Czech technical university in Prague
Faculty of Nuclear Sciences and Physical Engineering

Delonovské množiny uzavřené vůči afinním zobrazením

Delone sets closed under afine mappings

Master's Thesis

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Abstrakt: Vhodným modelem pro kvazikrystaly – nekrystalografické materiály s uspořádáním na dlouhou vzdálenost – je diskrétní množina vzniklá pomocí tzv. cut and project metody.

Tato metoda využívá projekce vícedimenzionálních mříží na vhodně zvolené podprostory. Umožňuje tak vytvořit diskrétní množiny se symetriemi, jež nemohou mít mřížky v nízkých dimenzích. V práci se zaměřujeme na následující otázku, totiž zda k zadanému lineárnímu zobrazení A existuje cut and project schéma takové, že jeho první projekce mříže je invariantní vůči tomuto zobrazení. V práci je dána odpověď pro libovolné zobrazení společně s popisem konstrukce příslušného schématu. Pro diagonalizovatelná zobrazení navíc určujeme minimální potřebnou dimenzi mříže. Celá práce využívá maticový formalismus založený na Jordanových formách matic.

Pro kvazikrystaly s pětičetnou symetrií s kruhovým oknem dále popíšeme všechny jejich možné lineární soběpodobnosti.

Klíčová slova: cut and project množina, cut and project schéma, kvazikrystal, soběpodobnost

Title: **Delone sets closed under affine mappings**

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Abstract: A suitable model for quasicrystals – non-crystallographic materials with long range order – is provided by discrete sets constructed using the so-called cut and project method.

The method uses projections of a higher-dimensional lattice to suitable subspaces, thus allowing to create discrete sets having symmetries forbidden in lattices of lower dimension. In this work we answer the following question: For a given linear mapping A, does there exist a cut and project scheme such that the first projection of a lattice is invariant under this mapping? We give the answer for all possible linear mappings and we give a construction of such a scheme. In the case of diagonalizable mappings we determine the minimal dimension of the scheme. For our study we use a matrix formalism based on Jordan forms of matrices.

We then focus on pentagonal cut and project sets with circular window and provide a description of all their self-similarities.

Key words: cut and project set, cut and project scheme, quasicrystal, self-similarity.

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Chapter 1

Introduction

Quasicrystals and their mathematical description have been intensively studied since 1984 when Dan Schechtman and his colleagues published their groundbreaking article [21] describing an alloy, whose X-ray diffraction pattern exhibits 5-fold symmetry – symmetry that had been forbidden according to theories which were accepted at the time. Since then many articles on the subject have been published as well as monographs, see for example [1], [2], [18], [16], and materials with other symmetries (8-, 12-, 18-fold) have been discovered and described [8], [23], [22].

It was shown already in [12], that a suitable mathematical approach to the description of such structures is the so-called cut and project method consisting in projecting of high-dimensional lattice points onto two suitable subspaces. In this way, one is able to construct discrete sets having the forbidden symmetries. In the cut and project method, one starts with a high-dimensional lattice $\mathcal{L} \subset \mathbb{R}^s$ and two orthogonal subspaces, which are usually called the physical and the inner subspace, and two orthogonal projections $\pi_\parallel$, $\pi_\perp$ on these subspaces. Taking only the $\pi_\parallel$-images of those lattice points whose $\pi_\perp$-images fit into a chosen bounded set Ω, the so-called window, one gets a discrete set $\Sigma(\Omega) \subset \mathbb{R}^n$, $n < s$, called cut and project set, which is commonly accepted as a suitable model for a quasicrystalline material. For, it has some non-trivial properties such as uniform discreteness and relative density, see Chapter 5 for more details. The quasicrystal models usually reveal all kinds of symmetries and self-similarities, i.e. affine mappings, under which the model is closed. Such self-similarities have been studied, in particular in the case when the affine mapping is just a scaling or rotation.

In this work we mainly focus on the following question: For a given general linear mapping A decide if there exists a suitable cut and project set $\Sigma(\Omega)$ such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$ and if so, provide a construction. Answering this question naturally implies finding suitable lattice $\mathcal{L} \subset \mathbb{R}^s$ and projections $\pi_\parallel$ and $\pi_\perp$ such that $A\pi_\parallel(\mathcal{L}) \subset \pi_\parallel(\mathcal{L})$. Our main result is

Theorem. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix whose spectrum is composed of algebraic integers. Then there exists a generic cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that $A\pi_\parallel(\mathcal{L}) \subset \pi_\parallel(\mathcal{L})$.*

We provide the construction for both diagonalizable and non-diagonalizable linear mappings A. In the diagonalizable case, we prove that our construction uses a lattice $\mathcal{L} \subset \mathbb{R}^s$ of minimal dimension s possible. By this, we extend the well known crystallographic restriction determining minimal dimension for obtaining n-fold symmetry in planar quasicrystal. Secondly, we formulate conditions which ensure the existence of an acceptance window Ω so that $A\Sigma(\Omega) \subset \Sigma(\Omega)$, see Theorem 10.4. This generalizes the known results by Lagarias, see [13], that give conditions on scaling factors of cut and project sets. In the literature we did not find any systematic research in this field. Some partial results are known and they will be mentioned in Section 6.2.

This work is divided into three main parts. Part I gives a thorough theoretical background needed for our study. We summarize number theoretic and algebraic notions as well as facts from matrix theory in chapters 2 - 4. Chapters 5 - 6 follow with an introduction to discrete geometry and mathematical models of quasicrystals.

Part II contains our findings and results on linear mappings as self-similarities of cut and project sets. In Chapter 7 we firstly give a brief introduction into a matrix formalism that is used in this work. We derive a criterion which enables us to construct a CPS with some, let us say generic, properties. Next chapter is dedicated to a construction of a scheme for a diagonalizable mapping A. We use a decomposition of $\sigma(A)$ into conjugacy classes and using corresponding elementary cut and project schemes. Then we define an operation called the direct sum of cut and project schemes that allows us to construct compound schemes from the elementary ones. We give some estimation on the dimension of the lattice $\mathcal{L}$ through a combinatorial approach. Then we show how to obtain a scheme with minimal dimension of the lattice. Chapter 9 brings a construction for the non-diagonalizable mappings. The construction needs to define an elementary non-diagonalizable cut and project scheme. The rest of the construction is based on similar ideas to those in Chapter 8 for diagonalizable mappings. In Chapter 10 we reformulate conditions on linear mapping A so that not only A is a self-similarity of some cut and project scheme, but also one can find an acceptance window Ω such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$.

In Part III we focus on decagonal quasicrystal models. We recall the construction of such a cut and project scheme. As a continuation to the bachelor's project [15], [14] where self-similarities of 5-fold cut and project schemes were studied, Chapter 11 is devoted to a slightly different approach to the problem. We then restrict ourselves to cut and project sets with a circular window Ω and we find conditions on the lattice transformation, that induces a mapping A in the physical space such that A is a self-similarity of the cut and project set $\Sigma(\Omega)$. For a subclass of such mappings we provide a complete classification according to their eigenvalues.

Part I

Theoretical background

Chapter 2

Basics of number theory

In this chapter basic tools and important results will be introduced. The mentioned theorems are well known facts from algebraic number theory, for proofs see for example [9].

2.1 Algebraic numbers and algebraic number fields

Definition 2.1. *We say that a number $\eta \in \mathbb{C}$ is an algebraic number, if there exists a monic polynomial $f \in \mathbb{Q}[X]$ such that $f(\eta) = 0$. The set of all algebraic numbers is usually denoted by $\mathbb{A}$.*

Definition 2.2. *Let $\eta \in \mathbb{A}$. The monic polynomial $f \in \mathbb{Q}[X]$ of minimal degree such that $f(\eta) = 0$ is called the minimal polynomial of η. The other roots of the polynomial f are called algebraic conjugates of η. The degree of η is defined as the degree of f.*

Definition 2.3. *Let $\eta \in \mathbb{A}$ be an algebraic number of degree n. Let $f(X) = X^n + \sum_{i=0}^{n-1} a_i X^i \in \mathbb{Q}[X]$ be its minimal polynomial. We define the companion matrix $C_f \in \mathbb{Q}^{n\times n}$ related to the polynomial f, or the number η respectively, as follows:*

$$C_f = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

Definition 2.4. *Number $\eta \in \mathbb{C}$ is an algebraic integer, if there exists a monic polynomial $g \in \mathbb{Z}[X]$ such that $g(\eta) = 0$. The set of all algebraic integers is usually denoted by $\mathbb{B}$.*

The following theorem shows that one can recognize an algebraic integer knowing only its minimal polynomial.

Theorem 2.5. *An algebraic number η is an algebraic integer if and only if the coefficients of its minimal polynomial are integers.*

Definition 2.6. *An algebraic number field $\mathbb{Q}(\alpha)$ for $\alpha \in \mathbb{A}$ is a finite field extension of $\mathbb{Q}$, i.e. the smallest subfield of $\mathbb{C}$ containing α. The degree of $\mathbb{Q}(\alpha)$ is the dimension of $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$.*

Note that $\mathbb{Q}(\alpha)$ is according to the definition equal to

$$\mathbb{Q}(\alpha) = \bigcap \{T \mid T \text{ is subfield of } \mathbb{C}, \alpha \in T\}.$$

Theorem 2.7. *Let $\alpha \in \mathbb{A}$ and let n be the degree of α. Then*

$$\mathbb{Q}(\alpha) = \left\{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mid a_i \in \mathbb{Q}\right\}.$$

One can think about number fields generated by other roots of the same minimal polynomial. These fields are isomorphic through special isomorphisms called the field isomorphisms.

Definition 2.8. *Let $\alpha \in \mathbb{A}$ and let α_i be its algebraic conjugates. The mapping*

$$\psi_i : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha_i) : a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mapsto a_0 + a_1\alpha_i + \cdots + a_{n-1}\alpha_i^{n-1}$$

is called the i-th field isomorphism.

Let us demonstrate the notions above on an example. Consider the *golden ratio*, i.e. the real number τ defined as

$$\tau = \frac{1+\sqrt{5}}{2}.$$

It can be checked that τ satisfies

$$\tau^2 = \tau + 1$$

thus it is a root of the polynomial $f_\tau(X) = X^2 - X - 1$ which means that $\tau \in \mathbb{A}$. Since the polynomial f_τ is irreducible over $\mathbb{Q}$ it is the minimal polynomial of τ and by Definition 2.4 $\tau \in \mathbb{B}$. The companion matrix of τ is

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The second root of f_τ is

$$\tau' = \frac{1-\sqrt{5}}{2} = -\frac{1}{\tau}$$

and it is the only algebraic conjugate to τ. The field extension $\mathbb{Q}(\tau)$ is of the form

$$\mathbb{Q}(\tau) = \{a + b\tau : \ a, b \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{5}).$$

There are two isomorphisms of this field. The first one is the identity and the second one, usually denoted by $'$ has the action

$$(a + b\tau)' = a + b\tau' \in \mathbb{Q}(\tau),$$

thus it is an automorphism of the field $\mathbb{Q}(\sqrt{5})$.

More details about field isomorphisms especially field automorphisms can be found in Chapter 3.

Definition 2.9. *Let $\alpha \in \mathbb{A}$ and let the degree of α be n. For $\beta \in \mathbb{Q}(\alpha)$ we define its field polynomial as*

$$\prod_{i=1}^{n}(X - \psi_i(\beta)).$$

As follows from Theorem 2.5 among rational numbers, algebraic integers are precisely elements of $\mathbb{Z}$. The following definition generalizes the notion of integers to general number fields.

Definition 2.10. *Let $\alpha \in \mathbb{A}$ and let $\mathbb{Q}(\alpha)$ be a number field. The set of all algebraic integers in the field $\mathbb{Q}(\alpha)$ is called the ring of integers $\mathcal{O}_{\mathbb{Q}(\alpha)}$, i.e.*

$$\mathcal{O}_{\mathbb{Q}(\alpha)} = \mathbb{Q}(\alpha) \cap \mathbb{B}.$$

As an example, it can be shown that for $\mathbb{Q}(\sqrt{5})$ it holds that

$$\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \left\{a + b\frac{1+\sqrt{5}}{2} : \ a, b \in \mathbb{Z}\right\} = \mathbb{Z}[\tau].$$

There exists a useful criterion to recognize integers in a given number field.

Theorem 2.11. *Let* $\alpha \in \mathbb{A}$ *and* $\beta \in \mathbb{Q}(\alpha)$ *. Then* $\beta \in \mathcal{O}_{\mathbb{Q}(\alpha)}$ *if and only if all coefficients of the field polynomial of* β *are integers.*

When studying algebraic integers one often deals with their minimal polynomials etc. It is therefore useful to know the following lemma on the factorization of polynomials with integer coefficients which is a consequence of the well known Gauss lemma.

Lemma 2.12. *Let* $f \in \mathbb{Q}[X]$ *and* $h \in \mathbb{Z}[X]$ *be monic polynomials. If* $f|h$*, then* $f \in \mathbb{Z}[X]$.

2.2 Cyclotomic polynomials, cyclotomic fields and roots of unity

We briefly introduce the so-called cyclotomic fields that are closely related to rotational symmetries which are a special type of self-similarity that has been studied most extensively. For that we recall the following notions:

Definition 2.13. *The* n*-th cyclotomic polynomial* Φ_n *is*

$$\Phi_n(X) = \prod_{1 \leq k \leq n,\ k \perp n} (X - \xi^k),$$

where $\xi = e^{\frac{2\pi i}{n}}$*.*

Note that we write $k \perp n$ iff $\gcd(k, n) = 1$.

Proposition 2.14 (Properties of cyclotomic polynomials)**.** *Let* $\Phi_n(X)$ *be the* n*-th cyclotomic polynomial. The following statements hold:*

- *The degree of* Φ_n *is* $\phi(n)$*, where* $\phi(n)$ *is the Euler function defined as* $\phi(n) = \left| \{1 \leq k \leq n : k \perp n\} \right|$.
- *It holds that*
$$X^n - 1 = \prod_{k|n} \Phi_k(X).$$
- $\Phi_n(X) \in \mathbb{Z}[X]$ *for all* $n \in \mathbb{N}$ *and its absolute term is equal to* ± 1.
- $\Phi_n(X)$ *is irreducible over* $\mathbb{Q}$ *for all* $n \in \mathbb{N}$.

In particular, if $n = p$ is prime, we obtain

$$\Phi_p(X) = \frac{X^p - 1}{X - 1} = 1 + X + \cdots + X^{p-1}$$

and for $\xi = e^{\frac{2\pi i}{p}}$ we get

$$\mathbb{Q}(\xi) = \left\{ a_0 + a_1\xi + \cdots + a_{p-2}\xi^{p-2} \mid a_i \in \mathbb{Q} \right\}.$$

For roots of unity the following construction can be introduced and we will use it later. Suppose that $p \in \mathbb{P}$ and let ξ satisfy

$$\xi^p = 1.$$

Note that $\xi_2 = -1$ is rational for $p = 2$. We thus consider p to be an odd prime. From the considerations above we can factorise this polynomial into

$$(\xi - 1)(\xi^{p-1} + \cdots + \xi + 1) = 0$$

and if $\xi \neq 1$, then the second term can be rewritten into the following form

$$1 + (\xi + \overline{\xi}) + (\xi^2 + \overline{\xi}^2) + \cdots + (\xi^{\frac{p-1}{2}} + \overline{\xi}^{\frac{p-1}{2}}) = 0. \tag{2.1}$$

This is well defined since p is odd. Let us define

$$r := \xi + \overline{\xi} = \xi + \frac{1}{\xi} = 2\cos\frac{2\pi}{p}. \tag{2.2}$$

Our aim is to rewrite equation (2.1) using r. It can be shown [3] that the following relation holds for every m

$$\xi^m + \overline{\xi}^m = \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \, \frac{m(m-j-1)!}{j!(m-2j)!} r^{m-2j}.$$

Therefore the equation (2.1) can be rewritten in the form

$$1 + \sum_{m=1}^{\frac{p-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} (-1)^j \, \frac{m(m-j-1)!}{j!(m-2j)!} r^{m-2j} = 0.$$

This expression can be used (as it is proven in [3]) that r is a root of a monic polynomial with integer coefficients and thus it is an algebraic integer.

2.3 Pisot numbers etc.

In the following section special classes of algebraic integers will be introduced. These numbers and their properties are widely studied and they occur in various mathematical problems related to non-standard number systems, approximations etc. They play an important role in our study, too.

Definition 2.15. *Let α be a real algebraic integer such that $\alpha > 1$. If all conjugates of α are in modulus strictly smaller than one, the number α is called Pisot number.*

The already mentioned golden ratio τ is a classical example of Pisot number since $\tau > 1$ and $|\tau'| \approx 0,618$.

Another example is the so-called minimal Pisot number or plastic number that is the real root ρ of polynomial $X^3 - X - 1$. Its numerical value is approximately $\rho \approx 1.3247179\ldots$ or precisely

$$\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}$$

whereas its algebraic conjugates are complex numbers of absolute value approximately $0.868837\ldots$.

Definition 2.16. *Let α be a non-real complex algebraic integer such that $\alpha > 1$. If all conjugates of α except its complex conjugate $\bar{\alpha}$ are in modulus strictly smaller than one, the number α is called complex Pisot number.*

An example of a pair of complex Pisot numbers can be directly derived from example ofthe minimal Pisot number ρ. It is sufficient to take the inverses of all roots of $X^3 - X - 1$. They are roots of $X^3 + X^2 - 1$ and the two complex roots are of absolute value clearly greater then 1 whereas the real one is equal to $\frac{1}{\rho} \in (0, 1)$.

Definition 2.17. *Let α be a real algebraic integer such that $\alpha > 1$. If all conjugates of α are in modulus smaller than or equal to one and at least one of them has modulus equal to one, the number α is called Salem number.*

The smallest known Salem number is the largest real root of polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

the so-called Lehmer's polynomial, and its numerical value is approximately $1.17628\ldots$.

Among all Pisot numbers there exists a special class of them, which are related to cyclotomic numbers introduced earlier.

Definition 2.18. *Let $r_p = 2\cos\frac{2\pi}{p}$ for a given $p > 4$, $p \in \mathbb{N}$. Denote by $\mathbb{Z}[r]$ its extension ring of order m, where m is the degree of the minimal polynomial of r. A Pisot-cyclotomic number β of degree m and order p associated to r is a Pisot number $\beta \in \mathbb{Z}[r]$ such that*

$$\mathbb{Z}[r] = \mathbb{Z}[\beta].$$

An example of Pisot-cyclotomic number of order 5 (resp. 10) is the golden ratio. To demonstrate it, suppose ω to be the fifth primitive root of 1, i.e. $\omega^5 = 1$. Then it holds that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0. \tag{2.3}$$

Denote $r = \omega + \omega^4$. Then it holds that $\omega^2 + \omega^3 = r^2 - 2$. Using this, Equation (2.3) will have the following form

$$r^2 + r - 1 = 0.$$

Its roots are $\frac{-1-\sqrt{5}}{2} = -\tau$ and $\frac{-1+\sqrt{5}}{2} = -\tau'$ and thus τ is Pisot-cyclotomic number of order 5.

On the other hand, for example the minimal Pisot number is not a Pisot-cyclotomic number of any order. Pisot-cyclotomic numbers were used in [3] for construction of quasicrystal models with given rotational symmetry.

The following results show that in every number field there exists a Pisot or a complex Pisot generator.

Theorem 2.19 (Salem, 1963)**.** *In every real number field K there is a Pisot number β such that $K = \mathbb{Q}(\beta)$.*

Theorem 2.20 (Vávra, Veneziano, 2017)**.** *Let $K \not\subseteq \mathbb{R}$ be a number field. Then there exists a complex Pisot number β such that $K = \mathbb{Q}(\beta)$.*

Chapter 3

Basics of Galois theory

3.1 Basic remarks on Galois theory

This chapter is dedicated to a part of algebra, named after a French mathematician Évariste Galois, which plays an important role when one is studying number fields and their properties. This theory enables us for example to describe structures of subfields of a given field. We will give a brief introduction to this part of algebra based on [6] where all details and proofs are given. In the end of the chapter we use the results of this theory to describe the structure of the cyclotomic field $\mathbb{Q}\left(e^{\frac{2\pi i}{20}}\right)$.

Definition 3.1. *Let K be a field and let ψ be its isomorphism. If $\psi : K \to K$ we say that ψ is a field automorphism. The set of all automorphisms of a given field K is denoted by $\mathbf{Aut}(K)$.*

It is easy to see that the set $\mathbf{Aut}(K)$ is non-empty for arbitrary field K, because each field has at least one automorphism. This one is the so-called trivial automorphism, the identity map.

Definition 3.2. *We say that $\psi \in \mathbf{Aut}(K)$ fixes an element $\alpha \in K$ if $\psi(\alpha) = \alpha$. We say that ψ fixes $F \subset K$ if ψ fixes all elements of F.*

We know that each field K has its prime field which is generated by the unit element $e \in K$. Moreover the defining property of an isomorphism $\psi \in \mathbf{Aut}(K)$ says that $\psi(e) = e$. Therefore it holds that $\psi(\alpha) = \alpha$ for every element α in the prime field of K. This observation can be summarized as follows: Each automorphism $\psi \in \mathbf{Aut}(K)$ fixes the prime field of K.

Definition 3.3. *Let K/F be a field extension, i.e. $F \subset K$ and F is a subfield of K. The set of all automorphisms $\psi \in \mathbf{Aut}(K)$ fixing F is denoted by $\mathbf{Aut}(K/F)$.*

Note that if F is the prime field of K then - in the sense of the previous remark - it holds $\mathbf{Aut}(K/F) = \mathbf{Aut}(K)$.

The following two propositions may look trivial but they are important in building up the Galois theory.

Proposition 3.4. *Let K/F be a field extension. Then $\mathbf{Aut}(K)$ is a group and $\mathbf{Aut}(K/F)$ is its subgroup.*

Proposition 3.5. *Let K be a field and let $\mathbf{H}$ be a subgroup of $\mathbf{Aut}(K)$, denote it by $\mathbf{H} \leq \mathbf{Aut}(K)$. Then the set F of all elements of K fixed by $\mathbf{H}$ is subfield of K.*

Definition 3.6. *Let $\mathbf{H} \leq \mathbf{Aut}(K)$. Then the subfield F (from the previous Proposition) of K fixed by $\mathbf{H}$ is called a fixed field of $\mathbf{H}$.*

Let us recall that the field extension K/F can be interpreted as a vector space K over F. Thus one can define the *degree* (or *index*) of K/F, denoted by $[K : F]$, as the dimension of K as a vector space over F. If $[K : F] < +\infty$ we say that K/F is *finite extension.*

Definition 3.7. *Let K/F be a finite extension. If $|\mathbf{Aut}(K/F)| = [K : F]$ then we say that K is Galois over F and K/F is a Galois extension. For a Galois extension K/F we call the group $\mathbf{Aut}(K/F)$ of automorphisms Galois group and denote it by $\mathbf{Gal}(K/F)$.*

Among all finite field extensions over $\mathbb{Q}$ one often use splitting fields of polynomials with coefficients in $\mathbb{Q}$. They satisfy the conditions above, thus the following proposition holds.

Proposition 3.8. *The splitting field of any polynomial over $\mathbb{Q}$ is Galois.*

We demonstrate these notions on few examples. Firstly, suppose that

$$K = \mathbb{Q}\left(\sqrt{5}\right) = \left\{a + b\sqrt{5} : a, b \in \mathbb{Q}\right\}.$$

Let ψ be an automorphism of this field. We know that this automorphism must fix the prime field of K which is $\mathbb{Q}$. Then it holds that

$$\psi\left(a + b\sqrt{5}\right) = a + b\psi\left(\sqrt{5}\right).$$

Since $\psi(\sqrt{5})$ must be a root of the minimal polynomial of $\sqrt{5}$, i.e. $x^2 - 5$, we have only two options: $\psi(\sqrt{5}) = \pm\sqrt{5}$, i.e.

$$\begin{aligned} \psi_1 &\equiv \mathrm{id} : \sqrt{5} \mapsto \sqrt{5}, \\ \psi_2 &\equiv \psi : \sqrt{5} \mapsto -\sqrt{5}, \end{aligned}$$

thus $\mathbf{Aut}(K) = \{\mathrm{id}, \psi\}$. Since $\mathbb{Q}(\sqrt{5})$ is the splitting field of $x^2 - 5$ due to Proposition 3.8 it holds that $\mathbf{Aut}(K) = \mathbf{Gal}(K)$ and $\mathbb{Q}(\sqrt{5})$ is Galois extension.

Now let us consider $K = \mathbb{Q}(\sqrt[3]{7}) = \left\{a + b\sqrt[3]{7} + c(\sqrt[3]{7})^2 : a, b, c \in \mathbb{Q}\right\} \subset \mathbb{R}$. The minimal polynomial of $\sqrt[3]{7}$ is $x^3 - 7$. The prime field of this K is, as in the previous example, equal to $\mathbb{Q}$. Suppose that $\psi \in \mathbf{Aut}(K)$. Applying this automorphism to an arbitrary element of K and using the fact that ψ fixes the prime field of K we get

$$a + b\psi\left(\sqrt[3]{7}\right) + c\left(\psi\left(\sqrt[3]{7}\right)\right)^2 \in K.$$

Thus $\psi(\sqrt[3]{7})$ must be a root of the minimal polynomial $x^3 - 7$ and must be an element of K. But the other two roots of this polynomial are complex numbers and therefore do not belong to K. Thus we conclude that $\psi(\sqrt[3]{7}) = \sqrt[3]{7}$, i.e. $\psi = \mathrm{id}$. So the group of all automorphisms of K is trivial and contains only one element, i.e. $\mathbf{Aut}(\mathbb{Q}(\sqrt[3]{7})) = \mathbf{Aut}(\mathbb{Q}(\sqrt[3]{7})/\mathbb{Q}) = \{\mathrm{id}\}$. But since $[\mathbb{Q}(\sqrt[3]{7}) : \mathbb{Q}] = 3$, we claim that $\mathbb{Q}(\sqrt[3]{7})$ is not a Galois extension.

Proposition 3.9. *Let K/F be any finite field extension. Then K/F is Galois if and only if F is a fixed field of $\mathbf{Aut}(K)$.*

Corollary 3.10. *Let $\mathbf{G} \leq \mathbf{Aut}(K)$ and let $\mathbf{G}$ be finite. Let F be the fixed field of $\mathbf{G}$. Then K/F is Galois with Galois group $\mathbf{G}$.*

The following theorem widely extends the statement of Proposition 3.8. It enables us to keep in mind the fact, that Galois field extension and splitting field of some polynomial with given property are the same objects.

Theorem 3.11. *Let K/F be a field extension. Then K/F is Galois if and only if K is a splitting field of some separable polynomial $f \in F[X]$, i.e. $f \in F[X]$ has no multiple roots and it splits completely in linear factors in $K[X]$ and there is no proper subfield of K containing F with this property.*

Moreover, if this is the case, then every irreducible polynomial with coefficients in F which has one root in K has all its roots in K and their multiplicity is equal to one.

Definition 3.12. *Let K/F be a Galois extension. If $\alpha \in K$ then for $\psi \in \mathbf{Gal}(K/F)$ we call $\psi(\alpha)$ a Galois conjugate or, simply, conjugate. If E is a subfield of K such that $F \subset E \subset K$ then the field $\psi(E)$ is called the conjugate field of E over F.*

Now we can introduce the Fundamental theorem of Galois theory which claims there is a bijection between the subgroups of $\mathbf{Gal}(K/F)$ and the subfields of K/F. Thus it is sufficient to determine the structure of subgroups and one knows the structure of subfields and vice versa.

Theorem 3.13 (Fundamental theorem of Galois theory)**.** *Let K/F be a Galois extension, set $\mathbf{G} = \mathbf{Gal}(K/F)$. Then there is a bijection between subfields of K containing F and subgroups $\mathbf{H} \leq \mathbf{G}$, i.e.*

$$\left\{\begin{array}{c} K \\ | \\ E \\ | \\ F \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \mathbf{I} \\ | \\ \mathbf{H} \\ | \\ \mathbf{G} \end{array}\right\},$$

given by

$$E \longmapsto \{\mathbf{H} = \textit{group of automorphisms fixing } E\}\,,$$
$$\mathbf{H} \longmapsto \{\textit{fixed field of } \mathbf{H}\}\,.$$

The following theorem is useful when one needs to describe the Galois group of n-th cyclotomic field because it states it is isomorphic to the multiplicative group of integers modulo n which consists of all integers from set $\{0, 1, \ldots, n-1\}$ that are coprime to n together with standard multiplication modulo n, denote this group by $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

Theorem 3.14. *Let ξ_n be the n-th root of unity. Then*

$$\mathbf{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^{\times}\,.$$

This isomorphism is given by

$$(\mathbb{Z}/n\mathbb{Z})^{\times} \longrightarrow \mathbf{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$$
$$a \bmod n \mapsto \psi_a$$

for a relatively prime to n such that $\psi_a(\xi_n) = \xi_n^a$.

Combining this theorem together with some basic results about the group $(\mathbb{Z}/n\mathbb{Z})^{\times}$ it is easily to verify that the following proposition holds.

Corollary 3.15. *Let $n = p_1^{a_1} \ldots p_k^{a_k}$ be a prime factorization. Then it holds that:*

- $\bigcap_i \mathbb{Q}\left(\xi_{p_i^{a_i}}\right) = \mathbb{Q}$,
- $\mathbb{Q}(\xi_n) = \mathbb{Q}\left(\xi_{p_1^{a_1}}\right) \ldots \mathbb{Q}\left(\xi_{p_k^{a_k}}\right)$, *i.e. $\mathbb{Q}(\xi_n)$ is the smallest field containing all $\mathbb{Q}\left(\xi_{p_i^{a_i}}\right)$. we say that $\mathbb{Q}(\xi_n)$ is the so-called composite field of all $\mathbb{Q}\left(\xi_{p_i^{a_i}}\right)$,*
- $\mathbf{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q}) \simeq \mathbf{Gal}\Big(\mathbb{Q}\left(\xi_{p_1^{a_1}}\right)/\mathbb{Q}\Big) \times \cdots \times \mathbf{Gal}\Big(\mathbb{Q}\left(\xi_{p_k^{a_k}}\right)/\mathbb{Q}\Big)$, *where $\times$ stands for a direct product of groups.*

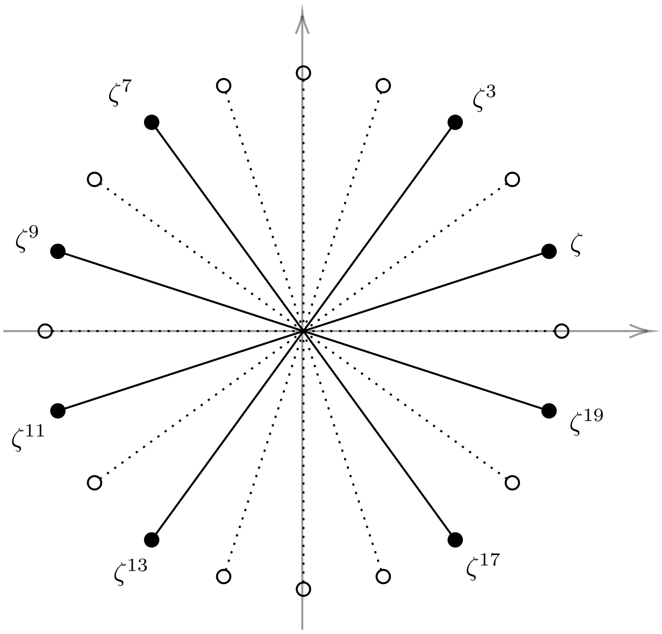
3.2 The 20-th cyclotomic field and its properties

In this section we study the structure of the 20-th cyclotomic field because it is the smallest field containing elements $\sin\frac{2\pi}{5}$, $\cos\frac{2\pi}{5}$ which are crucial for describing discrete sets with five-fold symmetry. For the sake of simplicity of notation we write ζ instead of ξ_{20} for the 20-th primitive root of unity.

According to the results of number theory (Proposition 2.14) the field we are dealing with is of degree $\phi(20) = 8$ and it can be written as follows:

$$\mathbb{Q}(\zeta) = \left\{a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6 + a_7\zeta^7 : a_i \in \mathbb{Q}\right\}.$$

The minimal polynomial of ζ is the 20-th cyclotomic polynomial $\Phi_{20}(X) = X^8 - X^6 + X^4 - X^2 + 1$. Its roots are $\zeta, \zeta^3, \zeta^7, \zeta^9, \zeta^{11}, \zeta^{13}, \zeta^{17}, \zeta^{19}$. The following picture visualizes them (full black lines) together with the other powers of ζ (dotted lines).



Since it holds that

$$\begin{array}{rcl} \zeta^9 &=& \zeta^7 - \zeta^5 + \zeta^3 - \zeta, \\ \zeta^{11} &=& -\zeta, \\ \zeta^{13} &=& -\zeta^3, \\ \zeta^{17} &=& -\zeta^7, \\ \zeta^{19} &=& -\zeta^7 + \zeta^5 - \zeta^3 + \zeta. \end{array} \tag{3.1}$$

we can see that $\mathbb{Q}(\zeta)$ is the splitting field of Φ_{20} and thus according to Proposition 3.8, $\mathbb{Q}(\zeta)$ is Galois. We can thus denote by

$$\mathbf{Gal}(\mathbb{Q}(\zeta)) = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8\}$$

the group of all automorphisms where these automorphisms are given by

$$\begin{array}{rclrcl} \psi_1(\zeta) &=& \zeta, & \psi_2(\zeta) &=& \zeta^3, \\ \psi_3(\zeta) &=& \zeta^7, & \psi_4(\zeta) &=& \zeta^9, \\ \psi_5(\zeta) &=& \zeta^{11}, & \psi_6(\zeta) &=& \zeta^{13}, \\ \psi_7(\zeta) &=& \zeta^{17}, & \psi_8(\zeta) &=& \zeta^{19}. \end{array}$$

The multiplication table of this group is the following

	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8
ψ_1	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8
ψ_2	ψ_2	ψ_4	ψ_1	ψ_3	ψ_6	ψ_8	ψ_5	ψ_7
ψ_3	ψ_3	ψ_1	ψ_4	ψ_2	ψ_7	ψ_5	ψ_8	ψ_6
ψ_4	ψ_4	ψ_3	ψ_2	ψ_1	ψ_8	ψ_7	ψ_6	ψ_5
ψ_5	ψ_5	ψ_6	ψ_7	ψ_8	ψ_1	ψ_2	ψ_3	ψ_4
ψ_6	ψ_6	ψ_8	ψ_5	ψ_7	ψ_2	ψ_4	ψ_1	ψ_3
ψ_7	ψ_7	ψ_5	ψ_8	ψ_6	ψ_3	ψ_1	ψ_4	ψ_2
ψ_8	ψ_8	ψ_7	ψ_6	ψ_5	ψ_4	ψ_3	ψ_2	ψ_1

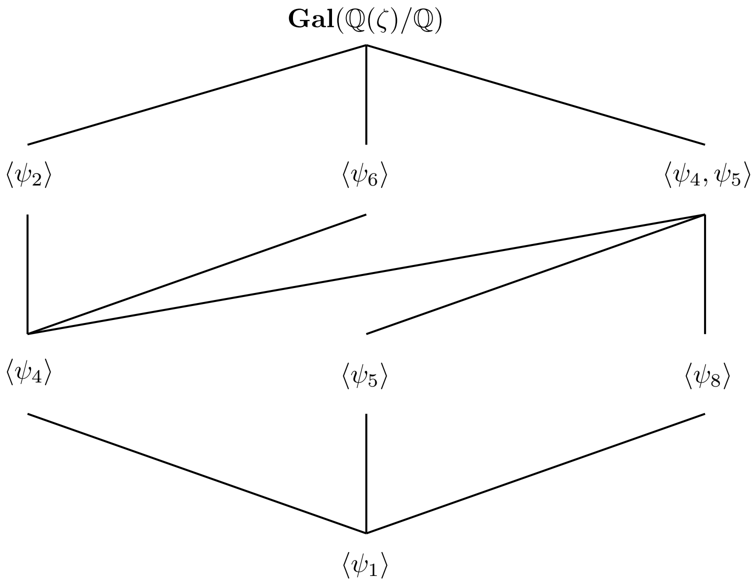
(3.2)

This fully corresponds with the statement of Theorem 3.14, i.e. that there is an isomorphism $\mathbf{Gal}(\mathbb{Q}(\zeta)) \simeq (\mathbb{Z}/20\mathbb{Z})^\times$. In order to determine the structure of $\mathbb{Q}(\zeta)$ it is, due to the fundamental theorem of Galois theory (Theorem 3.13), sufficient to examine the structure of subgroups of $\mathbf{Gal}(\mathbb{Q}(\zeta))$ and the corresponding fixed fields.

We identify subgroups of $\mathbf{Gal}(\mathbb{Q}(\zeta))$ using the table (3.2) as follows (and we rewrite them in terms of their generators)

$$\begin{array}{lclclcl} \{\psi_1,\psi_2,\psi_3,\psi_4\} & = & \langle\psi_2\rangle, & \{\psi_1,\psi_4\} & = & \langle\psi_4\rangle, \\ \{\psi_1,\psi_4,\psi_6,\psi_7\} & = & \langle\psi_6\rangle, & \{\psi_1,\psi_5\} & = & \langle\psi_5\rangle, \\ \{\psi_1,\psi_4,\psi_5,\psi_8\} & = & \langle\psi_4,\psi_5\rangle, & \{\psi_1,\psi_8\} & = & \langle\psi_8\rangle, \end{array}$$

and the trivial subgroup consisting of the identity element ψ_1. The following diagram shows the structure of subgroups of $\mathbf{Gal}(\mathbb{Q}(\zeta))$.



Note, that the subgroups of order 4 are either isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. To show that $\{\psi_1,\psi_2,\psi_3,\psi_4\}$ and $\{\psi_1,\psi_4,\psi_6,\psi_7\}$ represents the first case consider the following mapping

$$\begin{array}{ll} \psi_1 \mapsto 0, & \psi_4 \mapsto 2, \\ \psi_6 \mapsto 1, & \psi_7 \mapsto 3. \end{array}$$

This is an isomorphism of groups $\{\psi_1,\psi_4,\psi_6,\psi_7\}$ and $\mathbb{Z}/4\mathbb{Z}$. This holds since the tables of group multiplication (which is in the first case a mapping composition and in the second case an addition) are of the same form, i.e.

	ψ_1	ψ_6	ψ_4	ψ_7
ψ_1	ψ_1	ψ_6	ψ_4	ψ_7
ψ_6	ψ_6	ψ_4	ψ_7	ψ_1
ψ_4	ψ_4	ψ_7	ψ_1	ψ_6
ψ_7	ψ_7	ψ_1	ψ_6	ψ_4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

.

Similarly one could show the isomorphism $\{\psi_1,\psi_2,\psi_3,\psi_4\} \simeq \mathbb{Z}/4\mathbb{Z}$. On the other hand the group $\{\psi_1,\psi_4,\psi_5,\psi_8\}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ as the following multiplication table shows

	ψ_1	ψ_4	ψ_5	ψ_8
ψ_1	ψ_1	ψ_4	ψ_5	ψ_8
ψ_4	ψ_4	ψ_1	ψ_8	ψ_5
ψ_5	ψ_5	ψ_8	ψ_1	ψ_4
ψ_8	ψ_8	ψ_5	ψ_4	ψ_1

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

.

It is obvious that the subgroups of order 2 are isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Now we have to calculate for each subgroup of $\mathbf{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ its fixed subfield.

- $\{\psi_1, \psi_4\}$

  Let $\alpha = a + b\zeta + c\zeta^2 + d\zeta^3 + e\zeta^4 + f\zeta^5 + g\zeta^6 + h\zeta^7$ be an arbitrary element of $\mathbb{Q}(\zeta)$. Applying the isomorphism ψ_4 on α one obtains

$$\psi_4(\alpha) = a + b\zeta^9 + c\zeta^{18} + d\zeta^7 + e\zeta^{16} + f\zeta^5 + g\zeta^{14} + h\zeta^3 =$$

$$= a + c - b\zeta - c\zeta^2 + (b+h)\zeta^3 + (c-g)\zeta^4 + (f-b)\zeta^5 + (-c-e)\zeta^6 + (b+d)\zeta^7.$$

  We used relations (3.1) to rewrite the element unambiguosly. We impose the condition $\psi_4(\alpha) = \alpha$ and get the following restrictions on coefficients a, b, c, d, e, f, g, h:

$$\begin{array}{rclrcl} a &=& a+c, & e &=& c-g, \\ b &=& -b, & f &=& f-b, \\ c &=& -c, & g &=& -c-e, \\ d &=& b+h, & h &=& b+d. \end{array}$$

  Thus we conclude that an element fixed by ψ_4 and, of course, by the identity ψ_1 can be written as

$$a + d\zeta^3 + e\zeta^4 + f\zeta^5 - e\zeta^6 + d\zeta^7 =$$

$$a + e(\zeta^4 - \zeta^6) + \zeta^5\left(f + d(\zeta^2 - \zeta^8)\right) =$$

$$= a + e\frac{1}{\tau} + i(f + d\tau).$$

  In the second step we used the facts that $\zeta^4 - \zeta^6 = \frac{1}{\tau}$, $\zeta^5 = i$ and $\zeta^2 - \zeta^8 = \tau$. There are thus four independent elements $1, \zeta^4 - \zeta^6$, i, $i(\zeta^2 - \zeta^8)$.

  As it holds that $\frac{1}{\tau} = \frac{\sqrt{5}-1}{2}$ and $\tau = \frac{\sqrt{5}+1}{2}$, we finally conclude that the fixed field of $\{\psi_4, \psi_1\}$ is

$$\mathbb{Q}\left(\sqrt{5}\right) + i\mathbb{Q}\left(\sqrt{5}\right) = \mathbb{Q}\left(\sqrt{5}, i\right).$$

- $\{\psi_1, \psi_5\}$

  As in the previous case let us consider $\alpha = a + b\zeta + c\zeta^2 + d\zeta^3 + e\zeta^4 + f\zeta^5 + g\zeta^6 + h\zeta^7$ to be an arbitrary element of $\mathbb{Q}(\zeta)$. Applying the ψ_5 one gets

$$\psi_5(\alpha) = a + b\zeta^{11} + c\zeta^2 + d\zeta^{13} + e\zeta^4 + f\zeta^{15} + g\zeta^6 + h\zeta^{17} =$$

$$= a - b\zeta + c\zeta^2 - d\zeta^3 + e\zeta^4 - f\zeta^5 + g\zeta^6 - h\zeta^7.$$

  It can be easily seen that if $\psi(\alpha) = \alpha$, then it holds $b = 0,\ d = 0,\ f = 0,\ h = 0$. Thus each element of the fixed field can be written in the form

$$a + c\zeta^2 + e\zeta^4 + g\zeta^6.$$

  There are four independent elements 1, ζ^2, ζ^4, ζ^6.

  Denote by ω the primitive fifth root of unity, i.e. $\omega = e^{\frac{2\pi i}{5}}$. Then it can be easily checked that the following relations holds:

$$\begin{array}{rcl} \zeta^4 &=& \omega, \\ -\zeta^2 &=& \omega^3, \\ -\zeta^6 &=& \omega^4. \end{array}$$

  Using the fact, that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, we can express $\omega^2 = \zeta^6 - \zeta^4 + \zeta^2 - 1$ and this shows, that the resulting fixed field of $\{\psi_1, \psi_5\}$ is

$$\mathbb{Q}(\omega).$$

- $\{\psi_1, \psi_8\}$

  As in the previous case let us consider $\alpha = a + b\zeta + c\zeta^2 + d\zeta^3 + e\zeta^4 + f\zeta^5 + g\zeta^6 + h\zeta^7$ to be an arbitrary element of $\mathbb{Q}(\zeta)$. Then $\psi_8(\alpha)$ gives

$$\psi_8(\alpha) = a + b\zeta^{19} + c\zeta^{18} + d\zeta^{17} + e\zeta^{16} + f\zeta^{15} + g\zeta^{14} + h\zeta^{13} =$$

$$= a + c + b\zeta - c\zeta^2 + (-b - h)\zeta^3 + (c - g)\zeta^4 + (b - f)\zeta^5 + (-c - e)\zeta^6 + (-b - d)\zeta^7.$$

  Comparing the coefficients corresponding to the same independent elements gives

$$\begin{array}{rclrcl} a &=& a + c, & e &=& c - g, \\ b &=& b, & f &=& b - f, \\ c &=& -c, & g &=& -c - e, \\ d &=& -b - h, & h &=& -b - d. \end{array}$$

  This gives us the form of an arbitrary element fixed by ψ_8:

$$a + b\left(\zeta + \frac{\zeta^5}{2} - \zeta^7\right) + d(\zeta^3 - \zeta^7) + e(\zeta^4 - \zeta^6).$$

  Since it holds that

$$\begin{array}{rcl} \zeta^4 - \zeta^6 &=& 2\cos\frac{2\pi}{5} = \frac{1}{\tau}, \\ \zeta^3 - \zeta^7 &=& 2\sin\frac{4\pi}{5} = \sqrt{4 - \tau^2} = \frac{1}{\tau}\sqrt{\tau^2 + 1}, \\ \zeta + \frac{\zeta^5}{2} - \zeta^7 &=& \tau\zeta^{-1} + \frac{i}{2} = \sqrt{\tau^2 - \frac{1}{4}} = \frac{\tau}{2}\sqrt{\tau^2 + 1} \end{array}$$

  we conclude that these elements are real and thus one can think of this field K as of the smallest real subfield of $\mathbb{Q}(\zeta)$ in the sense of inclusion, i.e. $K = \mathbb{Q}(\zeta) \cap \mathbb{R}$.

  So the resulting field can be written as

$$\mathbb{Q}\left(\sin\frac{2\pi}{5}, \cos\frac{2\pi}{5}\right) = \mathbb{Q}\left(\tau, \tau\sqrt{\tau^2 + 1}, \frac{1}{\tau}\sqrt{\tau^2 + 1}\right).$$

- $\{\psi_1, \psi_2, \psi_3, \psi_4\}$

  Let us start with an arbitrary element $\beta = a + d\zeta^3 + e\zeta^4 + f\zeta^5 - e\zeta^6 + d\zeta^7$ of $\mathbb{Q}(\sqrt{5}, i)$ which is the fixed field of $\{\psi_1, \psi_4\}$. Applying firstly ψ_2 one obtains

$$\psi_2(\beta) = a + d\zeta^9 + e\zeta^{12} + f\zeta^{15} - e\zeta^{18} + d\zeta =$$

$$= a - e + e\zeta^2 + d\zeta^3 - e\zeta^4 + (-f - d)\zeta^5 + e\zeta^6 + d\zeta^7.$$

  Requiring β to be fixed by ψ_2 results in the following form of β

$$\beta = a + f\left(-2(\zeta^3 + \zeta^7) + \zeta^5\right).$$

  It can be easily verified that β is fixed by ψ_3 as well since $\psi_3 = \psi_2\psi_4$. So the resulting fixed field has its elements of the form

$$\beta = a + f\zeta^5\left(1 - 2(\zeta^2 + \zeta^{-2})\right) = a + fi(-2\tau + 1) = a - fi\sqrt{5}.$$

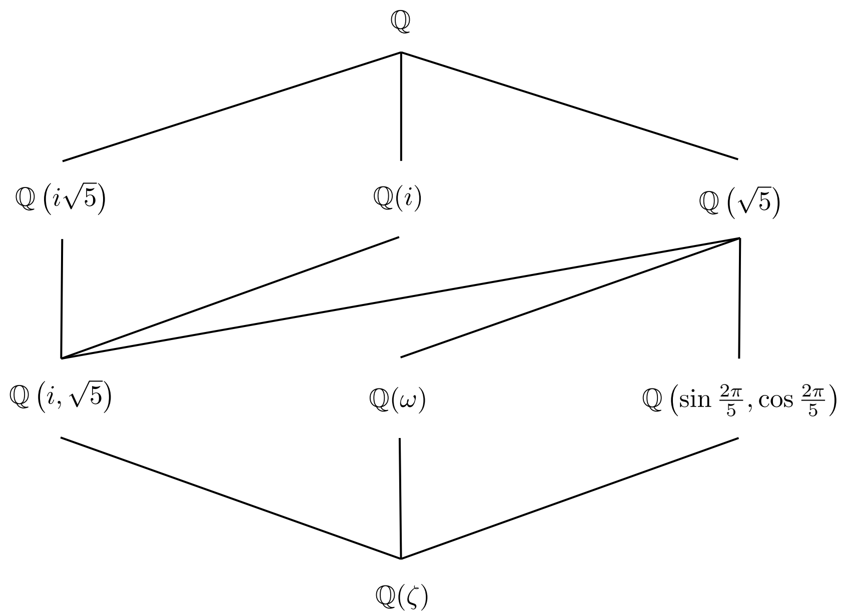
  This implies that the fixed field of $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ is

$$\mathbb{Q}(i\sqrt{5}).$$

- $\{\psi_1, \psi_4, \psi_6, \psi_7\}$
  For calculating the fixed field let us start with the same $\beta \in \mathbb{Q}(\sqrt{5}, i)$ as in the previous case. Applying ψ_6 one gets
  $$\psi_6(\beta) = a + d(\zeta^{19} + \zeta^{11}) + e(\zeta^{12} - \zeta^{18}) + f\zeta^5.$$
  Comparing this to β one finds out, that the fixed element is $a + if$ and this one is fixed by ψ_7 as well since $\psi_7 = \psi_4\psi_6$. Thus the fixed field of $\{\psi_1, \psi_4, \psi_6, \psi_7\}$ is
  $$\mathbb{Q}(i).$$

- $\{\psi_1, \psi_4, \psi_5, \psi_8\}$
  Applying ψ_4 on $\gamma = a + b\zeta^2 + c\zeta^4 + d\zeta^6$, an element of $\mathbb{Q}(\omega)$, one gets
  $$\psi_5(\gamma) = a + b\zeta^{18} + c\zeta^{16} + d\zeta^{14} = a - b\zeta^2 + (b - d)\zeta^4 + (-c - b)\zeta^6.$$
  So an element fixed by ψ_4, ψ_5, ψ_1 is
  $$a + c(\zeta^4 - \zeta^6).$$
  Calculation shows, that this element is fixed by ψ_8 as well because $\psi_8 = \psi_4\psi_5$. Since $\zeta^4 - \zeta^6 = 2\cos\frac{2\pi}{5}$, the resulting field is
  $$\mathbb{Q}(\tau) = \mathbb{Q}\left(\sqrt{5}\right).$$

So finally we can conclude that due to the Fundamental theorem of Galois theory, Theorem 3.13, the structure of subfields of $\mathbb{Q}(\zeta)$ is the following:



Chapter 4

Advanced tools from linear algebra

In this chapter we mention several tools from advanced linear algebra crucial for our study. First, we recall the Jordan canonical form of matrices, both in the classical sense and modifications to real Jordan and rational Jordan form. Then we introduce matrix norms and the Kronecker matrix product. Finally, we discuss various properties of some special polynomials related to a given matrix. Proofs of the following theorems and statements can be found in e.g. [25], [7], [4] etc.

4.1 Jordan canonical forms of matrices

First, we introduce canonical forms of matrices which play an important role in our construction.

4.1.1 Copmlex and real Jordan canonical form

Definition 4.1. *Let $\lambda_1, \cdots, \lambda_k$ be all eigenvalues (not necessarily distinct) of a matrix J. We say that matrix J is in the Jordan canonical form if it has a block diagonal form:*

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & O & O & \cdots & O \\ O & J_{n_2}(\lambda_2) & O & \cdots & O \\ O & O & J_{n_3}(\lambda_3) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & J_{n_k}(\lambda_k) \end{pmatrix} = \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i)\,, \tag{4.1}$$

where

$$J_n(\lambda) = \left.\begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}\right\} n$$

denotes the so-called Jordan block corresponding to eigenvalue λ and O is a zero matrix of corresponding order.

Theorem 4.2 (Jordan)**.** *Every matrix $M \in \mathbb{C}^{n\times n}$ is similar to a matrix in Jordan canonical form J.*

Let us denote $\mathrm{ind}_J(\lambda)$ the size of the biggest Jordan block corresponding to a given λ. For a general matrix M one can define $\mathrm{ind}_M(\lambda) := \mathrm{ind}_J(\lambda)$ where J is the Jordan canonical form of M.

Note that the geometric multiplicity $\mu_g(\lambda)$ of the eigenvalue λ of the matrix M gives us the number of Jordan blocks corresponding to λ in J.

If the matrix M is real, one can introduce *real Jordan canonical form.*

Theorem 4.3. *Every real matrix $M \in \mathbb{R}^{n\times n}$ is similar to its real Jordan canonical form $J^{\mathbb{R}}$, i.e. there exists $W \in \mathbb{R}^{n\times n}$ such that*

$$W^{-1}MW = J^{\mathbb{R}} = \begin{pmatrix} J^{\mathbb{R}}_{n_1}(\lambda_1) & O & O & \cdots & O \\ O & J^{\mathbb{R}}_{n_2}(\lambda_2) & O & \cdots & O \\ O & O & J^{\mathbb{R}}_{n_3}(\lambda_3) & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & J^{\mathbb{R}}_{n_k}(\lambda_k) \end{pmatrix} = \bigoplus_{i=1}^{k} J^{\mathbb{R}}_{n_i}(\lambda_i)\,,$$

where $\lambda_i \in \sigma(M)$ are eigenvalues of M and $J^{\mathbb{R}}_n(\lambda)$ is the real Jordan block corresponding to λ defined as

$$J^{\mathbb{R}}_n(\lambda) = \begin{pmatrix} R(\lambda) & I & O & O & \cdots & O \\ O & R(\lambda) & I & O & \cdots & O \\ O & O & R(\lambda) & I & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & \cdots & R(\lambda) & I \\ O & O & O & \cdots & O & R(\lambda) \end{pmatrix}$$

with I being the identity matrix of corresponding order and

$$R(\lambda) = \begin{cases} \left(\, \lambda \,\right) & \text{if } \lambda \in \mathbb{R} \\ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) & \text{if } \lambda = a + ib \in \mathbb{C}\backslash\mathbb{R} \end{cases}.$$

4.1.2 Rational Jordan canonical form

Similarly, we will use the so-called rational Jordan canonical form of a rational matrix C which reflects decomposition into the maximal number of cyclic subspaces (spanned by some vector and its repeated images under the mapping C) over $\mathbb{Q}$.

First, we recall that given a polynomial $f(x) = x^d - \sum_{j=0}^{d-1} a_j x^j$, $a_j \in \mathbb{Q}$ we define its companion matrix C_f by

$$C_f = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_{d-1} \end{pmatrix}.$$

Definition 4.4. *We say that a matrix is in the Jordan rational canonical form if it has the following block diagonal structure*

$$\bigoplus_k J^{\mathbb{Q}}_{n_k}(f_k)$$

where the rational Jordan block $J^{\mathbb{Q}}_n(f)$ corresponding to the polynomial $f \in \mathbb{Q}[X]$ is defined as

$$J^{\mathbb{Q}}_n(f) = \left.\begin{pmatrix} C_f & I & O & O \\ O & C_f & \ddots & O \\ \vdots & \ddots & \ddots & I \\ O & \cdots & O & C_f \end{pmatrix}\right\} n.$$

Let us mention that there are different possibilities how to define rational canonical forms in literature under various names (Frobenius normal form, Smith normal form, etc.). The following analogy of Theorems 4.2 and 4.3 can be proven.

Theorem 4.5. *Let $C \in \mathbb{Q}^{s\times s}$ and let $\chi_C \in \mathbb{Q}[X]$ be its characteristic polynomial. Denote by $f_1, f_2, \ldots, f_k$ the (not necessary distinct) polynomials with rational coefficients irreducible over $\mathbb{Q}$ such that $\chi_C = f_1 \cdot \cdots \cdot f_k$. Then there exists a non-singular matrix $W \in \mathbb{Q}^{s\times s}$ such that*

$$W^{-1}CW = \bigoplus_k J^{\mathbb{Q}}_{n_k}(f_k).$$

4.2 Jordan decomposition and generalized eigenspaces

For the purposes of further construction let us recall some notions and theorems about Jordan form or Jordan decomposition. As it was mentioned before (Theorem 4.2), each matrix A can be brought into the Jordan canonical form, i.e.

$$A \sim \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i)$$

where we use the same notation as in Theorem 4.2. In this section we focus on some non-trivial properties of matrices in this form and their consequences resulting in one concrete space decomposition.

The advantages of diagonalizability like decomposition into eigenspaces or the same value of geometric and algebraic multiplicity of each eigenvalue are not available when one works with a general matrix. Therefore it is natural to extend and generalize terms like eigenspace, eigenvector etc. in order to obtain new objects with diagonalizable-like properties.

Definition 4.6. *Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \sigma(A)$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is a generalized eigenvector of rank r corresponding to eigenvalue λ if the following two conditions are satisfied:*

(i) $(A - \lambda I)^r \boldsymbol{x} = \mathbf{0}$,

(ii) $(A - \lambda I)^{r-1} \boldsymbol{x} \neq \mathbf{0}$.

For $\lambda \in \sigma(A)$ we define the generalized eigenspace V_λ as follows:

$$V_\lambda = \left\{\boldsymbol{y} \in \mathbb{R}^n : \exists k \in \mathbb{N},\ (A - \lambda I)^k \boldsymbol{y} = \mathbf{0}\right\}.$$

It is clear that a generalized eigenvector of rank 1 is nothing but an ordinary eigenvector. Moreover, the following claims hold and their proofs can be found e.g. in [4], [25]:

Proposition 4.7. *Let $A \in \mathbb{C}^{n\times n}$. Then it holds that*

(i) there exist n linearly independent generalized eigenvectors of A,

(ii) $\mathbb{C}^n = \bigoplus_{\lambda\in\sigma(A)} V_\lambda$,

(iii) the generalized eigenspaces are A-invariant, i.e. $AV_\lambda \subset V_\lambda$,

(iv) if $A \in \mathbb{Z}^{n\times n}$ and if χ_A is irreducible over $\mathbb{Q}$, then generalized eigenvectors corresponding to λ, a root of χ_A, have their components in $\mathbb{Q}(\lambda)$ and generalized eigenvectors corresponding to an eigenvalue $\psi(\lambda)$ are (componentwise) images of generalized eigenvectors corresponding to λ under the field automorphism ψ that belongs to the group $\mathbf{Aut}(K)$ of the splitting field K of χ_A.

Let us demonstrate these properties on an example.

Example 1. *Suppose that we have a matrix $A \in \mathbb{R}^{6\times 6}$ with its Jordan form $J_A = J_3(7) \oplus J_2(3) \oplus J_1(11)$, i.e. $A = FJ_AF^{-1}$ for a regular matrix $F \in \mathbb{R}^{6\times 6}$. Denote its i-th column by $\boldsymbol{f}_i$. Then it holds that*

$$A\Big(\boldsymbol{f}_1,\ \boldsymbol{f}_2,\ \boldsymbol{f}_3,\ \boldsymbol{f}_4,\ \boldsymbol{f}_5,\ \boldsymbol{f}_6\Big) = \Big(\boldsymbol{f}_1,\ \boldsymbol{f}_2,\ \boldsymbol{f}_3,\ \boldsymbol{f}_4,\ \boldsymbol{f}_5,\ \boldsymbol{f}_6\Big)\begin{pmatrix} 7 & 1 & 0 & 0 & 0 & 0\\ 0 & 7 & 1 & 0 & 0 & 0\\ 0 & 0 & 7 & 0 & 0 & 0\\ 0 & 0 & 0 & 3 & 1 & 0\\ 0 & 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 0 & 11 \end{pmatrix},$$

i.e.

$$\Big(A\boldsymbol{f}_1,\ A\boldsymbol{f}_2,\ A\boldsymbol{f}_3,\ A\boldsymbol{f}_4,\ A\boldsymbol{f}_5,\ A\boldsymbol{f}_6\Big) = \Big(7\boldsymbol{f}_1,\ 7\boldsymbol{f}_2 + \boldsymbol{f}_1,\ 7\boldsymbol{f}_3 + \boldsymbol{f}_2,\ 3\boldsymbol{f}_4,\ 3\boldsymbol{f}_5 + \boldsymbol{f}_4,\ 11\boldsymbol{f}_6\Big).$$

Comparing the same columns on both sides of the equation one gets after simple manipulation

$$\begin{array}{rcl} (A-7I)\mathbf{f}_1 & = & \mathbf{0} \\ (A-7I)\mathbf{f}_2 & = & \mathbf{f}_1 \\ (A-7I)\mathbf{f}_3 & = & \mathbf{f}_2 \\ (A-3I)\mathbf{f}_4 & = & \mathbf{0} \\ (A-3I)\mathbf{f}_5 & = & \mathbf{f}_4 \\ (A-11I)\mathbf{f}_6 & = & \mathbf{0} \end{array}.$$

Clearly $(A-7I)^2\mathbf{f}_2 = (A-7I)\mathbf{f}_1 = \mathbf{0}$, $(A-7I)^3\mathbf{f}_3 = (A-7I)^2\mathbf{f}_2 = \mathbf{0}$ *and* $(A-3I)^2\mathbf{f}_5 = (A-3I)\mathbf{f}_4 = \mathbf{0}$. *This implies that we have found three generalized eigenvectors to 7, two generalized eigenvectors to* 3 *and one eigenvector to 11. Since* F *is non-singular its columns are linearly independent and we have*

$$V_7 = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{f}_1, \boldsymbol{f_2}, \boldsymbol{f_3}\},$$

$$V_3 = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{f}_4, \boldsymbol{f}_5\},$$

$$V_{11} = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{f}_6\},$$

and consequently we have the decomposition

$$\mathbb{R}^6 = V_7 \oplus V_3 \oplus V_{11}.$$

In the example one could see that a basis of generalized eigenspaces can be found in a special form.

Definition 4.8. *Let* $A \in \mathbb{C}^{n\times n}$ *and let* $\lambda \in \sigma(A)$. *Let further* $\boldsymbol{e}_m$ *be a generalized eigenvector of rank* m *corresponding to* A *and to* λ. *Then the set of vectors* $\{\boldsymbol{e}_m, \ \boldsymbol{e}_{m-1}, \dots, \ \boldsymbol{e}_1\}$ *satisfying*

$$\begin{array}{rcl} \boldsymbol{e}_{m-1} & = & (A-\lambda I)\boldsymbol{e}_m \\ \boldsymbol{e}_{m-2} & = & (A-\lambda I)\boldsymbol{e}_{m-1} = (A-\lambda I)^2\boldsymbol{e}_m, \\ \boldsymbol{e}_{m-3} & = & (A-\lambda I)\boldsymbol{e}_{m-2} = (A-\lambda I)^3\boldsymbol{e}_m, \\ & \vdots & \\ \boldsymbol{e}_1 & = & (A-\lambda I)\boldsymbol{e}_2 = (A-\lambda I)^{m-1}\boldsymbol{e}_m, \end{array}$$

generally

$$\boldsymbol{e}_l = (A-\lambda I)^{m-l}\boldsymbol{e}_m \quad \text{for } l \in \{1, \cdots, m-1\},$$

is called Jordan chain generated by generalized eigenvector $\boldsymbol{e}_m$.

Proposition 4.9. *Let* $A \in \mathbb{C}^{n\times n}$ *and let* $\lambda \in \sigma(A)$. *Let further* $\{\boldsymbol{e}_m, \ \boldsymbol{e}_{m-1}, \dots, \ \boldsymbol{e}_1\}$ *be a Jordan chain. Then the vectors* $\boldsymbol{e}_m, \ \boldsymbol{e}_{m-1}, \dots, \ \boldsymbol{e}_1$ *are linearly independent. Moreover, vector* $\boldsymbol{e}_l$ *is a generalized eigenvector of rank* l *corresponding to* λ.

Since Jordan chains are formed by linearly independent vectors they can form a basis of a vector space. If we can write basis of each generalized eigenspace only in terms of Jordan chains we call this basis *canonical.*

An obvious disadvantage of the concept of generalized eigenspaces is the fact that they are complex in general even if the considered matrix is real. For our purposes we need to find their real form. The proof of the following proposition gives a way how to construct it.

Proposition 4.10. *Let* C *be a real matrix and let* $\lambda, \overline{\lambda} \in \sigma(C)$. *Let* $\{\boldsymbol{e}_1, \ \dots, \ \boldsymbol{e}_k\} \subset \mathbb{C}^{kd}$ *be a Jordan chain corresponding to* $\lambda \in \mathbb{C}\backslash\mathbb{R}$ *and let* $\{\overline{\boldsymbol{e}}_1, \ \dots, \ \overline{\boldsymbol{e}}_k\} \subset \mathbb{C}^{kd}$ *be a Jordan chain corresponding to* $\overline{\lambda} \in \mathbb{C}\backslash\mathbb{R}$. *Then there exists a set of real generalized eigenvectors corresponding to a matrix* $R(\lambda) = \begin{pmatrix} \mathtt{Re}\,\lambda & \mathtt{Im}\,\lambda \\ -\mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix}$ *and it is of the form*

$$V_\lambda^{\mathbb{R}} = \left\{\boldsymbol{y} \in \mathbb{R}^{kd} : \exists j \in \mathbb{N}, \ \left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right)^j \boldsymbol{y} = \mathbf{0}\right\}.$$

Proof. We can derive from Proposition 4.9 that elements the Jordan chain $\{\boldsymbol{e}_1,\ \ldots,\ \boldsymbol{e}_k\}$ form columns of matrix $E \in \mathbb{C}^{kd\times k}$ such that

$$CE = E \left.\begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}\right\} k.$$

The same holds for Jordan chain $\{\overline{\boldsymbol{e}}_1,\ \ldots,\ \overline{\boldsymbol{e}}_k\}$ and for $\overline{\lambda}$. Our aim is to use both of them to find a suitable matrix $X \in \mathbb{R}^{kd\times 2k}$ such that

$$CX = X \left.\begin{pmatrix} R(\lambda) & I & 0 & \ldots & 0 \\ 0 & R(\lambda) & I & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & R(\lambda) & I \\ 0 & 0 & \cdots & 0 & R(\lambda) \end{pmatrix}\right\} 2k.$$

Let us start with

$$C\Big(\boldsymbol{e}_1,\ \ldots,\ \boldsymbol{e}_k,\ \overline{\boldsymbol{e}}_1,\ \ldots,\ \overline{\boldsymbol{e}}_k\Big) = \Big(\boldsymbol{e}_1,\ \ldots,\ \boldsymbol{e}_k,\ \overline{\boldsymbol{e}}_1,\ \ldots,\ \overline{\boldsymbol{e}}_k\Big) \begin{pmatrix} J_k(\lambda) & O \\ O & J_k\left(\overline{\lambda}\right) \end{pmatrix}.$$

Firstly we need to bring the matrix $\begin{pmatrix} J_k(\lambda) & O \\ O & J_k\left(\overline{\lambda}\right) \end{pmatrix}$ into a real Jordan form. For this purpose we define the matrix P

$$P = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & -i & & & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & -i \\ 1 & -i & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & -i & & & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & -i \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} k \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} k \end{matrix}$$

which gives the similarity transformation between Jordan canonical and real Jordan form. Then we have

$$C\Big(\boldsymbol{e}_1,\ \ldots,\ \boldsymbol{e}_k,\ \overline{\boldsymbol{e}}_1,\ \ldots,\ \overline{\boldsymbol{e}}_k\Big)P = \Big(\boldsymbol{e}_1,\ \ldots,\ \boldsymbol{e}_k,\ \overline{\boldsymbol{e}}_1,\ \ldots,\ \overline{\boldsymbol{e}}_k\Big)PP^{-1} \begin{pmatrix} J_k(\lambda) & O \\ O & J_k\left(\overline{\lambda}\right) \end{pmatrix} P. \tag{4.2}$$

Using $\texttt{Re}$ and $\texttt{Im}$ componentwise one obtains

$$\Big(\boldsymbol{e}_1,\ \ldots,\ \boldsymbol{e}_k,\ \overline{\boldsymbol{e}}_1,\ \ldots,\ \overline{\boldsymbol{e}}_k\Big)P = \Big(\texttt{Re}\,\boldsymbol{e}_1, \texttt{Im}\,\boldsymbol{e}_1, \texttt{Re}\,\boldsymbol{e}_2, \texttt{Im}\,\boldsymbol{e}_2, \ldots, \texttt{Re}\,\boldsymbol{e}_k, \texttt{Im}\,\boldsymbol{e}_k\Big).$$

Denote by $E_j = \Big(\texttt{Re}\,\boldsymbol{e}_j, \texttt{Im}\,\boldsymbol{e}_j\Big)$. Then we get from (4.2)

$$C\Big(E_1\ E_2\ \ldots\ E_k\Big) = \Big(E_1\ E_2\ \ldots\ E_k\Big) \begin{pmatrix} R(\lambda) & I & 0 & \ldots & 0 \\ 0 & R(\lambda) & I & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & R(\lambda) & I \\ 0 & 0 & \cdots & 0 & R(\lambda) \end{pmatrix}$$

which gives a set of matrix equations we have to solve:

$$\begin{aligned} CE_1 &= E_1 R(\lambda), \\ CE_2 &= E_1 + E_2 R(\lambda), \\ &\ \vdots \\ CE_k &= E_{k-1} + E_k R(\lambda). \end{aligned} \tag{4.3}$$

When solving this system of equations we will proceed using mathematical induction. The first equation in (4.3) gives after a rearrangement

$$\begin{aligned}(C - \mathtt{Re}\,\lambda\, I)\mathtt{Re}\,\boldsymbol{e}_1 &= -\mathtt{Im}\,\lambda\ \mathtt{Im}\,\boldsymbol{e}_1,\\ (C - \mathtt{Re}\,\lambda\, I)\mathtt{Im}\,\boldsymbol{e}_1 &= \mathtt{Im}\,\lambda\ \mathtt{Re}\,\boldsymbol{e}_1.\end{aligned}$$

Applying $(C - \mathtt{Re}\,\lambda I)$ on the second row and using the first one one obtains

$$(C - \mathtt{Re}\,\lambda\, I)^2 \mathtt{Im}\,\boldsymbol{e}_1 = \mathtt{Im}\,\lambda\ (C - \mathtt{Re}\,\lambda\, I)\mathtt{Re}\,\boldsymbol{e}_1 = -(\mathtt{Im}\,\lambda)^2 \mathtt{Im}\,\boldsymbol{e}_1,$$

i.e. the required form $\left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right) \mathtt{Im}\,\boldsymbol{e}_1 = \mathbf{0}$ and a similar procedure results in

$$\left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right) \mathtt{Re}\,\boldsymbol{e}_1 = \mathbf{0}.$$

Now let us suppose that

$$\begin{aligned}\left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right)^{j} \mathtt{Im}\,\boldsymbol{e}_j &= \mathbf{0},\\ \left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right)^{j} \mathtt{Re}\,\boldsymbol{e}_j &= \mathbf{0}\end{aligned}$$

holds for some $j < k$ and for all $j' \leq j$. We want to proceed using the induction and we want to show that $\mathtt{Re}\,\boldsymbol{e}_{j+1}$ and $\mathtt{Im}\,\boldsymbol{e}_{j+1}$ satisfy analogous conditions. Equation (4.3) gives

$$CE_{j+1} = E_j + E_{j+1}R(\lambda),$$

i.e.

$$\begin{aligned}(C - \mathtt{Re}\,\lambda\, I)\mathtt{Re}\,\boldsymbol{e}_{j+1} &= \mathtt{Re}\,\boldsymbol{e}_j - \mathtt{Im}\,\lambda\ \mathtt{Im}\,\boldsymbol{e}_{j+1},\\ (C - \mathtt{Re}\,\lambda\, I)\mathtt{Im}\,\boldsymbol{e}_{j+1} &= \mathtt{Im}\,\boldsymbol{e}_j + \mathtt{Im}\,\lambda\ \mathtt{Re}\,\boldsymbol{e}_{j+1}.\end{aligned}$$

As before, multiplying the second row by $(C - \lambda\, I)$ and using the first one, one has

$$\left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right) \mathtt{Im}\,\boldsymbol{e}_{j+1} = 2\mathtt{Im}\,\lambda\ \mathtt{Re}\,\boldsymbol{e}_j + \mathtt{Im}\,\boldsymbol{e}_{j-1}.$$

Applying $\left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right)^{j}$ on the equality and using the assumptions results in

$$\left((C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I\right)^{j+1} \mathtt{Im}\,\boldsymbol{e}_{j+1} = \mathbf{0}$$

and the same holds for $\mathtt{Re}\,\boldsymbol{e}_{j+1}$, thus the proof is complete. □

Note that the operator $(C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I$ can be decomposed (we use the commutativity of C and I) into

$$(C - \mathtt{Re}\,\lambda\, I)^2 + (\mathtt{Im}\,\lambda)^2 I = (C - \mathtt{Re}\,\lambda\, I - i\mathtt{Im}\,\lambda\, I)(C - \mathtt{Re}\,\lambda\, I + i\mathtt{Im}\,\lambda\, I) = (C - \lambda\, I)(C - \overline{\lambda}\, I)$$

which illustrates the fact that we have to take into an account both λ as well as its complex conjugate $\overline{\lambda}$.

Note that these generalized real eigenspaces have the same properties as the normal ones, i.e. their generators as described above are linearly independent. This follows from the fact that we have started with a set of linearly independent vectors $\boldsymbol{e}_1, \cdots, \overline{\boldsymbol{e}}_k$ and we transformed them using a regular transformation P that does not change their linear dependency.

We could as well extend the term Jordan chain into real Jordan chain that would be straightforward and in full correspondence with the proof of Proposition 4.10.

4.3 Matrix norms

This section is devoted to matrix norms, the relation between a norm on a vector space and the so-called induced norm on a space of all linear operators on the space, i.e. matrices. We are referencing to [10, Section 1] for more details. It is a well known fact that matrices with entries in a field $\mathbb{T} \subset \mathbb{C}$ of a fixed order, say $n \times m$, form a vector space over $\mathbb{T}$. Dimension of this space is clearly nm. Thus on this vector space we can introduce a norm as usual. Recall the definition of a norm on a vector space V.

Definition 4.11. *A mapping* $\|\cdot\| : V \to [0, +\infty)$ *is called a norm if*

(i) $\|\boldsymbol{x}\| \geq 0$ *for all* $\boldsymbol{x} \in V$*,*

(ii) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \varrho(\boldsymbol{x}) + \varrho(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in V$*,*

(iii) $\|\alpha \boldsymbol{x}\| = |\alpha| \, \|\boldsymbol{x}\|$ *for all* $\boldsymbol{x} \in V$ *and for all* $\alpha \in \mathbb{T}$*,*

(iv) equality $\|\boldsymbol{x}\| = 0$ *implies* $\boldsymbol{x} = \boldsymbol{0}$*.*

Now let us suppose that we have a vector space of all matrices of order n, i.e. square matrices $n \times n$. Together with a standard matrix multiplication it is n^2-dimensional an algebra and therefore a norm on the space should be compatible with this structure. We add additional condition, namely that

$$(v) \quad \|AB\| \leq \|A\| \, \|B\| \text{ for all matrices } A,\ B.$$

There is another way how to define a norm on a vector space of all operators that comes from a functional analysis, in particular

$$\|A\| := \sup_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|}.$$

This norm is usually called a *natural norm* or a *matrix norm induced by a vector norm.* It can be easily shown that this definition is equivalent to

$$\|A\| = \sup_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\|.$$

Since we are dealing only with finite-dimensional spaces, any norm on these spaces is a continuous mapping and since the set $\{\boldsymbol{x} \in V : \ \|\boldsymbol{x}\| = 1\}$ is compact, we get that there exists a vector $\boldsymbol{x}_0$ such that

$$\|A\| = \max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\| = \|A\boldsymbol{x}_0\|.$$

These properties are used for the demonstration of the following theorem about matrix norms induced by vector norms. Proof of this theorem can be found in [10].

Theorem 4.12. *Let* V *be a vector space over* $\mathbb{C}$ *of dimension* $\dim V = n$*. Let* $A \in \mathbb{C}^{n\times n}$ *and* $\boldsymbol{x} = (x_1, \ \cdots, \ x_n)^T \in V$*. Then*

- *vector norm* $\|\cdot\|_\infty$ *defined as* $\|\boldsymbol{x}\|_\infty = \max_i |x_i|$ *induces a matrix norm called maximum norm defined as*
$$\|A\|_\infty = \max_j \sum_{k=1}^{n} \left|[A]_{jk}\right|,$$
- *vector norm* $\|\cdot\|_1$ *defined as* $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|$ *induces a matrix norm defined as*
$$\|A\|_1 = \max_k \sum_{j=1}^{n} \left|[A]_{jk}\right|,$$
- *vector norm* $\|\cdot\|_2$ *defined as* $\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ *induces a matrix norm usually called spectral norm defined as*
$$\|A\|_2 = \sqrt{\varrho(A^*A)}$$
where $\varrho(M)$ *denotes the spectral radius of* M *and* A^* *is a Hermitian conjugate to* A*.*

4.4 Kronecker product of matrices

This section brings basic informations about a binary matrix called the Kronecker (or tensor) product. We can suppose matrices to have their entries all in a given field $\mathbb{T}$.

Definition 4.13. *Let $A \in \mathbb{T}^{n\times m}$ and $B \in \mathbb{T}^{p\times q}$. We define the Kronecker product $A \otimes B$ of matrices as a matrix of order $np \times mq$ as follows:*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix},$$

or precisely

$$A \otimes B = \begin{pmatrix}
a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1m}b_{11} & a_{1m}b_{12} & \cdots & a_{1m}b_{1q} \\
a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1m}b_{21} & a_{1m}b_{22} & \cdots & a_{1m}b_{2q} \\
\vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\
a_{11}b_{q1} & a_{11}b_{q2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1m}b_{q1} & a_{1m}b_{q2} & \cdots & a_{1m}b_{pq} \\
\vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\
a_{n1}b_{11} & a_{n1}b_{12} & \cdots & a_{n1}b_{1q} & \cdots & \cdots & a_{nm}b_{11} & a_{nm}b_{12} & \cdots & a_{nm}b_{1q} \\
a_{n1}b_{21} & a_{n1}b_{22} & \cdots & a_{n1}b_{2q} & \cdots & \cdots & a_{nm}b_{21} & a_{nm}b_{22} & \cdots & a_{nm}b_{2q} \\
\vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\
a_{n1}b_{q1} & a_{n1}b_{q2} & \cdots & a_{n1}b_{pq} & \cdots & \cdots & a_{nm}b_{q1} & a_{nm}b_{q2} & \cdots & a_{nm}b_{pq}
\end{pmatrix}.$$

The Kronecker product is closely related to the tensor product of vectors (thus such a symbol). In particular, if $A : U_1 \to V_1$ and $B : U_2 \to V_2$ are linear mappings between vector spaces U_i and V_i, then $A \otimes B : U_1 \otimes U_2 \to V_1 \otimes V_2$ is a linear mapping between the tensor products of original spaces. The mapping $A \otimes B$ can be expressed in a suitable basis and it can be shown that the resulting matrix is a Kronecker product of matrices of mappings A and B in certain basis.

Let us mention few properties of Kronecker product:

Proposition 4.14. *Let A, B, C, D be matrices with entries in $\mathbb{T}$ of a suitable order. Let $p \in \mathbb{T}$. Then it holds:*

(i) $A \otimes (B + C) = A \otimes B + A \otimes C$,

(ii) $(A + B) \otimes C = A \otimes C + B \otimes C$,

(iii) $(pA) \otimes B = A \otimes (pB) = p(A \otimes B)$,

(iv) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$,

(v) $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$,

(vi) if A and B are non-singular square matrices, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,

(vii) $\sigma(A \otimes B) = \{\lambda_i\mu_j : \lambda_i \in \sigma(A),\ \mu_j \in \sigma(B)\}$,

(viii) the so-called Kronecker sum $A \otimes I + I \otimes B$ satisfies

$$\sigma(A \otimes I + I \otimes B) = \{\lambda_i + \mu_j : \lambda_i \in \sigma(A),\ \mu_j \in \sigma(B)\}.$$

These properties are easy to prove, see [7].

4.5 Matrices annihilating polynomials

Definition 4.15. *We say that the polynomial $f \in \mathbb{T}[X], f = \sum_{i=0}^{k} a_i X^i$ annihilates a square matrix A (or the matrix A is annihilated by f), if*

$$O = \sum_{i=0}^{k} a_i A^i \equiv f(A),$$

where O denotes the zero matrix of corresponding order.

The following well-known theorem shows that characteristic polynomial χ_A of A is an example of a polynomial annihilating A.

Theorem 4.16 (Hamilton-Cayley)**.** *An arbitrary matrix A is annihilated by its characteristic polynomial, i.e. $\chi_A(A) = O$.*

4.5.1 Minimal polynomial

Definition 4.17. *The minimal polynomial μ_M of a given matrix $M \in \mathbb{C}^{n\times n}$ is a monic polynomial of the smallest degree such that $\mu_M(M) = O$.*

It can be easily seen that the degree of μ_M is equal or lower to the order of the matrix M and that minimal polynomials of two similar matrices coincide. For more details about minimal polynomials see for example [25].

If one needs to determine the minimal polynomial of a given matrix M, two basic approaches can be used. The first one is related to Jordan canonical form whereas the second one uses Smith normal form and invariant factor decomposition. We focus only on the former

Proposition 4.18. *Let K be a subfield of $\mathbb{C}$ of finite dimension over $\mathbb{Q}$. Using the notation above the minimal polynomial μ_M of $M \in K^{r\times r}$ is given by*

$$\mu_M(X) = \prod_{\lambda\in\sigma(M)} (X-\lambda)^{\mathrm{ind}_M \lambda}$$

where $\mathrm{ind}_M \lambda$ *denotes the size of the largest Jordan block with λ on its diagonal in the Jordan form of M.*

Proof. It is sufficient to show that the term $(X-\lambda)^{\mathrm{ind}_M \lambda}$ causes annihilation of all Jordan blocks corresponding to the eigenvalue λ. It is clear that

$$k\left\{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}^{l}\right. = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}}\right\} l \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} k-l \end{matrix},$$

so

$$(J_k(\lambda) - \lambda I_k)^k = O$$

and k is the minimal exponent that causes that the term above is annihilated. Therefore $(X-\lambda)^{\mathrm{ind}_M \lambda}$ causes annihilation of all blocks with the eigenvalue λ. Finally, the product $\prod_{\lambda\in\sigma(M)} (X-\lambda)^{\mathrm{ind}_J \lambda}$ annihilates whole matrix J and from the construction it follows that this polynomial is minimal. Since similar matrices have the same minimal polynomial with this property, the statement holds. □

One has to mention one important result about the minimal polynomial that follows directly from the second construction using the Smith normal form / invariant factor decomposition.

Theorem 4.19. *Let $M \in \mathbb{T}^{n\times n}$. Then $\mu_M \in \mathbb{T}[X]$.*

4.5.2 Minimal polynomial over $\mathbb{Q}$

In this work, we will consider matrices M whose entries are algebraic numbers. In that case, there exists a rational polynomial that annihilates M.

Proposition 4.20. *Let $\mathbb{T}$ be a subfield of $\mathbb{C}$ of finite dimension over $\mathbb{Q}$. Let $M \in \mathbb{T}^{n\times n}$. Then there exists a monic polynomial $f \in \mathbb{Q}[X]$ such that $f(M) = O$ where O denotes the zero matrix of the corresponding order.*

Proof. Denote by $\lambda_1, \dots, \lambda_k$ all (not necessary distinct) eigenvalues of M. These numbers are algebraic numbers, since λ_i are roots of the polynomial $\chi_M \in \mathbb{T}[X]$ which has algebraic coefficients. The field $\mathbb{A}$ of all algebraic numbers is algebraically closed, so λ_i are algebraic numbers. Therefore one can denote by μ_{λ_i} their minimal polynomials. Set

$$f(X) := \prod_{i=1}^{k} \mu_{\lambda_i}(X) = \prod_{i=1}^{k} (X - \lambda_i) p_i(X).$$

.

One can easily see that f is divisible by the characteristic polynomial χ_M since χ_M can be written in the form

$$\chi_M(X) = \prod_{i=1}^{k} (X - \lambda_i)$$

and from Hamilton-Cayley theorem it follows that $f(M) = O$. □

Definition 4.21. *The polynomial in $\mathbb{Q}[X]$ of smallest degree satisfying the above properties is called the minimal polynomial of the matrix M over $\mathbb{Q}$ and denoted by $\mu_{\mathbb{Q},M}$.*

The notion of a minimal polynomial over $\mathbb{Q}$ is not a standard one. Therefore we provide its basic properties with proofs.

Proposition 4.22 (Properties of $\mu_{\mathbb{Q},M}$)**.** *Let K be a subfield of $\mathbb{C}$ of finite dimension over $\mathbb{Q}$. Let $M \in \mathbb{T}^{n\times n}$. Then it holds:*

1. *The polynomial $\mu_{\mathbb{Q},M}$ is uniquely defined.*
2. *The polynomial $\mu_{\mathbb{Q},K}$ divides over $\mathbb{Q}[X]$ every polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(M) = O$.*
3. *If μ_M is the classical minimal polynomial of M, then μ_M divides $\mu_{\mathbb{Q},M}$ over $\mathbb{T}[X]$, in particular if $\mathbb{T} = \mathbb{Q}$, then $\mu_M = \mu_{\mathbb{Q},M}$.*
4. *The polynomial $\mu_{\mathbb{Q},M}$ is the smallest (with respect to degree) polynomial over $\mathbb{Q}$ divisible by μ_M.*
5. *If $M \in \mathbb{Q}^{n\times n}$, then $\mu_{\mathbb{Q},M}$ divides the characteristic polynomial χ_M of M.*
6. *If $M \in \mathbb{Z}^{n\times n}$, then $\mu_{\mathbb{Q},M} \in \mathbb{Z}[X]$.*
7. *Minimal polynomials over $\mathbb{Q}$ of similar matrices coincide.*

Proof. 1. For a contradiction suppose that there exists two polynomials $f, g \in \mathbb{Q}[X]$ of the same minimal degree such that

$$f(M) = O, \quad g(M) = O.$$

Denote $h(X) = f(X) - g(X) \in \mathbb{Q}[X]$. Then

$$h(M) = f(M) - g(M) = O,$$

which is a contradiction because $\deg h$ is strictly lower then $\deg f = \deg g$.

2. Let $f \in \mathbb{Q}[X]$ and $f(M) = O$. Suppose that $\mu_{\mathbb{Q},M}$ does not divide f over $\mathbb{Q}$. Then (according to Euclidean division algorithm) there exist $r, s \in \mathbb{Q}[X]$ such that

$$f(X) = r(X)\mu_{\mathbb{Q},M}(X) + s(X)$$

and $\deg s < \deg \mu_{\mathbb{Q},M}$. From

$$O = f(M) = r(M)\mu_{\mathbb{Q},M}(M) + s(M) = s(M)$$

it follows that $s(M) = O$ which is a contradiction.

3. The statement holds because $\mu_{\mathbb{Q},M}(M) = O$ and $\mathbb{Q}[X] \subset \mathbb{T}[X]$.

   In particular, if $M = \mathbb{Q}$ then $\mu_M \in \mathbb{Q}[X]$ is the minimal (w. r. t. degree) polynomial over $\mathbb{Q}$ annihilating M and so $\mu_M = \mu_{\mathbb{Q},M}$.

4. The statement follows from the definition of $\mu_{\mathbb{Q},M}$ and item 3.

5. Since $\chi_M \in \mathbb{Q}[X]$ and $\chi_M(M) = O$ due to Hamilton-Cayley's theorem, the statement follows from item 2.

6. If $M \in \mathbb{Z}^{n\times n}$ then $\chi_M \in \mathbb{Z}[X]$. If χ_M is irreducible over $\mathbb{Q}$ then $\chi_M = \mu_M = \mu_{\mathbb{Q},M}$. If it is reducible over $\mathbb{Q}$ then $\mu_{\mathbb{Q},M}$ can be of those factors. These factors are due to Gauss's lemma polynomials with integer coefficients, so the statement holds.

7. Let $A, B \in \mathbb{T}^{n\times n}$ and let there exist a regular matrix $P \in \mathbb{T}^{n\times n}$ such that

$$A = P^{-1}BP.$$

We know that $\mu_{\mathbb{Q},A}(A) = O$, i.e.

$$\mu_{\mathbb{Q},A}(A) = \sum_{i=0}^{k} a_i A^i = O.$$

So

$$O = \sum_{i=0}^{k} a_i A^i = \sum_{i=0}^{k} a_i (P^{-1}BP)^i = P^{-1}\left(\sum_{i=0}^{k} a_i B^i\right) P$$

from which it follows that $\sum_{i=0}^{k} a_i B^i = \mu_{\mathbb{Q},A}(B) = O$ and therefore $\mu_{\mathbb{Q},A}$ divides $\mu_{\mathbb{Q},B}$. Interchanging matrices A and B one obtains that $\mu_{\mathbb{Q},B}$ divides $\mu_{\mathbb{Q},A}$ and thus $\mu_{\mathbb{Q},A} = \mu_{\mathbb{Q},B}$.

□

Using the same technique as in the case of the standard minimal polynomial we can derive a way how to calculate the minimal polynomial over $\mathbb{Q}$

Proposition 4.23. *Let M be a matrix. Then the minimal polynomial of M over $\mathbb{Q}$ is given as follows:*

$$\mu_{\mathbb{Q},M} = \prod_{i=1}^{t} g_i^{\operatorname{ind}_C(g_i)}$$

where $\operatorname{ind}_C(g_i)$ is the maximal number of companion matrices used in a single Jordan rational block corresponding to a polynomial g_i.

Chapter 5

Introduction into discrete geometry

5.1 Basics of discrete geometry

In this chapter we briefly introduce basic concepts of discrete geometry. We define necessary objects related to this topic, we mention relations between them and moreover we compare concepts used by different authors. In particular we compare the way used by Lagarias in [13], Baake and Grimm in [1] and Pleasants in [19].

Definition 5.1. *A point set $X \subset \mathbb{R}^n$ is called discrete if for every $\mathbf{x} \in X$ there exists an open neighbourhood $U_{\mathbf{x}} \subset \mathbb{R}^n$ such that $U_{\mathbf{x}} \cap X = \{\mathbf{x}\}$.*

This implies that for each point $\mathbf{x} \in X$ there is a radius $r_{\mathbf{x}}$ such that $B_{r_{\mathbf{x}}}(\mathbf{x}) \cap X$ contains only one point $\mathbf{x}$.

For purposes of further use let us for sets A, B denote by $A + B$ the following set

$$A + B = \{a + b : a \in A, b \in B\} .$$

Definition 5.2. *A discrete set $X \subset \mathbb{R}^n$ is called uniformly discrete if there exists a radius $r > 0$ such that for all $\mathbf{x}, \mathbf{y} \in X$ it holds that $B_r(\boldsymbol{x}) \cap B_r(\boldsymbol{y}) = \emptyset$.*

Definition 5.3. *A discrete set $X \subset \mathbb{R}^n$ is called relatively dense if there exists a compact set $K \subset \mathbb{R}^n$ such that $X + K = \mathbb{R}^n$.*

These definitions come from [1]. Other authors prefer to use the definition using two radii r and R – the so-called *packing radius* and *covering radius*. Then the discrete set X is uniformly discrete if there is a packing radius r such that $X \cap B_r(\mathbf{x}) = \{\mathbf{x}\}$. This is equivalent to the statement that the function r_x defined under Definition 5.1 has a finite supremum strictly greater then 0. On the other hand the property of relative density can be formulated as follows: A discrete set X is relatively dense if there is a covering radius R such that $B_R(\mathbf{x}) \cap X \backslash \{\mathbf{x}\} \neq \emptyset$ for any $\mathbf{x} \in X$.

Definition 5.4. *Let $X \subset \mathbb{R}^n$ be a discrete point set. We say that X is a Delone set if it is both relatively dense and uniformly discrete.*

As a trivial example of a Delone set one can mention a lattice.

Definition 5.5. *Let $\boldsymbol{\ell}_1, \cdots, \boldsymbol{\ell}_n$ be linearly independent vectors in $\mathbb{R}^n$. We define a lattice $\mathcal{L}$ as an integer span of these vectors, i.e.*

$$\mathcal{L} = \operatorname{span}_{\mathbb{Z}} \{\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_n\} .$$

Every lattice is clearly relatively dense in $\mathbb{R}^n$ because for the compact set K from Definition 5.3 one can choose polytope generated by vectors $\boldsymbol{\ell}_1, \cdots, \boldsymbol{\ell}_n$, i.e. the set

$$\left\{ \sum_{i=1}^{n} \alpha_i \boldsymbol{\ell}_i : \alpha_i \in [0, 1] \right\} .$$

Uniform discreteness can be shown by taking the packing radius as e.g. $r = \frac{1}{3}\min\{||\boldsymbol{\ell}_1||, \dots, ||\boldsymbol{\ell}_n||\}$.

The following terms are defined according to Lagarias in [13].

Definition 5.6. *Let $X \subset \mathbb{R}^n$ be Delone set. We say that*

- *X is a finitely generated Delone set if the group*

$$\mathrm{span}_{\mathbb{Z}} \{\mathbf{x} - \mathbf{y} : \mathbf{x} - \mathbf{y} \in X - X\}$$

  is finitely generated and the number of generators is called rank of X, denoted usually by rank X.

- *X is a Delone set of finite type if $(X - X) \cap B_\varrho(\mathbf{0})$ is a finite set for all $\varrho \in \mathbb{R}$.*

- *X is a Meyer set if $X - X$ is a Delone subset of $\mathbb{R}^n$.*

The property being a Delone set of finite type is often called finite local complexity (FLC) and other authors (Baake, Pleasants) prefer to use this terminology for all discrete sets, i.e. discrete point set $X \subset \mathbb{R}^n$ has finite local complexity if $(X - X) \cap B_\varrho(\mathbf{0})$ is a finite set for all $\varrho \in \mathbb{R}$. Lagarias in [13] proves that the following chain of inclusions holds:

$$\text{Meyer sets} \;\subset\; \text{Delone sets of finite type} \;\subset\; \text{Finitely generated Delone sets.}$$

At this place one should mention that the above definition of a Meyer set is not the original one. Meyer's original definition says that $X \subset \mathbb{R}^n$ is Meyer if $X - X \subset X + F$ for some finite set $F \subset \mathbb{R}^n$. Lagarias shows that the two definitions are equivalent for a discrete set X in $\mathbb{R}^n$. According to [1] this equivalence holds not only for $X \subset \mathbb{R}^d$ but for X being a Delone subset in an arbitrary compactly generated[1] locally compact abelian group.

As an example of a Meyer set one can mention an arbitrary lattice $\mathcal{L}$. This holds since $\mathcal{L} - \mathcal{L} = \mathcal{L}$ thus, according to Meyer's definition, $\mathcal{L}$ is a Meyer set with finite set $F = \{\mathbf{0}\}$.

5.2 Mathematical quasicrystals

Mathematical quasicrystals can be seen as discrete point sets with some extra properties. Unfortunately there is no unambiguous definition of these structures. This is usually pointed out in papers related to this topic and Pleasants in his paper [19] lists 15 properties related to these sets. As the author says "a set does not need to have all these properties to be regarded as a mathematical quasicrystal, but they represent the sort of behaviour we have in mind when we think of quasicrystals." We list some of them (their numbering corresponds to [19]). Let $\Sigma \subset \mathbb{R}^n$ be a discrete point set. Then the following properties are to be considered:

(I) Σ has a *finite local complexity*.

(II) Σ is *repetitive*, i.e. for all $\varrho > 0$ there exists a radius $R_\varrho > 0$ such that for each *patch* $P \in \{P_\varrho(\mathbf{y}) = (\Sigma - \mathbf{y}) \cap B_\varrho(\mathbf{0}) : \mathbf{y} \in \Sigma\}$ and for all open balls with radius R_ϱ there is $\mathbf{x} \in B_{R_\varrho}(\mathbf{0}) \cap \Sigma$ such that $P = P(\mathbf{x})$.

(III) Every patch $P_\varrho(x)$ of Σ occurs with *positive uniform frequency* μ, i.e. for every $P_\varrho(x)$ there is μ such that for all $\varepsilon > 0$ there exists N such that

$$(\mu - \varepsilon)\mathrm{vol}(B) < \Big|\{\mathbf{y} \in \Sigma : P_\rho(\mathbf{x}) = P_\rho(\mathbf{y})\} \cap B\Big| < (\mu + \varepsilon)\mathrm{vol}(B)$$

for every ball B with volume $\mathrm{vol}(B) > N$.

(V) Σ is *aperiodic*, i.e. there is no $\mathbf{y} \in \mathbb{R}^n$ such that $\Sigma + \mathbf{y} = \Sigma$.

(IX) Σ is *finitely generated.*

[1] Locally compact abelian group G is compactly generated if there is a compact neighbourhood of $\mathbf{0} \in G$ such that this neighbourhood generates the entire group G.

(X) Σ is a *Meyer set.*

(XII) Σ has a *flation property*, i.e. there exists a real number $\lambda \neq \pm 1$ such that $(\Sigma + \mathbf{x}) \cap \lambda\Sigma$ is relatively dense for some $\mathbf{x} \in \mathbb{R}^n$.

(XIII) Σ has a *refining inflation property*, i.e. there exists a real number λ, $|\lambda| > 1$ such that $\lambda\Sigma \subset \Sigma + \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

The other properties which are not listed above are related to mathematical diffraction, to the so-called local rules etc. Properties above are not "independent"; there can be found some relation between them, see [19].

Proposition 5.7. *Let $\Sigma \subset \mathbb{R}^n$ be a discrete point set. With the notation above it holds that*

(i) Property (II) $\Rightarrow$ Property (I) $\Rightarrow$ Σ is uniformly discrete,

(ii) Property (I) and Property (III) $\Rightarrow$ Property (II) $\Rightarrow$ Σ is relatively dense,

(iii) Property (X) $\Rightarrow$ Property (I),

(iv) Property (II) and Property (XIII) $\Rightarrow$ Property (XII).

Since we listed some properties of the conceptional quasicrystal, let us describe some ways how to obtain such an aperiodic structure. Various procedures resulting in sets with desired properties have been invented. An introduction into these methods can be found e.g. in [1] where many useful references are given. We will just specify some of these methods and we will be mainly interested in the so-called cut-and-project method which will be described later in detail.

Planar quasicrystals can be constructed through aperiodic tilings of the plane. This is, in simple words, a way how to cover the plane with prototiles such that the resulting pattern is not periodic. From such a tiling one can get a discrete point set for example by labeling some vertices of given tiles etc. Tilings are described by certain local rules which specify how to compose the prototiles together. Probably the most studied tilings are the *Penrose tilings*, a family of tilings with five-fold symmetry consisting of many tilings like Penrose-Robinson tiling, Penrose pentagon tiling, Penrose rhombic tiling, Penrose darts and kites tiling etc. *Ammann-Beenker tiling* is well known tiling with octagonal symmetry. *Tübingen triangle tiling* has almost the same local rules as a Penrose pentagon tiling, but the resulting set is different and it can be shown that they are in some way "nonequivalent".

A closely related approach to the local rules is the so-called inflation method which is based on a mapping called *inflation rule*. It describes the way how a prototile is replaced by a cluster of prototiles. These mappings can have a fixed point and if so, then the fixed point defines a tiling of the space, thus the relation between the approaches is obvious. One can find a well described example of the similarity of these approaches in [1].

5.3 Cut and project method

This section is dedicated to another method that can be used for obtaining a mathematical quasicrystal. It is called cut and project method and it is based on projections of a high-dimensional lattice onto two suitable lower-dimensional subspaces. This method has been widely studied and there are different definitions based on the generality required. For our purposes we will need only the basic definition (in $\mathbb{R}^s$) however we will include the generalizations for coherence reasons.

Definition 5.8. *Let $\mathcal{L} \subset \mathbb{R}^s$ be an s-dimensional lattice. Denote by $\mathbf{e}_1, \ldots, \mathbf{e}_s$ the standard basis vectors of $\mathbb{R}^s$. Let further define the physical space as*

$$V_1 = \mathrm{span}_{\mathbb{R}} \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$$

and the inner space as

$$V_2 = \mathrm{span}_{\mathbb{R}} \{\mathbf{e}_{n+1}, \ldots, \mathbf{e}_s\}.$$

Let $\pi_{\parallel} : \mathbb{R}^s \to V_1$ and $\pi_{\perp} : \mathbb{R}^s \to V_2$ be projections on V_1 and V_2, respectively. The pair $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is called a cut-and-project scheme (CPS).

From now on, we suppose the above defined framework. We need to impose some conditions on a given scheme.

Definition 5.9. *We say that the cut-and-project scheme* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *is*

- *non-degenerated if* $\pi_{\|}\big|_{\mathcal{L}}$ *is injective,*
- *irreducible if* $\pi_{\perp}(\mathcal{L})$ *is dense in* V_2*,*
- *aperiodic if* $\pi_{\perp}\big|_{\mathcal{L}}$ *is injective, otherwise we call it periodic,*
- *generic if it is non-degenerate, irreducible and aperiodic.*

It is clear that with this notation we have $\mathbb{R}^s = V_1 \oplus V_2$ and $\pi_{\|}$ and $\pi_{\perp}$ can be seen as natural projections onto these two subspaces. If a given CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is non-degenerated one can define the so-called *star map* * as follows

$$\mathbf{x}^\star = (\pi_{\perp} \circ \pi_{\|}^{-1})(\mathbf{x}) \quad \text{for } \mathbf{x} \in \pi_{\|}(\mathcal{L}).$$

Thus the star map is a natural mapping between the physical and inner space, $^\star : \pi_{\|}(\mathcal{L}) \to V_2$.

The following proposition points out the algebraic structure of the lattice projections and explains the relation between them through the star map. Its proof is obvious and we will skip it.

Proposition 5.10. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a CPS. Then* $\pi_{\|}(\mathcal{L})$ *and* $\pi_{\perp}(\mathcal{L})$ *are* $\mathbb{Z}$*-modules of rank at most s respectively. Moreover if the CPS is non-degenerated and aperiodic the star map is a natural isomorphism of* $\mathbb{Z}$*-modules* $\pi_{\|}(\mathcal{L})$ *and* $\pi_{\perp}(\mathcal{L})$*.*

Having a generic CPS one can define a model set as a suitable subset of $\pi_{\|}(\mathcal{L})$ as follows:

Definition 5.11. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a generic CPS. Let* $\Omega \in \mathbb{R}^{s-n}$ *be a bounded set with non-empty interior called a window. We define the cut and project set (model set)* $\Sigma(\Omega)$ *related to the window* Ω *as*

$$\Sigma(\Omega) = \left\{ \pi_{\|}(\mathbf{x}) \in \mathbb{R}^n : \mathbf{x} \in \mathcal{L}, \pi_{\perp}(\mathbf{x}) \in \Omega \right\}.$$

Using the star-map one can rewrite this definition in a more elegant way

$$\Sigma(\Omega) = \left\{ \mathbf{y} \in \pi_{\|}(\mathcal{L}) : \mathbf{y}^\star \in \Omega \right\}.$$

Various restrictive conditions on the window may be given so that the resulting model set has certain given properties. We will list them later.

Definition 5.12. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a CPS. Denote by* $\Sigma(\Omega)$ *the cut and project set (model set) corresponding to the window* Ω

If $\mu(\partial\Omega) = 0$*, i.e. the Lebesgue measure of* $\partial\Omega$ *is zero, we call the model set regular.*

If $\pi_{\perp}(\mathcal{L}) \cap \partial\Omega = \emptyset$*, we call the model set generic [Baake et al.] / non-singular [Pleasants et al.]. Otherwise it is called singular.*

Requiring the regularity of $\Sigma(\Omega)$ one excludes for example fractal windows which may be seen as a generic demand. But there are several approaches to construct quasicrystals which turn out to be equivalent with constructing cut and project set with fractal window.

The term non-singular refers to the fact that in most situations the position of a window is such that the cut and project set has this property.

5.4 Comments on generalizations

In this section let us mention the generalizations of the cut and project construction concept that can be found in literature.

Pleasants in [19] starts with a lattice in a general Euclidean space $\mathbb{E}^s$ which can be decomposed into a direct sum of two subspaces U, V, i.e. $\mathcal{L} \subset \mathbb{E}^s = U \oplus V$. He also imposes some additional condition on the window Ω. It is required for Ω to be bounded and Riemann measurable, all sections of Ω by affine subspaces are Riemann measurable, all projections of $\mathrm{int}(\Omega)$ onto affine subspaces are Riemann measurable, and Ω is *half-open*.[2] The projections are natural projections on U, V respectively and both are surjective. Pleasants calls the resulting cut and project set *plain model set*.

Another extension of this concept is proposed by Pleasants as well. If the lattice $\mathcal{L}$ can be decomposed into several cosets (due to an equivalence relation on $\mathcal{L}$) one can assign to each coset, i.e. sublattice $\mathcal{L}_i$, a window Ω_i. Then one has to replace the Euclidean space $\mathbb{E}^s$ with a locally compact group $\mathbb{E}^s \times F$, where $F = \{f_1, \ldots, f_k\}$ refers to a finite group which represents the quotient group of the lattice $\mathcal{L}$ under the equivalence relation. Finally one needs to replace the inner space as well and use $V \times F$ instead of V. Then the window Ω is given as $\bigcup_{i=1}^{k}(\Omega_i \times f_i)$.

Baake and Grimm in [1] define a CPS as a triple $(\mathbb{R}^s, H, \mathcal{L})$ with a compactly generated locally compact abelian group H, a lattice $\mathcal{L} \subset \mathbb{R}^s \times H$ and two natural projections π, π_{int} onto $\mathbb{R}^s$ and H respectively. The physical space is $\mathbb{R}^s$ and the inner H. They also impose the condition so that π restricted to $\mathcal{L}$ is injective and $\pi_{int}(\mathcal{L})$ is dense in H. They define the so-called *Euclidean model sets* where $H = \mathbb{R}^m$ for some $m \in \mathbb{N}$, which fully corresponds with our approach in Section 5.3.

Note that a lattice $\mathcal{L}$ can be viewed as a discrete co-compact abelian subgroup of locally compact abelian group $\mathbb{R}^s$. The co-compactness means that $\mathbb{R}^s/\mathcal{L}$ is a compact abelian group, namely an s-dimensional torus. This is clear since $\mathbb{R}^s/\mathcal{L}$ can be seen as a fundamental cell of $\mathcal{L}$ with the correspondence of opposite facets of the fundamental polytope. This point of view enables us to introduce the concept of a lattice in a general locally compact abelian group.

Hence the most general case can be obtained from the previous one by replacing the physical space $\mathbb{R}^s$ with a σ-compact locally compact abelian group G and the resulting scheme is the triple $(G, H, \mathcal{L})$ where the lattice $\mathcal{L}$ is a discrete subgroup of $G \times H$ such that the quotient $G \times H/\mathcal{L}$ is compact in $G \times H$. Baake and Grimm claim that most of the theory can be formulated in such generality (and give references for further information), but we will restrict ourselves to the definition above.

This was a brief summary of standard approaches. During the last 30 years many extensions of this idea have been published which go beyond the scope of this work. Basic overview of some of these techniques can be found in Chapter 7.5. in [1].

[2] A subset K of Euclidean space is half-open in the direction $\mathbf{k}$, $\|\mathbf{k}\| = 1$ if it holds that

- $\mathbf{x} \in K \Rightarrow \exists \delta > 0$ such that $\mathbf{x} + d\mathbf{k} \in \mathrm{int}(K)$ for all $0 < d < \delta$,
- $\mathbf{x} \notin K \Rightarrow \exists \delta > 0$ such that $\mathbf{x} + d\mathbf{k} \in \mathrm{ext}(K)$ for all $0 < d < \delta$,

Chapter 6

Basic properties of cut and project sets and known results

6.1 General properties

Various properties of cut and project sets are somehow generic. They do not depend on definitions or properties of the window. Nevertheless we will use Definition 5.11 of cut and project sets.

Proposition 6.1. *Each cut and project set $\Sigma(\Omega)$ satisfies*

(i) $\Sigma(\Omega)$ *has finite local complexity,*

(ii) $\Sigma(\Omega)$ *is finitely generated.*

Proof. This proof comes from [19].

For the first claim we want to show that $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B_\varrho(\mathbf{0})$ is finite for every $\varrho > 0$. Let $\mathbf{y} \in \Sigma(\Omega) - \Sigma(\Omega)$. Then there exist lattice vectors $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{L}$ such that $\boldsymbol{y} = \pi_{\|}(\boldsymbol{m}_1) - \pi_{\|}(\boldsymbol{m}_2)$ and $\pi_\perp(\boldsymbol{m}_1)$, $\pi_\perp(\boldsymbol{m}_2) \in \Omega$. Both projections $\pi_{\|}$, $\pi_\perp$ are linear mappings thus there exists $\boldsymbol{m} \in \mathcal{L}$ such that $\boldsymbol{y} = \pi_{\|}(\boldsymbol{m})$ and $\pi_\perp(\boldsymbol{m}) \in \Omega - \Omega$. Since $B_\varrho(\mathbf{0}) + (\Omega - \Omega)$ is a bounded set in $\mathbb{R}^s$ it contains only finitely many lattice points. This implies that $(\Sigma(\Omega) - \Sigma(\Omega)) \cap B_\varrho(\mathbf{0})$ is finite for every $\varrho > 0$.

The second claim is obvious since each point of $\Sigma(\Omega)$ is an integer combination of $\pi_{\|}(\boldsymbol{\ell}_1), \ldots, \pi_{\|}(\boldsymbol{\ell}_n)$ where $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_n$ are the lattice generators. □

The following lemma and proposition come from [17] and show the Delone (and Meyer) property of cut and project sets.

Lemma 6.2. *Let $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ be a cut and project scheme. Let $U \subset \mathbb{R}^n$ be a non-empty open set. Then there exists a compact set $K \subset \mathbb{R}^{s-n}$ such that*

$$\mathbb{R}^n \times \mathbb{R}^{s-n} = \mathcal{L} + (U \times K).$$

Proposition 6.3. *With the notation from Lemma 6.2 it holds that,*

(i) the cut and project set $\Sigma(\Omega)$ derived from $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ using bounded window Ω with $\operatorname{int}(\Omega) \neq \emptyset$ *is a Delone set.*

(ii) the abelian group $\operatorname{span}_{\mathbb{Z}} \{\Sigma(\Omega) - \Sigma(\Omega)\}$ *is equal to* $\pi_{\|}(\mathcal{L})$*.*

Moreover $\Sigma(\Omega)$ *is a Meyer set.*

From the proof of this proposition it arises that uniform discreteness is given by boundeness of the window Ω. On the other hand the non-emptiness of $\operatorname{int}(\Omega)$ implies the relative density of the resulting cut and project set.

Proposition 6.4. *If the CPS* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *is aperiodic then for all bounded windows* $\Omega \subset \mathbb{R}^{s-n}$ *and for all* $\mathbf{t} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$

$$\Sigma(\Omega) \neq \Sigma(\Omega) + \mathbf{t},$$

i.e. $\Sigma(\Omega)$ *has Property V.*

Pleasants in his paper [19] defines the so-called Orientation Conditions. They impose some additional condition on the mutual position of the lattice $\mathcal{L}$ and two fixed subspaces of the Euclidean space $\mathbb{E}^s$. These conditions applied on the subspaces U, V from the decomposition $\mathbb{E}^s = U \oplus V$ gives sufficient and necessary conditions for $\pi_{\|}$ restricted to $\mathcal{L}$ to be bijection (i.e. the non-degeneracy of the scheme) and $\pi_{\perp}(\mathcal{L})$ to be dense in V (i.e. the irreducibility of the scheme). We will mention only two of these conditions and some of their implications. Later, we introduce our requirements on the orientation of the lattice and we will clearly show the equivalence of these conditions.

Definition 6.5. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a CPS. We say that the scheme satisfies*

- *W-condition if* $\mathbb{R}^{s-n} \cap \mathcal{L} = \{\mathbf{0}\}$*, i.e. the inner space* $\mathbb{R}^{s-n}$ *does not contain any lattice points except the origin.*
- *V-condition if* $\mathbb{R}^n$ *is not contained in any* $\mathcal{L}$*-hyperplane, i.e. the physical space* $\mathbb{R}^n$ *is not contained in any space generated by* $n-1$ *linearly independent lattice vectors.*

Now let us summarize the consequences of these conditions.

Proposition 6.6. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a CPS. Let* Ω *be a bounded window with* $\Omega \subset \overline{\mathrm{int}(\Omega)}$*. Then*

(i) V-condition is necessary and sufficient for $\pi_{\perp}(\mathcal{L})$ *to be dense in* $\mathbb{R}^{s-n}$*,*

(ii) W-condition is necessary and sufficient for $\pi_{\|}\big|_{\mathcal{L}}$ *to be a bijection,*

(iii) V-condition implies that $\Sigma(\Omega)$ *is repetitive and it is a Meyer set,*

(iv) W-condition implies that $\Sigma(\Omega)$ *is aperiodic if and only if* $\mathbb{R}^n \cap \mathcal{L} = \{\mathbf{0}\}$*.*

Note that the assumption on the window $\Omega \subset \overline{\mathrm{int}(\Omega)}$is required only for property (*iii*).

6.2 Self-similarities of cut and project sets

This section brings basic definitions related to self-similarities and shows various examples. Finally in this section we formulate two main questions that will be primarily studied.

Definition 6.7. *Let* A *be an affine mapping acting on* $\mathbb{R}^n$*. Let* $\Sigma(\Omega) \subset \mathbb{R}^n$ *be a cut and project set. We say that* A *is an affine self-similarity of* $\Sigma(\Omega)$ *if*

$$A\Sigma(\Omega) \subset \Sigma(\Omega).$$

We will restrict ourselves on linear mappings only. Thus we will use term *linear self-similarity* or simply *self-similarity.* Typical self-similarities S are listed below:

- Scaling, i.e. $S = \lambda I$ for $\lambda \in \mathbb{R}$,
- Scaled rotation, i.e. $S = \lambda R$ for $\lambda \in \mathbb{R}$ and $R \in \mathrm{O}(n, \mathbb{R})$,
- Translation i.e $S = I + \mathbf{t}$ where $\mathbf{t} \in \mathbb{R}^n$,
- General linear transformation, i.e. $S = A$ for $A \in \mathrm{Gl}(n, \mathbb{R})$,
- General affine transformation, i.e. $S = A + \mathbf{t}$ where $\mathbf{t} \in \mathbb{R}^n$ and $A \in \mathrm{Gl}(n, \mathbb{R})$.

Remark. *We use the standard notation for matrix groups, i.e.*

- $\mathrm{Gl}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$,
- $\mathrm{O}(n,\mathbb{R}) = \left\{A \in \mathbb{R}^{n\times n} : AA^T = A^T A = I\right\}$,
- $\mathrm{SO}(n,\mathbb{R}) = \{A \in \mathrm{O}(n,\mathbb{R}) : \det A = 1\}$.

The main question which we plan to solve is the following.

Question 1. *To a given linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, decide whether there exists a cut-and-project set $\Sigma(\Omega) \subset \mathbb{R}^n$ with A as its self-similarity. If yes, determine the minimal dimension s of the corresponding cut-and-project scheme and describe the construction.*

Firstly, according to the point (ii) of Proposition 6.3 it holds that

$$\mathrm{span}_{\mathbb{Z}}\{\Sigma(\Omega) - \Sigma(\Omega)\} = \pi_{\|}(\mathcal{L}).$$

Consequently, it holds that

Proposition 6.8. *Let $\Sigma(\Omega) \subset \mathbb{R}^n$ be constructed from a cut-and-project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$, then also $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.*

This trivial observation allows us to divide our study into two steps. First, we inspect self-similarities of $\mathbb{Z}$-module $\pi_{\|}(\mathcal{L})$ and afterwards we restrict ourselves only to a subset of them, which preserve a cut and project set $\Sigma(\Omega)$ for some window Ω.. For conciseness, we will say that a CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ has self-similarity $A \in \mathbb{R}^{n\times n}$, if A is a self-similarity of the $\mathbb{Z}$-module $\pi_{\|}(\mathcal{L})$, i.e. if $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$. The restricted question is therefore

Question 2. *To a given linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, decide whether there exists a CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that A is the self-similarity of the $\mathbb{Z}$-module $\pi_{\|}(\mathcal{L})$. If yes, determine the minimal dimension s of the corresponding CPS and describe the construction.*

6.3 Summarization of known results in literature

In literature there exist certain results on these questions. The upcoming text summarizes the most relevant results concerning self-similarities of quasicrystals. Each subsection is dedicated to a specific article and brings its most important statements.

6.3.1 Crystallographic restriction

Firstly, we recall some results from Baake and Grimm in [1], Chapter 3, estimating the minimal dimension of a lattice needed for a construction of a planar quasicrystal with n-fold symmetry. Their approach uses only basic tools of discrete geometry together with some elementary knowledge of linear algebra. We skip most of details and emphasize only the lemmas and theorems related to the minimal embedding dimension.

Note that $(\mathrm{S})\mathrm{O}(s,\mathbb{R})$ stands for (special) orthogonal group in s dimensions. We use ϕ for the Euler's totient function.

The following two definitions are related only to this Section.

Definition 6.9. *We say that a mapping R is a (point) symmetry of discrete set X if $RX = X$.*

Proposition 6.10. *The set of all symmetries R of discrete set X together with a composition of mappings form a group, we call it (point) symmetry group.*

Lemma 6.11 (Crystallographic restriction for lattices)**.** *Let $\mathcal{L} \subset \mathbb{R}^s$ be a lattice. If $R \in \mathrm{O}(s,\mathbb{R})$ satisfies $R\mathcal{L} \subset \mathcal{L}$, then it implies $R\mathcal{L} = \mathcal{L}$. The corresponding characteristic polynomial $\chi_R(X)$ has integer coefficients only, i.e. $\chi_R(X) \in \mathbb{Z}[X]$.*

As a direct corollary of this lemma one has the list of possible symmetries of lattices in the plane or in the 3-dimensional space.

Corollary 6.12. *If a lattice $\mathcal{L}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ has n-fold symmetry then $n \in \{1, 2, 3, 4, 6\}$.*

This restriction can be extended for more general discrete point sets than lattices. It turns out that the result gives condition on the group generated by the mapping R.

Lemma 6.13. *Let $\mathcal{L}$ be a lattice with a point symmetry group that contains an element of order p^r for p a prime and $r \geq 1$. Then, the minimal dimension of $\mathcal{L}$ is $s = \phi(p^r) = p^{r-1}(p-1)$.*

If one needs to generalize this lemma for an element of arbitrary order, it is necessary to introduce the so-called *additive counterpart of Euler's totient function* ϕ_{a}. This is defined as a function $\phi_{\mathsf{a}} : \mathbb{N}_0 \to \mathbb{N}$ such that

- $\phi_{\mathsf{a}}(0) = 1$,
- $\phi_{\mathsf{a}}(p^r) = \phi(p^r)$ for all primes p and $r \geq 1$,
- $\phi_{\mathsf{a}}(2n) = \phi_{\mathsf{a}}(n)$ for all odd $n > 1$,
- $\phi_{\mathsf{a}}(mn) = \phi_{\mathsf{a}}(m) + \phi_{\mathsf{a}}(n)$ for remaining coprime integers m, n.

Theorem 6.14. *Let $\mathcal{L}$ be a lattice with a point symmetry R of order n. Then the minimal dimension of $\mathcal{L}$ is $s_n = \phi_{\mathsf{a}}(n)$.*

The argument is very simple. It is based on the degree of minimal polynomial of roots of the characteristic polynomial of the symmetry R. We have seen in Corollary 6.12 that a lattice in low dimensions can have only symmetry of order 1,2,3,4, and 6. In order to obtain discrete sets with other symmetries, one uses the cut and project method which allows us to transfer symmetries of higher-dimensional to lower-dimensional structures For deriving the minimal dimension of lattice needed for construction of a planar quasicrystal with given n-fold symmetry one needs to add one constraint. One needs the invariance of a 2-dimensional subspace of $\mathbb{R}^s$ under the lattice symmetry.

Theorem 6.15. *Consider a locally finite planar point set X, i.e. the intersection of X with any ball in $\mathbb{R}^2$ contains only finite number of elements of X. Suppose that X is a discrete point set with n-fold symmetry that is constructed from a lattice in $\mathbb{R}^s$ by a symmetry preserving projection. Then $d \geq \phi(n)$ with the lower bound being sharp.*

6.3.2 Geometric Models for Quasicrystals, I. Delone Sets of Finite Type

J.C. Lagarias in his paper [13] extensively studies Delone sets. He provides their classification into several classes and determines relation between them. Moreover he describes their geometrical properties in detail and for Delone sets of finite type and Meyer sets gives many useful equivalences. In his paper there is a chapter about inflation symmetry which Lagarias defines as follows:

Definition 6.16. *A Delone set $X \subset \mathbb{R}^n$ has an inflation symmetry by the real number $\eta > 1$ if $\eta X \subset X$.*

we recall here his theorem about Delone sets with inflation symmetries which gives some conditions on the scaling parameter η.

Theorem 6.17. *Let X be a Delone set in $\mathbb{R}^d$ such that $\eta X \subset X$ for a real number $\eta > 1$.*

(i) If X is finitely generated, then η is an algebraic integer. If X has rank s, then the degree of η divides s.

(ii) If X is a Delone set of finite type, then η is an algebraic integer with all algebraic conjugates $|\eta'| \leq \eta$.

(iii) If X is a Meyer set, then η is an algebraic integer with all algebraic conjugates $|\eta'| \leq 1$. That is, η is a Pisot or a Salem number.

Techniques used in the proof of this theorem were inspiring for us. In the demonstration of (i) Lagarias rewrites the action of the inflation symmetry as a matrix multiplication and use the rational canonical form of this matrix to finish the proof. Slightly different techniques are used to prove (ii) and (iii). They require to define terms like almost linear mapping, shadow map etc. Under slightly stronger assumptions one can avoid such notions and use the matrix approach for the proof of (iii) as well, see [15]. The key moment in the proof is a decomposition of the whole space into eigenspaces corresponding to algebraic conjugates of η. For a coherence of the text we include the statement of Theorem 3.2. proven in [15] as the following corollary to Theorem 6.17.

Corollary 6.18. *Let $\Sigma(\Omega)$ be an n-dimensional cut-and-project set derived from CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that there exists $\eta \in \mathbb{R}$, $\eta > 1$ such that $\eta\Sigma(\Omega) \subset \Sigma(\Omega)$. Then it holds that*

(i) η is an algebraic integer and the degree of η divides s,

(ii) η is a Pisot or a Salem number.

6.3.3 Designer Quasicrystals: Cut-and-Project Sets with Pre-Assigned Properties

Pleasants' article [19] presents a comprehensive overview of certain properties of discrete sets that should be kept in mind when one models a mathematical quasicrystal. We have already presented a shortened list of these properties in Section 5.2 as well as conditions on inner and physical spaces and their subspaces that have to be satisfied in order to obtain a cut and project set with desired properties.

Let us focus on the second part of the article where quasicrystals with self-similarities are discussed and their construction is described. Requiring a quasicrystal with a symmetry or a flation property (see Section 5.2) Pleasants suggests to use operators.

Suppose that Σ is the cut and project set (or model set) derived form the cut and project scheme

$$\begin{array}{ccc} & \mathcal{L} \subset \mathbb{E}^s = V \oplus W & \\ \swarrow^{\pi_V} & & \searrow^{\pi_W} \\ \Sigma \subset \pi_V(\mathcal{L}) & & \Omega \subset \pi_W(\mathcal{L}) \end{array}$$

Proposition 6.19. *If Σ satisfies the V-condition and there is a linear operator T acting on $\mathbb{E}^s$ such that T restricted on V is a scaling by factor $\eta \neq \pm 1$, i.e. $T\Big|_V = \eta\, I$, W is T-invariant, $\mathcal{L} \cap T\mathcal{L}$ is a sublattice of $\mathcal{L}$ of full dimension N, and $\Omega \cap T\Omega$ has a non-empty interior, then Σ has the so-called flation property (XII), i.e. there is $\boldsymbol{t} \in V$ such that $(\Sigma + \boldsymbol{t}) \cap \eta\Sigma$ is dense.*

Note that Pleasants does not require $T\mathcal{L} = \mathcal{L}$ but on the other hand by requiring $\mathcal{L} \cap T\mathcal{L}$ to be a sublattice of $\mathcal{L}$ of full dimension s he also imposes the condition on T such that $T\mathcal{L}$ and $\mathcal{L}$ are commensurate.

Proposition 6.20. *Using the same notation as in Proposition 6.19, suppose that V-condition and W-condition are both satisfied. Let $S = \Omega \oplus V$. Then η gives a refining inflation (i.e. $|\eta| > 1$, $\eta \in \mathbb{R}$, and $\exists \boldsymbol{t} \in V$ such that $\eta\Sigma \subset \Sigma + \boldsymbol{t}$), if $T\mathcal{L} \subset \mathcal{L}$, $TS \subset S + \boldsymbol{l}$ for some $\boldsymbol{l} \in \mathcal{L}$.*

If $T\mathcal{L} \nsubseteq \mathcal{L}$ or $TS \setminus (S + \boldsymbol{l})$ has a non-empty interior for all $\boldsymbol{l} \in \mathcal{L}$, then η does not give a refining inflation.

Pleasants uses the following algebraic lemma to establish necessary and sufficient conditions for the existence of a suitable strip Σ such that $T\Sigma \subset \Sigma + \boldsymbol{l}$ for some $\boldsymbol{l}$.

Lemma 6.21. *With the same notation as above suppose that W-condition holds. Let T be a linear mapping on $V \oplus W$ such that V is an eigenspace of T corresponding to η, W is T-invariant and $\mathbb{Q}(T\mathcal{L} \cap \mathcal{L}) = \mathbb{Q}\mathcal{L}$. Then the characteristic polynomial χ_T satisfies $\chi_T = \mu^m$ where μ is the minimal polynomial of η over $\mathbb{Q}$ and $m \in \mathbb{N}$.*

Having this lemma one can establish necessary and sufficient conditions on a multiplier η giving a refining inflation. Suppose that Σ was derived using a polytopic window. Then one gets the following restriction.

Proposition 6.22. *Use the notation as in Proposition 6.19 with both V- and W-condition being satisfied as well as $T\mathcal{L} \subset \mathcal{L}$. Then necessary and sufficient condition for the existence of a polytope $\Omega \subset W$ such that T gives a refining inflation with multiplier η on Σ is that η is a Pisot number.*

It is important to point out that we are requiring Ω to be a polytope. If we relax this condition, refining inflation with a Salem number multiplier becomes possible.

In the next part of his paper, Pleasants gives an algebraic way to construct lattices using modules over algebraic number fields. Denote by K a real algebraic number field of degree d. Clearly, K^n has dimension nd as a vector space over $\mathbb{Q}$.

Definition 6.23. *A $\mathbb{Z}$-module generated by nd vectors in K^n that are linearly independent over $\mathbb{Q}$ is called a full module $\mathcal{Z}$ in K^n.*

As an example of a full module in K one has the ring of integers $\mathcal{O}_K$. Recall that $\mathcal{O}_K$ has the so-called integral basis, i.e. elements $\{\gamma_1, \ldots, \gamma_d\}$ such that each $\alpha \in \mathcal{O}_K$ can uniquely be written as $\alpha = a_1\gamma_1 + \cdots + a_d\gamma_d$ with integer coefficients a_i. It is clear that $\mathcal{O}_K$ is isomorphic to a d-dimensional lattice where the isomorphism is given as

$$\alpha \mapsto (a_1, a_2, \ldots, a_d).$$

This approach can be generalized and allows ones to construct a cut and project set. Since K is of degree d it can be written as $\mathbb{Q}(\theta)$ where θ is a root of a polynomial $f \in \mathbb{Q}[X]$ of degree d irreducible over $\mathbb{Q}$. Assume that f has r real and t pairs of complex conjugated roots. Then there exist d distinct isomorphisms of K, r of them into $\mathbb{R}$ and $2t$ into $\mathbb{C}$. Denote them as

$$\psi_1, \ldots, \psi_r, \psi_{r+1}, \overline{\psi}_{r+1}, \ldots, \psi_{r+t}, \overline{\psi}_{r+t}.$$

For $\boldsymbol{\alpha} \in K^n$ define a mapping κ as follows

$$\kappa(\boldsymbol{\alpha}) = (\psi_1(\boldsymbol{\alpha}), \ldots, \psi_r(\boldsymbol{\alpha}), \psi_{r+1}(\boldsymbol{\alpha}), \ldots, \psi_{r+t}(\boldsymbol{\alpha})) \tag{6.1}$$

which maps into $(\mathbb{E}^n)^r \oplus (\mathbb{C}^n)^t \simeq \mathbb{E}^{nd}$ and is one to one. Pleasants shows that the image of $\mathcal{O}_K^n$ is a lattice of full dimension in $\mathbb{E}^{nd}$. Since any full module $\mathcal{Z}$ satisfies that $\mathcal{Z} \cap \mathcal{O}_K^n$ has a finite index in $\mathcal{Z}$ (which is a well known number-theoretical fact) the κ-image of $\mathcal{Z}$ is also a lattice in $\mathbb{E}^{nd}$.

Using this construction one can for each full module $\mathcal{Z}$ in K^n obtain a model set Σ by splitting $\mathbb{E}^{nd}$ into the physical and the inner space as

$$\begin{aligned} V &= \mathbb{E}^n, \\ W &= (\mathbb{E}^n)^{r-1} \oplus \mathbb{C}^{nt} \simeq \mathbb{E}^{n(d-1)}, \end{aligned}$$

taking $\kappa(\mathcal{Z}) = \mathcal{L}$ as the lattice. As a projection π_V one takes the first d components of (6.1) and the remaining ones as π_W.

He shows other generic properties of such schemes. Let us list some of them

Lemma 6.24. *Let V, W, and $\mathcal{L}$ be derived from a full module $\mathcal{Z}$ in K^n as described above. Then*

(i) V, W satisfy the V- and W-condition,

(ii) if $K \neq \mathbb{Q}$ then $\mathcal{L} \cap V = \{\mathbf{0}\}$.

These properties imply that the constructed cut and project scheme is generic in the sense of Definition 5.12.

Then Pleasants demonstrates how to obtain linear operators on $V \oplus W$ such V and W are invariant under the action of these operators. They are derived from linear operators on K^n under which $\mathcal{Z}$ is invariant. If T is a linear operator on K^n it can be represented with respect to a suitable basis of K^n by a matrix M with entries in K. Then we can define T as a mapping that acts on $(\mathbb{E}^n)^r \oplus \mathbb{C}^{nt}$ as follows

$$(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_t) \mapsto \big(\psi_1(M)\boldsymbol{x}_1, \ldots, \psi_r(M)\boldsymbol{x}_r, \psi_{r+1}(M)\boldsymbol{z}_1, \ldots, \psi_{r+t}(M)\boldsymbol{z}_t\big).$$

It is clear that V and W are T-invariant and it can be shown that the lattice $\mathcal{L}$ as well. As an example of this mapping is scaling of V. This mapping can be derived from multiplication on K^n by a factor $\eta \in K$. Then the action of T is given by

$$(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_t) \mapsto \big(\psi_1(\eta)\boldsymbol{x}_1, \ldots, \psi_r(\eta)\boldsymbol{x}_r, \psi_{r+1}(\eta)\boldsymbol{z}_1, \ldots, \psi_{r+t}(\eta)\boldsymbol{z}_t\big).$$

The following proposition shows that there exists a way how to obtain a model set $\Sigma \subset \mathbb{E}^n$ that is invariant under an action of a group G of isometries of $\mathbb{E}^n$.

Proposition 6.25. *For every finite group isometry group G acting on $\mathbb{E}^n$ there is a plain model set $\Sigma \subset \mathbb{E}^n$ such that every $g \in G$ is a symmetry of Σ.*

The argument is based on the result from the theory of representation, that for G one can find a suitable real algebraic number field K that can be used as representation of G. Next, one has to define a G-invariant lattice. Pleasants starts with an arbitrary full module $\mathcal{Y}$ in K^n and defines a G-invariant full module $\mathcal{Z}$ in K^n as follows

$$\mathcal{Z} = \bigcap_{g \in G} g(\mathcal{Y}).$$

Taking the κ-image of $\mathcal{Z}$ one gets a full module that is G invariant. Choosing the acceptance window Ω to be G-invariant results in G-invariant model set Σ.

Pleasants summarizes the construction above in Theorem 3.5 in [19]. We provide a shortened version of this theorem related to the self-similarities.

Theorem 6.26. *Let G be an arbitrary finite isometry group acting on $\mathbb{E}^n$ and $K \neq \mathbb{Q}$ any real algebraic number field over which G has an equivalent representation. Then there exists a plain model set Σ such that Σ has*

- *properites (I) - (VII), (IX), (X), (XV),*
- *each $g \in G$ as a symmetry,*
- *a refining inflation with a multiplier η such that $\mathbb{Q}(\eta) = K$.*

6.3.4 On the self-similarities of model set

Cotfas in his paper [5] presents a method how to determine some self-similarities of a given cut and project set. Cotfas restricts himself only to cut and project sets derived from a scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ that is non-degenerated and irreducible. He does not require aperiodicity. When choosing a window Ω, Cotfas uses a compact set Ω satisfying $\Omega = \overline{\mathrm{int}(\Omega)} \neq \emptyset$. As usual Cotfas defines a star map between $\pi_{\parallel}(\mathcal{L})$ and $\pi_{\perp}(\mathcal{L})$. Note that in this case * is not a bijection.

As a self-similarity Cotfas considers all mappings A preserving $\Sigma(\Omega)$ that are of the form

$$A\boldsymbol{x} = \lambda R\boldsymbol{x} + \boldsymbol{v} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $R \in \mathrm{O}(n)$ and $\boldsymbol{v} \in \mathbb{R}^n$.

Let $L \in \mathbb{R}^{s \times s}$ be a matrix corresponding to the lattice $\mathcal{L}$, i.e. columns $\boldsymbol{\ell}_i$ of L satisfy that

$$\mathcal{L} = \sum_{i=1}^{s} \mathbb{Z}\boldsymbol{\ell}_i.$$

The set $\big\{\boldsymbol{\ell}_i : i \in \{1, \ldots, s\}\big\}$ forms a basis of $\mathbb{R}^s$, denote this basis as $\mathcal{L}$ (from the context it will always be clear what we understand under this notation).

For both projections Cotfas suggests to assign matrices $\Pi_{\|},\ \Pi_{\perp} \in \mathbb{R}^{s\times s}$ such that

$$\Pi_{\|} := {}^{\mathcal{L}}\!\left(\pi_{\|}\right), \qquad \Pi_{\perp} := {}^{\mathcal{L}}\!\left(\pi_{\perp}\right),$$

i.e. the matrices of the mappings in basis $\mathcal{L}$.

With this notation, Cotfas formulates and proves the following statement.

Theorem 6.27. *Using the notation defined above, if* $\lambda \in \mathbb{R} \setminus \{\mathbf{0}\}$, $\lambda' \in [-1, 1]$, $\boldsymbol{v} \in \pi_{\|}(\mathcal{L})$ *are such that*

$$M = \lambda \Pi_{\|} + \lambda' \Pi_{\perp}$$

is an integer matrix and

$$\lambda' \Omega + \boldsymbol{v}^{\star} \subset \Omega$$

then a mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ *defined as*

$$A\boldsymbol{x} = \lambda \boldsymbol{x} + \boldsymbol{v} \qquad \textit{for all } \boldsymbol{x} \in \mathbb{R}^n$$

is a self-similarity of $\Sigma(\Omega)$.

Note that Cotfas claims that using Theorem 6.27 one can not find all possible self-similarities of a given CPS and gives a reference to a counterexample.

Part II

Construction of schemes

Chapter 7

Matrix formalism for cut and project method

7.1 Formalism and its properties

It is natural, for our purposes, to use matrix formalism in the description of CPS. Let $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ be a CPS. With each lattice $\mathcal{L} \subset \mathbb{R}^s$ generated by vectors $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_s$ one can simply associate a regular matrix $L \in \mathbb{R}^{s \times s}$ such that the j-th column of L is formed by components of $\boldsymbol{\ell}_j$.

Definition 7.1. *We say that the matrix L described above corresponds to the lattice $\mathcal{L}$ and vice versa.*

Note that this correspondence is not one-to-one because one can change the basis of the lattice and thus change the matrix. These transformations are realized by integer matrices of order s with their determinant equal to ± 1 which can be easily seen.

Given a matrix L corresponding to the lattice $\mathcal{L}$ every lattice vector $\boldsymbol{l}$ can be written as

$$\boldsymbol{l} = L\mathbf{x}$$

where $\mathbf{x} \in \mathbb{Z}^s$. Using this formalism one can associate matrices to projections, i.e. rewrite the projections in terms of matrices as follows

$$\begin{aligned} \pi_{\parallel}(\boldsymbol{l}) &= (I_n, O)\,\boldsymbol{l}, \\ \pi_{\perp}(\boldsymbol{l}) &= (O, I_{s-n})\,\boldsymbol{l} \end{aligned}$$

for arbitrary vector $\boldsymbol{l} \in \mathbb{R}^s$. Here I_k stands for the identity matrix of order k and O stands for the zero matrix of order $n \times (s-n)$ or $(s-n) \times n$, respectively.

For a general regular matrix Y we introduce the following notation. We write $Y = (Y_1,\ Y_2)$ where

$$Y_1 := \big(Y_{\bullet 1}\, Y_{\bullet 2} \ldots Y_{\bullet n}\big) \in \mathbb{C}^{s \times n}, \quad Y_2 := \big(Y_{\bullet n+1}\, Y_{\bullet n+2} \ldots Y_{\bullet s}\big) \in \mathbb{C}^{s \times (s-n)}$$

where $Y_{\bullet j}$, denotes the j-th column Y_1 and $Y_{\bullet n+j}$, denotes the j-th column of Y_2. These column vectors are linearly independent since the matrix Y is regular and therefore there exists $V \in \mathbb{C}^{s \times s}$ such that $I = VY$. In the same way denote

$$V_1 := \begin{pmatrix} V_{1\bullet} \\ V_{2\bullet} \\ \vdots \\ V_{n\bullet} \end{pmatrix} \in \mathbb{C}^{n \times s}, \quad V_2 := \begin{pmatrix} V_{n+1\bullet} \\ V_{n+2\bullet} \\ \vdots \\ V_{s\bullet} \end{pmatrix} \in \mathbb{C}^{(s-n) \times s},$$

where $V_{i\bullet}$ denotes the i-th row of the matrix V. In the following propositions we will use the notation introduced above. For future purposes we reformulate properties of CPS in different way, mostly in terms of matrices. Note that the pair of subspaces that will be introduced in the following claim plays crucial role in our further construction.

Proposition 7.2. *Let $\mathcal{L} \subset \mathbb{R}^s$ be a lattice and L an associated matrix. Denote $Y = L^{-1}$. Let $1 \leq n < s$. The cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is*

(i) non-degenerate if and only if $\operatorname{span}_{\mathbb{R}}\{Y_{\bullet n+1}, \ldots, Y_{\bullet s}\} \cap \mathbb{Q}^s = \{\mathbf{0}\}$,

(ii) aperiodic if and only if $\operatorname{span}_{\mathbb{R}}\{Y_{\bullet 1}, \ldots, Y_{\bullet n}\} \cap \mathbb{Q}^s = \{\mathbf{0}\}$,

(iii) irreducible if and only if for any $\varepsilon > 0$ there exists elements in $\pi_\perp(\mathcal{L})$ that form a basis of $\mathbb{R}^{s-n}$ and each of them has norm $< \varepsilon$.

Proof. For the sake of the proof, write $Y = (Y_1, Y_2)$, where $Y_1 \in \mathbb{R}^{s \times n}$, $Y_2 \in \mathbb{R}^{s \times (s-n)}$. Let $\boldsymbol{r} \in \operatorname{span}_{\mathbb{R}}\{Y_{\bullet n+1}, \ldots, Y_{\bullet s}\} \cap \mathbb{Q}^s$. This is equivalent to the fact that there exist $\boldsymbol{\alpha} \in \mathbb{R}^{s-n}$ such that

$$\mathbb{Q}^s \ni \boldsymbol{r} = Y_2 \boldsymbol{\alpha} = (Y_1, Y_2) \begin{pmatrix} O \\ I_{s-n} \end{pmatrix} \boldsymbol{\alpha} = Y \begin{pmatrix} O \\ I_{s-n} \end{pmatrix} \boldsymbol{\alpha}.$$

Without loss of generality, we can assume that $\boldsymbol{r} \in \mathbb{Z}^s$. Since L is a regular matrix we have $\boldsymbol{r} \neq \mathbf{0}$ if and only if $L\boldsymbol{r} \neq \mathbf{0}$. But $L\boldsymbol{r}$ is a lattice vector such that

$$\pi_{\|}(L\boldsymbol{r}) = (I_n, O)L\boldsymbol{r} = (I_n, O)LY \begin{pmatrix} O \\ I_{s-n} \end{pmatrix} \boldsymbol{\alpha} = (I_n, O) I_s \begin{pmatrix} O \\ I_{s-n} \end{pmatrix} \boldsymbol{\alpha} = \mathbf{0} \in \mathbb{R}^n \,,$$

where we have used that $L = Y^{-1}$. This means that $\pi_{\|}$ restricted to $\mathcal{L}$ is not injective, which proves claim (i). Analogically, we demonstrate claim (ii).

Let us focus on claim (iii). Since the set $\pi_\perp(\mathcal{L})$ is a $\mathbb{Z}$-module, existence of an arbitrarily small basis of $\mathbb{R}^{s-n}$ ensures that $\pi_\perp(\mathcal{L})$ contains an $(s-n)$-dimensional lattice with arbitrarily small diameter of its unit cell. Since any point of $\mathbb{R}^{s-n}$ belongs to a unit cell, we find an element of $\pi_\perp(\mathcal{L})$ arbitrarily close to it. This proves the claim. □

Proposition 7.2 can be understood as a reformulation of Pleasant's Proposition 6.6 (claims *(i)*, *(ii)*, *(iv)*) for CPS.

Now let us assume that a window $\Omega \subset \mathbb{R}^{s-n}$ satisfies all conditions mentioned in Definition 5.11. Then one can rewrite a cut and project set derived from CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ through the window Ω in terms of matrices as

$$\Sigma(\Omega) = \{(I_n,\, O)L\boldsymbol{x} : \boldsymbol{x} \in \mathbb{Z}^s, (O,\, I_{s-n})L\boldsymbol{x} \in \Omega\}\,.$$

Let us now give some propositions related to cut and project schemes with self-similarities based on the matrix formulation. These mappings can be easily described through square matrices and we will not distinguish between the mapping and its matrix representation. Note that each linear self-similarity A of a generic CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is associated with some integer matrix C through the following relation

$$A(I_n, O)L = (I_n, O)LC. \tag{7.1}$$

The matrix C is unique due to the injectivity of $\pi_{\|}$. A acts only in the physical space and since it is a self-similarity, it maps a projection of a lattice point to the projection of another lattice point. We can say that the action of mapping A on $\pi_{\|}(\mathcal{L})$ induces an action of C on the whole lattice $\mathcal{L}$. Then there exists an induced mapping $B \in \mathbb{R}^{(s-n)\times(s-n)}$ acting on the inner space. The map B is given in the following way:

$$B(O, I_{s-n})L := (O, I_{s-n})LC. \tag{7.2}$$

Using the formalism above one can simply derive the following propositions. Their proofs can be found in [14].

Proposition 7.3. *Let $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ be a non-degenerate irreducible CPS. Let $A \in \mathbb{R}^{n \times n}$ satisfy $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$. Then there exists a matrix $C \in \mathbb{Z}^{s \times s}$, similar to a matrix*

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

where $B \in \mathbb{R}^{(s-n)\times(s-n)}$. *In particular*

$$C = L^{-1}\begin{pmatrix} A & O \\ O & B \end{pmatrix} L, \tag{7.3}$$

where $L \in \mathbb{R}^{s\times s}$ *is a matrix corresponding to the lattice* $\mathcal{L}$.

This proposition has a direct and obvious corollary.

Corollary 7.4. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a non-degenerate irreducible CPS. Let* $A \in \mathbb{R}^{n\times n}$ *satisfy* $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$. *Then the eigenvalues of the matrix* A *are algebraic integers and their minimal polynomial divides the characteristic polynomial of the matrix* C *over* $\mathbb{Q}$.

The next proposition shows that one can use integer matrices for constructing CPS with self-similarity induced by C. This will play crucial role in further construction.

Proposition 7.5. *Let for* $C \in \mathbb{Z}^{s\times s}$ *there exist a regular matrix* $L \in \mathbb{R}^{s\times s}$ *such that* LCL^{-1} *is block diagonal, i.e.*

$$LCL^{-1} = \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{(s-n)\times(s-n)}$. *Denote* $\mathcal{L} = \{\sum_{i=1}^{s} a_i \boldsymbol{l}_i : a_i \in \mathbb{Z}\}$ *the lattice corresponding to* L *and for a lattice vector* $\boldsymbol{l} \in \mathcal{L}$ *set the projections* $\pi_{\|} : \mathbb{R}^s \to \mathbb{R}^n$, $\pi_{\perp} : \mathbb{R}^s \to \mathbb{R}^{s-n}$ *as*

$$\pi_{\|}(\boldsymbol{l}) = (I_n, O)\boldsymbol{l}, \qquad \pi_{\perp}(\boldsymbol{l}) = (O, I_{s-n})\boldsymbol{l}.$$

(i) Then the CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *is self-similar with self-similarity* A, *i.e.* $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.

(ii) If $Z \in \mathbb{Z}^{s\times s}$ *is another matrix satisfying*

$$LZL^{-1} = \begin{pmatrix} S & O \\ O & T \end{pmatrix},$$

for some $S \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{(s-n)\times(s-n)}$, *then the CPS* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *has also self-similarity* S, *i.e.* $S\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.

It is important to keep in mind that Proposition 7.5 does not ensure that the constructed CPS is non-degenerate, aperiodic etc. For that we need Proposition 7.2 and verify the assumptions therein. The next lemma will enable us to consider matrices A, B, and C from Propositions 7.3 and 7.5 in special forms.

Lemma 7.6. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a CPS and let* $L \subset \mathbb{R}^{s\times s}$ *be a matrix corresponding to the lattice* $\mathcal{L}$. *Let further* $W_A \in \mathbb{R}^n$, $W_B \in \mathbb{R}^{(s-n)\times(s-n)}$, $Q \in \mathbb{Q}^{s\times s}$ *be regular matrices. Define*

$$\tilde{L} = \begin{pmatrix} W_A & O \\ O & W_B \end{pmatrix} LQ^{-1}$$

and $\tilde{\mathcal{L}} = \operatorname{span}_{\mathbb{Z}}\{\tilde{L}_{\bullet 1}, \dots, \tilde{L}_{\bullet s}\}$ *the corresponding lattice to* $\tilde{L}$. *Then* $(\tilde{\mathcal{L}} \subset \mathbb{R}^s, \mathbb{R}^n)$ *is a generic CPS if and only if the CPS* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *is generic.*

Proof. Firstly, we show the non-degenerancy. With respect to Proposition 7.2 we want to show that for $\tilde{Y} := \tilde{L}^{-1}$ it holds that $\operatorname{span}_{\mathbb{R}}\{\tilde{Y}_{\bullet n+1}, \dots, \tilde{Y}_{\bullet s}\} \cap \mathbb{Q}^s = \{\boldsymbol{0}\}$ if and only if $\operatorname{span}_{\mathbb{R}}\{Y_{\bullet n+1}, \dots, Y_{\bullet s}\} \cap \mathbb{Q}^s = \{\boldsymbol{0}\}$. Let $\boldsymbol{r} \in \mathbb{Q}^s$ and let $\boldsymbol{r} = \tilde{Y}\binom{O}{I_{s-n}}\boldsymbol{\alpha}$ for $\boldsymbol{\alpha} \in \mathbb{R}^{s-n}$.

$$\boldsymbol{r} = \tilde{Y}\begin{pmatrix} O \\ I_{s-n} \end{pmatrix}\boldsymbol{\alpha} = \tilde{L}^{-1}\begin{pmatrix} O \\ I_{s-n} \end{pmatrix}\boldsymbol{\alpha} = QL^{-1}\begin{pmatrix} W_A^{-1} & O \\ O & W_B^{-1} \end{pmatrix}\begin{pmatrix} O \\ I_{s-n} \end{pmatrix}\boldsymbol{\alpha} =$$

$$= QY\begin{pmatrix} O \\ W_B^{-1} \end{pmatrix}\boldsymbol{\alpha} = QY_2W_B^{-1}\boldsymbol{\alpha}.$$

Since matrices Q, W_B are regular matrices, it holds that $\boldsymbol{r} = \boldsymbol{0}$ if and only if $Y_2 W_B^{-1}\boldsymbol{\alpha} = \boldsymbol{0}$ for $\boldsymbol{\beta} \in \mathbb{R}^{s-n}$. This holds if and only if $Y_2\boldsymbol{\beta} = \boldsymbol{0}$ but this is equivalent to the fact that $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is non-degenerate.

Analogously one shows the aperiodicity of schemes.

In order to prove the irreducibility let us invert the transformation and obtain an expression of the lattice $\mathcal{L}$ in terms of $\tilde{\mathcal{L}}$ and then let us apply the second projection $\pi_\perp$ on $\tilde{\mathcal{L}}$. we have

$$\pi_\perp(\mathcal{L}) = W_B^{-1}\pi_\perp(\tilde{\mathcal{L}}Q).$$

Since Q is a rational matrix, there exists $p \in \mathbb{N}$ such that $pQ \in \mathbb{Z}^{s\times s}$. Therefore it holds that

$$\tilde{\mathcal{L}}Q = \frac{1}{p}\tilde{\mathcal{L}}pQ \subset \frac{1}{p}\tilde{\mathcal{L}}.$$

So we have

$$\pi_\perp(\mathcal{L}) = W_B^{-1}\pi_\perp(\tilde{\mathcal{L}}Q) \subset \frac{1}{p}W_B^{-1}\pi_\perp(\tilde{\mathcal{L}}).$$

Multiplying this equation from the left by W_B and by p one gets

$$pW_B^{-1}\pi_\perp(\mathcal{L}) \subset \pi_\perp(\tilde{\mathcal{L}}).$$

Since the regular transformation pW_B does not change the density of the projection $\pi_\perp(\mathcal{L})$, the equation above implies the density of $\pi_\perp(\tilde{\mathcal{L}})$ which proves one implication. The second one can be proven in the same way by interchanging the role of $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

□

The above lemma allows us to define a relation of equivalence on all possible cut and project schemes.

Definition 7.7. *We say that cut and project schemes* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *and* $(\tilde{\mathcal{L}} \subset \mathbb{R}^s, \mathbb{R}^n)$ *from Proposition 7.6 are equivalent.*

Note that this equivalence can be understood as an extension of term *commensurateness*. Recall that two lattices $\mathcal{L}$, $\tilde{\mathcal{L}}$ are commensurate if $\mathbb{Q}\mathcal{L} = \mathbb{Q}\tilde{\mathcal{L}}$.

Note also that the equivalence between CPS allows us when solving Questions 1 and 2 to restrict ourselves only to matrices in a special form.

Proposition 7.8. *When solving Questions 1 and 2, we can, without loss of generality, consider* (7.3) *with matrices* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{(s-n)\times(s-n)}$ *taken in real Jordan form and matrix* $C \in \mathbb{Z}^{s\times s}$ *in rational Jordan form.*

Proof. Given a cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$, let us assume that $A \in \mathbb{R}^{n\times n}$ is a self-similarity of the $\mathbb{Z}$-module $\pi_\parallel(\mathcal{L})$, i.e. $A\pi_\parallel(\mathcal{L}) \subset \pi_\parallel(\mathcal{L})$. By Proposition 7.3 there exist matrices $B \in \mathbb{R}^{(s-n)\times(s-n)}$, $C \in \mathbb{Z}^{s\times s}$, such that (7.3)holds.

Suppose we have regular matrices $W_A \in \mathbb{R}^{n\times n}$, $W_B \in \mathbb{R}^{(s-n)\times(s-n)}$, $Q \in \mathbb{Q}^{s\times s}$ which bring matrices A, B, C into the corresponding Jordan forms $J_A = W_A A W_A^{-1}$, $J_B = W_B B W_B^{-1}$, $J_C = QCQ^{-1}$. We have

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} L = \begin{pmatrix} W_A^{-1}J_A W_A & O \\ O & W_B^{-1}J_B W_B \end{pmatrix} L = \begin{pmatrix} W_A^{-1} & O \\ O & W_B^{-1} \end{pmatrix}\begin{pmatrix} J_A & O \\ O & J_B \end{pmatrix}\begin{pmatrix} W_A & O \\ O & W_B \end{pmatrix} L \quad (7.4)$$

and $LC = LQ^{-1}J_C Q$. Substituting into (7.3), one obtains

$$\begin{pmatrix} J_A & O \\ O & J_B \end{pmatrix}\begin{pmatrix} W_A & O \\ O & W_B \end{pmatrix} LQ^{-1} = \begin{pmatrix} W_A & O \\ O & W_B \end{pmatrix} LQ^{-1}J_C\,.$$

Denoting $\tilde{L} = \left(\begin{smallmatrix} W_A & O \\ O & W_B \end{smallmatrix}\right) LQ^{-1}$, we obtain

$$\begin{pmatrix} J_A & O \\ O & J_B \end{pmatrix}\tilde{L} = \tilde{L}J_C\,.$$

Define $\tilde{\mathcal{L}}$ as the lattice corresponding to the matrix $\tilde{L}$. Now the CPS $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ and $(\tilde{\mathcal{L}} \subset \mathbb{R}^s, \mathbb{R}^n)$ are equivalent in the sense of Definition 7.7 and thus the statement of the corollary follows from Lemma 7.6.

□

7.2 Necessary conditions through minimal polynomials

This section shows how minimal polynomials over $\mathbb{Q}$ of matrices A, B and C defining a self-similarity of a CPS by (7.3) are related. This turns out to be helpful if one is interested in some estimations on the dimension of the lattice needed in the construction.

Proposition 7.9. *Suppose the cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is non-degenerated. Then for every polynomial $p \in \mathbb{Z}[X]$ it holds that if $p(A) = O$, then $p(B) = O$ where A, B are non-zero matrices from Proposition 7.3.*

Proof. According to Proposition 7.3 for any polynomial p it holds that

$$Lp(C)L^{-1} = p(LCL^{-1}) = \begin{pmatrix} p(A) & O \\ O & p(B) \end{pmatrix}.$$

One realizes that $p(C) = O$ if and only if both $p(A) = O$ and $p(B) = O$.

For a contradiction suppose that $p(A) = O$ and $p(B) \neq O$ (so consequently $p(C) \neq O$). Then

$$Lp(C) = \begin{pmatrix} O & O \\ O & p(B) \end{pmatrix} L.$$

Applying $\pi_{\|}$ projection on both sides of the equality one obtains

$$(I_n,\, O)\, Lp(C) = (I_n,\, O) \begin{pmatrix} O & O \\ O & p(B) \end{pmatrix} V = (O,\, O).$$

Since $p(C) \in \mathbb{Z}^{s\times s}$ is a non-zero matrix there exists a non-zero column in $p(C)$, denote it $\mathbf{x}$. Then the vector $\boldsymbol{\ell} := L\mathbf{x} \in \mathcal{L}$ is a non-zero lattice vector whose first projection gives zero vector, i.e. $\pi_{\|}(\boldsymbol{\ell}) = \mathbf{0}$. Thus $\pi_{\|}\big|_{\mathcal{L}}$ is not injective and this is a contradiction. □

Proposition 7.10. *Suppose the cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is non-degenerated. Then $\mu_{\mathbb{Q},B}$ divides $\mu_{\mathbb{Q},A}$ and $\mu_{\mathbb{Q},A} = \mu_{\mathbb{Q},C} = \mu_C$.*

Proof. According to Proposition 7.9 $\mu_{\mathbb{Q},A}(A) = O$ implies $\mu_{\mathbb{Q},A}(B) = O$. Since $\mu_{\mathbb{Q},B}(B) = O$ we get from the basic properties of minimal polynomials over $\mathbb{Q}$ that $\mu_{\mathbb{Q},B}$ divides $\mu_{\mathbb{Q},A}$.

Further we get $\mu_{\mathbb{Q},A}(C) = L^{-1}OL = O$ and therefore $\mu_{\mathbb{Q},C}$ divides $\mu_{\mathbb{Q},A}$. To show that $\mu_{\mathbb{Q},A}$ divides $\mu_{\mathbb{Q},C}$ is sufficient to verify that $\mu_{\mathbb{Q},C}(A) = O$. Since we know that minimal polynomials over $\mathbb{Q}$ of similar matrices coincide and from the fact that $\mu_{\mathbb{Q},C}(C) = O$ we obtain

$$O = \mu_{\mathbb{Q},C}(C) = \mu_{\mathbb{Q},C}\left(L^{-1}\begin{pmatrix} A & O \\ O & B \end{pmatrix} L\right) = L^{-1}\begin{pmatrix} \mu_{\mathbb{Q},C}(A) & O \\ O & \mu_{\mathbb{Q},C}(B) \end{pmatrix} L.$$

From that it follows that $\mu_{\mathbb{Q},C}(A) = O$. □

In the same way one may prove the following proposition.

Proposition 7.11. *Suppose the cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is aperiodic. Then $\mu_{\mathbb{Q},A}$ divides $\mu_{\mathbb{Q},B}$ and $\mu_{\mathbb{Q},B} = \mu_{\mathbb{Q},C} = \mu_C$.*

Combining the above propositions together we obtain the necessary conditions for creating non-degenerate aperiodic cut and project schemes.

Theorem 7.12. *Let $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ be a non-degenerated and aperiodic cut and project scheme with self-similarity A. Then $\mu_{\mathbb{Q},A} = \mu_{\mathbb{Q},B} = \mu_{\mathbb{Q},C} = \mu_C$.*

Requiring a CPS with n-fold rotation symmetry A, then necessary $A^n = I$. This implies that A is annihilated by the polynomial $X^n - 1$. The polynomial $X^n - 1$ is divisible by the cyclotomic polynomial $\Phi_n(X) \in \mathbb{Z}[X]$. $\Phi_n(X)$ is irreducible and minimal one that annihilates A. Using the condition derived in Theorem 7.12 one gets

$$\mu_{\mathbb{Q},A} = \deg \Phi_n = \mu_C.$$

Taking C such that the minimal polynomial μ_C is the characteristic polynomial χ_C (for example the companion matrix to Φ_n) we have an estimation on the minimal dimension of C given by $\phi(n)$, which fully corresponds with Theorem 6.15.

7.3 Composition of schemes

This section presents a binary operation over all generic CPS. We show that a set of all generic CPS is closed under this operation. For a composition of two schemes with self-similarities we show how the self-similarity of the resulting scheme looks like.

Definition 7.13. *Let $\hat\Lambda = (\hat{\mathcal{L}} \subset \mathbb{R}^{\hat s}, \mathbb{R}^{\hat n})$, $\check\Lambda = (\check{\mathcal{L}} \subset \mathbb{R}^{\check s}, \mathbb{R}^{\check n})$ be cut and project schemes and let $\hat L$, $\check L$ be matrices in $\mathbb{R}^{\hat s\times\hat s}$, $\mathbb{R}^{\check s\times\check s}$ corresponding to lattices $\hat{\mathcal{L}}$, $\check{\mathcal{L}}$, respectively. Set*

$$L = P\begin{pmatrix}\hat L & O\\ O & \check L\end{pmatrix} \qquad \text{where} \qquad P = \begin{pmatrix} I_{\hat n} & O & O & O\\ O & O & I_{\hat s-\hat n} & O\\ O & I_{\check n} & O & O\\ O & O & O & I_{\check s-\check n}\end{pmatrix}. \tag{7.5}$$

Denote $s = \hat s + \check s$, $n = \hat n + \check n$. Define the lattice $\mathcal{L} \in \mathbb{R}^s$ by

$$\mathcal{L} := \operatorname{span}_{\mathbb{Z}}\{L_{\bullet 1}, \dots, L_{\bullet s}\}$$

and projections $\pi_{\|} : \mathbb{R}^s \to \mathbb{R}^n$, $\pi_\perp : \mathbb{R}^s \to \mathbb{R}^{s-n}$ by

$$\begin{aligned}\pi_{\|}(\mathcal{L}) &= (I_n, O)\mathcal{L},\\ \pi_\perp(\mathcal{L}) &= (O, I_{s-n})\mathcal{L}.\end{aligned}$$

Then we call the scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ the direct sum of schemes $\hat\Lambda$, $\check\Lambda$. We denote it by $\Lambda = \hat\Lambda \oplus \check\Lambda$.

Inductively, we define direct sum of more than two CPS, i.e.

$$\Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_n := (\Lambda_1 \oplus \cdots \Lambda_{n-1}) \oplus \Lambda_n.$$

Proposition 7.14. *Let $\Lambda = \hat\Lambda \oplus \check\Lambda$. Then*

- *Λ is non-degenerate if and only if $\hat\Lambda$ and $\check\Lambda$ are both non-degenerate;*
- *Λ is irreducible if and only if $\hat\Lambda$ and $\check\Lambda$ are both irreducible;*
- *Λ is aperiodic if and only if $\hat\Lambda$ and $\check\Lambda$ are both aperiodic.*

Proof. Let $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$, $\hat\Lambda = \left(\hat{\mathcal{L}} \subset \mathbb{R}^{\hat s}, \mathbb{R}^{\hat n}\right)$, $\check\Lambda = \left(\check{\mathcal{L}} \subset \mathbb{R}^{\check s}, \mathbb{R}^{\check n}\right)$, and let $L, \hat L, \check L$ be the matrices corresponding to the lattices $\mathcal{L}, \hat{\mathcal{L}}, \check{\mathcal{L}}$, respectively.

One can write $L = \binom{L_1}{L_2}$ where $L_1 \in \mathbb{R}^{n\times s}$, $L_2 \in \mathbb{R}^{(s-n)\times s}$, $Y = (Y_1, Y_2)$, where $Y_1 \in \mathbb{R}^{s\times n}$, $Y_2 \in \mathbb{R}^{s\times(s-n)}$. Similarly, we introduce notation for $\hat L = \binom{\hat L_1}{\hat L_2}$, $\hat Y = (\hat Y_1, \hat Y_2)$, and $\check L = \binom{\check L_1}{\check L_2}$, $\check Y = (\check Y_1, \check Y_2)$.

The proof of the proposition stems in the fact that from (7.5),

$$L = P\begin{pmatrix}\hat L & O\\ O & \check L\end{pmatrix} = \begin{pmatrix}\hat L_1 & O\\ O & \check L_1\\ \hat L_2 & O\\ O & \check L_2\end{pmatrix}, \quad \text{i.e.} \quad L_1 = \begin{pmatrix}\hat L_1 & O\\ O & \check L_1\end{pmatrix},\ L_2 = \begin{pmatrix}\hat L_2 & O\\ O & \check L_2\end{pmatrix},$$

where we have taken into account the dimensions of the matrices. Since the matrix P is a permutation matrix it holds that $P^{-1} = P^T$. Hence we have

$$I_{\hat s+\check s} = YL = \begin{pmatrix}\hat Y & O\\ O & \check Y\end{pmatrix}\begin{pmatrix}\hat L & O\\ O & \check L\end{pmatrix} = \begin{pmatrix}\hat Y & O\\ O & \check Y\end{pmatrix} P^T P \begin{pmatrix}\hat L & O\\ O & \check L\end{pmatrix} =$$

$$= \begin{pmatrix}\hat Y_1 & O & \hat Y_2 & O\\ O & \check Y_1 & O & \check Y_2\end{pmatrix}\begin{pmatrix}\hat L_1 & O\\ O & \check L_1\\ \hat L_2 & O\\ O & \check L_2\end{pmatrix}.$$

Consequently

$$Y_1 = \begin{pmatrix} \hat{Y}_1 & O \\ O & \check{Y}_1 \end{pmatrix}, \; Y_2 = \begin{pmatrix} \hat{Y}_2 & O \\ O & \check{Y}_2 \end{pmatrix}.$$

Properties (i)–(iii) are now checked with the help of Proposition 7.2 because the real span of columns of Y_1 does not have any rational subspaces if and only if the real spans of columns of $\hat{Y}_1$ and $\check{Y}_1$ do not have any rational subspaces. For irreducibility, we use the fact that the cartesian product of the sets $\pi_\perp(\hat{\mathcal{L}})$, $\pi_\perp(\check{\mathcal{L}})$ that are dense in $\mathbb{R}^{\hat{s}-\hat{n}}$, $\mathbb{R}^{\check{s}-\check{n}}$ resp., is dense in $\mathbb{R}^{s-n}$. □

Proposition 7.15. *Let $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\check{A} \in \mathbb{R}^{\check{n}\times\check{n}}$ be linear mappings. If there exist cut and project schemes $\hat{\Lambda} = (\hat{\mathcal{L}} \subset \mathbb{R}^{\hat{s}}, \mathbb{R}^{\hat{n}})$, $\check{\Lambda} = (\check{\mathcal{L}} \subset \mathbb{R}^{\check{s}}, \mathbb{R}^{\check{n}})$, with self-similarities $\hat{A}$, $\check{A}$, respectively, than $A = \begin{pmatrix} \hat{A} & O \\ O & \check{A} \end{pmatrix}$, is a self-similarity of the cut and project scheme $\Lambda = \hat{\Lambda} \oplus \check{\Lambda}$.*

Proof. By Proposition 7.3, there exist matrices $\hat{B} \in \mathbb{R}^{(\hat{s}-\hat{n})\times(\hat{s}-\hat{n})}$, $\check{B} \in \mathbb{R}^{(\check{s}-\check{n})\times(\check{s}-\check{n})}$, $\hat{C} \in \mathbb{Z}^{\hat{s}\times\hat{s}}$, $\check{C} \in \mathbb{Z}^{\check{s}\times\check{s}}$ such that

$$\begin{pmatrix} \hat{A} & O \\ O & \hat{B} \end{pmatrix} \hat{L} = \hat{L}\hat{C}, \qquad \begin{pmatrix} \check{A} & O \\ O & \check{B} \end{pmatrix} \check{L} = \check{L}\check{C},$$

where $\hat{L} \in \mathbb{R}^{\hat{s}\times\hat{s}}$, $\check{L} \in \mathbb{R}^{\check{s}\times\check{s}}$ are matrices associated to the lattices $\hat{\mathcal{L}} \subset \mathbb{R}^{\hat{s}}$, $\check{\mathcal{L}} \subset \mathbb{R}^{\check{s}}$. Denoting as above $\hat{L} = \begin{pmatrix} \hat{L}_1 \\ \hat{L}_2 \end{pmatrix}$, $\check{L} = \begin{pmatrix} \check{L}_1 \\ \check{L}_2 \end{pmatrix}$, we derive

$$\begin{pmatrix} \hat{A} & O & O & O \\ O & \check{A} & O & O \\ O & O & \hat{B} & O \\ O & O & O & \check{B} \end{pmatrix} \begin{pmatrix} \hat{L}_1 & O \\ O & \check{L}_1 \\ \hat{L}_2 & O \\ O & \check{L}_2 \end{pmatrix} = \begin{pmatrix} \hat{L}_1 & O \\ O & \check{L}_1 \\ \hat{L}_2 & O \\ O & \check{L}_2 \end{pmatrix} \begin{pmatrix} \hat{C} & O \\ O & \check{C} \end{pmatrix}.$$

Setting $B = \begin{pmatrix} \hat{B} & O \\ O & \check{B} \end{pmatrix}$, $C = \begin{pmatrix} \hat{C} & O \\ O & \check{C} \end{pmatrix}$, we have $\begin{pmatrix} A & O \\ O & B \end{pmatrix} L = LC$, for the matrix $L = P\begin{pmatrix} \hat{L} & O \\ O & \check{L} \end{pmatrix}$ corresponding to the direct sum of schemes in Definition 7.13. By Proposision 7.5, the statement follows. □

The following theorem proves that we can, without loss of generality, proceed in our construction with matrices having only one conjugacy class in $\sigma(A)$, i.e. only matrices A whose eigenvalues are roots of the same polynomial f irreducible over $\mathbb{Q}$. A general case can be then obtained through the direct sum of CPS.

Theorem 7.16. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that all its eigenvalues are algebraic integers. Let $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\check{A} \in \mathbb{R}^{\check{n}\times\check{n}}$ be such that*

$$A = \begin{pmatrix} \hat{A} & O \\ O & \check{A} \end{pmatrix}.$$

Suppose that no eigenvalue of $\hat{A}$ is an algebraic conjugate of any eigenvalue of $\check{A}$. Let A be a self-similarity of a cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$. Then there exist cut and project schemes $\hat{\Lambda} = (\hat{\mathcal{L}} \subset \mathbb{R}^{\hat{s}}, \mathbb{R}^{\hat{n}})$, $\check{\Lambda} = (\check{\mathcal{L}} \subset \mathbb{R}^{\check{s}}, \mathbb{R}^{\check{n}})$ such that Λ is equivalent to the direct sum $\hat{\Lambda} \oplus \check{\Lambda}$.

Proof. Let $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ and let $L \subset \mathbb{R}^{s\times s}$ be a regular matrix corresponding to $\mathcal{L}$. Let $A = \left(\begin{smallmatrix} \hat{A} & O \\ O & \check{A} \end{smallmatrix}\right)$ with mappings $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\check{A} \in \mathbb{R}^{\check{n}\times\check{n}}$. If A is a self-similarity of $\pi_\parallel(\mathcal{L})$, then by Proposition 7.3, there exist $C \in \mathbb{Z}^{s\times s}$, $B \in \mathbb{R}^{(s-n)\times(s-n)}$ such that $\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right) L = LC$. Since we study the scheme Λ up to equivalence, we can consider without loss of generality $B = \left(\begin{smallmatrix} \hat{B} & O \\ O & \check{B} \end{smallmatrix}\right)$, $\hat{B} \subset \mathbb{R}^{(\hat{s}-\hat{n})\times(\hat{s}-\hat{n})}$, $\check{B} \subset \mathbb{R}^{(\check{s}-\check{n})\times(\check{s}-\check{n})}$ and $C = \left(\begin{smallmatrix} \hat{C} & O \\ O & \check{C} \end{smallmatrix}\right)$, $\hat{C} \subset \mathbb{Z}^{\hat{s}\times\hat{s}}$, $\check{C} \subset \mathbb{Z}^{\check{s}\times\check{s}}$. Moreover, we can decompose the spectra $\sigma(\hat{C}) = \sigma(\hat{A}) \cup \sigma(\hat{B})$, $\sigma(\check{C}) = \sigma(\check{A}) \cup \sigma(\check{B})$. By assumption, no eigenvalue of $\hat{C}$ is an algebraic conjugate of no eigenvalue of $\check{C}$.

Consider $Y = L^{-1}$ and write Y as a block matrix $Y = \begin{pmatrix} \hat{Y}_1 & \check{W}_1 & \hat{Y}_2 & \check{W}_2 \\ \hat{W}_1 & \check{Y}_1 & \hat{W}_2 & \check{Y}_2 \end{pmatrix}$, where matrices in the first row $\hat{Y}_1, \check{W}_1, \hat{Y}_2, \check{W}_2$ have $\hat{s}$ rows and $\hat{n}, \check{n}, \hat{s}-\hat{n}, \check{s}-\check{n}$ columns respectively. Similarly, matrices $\hat{W}_1, \check{Y}_1, \hat{W}_2, \check{Y}_2$ in the second row have $\check{s}$ rows and again $\hat{n}, \check{n}, \hat{s}-\hat{n}, \check{s}-\check{n}$ columns respectively. Our aim is to show that all of the W matrices vanish. We have

$$\begin{aligned} Y \begin{pmatrix} A & O \\ O & B \end{pmatrix} &= \begin{pmatrix} \hat{Y}_1 & \check{W}_1 & \hat{Y}_2 & \check{W}_2 \\ \hat{W}_1 & \check{Y}_1 & \hat{W}_2 & \check{Y}_2 \end{pmatrix} \begin{pmatrix} \hat{A} & O & O & O \\ O & \check{A} & O & O \\ O & O & \hat{B} & O \\ O & O & O & \check{B} \end{pmatrix} = \\ &= \begin{pmatrix} \hat{Y}_1\hat{A} & \check{W}_1\check{A} & \hat{Y}_2\hat{B} & \check{W}_2\check{B} \\ \hat{W}_1\hat{A} & \check{Y}_1\check{A} & \hat{W}_2\hat{B} & \check{Y}_2\check{B} \end{pmatrix} = \begin{pmatrix} \hat{C} & O \\ O & \check{C} \end{pmatrix} \begin{pmatrix} \hat{Y}_1 & \check{W}_1 & \hat{Y}_2 & \check{W}_2 \\ \hat{W}_1 & \check{Y}_1 & \hat{W}_2 & \check{Y}_2 \end{pmatrix} \end{aligned} \tag{7.6}$$

For the first $\hat{n}$ columns of (7.6), we have

$$\begin{pmatrix} \hat{Y}_1 \\ \hat{W}_1 \end{pmatrix} \hat{A} = \begin{pmatrix} \hat{C} & O \\ O & \check{C} \end{pmatrix} \begin{pmatrix} \hat{Y}_1 \\ \hat{W}_1 \end{pmatrix}. \tag{7.7}$$

From the lower $\check{s}$ rows of (7.7) we extract that $\hat{W}_1\hat{A} = \check{C}\hat{W}_1$. There exists a matrix $R \in \mathbb{C}^{\hat{n}\times\hat{n}}$ such that $\hat{A} = R^{-1}J_{\hat{A}}R$ where $J_{\hat{A}}$ is the Jordan form of $\hat{A}$. We derive that $\hat{W}_1R^{-1}J_{\hat{A}} = \check{C}\hat{W}_1R^{-1}$, i.e. the columns of $\hat{W}_1R^{-1}$, if non-zero, are generalized eigenvectors of $\check{C}$ corresponding to the Jordan blocks of $\hat{A}$. This is impossible, since by assumption, $\sigma(\check{C}) \cap \sigma(\hat{A}) = \emptyset$.

Thus $\hat{W}_1$ must be the zero matrix, i.e. $\hat{W}_1 = O$. By a similar approach, we can deduce from (7.6) that $\hat{W}_2 = O$, $\check{W}_1 = O$, and $\check{W}_2 = O$, i.e. the matrix Y is of the form

$$Y = \begin{pmatrix} \hat{Y}_1 & O & \hat{Y}_2 & O \\ O & \check{Y}_1 & O & \check{Y}_2 \end{pmatrix} = \begin{pmatrix} \hat{Y}_1 & \hat{Y}_2 & O & O \\ O & O & \check{Y}_1 & \check{Y}_2 \end{pmatrix} \begin{pmatrix} I_{\hat{n}} & O & O & O \\ O & O & I_{\hat{s}-\hat{n}} & O \\ O & I_{\check{n}} & O & O \\ O & O & O & I_{\check{s}-\check{n}} \end{pmatrix} = \begin{pmatrix} \hat{Y} & O \\ O & \check{Y} \end{pmatrix} P$$

where P is the permutation matrix as in Definition 7.13, and we denote $\hat{Y} = (\hat{Y}_1, \hat{Y}_2)$, $\check{Y} = (\check{Y}_1, \check{Y}_2)$, as before.

Let us derive the matrix $L = Y^{-1}$. We have

$$L = \left(\begin{pmatrix} \hat{Y} & O \\ O & \check{Y} \end{pmatrix} P \right)^{-1} = P^{-1} \begin{pmatrix} \hat{Y}^{-1} & O \\ O & \check{Y}^{-1} \end{pmatrix} = P^T \begin{pmatrix} \hat{L} & O \\ O & \check{L} \end{pmatrix},$$

where we use $P^{-1} = P^T$ and denote $\hat{L} = \hat{Y}^{-1}$, $\check{L} = \check{Y}^{-1}$. If $\hat{L}$, $\check{L}$ are taken to define lattices $\hat{L}$, $\check{L}$, comparing with Definition 7.13, we see that the cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ is the direct sum of the schemes $\hat{\Lambda} = (\hat{\mathcal{L}} \subset \mathbb{R}^{\hat{s}}, \mathbb{R}^{\hat{n}})$, $\check{\Lambda} = (\check{\mathcal{L}} \subset \mathbb{R}^{\check{s}}, \mathbb{R}^{\check{n}})$. □

Note that the assumption on the spectra of $\hat{A}$ and $\check{A}$ is important. If the spectra $\sigma(\hat{A})$, $\sigma(\check{A})$ contain algebraic conjugates, then a scheme with self-similarity $A = \begin{pmatrix} \hat{A} & O \\ O & \check{A} \end{pmatrix}$ may not be a direct sum of the schemes with self-similarities $\hat{A}$, $\check{A}$, respectively. As a trivial counterexample let us suppose $\hat{A} = I_n$ and $\check{A} = I_m$. The resulting mapping A is identity I_{n+m} which is a (trivial) self-similarity of all CPS in $n+m$ dimensions, not only of those in the form of direct sums of two schemes in dimensions n and m.

Chapter 8

Construction of a scheme with a given diagonalizable self-similarity

This chapter contains a construction of a CPS with desired diagonalizable self-similarity A. As we have already proven in Proposition 7.8 it is sufficient to restrict ourselves only on matrices in Jordan forms. When constructing a CPS with a given self-similarity, we will proceed by analyzing the spectrum of A. Firstly we determine CPS corresponding to the minimal polynomials of the individual eigenvalues from the spectrum of A. Such *elementary CPS* are described in Section 8.1 together with their construction. In the next step we compose them together using the operation described in Section 7.3.

8.1 Elementary schemes

As it was mentioned before this section is dedicated to the so-called elementary blocks that play a crucial role in the construction of a cut and project scheme to a given self-similarity A.

The spectrum $\sigma(A)$ of the matrix A can be splitted into the so-called conjugacy classes, i.e. so that eigenvalues in each class are algebraic conjugates. According to Proposition 7.3, there exists an integer matrix C satisfying $\sigma(C) = \sigma(A) \cup \sigma(B)$. This allows us to factorise the characteristic polynomial of C and create the cut and project scheme from elementary cut and project schemes whose matrices C_e will be companion matrices to the irreducible factors of the characteristic polynomial of C.

Suppose that we have a polynomial f of degree s irreducible over $\mathbb{Q}$ and its companion matrix C_f. Denote the roots of f by $\beta = \beta_1, \ \ldots, \ \beta_s$. Consider the isomorphisms $\psi_i : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_i)$ defined as in Chapter 2.

For this polynomial f let us define the following matrix W_f:

$$W_f = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta & \psi_2(\beta) & \cdots & \psi_s(\beta) \\ \vdots & \vdots & \ddots & \vdots \\ \beta^{s-1} & \psi_2(\beta^{s-1}) & \cdots & \psi_s(\beta^{s-1}) \end{pmatrix}. \tag{8.1}$$

Definition 8.1. *The matrix W_f defined in* (8.1) *is called Vandermonde matrix corresponding to numbers $\beta, \psi_2(\beta), \ \ldots, \ \psi_s(\beta)$.*

Vandermonde matrices have interesting properties; first, we use the fact that its columns are eigenvectors of the matrix C_f.

Lemma 8.2. *Let C_f be the companion matrix to a polynomial f irreducible over $\mathbb{Q}$. Denote by β one of its roots. Then the i-th column of the matrix W_f defined in* (8.1) *is an eigenvector of C_f corresponding to $\psi_i(\beta)$.*

Proof. This claim can be shown directly. □

Moreover it turns out to be useful to prove the following lemma about the matrix W_f:

Lemma 8.3. *Let β be an algebraic number of degree s (i.e. the root of polynomial f) and let W_f be the corresponding Vandermonde matrix* (8.1)*. Then the i-th row of the matrix W_f^{-1} is the image of its first row under the i-th isomorphism ψ_i of K.*

Proof. According to Lemma 8.2 the columns of the matrix W_f are eigenvectors of the matrix C corresponding to eigenvalues $\beta, \psi_2(\beta), \cdots, \psi_s(\beta)$. Thus one can write

$$C_f W_f = W_f D$$

with $D = \operatorname{diag}\{\beta, \psi_2(\beta), \cdots, \psi_s(\beta)\}$. Multiplying this equation from both sides by the matrix W^{-1} one obtains

$$W_f^{-1} C_f = D W_f^{-1}.$$

Let us study the following matrix equation for some matrix $V \in \mathbb{C}^{s \times s}$:

$$V C_f = D V$$

Transposing this equation one gets

$$C_f^T V^T = V^T D$$

which is an equation for (right) eigenvectors of the matrix C_f^T. Each eigenvector can be chosen so that its components are numbers from $\mathbb{Q}(\beta)$ or some of its isomorphic copy $\mathbb{Q}(\psi_m(\beta))$. Since the j-th column of the matrix V^T is the image of the first column under the j-th field isomorphism ψ_j, the j-th row of V is the image of the first row of V under ψ_j. The eigenvectors are unique up to a scaling factor so one can fix these factors so that $V W_f = I$ and therefore $V = W_f^{-1}$. □

The above lemma has a simple corollary. Having a rational vector $\boldsymbol{r}$, the components of the vector $W_f^{-1}\boldsymbol{r}$ are images under the Galois automorphisms in the field extension $K/\mathbb{Q}$. Therefore the components must be all non-zero or all equal to 0.

Corollary 8.4. *Let W_f be as above and let $\boldsymbol{r} \in \mathbb{Q}^s$. If any component of the vector $W_f^{-1}\boldsymbol{r}$ vanishes, then $\boldsymbol{r} = \boldsymbol{0}$.*

Lemma 8.5. *Let W be matrix of the form* (8.1) *for β a root of a polynomial f irreducible over $\mathbb{Q}$. Then the entries in the i-th row of W^{-1} are linearly independent over $\mathbb{Q}$ for any i.*

Proof. For a contradiction suppose that there exist $\gamma_1, \ldots, \gamma_s \in \mathbb{Q}$ not all zero such that a linear combination of entries of a given row with γ_i as coefficients vanishes. But then it holds that

$$W^{-1} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{pmatrix} = \boldsymbol{0}$$

because each row is only an image of the first one under an automorphism of splitting field of f (according to Lemma 8.3).

Multiplying this equality by W from the left side one gets

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{pmatrix} = \boldsymbol{0},$$

a contradiction. □

Since C_f is a companion matrix to a polynomial f irreducible over $\mathbb{Q}$, all eigenvalues of C_f have multiplicity 1. Using the eigenvectors $\boldsymbol{y}_j$, one can form two kinds of elementary real invariant subspaces of C_f, namely

$$\begin{aligned} \operatorname{span}_{\mathbb{R}}\{\boldsymbol{y}_j\} &\quad \text{if } \beta_j \in \mathbb{R} \\ \operatorname{span}_{\mathbb{R}}\{\mathtt{Re}\,\boldsymbol{y}_j, \mathtt{Im}\,\boldsymbol{y}_j\} &\quad \text{if } \beta_j \in \mathbb{C}\setminus\mathbb{R}, \end{aligned} \tag{8.2}$$

where the real part $\mathtt{Re}$, and the imaginary part $\mathtt{Im}$ are taken componentwise. It is obvious that every real invariant subspace of C_f is a direct sum of a selection of the above elementary subspaces.

Let us study an arbitrary real invariant subspace of C_f of dimension $1 \le n < s$. Since the subspace is real there must be a basis of it formed by basis vectors of the trivial subspaces (8.2) for some indices j. The set of all indices is denoted by $\mathcal{J}$. Note that vectors $\boldsymbol{y}_j$, $\mathtt{Re}\,\boldsymbol{y}_j$, and $\mathtt{Im}\,\boldsymbol{y}_j$ for $j \in \{1,\dots,s\}\setminus\mathcal{J}$ form a basis of another real invariant subspace of C_f.

For purposes of further proofs denote by Y_1 a matrix formed from the column vectors of basis of the real invariant subspace corresponding to the set of indices $\mathcal{J}$. In the same way denote by Y_2 the matrix corresponding to generators with indices from $\{1,\dots,s\}\setminus\mathcal{J}$.With an abuse of terminology and notation we will not distinguish the matrix Y_i, $i = 1,2$ and the subspace $\mathbb{R}$-spanned by its columns As before denote by $Y := (Y_1,\ Y_2) \in \mathbb{R}^{s\times s}$ and let us define $L := Y^{-1} \in \mathbb{R}^{s\times s}$. Moreover denote first n rows of the matrix L as $L_1 \in \mathbb{R}^{n\times s}$ and the remaining rows as $L_2 \in \mathbb{R}^{(s-n)\times s}$. Thus one can write $L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$. Let us mention that one can obtain the matrix Y from the matrix W_f of (8.1) by multiplying

$$Y = W_f P,$$

with P, a block diagonal matrix $P = \bigoplus_{j=1}^{s} P_j$ with $P_j = 1$ if the corresponding β_j from (8.2) is real, or $P_j = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$ in a complex case.

Having a real matrix L one can define a CPS

$$\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n) \tag{8.3}$$

with the lattice $\mathcal{L}$ corresponding to the matrix L and projections $\pi_{\|}$ and $\pi_{\perp}$ defined as usual.

In what will follow we show that the scheme is generic for an arbitrary choice of subspaces Y_1, Y_2. Thus the scheme is generic for an arbitrary allowed dimension of the resulting $\mathbb{Z}$-module $\pi_{\|}(\mathcal{L})$.

Theorem 8.6. *The scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ defined above is generic.*

The proof of this statement is divided into several statements. The following proposition shows that the scheme is non-degenerate and aperiodic.

Proposition 8.7. *The companion matrix C_f of an algebraic integer does not have any non-trivial rational invariant subspaces.*

Proof. Using the notation mentioned before for a contradiction let us suppose that there exists $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$ such that

$$Y_1 \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \underbrace{\begin{pmatrix} q_1 \\ \vdots \\ q_s \end{pmatrix}}_{\in\mathbb{Q}^s} = \frac{1}{r} \underbrace{\begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix}}_{\in\mathbb{Z}^s} \neq 0.$$

Then

$$Y^{-1}Y_1 \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_s \end{pmatrix} = \frac{1}{r} Y^{-1} \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix}, \quad \text{with } \tilde{\alpha}_j = \begin{cases} \alpha_j & j \in \mathcal{J}, \\ 0 & j \notin \mathcal{J}. \end{cases} \tag{8.4}$$

By Lemma 8.3, the components of the vector on the right hand side of Equation (8.4) are images under the field isomorphisms. Since $\mathcal{J} \neq \{1, \ldots, s\}$ there exists one coordinate, say k, such that $\tilde{\alpha}_k = 0$. Since all components are conjugated, $\tilde{\alpha}_j = 0$ for all j and therefore $\alpha_j = 0$ for all $j \in \mathcal{J}$, which is a contradiction. In the same way aperiodicity can be shown. □

The next claims are formulated with the aim to prove irreducibility of the scheme. They are based on trivial observation that Y_1, Y_2 are invariant subspaces of C_f. Then we use some facts about Pisot numbers in a finite field extensions.

Lemma 8.8. *The above defined lattice $\mathcal{L}$ is closed under the linear mapping given by matrix*

$$M := \begin{pmatrix} M_1 & O \\ O & M_2 \end{pmatrix},$$

where M_1, M_2 have on diagonal either the eigenvalues $\beta_j \in \mathbb{R}$, or the corresponding blocks $\begin{pmatrix} \mathrm{Re}\,\beta_j & \mathrm{Im}\,\beta_j \\ -\mathrm{Im}\,\beta_j & \mathrm{Re}\,\beta_j \end{pmatrix}$ if $\beta_j \notin \mathbb{R}$. In other words, $M\mathcal{L} \subset \mathcal{L}$.

Proof. Recall that matrix Y can be obtained from the Vandermonde matrix (8.1) as

$$Y = W_f P,$$

where P is a block diagonal matrix $P = \bigoplus_{j=1}^{s} P_j$ with $P_j = 1$ if the corresponding β_j from (8.2) is real, or $P_j = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ in a complex case. It holds that

$$C_f W_f = W_f D$$

with D being a diagonal matrix with eigenvalues of C_f on the diagonal. Multiplying this equation from the right by P one gets

$$C_f W_f P = W_f P P^{-1} D P.$$

Define $M := P^{-1}DP$. This matrix has clearly the form required above thus the statement holds because we have

$$C_f Y = Y M$$

which is equivalent to

$$L C_f = M L.$$

□

Since the lattice $\mathcal{L}$ is a $\mathbb{Z}$-module we obtain a direct corollary of this lemma.

Corollary 8.9. *For every $a_1, \cdots, a_{s-1} \in \mathbb{Z}$ it holds that*

$$\left(\sum_{j=1}^{s-1} a_j M^j \right) \mathcal{L} \subset \mathcal{L}.$$

In the following lemma we use the action of C_f on the invariant subspace Y_2, in order to find a regular contracting map Δ on Y_2.

Lemma 8.10. *There exist $c_0, \ldots, c_{s-1} \in \mathbb{Z}$ such that $\Delta = \sum_{k=0}^{s-1} c_k M_2^k \in \mathbb{R}^{(s-n)\times(s-n)}$ is non-singular and is of spectral radius in modulus smaller than 1.*

Proof. Let us start with a root β of the polynomial f defined earlier. Take such a β that is not an eigenvalue of M_2. If β is real, then according to Salem [20], in the extension field $\mathbb{Q}(\beta)$ (of degree s over $\mathbb{Q}$) there exists a Pisot number δ of degree s. On the other hand if β is complex, there is a complex Pisot number of full degree which was proved by [24]. For simplicity, consider the real case, the complex one being very similar.

The number δ is an algebraic integer thus it is contained in $\mathcal{O}_{\mathbb{Q}(\beta)}$, i.e. in the ring of all algebraic integers in $\mathbb{Q}(\beta)$. This is a $\mathbb{Z}$-module of rank s. It is obvious that the set $\mathbb{Z}[\beta] = \left\{ \sum_{j=1}^{s-1} c_j \beta^j : c_j \in \mathbb{Z} \right\}$ is a $\mathbb{Z}$-module of rank s as well. Moreover it is a known result from number theory that $\mathbb{Z}[\beta] \subset \mathcal{O}_{\mathbb{Q}(\beta)}$ and one can consider $\mathbb{Z}[\beta]$ as a submodule of $\mathcal{O}_{\mathbb{Q}(\beta)}$.

On the ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ one can define a relation $\sim$ using the set $\mathbb{Z}[\beta]$ as follows:

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}[\beta].$$

It is obvious that this relation is reflexive, symmetric and transitive, so it is an equivalence relation. Since the quotient set of $\mathcal{O}_{\mathbb{Q}(\beta)}$ by $\sim$ is a finite set, each $x \in \mathcal{O}_{\mathbb{Q}(\beta)}$ can be decomposed in two summands

$$x = w + z, \text{ where } \; z \in \mathbb{Z}[\beta], \; w \in F$$

where F is finite.

For every $n \in \mathbb{N}$ we have

$$\delta^n = w_n + z_n$$

for some $w_n \in F, z_n \in \mathbb{Z}[\beta]$. Because of the finiteness of the set F there exist $n_1, n_2 \in \mathbb{N}$ such that $w_{n_1} = w_{n_2}$. These can be chosen so that

$$\delta^{n_1} - \delta^{n_2} = z_{n_1} - z_{n_2} > 1,$$

$$|\psi(\delta^{n_1} - \delta^{n_2})| < 1$$

for any non-identical embedding $\psi : \mathbb{Q}(\beta) \to \mathbb{C}$, i.e. an isomorphism $\mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j)$ for some $j \in \{2, \dots, s\}$, where $\beta = \beta_1, \beta_2, \dots, \beta_s$ are the roots of the polynomial f.

Denote by $\nu = \delta^{n_1} - \delta^{n_2}$. This number is a Pisot number from construction. We will show that the degree of ν is s. From Definition 2.9 it follows that $\psi_j(\nu)$ are roots of the field polynomial of ν in $\mathbb{Q}(\beta)$. Since this polynomial is a power of the minimal polynomial of ν, it holds that the degree of ν is s. For a contradiction suppose that the field polynomial is the r-th power of the minimal polynomial. Therefore there are r different s/r-tuples of the same roots among the field polynomial roots $\nu, \psi_j(\nu)$. But from construction it follows that there is only one root of modulus greater then 1, hence a contradiction. Thus ν can be written as $\nu = \sum_{i=0}^{s-1} c_i \beta^i$ for some $c_i \in \mathbb{Z}$.

The matrix $\left(\sum_{k=0}^{s-1} c_k M^k \right)$ has for eigenvalues the Pisot number ν and its algebraic conjugates, and thus it is non-singular. Therefore the matrix $\Delta \in \mathbb{R}^{(s-n)\times(s-n)}$ defined as

$$\Delta := (O, I_{s-n}) \left(\sum_{k=0}^{s-1} c_k M^k \right) \begin{pmatrix} O \\ I_{s-n} \end{pmatrix} = \sum_{k=0}^{s-1} c_k M_2^k$$

is also non-singular. Moreover, among its eigenvalues there are only the conjugates of ν different from ν. Thus the spectral radius of Δ is smaller than 1 what was to be shown.

□

Proof of Theorem 8.6. According to Propositions 8.7 and 7.2, the constructed CPS (8.3) is non-degenerate and aperiodic. In order to prove irreducibility, we use the criterion (c) in Proposition 7.2.

It is clear that since the real span of $\pi_\perp(\mathcal{L}) \subset \mathbb{R}^{s-n}$ is $\mathbb{R}^{s-n}$ the $\mathbb{Z}$-module must contain $s - n$ vectors linearly independent over $\mathbb{R}$. As Δ from Lemma 8.10 is a regular transformation, the image of the family of $s - n$ linearly independent vectors by Δ^k is again a family of linearly independent vectors, for any $k \in \mathbb{N}$. Since the spectral radius of Δ is strictly smaller than 1, Δ^k tends to the zero matrix with $k \to \infty$. Thus it is sufficient to take k large enough to obtain the condition of item (c) in Proposition 7.2 and hence the scheme is irreducible which completes the proof. □

Note, that we have proven a stronger statement, namely

Remark 8.11. *Consider the matrix $\widetilde{L} \in \mathbb{R}^{(s-n)\times s}$ obtained from Y^{-1} by erasing $n \geq 1$ rows. Then the set $\left\{ \widetilde{L}\vec{x} : \vec{x} \in \mathbb{Z}^s \right\}$ is dense in $\mathbb{R}^{s-n}$.*

We will use this fact later.

8.2 Construction of CPS for trivial self-similarities

In this section we present construction of a cut and project scheme with self-similarities A which have in some sense trivial spectrum. Recall the self-similarity is given by a matrix $A \in \mathbb{R}^{n\times n}$ whose spectrum is formed by algebraic integers. The situation is trivial, if eigenvalues of A are rational integers, or non-real quadratic numbers. In fact, for such A, we can find a discrete structure in $\mathbb{R}^n$ (namely a lattice), with this self-similarity. If, for some reason, we need a generic cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$, $s > n$, such that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$, the construction goes as follows.

If $\sigma(A) \subset \mathbb{Z}$, it is obvious that any generic scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ satisfies the required properties. For, any $\mathbb{Z}$-module is closed under multiplication by integers.

Suppose now that the spectrum $\sigma(A)$ contains only complex conjugated pairs of the same non-real quadratic numbers $\lambda, \bar{\lambda}$.

Proposition 8.12. *Let $A \in \mathbb{R}^{n\times n}$ be a diagonalizable mapping such that its eigenvalues are roots of the same quadratic polynomial $\lambda^2 - p\lambda - q$, $p, q \in \mathbb{Z}$ with negative discriminant.Then there exists a generic CPS $(\mathcal{L} \subset \mathbb{R}^{n+2}, \mathbb{R}^n)$ such that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.*

Proof. It is obvious that n is even, i.e. $n = 2m$ for some m. The mapping A can be assumed to be in the form

$$A = I_m \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & -\mathtt{Im}\,\lambda \\ \mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix}.$$

Let us define the lattice $\mathcal{L}$ through the following matrix

$$L = H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}$$

for some matrix $H \in \mathbb{R}^{(m+1)\times(m+1)}$. The conditions on this matrix will be imposed later. At the same time let us define

$$C = I_{m+1} \otimes \begin{pmatrix} 0 & q \\ 1 & p \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mathtt{Re}\,\lambda & -\mathtt{Im}\,\lambda \\ \mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix}.$$

We verify that equation (7.3) holds:

$$\begin{aligned} LC &= \begin{pmatrix} A & O \\ O & B \end{pmatrix} L \\ \left(H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}\right)\left(I_{m+1} \otimes \begin{pmatrix} 0 & q \\ 1 & p \end{pmatrix}\right) &= \left(I_{m+1} \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & -\mathtt{Im}\,\lambda \\ \mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix}\right)\left(H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}\right), \\ H \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & q + p\mathtt{Re}\,\lambda \\ \mathtt{Im}\,\lambda & p\mathtt{Im}\,\lambda \end{pmatrix} &= H \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & \mathtt{Re}^2\lambda - \mathtt{Im}^2\lambda \\ \mathtt{Im}\,\lambda & 2\mathtt{Im}\,\lambda\mathtt{Re}\,\lambda \end{pmatrix}, \end{aligned}$$

which is true, because of $\mathtt{Re}^2\lambda - \mathtt{Im}^2\lambda = q + p\,\mathtt{Re}\,\lambda$ and $2\mathtt{Im}\,\lambda\mathtt{Re}\,\lambda = p\,\mathtt{Im}\,\lambda$. This proves by Proposition 7.5 that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.

The projection in the inner space $\mathbb{R}^2$ can be expressed as

$$\pi_{\perp}(\mathcal{L}) = \sum_{i=1}^{m+1} a_i \begin{pmatrix} H_{m+1\,i} \\ 0 \end{pmatrix} + \sum_{i=1}^{m+1} b_i \begin{pmatrix} H_{m+1\,i}\mathtt{Re}\,\lambda \\ H_{m+1\,i}\mathtt{Im}\,\lambda \end{pmatrix}, \ a_i, b_i \in \mathbb{Z}.$$

The second row of the projection can be expressed a set $\mathcal{H} = \{b_i H_{m+1\,i}\mathtt{Im}\,\lambda : b_i \in \mathbb{Z}\}$. Let us choose the set $\{H_{m+1\,i} : i = 1, 2, \ldots, m+1\}$ to be linearly independent over $\mathbb{Q}$. This assures that the set $\mathcal{H}$ is dense in $\mathbb{R}$. The first row of the projection $\pi_{\perp}(\mathcal{L})$ be then written as $\frac{1}{\mathtt{Im}\,\lambda}\mathcal{H} \oplus \frac{\mathtt{Re}\,\lambda}{\mathtt{Im}\,\lambda}\mathcal{H}$. This set is clearly dense in $\mathbb{R}$ as well. Finally we can conclude that $\pi_{\perp}(\mathcal{L})$ is dense in $\mathbb{R}^2$ because for each point of $\mathbb{R}^2$ we can find a point of projection that will be sufficiently close.

The non-degeneracy of the scheme can be assured by choosing $\{H_{m\,i} : i = 1, 2, \ldots, m+1\}$ to be linearly independent over $\mathbb{Q}$. Indeed, suppose that

$$\mathbf{0} = \sum_{i=1}^{m+1} a_i \begin{pmatrix} H_{1\,i} \\ 0 \\ \vdots \\ H_{m\,i} \\ 0 \end{pmatrix} + \sum_{i=1}^{m+1} b_i \begin{pmatrix} H_{1\,i}\texttt{Re}\,\lambda \\ H_{1\,i}\texttt{Im}\,\lambda \\ \vdots \\ H_{m\,i}\texttt{Re}\,\lambda \\ H_{m\,i}\texttt{Im}\,\lambda \end{pmatrix}, \ a_i, b_i \in \mathbb{Z}.$$

But then it must holds that

$$\sum_{i=1}^{m+1} b_i H_{m\,i}\texttt{Im}\,\lambda = 0.$$

From the requirement on linear independence of $H_{m\,i}$ one gets $b_i = 0$ for $i = 1, \ldots, m+1$. Similarly it can be checked that $a_i = 0$ for $i = 1, \ldots, m+1$. This completes the proof of genericity of the constructed scheme. □

8.3 CPS with a diagonalizable self-similarity A

This section gives the answer to Question 2 in case of a diagonalizable mapping A. The construction is based on the spectrum $\sigma(A)$. We divide this spectrum into conjugacy classes, i.e. subsets of $\sigma(A)$ containing only algebraic conjugates related to one certain minimal polynomial. For each such class we derive the corresponding CPS using the elementary schemes introduced in Section 8.1. Finally we compose them into a CPS having A as a self-similarity.

Let us briefly explain what will follow. According to Theorem 7.16 we can consider a mapping A with eigenvalues being roots of an irreducible polynomial, say f. Propositions 7.5 and 7.8 say that the matrix B must contain at least the remaining roots of f and matrix C is at least a block diagonal matrix with companion matrices C_f on its diagonal. We show that this is satisfied and we demonstrate a way how to break the real eigenspaces corresponding to C_f in order to obtain a generic CPS. This involves a combinatorial consideration that will be given as a lemma. Let us illustrate this approach on an example.

Example 2. *Let us study the matrix*

$$A = \begin{pmatrix} \delta_S & 0 & 0 \\ 0 & \delta_S & 0 \\ 0 & 0 & \delta'_S \end{pmatrix},$$

where $\delta_S = 1 + \sqrt{2}$ is the so-called silver mean and $\delta'_S = 1 - \sqrt{2}$ its algebraic conjugate. The minimal polynomial of δ_S, δ'_S is $f(x) = x^2 - 2x - 1$. Its companion matrix is

$$C_f = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \text{ with eigenvactors } \begin{pmatrix} 1 \\ \delta_S \end{pmatrix}, \begin{pmatrix} 1 \\ \delta_S \end{pmatrix}$$

corresponding to the eigenvalues δ_S, δ'_S, respectively.

We will construct a cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^3)$ with self-similarity $A\pi_{\parallel}(\mathcal{L}) \subset \pi_{\parallel}(\mathcal{L})$ as a direct sum of several elementary schemes. Setting

$$C = \begin{pmatrix} C_f & O & O \\ O & C_f & O \\ O & O & C_f \end{pmatrix} \in \mathbb{R}^{6\times 6}, \qquad B = \begin{pmatrix} \delta'_S & 0 & 0 \\ 0 & \delta'_S & 0 \\ 0 & 0 & \delta_S \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \delta_S & \delta'_S & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \delta_S & \delta'_S & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \delta_S & \delta'_S \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{P} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \delta_S & 0 & 0 & \delta'_S & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & \delta_S & 0 & 0 & \delta'_S & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \delta'_S & 0 & 0 & \delta_S \end{pmatrix},$$

we have

$$L = Y^{-1} = \frac{1}{4} \begin{pmatrix} \delta'_S + 1 & \delta_S - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta'_S + 1 & \delta_S - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta'_S + 1 & \delta_S - 1 \\ \delta_S + 1 & \delta'_S - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_S + 1 & \delta'_S - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_S + 1 & \delta'_S - 1 \end{pmatrix}.$$

Then $CY = Y \left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix} \right)$*. Taking columns of the matrix* $L = Y^{-1}$ *as generators of the lattice* $\mathcal{L}$*, we have created a cut and project scheme* $(\mathcal{L} \subset \mathbb{R}^6, \mathbb{R}^3)$*, which, by Proposition 7.5, has self-similarity* A*.*

The ordering of columns in the matrix Y *was chosen so that nor the physical space* $\operatorname{span}_{\mathbb{R}}\{Y_{\bullet 1}, Y_{\bullet 2}, Y_{\bullet 3}\}$*, nor the inner space* $\operatorname{span}_{\mathbb{R}}\{Y_{\bullet 4}, Y_{\bullet 5}, Y_{\bullet 6}\}$ *has a rational subspace so the resulting CPS is generic.*

To ensure that the eigenvalues of each copy of the companion matrix were distributed to the spectrum of both A *and* B*. We have three companion matrices* C_f*. Write their eigenvalues in columns as*

$$\begin{matrix} \textcircled{δ_S} & \textcircled{δ_S} & \delta_S \\ \delta'_S & \delta'_S & \textcircled{δ'_S} \end{matrix}$$

In each column we must have an eigenvalue for the spectrum of A *and an eigenvalue for the spectrum of* B*. The eigenvalues of* A *are marked with a circle. In fact, the problem can be formulated as a searching for a matrix with two rows and* 3 *columns having only elements "0" or "1", where "1" stands instead of the eigenvalue in a circle. The matrix must have twice the value* 1 *at the first row, once "1" at the second row, and in each column at least one "1" and at least one "0". In our example,*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One could ask why only two copies of the companion matrix are not sufficient for such a construction. The reason is that no 2×2 *matrix with the above properties exists. This can be rephrased by saying that no ordering of the eigenvectors of the matrix* $C = I_2 \otimes C_f \in \mathbb{R}^{4 \times 4}$ *would ensure absence of a rational subspace in the physical space.*

The consideration made in the example above can be generalized and one can pose a question whether there exists a matrix satisfying certain conditions. The following lemma states such requirements and demonstrates the necessary and sufficient condition on the dimension of the considered matrix.

Lemma 8.13. *Let* $K, N \in \mathbb{N}$ *and let* $l_1, \ldots, l_N$ *be a non-negative integers. Then there exists a matrix* $\mathcal{R} \in \{0,1\}^{N \times K}$ *satisfying conditions*

(i) $\displaystyle\sum_{j=1}^{K} \mathcal{R}_{ij} = l_i$ *for all* $i \in \{1, \ldots, N\}$*.*

(ii) $N-1 \geq \sum_{i=1}^{M} \mathcal{R}_{ij} \geq 1$ *for all* $j \in \{1,\ldots,K\}$

if and only if $N \geq 2$, $K \geq \max\{l_1,\ldots,l_N\}$ *and the following inequality holds:*

$$\frac{1}{N-1}\sum_{i=1}^{N} l_i \leq K \leq \sum_{i=1}^{N} l_i. \tag{8.5}$$

Proof. First, the condition (ii) cannot be fulfilled for $N=1$. This implies $N \geq 2$. To satisfy the condition (i) it is necessary for K to be greater or equal to the maximal value among l_i. Let us sum the condition (ii) over all columns, i.e. over all $j \in \{1,\cdots,K\}$.

$$\sum_{i=1}^{K}(N-1) \geq \sum_{i=1}^{K}\sum_{j=1}^{N}\mathcal{R}_{ij} \geq \sum_{i=1}^{K} 1$$

$$K(N-1) \geq \sum_{j=1}^{N}\sum_{i=1}^{K}\mathcal{R}_{ij} = \sum_{j=1}^{N} l_j \geq K$$

We used the condition (i) and this inequality can be rewritten as

$$\frac{1}{N-1}\sum_{i=1}^{N} l_i \leq K \leq \sum_{i=1}^{N} l_i$$

which shows that conditions (i) and (ii) are necessary.

Now we constructively demonstrates that condition (8.5) is sufficient to obtain a matrix meeting conditions (i) and (ii). First let $l_1 + l_2 \leq K$. Let us define matrix $\mathcal{R}$ as follows. Put into the first row l_1 "1" on positions $1,\ldots,l_1$ and "0" elsewhere. The second row le be chosen so that "1" are on positions $l_1+1,\ldots,l_2$ and "0" on the remaining positions. This choice ensures that (i) is satisfied by these two rows. The other rows can be chosen independently and the condition $\sum_{i=1}^{N}\mathcal{R}_{ij} \leq N-1$.

If $N=2$, then according to assumption it holds that

$$\sum_{i=1}^{2} l_i \leq K \leq \sum_{i=1}^{2}.$$

Thus it holds that $K = l_1 + l_2$ and the matrix $\mathcal{R}$ is of the form

$$\mathcal{R} = \begin{pmatrix} \overbrace{1\,1\,\cdots\,1}^{l_1} & 0\,0\,\cdots\,0 \\ 0\,0\,\cdots\,0 & \underbrace{1\,1\,\cdots\,1}_{l_2} \end{pmatrix}.$$

This matrix also satisfies the condition $\sum_{i=1}^{2}\mathcal{R}_{ij} \geq 1$.

If $N \geq 3$ then from Condition (8.5) it follows that there exists $i_0 \geq 2$ such that

$$\sum_{i=1}^{i_0} l_i \leq K < \sum_{i=i_0+1}^{N}.$$

For $i = 3,\cdots,\ i_0$ we define

$$\mathcal{R}_{ij} = \begin{cases} 1 & \text{for} \qquad j = \sum_{k=1}^{i-1} l_k + 1,\cdots,\sum_{k=1}^{i} l_k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{R}_{i_0+1\ j} = \begin{cases} 1 & \text{for } j = \sum_{k=1}^{i_0} l_k + 1, \cdots, K, \\ 0 & \text{otherwise.} \end{cases}$$

The remaining rows may be defined arbitrary just satisfying the condition (i). Then the condition (ii), i.e $\sum_{i=1}^{N} \mathcal{R}_{ij} \geq 1$, holds for every column. The matrix $\mathcal{R}$ is then of the form

$$\mathcal{R} = \begin{pmatrix} \underbrace{1 \ \ldots \ 1}_{l_1} & & & & & \\ & \underbrace{1 \ \ldots \ 1}_{l_2} & & & & \\ & & \underbrace{1 \ \ldots \ 1}_{l_3} & & & \\ & & & \cdots & \overbrace{1 \ \ldots \ 1}^{l_{i_0}} & \\ & & & & & 1 \ \ldots \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}.$$

Now let us suppose that $l_1 + l_2 > K$. This is equivalent to condition

$$(K - l_1) + (K - l_2) < K.$$

But this is equivalent to the task being solved in the previous demonstration with "0" instead of "1" and vice versa. To finish the proof it is sufficient to show that it holds

$$K \leq \sum_{i=1}^{N} (K - l_i).$$

But this can be rewrite as

$$K \leq KN - \sum_{i=1}^{N} l_i,$$

$$\frac{1}{N-1} \sum_{i=1}^{N} l_i \leq K,$$

which is our assumption (8.5). □

Finally we can give the proof of the fact that for each diagonalizable mapping A there exists a generic CPS with self-similarity A.

Theorem 8.14. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix diagonalizable over $\mathbb{C}$, $\sigma(A) \subset \mathbb{B}$. Then there exists a generic cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.*

Proof. The proof is constructive. We demonstrate that a suitable CPS can be obtained as a direct sum of a sufficient number of elementary schemes. As a consequence of Theorem 7.16, it suffices to assume that all eigenvalues of the matrix A are roots of the same minimal polynomial f, let us say of degree d. From Proposition 7.5 and Corollary 7.4, it is clear that the dimension s of the lattice must be divisible by d. The corresponding integer matrix C will be a tensor product of an identity matrix of order, say K, and the companion matrix C_f of the polynomial f. Thus its dimension is $s = Kd$. This can be seen for example using the fact that since $C \in \mathbb{Z}^{s \times s}$ it implies that $\chi_C \in \mathbb{Z}[X]$ and $\chi_C = \chi_A \chi_B$. We are interested in the minimal s and since all eigenvalues are conjugate, χ_C is necessary a power of the minimal polynomial of these numbers, $\chi_C = f^K$.

Since all eigenvalues of A are eigenvalues of C, the number K of such companion matrices is at least equal to the maximum M of multiplicities of the eigenvalues of A. Formally one can suppose that the polynomial f has r real and t pairs of complex conjugate roots, i.e.

$$f(x) = \prod_{j=1}^{r}(x-\lambda_j)\prod_{k=1}^{t}(x-\mu_k)(x-\overline{\mu_k})\,, \quad r+t \geq 2\,.$$

Define $M := \max_{j,k}\{l_j, m_k\}$, where l_j, m_k are multiplicities of λ_j, μ_k in the spectrum of A. Set $u := r+t$. Since the trivial cases have been already solved in Section 8.2, we can without loss of generality suppose that the eigenvalues of A are neither integers, nor non-real quadratic numbers, i.e. $u \geq 2$.

In order to obtain a generic cut and project scheme by such a simple construction, it may not be sufficient to use M companion matrices C_f. For, we have to ensure validity of conditions of Proposition 7.2. Both the physical and inner spaces are real invariant subspaces of the matrix C. But neither of them has a rational subspace. Each set of eigenvectors generating a rational invariant subspace corresponding to the decomposition $C = \bigoplus_{j\in\{1,\ldots,K\}} C_f$ must be broken so that certain eigenvectors belong to the physical and the other to the inner spaces. Consequently, in other words, each list of algebraic conjugated eigenvalues must be divided between the spectrum of A and B.

This algebraic situation can be viewed as a combinatorial task solved in Lemma 8.13. Each matrix C_f in C can be considered as a family of K identical lists, whose items are

$$\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{r+t},$$

where

$$\alpha_j = \begin{cases} \lambda_j, & \text{for } j \in \{1,2,\cdots,r\}, \\ \{\mu_{j-r}, \overline{\mu_{j-r}}\}, & \text{for } j \in \{r+1,\ldots,r+t\}. \end{cases}$$

For simplicity, denote $l_j = m_{j-r}$ for $j \in \{r+1,\ldots,r+t\}$.

Write the lists into K columns, our task is to find K such that the following has a solution: Put in circle certain items in the lists so that

(i) the circled items form the spectrum of the matrix A,

(ii) each list contains both circled and non-circled items.

Conditions (i) and (ii) are clearly a reformulation of the demand on the absence of rational subspaces. If an circled item is marked by "1" and non-circled item by "0", in such a way we obtain a matrix $\mathcal{R}$ which we call a good coloring where we set $N = r+t$. Existence of such a good coloring is solved in Lemma 8.13. It suffices to verify that the inequalities (8.5) have a solution $K \geq M = \max\{l_i : i = 1,\ldots,r+t\}$. Such K is for example $K := \max\left\{M, \left\lceil \frac{\sum_{j=1}^{N} l_j}{N-1} \right\rceil\right\}$. □

8.4 Minimality of dimension

The construction of a CPS with a given diagonalizable self-similarity A described in the previous section was a simple one. The main advantage was that the resulting scheme is obtained as a direct sum of elementary ones, and the corresponding lattice has generators with components in the algebraic extension field given by eigenvalues of A. However, the dimension of the constructed scheme is not a minimal one. In this section we determine a better bound on the minimal dimension and show that it is actually achieved. On the other hand we will lose the property that the components of the lattice generators are in a certain algebraic field extension determined by eigenvalues of A.

Theorem 8.15. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix diagonalizable over $\mathbb{C}$ such that $\sigma(A) \subset \mathbb{B}$, and all eigenvalues $\lambda \in \sigma(A)$ have the same minimal polynomial f of degree d. Denote $l_1,\ldots,l_d$ the multiplicities of the* *roots $\beta_1, \dots \beta_d$ of f in the spectrum $\sigma(A)$, and set $M := \max_j\{l_j\}$, $m := \min_j\{l_j\}$. Then there exists $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$, a generic cut and project scheme with the dimension s satisfying*

$$s = \begin{cases} Md & \text{if } m < M, \\ (M+1)d, & \text{otherwise.} \end{cases}$$

The s is minimal possible.

Before presenting the proof, we demonstrate the idea on the self-similarity A taken from Example 2.

Example 3. *Let A, B, C_f, C, Y be as in Example 2. Note that for the matrix Y can be written as*

$$Y = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \delta_S & \delta'_S & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \delta_S & \delta'_S & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \delta_S & \delta'_S \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{P} = \left(I_3 \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) P,$$

where $\otimes$ stands for the Kronecker product which is together eith its properties described in Appendix 4. Using the same permutation matrix P (and the fact that $P^{-1} = P^T$ were P^T stands for a transposed matrix), we can transform

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} = P^T \begin{pmatrix} \delta_S & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta'_S & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_S & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta'_S & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_S & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta'_S \end{pmatrix} P = P^T \left(I_3 \otimes \begin{pmatrix} \delta_S & 0 \\ 0 & \delta'_S \end{pmatrix}\right) P$$

Obviously it also holds that $C = I_3 \otimes C_f$. Substituting these into $CY = Y \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, we obtain

$$(I_3 \otimes C_f) \left(I_3 \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) P = \left(I_3 \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) P P^T \left(I_3 \otimes \begin{pmatrix} \delta_S & 0 \\ 0 & \delta'_S \end{pmatrix}\right) P.$$

This results in

$$(I_3 \otimes C_f) \left(I_3 \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) = \left(I_3 \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) \left(I_3 \otimes \begin{pmatrix} \delta_S & 0 \\ 0 & \delta'_S \end{pmatrix}\right),$$

which can be reduced into the obvious

$$C_f \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix} \begin{pmatrix} \delta_S & 0 \\ 0 & \delta'_S \end{pmatrix}. \tag{8.6}$$

The composition of the elementary CPS into their direct sum can be, as it was shown, formulated using Kronecker product with the identity matrix. One can ask whether the Kronecker product with another regular matrix, say H, could not produce better results. In particular with respect to the number of elementary schemes needed. Indeed, it turns out that in many cases, a suitable choice of the matrix H does reduce the dimension of the resulting cut and project scheme.

For the matrix A from Example 2, we will try to take only two elementary schemes, but instead of the identity matrix I_2, we put a general regular matrix $H \in \mathbb{R}^{2\times 2}$, thus we obtain

$$(I_2 \otimes C_f) \left(H \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) = \left(H \otimes \begin{pmatrix} 1 & 1 \\ \delta_S & \delta'_S \end{pmatrix}\right) \left(I_2 \otimes \begin{pmatrix} \delta_S & 0 \\ 0 & \delta'_S \end{pmatrix}\right).$$

For $H = I_2$, as it was explained in Example 2 the constructed scheme is not generic. Taking a different H may produce a generic scheme.

The following three lemmas will be used in the proof of Theorem 8.15 for verifying that the constructed scheme is generic.

Lemma 8.16. *Let $H \in \mathbb{C}^{K\times K}$ be a matrix of the following form*

$$H = \begin{pmatrix} 1 & -t_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -t_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -t_{K-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{8.7}$$

Then

$$H^{-1} = \begin{pmatrix} 1 & t_1 & t_1t_2 & \dots & t_1t_2\cdots t_{K-2} & t_1t_2\cdots t_{K-1} \\ 0 & 1 & t_2 & \dots & t_2t_3\cdots t_{K-2} & t_2t_3\cdots t_{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & t_{K-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The proof of this lemma is straightforward and we skip it.

Lemma 8.17. *Let $W_f \in \mathbb{C}^{d\times d}$ be a Vandermonde matrix of the form (8.1) with columns $\vec{w}_j = (1, \beta_j, \dots, \beta_j^{d-1})^T$ for $j = 1, \dots, d$. Let*

$$H = \begin{pmatrix} 1 & -t_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -t_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -t_{K-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Define $\mathcal{W} = (H \otimes W_f)\binom{O}{I_{Kd-1}}$. Then for any $\boldsymbol{x} \in \mathbb{C}^{Kd-1}$ we have $\mathcal{W}\boldsymbol{x} \notin \mathbb{Q}^{Kd} \setminus \{\mathbf{0}\}$.

Proof. Assume that there is a non-zero vector $\boldsymbol{r} \in \mathbb{Q}^{Kd}$ such that $\mathcal{W}_f\boldsymbol{x} = (H \otimes W)\binom{O}{I_{Kd-1}}\boldsymbol{x} = \boldsymbol{r}$ for some $\boldsymbol{x} \in \mathbb{C}^{Kd-1}$. Then we can rewrite the projection of $\boldsymbol{x}$ as follows:

$$\begin{pmatrix} O \\ I_{Kd-1} \end{pmatrix}\boldsymbol{x} = (H^{-1} \otimes W_f^{-1})(H \otimes W_f)\begin{pmatrix} O \\ I_{Kd-1} \end{pmatrix}\boldsymbol{x} = (H^{-1} \otimes W_f^{-1})\boldsymbol{r}\,. \tag{8.8}$$

We will only use first d rows of the above equation. For simplicity, denote

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{Kd} \end{pmatrix}, \quad \boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_K \end{pmatrix}, \quad \text{for } \boldsymbol{r}_j \in \mathbb{Q}^d\,.$$

According to Lemma 8.16 the first row of the matrix H^{-1} is $(H^{-1})_{1\boldsymbol{\cdot}} = (1, t_1, t_1t_2, \dots, t_1t_2\cdots t_{K-1})$. One derives from (8.8) that for the first d entries of $\binom{O}{I_{Kd-1}}\boldsymbol{x}$ holds

$$\begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = (I_d\ O)\begin{pmatrix} W_f^{-1} & t_1W_f^{-1} & t_1t_2W_f^{-1} & \dots & t_1\cdots t_{K-2}W_f^{-1} & t_1\cdots t_{K-1}W_f^{-1} \\ O & W_f^{-1} & t_2W_f^{-1} & \dots & t_2\cdots t_{K-2}W_f^{-1} & t_2\cdots t_{K-1}W_f^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & W_f^{-1} & t_{K-1}W_f^{-1} \\ O & O & O & \dots & O & W_f^{-1} \end{pmatrix}\begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_K \end{pmatrix} =$$

$$= W^{-1}\boldsymbol{r}_1 + t_1W^{-1}\boldsymbol{r}_2 + t_1t_2W^{-1}\boldsymbol{r}_3 + \dots + t_1t_2\cdots t_{K-1}W^{-1}\boldsymbol{r}_K\,. \tag{8.9}$$

For $j = 1, \dots, K$, the components of vectors $W^{-1}\boldsymbol{r}_j$ belong to the extension field $T = \mathbb{Q}(\beta_1, \dots, \beta_d)$. It can be easily seen that one can chose $t_1, \dots, t_{K-1}$ such that the coefficients $1, t_1, t_1t_2, \dots, t_1 \cdots t_{K-1}$ are linearly independent over T. Inspecting the first row of (8.9), we derive that the first components of the vectors $W^{-1}\boldsymbol{r}_j$ vanish. But by Corollary 8.4 we conclude that $\boldsymbol{r}_1 = \dots = \boldsymbol{r}_K = \boldsymbol{0}$, which is a contradiction. □

Lemma 8.18. *Let $U \in \mathbb{R}^{u\times t}$, $V \in \mathbb{R}^{u\times v}$, $Z \in \mathbb{R}^{w\times v}$ be matrices such that $\{U\boldsymbol{x} : \boldsymbol{x} \in \mathbb{Z}^t\}$ is dense in $\mathbb{R}^u$ and $\{Z\boldsymbol{y} : \boldsymbol{y} \in \mathbb{Z}^v\}$ is dense in $\mathbb{R}^w$. Then the set*

$$\left\{\begin{pmatrix} U & V \\ O & W\end{pmatrix}\boldsymbol{z} : \boldsymbol{z} \in \mathbb{Z}^{t+v}\right\} \tag{8.10}$$

is dense in $\mathbb{R}^{u+w}$.

Proof. In order to prove the claim, realize first that if $\{U\boldsymbol{x} : \boldsymbol{x} \in \mathbb{Z}^t\}$ is dense in $\mathbb{R}^u$, then also

$$\left\{U\boldsymbol{x} : \boldsymbol{x} \in \mathbb{Z}^t\right\} + \{V\boldsymbol{y} : \boldsymbol{y} \in \mathbb{Z}^v\}$$

is dense in $\mathbb{R}^u$. Since the set (8.10) is a cartesian product of $\{U\boldsymbol{x} : \boldsymbol{x} \in \mathbb{Z}^t\} + \{V\boldsymbol{y} : \boldsymbol{y} \in \mathbb{Z}^v\} \subset \mathbb{R}^u$ and $\{W\boldsymbol{y} : \boldsymbol{y} \in \mathbb{Z}^v\} \subset \mathbb{R}^w$, it is obviously a dense set in $\mathbb{R}^{u+w}$. □

Proof of Theorem 8.15. Suppose first that $m < M$. The fact that $s = Kd \geq Md$, where M is the maximum of multiplicities of eigenvalues of A, is obvious, since s is the dimension of the integer matrix C. We will show that the bound is reached by constructing a generic scheme of dimension Md. Denote C_f the companion matrix of the polynomial f. Set $J_f^{\mathbb{R}}$ to be the real Jordan form of C_f. Recall that the real Jordan form is the block diagonal matrix that has on the diagonal the real roots β_j of f or the blocks $\left(\begin{smallmatrix} a_k & b_k \\ -b_k & a_k\end{smallmatrix}\right)$ for a pair of complex conjugated eigenvalues $\beta_k = a + bi$, $\overline{\beta}_k = a - bi$.

Further denote Y_f the matrix whose columns are vectors

$$\boldsymbol{y}_j = \begin{pmatrix} 1 \\ \beta_j \\ \vdots \\ \beta_j^{d-1}\end{pmatrix} \text{ if } \beta_j \in \mathbb{R}, \text{ or } \boldsymbol{y}_j = \begin{pmatrix} 1 \\ \mathtt{Re}\,\beta_j \\ \vdots \\ \mathtt{Re}\,\beta_j^{d-1}\end{pmatrix}, \ \boldsymbol{y}_j = \begin{pmatrix} 1 \\ \mathtt{Im}\,\beta_j \\ \vdots \\ \mathtt{Im}\,\beta_j^{d-1}\end{pmatrix} \text{ in cases that } \beta_j \in \mathbb{C}\setminus\mathbb{R}.$$

We have the equality $C_fY_f = Y_fJ_f^{\mathbb{R}}$.

Let $K \in \mathbb{N}$ be the number of elementary schemes to be composed. Then for any matrix $H \in \mathbb{R}^{K\times K}$, we obtain using the equality above

$$(I_K \otimes C_f)(H \otimes Y_f) = (I_KH) \otimes (C_fY_f) = (H \otimes Y_f)(I_K \otimes J_f^{\mathbb{R}})$$

and consequently

$$(I_K \otimes C_f)(H \otimes Y_f)P = (H \otimes Y_f)PP^T(I_K \otimes J_f^{\mathbb{R}})P = (H \otimes Y_f)P\begin{pmatrix} A & O \\ O & B\end{pmatrix}$$

for any $Kd \times Kd$ permutation matrix P such that

$$P^T(I_K \otimes J_f^{\mathbb{R}})P = \begin{pmatrix} A & O \\ O & B\end{pmatrix}.$$

Setting $Y = (H \otimes Y_f)P$, we can define the lattice $\mathcal{L}$ as the $\mathbb{Z}$-span of the columns of the corresponding matrix $L = Y^{-1} = P^{-1}(H \otimes Y_f)^{-1} = P^T\left(H^{-1} \otimes Y_f^{-1}\right)$. Keep in mind that the permutation matrix P^T only reorganizes the rows of L, i.e. it determines the projections $\pi_\parallel, \pi_\perp$. This amounts to determining which columns of the matrix $H \otimes Y_f$ belong to the physical and which to the inner spaces. In other words, it determines which columns of $H \otimes Y_f$ form the matrix Y_1 and which columns form Y_2.

The requirement of non-degeneracy, resp. aperiodicity is translated to the choice of P such that no rational vector can be obtained by real linear combination from the columns of Y_2, resp. from the columns of Y_1, see Proposition 7.2. Whether this is possible depends on the parameter K and the chosen matrix $H \in \mathbb{R}^{K \times K}$. We will set $K = M = \max\{l_j : j = 1, \dots, d\}$ and H as in (8.7). Then

$$H \otimes Y_f = \begin{pmatrix} Y_f & -t_1 Y_f & O & \dots & O & O \\ O & Y_f & -t_2 Y_f & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & Y_f & -t_{K-1} Y_f \\ O & O & O & \dots & O & Y_f \end{pmatrix}. \tag{8.11}$$

Since the first d columns of $H \otimes Y_f$ can be simply combined to a rational vector (e.g. their sum), necessarily, the choice of columns to create Y_1, Y_2 must be such that at least one of the first d columns of $H \otimes Y_f$ belongs to Y_1 and at least one to Y_2. Note that the columns of Y_f are interchangeable, thus it suffices to show that even if Y_1 is composed of all $H \otimes Y_f$ but its first column, no real combination of its column is a non-zero rational vector. The same will hold for Y_2. We will show an even stronger result, namely considering not only real but complex linear combinations.

From Lemma 8.17, we can instantly derive that the cut and project scheme constructed above is non-degenerated and aperiodic.

It remains to discuss irreducibility. We have to show that $\pi_\perp(\mathcal{L})$ is dense in $\mathbb{R}^{s-n}$. The lattice $\mathcal{L}$ has been defined as the $\mathbb{Z}$-span of columns of the matrix $L = Y^{-1} = P^{-1}(H \otimes Y_f)^{-1}$. We explicitly write its form, i.e.

$$(H \otimes Y_f)^{-1} = H^{-1} \otimes Y_f^{-1} = \begin{pmatrix} Y_f^{-1} & t_1 Y_f^{-1} & t_1 t_2 Y_f^{-1} & \dots & t_1 \cdots t_{K-2} Y_f^{-1} & t_1 \cdots t_{K-1} Y_f^{-1} \\ O & Y_f^{-1} & t_2 Y_f^{-1} & \dots & t_2 \cdots t_{K-2} Y_f^{-1} & t_2 \cdots t_{K-1} Y_f^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & Y_f^{-1} & t_{K-1} Y_f^{-1} \\ O & O & O & \dots & O & Y_f^{-1} \end{pmatrix}.$$

The permutation matrix P^{-1} only chooses the rows to define projections $\pi_\parallel$, $\pi_\perp$, keeping in mind that $P^T(I_K \otimes D_f)P = \left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$.

Since the maximal multiplicity M is strictly greater than the minimal multiplicity m of eigenvalues in A, the permutation P takes from each of the $K = M$ matrices $J_f^{\mathbb{R}}$ at least one eigenvalue to A. In other words, in the matrix $H^{-1} \otimes Y_f^{-1}$, from each block of d rows, at least one is missing in the projection $\pi_\perp$. Now, it is sufficient to combine Remark 8.11 with Lemma 8.18 applied inductively.

It remains to show the statement of the theorem for the case when $m = M$, i.e. when all roots of the polynomial f are eigenvalues of the matrix A, and, moreover, have the same multiplicity. This is equivalent to saying that the characteristic polynomial χ_A of A is equal to $\chi_A = f^M$. In order that the constructed scheme be non-degenerated and aperiodic, the matrix B corresponding to A by Proposition 7.5, must be non-trivial (at least of order 1). Since for the characteristic polynomials we have $\chi_A \chi_B = \chi_C \in \mathbb{Z}[X]$, necessarily, χ_B is a multiple of the polynomial f. When searching for the minimal dimension, we consider $\chi_B = f$. For the construction of the scheme, we will thus need a matrix H of (8.7) where $K = M + 1$. The rest of the proof follows the same lines as in case that $m < M$. □

Chapter 9

Construction of a scheme with a given non-diagonalizable self-similarity

In this chapter we present a construction of a generic CPS with non-diagonalizable self-similarity A. Basic techniques are similar to those introduced in Chapter 8, i.e. the concept of elementary CPS and the direct sum of CPS. Similarly as before we construct an *elementary non-diagonalizable CPS* being invariant under a mapping in the form of a direct sum of real Jordan block $J_d^{\mathbb{R}}(\lambda_i)$ for λ_i being roots of the same irreducible polynomial, say f, i.e. if $\lambda \in \mathbb{R}$ then

$$J_d^{\mathbb{R}}(\lambda) = \underbrace{\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}}_{d},$$

if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then for a pair of complex conjugates one has $\lambda, \overline{\lambda}$

$$J_d^{\mathbb{R}}(\lambda) = \underbrace{\begin{pmatrix} R(\lambda) & I & O & \cdots & O \\ O & R(\lambda) & I & \cdots & O \\ \vdots & & \ddots & \ddots & \vdots \\ O & \cdots & \cdots & R(\lambda) & I \\ O & \cdots & \cdots & O & R(\lambda) \end{pmatrix}}_{2d},$$

with $R(\lambda) = \begin{pmatrix} \mathtt{Re}\,\lambda & \mathtt{Im}\,\lambda \\ -\mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix}$ and $I = I_2$.

First, the construction of an elementary non-diagonalizable CPS will be based on the elementary CPS introduced in Section 8.1. As in the previous chapter we will proceed by analysing the spectrum $\sigma(A)$ and by analysing the real Jordan blocks of A. For a given A we claim that a generic CPS having A as a self-similarity can be obtained as a direct sum of elementary non-diagonalizable CPS corresponding to Jordan blocks in the real Jordan form of A. The number of elementary non-diagonalizable CPS needed will be derived in the same way as in Theorem 8.6.

9.1 Elementary non-diagonalizable cut and project schemes

In what will follow we introduce an integer matrix C whose real Jordan form has all Jordan blocks of the same dimension, say k. The non-trivial real C-invariant subspaces are of the form

$$
\begin{aligned}
V_\lambda &= \left\{ \boldsymbol{y} : \exists j, \in \mathbb{N}\ (C-\lambda I)^j \boldsymbol{y} = \boldsymbol{0} \right\} \quad \text{if } \lambda \in \mathbb{R}, \\
V_\lambda^{\mathbb{R}} &= \left\{ \boldsymbol{y} \in \mathbb{R}^{kd} : \exists j \in \mathbb{N},\ \left((C - \mathtt{Re}\,\lambda I)^2 + (\mathtt{Im}\,\lambda)^2 I \right)^j \boldsymbol{y} = \boldsymbol{0} \right\} \quad \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{aligned}
\tag{9.1}
$$

which are the generalized eigenspaces introduced in Section 4.2. It is obvious that every real invariant subspace of C related to a certain direct sum of Jordan blocks of the same dimension k is a direct sum of a selection of the above elementary subspaces. We necessarily focus on studying an arbitrary real invariant subspace of C of dimension $k,\ 2k,\ 3k, \cdots, (d-1)k$. We restrict ourselves only to these dimensions, see Proposition 9.1 .

Since the subspace is real it must have a basis formed by basis vectors of the subspaces (9.1). Using these basis vectors one could define matrix $Y = (Y_1, Y_2)$ as in Section 8.1 and proceed in a similar way to the construction presented for the diagonalizable case.

We decided to present a different approach that uses certain facts that were proved earlier. Let us start with λ, an algebraic integer of degree d. Denote by f its minimal polynomial, C_f the corresponding companion matrix and let Y_f be defined as in Theorem 8.6, i.e. satisfying

$$
C_f Y_f = Y_f D
$$

for $D \in \mathbb{R}^{d\times d}$ a diagonal matrix with λ on its diagonal (in case of $\lambda \in \mathbb{R}$), or a block diagonal matrix with blocks $R(\lambda)$ in case of $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Let us define an integer matrix C and a real matrix Y by

$$
C = C_f \otimes I_k + I_d \otimes \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}}_{k} \quad \text{and} \quad Y = Y_f \otimes I_k. \tag{9.2}
$$

The spectrum $\sigma(C)$ coincides with $\sigma(C_f)$ because due to Proposition 4.14 it holds that

$$
\sigma(C) = \left\{ \lambda_i + \mu_j : \lambda_i \in \sigma(C_f), \mu_j \in \sigma\left(\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \right) \right\} = \{\lambda_i : \lambda_i \in \sigma(C_f)\} = \sigma(C_f).
$$

We used an obvious fact that $\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$ has a trivial spectrum. Moreover this calculation also shows that $\lambda_i \in \sigma(C)$ has its algebraical multiplicity equal to k for all $i \in \{1, \cdots, d\}$. In order to satisfy this requirement for the characteristic polynomial χ_C and for the minimal polynomial μ_C it has to hold that

$$
\mu_C = \chi_C = \left(\mu_{C_f} \right)^k .
$$

We show that $Y^{-1}CY$ has a block diagonal form and this allows us to discuss the possible mappings A, B.

$$
Y^{-1}CY = \left(Y_f^{-1} \otimes I_d \right) \left(C_f \otimes I_d + I_k \otimes \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix} \right) (Y_f \otimes I_d) =
$$

$$= D \otimes I_d + I_k \otimes \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix} = \bigoplus_{\lambda \in \sigma(C_f)} J_d^{\mathbb{R}}(\lambda).$$

Set $L = Y^{-1}$ and let $\mathcal{L}$ be the corresponding lattice. The block diagonal structure implies, that the mapping A can be chosen so that it is a direct sum of Jordan blocks of dimension k corresponding to different roots of f. Therefore, according to Proposition 7.5, the following proposition holds.

Proposition 9.1. *With the notation above for each $j \in \{1, \cdots, k-1\}$ we have a CPS*

$$\Lambda = \left(\mathcal{L} \subset \mathbb{R}^{kd}, \mathbb{R}^{jd}\right)$$

with the self-similarity

$$A = \bigoplus_{l \in \mathcal{J}} J_k^{\mathbb{R}}(\lambda_l),$$

defined up to a permutation of the blocks, where $|\mathcal{J}| = j$ and $\mathcal{J} \subset \{1, \cdots, k\}$.

The matrix $L = Y^{-1} = Y_f^{-1} \otimes I_k$ has the following form

$$\begin{pmatrix} \left[Y_f^{-1}\right]_{11} I_k & \cdots & \left[Y_f^{-1}\right]_{1d} I_k \\ \vdots & \ddots & \vdots \\ \left[Y_f^{-1}\right]_{d1} I_k & \cdots & \left[Y_f^{-1}\right]_{dd} I_k \end{pmatrix}. \tag{9.3}$$

and therefore according to Lemma 8.3 it holds that the

$$(gk+1)^{\text{th}},\ (gk+2)^{\text{th}},\ \ldots,\ (gk+k)^{\text{th}}$$

rows of the matrix L are images of

$$1^{\text{st}},\ 2^{\text{nd}},\ \ldots,\ k^{\text{th}}$$

rows of L, respectively, under the g-th field automorphism of the splitting field of f. As a direct consequence of this we have the following corollary.

Corollary 9.2. *Let $\boldsymbol{\alpha} \in \mathbb{Q}^{kd}$. If there exists $g \in \{0, 1, \cdots d-1\}$ such that*

$$\left[Y^{-1}\boldsymbol{\alpha}\right]_{gk+1} = \left[Y^{-1}\boldsymbol{\alpha}\right]_{gk+2} = \cdots = \left[Y^{-1}\boldsymbol{\alpha}\right]_{gk+k} = \mathbf{0}$$

then $Y^{-1}\boldsymbol{\alpha} = \mathbf{0}$.

We will use it when proving the genericity of the constructed CPS.

Theorem 9.3. *The cut and project scheme*

$$\Lambda = \left(\mathcal{L} \subset \mathbb{R}^{kd}, \mathbb{R}^{jd}\right)$$

defined in Proposition 9.1 is generic.

Proof. We will proceed similarly as in the proof of Theorem 8.6. Reorganize the matrix $Y = \left(Y_1,\ Y_2\right)$ such that Y_1 contains the k-tuples of vectors corresponding to Jordan blocks $J^{\mathbb{R}}(\lambda_i)$, $i \in \mathcal{J}$. First, we show that C does not contain any non-trivial rational subspaces. Suppose that there exists $\boldsymbol{\alpha} \in \mathbb{Q}^{kd}$ such that

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_d \end{pmatrix}, \quad \boldsymbol{\alpha}_i \in \mathbb{Q}^k \setminus \{\mathbf{0}\} \quad \text{and } Y_1\boldsymbol{\alpha} = \boldsymbol{z} \in \mathbb{Z}^{kd} \setminus \{\mathbf{0}\}.$$

Note that we can assume $\boldsymbol{z}$ to be an integer vector without loss of generality. Then we have

$$Y^{-1}Y_1\boldsymbol{\alpha} = \begin{pmatrix} \widetilde{\boldsymbol{\alpha}}_1 \\ \widetilde{\boldsymbol{\alpha}}_2 \\ \vdots \\ \widetilde{\boldsymbol{\alpha}}_d \end{pmatrix} = Y^{-1}\boldsymbol{z} \quad \text{such that } \widetilde{\boldsymbol{\alpha}}_i = \begin{cases} \boldsymbol{\alpha}_i & \text{if } i \in \mathcal{J}, \\ \mathbf{0} & \text{if } i \notin \mathcal{J}. \end{cases}.$$

Since $\{1, \dots, k\} \setminus \mathcal{J}$ is non-empty, Corollary 9.2 gives that $\boldsymbol{\alpha} = \mathbf{0}$, which is a contradiction. Combining that C does not contain any rational subspaces together with Proposition 7.2 gives us that the CPS is non-degenerate and aperiodic.

In order to prove irreducibility of the scheme, denote by $\left\{\hat{\imath}_1, \ \dots, \ \hat{\imath}_{k-j}\right\} = \{1, \ \dots, \ k\} \setminus \mathcal{J}$. Then the projection $\pi_\perp$ of the generators of the lattice $\mathcal{L}$ can be written as

$$\pi_\perp(\mathcal{L}) = \begin{pmatrix} \left[Y_f^{-1}\right]_{\hat{\imath}_1 1} I_k & \cdots & \left[Y_f^{-1}\right]_{\hat{\imath}_1 d} I_k \\ \vdots & \ddots & \vdots \\ \left[Y_f^{-1}\right]_{\hat{\imath}_{k-j} 1} I_k & \cdots & \left[Y_f^{-1}\right]_{\hat{\imath}_{k-j} d} I_k \end{pmatrix}.$$

Each row m of the $\pi_\perp$-projection of the lattice $\mathcal{L}$ has therefore form

$$\mathcal{S}_{\hat{\imath}} = \left\{ \sum_{l=1}^{d} a_l^{(m)} \left[Y_f^{-1}\right]_{\hat{\imath} l} : a_l^{(m)} \in \mathbb{Z} \right\},$$

which is an integer span of d linearly independent numbers over $\mathbb{Q}$ (by Lemma 8.5) which is a dense set in $\mathbb{R}$. Therefore the entire projection $\pi_\perp(\mathcal{L})$ can be written as

$$\pi_\perp(\mathcal{L}) = \bigoplus_{l=1}^{k-j} \underbrace{\mathcal{S}_{\hat{\imath}_l} \oplus \mathcal{S}_{\hat{\imath}_l} \oplus \cdots \oplus \mathcal{S}_{\hat{\imath}_l}}_{d}$$

which shows the density of $\pi_\perp(\mathcal{L})$ in $\mathbb{R}^{(k-j)d}$ and proves irreducibility of the scheme. □

9.2 Construction of CPS for trivial self-similarities

In this section we present a construction of a CPS with non-diagonalizable self-similarities A which have in some sense trivial spectrum. As in Section 8.2 the situation is trivial, if eigenvalues of A are rational integers, or non-real quadratic numbers. For such A, we can find a discrete structure in $\mathbb{R}^n$ (namely a lattice), with this self-similarity. If, for some reason, we need a generic cut and project scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$, $s > n$, such that $A\pi_\parallel(\mathcal{L}) \subset \pi_\parallel(\mathcal{L})$, the construction is described here. Before the construction itself we formulate one useful lemma.

Lemma 9.4. *Let $k \geq l$ and let us define*

$$I_{k,l} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}}_{k} \oplus \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}}_{l} \in \{0,1\}^{(k+l)\times(k+l)}.$$

The set of all matrices that commute with $I_{k,l}$ contains only block matrices $H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$ such that $H_1 \in \mathbb{C}^{k\times k}$, $H_2 \in \mathbb{C}^{k\times l}$, $H_3 \in \mathbb{C}^{l\times k}$, and $H_4 \in \mathbb{C}^{l\times l}$ and it holds that

$$H_1 = \begin{pmatrix} h_1^{(1)} & h_2^{(1)} & h_3^{(1)} & \cdots & h_{k-1}^{(1)} & h_k^{(1)} \\ 0 & h_1^{(1)} & \ddots & \ddots & & h_{k-1}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & h_3^{(1)} \\ 0 & & & 0 & h_1^{(1)} & h_2^{(1)} \\ 0 & \cdots & \cdots & 0 & 0 & h_1^{(1)} \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_1^{(2)} & h_2^{(2)} & \cdots & h_l^{(2)} \\ 0 & h_1^{(2)} & & h_{l-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_1^{(2)} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{9.4}$$

$$H_3 = \begin{pmatrix} 0 & \cdots & 0 & h_1^{(3)} & h_2^{(3)} & \cdots & h_l^{(3)} \\ 0 & \cdots & 0 & 0 & h_1^{(3)} & & h_{l-1}^{(3)} \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & h_1^{(3)} \end{pmatrix}, \quad H_4 = \begin{pmatrix} h_1^{(4)} & h_2^{(4)} & \cdots & h_l^{(4)} \\ 0 & h_1^{(4)} & & h_{l-1}^{(4)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_1^{(4)} \end{pmatrix}.$$

Proof. Without loss of generality we can assume that H can be written in a block form, namely

$$H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$$

with $H_1 \in \mathbb{C}^{k\times k}$, $H_2 \in \mathbb{C}^{k\times l}$, $H_3 \in \mathbb{C}^{l\times k}$, and $H_4 \in \mathbb{C}^{l\times l}$. Requiring the commutativity $I_{k,l}H = HI_{k,l}$ one gets

$$\begin{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{k} H_1 & \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{k} H_2 \\ \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{l} H_3 & \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{l} H_4 \end{pmatrix} = \begin{pmatrix} H_1 \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{k} & H_2 \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{l} \\ H_3 \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{k} & H_4 \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{l} \end{pmatrix}.$$

Comparing both sides of the equation gives conditions on matrices H_1, H_2, H_3, and H_4 and they can be treated separately. Let us proceed with H_1. Denote by

$$H_1 = \begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & \cdots & h_{1k}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)} & & \vdots \\ \vdots & & \ddots & \vdots \\ h_{k1}^{(1)} & \cdots & \cdots & h_{kk}^{(1)} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & \cdots & h_{1k}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)} & & \vdots \\ \vdots & & \ddots & \vdots \\ h_{k1}^{(1)} & \cdots & \cdots & h_{kk}^{(1)} \end{pmatrix} = \begin{pmatrix} h_{11}^{(1)} & h_{12}^{(1)} & \cdots & h_{1k}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)} & & \vdots \\ \vdots & & \ddots & \vdots \\ h_{k1}^{(1)} & \cdots & \cdots & h_{kk}^{(1)} \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} h_{21}^{(1)} & h_{22}^{(1)} & \cdots & h_{2k}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k1}^{(1)} & h_{k2}^{(1)} & \cdots & h_{kk}^{(1)} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & h_{11}^{(1)} & \cdots & h_{1\ k-1}^{(1)} \\ 0 & h_{21}^{(1)} & \cdots & h_{2\ k-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{k1}^{(1)} & \cdots & h_{k\ k-1}^{(1)} \end{pmatrix}.$$

This can be finally summarized in the following system of equations

$$\begin{aligned} h_{i1}^{(1)} &= 0 && \text{for } i \in \{2, \cdots, k\}, \\ h_{kj}^{(1)} &= 0 && \text{for } j \in \{1, \cdots, k-1\}, \\ h_{ij}^{(1)} &= h_{i-1\ j-1}^{(1)} && \text{for } i, j \in \{2, \cdots, k\}. \end{aligned}$$

It has a solution that results in matrix H_1 having an upper triangular form

$$H_1 = \begin{pmatrix} h_1^{(1)} & h_2^{(1)} & h_3^{(1)} & \cdots & h_{k-1}^{(1)} & h_k^{(1)} \\ 0 & h_1^{(1)} & \ddots & \ddots & & h_{k-1}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & h_3^{(1)} \\ 0 & & & 0 & h_1^{(1)} & h_2^{(1)} \\ 0 & \cdots & \cdots & 0 & 0 & h_1^{(1)} \end{pmatrix}.$$

The same procedure gives the form of matrices H_2, H_3, H_4 and completes the proof. □

Let us return to the construction. If $\sigma(A) \subset \mathbb{Z}$, it is obvious that any generic scheme $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ satisfies the required properties. For, any $\mathbb{Z}$-module is closed under multiplication by integers. Suppose now that the spectrum $\sigma(A)$ is formed only by complex conjugated pairs of the same non-real quadratic numbers $\lambda, \bar{\lambda}$.

We show that this case can be treated in the same way as in Section 8.2. Without loss of generality we can suppose that A has only real Jordan blocks of the same size in its real Jordan form. If it does not, we can use the direct sum of CPS for Jordan blocks of different sizes. Then we show that we need to add only one real Jordan block in order to have a generic CPS. Suppose that A has r real Jordan blocks of dimension k. Then A can be assumed to have the following form

$$A = I_r \otimes \left(I_k \otimes R(\lambda) + \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}}_{k} \otimes I_2 \right). \tag{9.5}$$

We present the construction for $r = 1$. The extension for an arbitrary r can be done easily and it is discussed after the proof of the proposition.

Proposition 9.5. *Let $A \in \mathbb{R}^{n\times n}$ be a non-diagonalizable mapping such that its eigenvalues are roots of the same quadratic polynomial $\lambda^2 - p\lambda - q$, $p, q \in \mathbb{Z}$ with negative discriminant. Then there exists a generic scheme $(\mathcal{L} \subset \mathbb{R}^{2n}, \mathbb{R}^n)$ such that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.*

Proof. Clearly n is even, i.e. $n = 2k$ for some $k \geq 2$. The mapping A can be assumed to be in the form

$$A = \begin{pmatrix} R(\lambda) & I & O & O & \cdots & O \\ O & R(\lambda) & I & O & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & \cdots & O & R(\lambda) & I & O \\ O & \cdots & O & O & R(\lambda) & I \\ O & \cdots & O & O & O & R(\lambda) \end{pmatrix} = I_k \otimes R(\lambda) + \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}}_{k} \otimes I_2$$

with $R(\lambda) = \begin{pmatrix} \mathtt{Re}\,\lambda & -\mathtt{Im}\,\lambda \\ \mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix}$. We define the lattice $\mathcal{L}$ through its corresponding matrix L defined by

$$L = H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}$$

for a matrix $H \in \mathbb{R}^{(k+l)\times(k+l)}$ for some $l \in \mathbb{N}$. The conditions on the matrix H will be imposed later. At the same time let us define (using the notation from Lemma 9.4)

$$C = I_{k+l} \otimes \begin{pmatrix} 0 & q \\ 1 & p \end{pmatrix} + I_{k,l} \otimes I_2 \quad \text{and} \quad B = I_l \otimes R(\lambda) + \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}}_{l} \otimes I_2.$$

In order to obtain a CPS with A as its self-similarity, equation (7.3) has to hold:

$$LC = \begin{pmatrix} A & O \\ O & B \end{pmatrix} L$$

$$\left(H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}\right)\left(I_{k+l} \otimes \begin{pmatrix} 0 & q \\ 1 & p \end{pmatrix} + I_{k,l} \otimes I_2\right) = \left(I_{k+l} \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & -\mathtt{Im}\,\lambda \\ \mathtt{Im}\,\lambda & \mathtt{Re}\,\lambda \end{pmatrix} + I_{k,l} \otimes I_2\right)\left(H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}\right),$$

$$H \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & q + p\mathtt{Re}\,\lambda \\ \mathtt{Im}\,\lambda & p\mathtt{Im}\,\lambda \end{pmatrix} + HI_{k,l} \otimes I_2 = H \otimes \begin{pmatrix} \mathtt{Re}\,\lambda & \mathtt{Re}^2\lambda - \mathtt{Im}^2\lambda \\ \mathtt{Im}\,\lambda & 2\mathtt{Im}\,\lambda\mathtt{Re}\,\lambda \end{pmatrix} + I_{k,l}H \otimes I_2.$$

Since $\mathtt{Re}^2\lambda - \mathtt{Im}^2\lambda = q + p\,\mathtt{Re}\,\lambda$ and $2\mathtt{Im}\,\lambda\,\mathtt{Re}\,\lambda = p\,\mathtt{Im}\,\lambda$, it is sufficient to fulfil the condition (7.3) to take as the matrix H the one described in Lemma 9.4. This proves by Proposition 7.5 that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.

Requiring aperiodicity of the resulting CPS one has to take $l \geq k$. Otherwise one would have a zero column in the matrix H_3 from (9.4). This implies that there is a lattice vector with $\pi_\perp$-projection being a zero vector and hence the scheme would not be aperiodic. On the other hand requiring non-degeneracy gives for the same reason a condition $k \leq l$. Thus one has to take $l = k$.

In order to prove non-degeneracy of the scheme let us study the projection of generators of lattice $\mathcal{L}$ to the physical space (keep in mind the notation from (9.4)):

$$\pi_\parallel(L) = \begin{pmatrix} h_1^{(1)} & h_2^{(1)} & \cdots & h_k^{(1)} & h_1^{(2)} & h_2^{(2)} & \cdots & h_k^{(2)} \\ 0 & h_1^{(1)} & & h_{k-1}^{(1)} & 0 & h_1^{(2)} & & h_{k-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_1^{(1)} & 0 & \cdots & 0 & h_1^{(2)} \end{pmatrix} \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix}.$$

Clearly, choosing $h_1^{(1)}, \ldots, h_k^{(1)}, h_1^{(2)}, \ldots, h_k^{(2)}$ to be linearly independent over $\mathbb{Q}$ one ensures the non-degeneracy of the scheme. Similarly, choosing $h_1^{(3)}, \ldots, h_k^{(3)}, h_1^{(4)}, \ldots, h_k^{(4)}$ to be linearly independent over $\mathbb{Q}$ results in the aperiodicity of the scheme.

The last row of the $\pi_\perp$ projection of the whole lattice can be expressed as the set

$$\mathcal{H} = \left\{ \left(b_k h_k^{(3)} + b_{2k} h_k^{(4)} \right) \mathtt{Im}\,\lambda : b_i \in \mathbb{Z} \right\}.$$

From the requirement of the linear independence on the elements of H it follows that $\mathcal{H}$ is dense in $\mathbb{R}$. Using the same argument as in proof of Proposition 8.12, i.e. the fact that the $(2k-1)$-th row of the projection $\pi_\perp(\mathcal{L})$ can be then written as $\frac{1}{\mathtt{Im}\,\lambda}\mathcal{H} \oplus \frac{\mathtt{Re}\,\lambda}{\mathtt{Im}\,\lambda}\mathcal{H}$, one gets that the $(2k-1)$-th row of the projection $\pi_\perp(\mathcal{L})$ is clearly dense in $\mathbb{R}$ as well. Now we can apply Lemma 8.18 and the projection is thus dense in the whole inner space. With this claim the proof of genericity of the scheme is complete. □

As it was mentioned before, the extension for an arbitrary r is straightforward. If A is of the form (9.5), one has to use

$$L = H \otimes \begin{pmatrix} 1 & \mathtt{Re}\,\lambda \\ 0 & \mathtt{Im}\,\lambda \end{pmatrix} \qquad \text{with } H \in \mathbb{R}^{(k(r+1)) \times (k(r+1))},$$

$$C = I_{k(r+1)} \otimes \begin{pmatrix} 0 & q \\ 1 & p \end{pmatrix} + I_{r+1} \otimes I_{k,0} \otimes I_2.$$

We claim that Lemma 9.4 can be extended for matrices of the form $I_{r+1} \otimes I_{k,0}$ and the commuting matrices H will have a block structure with submatrices in upper triangular form.

9.3 CPS with a non-diagonalizable self-similarity A

In the same way as in the diagonalizable case we can restrict ourselves only to those mappings that have eigenvalues among roots of one certain irreducible polynomial with integer coefficients. According to Proposition 7.8 we can assume A to be in a real Jordan form. Without loss of generality we can also assume that the Jordan blocks are ordered by their dimension from the highest to the lowest one. We claim that for each dimension of Jordan block, say d_i, we can construct a generic CPS. The final CPS that has A as a self-similarity is given by a direct sum of such CPS over all sizes d_i of Jordan blocks in A. Let us demonstrate this on an example:

Example 4. *Consider β to be a root of polynomial $x^3 - 2x^2 - x + 1$. Denote by β', β'' the remaining roots. Let*

$$A = \begin{pmatrix} \beta & 1 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta' \end{pmatrix}.$$

We need to estimate a dimension s such that there is a matrix $C \in \mathbb{Z}^{s\times s}$ satisfying $C \sim \left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$. First, note that in the Jordan canonical form of C there must be at least two Jordan blocks, namely $\begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}$ and (β'). In order to find a suitable dimension let us study the generalized eigenspaces corresponding to eigenvalues β, β' and their dimensions. Let us start with β. Recall that the generalized eigenspace V_β is defined as follows:

$$V_\beta = \left\{ \boldsymbol{x} \in \mathbb{R}^d : (C - \beta I)^k \boldsymbol{x} = \boldsymbol{0} \text{ for some } k \in \mathbb{N} \right\}.$$

This space contains at least one eigenvector corresponding to β, say $\boldsymbol{e}_1$. Another generalized eigenvector, linearly independent to $\boldsymbol{e}_1$, say $\boldsymbol{e}_2$, that belongs to V_β satisfies the following equation

$$(C - \beta I)\boldsymbol{e}_2 = \boldsymbol{e}_1.$$

This implies that $\dim V_\beta \geq 2$.

Moreover, $V_{\beta'}$ contains at least one vector, say $\boldsymbol{e}_3$, the eigenvector to β'.

Since C has to be an integer matrix, its generalized eigenvectors are only (componentwise) images under automorphisms of the splitting field of the characteristic polynomial χ_C, namely identity, $\beta \to \beta'$, and $\beta \to \beta''$. This implies, that the dimensions of generalized eigenspaces are the same and equal to 3. Thus the minimal dimension d of C is $d = 9$ and the Jordan form of C can be chosen for example as

$$J_C = \begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta' & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta'' & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta'' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix} \sim \begin{pmatrix} J_2(\beta) & O & O & O & O & O \\ O & J_2(\beta') & O & O & O & O \\ O & O & J_2(\beta'') & O & O & O \\ O & O & O & J_1(\beta) & O & O \\ O & O & O & O & J_1(\beta') & O \\ O & O & O & O & O & J_1(\beta'') \end{pmatrix}.$$

So the matrix C can be chosen for example as

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 \end{pmatrix}.$$

When proving the following theorem we follow the same as in the demonstration of Theorem 8.14. The only difference is the fact, that instead of $\lambda \in \sigma(A)$ we put into the matrix $\mathcal{R}$ "1" for each real Jordan block $J_d^{\mathbb{R}}(\lambda)$. This finally results in a slightly different estimation on the dimension of the scheme.

Theorem 9.6. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix non-diagonalizable over $\mathbb{C}$, $\sigma(A) \subset \mathbb{B}$. Then there exists a generic cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$.*

Proof. The proof is constructive. We demonstrate that a suitable CPS can be obtained as a direct sum of a sufficient number of elementary non-diagonalizable schemes. Without loss of generality we can assume that $\sigma(A)$ has its entries among roots of an irreducible polynomial of degree d, say f. Moreover, we can, without loss of generality, restrict ourselves only on those matrices that have in their Jordan real form Jordan blocks of one dimension, say k. This can be assumed because otherwise we take a direct sum of schemes corresponding to mappings having in their Jordan decomposition blocks of only one size.

As in the proof of Theorem 8.14, we compose the scheme from elementary non-diagonalizable CPS. This implies that the corresponding integer matrix will be a Kronecker product of an identity matrix of order, say K, and the matrix C defined in Section 9.1 by (9.2), i.e.

$$I_K \otimes \left(C_f \otimes I_k + I_d \otimes \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix} \right). \tag{9.6}$$

Thus its dimension is $s = Kkd$. Since all eigenvalues of A are eigenvalues of C and thus of (9.6), the number K of such matrices is at least equal to the maximum M of multiplicities of Jordan blocks corresponding to the eigenvalues of A. Formally one can suppose that the polynomial f has r real and t pairs of complex conjugate roots, i.e.

$$f(x) = \prod_{i=1}^{r}(x - \lambda_i) \prod_{j=1}^{t}(x - \mu_j)(x - \overline{\mu_j}), \quad r + t \geq 2.$$

Define $M := \max_{i,j}\{l_i, m_j\}$, where l_i, m_j are numbers of Jordan blocks $J_d^{\mathbb{R}}(\lambda)$ for λ_j, μ_k in the spectrum of A. Set $N = r + t$. Since the trivial cases have been already solved in Section 9.2, without loss of generality we can suppose that the eigenvalues of A are neither integers, nor non-real quadratic numbers, i.e. $N \geq 2$. The rest of the proof, i.e. the combinatorial task and its solution using Lemma 8.13 is exactly the same as in the diagonalizable case. The dimension is then given by

$$K := \max\left\{ M, \left\lceil \frac{\sum_{i=1}^{r} l_i + \sum_{j=1}^{t} m_j}{N-1} \right\rceil \right\}.$$

□

Chapter 10

Cut and project sets with a given self-similarity

Let us come back to the original question, namely of the existence of a cut-and-project set $\Sigma(\Omega) \subset \mathbb{R}^n$ with a given self-similarity $A \in \mathbb{R}^{n\times n}$. Having a generic scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ with $A\pi_{\parallel}(\mathcal{L}) \subset \pi_{\parallel}(\mathcal{L})$, it suffices to decide whether there exists a suitable window $\Omega \subset \mathbb{R}^{s-n}$. It is not difficult to show that the window must be closed under the mapping B associated with A by Proposition 7.3.

Proposition 10.1. *Let $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ be a non-degenerated irreducible cut-and-project scheme with a self-similarity A. If there exists a window $\Omega \subset \mathbb{R}^{s-n}$, such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$, then the eigenvalues of the matrix B from Proposition 7.3 are in modulus smaller or equal to* 1.

Proof. Assume that for some $\Omega \subset \mathbb{R}^{s-n}$ we have $A\Sigma(\Omega) \subset \Sigma(\Omega)$. This means that $A^k\boldsymbol{z} \in \Sigma(\Omega)$ for any $\boldsymbol{z} \in \Sigma(\Omega)$ and $k \in \mathbb{N}$. For the integer matrix $C \in \mathbb{Z}^{s\times s}$ and the real matrix $B \in \mathbb{R}^{(s-n)\times(s-n)}$ from Proposition 7.3 we have, by iterating (7.3), that for any $k \in \mathbb{N}$

$$\begin{pmatrix} A^k & O \\ O & B^k \end{pmatrix} = LC^kL^{-1}.$$

Realize that if $\boldsymbol{z} \in \Sigma(\Omega)$, then $\boldsymbol{z} = \pi_{\parallel}(\boldsymbol{l})$ for some $\boldsymbol{l} \in \mathcal{L}$ and $\pi_{\perp}(\boldsymbol{l}) \in \Omega$. Thus for any k there exist $\boldsymbol{l}' \in \mathcal{L}$ such that $A^k\boldsymbol{z} = \pi_{\parallel}(\boldsymbol{l}')$ and $\pi_{\perp}(\boldsymbol{l}') \in \Omega$. Rewriting in the matrix formalism,

$$A^k\pi_{\parallel}(\boldsymbol{l}) = A^k(I_n,\, O)\boldsymbol{l} = (I_n,\, O)\begin{pmatrix} A^k & O \\ O & B^k \end{pmatrix}\boldsymbol{l} = (I_n,\, O)LC^kL^{-1}\boldsymbol{l} = \pi_{\parallel}(\boldsymbol{l}').$$

Since the scheme is non-degenerate, the projection $\pi_{\parallel}$ is injective, and thus we can derive that $\boldsymbol{l}' = LC^kL^{-1}\boldsymbol{l}$. The condition $\pi_{\perp}(\boldsymbol{l}') \in \Omega$ is therefore equivalent to

$$\pi_{\perp}(\boldsymbol{l}') = (O,\, I_{s-n})\boldsymbol{l}' = (O,\, I_{s-n})LC^kL^{-1}\boldsymbol{l} = (O,\, I_{s-n})\begin{pmatrix} A^k & O \\ O & B^k \end{pmatrix}\boldsymbol{l} = B^k(O,\, I_{s-n})\boldsymbol{l} = B^k\pi_{\perp}(\boldsymbol{l}).$$

Now realize that by irreducibility the set $\{\pi_{\perp}(\boldsymbol{l}) : \boldsymbol{l} \in \mathcal{L},\ \pi_{\parallel}(\boldsymbol{l}) \in \Sigma(\Omega)\}$ is dense in the bounded window Ω. By linearity of B, we must have for the closure of the window that $B^k\overline{\Omega} \subset \overline{\Omega}$. Suppose that B has a real eigenvalue λ of modulus strictly exceeding 1. As $B \in \mathbb{R}^{(s-n)\times(s-n)}$, we have a real eigenvector $\boldsymbol{w}$ of B corresponding to the eigenvalue λ. Iterating, we obtain a contradiction $B^k\boldsymbol{w} = \lambda^k\boldsymbol{w} \notin \Omega$ for sufficiently large $k \in \mathbb{N}$. If λ is a non-real eigenvalue of B with $|\lambda| > 1$ with a non-real eigenvector $\boldsymbol{w}$, then $\overline{\lambda}$ is an eigenvalue of B corresponding to the eigenvector $\overline{\boldsymbol{w}}$. Over the real space of dimension 2, spanned by the vectors $\boldsymbol{w} + \overline{\boldsymbol{w}}$, $i(\boldsymbol{w} - \overline{\boldsymbol{w}})$, the mapping B acts as multiplication by $|\lambda|$ and rotation by the argument of λ. We obtain a similar contradiction with boundedness of the window Ω as before. □

Proposition 10.1 gives a necessary condition in order that $\mathbb{A}\Sigma(\Omega) \subset \Sigma(\Omega)$ for a cut and project set $\Sigma(\Omega)$. The following two statements provide sufficient conditions for existence of a self-similar cut and project set.

Proposition 10.2. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a non-degenerated irreducible cut and project scheme with a self-similarity* A*. If the matrix* B *from Proposition 7.3 has all eigenvalues in modulus strictly smaller than 1, then there exists a window* Ω *such that* A *is a self-similarity of the cut and project set* $\Sigma(\Omega)$*.*

Proof. Since the eigenvalues of the matrix B are strictly smaller than 1, by [11, Corollary 1.2.3] there exists a metric ρ in $\mathbb{R}^{s-n}$ such that the mapping B is in that metric contracting, i.e. there exists $\delta < 1$ such that for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{s-n}$ we have

$$\delta\rho(\boldsymbol{x}, \boldsymbol{y}) > \rho(B\boldsymbol{x}, B\boldsymbol{y}).$$

Choosing for the window the set

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^m : \rho(\boldsymbol{x}, \boldsymbol{0}) \leq 1\},$$

we have for any $\boldsymbol{l} \in \mathcal{L}$ with $\pi_\perp(\boldsymbol{l}) \in \Omega$, that $B\pi_\perp(\boldsymbol{l}) \in \Omega$. Therefore $A\pi_\parallel(\boldsymbol{l}) \in \Sigma(\Omega)$ for any $\boldsymbol{l} \in \mathcal{L}$ such that $\pi_\parallel(\boldsymbol{l}) \in \Sigma(\Omega)$. Thus $A\Sigma(\Omega) \subset \Sigma(\Omega)$. □

Restricting our considerations to diagonalizable B, we can weaken the assumptions on the eigenvalues of B.

Proposition 10.3. *Let* $(\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ *be a non-degenerated irreducible cut and project scheme with a self-similarity* A*. If the matrix* B *from Proposition 7.3 is diagonalizable and all its eigenvalues are in modulus smaller than or equal to 1, then there exists a window* Ω *such that* A *is a self-similarity of the cut and project set* $\Sigma(\Omega)$*.*

Proof. We will construct a positive semi-definite matrix which induces an inner product on $\mathbb{R}^{s-n}$ (and consequently a metric) in which the mapping B is non-expanding, i.e. does not enlarge the distances. First we define an inner product in which the eigenvectors of B form an orthonormal basis of $\mathbb{R}^{s-n}$. Denote by $\eta_1, \eta_2, \ldots, \eta_m$, $m := s - n$, the eigenvalues of B and the corresponding eigenvectors by $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$. The inner product is defined using the hermitian matrix $H = G^*G$ where G^* stands for conjugate transpose. For G we take the matrix transferring the eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ into the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$. Then the inner product for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_H := \boldsymbol{x}^* G^* G \boldsymbol{y}.$$

Consider a general vector $\boldsymbol{x} = \sum_{i=1}^m \alpha_i \boldsymbol{w}_i$. Then

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_H = \left\langle \sum_{i=1}^m \alpha_i \boldsymbol{w}_i, \sum_{j=1}^m \alpha_j \boldsymbol{w}_j \right\rangle_H = \sum_{i,j=1}^m \overline{\alpha_i} \alpha_j \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_H = \sum_{i=1}^m |\alpha_i|^2,$$

$$\langle B\boldsymbol{x}, B\boldsymbol{x} \rangle_H = \left\langle \sum_{i=1}^m \alpha_i B\boldsymbol{w}_i, \sum_{j=1}^m \alpha_j B\boldsymbol{w}_j \right\rangle_H = \sum_{i,j=1}^m \overline{\alpha_i \eta_i} \alpha_j \eta_j \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_H = \sum_{i=1}^m |\eta_i|^2 |\alpha_i|^2.$$

As for all eigenvalues $|\eta| \leq 1$, we thus have

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_H \geq \langle B\boldsymbol{x}, B\boldsymbol{x} \rangle_H$$

for any $\boldsymbol{x} \in \mathbb{R}^m$, and therefore the mapping B is not expanding. Setting for the acceptance window Ω a ball in the metric induced by this inner product, i.e.

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^m : \langle \boldsymbol{x}, \boldsymbol{x} \rangle_H \leq 1\},$$

it can be again easily derived that the cut-and-project $\Sigma(\Omega)$ has self-similarity A. □

In our case, Proposition 10.1 implies that the eigenvalues of the matrix B must all be in modulus smaller or equal to 1. According to our construction for diagonalizable mappings (Chapter 8) the diagonalizability of B implies the diagonalizability of A and vice versa. Thus the following theorem reformulates this necessary and sufficient condition in terms that do not require the knowledge of the mapping B.

Theorem 10.4. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix diagonalizable over $\mathbb{C}$, $\sigma(A) \subset \mathbb{B}$, all $\lambda \in \sigma(A)$ have the same minimal polynomial f of degree d. Denote $l_1, \ldots, l_d$ the multiplicities of the roots $\beta_1, \ldots \beta_d$ of f in the spectrum $\sigma(A)$, and set $M := \max_j\{l_j\}$, $m := \min_j\{l_j\}$.*

If all roots of f are of modulus 1, then there exists a model set $\Sigma(\Omega)$ with self-similarity A.

Otherwise, there exists a model set $\Sigma(\Omega)$ with self-similarity A if and only if $m < M$, and for every root λ of f with $|\lambda| > 1$ we have $\lambda \in \sigma(A)$ and its multiplicity in $\sigma(A)$ is equal to M.

Proof. According to Theorem 8.14, from the assumptions, we can construct a cut-and-project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that the $\mathbb{Z}$-module $\pi_{\|}(\mathcal{L})$ has self-similarity A. By Proposition 7.3 we have the corresponding matrices B, C such that

$$LC = \begin{pmatrix} A & O \\ O & B \end{pmatrix} L$$

, where L is a matrix corresponding to the lattice $\mathcal{L}$. Moreover, for the spectra of the matrices, it holds that $\sigma(C) = \sigma(A) \cup \sigma(B)$. From our construction, if A is diagonalizable over $\mathbb{C}$, then C and hence also B are diagonalizable over $\mathbb{C}$.

If f has all eigenvalues of absolute value 1, then by the result of Proposition 10.2, an acceptance window $\Omega \subset \mathbb{R}^{s-n}$ can be found so that $B\Omega \subset \Omega$, whence $A\Sigma(\Omega) \subset \Sigma(\Omega)$.

In case that the polynomial f has some roots of modulus strictly greater than 1, then the condition stated in the theorem expresses the fact that all the eigenvalues of the matrix C in modulus strictly greater than 1 belong to the spectrum of A and not to the spectrum of B and the window can be found using Proposition 10.3. □

Suppose that a mapping A is a scaling by an algebraic integer λ of degree k, i.e. $A = \lambda I_n$. Then, using the notation defined above, we have $M = n$ and $m = 0$. According to Theorem 10.4 there exists $\Sigma(\Omega)$ with A as its self-similarity if and only if $\lambda \in \sigma(A)$ is the only root of its minimal polynomial that is strictly greater then 1. This is equivalent to the fact that the algebraic conjugates to λ are smaller then or equal to one, i.e. λ is a Pisot or a Salem number. This fully corresponds to the point *(iii)* of Theorem 6.18 and therefore the Theorem 10.4 can be understood as its extension from scalings to general linear mappings.

Part III

Quasicrystals with five-fold symmetry

Chapter 11

Self-similarities of cut and project set with five-fold symmetry

In this chapter we firstly recall one specific CPS, namely $\Lambda_5 = (\mathcal{L} \subset \mathbb{R}^4, \mathbb{R}^2)$ with 5-fold symmetry. Then we recall the set of all mappings preserving the $\mathbb{Z}$-modules $\pi_\parallel(\mathcal{L})$ and $\pi_\perp(\mathcal{L})$ as it was derived in [14]. Next we describe all possible mapping which are self-similarities of a cut and project set $\Sigma(\Omega)$ derived from Λ_5. We show that the set of all such mappings is in one-to-one correspondence with a 2-dimensional cut and project set. Finally, we describe self-similarities of a pentagonal cut and project set with circular acceptance window.

11.1 Cut and project set with five-fold symmetry

The construction of Λ_5 is described in detail in [14]. Let us recall that the CPS is derived using the following matrix C,

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix},$$

which is the companion matrix of the cyclotomic polynomial $\Phi_5(X) = X^4 + X^3 + X^2 + X + 1$. The corresponding lattice $\mathcal{L}$ can be generated with the basis vectors $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_4$, namely

$$\mathcal{L} = \mathbb{Z}\begin{pmatrix} 1-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} \\ 1-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5} \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 \\ 2\sin\frac{2\pi}{5} \\ 0 \\ 2\sin\frac{4\pi}{5} \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \\ \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5}-\sin\frac{2\pi}{5} \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5}-\sin\frac{4\pi}{5} \\ \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \end{pmatrix} \equiv \sum_{i=1}^{4}\mathbb{Z}\boldsymbol{\ell}_i. \quad (11.1)$$

Let us depict the projections of the lattice generators in physical, and inner space, i.e. $\pi_\parallel(\boldsymbol{\ell}_i)$ and $\pi_\perp(\boldsymbol{\ell}_i)$, in order to underline the naturally arising pentagonal structure.

In [14] it is shown that Λ_5 is a generic CPS and therefore one can use it to get a cut and project set. Let Ω be a bounded window with non-empty interior. The resulting cut and project set $\Sigma(\Omega)$ can be written as

$$\Sigma(\Omega) = \left\{ \sum_{i=1}^{4} a_i \pi_\parallel(\boldsymbol{\ell}_i) : a_i \in \mathbb{Z},\ \sum_{i=1}^{4} a_i \pi_\perp(\boldsymbol{\ell}_i) \in \Omega \right\}.$$

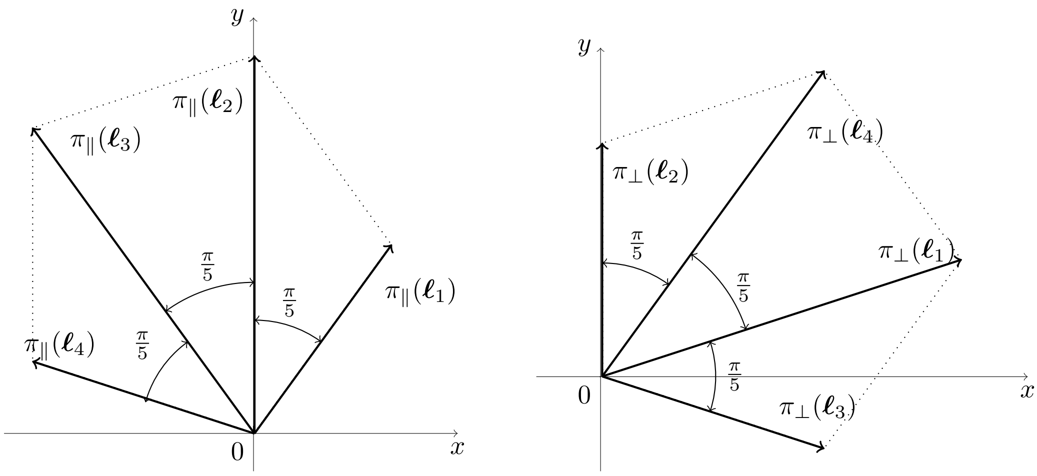


Figure 11.1: Projections $\pi_\parallel$ and $\pi_\perp$ of lattice generators.

It can be shown that we can rewrite $\Sigma(\Omega)$ into

$$\Sigma(\Omega) = \Bigg\{ (a+b\tau)\begin{pmatrix}1-\cos\frac{2\pi}{5}\\ \sin\frac{2\pi}{5}\end{pmatrix} + (c+d\tau)\begin{pmatrix}\cos\frac{4\pi}{5}-\cos\frac{2\pi}{5}\\ \sin\frac{2\pi}{5}-\sin\frac{4\pi}{5}\end{pmatrix} : a,b,c,d\in\mathbb{Z},$$
$$(a+b\tau')\begin{pmatrix}1-\cos\frac{4\pi}{5}\\ \sin\frac{4\pi}{5}\end{pmatrix} + (c+d\tau')\begin{pmatrix}\cos\frac{2\pi}{5}-\cos\frac{4\pi}{5}\\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5}\end{pmatrix}\in\Omega\Bigg\}.$$

11.2 Self-similarities of Λ_5

It was also proved in [14] and in [15] that each transformation Z of the lattice that preserves $\mathbb{Z}$-modules $\pi_\parallel(\mathcal{L})$ and $\pi_\perp(\mathcal{L})$ (which are in fact $\mathbb{Z}[\tau]$-modules of rank 2) is of the form

$$Z = \begin{pmatrix} a & b & c & d\\ e & f & g & h\\ -a-e+f-h & -b+g-h & -c-e+f & -d-e+g-h\\ a-b+d+e-f+h & -c+d-g+h & a-b+e-f & a-c+d+e-g+h\end{pmatrix}$$

where $a,b,c,d,e,f,g,h\in\mathbb{Z}$. This matrix can be brought into a block diagonal form with mappings A and B on the diagonal:

$$\begin{pmatrix}A & O\\ O & B\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22} & 0 & 0\\ 0 & 0 & B_{11} & B_{12}\\ 0 & 0 & B_{21} & B_{22}\end{pmatrix} \tag{11.2}$$

with coefficients

$$
\begin{aligned}
A_{11} &= a + b\cos\frac{2\pi}{5} + (c+d)\cos\frac{4\pi}{5},\\
A_{12} &= b\sin\frac{2\pi}{5} + (c-d)\sin\frac{4\pi}{5},\\
A_{21} &= \frac{1}{5}\left((2a-3b+2c+2d+8e-2f-2g-2h)\sin\frac{2\pi}{5}+\right.\\
&\quad\left.+(-6a+4b-c-d-4e+6f-4g-4h)\sin\frac{4\pi}{5}\right),\\
A_{22} &= -c+d+f+h+(2g-b)\cos\frac{2\pi}{5} + (-c+d+2h)\cos\frac{4\pi}{5},\\
B_{11} &= a + (c+d)\cos\frac{2\pi}{5} + b\cos\frac{4\pi}{5},\\
B_{12} &= (d-c)\sin\frac{2\pi}{5} + b\sin\frac{4\pi}{5},\\
B_{21} &= \frac{1}{5}\left((6a-4b+c+d+4e-6f+4g+4h)\sin\frac{2\pi}{5}+,\right.\\
&\quad\left.+(2a-3b+2c+2d+8e-2f-2g-2h)\sin\frac{4\pi}{5}\right).\\
B_{22} &= -c+d+f+h+(-c+d+2h)\cos\frac{2\pi}{5} + (2g-b)\cos\frac{4\pi}{5}.
\end{aligned}
\tag{11.3}
$$

In what will follow we rewrite the condition on the mapping Z (i.e. it has to preserve the two modules) in terms of its characteristic polynomial. Then we show that any such mapping Z corresponds to an element of a cut and project set with triangle window.

Proposition 11.1. *Let $\Lambda_5 = (\mathcal{L} \subset \mathbb{R}^4, \mathbb{R}^2)$ be a CPS with five-fold symmetry and let $Z \in \mathbb{Z}^{4\times4}$. Then Z is a self-similarity of $\mathcal{L}$ inducing linear mappings A, B, which preserve modules $\pi_{\parallel}(\mathcal{L})$ and $\pi_{\perp}(\mathcal{L})$ respectively if and only if $\chi_Z \in \mathbb{Z}[X]$ and $\chi_Z(X) = \chi(X)\cdot\chi'(X)$, where $\chi \in \mathbb{Z}[\tau][X]$ and $'$ is the non-trivial field automorphism of $\mathbb{Q}(\tau)$, i.e. $\chi(X) = X^2+pX+q$, $p, q \in \mathbb{Z}[\tau]$ and $\chi'(X) = X^2+p'X+q'$.*

Proof. Firstly, suppose that Z is a self-similarity of $\mathcal{L}$ preserving the modules. Then according to Proposition 7.5 it holds that

$$Z \sim \begin{pmatrix} A & O\\ O & B\end{pmatrix}$$

and therefore $\chi_Z(X) = \chi_A(X)\cdot\chi_B(X)$. Since $\pi_{\parallel}(\mathcal{L})$ can be expressed as $\operatorname{span}_{\mathbb{Z}[\tau]}\{v_1, v_2\}$ where

$$v_1 = \begin{pmatrix} 1-\cos\frac{2\pi}{5}\\ \sin\frac{2\pi}{5}\end{pmatrix}, \qquad v_2 = \begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5}\\ \sin\frac{2\pi}{5}-\sin\frac{4\pi}{5}\end{pmatrix}$$

and A is a self-similarity of this module, it holds that

$$A(v_1) = \alpha v_1 + \beta v_2, \qquad A(v_2) = \gamma v_1 + \delta v_2$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[\tau]$. It is clear that the entries of the matrix A written in the basis (v_1, v_2) belong to $\mathbb{Z}[\tau]$ and therefore $\chi_A \in \mathbb{Z}[\tau][X]$.

The same argument can be used for B. From the construction it also follows that $A'_{ij} = B_{ij}$ so $\chi_B(X) = \chi'_A(X)$.

On the other hand suppose that $\chi_Z(X) = \chi(X)\cdot\chi'(X)$ with $\chi(X) \in \mathbb{Z}[\tau][X]$ Without loss of generality we can assume that

$$\chi(X) = X^2 + pX + q, \qquad p, q \in \mathbb{Z}[\tau].$$

Then we define

$$A = \begin{pmatrix} 0 & 1\\ -q & -p\end{pmatrix},$$

which is clearly a self-similarity of the $\mathbb{Z}[\tau]$-module $\pi_{\|}(\mathcal{L})$. Analogously we define the mapping B preserving $\pi_{\perp}(\mathcal{L})$. Finally using Proposition 7.3 we obtain the matrix

$$L^{-1}\begin{pmatrix} A & O \\ O & B \end{pmatrix} L = Z \in \mathbb{Z}^{4\times 4}$$

where L is a matrix corresponding to the lattice $\mathcal{L}$. □

Proposition 11.2. *Let $\Lambda_5 = (\mathcal{L} \subset \mathbb{R}^4, \mathbb{R}^2)$ be a CPS with five-fold symmetry. Let Z be a self-similarity of $\mathcal{L}$ inducing linear mappings A, B, which preserve modules $\pi_{\|}(\mathcal{L})$ and $\pi_{\perp}(\mathcal{L})$ respectively. Let us denote $\chi_Z = \chi_A \cdot \chi_A'$ with $\chi_A(X) = X^2 + pX + q$. Then there is a window $\Omega \subset \mathbb{R}^2$ such that Z induces a mapping A that preserves $\Sigma(\Omega)$ if and only if*

$$(p, q) \in \left\{(x, y) \in (\mathbb{Z}[\tau])^2 \ : \ (x', y') \in \triangle\right\}$$

where $'$ is the non-trivial automorphism of the field $\mathbb{Q}(\tau)$ and $\triangle$ denotes the convex hull of vectors $(0, -1)$, $(2, 1)$, $(-2, 1)$.

Proof. The mapping Z induces the self-similarity A of a cut and project set $\Sigma(\Omega)$ if and only if the mapping A preserves the $\mathbb{Z}[\tau]$-module $\pi_{\|}(\mathcal{L})$ and $B\Omega \subset \Omega$. By Proposition 10.1 the eigenvalues of the corresponding mapping B must be in modulus smaller then or equal to 1. According to our assumptions $\chi_B(X) = X^2 + p'X + q'$ so it holds that the eigenvalues λ_i of B are of the form

$$\lambda_i = \frac{-p' \pm \sqrt{p'^2 - 4q'}}{2}. \tag{11.4}$$

Using Viète's formulas one gets the following condition: $1 \geq |\lambda_1 \lambda_2| = |q'|$. Now we add another constraint according to the sign of the discriminant of $\chi_B(X)$. If $p'^2 - 4q' < 0$, no other condition appears. If $p'^2 - 4q' \geq 0$, then we get from $-1 \leq \lambda_i \leq 1$ using (11.4) the following constraint:

$$-p' + \sqrt{p'^2 - 4q'} \leq 2, \qquad -p' - \sqrt{p'^2 - 4q'} \geq -2.$$

These can be rewritten as:

$$\sqrt{p'^2 - 4q'} \leq 2 + p', \tag{11.5}$$

$$\sqrt{p'^2 - 4q'} \leq 2 - p'. \tag{11.6}$$

If $p' > 0$ then we get from (11.6) inequalities $0 \leq p' \leq 2$ and $q' \geq p' - 1$. If $p' < 0$ then we obtain from (11.5) restrains $-2 \leq p' \leq 0$ and $q' \geq -p' - 1$. We get the set of inequalities which describe all possible pairs (p', q'), i.e. we obtain the desired cut and project set $\Sigma(\triangle)$ with window

$$\triangle = \left\{(x, y) \ : \ \frac{x^2}{4} < y \leq 1\right\} \cup \left\{(x, y) \ : \ -1 \leq y \leq \frac{x^2}{4}, y \geq |x| - 1\right\},$$

see Figure 11.2 □

Let us focus on the subclass of all mappings Z, Z' that satisfy $ZZ' = Z'Z$. It was shown [14] that this subclass contains all matrices of the form $Z = \wp(C)$ where $\wp \in \mathbb{Z}[X]$. Moreover it can be easily shown that it is sufficient to consider only the polynomial of degree smaller then or equal to 3, i.e.

$$Z = aI + bC + cC^2 + dC^3$$

for $a, b, c, d \in \mathbb{Z}$. It can be shown, that the resulting mappings A, B are of the form

$$A = \begin{pmatrix} \begin{matrix} a+b\cos\frac{2\pi}{5}+ \\ +(c+d)\cos\frac{4\pi}{5} \end{matrix} & \begin{matrix} b\sin\frac{2\pi}{5}+ \\ +(c-d)\sin\frac{4\pi}{5} \end{matrix} \\ \begin{matrix} -b\sin\frac{2\pi}{5}+ \\ +(d-c)\sin\frac{4\pi}{5} \end{matrix} & \begin{matrix} a+b\cos\frac{2\pi}{5}+ \\ +(c+d)\cos\frac{4\pi}{5} \end{matrix} \end{pmatrix},$$

$$B = \begin{pmatrix} \begin{matrix} a+b\cos\frac{4\pi}{5}+ \\ +(c+d)\cos\frac{2\pi}{5} \end{matrix} & \begin{matrix} b\sin\frac{4\pi}{5}+ \\ +(c-d)\sin\frac{2\pi}{5} \end{matrix} \\ \begin{matrix} -b\sin\frac{4\pi}{5}+ \\ +(d-c)\sin\frac{2\pi}{5} \end{matrix} & \begin{matrix} a+b\cos\frac{4\pi}{5}+ \\ +(c+d)\cos\frac{2\pi}{5} \end{matrix} \end{pmatrix}.$$

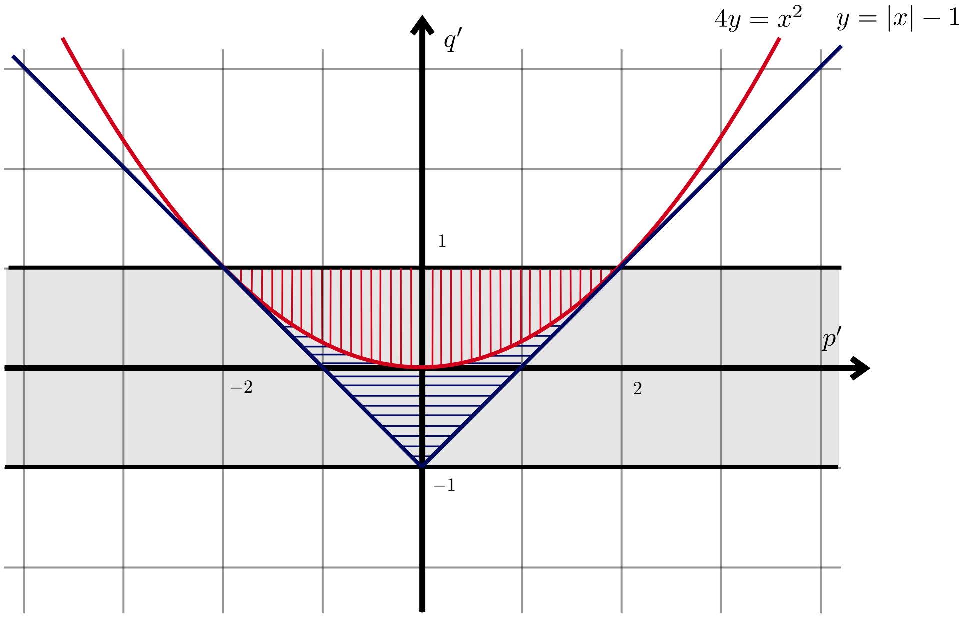


Figure 11.2: The bounded set $\triangle$ as the acceptance window for the cut and project set of self-similarities of the $\mathbb{Z}$-module $\pi_{\|}(\mathcal{L})$.

Their corresponding eigenvalues can be written in the following form

$$\lambda_1 = \alpha + \beta\omega + \gamma\omega^2 + \delta\omega^3$$
$$\lambda_4 = \psi_4(\lambda_1) = \alpha + \beta\omega^4 + \gamma\omega^3 + \delta\omega^2$$

$$\lambda_2 = \psi_2(\lambda_1) = \alpha + \beta\omega^2 + \gamma\omega^4 + \delta\omega$$
$$\lambda_3 = \psi_3(\lambda_1) = \alpha + \beta\omega^3 + \gamma\omega + \delta\omega^4$$

where $\omega = e^{\frac{2\pi i}{5}}$ and ψ_i denotes the i-th Galois automorphism of $\mathbb{Q}(\omega)$, $\psi_i : \omega \mapsto \omega^i$. It also holds that $\lambda_1 = \overline{\lambda_4}$ and $\lambda_2 = \overline{\lambda_3}$. More details about this field and its properties are described in Chapter 3.

The transformation Z is diagonalizable and according to [14] it induces A that is a self-similarity of some $\Sigma(\Omega)$ if and only if $|\lambda_2| = |\lambda_3| \leq 1$. In the following text we classify all possible cases according to the index $[\mathbb{Q}(\lambda_1) : \mathbb{Q}]$:

- If $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 4$, then χ_Z is irreducible over $\mathbb{Q}$ and the corresponding mappings A, B are scaled rotations of $\mathbb{Z}[\tau]$-modules $\pi_\perp(\mathcal{L})$ and $\pi_{\|}(\mathcal{L})$. Let us impose the condition $|\lambda_2| \leq 1$ and $|\lambda_3| \leq 1$. Denote by $r \in \mathbb{Z}$ the absolute coefficient of the polynomial $\chi_Z \in \mathbb{Z}[X]$. Using Viète's formulas it holds that $r = \lambda_1\lambda_2\lambda_3\lambda_4$. This gives us the following estimation

$$|\lambda_1\lambda_4| = \frac{|r|}{|\lambda_2\lambda_3|} \geq |r| \geq 1.$$

  Since $\lambda_1 = \overline{\lambda_4}$ we can derive that $|\lambda_1\overline{\lambda_1}| = |\lambda_4\overline{\lambda_4}| \geq 1$. Finally we can conclude that if $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 4$, then Z is self-similarity of $\Sigma(\Omega)$ if and only if its eigenvalue λ_1 is a complex Pisot number.

- If $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 2$, then according to Proposition 11.1 we get $\chi_Z = \chi_A \cdot \chi'_A$. Due to the assumption on the index of $\mathbb{Q}(\lambda_1)$ it holds that the roots of χ_A generate $\mathbb{Q}(\tau)$ which is the only non-trivial subfield of $\mathbb{Q}(\omega)$, see Chapter 3 for details. An easy calculation proves that $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_3$. This shows that in this case Z induces scaling of both modules $A = \lambda_1 I$, $B = \lambda_2 I$. Next, it is clear that $\chi_Z = \chi^2$, where $\chi \in \mathbb{Z}[X]$ is the minimal polynomial over $\mathbb{Q}$ of $\lambda_1 \in \mathbb{Z}[\tau]$. The image of λ_1

under the non-trivial field automorphism $'$ is λ_2. Imposing the condition $|\lambda_2| \leq 1$ and assuming that $\chi[X] = X^2 + pX + q$ we derive in the same way that $|\lambda_1| \geq 1$ and the equality is reached for $q = \pm 1$. Finally we can conclude that if $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 2$, then Z is self-similarity of $\Sigma(\Omega)$ if and only if its eigenvalue λ_1 is a Pisot (or negative Pisot) number in $\mathbb{Z}[\tau]$.

- If $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 1$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = p \in \mathbb{Z}$. If we impose the conditions as above we can conclude, that Z is self-similarity of $\Sigma(\Omega)$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \pm 1$.

Proposition 11.3. *Let $\Lambda_5 = (\mathcal{L} \subset \mathbb{R}^4, \mathbb{R}^2)$ be the cut and project scheme with five-fold symmetry. Let $Z = aI + bC + cC^2 + dC^3$ and let λ_1 be its eigenvalue. Then either*

- $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 4$, *and Z induces a self-similarity $A = \begin{pmatrix} \mathtt{Re}\,\lambda_1 & -\mathtt{Im}\,\lambda_1 \\ \mathtt{Im}\,\lambda_1 & \mathtt{Re}\,\lambda_1 \end{pmatrix}$ of $\Sigma(\Omega)$ if and only if λ_1 is a complex Pisot number, or*

- $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 2$, *and Z induces a self-similarity $A = \lambda_1 I$ of $\Sigma(\Omega)$ if and only if λ_1 is a Pisot (or negative Pisot) number in $\mathbb{Z}[\tau]$, or*

- $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] = 1$, *and Z induces a self-similarity $A = \pm I$ of $\Sigma(\Omega)$ if and only if $\lambda_1 = \pm 1$.*

It remains to discuss the transformations Z that are not of the form $\wp(C)$. Their characteristic polynomials χ_Z are of the degree 4 and their roots (i.e. the eigenvalues of Z) belong to some quadratic extension K of $\mathbb{Q}(\tau)$. This holds because due to Proposition 11.1 we know that χ_Z splits into the product of two conjugated polynomials $\chi, \chi' \in \mathbb{Z}[\tau][X]$. Opposite statement holds as well, i.e. for each quadratic extension K of $\mathbb{Q}(\tau)$ we can find a lattice transformation Z and thus a self-similarity of $\pi_{\|}(\mathcal{L})$.

Proposition 11.4. *Let K be a quadratic extension of $\mathbb{Q}(\tau)$, i.e. $[K : \mathbb{Q}(\tau)] = 2$. Denote by $\mathcal{O}_K = K \cap \mathbb{B}$ the ring of integers in this field. Then for all fields K satisfying the property above and for all $\lambda \in \mathcal{O}_K$ there exists $Z \in \mathbb{Z}^{4\times 4}$ such that Z induces a self-similarity of $\pi_{\|}(\mathcal{L})$.*

Proof. Let λ be a root of $X^2 + pX + q \in \mathbb{Q}(\tau)[X]$. Then $(X^2 + pX + q)(X^2 + p'X + q') \in \mathbb{Q}[X]$ is an irreducible polynomial over $\mathbb{Q}$. Since $\lambda \in \mathcal{O}_K$, i.e. λ is an algebraic integer, its minimal polynomial $f \in \mathbb{Z}[X]$. Since the minimal polynomial is unique we conclude that $f(X) = (X^2+pX+q)(X^2+p'X+q') \in \mathbb{Z}[X]$. This implies that $g(X) = X^2 + pX + q \in \mathbb{Z}[\tau][X]$.

As a lattice transformation Z let us choose a companion matrix to the polynomial f. This matrix satisfies $\chi_Z = f = g \cdot g'$ and according to Proposition 11.1 transformation Z is a self-similarity of $\mathcal{L}$ corresponding to Λ_5. $\square$

11.3 General self-similarity preserving circular window

In this subsection we derive a necessary and sufficient condition on matrix B so that it preserves a circular window $\Omega = B(\mathbf{0}, 1)$. We already know from Proposition 10.1 that if $B\Omega \subset \Omega$, then the spectral radius of B, i.e. the highest modulus of its eigenvalues, satisfies $\varrho(B) \leq 1$. But this condition is not sufficient as it was shown in detail in [14]. As a counterexample one takes the following lattice transformation T

$$T = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix can be brought into a block diagonal matrix of the form (11.2) using the corresponding matrix L to the lattice $\mathcal{L}$.. Then the corresponding mappings R, S related to T by $LTL^{-1} = \begin{pmatrix} R & O \\ O & S \end{pmatrix}$

are of the form

$$R = \begin{pmatrix} -\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} + 1 \end{pmatrix} = \begin{pmatrix} -\tau - \frac{1}{2\tau} & -\frac{1}{2}\sqrt{\tau^2+1} \\ \frac{1}{2}\sqrt{\tau^2+1} & \frac{1}{2\tau} + 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 2\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5} & -\sin\frac{4\pi}{5} \\ \sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} + 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau} + \frac{\tau}{2} & -\frac{1}{2\tau}\sqrt{\tau^2+1} \\ \frac{1}{2\tau}\sqrt{\tau^2+1} & 1 - \frac{\tau}{2} \end{pmatrix}.$$

According to Proposition 7.5 we get a self-similarity of CPS Λ_5 which is not a rotation or scaled rotation.

Let us study the action of the mapping S. Its eigenvalues are

$$\eta_1 = \frac{1}{\tau}, \qquad \eta_2 = 1,$$

with the corresponding eigenvectors

$$\boldsymbol{s}_1 = \begin{pmatrix} -\sin\frac{4\pi}{5} \\ \cos\frac{4\pi}{5} \end{pmatrix}, \qquad \boldsymbol{s}_2 = \begin{pmatrix} \cos\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} \end{pmatrix}.$$

Since the eigenvalues of the matrix S are in modulus ≤ 1, by Proposition 10.3, there exists a window Ω such that the cut and project set $\Sigma(\Omega)$ has self-similarity S. But it turns out that this window is not the circular one. As a suitable S-invariant window Ω one can choose for example an ellipse

$$\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,:\, x^2 - 2xy\sin\frac{2\pi}{5} + y^2 \leq 1 \right\}.$$

Thus requiring a preservation of the circular window one has to impose on B that every $\boldsymbol{x} \in \mathbb{R}^2$ of norm $\|\boldsymbol{x}\| \leq 1$ satisfies

$$\|B\boldsymbol{x}\| \leq 1. \tag{11.7}$$

Note that $\|\cdot\|$ is standard Euclidean norm on $\mathbb{R}^2$. But the condition (11.7) is equivalent to a requirement

$$\|B\| \leq 1, \tag{11.8}$$

i.e. the norm of the matrix B is smaller or equal to one. Recall the definition of matrix norm:

$$\|B\| := \sup_{\|\boldsymbol{x}\|=1} \|B\boldsymbol{x}\|.$$

More details about matrix norms can be found in Chapter 4 or in [7], [10].

Note that this approach can be slightly generalized as the following proposition states.

Proposition 11.5. *Let $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ be a CPS such that there is $A \in \mathbb{R}^{n\times n}$ satisfying $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$. Let Ω be of the form*

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^{s-n} : \|\boldsymbol{x}\|_\bullet \leq 1 \right\}$$

where $\|\cdot\|_\bullet$ denotes an arbitrary norm on $\mathbb{R}^{s-n}$. Then A is a self-similarity of $\Sigma(\Omega)$ if and only if

$$\|B\|_\bullet^{\text{ind}} \leq 1,$$

where $\|\cdot\|_\bullet^{\text{ind}}$ denotes a matrix norm induced by vector norm $\|\cdot\|_\bullet$.

Recall form Theorem 4.12 that the matrix norm induced by the Euclidean norm is the so-called *spectral norm* defined as

$$\|B\| = \sqrt{\varrho(B^\star B)},$$

where $B^\star$ is Hermitian conjugate of B. The considerations above can be summarized as follows

Proposition 11.6. *Let Λ_5 be a CPS with five-fold symmetry and let $A\pi_{\|}(\mathcal{L}) \subset \pi_{\|}(\mathcal{L})$. Then the mapping B preserves the ball $B_1(\mathbf{0})$ if and only if*

$$\|B\| = \sqrt{\varrho(B^\star B)} \leq 1. \tag{11.9}$$

We reformulate this condition in terms of traces and determinants of some matrices. Firstly let us calculate the eigenvalues of $B^\star B$. The characteristic polynomial of $B^\star B$ is of the form

$$\chi_{B^\star B}(X) = X^2 - (\operatorname{tr} B^\star B)X + \det B^\star B.$$

Thus its eigenvalues are

$$\lambda = \frac{\operatorname{tr} B^\star B \pm \sqrt{(\operatorname{tr} B^\star B)^2 - 4\det B^\star B}}{2}.$$

Since $B^\star B$ is obviously Hermitian matrix it holds (due to [25, Theorem 8.1]) that its spectrum $\sigma(B^\star B)$ is real. So the condition (11.9) says:

$$\begin{aligned} \frac{\operatorname{tr} B^\star B \pm \sqrt{(\operatorname{tr} B^\star B)^2 - 4\det B^\star B}}{2} &\leq 1, \\ (\operatorname{tr} B^\star B)^2 - 4\det B^\star B &\leq (2 - \operatorname{tr} B^\star B)^2. \end{aligned}$$

Finally we obtain an equivalent condition to requirement (11.9)

$$\operatorname{tr} B^\star B \leq 1 + \det B^\star B. \tag{11.10}$$

Denote the entries of B as follows

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We rewrite the last inequality for matrix B in terms of $\operatorname{tr} B$, $\det B$ and *antitrace* $\operatorname{atr} B$ which is defined as

$$\operatorname{atr} B := b_{12} - b_{21}.$$

Since $B \in \mathbb{R}^{2\times 2}$, it holds that $B^\star B = B^T B$ where B^T denotes the transpose of B. Thus for

$$B^\star B = B^T B = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}^2 + b_{21}^2 & b_{11}b_{12} + b_{22}b_{21} \\ b_{11}b_{12} + b_{22}b_{21} & b_{22}^2 + b_{12}^2 \end{pmatrix}$$

it holds that

$$\operatorname{tr} B^T B = b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 = (\operatorname{tr} B)^2 + (\operatorname{atr} B)^2 - 2\det B,$$

and we know that $\det B^T B = (\det B)^2$. Using these relations one gets an alternative to (11.10) in quite elegant form

$$(\operatorname{tr} B)^2 + (\operatorname{atr} B)^2 \leq (1 + \det B)^2.$$

We thus have derived the following statement.

Proposition 11.7. *The following conditions are equivalent:*

(i) $\|B\| = \sqrt{\varrho(B^\star B)} \leq 1$*,*

(ii) $(\operatorname{tr} B)^2 + (\operatorname{atr} B)^2 \leq (1 + \det B)^2$*,*

(iii) $\operatorname{tr} B^\star B \leq 1 + \det B^\star B$*.*

Chapter 12

Conclusion

In this work we focused mainly on the question to decide, for a given $A \in \mathrm{Gl}(n, \mathbb{R})$, whether there exists a cut and project set $\Sigma(\Omega) \subset \mathbb{R}^n$ such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$. This question is related to deciding whether there exists a cut and project scheme $\Lambda = (\mathcal{L} \subset \mathbb{R}^s, \mathbb{R}^n)$ such that $A\pi_{\parallel}(\mathcal{L}) \subset \pi_{\parallel}(\mathcal{L})$.

Let us summarize the content of this work. This work is divided into three parts. The first one provides a theoretical introduction into this topic together with some mathematical tools that are not commonly known. The first and the second chapter are dedicated to the number theory, and Galois theory respectively. Special attention is paid on description of Galois field $\mathbb{Q}(\xi_{20})$ and its subfields which is used later in Chapter 11 and enables a classification of certain mappings. Chapter 4 gives an introduction into the theory of Jordan forms. We recall different Jordan forms and their properties. Then the Jordan decomposition is described and the concept of generalized eigenspaces is presented. In the next section, Section 4.5 we defined a special matrix annihilating polynomial called the minimal polynomial over a field $\mathbb{Q}$. Using this polynomial we could set up some conditions on the mapping B in order to get generic cut and project scheme (Section 7.2). Chapter 5 provides a basic introduction into a discrete geometry. All necessary definitions and their properties age given as well as references to suitable literature. Section 5.2 is devoted to the term “mathematical quasicrystal". Our aim was to point out some properties of such structures as well as methods that are commonly used in order to construct them. We have decided to use the so-called cut and project method and therefore Section 5.3 brings the important information about this procedure. Its possible generalizations are discussed in Section 5.4. More detailed information related to the issue of cut and project sets, especially those with self-similarity, are described in Chapter 6. This chapter also brings a short summary of the most important articles related to this topic.

In the second part of this work a construction of a cut and project scheme with a given linear self-similarity A is built up. For this purpose we developed a matrix formalism that enables us to construct cut and project schemes with a given self-similarity (Proposition 7.5) and we also derived conditions that need to be met if one requires the scheme to have certain property, e.g. non-degeneracy, aperiodicity etc., (Proposition 7.2). In Section 7.3 we defined an operation on the set of all generic cut and project schemes that enables us to combine two and more schemes together, thus creating the new one. In this Section we proved that various assumptions can be posed without loss of generality on the mapping A.

Then, in Chapter 8, we built up a construction of cut and project scheme with given linear diagonalizable self-similarity A. This construction is based on the decomposition of the spectrum $\sigma(A)$ into conjugacy classes, i.e. sets whose elements are distinct algebraic conjugates. We showed that we can, for each of these conjugacy classes, find the so-called elementary cut and project scheme (Section 8.1) and we showed that these schemes are generic (Theorem 8.6). It was necessary to treat some special cases separately, which has been done in Section 8.2. We also derived an estimation on the dimension of the resulting lattice based on a combinatorial idea (Theorem 8.14). This estimation gives good results, but for some cases it is not sharp. For determining the minimal dimension s of the lattice we needed to consider a finer approach that is introduced in Section 8.4. In a similar way we proceeded in Chapter 9 where the construction of cut and project scheme with non-diagonalizable self-similarity A is given (Theorem 9.6). Thus we answered the question posed in the beginning of the work that is to decide whether there exists a cut and project scheme that is invariant under a given linear mapping A. We answered the question in positive for any A whose eigenvalues are algebraic integers (Theorems 8.14 and 9.6). For the diagonalizable self-similarity we provide a construction with the minimal dimension of the lattice (Theorem 8.15). It remains to find the minimal dimension for the non-diagonalizable case but it seem us that a different approach to that problem should be used.

Chapter 10 gives an answer to the second question, i.e. whether there exists a cut and project set invariant under a given self-similarity A. In the case of general mapping A (and thus through the construction induced mapping B) we obtained a sufficient condition on the mapping B (Proposition 10.2). In the case of diagonalizable mappings we derived necessary and sufficient condition on A (Theorem 10.4) and we showed that this theorem can be understood as an extension of Lagarias's result for a special subclass of Meyer sets.

The last part of this work is focused on cut and project scheme with 5-fold symmetry. After its brief description is given (Section 11.1), we use an algebraic approach to treat the possible self-similarities of this cut and project scheme and show that there exists a bijection between a certain two-dimensional cut and project set and the set of all linear self-similarities of the quasicrystal (Proposition 11.2). For a specific subclass of commuting self-similarities we provide full classification of them based on their eigenvalues (Proposition 11.3). The last subsection is dedicated to self-similarities that preserve a circular window. We provide a complete description (Proposition 11.6) of possible linear self-similarities of a cut and project set with circular acceptance window using matrix norms. This approach can be used slightly generally as it was shown in Proposition 11.5.

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