}{\mathrm{d}t}(3h\tau - J\tau) = 0,\\
&\dot{\tau} = (3h - J)\tau + c,
\end{aligned}
$$

where $c$ is an arbitrary constant. We obtain a first order linear ordinary differential equation. The general solution of this differential equation is

$$
\tau = c\left(\int e^{\int^t(-3h(\tilde{t})+J)\mathrm{d}\tilde{t}}\mathrm{d}t\right)e^{\int(3h(t)-J)\mathrm{d}t}.
$$

With

$$
\int h(t)\mathrm{d}t = \int \frac{\dot{\varphi}(t)}{\varphi(t)}\mathrm{d}t = \ln\varphi(t)
$$

we get

$$
\begin{aligned}
\tau &= c\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t,\\
\eta &= \left(2a - c - 3c\dot{\varphi}\varphi^2 e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t + Jc\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t+\right.\\
&\left.+\,2hc\varphi^3e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u\\
&= \left(2a - c - c[\dot{\varphi}\varphi^2 - J\varphi^3]e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u.
\end{aligned}
$$

Finally we can summarize the results as

$$
\begin{aligned}
\eta &= \left(2a - c - c[\dot{\varphi}\varphi^2 - J\varphi^3(t)]e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u,\\
\xi &= ax + b,\\
\tau &= c\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t,
\end{aligned}
$$

and thus the infinitesimal generators of the symmetries of equation (5.1) are spanned by

$$
\begin{aligned}
X_1 &= 2u\partial_u + x\partial_x,\\
X_2 &= \partial_x,\\
X_3 &= -\left(1 + [\dot{\varphi}\varphi^2 - J\varphi^3(t)]e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u\partial_u + \varphi^3(t)e^{-Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)\partial_t.
\end{aligned}
$$

Next we verify that the infinitesimal generators form Lie algebra:

$$
[X_1, X_2] = -\partial_x = -X_2,
$$

$$
\begin{aligned}
[X_1, X_3] &= -2u\left(1 + [\dot\varphi\varphi^2 - J\varphi^3(t)]e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)\partial_u + \\
&\quad + \left(1 + [\dot\varphi\varphi^2 - J\varphi^3(t)]e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u2\partial_u = 0, \\
[X_2, X_3] &= 0.
\end{aligned}
$$

Due to the complexity the generator $X_3$, we will write down the corresponding symmetries only for the generators $X_1, X_2$ :

1. $X_1$ :

$$
\begin{aligned}
\frac{\mathrm{d}\hat t}{\mathrm{d}\varepsilon} &= 0 \implies \hat t = t, \\
\frac{\mathrm{d}\hat x}{\mathrm{d}\varepsilon} &= x \implies \hat x = xe^{\varepsilon}, \\
\frac{\mathrm{d}\hat u}{\mathrm{d}\varepsilon} &= 2u \implies \hat u = ue^{2\varepsilon}.
\end{aligned}
$$

2. $X_2$ :

$$
\begin{aligned}
\frac{\mathrm{d}\hat t}{\mathrm{d}\varepsilon} &= 0 \implies \hat t = t, \\
\frac{\mathrm{d}\hat x}{\mathrm{d}\varepsilon} &= 1 \implies \hat x = x + \varepsilon, \\
\frac{\mathrm{d}\hat u}{\mathrm{d}\varepsilon} &= 0 \implies \hat u = u.
\end{aligned}
$$

## 5.2 Invariant solutions

We would like to find out a solution which is invariant under the symmetry generated by

$$aX_1 + bX_2 + X_3,$$

where $a, b$ are arbitrary parameters.The characteristic has form

$$
\begin{aligned}
\mathcal{Q} = &\left(2a - 1 - (\dot\varphi\varphi^2 - J\varphi^3(t))e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u - (ax+b)u_x - \\
&- \varphi^3(t)e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t u_t = 0.
\end{aligned}
$$

We write down the corresponding characteristic equations of the characteristic

$$\frac{\mathrm{d}u}{\left(2a - 1 - (\dot\varphi\varphi^2 - J\varphi^3(t))e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u} = \frac{\mathrm{d}x}{ax+b} = \frac{\mathrm{d}t}{\varphi^3(t)e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t}$$

and solve them to find suitable transformations of the original problem:

$$
\begin{aligned}
&\frac{\mathrm{d}x}{ax+b} = \frac{\mathrm{d}t}{\varphi^3(t)e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t} \\
&\implies \frac{\ln ax+b}{a} = \int \frac{\varphi^{-3}(t)e^{Jt}}{\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t}\mathrm{d}t = \ln\left(\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)
\end{aligned}
$$

$$
\begin{aligned}
&\implies p = \frac{ax+b}{\left(\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a},\\
&\frac{\mathrm{d}u}{\left(2a-1-(\dot{\varphi}\varphi^2 - J\varphi^3(t))e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)u} = \frac{\mathrm{d}t}{\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t}\\
&\implies \ln u = \int \frac{\left(2a-1-(\dot{\varphi}\varphi^2 - J\varphi^3(t))e^{-Jt}\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)}{\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t}\mathrm{d}t =\\
&= \int \frac{2a-1}{\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t}\mathrm{d}t - \int \frac{\dot{\varphi}}{\varphi(t)}\mathrm{d}t + \int \frac{J\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t}{\varphi^3(t)e^{-Jt}\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t}\mathrm{d}t =\\
&= (2a-1)\ln\left(\int \varphi^{-3}(t)e^{Jt}\mathrm{d}t\right) - \int h(t)\mathrm{d}t + Jt\\
&\implies u = \frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)} F(p).
\end{aligned}
$$

Now we substitute these expressions into the original PDE:

$$
\begin{aligned}
&-h\frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)}F(p) + J\frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)}F(p)+\\
&+(2a-1)\varphi^{-3}(t)e^{Jt}\frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-2)}F(p)+\\
&+\frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)}F'(p)\frac{ax+b}{\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a}\frac{-a\varphi^{-3}(t)e^{Jt}}{\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t}+\\
&+h\frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)}F(p) =\\
&=\frac{D}{\varphi^4}e^{2Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{2(2a-1)}F'^2(p)\left(\frac{a}{\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a}\right)^2+\\
&+\frac{D}{\varphi^4}e^{2Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{2(2a-1)}F''(p)F(p)\left(\frac{a}{\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a}\right)^2+\\
&+J\frac{1}{\varphi}e^{Jt}\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)}F(p)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&(2a-1)\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-2)}F(p)+\\
&+\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{(2a-1)}F'(p)\frac{ax+b}{\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a}\frac{-a}{\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t} =\\
&=D\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{2(2a-1)}F'^2(p)\left(\frac{a}{\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a}\right)^2+\\
&+D\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^{2(2a-1)}F''(p)F(p)\left(\frac{a}{\left(\int\varphi^{-3}(t)e^{Jt}\mathrm{d}t\right)^a}\right)^2
\end{aligned}
$$

or in a yet simpler form

$$a^2DF'^2(p) + a^2DF(p)F''(p) + apF'(p) - (2a-1)F(p) = 0. \tag{5.6}$$

Because of complexity of this ODE we shall employ the method of symmetries for solving it. The symmetries of the ordinary differential equations can be found similarly as the symmetries of PDEs. These symmetries of ODEs resp. their infinitesimal generators can be used to reduce their order in the case of high order ODEs and can be used to solve them in the case of first order ODEs, see [4] and [13].

### 5.2.1 Symmetries of corresponding ODE

We denote

$$F_o := a^2Dy''(x)y(x) + a^2Dy'^2(x) + axy'(x) - (2a-1)y(x). \tag{5.7}$$

The linearized symmetry condition has the following form

$$\begin{aligned}
&X^2(F_o) = a^2D\eta^{(2)}y + a^2D\eta y'' + 2a^2Dy'\eta^{(1)} + a\xi y' + ax\eta^{(1)} - (2a-1)\eta = \\
&= a^2Dy\eta_{xx} + a^2Dy(2\eta_{xy} - \xi_{xx})y' + a^2Dy(\eta_{yy} - 2\xi_{xy})y'^2 - a^2Dy\xi_{yy}y'^3 + \\
&+ a^2Dy(\eta_y - 2\xi_x - 3\xi_y y')\left(\frac{2a-1}{a^2D} - \frac{y'^2}{y} - \frac{xy'}{aDy}\right) + a^2D\eta\left(\frac{2a-1}{a^2D} - \frac{y'^2}{y} - \frac{xy'}{aDy}\right) + \\
&+ 2a^2D\eta_x y' + 2a^2D(\eta_y - \xi_x)y'^2 - 2a^2D\xi_y y'^3 + a\xi y' + ax\eta_x + ax(\eta_y - \xi_x)y' - ax\xi_y y'^2 - \\
&- (2a-1)\eta = 0.
\end{aligned}$$

Next we compare the terms multiplied by different powers of $y'$ :

$$\begin{aligned}
y'^3:\quad & -a^2Dy\xi_{yy} + 3a^2D\xi_y - 2a^2D\xi_y = 0 \\
& \Longleftrightarrow \frac{\partial_y \xi_y}{\partial_y} = \frac{\xi_y}{y} \\
& \Longleftrightarrow \ln \xi_y = \ln y + f(x) \implies \xi_y = A(x)y \implies \xi = \frac{1}{2}A(x)y^2 + B(x), \\
y'^2:\quad & a^2Dy(\eta_{yy} - 2\xi_{xy}) - a^2D(\eta_y - 2\xi_x) + 3ax\xi_y - a^2D\eta\frac{1}{y} + 2a^2D(\eta_y - \xi_x) - ax\xi_y = \\
= \quad & a^2Dy(\eta_{yy} - 2\xi_{xy}) + 2ax\xi_y - a^2D\eta\frac{1}{y} + a^2D\eta_y = 0, \\
y':\quad & a^2Dy(2\eta_{xy} - \xi_{xx}) - 3a^2y\xi_y\frac{2a-1}{a^2} - ax(\eta_y - 2\xi_x) - a\eta\frac{x}{y} + 2a^2D\eta_x + a\xi + ax(\eta_y - \xi_x) = \\
& = a^2Dy(2\eta_{xy} - \xi_{xx}) - 3y\xi_y(2a-1) - a\eta\frac{x}{y} + 2a^2D\eta_x + a\xi + ax\xi_x = 0, \\
1:\quad & a^2Dy\eta_{xx} + y(\eta_y - 2\xi_x)(2a-1) + \eta(2a-1) + ax\eta_x - (2a-1)\eta = \\
& = a^2Dy\eta_{xx} + y(\eta_y - 2\xi_x)(2a-1) + ax\eta_x = 0,
\end{aligned}$$

where $A(x)$ and $B(x)$ are arbitrary functions. We obtain the system of determining equations:

$$\xi = \frac{1}{2}A(x)y^2 + B(x), \tag{5.8}$$

$$a^2Dy(\eta_{yy} - 2\xi_{xy}) + 2ax\xi_y - a^2D\eta\frac{1}{y} + a^2D\eta_y = 0, \tag{5.9}$$

$$a^2Dy(2\eta_{xy} - \xi_{xx}) - 3y\xi_y(2a-1) - a\eta\frac{x}{y} + 2a^2D\eta_x + a\xi + ax\xi_x = 0, \tag{5.10}$$

$$a^2Dy\eta_{xx} + y(\eta_y - 2\xi_x)(2a-1) + ax\eta_x = 0. \tag{5.11}$$

By substituting expression (5.8) into equation (5.9) we obtain a condition

$$a^2Dy^2\eta_{yy} + a^2Dy\eta_y - a^2D\eta - 2a^2Dy^3A'(x) + 2ay^2xA(x) = 0. \tag{5.12}$$

To make analytical progress we follow an ansatz

$$\eta(x,y) = C(x)y^3 + \tilde{D}(x)y^2 + E(x)y + F(x),$$

where $C(x), \tilde{D}(x), E(x)$ and $F(x)$ are arbitrary functions and hence condition (5.12) reads

$$\begin{aligned}&6a^2DC(x)y^3 + 2a^2D\tilde{D}(x)y^2 + 3a^2DC(x)y^3 + 2a^2D\tilde{D}(x)y^2 + a^2DE(x)y - \\ &- a^2DC(x)y^3 - a^2D\tilde{D}(x)y^2 - a^2DE(x)y - a^2DF(x) - 2a^2Dy^3A'(x) + 2ay^2xA(x) = 0.\end{aligned}$$

Now we can compare the terms multiplied by different powers of $y$ :

$$\begin{aligned}y^3:&\quad 8a^2DC(x) - 2a^2DA'(x) = 0 \implies C(x) = \frac{1}{4}A'(x),\\ y^2:&\quad 3a^2D\tilde{D}(x) + 2axA(x) = 0 \implies \tilde{D}(x) = -\frac{2}{3aD}xA(x),\\ y:&\quad 0 = 0,\\ 1:&- a^2DF(x) = 0 \implies F(x) = 0\end{aligned}$$

and hence

$$\eta = \frac{1}{4}A'(x)y^3 - \frac{2}{3aD}xA(x)y^2 + E(x)y.$$

Next we substitute this result into condition (5.11) and follow the same procedure:

$$\begin{aligned}&\frac{a^2D}{4}A'''(x)y^4 - \frac{2a}{3}xA''(x)y^3 - \frac{4a}{3}A'(x)y^3 + a^2DE''(x)y^2 + \\ &+ (2a-1)\left(\frac{3}{4}A'(x)y^3 - \frac{4}{3aD}xA(x)y^2 + E(x)y - A'(x)y^3 - 2B'(x)y\right) + \frac{a}{4}xA''(x)y^3 - \\ &- \frac{2}{3D}xA(x)y^2 - \frac{2}{3D}x^2A'(x)y^2 + axE'(x)y = 0,\end{aligned}$$

$$\begin{aligned}y^4:&\quad \frac{a^2D}{4}A'''(x) = 0 \implies A(x) = \frac{1}{2}\tilde{a}x^2 + \tilde{b}x + \tilde{c},\\ y^3:&\quad -\frac{2a\tilde{a}}{3}x - \frac{4a\tilde{a}}{3}x - \frac{4a\tilde{b}}{3} + \frac{3(2a-1)}{4}\tilde{a}x + \frac{3(2a-1)}{4}\tilde{b} - (2a-1)\tilde{a}x - \\ &- (2a-1)\tilde{b} + \frac{a}{4}\tilde{a}x = 0,\\ x:&\quad \left(-\frac{6a}{3} - \frac{3(2a-1)}{4} - (2a-1) + \frac{a}{4}\right)\tilde{a} = 0\\ &\implies \tilde{a} = 0,\end{aligned}$$

$$
\begin{aligned}
1:\quad & \left(-\frac{4a}{3}+\frac{3(2a-1)}{4}-(2a-1)\right)\tilde{b}=0 \\
& \implies \tilde{b}=0, \\
y^2:\quad & a^2DE''(x)-\frac{4(2a-1)}{3Da}\tilde{c}x-\frac{2}{3D}\tilde{c}x=0 \\
& \implies E(x)=\frac{3a-2}{9a^3D^2}\tilde{c}x^3+\tilde{d}x+\tilde{e},
\end{aligned}
$$

$$
\begin{aligned}
y:\quad & (2a-1)E(x)-2(2a-1)B'(x)+axE'(x)= \\
&=(2a-1)\frac{3a-2}{9a^3D^2}\tilde{c}x^3+(2a-1)\tilde{d}x+(2a-1)\tilde{e}-2(2a-1)B'(x)+\frac{3a-2}{3a^2D^2}\tilde{c}x^3+a\tilde{d}x=0 \\
& \implies B(x)=\left(\frac{3a-2}{72a^3D^2}+\frac{3a-2}{24a^2D^2(2a-1)}\right)\tilde{c}x^4+\left(\frac{1}{4}+\frac{a}{4(2a-1)}\right)\tilde{d}x^2+\frac{1}{2}\tilde{e}x+\tilde{f}, \\
\xi(x,y)&=\frac{1}{2}\tilde{c}y^2+\left(\frac{3a-2}{72a^3D^2}+\frac{3a-2}{24a^2D^2(2a-1)}\right)\tilde{c}x^4+\left(\frac{1}{4}+\frac{a}{4(2a-1)}\right)\tilde{d}x^2+\frac{1}{2}\tilde{e}x+\tilde{f}, \\
\eta(x,y)&=\frac{2}{3a}\tilde{c}xy^2+\frac{3a-2}{9a^3D^2}\tilde{c}x^3y+\tilde{d}xy+\tilde{e}y.
\end{aligned}
$$

Finally we substitute these results into equation (5.10):

$$
\begin{aligned}
&2a^2D\frac{4}{3a}\tilde{c}y^2+2a^2\frac{3a-2}{3a^3D}\tilde{c}x^2y+2a^2D\tilde{d}y-a^2\left(\frac{3a-2}{6a^3D}+\frac{3a-2}{2a^2D(2a-1)}\right)\tilde{c}x^2y- \\
&-a^2D\left(\frac{1}{2}+\frac{a}{2(2a-1)}\right)\tilde{d}y-3(2a-1)\tilde{c}y^2-a\frac{2}{3a}\tilde{c}x^2y-a\frac{3a-2}{9a^3D^2}\tilde{c}x^4-a\tilde{d}x^2-a\tilde{e}x+\frac{4aD}{3}\tilde{c}y^2+ \\
&+2\frac{3a-2}{3aD}\tilde{c}x^2y+2a^2D\tilde{d}y+\frac{a}{2}\tilde{c}y^2+\left(\frac{3a-2}{72a^2D^2}+\frac{3a-2}{24aD^2(2a-1)}\right)\tilde{c}x^4+\left(\frac{a}{4}+\frac{a^2}{4(2a-1)}\right)\tilde{d}x^2+ \\
&+\frac{a}{2}\tilde{e}x+a\tilde{f}+\left(\frac{3a-2}{18a^2D^2}+\frac{3a-2}{6aD^2(2a-1)}\right)\tilde{c}x^4+\left(\frac{a}{2}+\frac{a^2}{2(2a-1)}\right)\tilde{d}x^2+\frac{a}{2}\tilde{e}x=0,
\end{aligned}
$$

$$
\begin{aligned}
x^2y:\quad & \left(2\frac{3a-2}{3aD}-\frac{3a-2}{6aD}-\frac{3a-2}{2D(2a-1)}-\frac{2}{3}+2\frac{3a-2}{3aD}\right)\tilde{c}=0 \\
& \implies \tilde{c}=0, \\
x^2:\quad & \left(-a+\frac{a}{4}+\frac{a^2}{4(2a-1)}+\frac{a}{2}+\frac{a^2}{2(2a-1)}\right)\tilde{d}=0 \\
& \implies \tilde{d}=0, \\
x:\quad & \left(-a+\frac{a}{2}+\frac{a}{2}\right)\tilde{e}=0.
\end{aligned}
$$

In summary, we obtained the infinitesimal generators

$$
\begin{aligned}
\xi(x,y)&=\frac{\tilde{e}}{2}x, \\
\eta(x,y)&=\tilde{e}y
\end{aligned}
$$

and thus the corresponding symmetry is

$$X=\frac{1}{2}x\partial_x+y\partial_y.$$

As a next step we have to find the so-called canonical coordinates. The canonical coordinates $(r, s)$ corresponding to the infinitesimal generator $X = \eta\partial_y + \xi\partial_x$ have to satisfy

$$\xi(x,y)r_x + \eta(x,y)r_y = 0,$$
$$\xi(x,y)s_y + \eta(x.y)s_y = 1.$$

We can use the method of characteristic to find these coordinates:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\eta(x,y)}{\xi(x,y)} = \frac{2y}{x}$$
$$\Longrightarrow r = \frac{y}{x^2},$$
$$s = \int \frac{\mathrm{d}x}{\xi(x)} = \ln x.$$

Using these canonical coordinates we may reduce the order of ODE (5.7):

$$\dot{s} = \frac{\mathrm{d}s}{\mathrm{d}r} = \frac{s_x + y's_y}{r_x + y'r_y} = \frac{\frac{1}{x}}{-2\frac{y}{x^3} + \frac{y'}{x^2}} = \frac{x}{y' - 2\frac{y}{x}}.$$

Next we denote a new coordinate

$$v = \frac{1}{\dot{s}} = \frac{y'}{x} - 2r$$

and compute its derivative with respect to $r$ :

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -2 + \frac{\frac{y''}{x} - \frac{y'}{x^2}}{-2\frac{y}{x^3} + \frac{y'}{x^2}} = -2 + \frac{y'' - \frac{y'}{x}}{-2\frac{y}{x^2} + \frac{y'}{x}} = -2 + \frac{y'' - v - 2r}{v}$$
$$\Longrightarrow y'' = \dot{v}v + 3v + 2r.$$

Now we substitute these results into original ODE (5.7):

$$a^2D(\dot{v}v + 3v + 2r)y + a^2D(v+2r)^2x^2 + ax^2(v+2r) - (2a-1)y = 0$$
$$\Longleftrightarrow a^2D(\dot{v}v + 3v + 2r)r + a^2D(v+2r)^2 + a(v+2r) - (2a-1)r = 0$$
$$\Longrightarrow \dot{v}vr + v^2 + 7vr + \frac{v}{aD} + \frac{r}{a^2D} + 6r^2 = 0.$$

Because of complexity of this first order differential equation we shall focus on finding a solution rather than all solutions at hand. If we set $a = \frac{1}{3}$ (recall that $a$ is a free parameter) in original ODE (5.6) we obtain

$$\frac{D}{9}F'^2(p) + \frac{D}{9}F''(p)F(p) + \frac{1}{3}pF'(p) + \frac{1}{3}F(p) = 0,$$

which is equivalent to

$$D(FF')' + 3(Fp)' = 0 \Longleftrightarrow DFF' + 3Fp = K,$$

where $K$ is an arbitrary constant. Further we set $K = 0$ and thus

$$DFF' + 3Fp = 0 \implies F(p) = 0 \text{ or } F(p) = -\frac{3}{2D}p^2 + \tilde{K},$$

where $\tilde{K}$ is an arbitrary constant. The invariant solution of original equation (5.1) with $\frac{1}{a}$ is

$$u(x,t) = \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{\frac{1}{3}x + b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right],$$

where $b$ and $\tilde{K}$ are arbitrary constants.

## 5.3 Boundary conditions

In this section we are interested in finding invariant solutions which satisfy the boundary conditions, see Section 2.6.

### 5.3.1 Dirichlet boundary conditions

We are interested in the solution $u_D(x,t)$ which satisfies the conditions

$$u_D(0,t) = u_D(1,t) = 0.$$

We use the invariant solution from above:

$$\begin{aligned} u(0,t) &= \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right] \overset{!}{=} 0, \\ u(1,t) &= \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{\frac{1}{3} + b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right] \overset{!}{=} 0 \\ \implies & u_D(x,t) = 0. \end{aligned}$$

### 5.3.2 Neumann boundary conditions

Next we are interested in the solution $u_N(x,t)$ which satisfies Neumann boundary conditions:

$$\begin{aligned} \partial_x u(0,t) &= \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{b}{D \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{2}{3}}} + \tilde{K} \right] \overset{!}{=} 0, \\ \partial_x u(1,t) &= \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{\left( \frac{1}{3} + b \right)}{D \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{2}{3}}} + \tilde{K} \right] \overset{!}{=} 0 \\ \implies & u_N(x,t) = 0. \end{aligned}$$

### 5.3.3 Robin boundary conditions

Now we would like to find out the solution $u_R(x,t)$ which satisfies Robin boundary conditions:

$$\frac{1}{Lr(t)} \partial_x u(0,t) - \alpha u(0,t) = \frac{1}{L\dot{r}(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{b}{D \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{2}{3}}} + \tilde{K} \right] -$$

$$- \alpha \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right] \stackrel{!}{=} 0,$$

$$\frac{1}{Lr(t)} \partial_x u(1,t) + \alpha u(1,t) = \frac{1}{L\dot{r}(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{\left(\frac{1}{3} + b\right)}{D \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{2}{3}}} + \tilde{K} \right] +$$

$$+ \alpha \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{\frac{1}{3} + b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right] \stackrel{!}{=} 0,$$

which yields

$$u_R(x,t) = 0.$$

### 5.3.4 Periodic boundary conditions

Finally we would like to find the solution $u_P(x,t)$ which satisfies periodic boundary conditions:

$$u(0,t) = u(1,t) \Longleftrightarrow$$

$$\frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right] =$$

$$= \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{3}{2D} \left( \frac{\frac{1}{3} + b}{\left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{1}{3}}} \right)^2 + \tilde{K} \right],$$

$$\partial_x u(0,t) = \partial_x u(1,t) \Longleftrightarrow$$

$$\frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{b}{D \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{2}{3}}} + \tilde{K} \right] =$$

$$= \frac{1}{\varphi(t)} e^{Jt} \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{(-\frac{1}{3})} \left[ -\frac{\left(\frac{1}{3} + b\right)}{D \left( \int \varphi^{-3}(t) e^{Jt} \mathrm{d}t \right)^{\frac{2}{3}}} + \tilde{K} \right],$$

which yields

$$u_P(x,t) = 0.$$

## 5.4 Spatially independent solution

In the case of apical growth we found an invariant spatially independent solution. Similarly we would like to find the spatially independent solution $u = u(t)$ for the uniform growth:

$$u_t + \frac{\dot{\varphi}(t)}{\varphi(t)} u = Ju,$$

which gives

$$u(t) = \frac{C}{\varphi(t)} e^{Jt},$$

where $C$ is an arbitrary constant. This solution is not invariant under the symmetries we found, however satisfies Neumann and periodic boundary conditions.

# Chapter 6

# Two component system of uniform growth with non-fickian diffusion

Finally we consider a two component system undergoing an uniform growth. It can be described by the set of two equations:

$$F_u = u_t + h(t)u - \frac{u_x^2}{\varphi^2(t)} - \frac{uu_{xx}}{\varphi^2(t)} - J_1 u - J_2 v = 0, \tag{6.1}$$

$$F_v = v_t + h(t)v - D\frac{u_x^2}{\varphi^2(t)} - D\frac{uu_{xx}}{\varphi^2(t)} - J_3 u - J_4 v = 0, \tag{6.2}$$

where $h(t) = \frac{\dot{\varphi}(t)}{\varphi(t)}$, see (2.12).

## 6.1 Symmetries and their infinitesimal generators

As for the scalar case, we would like to find the symmetries of this set of coupled equations. As the framework and procedure remain the same as before we rather list the results of the necessary steps then discuss everything in details. The linearized conditions have form

$$\begin{aligned}
X^{(2)}(F_u) &= \eta^t + h(t)\eta + \dot{h}(t)u\tau - \frac{2u_x}{\varphi^2(t)}\eta^x + \frac{2\dot{\varphi}(t)}{\varphi^3(t)}u_x^2\tau - \frac{u_{xx}}{\varphi^2(t)}\eta - \frac{u}{\varphi^2(t)}\eta^{xx} + \frac{2\dot{\varphi}(t)}{\varphi^3(t)}uu_{xx}\tau - \\
&- J_1\eta - J_2\vartheta = \\
&= \eta_t - \xi_t u_x + \eta_u\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right) - \tau_t\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right) - \\
&- \xi_u u_x\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right) - -\tau_u\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right)^2 + \eta h + \tau\dot{h}u - \\
&- \frac{2\eta_x}{\varphi^2}u_x + (\xi_x - \eta_u)\frac{2u_x^2}{\varphi^2} + 2\tau_x\frac{u_x}{\varphi^2}\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right) + \\
&+ \xi_u\frac{2u_x^3}{\varphi^2} + \tau_u\frac{2u_x^2}{\varphi^2}\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right) + \tau\frac{2\dot{\varphi}}{\varphi^3}u_x^2 - \frac{\eta}{\varphi^2}u_{xx} + \frac{2\tau\dot{\varphi}}{\varphi^3}uu_{xx} - \eta_{xx}\frac{u}{\varphi^2} - \\
&- (2\eta_{xu} - \xi_{xx})\frac{u}{\varphi^2}u_x + \tau_{xx}\frac{u}{\varphi^2}\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right) - (\eta_{uu} - 2\xi_{xu})\frac{u}{\varphi^2}u_x^2 + \\
&+ \tau_{xu}\frac{2u}{\varphi^2}\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1 u + J_2 v - hu\right)u_x + \xi_{uu}\frac{u}{\varphi^2}u_x^3 +
\end{aligned}$$

$$+ \tau_{uu}\frac{u}{\varphi^2}\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1u + J_2v - hu\right)u_x^2 - (\eta_u - 2\xi_x)\frac{u}{\varphi^2}u_{xx} + 2\tau_x\frac{u}{\varphi^2}u_{xt} + 3\xi_u\frac{u}{\varphi^2}u_xu_{xx} +$$
$$+ \tau_u\frac{u}{\varphi^2}\left(\frac{u_x^2}{\varphi^2} + \frac{uu_{xx}}{\varphi^2} + J_1u + J_2v - hu\right)u_{xx} + 2\tau_u\frac{u}{\varphi^2}u_xu_{xt} - J_1\eta - J_2\vartheta = 0,$$

$$X^{(2)}(F_v) = \vartheta^t + h(t)\vartheta + \dot{h}(t)v\tau - D\frac{2v_x}{\varphi^2(t)}\vartheta^x + D\frac{2\dot{\varphi}(t)}{\varphi^3(t)}v_x^2\tau - D\frac{v_{xx}}{\varphi^2(t)}\vartheta - D\frac{v}{\varphi^2(t)}\vartheta^{xx} +$$
$$+ D\frac{2\dot{\varphi}(t)}{\varphi^3(t)}vv_{xx}\tau - -J_3\eta - J_4\vartheta =$$
$$= \vartheta_t - \xi_tv_x + \vartheta_v\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right) - \tau_t\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right) -$$
$$- \xi_vv_x\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right) - \tau_v\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right)^2 + \vartheta h +$$
$$+ \tau\dot{h}v - D\frac{2\vartheta_x}{\varphi^2}v_x + D(\xi_x - \vartheta_v)\frac{2v_x^2}{\varphi^2} + 2D\tau_x\frac{v_x}{\varphi^2}\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right) +$$
$$+ D\xi_v\frac{2v_x^3}{\varphi^2} + D\tau_v\frac{2v_x^2}{\varphi^2}\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right) + D\tau\frac{2\dot{\varphi}}{\varphi^3}v_x^2 - D\frac{\vartheta}{\varphi^2}v_{xx} + D\frac{2\tau\dot{\varphi}}{\varphi^3}vv_{xx} -$$
$$- D\vartheta_{xx}\frac{v}{\varphi^2} - D(2\vartheta_{xv} - \xi_{xx})\frac{v}{\varphi^2}v_x + D\tau_{xx}\frac{v}{\varphi^2}\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right) -$$
$$- D(\vartheta_{vv} - 2\xi_{xv})\frac{v}{\varphi^2}v_x^2 + D\tau_{xv}\frac{2v}{\varphi^2}\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right)v_x + D\xi_{vv}\frac{v}{\varphi^2}v_x^3 +$$
$$+ D\tau_{vv}\frac{v}{\varphi^2}\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right)v_x^2 - D(\vartheta_v - 2\xi_x)\frac{v}{\varphi^2}v_{xx} +$$
$$+ 2D\tau_x\frac{v}{\varphi^2}v_{xt} + 3D\xi_v\frac{v}{\varphi^2}v_xv_{xx} + D\tau_v\frac{v}{\varphi^2}\left(D\frac{v_x^2}{\varphi^2} + D\frac{vv_{xx}}{\varphi^2} + J_3u + J_4v - hv\right)v_{xx} +$$
$$+ 2D\tau_v\frac{v}{\varphi^2}v_xv_{xt} - J_3\eta - J_4\vartheta = 0,$$

where we used formulas from Section 1.1 and the conditions $F_u = 0, F_v = 0$. Again as a next step we compare the terms multiplied by derivatives of $u$ and $v$ and their products independently:

$$u_xu_{xt}: \quad \frac{2u}{\varphi^2}\tau_u = 0,$$
$$v_xv_{xt}: \quad \frac{2Dv}{\varphi^2}\tau_v = 0,$$
$$u_{xt}: \quad \frac{2u}{\varphi^2}\tau_x = 0,$$
$$v_{xt}: \quad \frac{2Dv}{\varphi^2}\tau_x = 0,$$

which implies $\tau = \tau(t)$ and:

$$u_xu_{xx}: \quad 2\frac{u}{\varphi^2}\xi_u = 0 \implies \xi = \xi(x,t,v),$$
$$v_xv_{xx}: \quad 2D\frac{v}{\varphi^2}\xi_v = 0 \implies \xi = \xi(x,t),$$

$$
\begin{aligned}
u_x^2: &\quad -\eta_u\frac{1}{\varphi^2}-\dot\tau\frac{1}{\varphi^2}+2\xi_x\frac{1}{\varphi^2}+2\frac{\dot\varphi}{\varphi^3}\tau-\frac{u}{\varphi^2}\eta_{uu}=0,\\
v_x^2: &\quad -\vartheta_v\frac{D}{\varphi^2}-\dot\tau\frac{D}{\varphi^2}+2\xi_x\frac{D}{\varphi^2}+2D\frac{\dot\varphi}{\varphi^3}\tau-D\frac{v}{\varphi^2}\vartheta_{vv}=0,\\
u_{xx}: &\quad -\dot\tau\frac{u}{\varphi^2}-\frac{\eta}{\varphi^2}+2\frac{\dot\varphi}{\varphi^3}\tau u+2\frac{u}{\varphi^2}\xi_x=0,\\
v_{xx}: &\quad -\dot\tau\frac{Dv}{\varphi^2}-D\frac{\vartheta}{\varphi^2}+2D\frac{\dot\varphi}{\varphi^3}\tau v+2D\frac{v}{\varphi^2}\xi_x=0,\\
u_x: &\quad -\xi_t-2\frac{\eta_x}{\varphi^2}-2\frac{\eta_{xu}}{\varphi^2}u+\frac{\xi_{xx}}{\varphi^2}u=0,\\
v_x: &\quad -\xi_t-2D\frac{\vartheta_x}{\varphi^2}-2D\frac{\vartheta_{xv}}{\varphi^2}v+D\frac{\xi_{xx}}{\varphi^2}v=0,\\
1(F_u): &\quad \eta_t-\eta_u hu+\eta_u J_1u+\eta_u J_2 v+\eta h+\dot\tau hu-\dot\tau J_1 u-\dot\tau J_2 v+\tau\dot h u-\\
&\quad -\frac{u}{\varphi^2}\eta_{xx}-\eta J_1-\vartheta J_2=0,\\
1(F_v): &\quad \vartheta_t-\vartheta_v hv+\vartheta_v J_3u+\vartheta_v J_4 v+\vartheta h+\dot\tau hv-\dot\tau J_3 u-\dot\tau J_4 v+\tau\dot h v-\\
&\quad -D\frac{v}{\varphi^2}\vartheta_{xx}-\eta J_3-\vartheta J_4=0.
\end{aligned}
$$

As a next step we solve the remaining system of equations

$$\eta_u+u\eta_{uu}+\dot\tau-2h\tau-2\xi_x=0, \tag{6.3}$$
$$\vartheta_v+v\vartheta_{vv}+\dot\tau-2h\tau-2\xi_x=0, \tag{6.4}$$
$$2u\xi_x-\dot\tau u-\eta+2h\tau u=0, \tag{6.5}$$
$$2v\xi_x-\dot\tau v-\vartheta+2h\tau v=0, \tag{6.6}$$
$$\xi_t\varphi^2+2\eta_x+2\eta_{xu}u-u\xi_{xx}=0, \tag{6.7}$$
$$\xi_t\varphi^2+2D\vartheta_x+2D\vartheta_{xv}v-Dv\xi_{xx}=0, \tag{6.8}$$
$$\eta_t-\eta_u hu+\eta_u J_1u+\eta_u J_2v+\eta h+\dot\tau hu-\dot\tau J_1u-\dot\tau J_2v+\tau\dot hu-\frac{u}{\varphi^2}\eta_{xx}-\eta J_1-\vartheta J_2=0, \tag{6.9}$$
$$\vartheta_t-\vartheta_v hv+\vartheta_v J_3u+\vartheta_v J_4v+\vartheta h+\dot\tau hv-\dot\tau J_3u-\dot\tau J_4v+\tau\dot hv-D\frac{v}{\varphi^2}\vartheta_{xx}-\eta J_3-\vartheta J_4=0. \tag{6.10}$$

From equations (6.5) and (6.6) we obtain

$$
\begin{aligned}
\eta&=(2\xi_x-\dot\tau+2h\tau)u,\\
\vartheta&=(2\xi_x-\dot\tau+2h\tau)v.
\end{aligned}
$$

These results we substitute into equations (6.7) and (6.8):

$$
\begin{aligned}
&\xi_t\varphi^2+7\xi_{xx}u=0,\\
&\xi_t\varphi^2+7D\xi_{xx}u=0,\\
&\Longrightarrow\ \xi=ax+b,
\end{aligned}
$$

where $a, b$ are arbitrary constants. As a next step we verify that our results satisfy equations (6.3) and (6.4):

$$2a-\dot\tau+2h\tau+\dot\tau-2h\tau-2a=0.$$

Finally we substitute our results into the last two conditions (6.9) and (6.10):

$$
\begin{aligned}
&-\ddot{\tau}u + 2(\dot{h\tau})u + (J_1 - h)(2a - \dot{\tau} + 2h\tau)u + J_2(2a - \dot{\tau} + 2h\tau)v + h(2a - \dot{\tau} + 2h\tau)u + \dot{\tau}hu - \\
&- \dot{\tau}J_1u - \dot{\tau}J_2v + \tau\dot{h}u - J_1(2a - \dot{\tau} + 2h\tau)u - J_2(2a - \dot{\tau} + 2h\tau)v = \\
&= -\ddot{\tau}u + 3(\dot{h\tau})u - \dot{\tau}J_1u - \dot{\tau}J_2v = 0,
\end{aligned}
$$

$$
\begin{aligned}
&-\ddot{\tau}v + 2(\dot{h\tau})v + (J_4 - h)(2a - \dot{\tau} + 2h\tau)v + J_3(2a - \dot{\tau} + 2h\tau)u + h(2a - \dot{\tau} + 2h\tau)v + \dot{\tau}hv - \\
&- \dot{\tau}J_3u - \dot{\tau}J_4v + \tau\dot{h}v - J_3(2a - \dot{\tau} + 2h\tau)u - J_4(2a - \dot{\tau} + 2h\tau)v = \\
&= -\ddot{\tau}v + 3(\dot{h\tau})v - \dot{\tau}J_3u - \dot{\tau}J_4v = 0 \\
&\implies \dot{\tau} = 0 \implies 3\tau(\dot{h})u = 0 \wedge 3\tau(\dot{h})v = 0 \\
&\implies \tau = 0.
\end{aligned}
$$

We can summarize the results as

$$
\begin{aligned}
\eta &= 2au, \\
\vartheta &= 2av, \\
\xi &= ax + b, \\
\tau &= 0,
\end{aligned}
$$

and hence the infinitesimal generators are spanned by

$$
\begin{aligned}
X_1 &= 2u\partial_u + 2v\partial_v + x\partial_x, \\
X_2 &= \partial_x.
\end{aligned}
$$

It is easy to verify that the infinitesimal generators form Lie algebra:

$$[X_1, X_2] = -\partial_x = -X_2,$$

and similarly as in the previous chapter we write down the corresponding symmetries:

1. $X_1 = 2u\partial_u + 2v\partial_v + x\partial_x$ :

$$(\hat{u}, \hat{v}, \hat{x}, \hat{t}) = (ue^{2\varepsilon}, ve^{2\varepsilon}, xe^{\varepsilon}, t),$$

2. $X_2 = \partial_x$ :

$$(\hat{u}, \hat{v}, \hat{x}, \hat{t}) = (u, v, x + \varepsilon, t).$$

## 6.2 Invariant solutions

Now we would like to find the solution which is invariant under the generator

$$X_1 + bX_2,$$

where $b$ is an arbitrary parameter. The characteristics have a form

$$
\begin{aligned}
\mathcal{Q}_u &= 2u - (x + b)u_x = 0, \\
\mathcal{Q}_v &= 2v - (x + b)v_x = 0,
\end{aligned}
$$

which implies that the invariant solution read

$$u(x,t) = A(t)(x+b)^2,$$
$$v(x,t) = B(t)(x+b)^2,$$

where $A(t), B(t)$ are arbitrary functions of $t$. We substitute these expressions into original system (6.1)

$$\dot{A}(t)(x+b)^2 + h(t)A(t)(x+b)^2 - 4\frac{A^2(t)(x+b)^2}{\varphi^2(t)} - 2\frac{A^2(t)(x+b)^2}{\varphi^2(t)} -$$
$$- J_1A(t)(x+b)^2 - J_2B(t)(x+b)^2 = 0,$$
$$\dot{B}(t)(x+b)^2 + h(t)B(t)(x+b)^2 - 4D\frac{B^2(t)(x+b)^2}{\varphi^2(t)} - 2D\frac{B^2(t)(x+b)^2}{\varphi^2(t)} -$$
$$- J_3A(t)(x+b)^2 - J_4B(t)(x+b)^2 = 0.$$

We obtain a system of two ordinary differential equations for unknown functions $A(t), B(t)$ :

$$\dot{A}(t) + h(t)A(t) - 6\frac{A^2(t)}{\varphi^2(t)} - J_1A(t) - J_2B(t) = 0,$$
$$\dot{B}(t) + h(t)B(t) - 6D\frac{B^2(t)}{\varphi^2(t)} - J_3A(t) - J_4B(t) = 0,$$

however, we fail to solve this nonlinear non-autonomous coupled system of two ODEs and thus we did not succeed to find an invariant solution of the reaction-diffusion system undergoing growth with non-fickian diffusion. Note that an invariant solution might exist but is either inaccessible or not within the chosen ansatz.

## 6.3 Spatially independent solution

As in the scalar case we would like to find the spatially independent solution $(u,v) = (u(t), v(t))$ for the uniform growth:

$$u_t + \frac{\dot{\varphi}(t)}{\varphi(t)}u = J_1u + J_2v,$$
$$v_t + \frac{\dot{\varphi}(t)}{\varphi(t)}v = J_3u + J_4v,$$

which yields

$$u(t) = \frac{C}{\varphi(t)}e^{at},$$
$$v(t) = \frac{C}{\varphi(t)}\frac{(a-J_1)}{J_2}e^{at},$$

where $a$ is an eigenvalue of matrix $\mathbb{J}$ and $C$ is an arbitrary constant. This solution corresponds to the scalar case. Again it is not invariant solution under the symmetries we found but it satisfies Neumann and periodic boundary conditions.

# Conclusion

In the thesis we introduced the powerful concept of symmetries of the partial differential equations and showed how to use them to find a certain class of explicit solutions, the so-called invariant solutions.

The aim was to replace the standardly used spectral approach on static domains via other explicit solutions that would reveal some insight into large time behaviour of reaction-diffusion system on growing domains, which is a long-standing open problem.

We succeeded in finding large families of explicit solutions to these complex problems but failed to identify non-trivial ones which would also satisfy the relevant boundary conditions. Nevertheless this thesis contains a significant amount of novel results, particularly those concerning apical growth. We would like to inspect, if certain assumptions (like the used ansatzes) cannot be removed and also to use the symmetries in the case of apical growth to generate new solutions from known ones. This procedure was described in Section 1.4.1 and it could enable us to obtain lots of new solutions.

Finally we aim to publish these results, which even without having implications for large time behaviour have their merit as a pure mathematical result but also which can be used for checking numerical methods.

Next we would like

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