Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Lieb-Thirring inequalities for the damped wave equation

Lieb-Thirringovy nerovnosti pro vlnovou rovnici s útlumem

Master's Thesis

Author: **Bc. Tereza Kurimaiová**

Supervisor: **doc. Mgr. David Krejčiřík, PhD., DSc.**

Academic year: 2018/2019

- Zadání práce -

- Zadání práce (zadní strana) -

Acknowledgment:
I would like to thank doc. D. Krejčiřík for his enormous patience and expert guidance. I would also like to thank B. Bobsonson and M. Tichý for their huge support and love.

Prohlášení:
Prohlašuji, že jsem svou diplomovou práci vypracovala samostatně a použila jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.
Nemám závažný důvod proti použití tohoto školního díla ve smyslu § 60 Zákona č. 121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Praze dne 6. května 2019 Tereza Kurimaiová

Název práce:

Lieb-Thirringovy nerovnosti pro vlnovou rovnici s útlumem

Autor: Tereza Kurimaiová

Obor: Matematická fyzika

Druh práce: Diplomová práce

Vedoucí práce: doc. Mgr. David Krejčiřík, PhD., DSc., Katedra matematiky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Abstrakt: Operátorý přístup k vlnové rovnici s útlumem nám poskytuje jednoznačnost a regularitu jejích řešení, jelikož operátor asociovaný s touto rovnicí generuje C_0-semigrupu. Z chování spektra jsme navíc schopni získat informace o stabilitě těchto řešení. Díky korespondenci mezi spektrem tohoto operátoru a Schrödingerova operátoru obdržíme množství odhadů na vlastní čísla a kritéria pro jejich existenci a absenci i v případě komplexního útlumu. Tyto výsledky jsou demonstrovány na analyticky řešitelném případě kdy je útlum tvaru konečné obdélníkové jámy.

Klíčová slova: Birman-Schwingerův princip, C_0-semigrupa, Lieb-Thirringovy nerovnosti, operátor vlnové rovnice s útlumem

Title:

Lieb-Thirring inequalities for the damped wave equation

Author: Tereza Kurimaiová

Abstract: The operator approach for the damped wave equation provides us with uniqueness and regularity of its solutions since the operator generates a C_0-semigroup. Moreover the behavior of its spectrum gives us information about the stability of the solutions. Using the correspondence between the spectra of this operator and the Schrödinger operator we obtain numerous bounds on the eigenvalues and criteria for their existence or absence even in the case of complex damping. These results are demonstrated on the analytically computable case when the damping is a finite rectangular well.

Key words: Birman-Schwinger principle, C_0-semigroup, damped wave operator, Lieb-Thirring inequalities

Contents

Introduction

Consider the second order partial differential equation

$$u_{tt} + au_t - \Delta u = 0$$

for some function u where a is an arbitrary function and $-\Delta u = -\sum_{i=1}^{d} u_{x_i x_i}$ with d being the dimension of the space. Let us call it the damped wave equation with the damping a. It can be used to model various physical systems in a more realistic manner than can be achieved by the standard wave equation without any damping. It describes namely the vibrations of an elastic string, membrane or any other object in a viscous liquid or some other medium which affects the vibration. It is a special case of the telegraph equations which describe the current and voltage on an electrical transmission line and moreover it is being used in relativistic quantum mechanics and cosmology.

In this thesis we aim to provide spectral bounds for the non-self-adjoint operator associated with the damped wave equation, the damped wave operator. This will be done using the correspondence between this operator and the self-adjoint Schrödinger operator. The behavior of the spectrum the provides some information about the time evolution and stability of the solutions of the damped wave equation.

In the first chapter we properly define the damped wave operator. First we analyze the explicitly computable example of vibrations of a string with constant damping. We show that in this case the operator generates a C_0-semigroup which implies that the solutions of the equation are generated by this C_0-semigroup and also that they are unique and sufficiently regular. Moreover using the growth bound of the C_0-semigroup we are able to obtain a uniform optimal damping for which the system returns to equilibrium in the shortest time. This example serves as a motivation for the next part of the chapter where we define the damped wave operator on an arbitrary domain in $\mathbb{R}^d$ and with bounded damping. We again show that it generates a C_0-semigroup and finally we state some results on the stability of the solutions.

In the second chapter we define the Schrödinger operator as a bounded perturbation of the Dirichlet Laplacian by some real potential. We state some of its spectral properties which will be needed in the next chapter. Then we parameterize the potential of the Schrödinger operator and establish the formulas for the first and second derivative of the first eigenvalue with respect to the parameter. We provide an example of the behavior of the spectra with respect to the parameter when the potential is the finite rectangular well. At the end of this chapter we state Theorem 2.4.1 which gives us the connection between the spectrum of the Schrödinger operator and the damped wave operator.

In the third and most important chapter of the thesis we establish our own results for the damped wave operator using the well-known results for the Schrödinger operator. The obtained results consist of proving the absence of some part of the spectrum and of upper and lower bounds for the eigenvalues. In particular the Lieb-Thirring inequalities were used to prove Theorems 3.1.1 and 3.1.3. Moreover to obtain Theorems 3.2.1, 3.2.2 and 3.2.3 we employed the Buslaev-Faddeev-Zakharov trace formulae. Finally we lowered the assumptions on the damping enabling it to be complex but still bounded and we established the Birman-Schwinger principle for the damped wave operator, Theorem 3.3.1, using which we were able to obtain further results, namely Theorems 3.3.2 and 3.3.3, some of them generalizing the previous ones.

In the final part of the thesis we consider the damped wave operator with the potential being the finite potential well and compute the implicit equation for its eigenvalues. Using this we plot the behavior of the point spectra when the well becomes more deep or narrow. Moreover we plot the dependence of the bounds obtained in the previous chapter and compare them with the numerically computed values of the eigenvalues. Finally we provide a numerical evidence that the bound of Theorem 3.1.1, respectively 3.1.1 is sharp when the well is taken infinitely deep and narrow which can be interpreted as the Dirac delta function. This corresponds to the well-known result for the Schrödinger operator.

Chapter 1

Damped wave operator

1.1 Derivation

In this section we introduce the operator approach for the damped wave equation. Let $\Omega \subset \mathbb{R}^d$ be an arbitrary domain and let $a : \Omega \to \mathbb{R}$ be a damping function. Consider a system governed by the damped wave equation for some function u subject to the initial conditions, i.e.

$$\begin{aligned} u_{tt} + au_t - \Delta u = 0, &\quad \text{in}\, \Omega, \quad t > 0 \\ u = u_1, &\quad \text{in}\, \Omega, \quad t = 0 \\ u_t = u_2, &\quad \text{in}\, \Omega, \quad t = 0 \end{aligned} \tag{1.1}$$

where $-\Delta u = -\sum_{i=1}^{d} u_{x_i x_i}$. Moreover we impose the Dirichlet boundary condition

$$u = 0 \quad \text{on}\, \partial\Omega, \quad t > 0. \tag{1.2}$$

To cast this system into the operator form it is customary to use the Hilbert space

$$\mathcal{H} := \big(H_0^1(\Omega) \times L^2(\Omega),\, (\cdot,\cdot)_{\mathcal{H}}\big) \tag{1.3}$$

where the Sobolev space $H_0^1(\Omega)$ is the closure of the subset of smooth functions with compact support $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$, a member of the family of the Sobolev spaces

$$H^k(\Omega) := \big(\{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega),\, \forall |\alpha| \le k\},\, (\cdot,\cdot)_{H^k}\big)$$

where D^α stands for the weak derivative of order $k \in \mathbb{N}$. The inner product $(\cdot,\cdot)_{H^k}$ is defined as

$$(f,g)_{H^k} := \sum_{|\alpha| \le k} (D^\alpha f, D^\alpha g)$$

where $(\cdot,\cdot)$ denotes the standard inner product on $L^2(\Omega)$. Accordingly the norm on $L^2(\Omega)$ will be denoted by $\|\cdot\|$, i.e. without any subscript. However the same notation will also be used for the operator norm, the two to be distinguished from the context.

$H^k(\Omega)$ is complete and therefore a Hilbert space. Hence the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ are complete and equipped with the inner product

$$(f,g)_{H^1} = (\nabla f, \nabla g) + (f,g) = \int_\Omega \nabla \overline{f} \nabla g + \overline{f} g. \tag{1.4}$$

Note that $H_0^1(\mathbb{R}^d) = H^1(\mathbb{R}^d)$ (see [1, Corollary 3.23]). The inner product on the whole space $\mathcal{H}$ is just the inner product on the Cartesian product of the two Hilbert spaces, i.e.

$$(\Psi, \Phi)_{\mathcal{H}} := \left(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)_{\mathcal{H}} = \int_\Omega \nabla\overline{\psi}_1 \nabla\phi_1 + \overline{\psi}_1\phi_1 + \overline{\psi}_2\phi_2. \tag{1.5}$$

Next denoting

$$U_0 := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad U(t) := \begin{pmatrix} u \\ u_t \end{pmatrix} \tag{1.6}$$

we can formally write

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \begin{pmatrix} u_t \\ u_{tt} \end{pmatrix} = \begin{pmatrix} u_t \\ -au_t + \Delta u \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & -a \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix}$$

where I is the identity operator on $L^2(\Omega)$ and thus we obtain an evolution problem (or abstract Cauchy problem) with a matrix valued operator

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \begin{pmatrix} 0 & I \\ \Delta & -a \end{pmatrix} U(t), \quad U(0) = U_0.$$

Motivated by the formal derivation we define the damped wave operator $\mathcal{A}$ on $\mathcal{H}$ as

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ \Delta & -a \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}) := \mathrm{Dom}(-\Delta) \times H_0^1(\Omega) \tag{1.7}$$

where the definition by matrix means (and in the whole thesis would mean)

$$\mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \psi_2 \\ \Delta\psi_1 - a\psi_2 \end{pmatrix}$$

for $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathrm{Dom}(\mathcal{A})$. Moreover the differential Laplace operator Δ is meant in the weak sense and $\mathrm{Dom}(-\Delta)$ stands for the domain of the self-adjoint Dirichlet Laplacian $\mathcal{T}$ defined in Section 2.1 below. The evolution problem for $\mathcal{A}$ which is thanks to (1.6) associated with (1.1), (1.2) and in which we are thus interested is

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathcal{A}U(t), \quad U(0) = U_0. \tag{1.8}$$

1.2 Bounded $\Omega \subset \mathbb{R}$ and constant damping $a \geq 0$

As a motivation for the next section we will now analyze the explicitly computable case when Ω is a bounded domain in $\mathbb{R}$, to be precise an interval $(0, L)$ and where the damping coefficient a is a positive constant. This can be the model for example for the damped vibrations of a string of length L with fixed edges.

Our aim is to show that the operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup of contractions and then see what it implies primarily for the time evolution. For this it is necessary to take a different inner product in $\mathcal{H}$ than the standard (1.5). More precisely we define a new inner product in $H_0^1(0, L)$ as

$$(f, g)_{H_0^1} := \int_0^L \frac{\mathrm{d}\overline{f}}{\mathrm{d}x}\frac{\mathrm{d}g}{\mathrm{d}x} = \left(\frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}g}{\mathrm{d}x} \right). \tag{1.9}$$

The norm $\|\cdot\|_{H_0^1}$ corresponding to this inner product is equivalent to the norm $\|\cdot\|_{H^1}$ induced by inner product (1.4) which is inherited from $H^1(0,L)$. Indeed using the Poincaré inequality

$$\lambda_1 \|f\|^2 \leq \left\| \frac{\mathrm{d}f}{\mathrm{d}x} \right\|^2, \quad \forall f \in H_0^1(0,L) \tag{1.10}$$

where $\lambda_1 > 0$ is the first eigenvalue of the self-adjoint Dirichlet Laplacian $\mathcal{T}$ in one dimension and therefore the lowest eigenvalue λ of the following eigenvalue problem

$$\begin{aligned} -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi = \lambda\psi, &\quad x \in (0,L) \\ \psi = 0, &\quad x = 0, L \end{aligned} \tag{1.11}$$

whose spectrum is known to be

$$\sigma = \left\{ \left(\frac{n\pi}{L} \right)^2 \right\}_{n=1}^{+\infty}, \tag{1.12}$$

we can immediately see that these two norms are equivalent:

$$\begin{aligned} \|f\|_{H_0^1}^2 = \left\| \frac{\mathrm{d}f}{\mathrm{d}x} \right\|^2 &\leq \left\| \frac{\mathrm{d}f}{\mathrm{d}x} \right\|^2 + \|f\|^2 = \|f\|_{H^1}^2 \leq \left(1 + \frac{1}{\lambda_1}\right) \left\| \frac{\mathrm{d}f}{\mathrm{d}x} \right\|^2 \\ &= \left(1 + \frac{1}{\lambda_1}\right) \|f\|_{H_0^1}^2, \quad \forall f \in H_0^1(0,L). \end{aligned}$$

The space $H_0^1(0,L)$ equipped with inner product (1.9) is denoted as

$$\dot{H}_0^1(0,L) := \left(\overline{C_0^\infty(0,L)}^{\|\cdot\|_{H^1}}, \ (\cdot,\cdot)_{H_0^1} \right).$$

The two norms thus induce the same topology and the space $\dot{H}_0^1(0,L)$ is complete since $H_0^1(0,L)$ is. In the whole section we consider the Hilbert space being the same as $\mathcal{H}$ when considering vector spaces but with different inner product

$$(\Psi, \Phi)_{\dot{\mathcal{H}}} := \left(\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)_{\dot{\mathcal{H}}} = \int_0^L \frac{\mathrm{d}\overline{\psi}_1}{\mathrm{d}x} \frac{\mathrm{d}\phi_1}{\mathrm{d}x} + \overline{\psi}_2 \phi_2 \tag{1.13}$$

and denoted as

$$\dot{\mathcal{H}} := \left(\dot{H}_0^1(0,L) \times L^2(0,L), (\cdot,\cdot)_{\dot{\mathcal{H}}} \right). \tag{1.14}$$

1.2.1 Damped wave operator

Considering the domains of $\mathcal{T}$, Remark 2.1.1 below, the damped wave operator $\mathcal{A} : \mathrm{Dom}(A) \subset \dot{\mathcal{H}} \to \dot{\mathcal{H}}$ is thus defined as

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} & -a \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}) = \left(H^2(0,L) \cap H_0^1(0,L) \right) \times H_0^1(0,L) \tag{1.15}$$

where the second derivative is meant in the weak sense. Since $C_0^\infty(0,L)$ is dense in $L^2(0,L)$ the domain of this operator is dense in $\dot{\mathcal{H}}$ and thus $\mathcal{A}$ is densely defined. It is also unbounded since it has compact resolvent (which will be shown further) and since $\dot{\mathcal{H}}$ is an infinite-dimensional space. This is a trivial consequence of the fact that compact operators form an ideal in the algebra of bounded operators and of the fact that the identity operator is compact if and only if the space is finite-dimensional. The adjoint of $\mathcal{A}$ is

$$\mathcal{A}^* = \begin{pmatrix} 0 & -I \\ -\frac{\mathrm{d}^2}{\mathrm{d}x^2} & -a \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}^*) = \big(H^2(0,L) \cap H_0^1(0,L)\big) \times H_0^1(0,L). \tag{1.16}$$

Indeed we want to find $\Phi \in \dot{\mathcal{H}}$ such that there exists $\Phi^* \in \dot{\mathcal{H}}$ for which

$$(\mathcal{A}\Psi, \Phi)_{\dot{\mathcal{H}}} = (\Psi, \Phi^*)_{\dot{\mathcal{H}}}$$

holds for all $\Psi \in \mathrm{Dom}(\mathcal{A})$. Let $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{A},\ \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \dot{\mathcal{H}}$ then

$$\begin{aligned} (\mathcal{A}\Psi, \Phi)_{\dot{\mathcal{H}}} &= \left(\begin{pmatrix} \psi_2 \\ \frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} - a\psi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)_{\dot{\mathcal{H}}} = \left(\frac{\mathrm{d}\psi_2}{\mathrm{d}x}, \frac{\mathrm{d}\phi_1}{\mathrm{d}x} \right) + \left(\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} - a\psi_2, \phi_2 \right) \\ &= \left(\frac{\mathrm{d}\psi_2}{\mathrm{d}x}, \frac{\mathrm{d}\phi_1}{\mathrm{d}x} \right) - \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x}, \frac{\mathrm{d}\phi_2}{\mathrm{d}x} \right) - a(\psi_2, \phi_2) \end{aligned} \tag{1.17}$$

where we used integration by parts. This has to be equal to

$$(\Psi, \Phi^*)_{\dot{\mathcal{H}}} = \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x}, \frac{\mathrm{d}\phi_1^*}{\mathrm{d}x} \right) + (\psi_2, \phi_2^*)$$

for some and $\Phi^* = \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \in \dot{\mathcal{H}}$. By choosing $\phi_1^* := -\phi_2$ and $\phi_2^* := -\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2} - a\phi_2$ we get

$$\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x}, \frac{\mathrm{d}\phi_1^*}{\mathrm{d}x} \right) + (\psi_2, \phi_2^*) = -\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x}, \frac{\mathrm{d}\phi_2}{\mathrm{d}x} \right) + \left(\frac{\mathrm{d}\psi_2}{\mathrm{d}x}, \frac{\mathrm{d}\phi_1}{\mathrm{d}x} \right) - a(\psi_2, \phi_2)$$

which is equal to (1.17) and proves (1.16). Therefore we deal with a non-self-adjoint operator.

1.2.2 Basics of semigroup theory

Now we state some basics of the semigroup theory. Henceforth let X be a Banach space.

Definition 1.2.1 (C_0-semigroup of bounded linear operators)**.** *A one parameter family $T(t),\ 0 \leq t < +\infty$, of bounded linear operators on X is a C_0-semigroup of bounded linear operators on X (a C_0-semigroup) if*

1. $T(0) = I$ *(I is the identity operator on X)*
2. $T(t+s) = T(t)T(s), \quad \forall t, s \geq 0$ *(the semigroup property)*
3. $\lim_{t\to 0^+} T(t)x = x, \quad \forall x \in X$ *(the strong continuity).*

An infinitesimal generator of a C_0-semigroup $T(t)$ is a linear operator A satisfying

$$\begin{aligned} Ax &= \lim_{t\to 0^+} \frac{T(t)x - x}{t} = \left. \frac{\mathrm{d}^+ T(t)x}{\mathrm{d}t} \right|_{t=0} \\ \mathrm{Dom}(A) &= \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \ \text{exists} \right\}. \end{aligned}$$

Also for every C_0-semigroup $T(t)$ we have the following growth estimate [16, Theorem 1.2.2]. There exists constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}, \quad 0 \leq t < +\infty. \tag{1.18}$$

The value ω is called a growth rate. Whenever $\omega = 0$ we talk about uniformly bounded C_0-semigroup and moreover if $M = 1$ then we have a C_0-semigroup of contractions ($\|T(t)\| \leq 1$ is the so called contraction property). Also for all $\gamma > \omega$, γ is again the growth rate, i.e. $\|T(t)\| \leq Me^{\gamma t}$. The growth rate is therefore not unique.

Remark 1.2.2 (Motivation for the definition)**.** *To see the motivation for the definition of the C_0-semigroup let us consider an evolution problem with a complex matrix $\mathbb{A} \in \mathbb{C}^{d,d}$*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \mathbb{A}v(t), \quad v(0) = v_0 \tag{1.19}$$

where $v(t), v_0 \in \mathbb{C}^d$ for $t \in [0, +\infty)$ and v_0 is the initial state. We know that the solution of this problem is

$$v(t) = e^{\mathbb{A}t} v_0, \quad 0 \leq t < +\infty$$

where the term $e^{\mathbb{A}t}$ is the exponential of a matrix defined as

$$e^{\mathbb{A}t} = \sum_{n=0}^{+\infty} \frac{\mathbb{A}^n t^n}{n!}. \tag{1.20}$$

This sum always converges and satisfies the following properties (see for example [10, Section 1.4])

1. $e^{\mathbb{A}\cdot 0} = I$
2. $e^{\mathbb{A}(t+s)} = e^{\mathbb{A}t} e^{\mathbb{A}s}$
3. $\mathbb{A}v = \lim_{t \to 0^+} \dfrac{e^{\mathbb{A}t}v - v}{t} = \left.\dfrac{\mathrm{d}^+ e^{\mathbb{A}t} v}{\mathrm{d}t}\right|_{t=0}$
4. $\dfrac{\mathrm{d}}{\mathrm{d}t} e^{\mathbb{A}t} = \mathbb{A}e^{\mathbb{A}t} = e^{\mathbb{A}t}\mathbb{A}.$

In these we can see the semigroup property and the definition of the infinitesimal generator. It is also from this reason why the C_0-semigroup is being denoted using the exponential of its infinitesimal generator A, i.e. $T(t) = e^{At}$, even though the operator A can be unbounded (and therefore sum (1.20) *can be divergent in general). The C_0-semigroups whose infinitesimal generator is a bounded operator and therefore $T(t) = e^{At}$ is not only formal denotation are called uniformly continuous [16, Theorem 1.1.2]. An equivalent definition is that the C_0-semigroup $T(t)$ is uniformly continuous if* $\lim_{t \to 0^+} \|T(t) - I\| = 0$.

1.2.3 Generation of semigroup and its consequences

To show that $\mathcal{A}$ is an infinitesimal generator of some C_0-semigroup we will use the characterization by the Lumer-Phillips theorem instead of the direct computation. This requires for the operator to be m-dissipative.

Definition 1.2.3. *A linear operator A on a Hilbert space H with inner product $(\cdot,\cdot)_H$ is called m-dissipative if* $\Re(A\psi, \psi)_H \leq 0$, $\forall \psi \in \mathrm{Dom}(A)$ *and* $\mathrm{Ran}(I - A) = H$.

The notion of m-dissipativness can be defined for Banach spaces in general using the so called duality set [16, Definition 1.4.1]. Since our operator $\mathcal{A}$ is defined on the Hilbert space $\dot{\mathcal{H}}$ we are able to use its inner product instead.

Theorem 1.2.4 (Lumer-Phillips, [16, Theorem 1.4.3]). *A dense operator A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is m-dissipative.*

First we show that $\operatorname{Ran}(I - \mathcal{A}) = \dot{\mathcal{H}}$, i.e. that the operator $\mathcal{A} : \operatorname{Dom}(\mathcal{A}) \subset \dot{\mathcal{H}} \to \dot{\mathcal{H}}$ is surjective. Given $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \dot{\mathcal{H}}$ we want to find $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \operatorname{Dom}(\mathcal{A})$ such that $(I - \mathcal{A}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. This leads to the following system of equations

$$\psi_1 - \psi_2 = f_1 \tag{1.21a}$$

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 + (1+a)\psi_2 = f_2 \tag{1.21b}$$

$$\psi_1(0) = \psi_1(L) = 0 \tag{1.21c}$$

$$\psi_2(0) = \psi_2(L) = 0. \tag{1.21d}$$

Substituting from (1.21a) into (1.21b) we obtain

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 + (1+a)\psi_1 = f_2 + (1+a)f_1 \tag{1.22}$$

and the solution of this equation can be easily obtained using the method of variation of constants

$$\begin{aligned}\psi_1(x) &= C_1 \exp(\sqrt{1+a}x) + C_2 \exp(-\sqrt{1+a}x) \\ &- \frac{\exp(\sqrt{1+a}x)}{2\sqrt{1+a}} \int_0^x \exp(-\sqrt{1+a}s)((1+a)f_1(s) + f_2(s))\mathrm{d}s \\ &+ \frac{\exp(-\sqrt{1+a}x)}{2\sqrt{1+a}} \int_0^x \exp(\sqrt{1+a}s)((1+a)f_1(s) + f_2(s))\mathrm{d}s\end{aligned}$$

where the constants can be obtained from (1.21c) which is a non-homogeneous linear system of two linearly independent equations for C_1 and C_2 and thus it has exactly one solution. The function ψ_1 has the derivative

$$\begin{aligned}\frac{\mathrm{d}\psi_1}{\mathrm{d}x} &= C_1\sqrt{1+a}\exp(\sqrt{1+a}x) - C_2\sqrt{1+a}\exp(-\sqrt{1+a}x) \\ &- \frac{\exp(\sqrt{1+a}x)}{2} \int_0^x \exp(-\sqrt{1+a}s)((1+a)f_1(s) + f_2(s))\mathrm{d}s \\ &+ \frac{\exp(-\sqrt{1+a}x)}{2} \int_0^x \exp(\sqrt{1+a}s)((1+a)f_1(s) + f_2(s))\mathrm{d}s\end{aligned}$$

and since the integral as a function of the upper bound is an absolutely continuous function both ψ_1 and $\frac{\mathrm{d}}{\mathrm{d}x}\psi_1$ belong to $L^2(0,L)$. From (1.22) we see that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1$ also belongs to $L^2(0,L)$ and thus $\psi_1 \in H^2(0,L) \cap H_0^1(0,L)$. Equation (1.21d) and moreover the fact that $\psi_2 \in H_0^1(0,L)$ are satisfied by (1.21a), (1.21c), the fact that $\psi_1 \in H_0^1(0,L)$ and by using $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \dot{\mathcal{H}}$.

Finally we show that $\Re(\mathcal{A}\Psi, \Psi)_{\dot{\mathcal{H}}} \leq 0$ for all $\Psi \in \mathrm{Dom}(\mathcal{A}) \in \dot{\mathcal{H}}$. Let $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathrm{Dom}(\mathcal{A})$ then

$$\begin{aligned}
\Re\left(\mathcal{A}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right)_{\dot{\mathcal{H}}} &= \Re\left(\begin{pmatrix} \psi_2 \\ \frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} - a\psi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right)_{\dot{\mathcal{H}}} \\
&= \Re\int_0^L \frac{\mathrm{d}\overline{\psi}_2}{\mathrm{d}x}\frac{\mathrm{d}\psi_1}{\mathrm{d}x} + \frac{\mathrm{d}^2\overline{\psi}_1}{\mathrm{d}x^2}\psi_2 - a\overline{\psi}_2\psi_2 \\
&= \Re\int_0^L \frac{\mathrm{d}\overline{\psi}_2}{\mathrm{d}x}\frac{\mathrm{d}\psi_1}{\mathrm{d}x} - \frac{\mathrm{d}\overline{\psi}_1}{\mathrm{d}x}\frac{\mathrm{d}\psi_2}{\mathrm{d}x} - a\overline{\psi}_2\psi_2 \\
&= \Re\int_0^L \frac{\mathrm{d}\psi_1}{\mathrm{d}x}\frac{\mathrm{d}\overline{\psi}_2}{\mathrm{d}x} - \Re\int_0^L \overline{\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\frac{\mathrm{d}\overline{\psi}_2}{\mathrm{d}x}} - \Re\int_0^L a\overline{\psi}_2\psi_2 \\
&= -a\int_0^L |\psi_2|^2 \leq 0
\end{aligned}$$

where we used integration by parts and the fact that $\psi_2 \in H_0^1(0, L)$. Hence the linear operator $\mathcal{A}$ is m-dissipative and using the Lumer-Phillips theorem we obtain that it generates a C_0-semigroup of contractions which we will denote by $e^{\mathcal{A}t}$.

For such a generator (not necessarily of contractions) we have the following properties. Let $A : \mathrm{Dom}(A) \subset X \to X$ be an infinitesimal generator of C_0-semigroup. Then $\mathrm{Dom}(A)$ is dense in X and A is closed [16, Corollary 1.2.5]. Our operator $\mathcal{A}$ is thus closed and densely defined where the latter was already shown before. Next we have a result on continuity. It holds that for every C_0-semigroup $T(t)$ and for every $x \in X$ the function $t \mapsto T(t)x$ is continuous as a function from $[0, +\infty) \to X$ [16, Corollary 1.2.3]. This is a simple consequence of the existence of growth rate (1.18). Therefore for $U_0 \in \dot{\mathcal{H}}$,

$$U(t) := e^{\mathcal{A}t}U_0 \tag{1.23}$$

is a continuous function mapping $[0, +\infty) \to \dot{\mathcal{H}}$. Finally we state one of the most important theorems for the applications of the semigroup theory.

Theorem 1.2.5 ([16, Theorem 1.2.4]). *Let $A : \mathrm{Dom}(A) \subset X \to X$ be an infinitesimal generator of C_0-semigroup $T(t)$ then for every $x \in \mathrm{Dom}(A)$*

$$T(t)x \in \mathrm{Dom}(A) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}T(t)x = AT(t)x = T(t)Ax.$$

In our setting this means that for every initial condition $U_0 \in \mathrm{Dom}(\mathcal{A})$, the function $U(t)$ defined in (1.23) belongs to $\mathrm{Dom}(\mathcal{A})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathcal{A}U(t) \tag{1.24}$$

which is exactly evolution problem (1.8) stated in the beginning. We see that the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ generates the solution $U(t)$ of problem (1.8) using the initial value U_0. In other words by showing that $\mathcal{A}$ generates a C_0-semigroup we solved our evolution problem in the terms of $e^{\mathcal{A}t}$. However we do not yet know whether $U(t)$ is continuously differentiable or not. But fortunately we can say something about the regularity and uniqueness of the solution.

Theorem 1.2.6 ([16, Theorem 4.1.3]). *Let A be a densely defined linear operator with non-empty resolvent set $\rho(A)$. The initial value evolution problem for the operator A*

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = AY(t) \quad \text{and} \quad Y(0) = Y_0$$

has for every initial value $Y_0 \in \mathrm{Dom}(A)$ a unique continuously differentiable solution $Y(t)$ on $[0, +\infty)$, if and only if A is the infinitesimal generator of the C_0-semigroup $T(t)$.

To show the non-emptiness of the resolvent set $\rho(\mathcal{A})$ in our case we can use for example the famous Hille-Yosida theorem. In fact the Lumer-Phillips theorem 1.2.4 is only a consequence of this theorem.

Theorem 1.2.7 (Hille-Yosida, [16, Theorem 1.3.1]). *A linear operator $A : \mathrm{Dom}(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$ if and only if*

1. *A is closed and densely defined*
2. *The resolvent set $\rho(A)$ contains $(0, +\infty)$ and for every $\lambda > 0$*

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda}$$

where the family $R_\lambda(A) = (\lambda I - A)^{-1},\ \lambda \in \rho(A)$ denotes the resolvent of A.

Hence we see that the operator $\mathcal{A}$ has non-empty resolvent set. Thus the solution $U(t) = e^{\mathcal{A}t}U_0$ is continuously differentiable on $[0, +\infty)$ and unique. Returning back to damped wave equation (1.1) we see that using notation (1.6) we obtained a unique solution

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = e^{\mathcal{A}t}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

of the problem

$$\begin{aligned} u_{tt} + au_t - u_{xx} &= 0, \quad x \in (0, L),\ t > 0 \\ u &= 0, \quad x = 0, L, \quad t > 0 \\ u &= u_1, \quad x \in (0, L),\ t = 0 \\ u_t &= u_2, \quad x \in (0, L),\ t = 0 \end{aligned}$$

where

$$\begin{aligned} u(t, \cdot) \in C^0([0, +\infty), H^2(0, L) \cap H_0^1(0, L)) \cap C^1([0, +\infty), H_0^1(0, L)) \\ \cap\, C^2([0, +\infty), L^2(0, L)) \end{aligned}$$

for every initial value $u_0 \in H^2(0, L) \cap H_0^1(0, L)$ and $u_1 \in H_0^1(0, L)$. The derivatives with respect to t are thus classical but the derivative with respect to x is still meant in the weak sense.

Since this problem is explicitly solvable we are able to provide an explicit formula for the C_0-semigroup $e^{\mathcal{A}t}$. Indeed using the Fourier method and assuming $u(x, t) = X(x)\Phi(t)$ we get

$$X(x)\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + aX(x)\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \Phi(t)\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = 0$$

which can be separated into two equations by dividing it by $u(x,t)$ since both sides of the equation now depend on a different independent variable and therefore they are equal to some constant denoted by $-\lambda$:

$$-\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = \lambda X(x) \tag{1.25a}$$

$$\frac{\mathrm{d}^2 \Phi}{\mathrm{d}t^2} + a\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \lambda\Phi(t) = 0. \tag{1.25b}$$

Boundary condition (1.2) transforms to

$$X(0) = X(L) = 0 \tag{1.26}$$

and we can see that this equation together with (1.25a) is exactly the eigenvalue problem for Dirichlet Laplacian (1.11) for which the eigenvalues form a countable set indexed by $n \in \mathbb{N}$ and

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and the corresponding eigenfunctions are thus

$$X_n(x) = C\sin\left(\frac{n\pi x}{L}\right).$$

Equation (1.25b) can be solved using the assumption $\Phi(t) = e^{mt}$. We obtain the characteristic equation for m

$$m^2 + am + \lambda_n = 0 \tag{1.27}$$

with the solution

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4\lambda_n}}{2}.$$

This solution can be either negative or complex depending on the size of a corresponding to either under-damped or over-damped solutions. The whole solution $u(x,t)$ is hence

$$u(x,t) = \sum_{n=1}^{+\infty} \left(C_1 e^{m_1 t} + C_2 e^{m_2 t}\right)\sin\left(\frac{n\pi x}{L}\right)$$

where the constants can be determined from the initial conditions u_0 and u_1. Finally we get the action of the C_0-semigroup on $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathrm{Dom}(\mathcal{A})$:

$$e^{\mathcal{A}t}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{n=1}^{+\infty} \left(C_1 e^{m_1 t} + C_2 e^{m_2 t}\right)\sin\left(\frac{n\pi x}{L}\right) \\ \displaystyle\sum_{n=1}^{+\infty} \left(C_1 m_1 e^{m_1 t} + C_2 m_2 e^{m_2 t}\right)\sin\left(\frac{n\pi x}{L}\right) \end{pmatrix}.$$

1.2.4 Growth bound and the spectrum

The theory of semigroups can give us even more. Recall the growth rate ω for some C_0-semigroup $T(t)$ with the infinitesimal generator A defined in (1.18). We already know that ω is not unique. Therefore we define the lowest growth rate by

$$\omega_0(A) := \inf\{\omega : \exists M(\omega),\ \|T(t)\| \leq M(\omega)e^{\omega t}\}. \tag{1.28}$$

The lowest growth rate $\omega_0(A)$ is usually called a growth bound or a growth abscissa. It can be shown [14, Theorem 2.19] that $\omega_0(A)$ is equal to

$$\omega_0(A) = \lim_{t\to+\infty} \frac{\log \|T(t)\|}{t} = \inf_{t>0} \frac{\log \|T(t)\|}{t}. \tag{1.29}$$

One can imagine that such a quantity is not easily computable. Fortunately it can be shown that this quantity is related to the so called spectral abscissa. In particular for every C_0-semigroup $T(t)$ with the infinitesimal generator A we have

$$\omega_0(A) \geq \omega_\sigma(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\} \tag{1.30}$$

where $\sigma(A)$ is the spectrum of A and $\omega_\sigma(A)$ is the spectral abscissa [14, Theorem 2.20]. Furthermore for some special C_0-semigroups we can get equality in (1.30) and we say that the C_0-semigroup has a spectrum determined growth property. One of such cases is when the C_0-semigroup is actually an analytic semigroup [14, Theorem 2.21].

Definition 1.2.8 (Analytic semigroup). *Let $Z = \{z \in \mathbb{C} : \phi_1 < \arg(z) < \phi_2, \phi_1 < 0 < \phi_2\}$ and let $T(z)$ be a bounded linear operator on X for every $z \in Z$. Then the family $T(z)$, $z \in Z$ is an analytic semigroup if the following is satisfied*

1. *$z \mapsto T(z)$ is analytic function in Z*

2. *$T(0) = I$ and $\lim\limits_{\substack{z\to 0\\ z\in Z}} T(z)x = x$ for every $x \in X$*

3. *$T(z_1 + z_2) = T(z_1)T(z_2)$ for every $z_1, z_2 \in Z$.*

Another case when the equality in (1.30) holds is when the eigenvectors of the infinitesimal generator of the C_0-semigroup of bounded operators in a Hilbert space H form an orthonormal basis in H and the supremum of the set of the real parts of eigenvalues of the infinitesimal generator is less than infinity [14, Theorem 2.22].

The last and for us the most important case mentioned here when the equality holds is when the infinitesimal generator is a Riesz-spectral operator and again the supremum of the set of the real parts of eigenvalues of the infinitesimal generator is less than infinity [3, Theorem 2.3.5]. The Riesz-spectral operator is a linear, closed operator on the Hilbert space H with simple eigenvalues $\{\lambda_n : n \geq 1\}$ and the corresponding eigenvectors $\{\phi_n : n \geq 1\}$ which form a Riesz basis in H and moreover the closure of $\{\lambda_n : n \geq 1\}$ is totally disconnected, meaning that no two points from the closure can be joined by a segment lying entirely in it.

Definition 1.2.9 (Riesz basis). *A Riesz basis is a set of vectors $\{\phi_n : n \geq 1\}$ in a Hilbert space H for which the following conditions hold*

1. $\overline{\operatorname{span}\{\phi_n : n \geq 1\}} = H$

2. *There exists $m, M > 0$ such that for any $N \in \mathbb{N}$ and any numbers α_n, $n \in \{1, 2, \ldots, N\}$*

$$m \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|^2 \leq M \sum_{n=1}^{N} |\alpha_n|^2.$$

An equivalent definition would be that Riesz basis is an image of an orthonormal basis in H under a bounded linear operator with bounded inverse. The fact that the eigenvectors of our operator $\mathcal{A}$ form a Riesz basis and moreover that $\mathcal{A}$ is a Riesz-spectral operator and that the supremum of the set of the real parts of eigenvalues of $\mathcal{A}$ is less than infinity was shown by Cox and Zuazua in [2, Theorem 2.1]. Therefore

$$\omega_0(\mathcal{A}) = \omega_\sigma(\mathcal{A}).$$

To obtain the growth bound we thus have to calculate the spectrum of $\mathcal{A}$. First we show that $\mathcal{A}$ has purely discrete spectrum $\sigma_{\mathrm{disc}}(\mathcal{A})$. This in the non-self-adjoint setting means that $\lambda \in \sigma_{\mathrm{disc}}(A)$ if and only if it is an isolated eigenvalue with finite algebraic multiplicity and with $\operatorname{Ran}(A - \lambda I)$ closed. Moreover for the rest of the thesis we define the essential spectrum of the non-self-adjoint operator A to be $\sigma_{\mathrm{ess}}(A) = \rho(A) \setminus \sigma_{\mathrm{disc}}(A)$.

The discreteness of the spectrum of $\mathcal{A}$ follows from the fact that it has compact resolvent. Indeed we first show that the undamped operator, i.e. with $a = 0$, has compact resolvent and then we conclude that the bounded perturbation via the damping a does not affect the compactness.

Let $\mathcal{A}_0$ be the undamped operator, i.e.

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} & 0 \end{pmatrix}, \quad \operatorname{Dom}(\mathcal{A}_0) = \left(H^2(0,L) \cap H_0^1(0,L)\right) \times H_0^1(0,L).$$

It is a known fact that a closed operator A on a Hilbert space H has compact resolvent if and only if the embedding $(\operatorname{Dom}(A), \|\cdot\|_A) \hookrightarrow H$ is compact. Here $\|\cdot\|_A$ is the graph norm of the operator A, i.e. given $\psi \in \operatorname{Dom}(A)$, $\|\psi\|_A^2 := \|\psi\|_H^2 + \|A\psi\|_H^2$. Thus we have to show that the embedding of $(\operatorname{Dom}(\mathcal{A}_0), \|\cdot\|_{\mathcal{A}_0})$ which is the domain of $\mathcal{A}_0$ understood as a Hilbert space with the graph norm $\|\cdot\|_{\mathcal{A}_0}$ (it is complete provided that $\mathcal{A}_0$ is closed) into $\dot{\mathcal{H}}$ is compact.

First we show that there is a norm on $\operatorname{Dom}(\mathcal{A}_0)$ equivalent to the graph norm. It is the norm of the Cartesian product of the two Sobolev spaces $H^2(0,L) \cap H_0^1(0,L)$ and $H_0^1(0,L)$ denoted by $\|\cdot\|_c$. Let $\Psi \in \operatorname{Dom}(\mathcal{A}_0)$, $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ then

$$\|\Psi\|_c^2 = \|\psi_1\|_{H^2}^2 + \|\psi_2\|_{H^1}^2 = \left\|\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}\right\|^2 + \left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \|\psi_1\|^2 + \left\|\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\right\|^2 + \|\psi_2\|^2.$$

We can immediately see the first inequality

$$\begin{aligned} \|\Psi\|_{\mathcal{A}_0}^2 &= \left\|\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}\right\|^2 + \left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \left\|\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\right\|^2 + \|\psi_2\|^2 \\ &\leq \left\|\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}\right\|^2 + \left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \|\psi_1\|^2 + \left\|\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\right\|^2 + \|\psi_2\|^2 = \|\Psi\|_c^2 \end{aligned}$$

and the opposite inequality can be obtained using Poincaré inequality (1.10)

$$\begin{aligned} \|\Psi\|_{\mathcal{A}_0}^2 &= \left\|\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}\right\|^2 + \frac{1}{2}\left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \frac{1}{2}\left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \left\|\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\right\|^2 + \|\psi_2\|^2 \\ &\geq \left\|\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}\right\|^2 + \frac{1}{2}\left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \frac{1}{2}\lambda_1 \|\psi_1\|^2 + \left\|\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\right\|^2 + \|\psi_2\|^2 \\ &\geq C\|\Psi\|_c^2 \end{aligned} \tag{1.31}$$

where $C = \min\left\{\frac{1}{2}, \frac{\lambda_1}{2}\right\}$. Hence the norms are equivalent and thus the identity map $I : (\mathrm{Dom}(\mathcal{A}_0), \|\cdot\|_{\mathcal{A}_0}) \to (\mathrm{Dom}(\mathcal{A}_0), \|\cdot\|_c)$ is bounded. In fact for the boundedness of I it would be sufficient to show only the validity of second inequality (1.31).

Next we use the fact that the embeddings

$$H^2(0,L) \hookrightarrow H^1(0,L) \quad \text{and} \quad H_0^1(0,L) \hookrightarrow L^2(0,L)$$

are compact. This is a consequence of the Rellich-Kondrachov theorem [5, Theorem 5.7.1]. From the former we obtain that

$$\big(H^2(0,L) \cap H_0^1(0,L)\big) \hookrightarrow \dot{H}_0^1(0,L)$$

is also compact since $H_0^1(0,L)$ is by definition a closed subspace of $H^1(0,L)$ and the norms on $H_0^1(0,L)$ and $\dot{H}_0^1(0,L)$ are equivalent. Also the Cartesian product of the two maps

$$\big(H^2(0,L) \cap H_0^1(0,L)\big) \hookrightarrow \dot{H}_0^1(0,L) \quad \text{and} \quad H_0^1(0,L) \hookrightarrow L^2(0,L)$$

is compact, i.e. the embedding

$$J : \big(H^2(0,L) \cap H_0^1(0,L)\big) \times \dot{H}_0^1(0,L) \to H_0^1(0,L) \times L^2(0,L)$$

is compact.

Since the composition of a compact map with a bounded map is again compact we obtain that the embedding

$$J \circ I : (\mathrm{Dom}(\mathcal{A}_0), \|\cdot\|_{\mathcal{A}_0}) \to \dot{\mathcal{H}}$$

is compact which proves the statement that $\mathcal{A}_0$ has a compact resolvent.

Now we state the stability theorem.

Theorem 1.2.10 ([10, Theorem IV.1.16])**.** *Let T and A be operators on Banach space X. Let T^{-1} exist and be bounded. Let A be bounded and satisfying the inequality*

$$\|A\|\|T^{-1}\| < 1. \tag{1.32}$$

Then $S = T + A$ is closed and invertible and S^{-1} is bounded. If in addition T^{-1} is compact then S^{-1} is also compact.

Let z be an arbitrary point from the resolvent set of $\mathcal{A}_0$. Then $zI - \mathcal{A}_0$ plays the role of the operator T in the theorem (T^{-1} thus exists), and the perturbation of $\mathcal{A}_0$ denoted by $\mathcal{B}$ stands for $-A$. In particular

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{B}) := \dot{\mathcal{H}}.$$

To prove (1.32) we will use the following bound for the norm of $\mathcal{B}$. Let $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ then

$$\|\mathcal{B}\| = \sup_{\substack{\Psi \in \mathrm{Dom}(\mathcal{B}) \\ \Psi \neq 0}} \frac{\|\mathcal{B}\Psi\|}{\|\Psi\|} = \sup_{\substack{\Psi \in \mathrm{Dom}(\mathcal{B}) \\ \Psi \neq 0}} \frac{\|-a\psi_2\|}{\sqrt{\left\|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\right\|^2 + \|\psi_2\|^2}} \leq \sup_{\substack{\Psi \in \mathrm{Dom}(\mathcal{B}) \\ \Psi \neq 0}} \frac{a\|\psi_2\|}{\|\psi_2\|} = a$$

hence $\mathcal{B}$ is a bounded operator and inequality (1.32) transforms to

$$\|\mathcal{B}\|\|(zI - \mathcal{A}_0)^{-1}\| = a\|(zI - \mathcal{A}_0)^{-1}\| < 1.$$

Recall the Hille-Yoshida theorem 1.2.7. Since $\mathcal{A}_0$ is an infinitesimal generator of C_0-semigroup of contractions the theorem implies that for every $\lambda > 0,\ \lambda \in \rho(\mathcal{A}_0)$ and

$$\|(\lambda I - \mathcal{A}_0)^{-1}\| \leq \frac{1}{\lambda}.$$

For given $a > 0$ we choose $z > a$ and get

$$\|(zI - \mathcal{A}_0)^{-1}\| \leq \frac{1}{z} < \frac{1}{a}$$

which proves inequality (1.32). Hence the operator $(zI - \mathcal{A}_0 - \mathcal{B})^{-1} = (zI - \mathcal{A})^{-1}$ is compact. It holds that if $(zI - \mathcal{A})^{-1}$ is compact for some point $z \in \rho(\mathcal{A})$ then $(\lambda I - \mathcal{A})^{-1}$ is compact for all $\lambda \in \rho(\mathcal{A})$. Thus $\mathcal{A}$ has compact resolvent which we wanted to prove. Its spectrum is purely discrete.

We move on to determining the discrete spectrum. Let $\Psi \in \mathrm{Dom}(\mathcal{A}),\ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\mathcal{A}\Psi = \lambda\Psi$, i.e.

$$\psi_2 = \lambda\psi_1 \tag{1.33a}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 - a\psi_2 = \lambda\psi_2 \tag{1.33b}$$
$$\psi_1(0) = \psi_1(L) = 0 \tag{1.33c}$$
$$\psi_2(0) = \psi_2(L) = 0. \tag{1.33d}$$

Substituting from (1.33a) into (1.33b) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 - \lambda a\psi_1 - \lambda^2\psi_1 = 0. \tag{1.34}$$

After rearranging terms and denoting $\mu := -\lambda^2 - \lambda a$ we see that together with equation (1.33c) we obtained eigenvalue problem (1.11) for the eigenvalue μ

$$\begin{aligned} -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 &= \mu\psi_1, \quad x \in (0, L) \\ \psi_1 &= 0, \quad x = 0, L. \end{aligned} \tag{1.35}$$

But we know that the spectrum of this problem is (1.12) and so it holds

$$\left(\frac{n\pi}{L}\right)^2 = \mu_n = -\lambda^2 - \lambda a$$

which is a quadratic equation for λ which is actually the same equation as equation (1.27) for m. The solution, i.e. the spectrum of $\mathcal{A}$ is then

$$\sigma(\mathcal{A}) = \left\{ \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4\left(\frac{n\pi}{L}\right)^2} \right) \right\}_{n=1}^{+\infty}.$$

To find out the spectral abscissa we now analyze the behavior of the spectrum with respect to the damping a.

- For $a = 0$, i.e. no damping, all the eigenvalues are symmetrically and equidistantly distributed on the imaginary axis, see Figure 1.1.

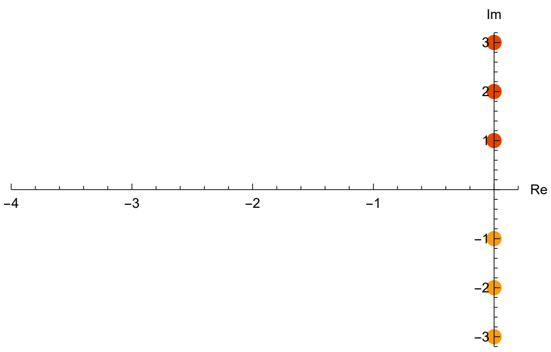

Figure 1.1: Plot of first six eigenvalues from $\sigma(\mathcal{A})$ with $a = 0$

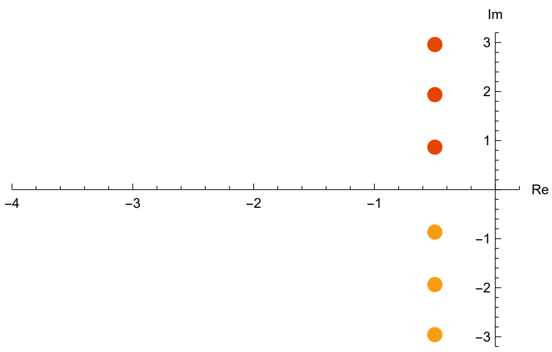

Figure 1.2: Plot of first six eigenvalues from $\sigma(\mathcal{A})$ with $a = 1$

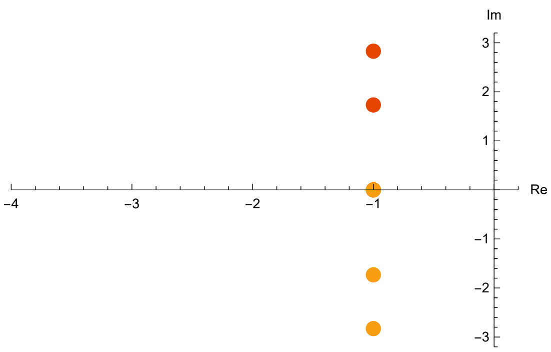

Figure 1.3: Plot of first six eigenvalues from $\sigma(\mathcal{A})$ with $a = 2$. One conjugate pair collides on the real axis and creates a degenerate eigenvalue

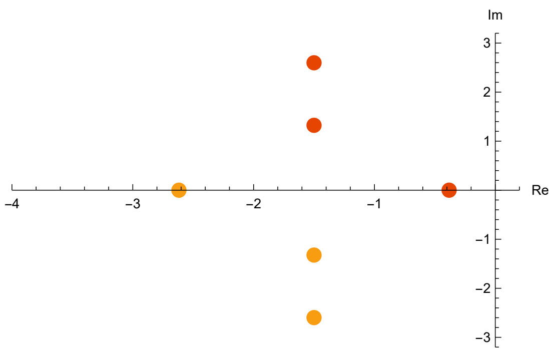

Figure 1.4: Plot of first six eigenvalues from $\sigma(\mathcal{A})$ with $a = 3$. The collided eigenvalues separate again each one moving to another side of the real axis

- For $a \in \left(0, \frac{2\pi}{L}\right)$ the eigenvalues form complex conjugate pairs located on a sphere originated in 0 of the complex plane with radius equal to $\frac{n\pi}{L}$ as shown on Figure 1.2. Indeed let $\lambda_n \in \sigma(\mathcal{A})$, then

$$(\Re\lambda_n)^2 + (\Im\lambda_n)^2 = \left(-\frac{a}{2}\right)^2 + \left(\pm\frac{1}{2}\sqrt{4\left(\frac{n\pi}{L}\right)^2 - a^2}\right)^2 = \left(\frac{n\pi}{L}\right)^2.$$

  The larger is the damping a, the larger is the distance from the imaginary axis.

- For the critical value $a = \frac{2\pi}{L}$ the two conjugate eigenvalues which were on the sphere with the smallest radius collide, see Figure 1.3.

- For $a \in \left(\frac{2\pi}{L}, \frac{4\pi}{L}\right)$ the two eigenvalues which previously collided are now separated again with one moving on the real line towards 0 and the second towards $-\infty$ while the others continue with the motion on the spheres until the next two eigenvalues collide when $a = \frac{4\pi}{L}$, see Figures 1.4 and 1.5.

- The whole process continues in analogous way.

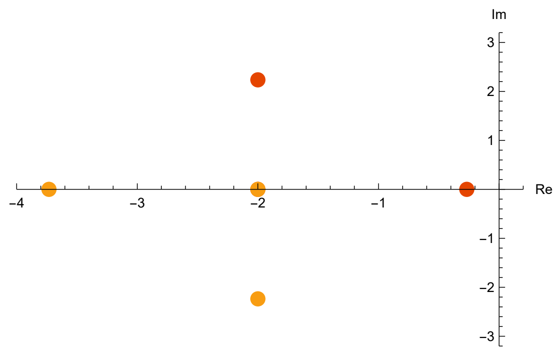

Figure 1.5: Plot of first six eigenvalues from $\sigma(\mathcal{A})$ with $a = 4$. Another conjugate pair collides

From this behavior it can be seen that the spectral abscissa and hence the growth bound is

$$\omega_0(\mathcal{A}) = \sup\{\Re\lambda : \lambda \in \sigma(\mathcal{A})\} = \begin{cases} -\frac{a}{2}, & a \leq \frac{2\pi}{L} \\ -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4\left(\frac{\pi}{L}\right)^2}, & a > \frac{2\pi}{L}. \end{cases} \tag{1.36}$$

This means that there exists $M = M(\omega_0(\mathcal{A})) > 0$ such that $\|e^{\mathcal{A}t}\| \leq Me^{\omega_0(\mathcal{A})t}$ and moreover from the contraction property ($\|e^{\mathcal{A}t}\| \leq 1$) we have $M \leq 1$. From (1.29) we get

$$\log \|e^{\mathcal{A}t}\| = \omega_0(\mathcal{A})t + o(t)$$

where o stands for the standard small o notation in the asymptotic regime $t \to +\infty$. Exponentiating, this leads to $\|e^{\mathcal{A}t}\| = e^{\omega_0(\mathcal{A})t}e^{o(t)}$ and thus

$$\|e^{\mathcal{A}t}\| \sim e^{\omega_0(\mathcal{A})t} \quad \text{as} \quad t \to +\infty \tag{1.37}$$

where $\sim$ has the meaning $f(t) \sim g(t),\ t \to +\infty$ if and only if $f(t) - g(t) \xrightarrow[t\to+\infty]{} 0$. Indeed

$$e^{\omega_0(\mathcal{A})t} - \|e^{\mathcal{A}t}\| = e^{\omega_0(\mathcal{A})t} - e^{\left(\omega_0(\mathcal{A}) + \frac{o(t)}{t}\right)t} \xrightarrow[t\to+\infty]{} 0.$$

Given the initial value $U_0 \in \mathrm{Dom}(\mathcal{A})$ we thus obtain for the solution $U(t) = e^{\mathcal{A}t}U_0$ the bound

$$\|U(t)\|_{\dot{\mathcal{H}}} \leq e^{\omega_0(\mathcal{A})t}\|U_0\|_{\dot{\mathcal{H}}}.$$

which is uniform (with respect to the initial condition). Next recall notation (1.6). We obtain

$$\|U(t)\|^2_{\dot{\mathcal{H}}} = \left\| \begin{pmatrix} u \\ u_t \end{pmatrix} \right\|^2_{\dot{\mathcal{H}}} = \int_0^L |u_x|^2 + |u_t|^2$$

which is a well-known expression for the energy of a string at time t, i.e. $E(t) = \|U(t)\|^2_{\dot{\mathcal{H}}}$, and

$$E(t) \leq E(0)e^{2\omega_0(\mathcal{A})t}.$$

We see that using the semigroup theory we obtained a uniform bound for the energy of the system.

Finally we can find an optimal damping for which the system returns to the equilibrium ($u = 0$) in the shortest time. Viewing $\omega_0(\mathcal{A})$ as a function of the damping a we can find its minimum which according to formula (1.36) is achieved at $a = \frac{2\pi}{L}$:

$$\min_{a \geq 0} \omega_0(\mathcal{A}) = -\frac{\pi}{L}.$$

Thus for $a = \frac{2\pi}{L}$ the solution has the fastest decay uniformly with respect to the initial condition

$$\|U(t)\|_{\dot{\mathcal{H}}} \leq e^{-\frac{\pi t}{L}} \|U_0\|_{\dot{\mathcal{H}}} \quad \text{as} \quad t \to +\infty.$$

1.3 Arbitrary $\Omega \subset \mathbb{R}^d$ and bounded damping $a \in L^\infty(\Omega)$

In this crucial section we move on to the definition of the damped wave operator for arbitrary domain $\Omega \subset \mathbb{R}^d$ and for bounded damping function a, in particular $a \in L^\infty(\Omega)$. We again show that this operator generates a C_0-semigroup but now without the contraction property in general since we work on $\mathcal{H}$ instead of $\dot{\mathcal{H}}$ and moreover without the spectrum determined growth property. Nevertheless we will still be able to deduce some consequences for the time evolution and stability of the system. Note that there is a recent paper [8] where the generation of a C_0-semigroup even in the case of unbounded damping is proved by working in a different Hilbert space.

1.3.1 Damped wave operator

We work in Hilbert space $\mathcal{H}$ (1.3) and consider damped wave operator (1.7). Since the Dirichlet Laplacian is densely defined (see Section 2.1) for arbitrary Ω and $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ the operator $\mathcal{A}$ is also densely defined. The operator is again non-self-adjoint as in the preceding case. Next we show that $\mathcal{A}$ is closed.

Let $\mathcal{A}_0$ denote the undamped operator $\mathcal{A}$ with $a = 0$. We first show that $\mathcal{A}_0$ is closed and then since the sum of bounded and closed operator is again a closed operator the damping does not violate the closedness. Given Ψ, $\Phi \in \mathcal{H}$ and a sequence $\Psi_n \in \operatorname{Dom}(\mathcal{A}_0)$, $n \in \mathbb{N}$, let

$$\begin{aligned} \Psi_n &\xrightarrow[n\to+\infty]{} \Psi \quad \text{in } \mathcal{H} \\ \mathcal{A}_0\Psi_n &\xrightarrow[n\to+\infty]{} \Phi \quad \text{in } \mathcal{H}. \end{aligned} \tag{1.38}$$

We have to show that $\Psi \in \operatorname{Dom}(\mathcal{A}_0)$ and $\mathcal{A}_0\Psi = \Phi$. Denoting $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, $\Psi_n = \begin{pmatrix}\psi_{n1}\\ \psi_{n2}\end{pmatrix}$, $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and rewriting limits (1.38) we get

$$\begin{aligned} \psi_{n1} &\xrightarrow[n\to+\infty]{} \psi_1 \quad \text{in } H^1(\Omega) \\ \psi_{n2} &\xrightarrow[n\to+\infty]{} \psi_2 \quad \text{in } L^2(\Omega) \\ \psi_{n2} &\xrightarrow[n\to+\infty]{} \phi_1 \quad \text{in } H^1(\Omega) \\ \Delta\psi_{n1} &\xrightarrow[n\to+\infty]{} \phi_2 \quad \text{in } L^2(\Omega) \end{aligned}$$

which means

$$\|\nabla\psi_{n1} - \nabla\psi_1\|^2 + \|\psi_{n1} - \psi_1\|^2 \xrightarrow[n\to+\infty]{} 0 \tag{1.39a}$$

$$\|\psi_{n2} - \psi_2\|^2 \xrightarrow[n\to+\infty]{} 0 \tag{1.39b}$$

$$\|\nabla\psi_{n2} - \nabla\phi_1\|^2 + \|\psi_{n2} - \phi_1\|^2 \xrightarrow[n\to+\infty]{} 0 \tag{1.39c}$$

$$\|\Delta\psi_{n1} - \phi_2\|^2 \xrightarrow[n\to+\infty]{} 0. \tag{1.39d}$$

Equation (1.39c) implies that also $\|\psi_{n2} - \phi_1\|^2 \xrightarrow[n\to+\infty]{} 0$ which together with (1.39b) gives $\psi_2 = \phi_1 \in H_0^1(\Omega)$ provided that the limit in $L^2(\Omega)$ is unique. As a next step we integrate by parts getting

$$(\Delta\varphi, \psi_1) = -(\nabla\varphi, \nabla\psi_1)$$

where $\varphi \in C_0^\infty(\Omega)$. From (1.39a) we get that $\nabla\psi_{n1} \xrightarrow[n\to+\infty]{} \nabla\psi_1$ in $L^2(\Omega)$ and thanks to the continuity of the inner product we have

$$-(\nabla\varphi, \nabla\psi_1) = -\lim_{n\to+\infty} (\nabla\varphi, \nabla\psi_{n1}).$$

Another integration by parts leads us to

$$-\lim_{n\to+\infty} (\nabla\varphi, \nabla\psi_{n1}) = \lim_{n\to+\infty} (\varphi, \Delta\psi_{n1})$$

where finally (1.39d) implies

$$\lim_{n\to+\infty} (\varphi, \Delta\psi_{n1}) = (\varphi, \phi_2).$$

Hence for all $\varphi \in C_0^\infty(\Omega)$ it holds $(\Delta\varphi, \psi_1) = (\varphi, \phi_2)$ which from the definition means that ϕ_2 is the distributional Laplacian of ψ_1 on $L^2(\Omega)$, i.e. $\Delta\psi_1 = \phi_2 \in L^2(\Omega)$ and the operator $\mathcal{A}_0$ is thus closed. Next we have to show that the perturbation of $\mathcal{A}_0$ denoted by $\mathcal{B}$ is bounded. In particular

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{B}) := \mathcal{H}$$

and $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$. Let $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ then

$$\begin{aligned} \|\mathcal{B}\| &= \sup_{\substack{\Psi\in\mathrm{Dom}(\mathcal{B}) \\ \Psi\neq 0}} \frac{\|\mathcal{B}\Psi\|}{\|\Psi\|} = \sup_{\substack{\Psi\in\mathrm{Dom}(\mathcal{B}) \\ \Psi\neq 0}} \frac{\|-a\psi_2\|}{\sqrt{\|\nabla\psi_1\|^2 + \|\psi_1\|^2 + \|\psi_2\|^2}} \\ &\leq \sup_{\substack{\Psi\in\mathrm{Dom}(\mathcal{B}) \\ \Psi\neq 0}} \frac{\|a\|_\infty \|\psi_2\|}{\|\psi_2\|} = \|a\|_\infty \end{aligned}$$

where $\|a\|_\infty$ is the norm in $L^\infty(\Omega)$ in particular the essential supremum of $|a|$. Since the sum of bounded and closed operator is again a closed operator we obtained that $\mathcal{A}$ is closed.

1.3.2 Generation of semigroup and consequences

Now we will move on to the proof that $\mathcal{A}$ is an infinitesimal generator of some C_0-semigroup. This will be done by entirely different means than in the previous chapter. We first show that it generates a C_0-semigroup in the case of zero damping, i.e. $a = 0$ and then we use a perturbation theorem which states that the generation is still preserved with the non-zero damping present.

Therefore let $\mathcal{A}_0$ denote the undamped operator $\mathcal{A}$ with $a = 0$. We will use the following characterization for the generator of the C_0-semigroup which will give us also the growth rate of the C_0-semigroup.

Theorem 1.3.1 ([16, Theorem 1.5.3]). *A linear operator A on Banach space X is an infinitesimal generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}$ if and only if*

1. *A is closed and densely defined*
2. *The resolvent set $\rho(A)$ contains $(\omega, +\infty)$ and*

$$\|R_\lambda(A)^n\| \leq \frac{M}{(\lambda-\omega)^n}, \quad \forall\, \lambda > \omega,\ n \in \mathbb{N}. \tag{1.40}$$

The first condition of this theorem is already proven so it remains to prove inequality (1.40) for some ω. For this we will need the following lemmas. This approach was used in [7] in a more general setting where they were inspired by [16, Section 7.4].

Lemma 1.3.2. *Let $\epsilon \in (0,1)$. Then for any $\phi \in L^2(\Omega)$ there exists a unique function $\psi \in \mathrm{Dom}(-\Delta)$ which satisfies the equation*

$$(1-\epsilon^2\Delta)\psi = \phi. \tag{1.41}$$

Proof. This proof is based on [11, Remark 2.3.2]. First we establish an a-priori bound. Let $\phi \in L^2(\Omega)$ and $\epsilon \in (0,1)$ and assume that there exists $\psi \in H_0^1(\Omega)$ satisfying (1.41) then

$$\|\psi\|_{H^1} \leq \frac{1}{\epsilon}\|\phi\|. \tag{1.42}$$

Indeed multiplying (1.41) with $\overline{\psi}$ and integrating over Ω we get

$$\|\psi\|^2 + \epsilon^2\|\nabla\psi\|^2 = (\psi, \phi)$$

where we used integration by parts. Moreover

$$\|\psi\|^2 \leq \|\psi\|^2 + \epsilon^2\|\nabla\psi\|^2 = (\psi,\phi) \leq \|\psi\|\|\phi\|$$

which implies $\|\psi\| \leq \|\phi\|$ and thus

$$\min\left\{1, \epsilon^2\right\}\|\psi\|_{H^1}^2 \leq \|\phi\|^2.$$

Finally for $\epsilon \in (0,1)$ we get (1.42).

Next we take a sequence $(\Omega_n)_{n\in\mathbb{N}}$ such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n\in\mathbb{N}}\Omega_n = \Omega$. From [11, Section 2.3] we know that there exists a unique weak solution $\psi \in H_0^1(\Omega_n) \cap H^2(\Omega_n)$ of (1.41) with $\Omega = \Omega_n$, i.e. for all $\varphi \in C_0^\infty(\Omega_n)$ it holds

$$(\varphi, \psi_n) - \epsilon^2(\varphi, \Delta\psi_n) = (\varphi, \phi). \tag{1.43}$$

We extend ψ_n to the whole domain Ω with zero outside Ω_n and retain the same notation for it. For all such ψ_n bound (1.42) holds and thus $(\psi_n)_{n\in\mathbb{N}}$ is uniformly bounded in $H_0^1(\Omega)$. This implies that there exists a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$ which converges weakly to some function [20, Theorem 4.25]. Let us denote this function by ψ. Fixing some $\varphi \in C_0^\infty(\Omega)$ being zero outside Ω_n and taking limit $k \to +\infty$ in the subsequence $(\psi_{n_k})_{n\in\mathbb{N}}$ in (1.43) with $n \le n_k$ we obtain

$$(\varphi, \psi) - \epsilon^2(\varphi, \Delta\psi) = (\varphi, \phi). \tag{1.44}$$

Since these φ form a dense set in $H_0^1(\Omega)$ the function ψ satisfies (1.44) for all $\varphi \in H_0^1(\Omega)$ and thus it is a weak solution of (1.41). The uniqueness follows from bound (1.42) and it implies that $(\psi_n)_{n\in\mathbb{N}} \xrightarrow[n\to+\infty]{} \psi$ in $H_0^1(\Omega)$. The fact that $\psi \in \mathrm{Dom}(-\Delta) = \left\{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\right\}$ follows from $\phi \in L^2(\Omega)$ since

$$\Delta\psi = \frac{1}{\epsilon^2}(\phi + \psi).$$

□

Lemma 1.3.3. *Let $\epsilon \in (0,1)$ and $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Then the equation*

$$\Psi - \epsilon\mathcal{A}_0\Psi = \Phi \tag{1.45}$$

has a unique solution $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in \mathrm{Dom}(\mathcal{A}_0)$ and moreover

$$\|\Psi\|_{\mathcal{H}} \le \frac{1}{1-\epsilon}\|\Phi\|_{\mathcal{H}}.$$

Proof. Let $\varphi_1 \in \mathrm{Dom}(-\Delta)$ be the unique (thanks to the previous lemma) solution of

$$(1 - \epsilon^2\Delta)\varphi_1 = \phi_1$$

and let $\varphi_2 \in \mathrm{Dom}(-\Delta)$ be the unique solution of

$$(1 - \epsilon^2\Delta)\varphi_2 = \phi_2.$$

Define

$$\psi_1 := \varphi_1 + \epsilon\varphi_2, \quad \psi_2 := \epsilon\Delta\varphi_1 + \varphi_2$$

such that $\Psi := \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ satisfies (1.45), i.e.

$$\psi_1 - \epsilon\psi_2 = \phi_1, \quad -\epsilon\Delta\psi_1 + \psi_2 = \phi_2$$

from which it can be seen that $\Psi \in \mathrm{Dom}(\mathcal{A}_0)$. Next let $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ then

$$\begin{aligned}
\|\Phi\|_{\mathcal{H}}^2 &= \|\phi_1\|_{H^1}^2 + \|\phi_2\|^2 = \|\psi_1 - \epsilon\psi_2\|_{H^1}^2 + \| - \epsilon\Delta\psi_1 + \psi_2\|^2 \\
&= \|\psi_1\|_{H^1}^2 + \| - \epsilon\psi_2\|_{H^1}^2 + 2\Re(\psi_1, -\epsilon\psi_2)_{H^1} \\
&\quad + \| - \epsilon\Delta\psi_1\|^2 + \|\psi_2\|^2 + 2\Re(-\epsilon\Delta\psi_1, \psi_2) \\
&\ge \|\psi_1\|_{H^1}^2 + \|\psi_2\|^2 - 2\epsilon\Re\left((\nabla\psi_1, \nabla\psi_2) + (\psi_1, \psi_2) + (\Delta\psi_1, \psi_2)\right).
\end{aligned}$$

Now we use integration by parts and the fact that $\psi_1, \psi_2 \in H_0^1(\Omega)$

$$\begin{aligned}&\|\psi_1\|_{H^1}^2 + \|\psi_2\|^2 - 2\epsilon\Re\left((\nabla\psi_1, \nabla\psi_2) + (\psi_1, \psi_2) + (\Delta\psi_1, \psi_2)\right)\\ &= \|\psi_1\|_{H^1}^2 + \|\psi_2\|^2 - 2\epsilon\Re(\psi_1, \psi_2)\end{aligned}$$

and using Cauchy-Schwarz and Young inequality we get

$$\begin{aligned}&\|\psi_1\|_{H^1}^2 + \|\psi_2\|^2 - 2\epsilon\Re(\psi_1, \psi_2) \geq \|\psi_1\|_{H^1}^2 + \|\psi_2\|^2 - \epsilon(\|\psi_1\|^2 + \|\psi_2\|^2)\\ &\geq (1-\epsilon)\|\Psi\|_{\mathcal{H}}^2.\end{aligned}$$

Since $1 - x \geq (1-x)^2$ for $x \in (0,1)$ we can conclude with

$$\|\Psi\|_{\mathcal{H}} \leq \frac{1}{1-\epsilon}\|\Phi\|_{\mathcal{H}}.$$

□

Corollary 1.3.4. *Let $\epsilon \in (0,1)$ and $\Phi \in \mathcal{H}$. Then the equation*

$$\Psi - \epsilon\mathcal{A}_0\Psi = \Phi$$

has a unique solution $\Psi \in \mathrm{Dom}(\mathcal{A}_0)$ and moreover

$$\|\Psi\|_{\mathcal{H}} \leq \frac{1}{1-\epsilon}\|\Phi\|_{\mathcal{H}}. \tag{1.46}$$

Proof. Let $\epsilon \in (0,1)$. Then the Lemma 1.3.3 implies that $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ lies in the range of the operator $I - \epsilon\mathcal{A}_0$. From the closedness of $\mathcal{A}_0$ it then follows that $I - \epsilon\mathcal{A}_0$ is surjective. □

Now we are able to prove the validity of inequality (1.40) for $\mathcal{A}_0$. Since from Corollary 1.3.4 we have that $I - \epsilon\mathcal{A}_0 : \mathrm{Dom}(\mathcal{A}_0) \subset \mathcal{H} \to \mathcal{H}$ is a bijection for $\epsilon \in (0,1)$ the operator $(I - \epsilon\mathcal{A}_0)^{-1}$ exists. This implies that also the operator $(zI - \mathcal{A}_0)^{-1}$ exists for $z \in (1, +\infty)$ where $z = 1/\epsilon$ with

$$\|(zI - \mathcal{A}_0)^{-1}\| \leq (z-1)^{-1}$$

provided that (1.46) holds for all functions from $\mathrm{Dom}(\mathcal{A}_0)$ and the norm of the inverse of some bounded operator T can be computed as

$$\|T^{-1}\| = \sup\left\{\frac{\|x\|}{\|Tx\|} : x \in \mathrm{Dom}(T),\ x \neq 0\right\}.$$

Since $\|AB\| \leq \|A\|\|B\|$ for arbitrary bounded operators A and B we have that

$$\|((zI - \mathcal{A}_0)^{-1})^n\| \leq (z-1)^{-n}, \quad n \in \mathbb{N} \tag{1.47}$$

for all $z \in (1, +\infty)$ which proves inequality (1.40) where $M = 1$. Thus according to Theorem 1.3.1 the undamped wave operator $\mathcal{A}_0$ is the infinitesimal generator of a C_0-semigroup denoted by $e^{\mathcal{A}_0 t}$ with growth rate $\omega = 1$, i.e.

$$\|e^{\mathcal{A}_0 t}\| \leq e^t, \quad 0 \leq t < +\infty.$$

Remark 1.3.5 ($\mathcal{A}_0$ as an infinitesimal generator of a C_0-group)**.** *Moreover it can be shown that $\mathcal{A}_0$ generates a C_0-group at least in the case $\Omega = \mathbb{R}^d$.*

Definition 1.3.6 (C_0-group). *Let X be a Banach space. A one parameter family $T(t)$, $-\infty < t < +\infty$ is a group of bounded linear operators on X is a C_0-group of bounded operators if the following is satisfied*

1. $T(0) = I$
2. $T(t+s) = T(t)T(s)$, $-\infty < t, s < +\infty$
3. $\lim_{t\to 0} T(t)x = x$, $x \in X$.

Being a C_0-group is a stronger property than being a C_0-semigroup, meaning that every C_0-group is a C_0-semigroup. The fact that $\mathcal{A}_0$ generates also a C_0-group can be obtained in a similar way using the following theorem.

Theorem 1.3.7 ([16, Theorem 1.6.3]). *A linear operator A on Banach space X is an infinitesimal generator of a C_0-group $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega|t|}$ if and only if*

1. *A is closed and densely defined*
2. *Every real λ, $|\lambda| > \omega$ lies in the resolvent set $\rho(A)$ and it satisfies*

$$\|R_\lambda(A)^n\| \leq \frac{M}{(|\lambda| - \omega)^n}, \quad \forall\, n \in \mathbb{N}. \tag{1.48}$$

This has been proven in [16, Section 7.4] by a similar method which was used here to prove the generation of the C_0-semigroup.

Finally we use a perturbation theorem to prove that also the damped wave operator $\mathcal{A}$ generates a C_0-semigroup.

Theorem 1.3.8 ([16, Theorem 3.1.1]). *Let X be a Banach space and let A be the infinitesimal generator af a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X then $A + B$ is an infinitesimal generator of a C_0-semigroup $S(t)$ satisfying $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$.*

Since we already know that $\mathcal{B}$ is a bounded operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $e^{\mathcal{A}t}$ with the growth rate

$$\|e^{\mathcal{A}t}\| \leq e^{(1+\|a\|_\infty)t}.$$

Remark 1.3.9. *For comparison we provide the result from [7, Theorem 5] where they obtained a different growth bound for the C_0-semigroup $e^{\mathcal{A}t}$ using a different method*

$$\|e^{\mathcal{A}t}\| \leq e^{2(1+|a_{min}|)t}, \quad 0 \leq t < +\infty.$$

Here a_{min} is the essential infimum of the damping a.

Remark 1.3.10 (Complex damping). *As can be seen from Theorem 1.3.8 we could have assumed that the damping function a is complex in general and we would still get the generation of the C_0-semigroup.*

Recalling Theorem 1.3.1 we see that $\rho(\mathcal{A}) \supset (1 + \|a\|_\infty, +\infty)$. Moreover [16, Remark 1.5.4] implies that every complex μ such that $\Re\mu > 1 + \|a\|_\infty$ lies in the resolvent set of $\mathcal{A}$. The following proposition summarizes the results obtained so far.

Proposition 1.3.11. *Let Ω be an arbitrary domain in $\mathbb{R}^d$. The damped wave operator $\mathcal{A}$ with bounded damping generates a C_0-semigroup $e^{\mathcal{A}t}$ with the growth bound $1 + \|a\|_\infty$. Moreover*

$$\sigma(\mathcal{A}) \subset \{\mu \in \mathbb{C} : \Re\mu \leq 1 + \|a\|_\infty\}.$$

Next we move on to the regularity and uniqueness of the solutions. Since $\mathcal{A}$ is the infinitesimal generator of $e^{\mathcal{A}t}$ we already know from the previous section that given $U_0 \in \mathrm{Dom}(\mathcal{A})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathcal{A}U(t), \quad U(0) = U_0$$

where $U(t) := e^{\mathcal{A}t}U_0$ and thus using the C_0-semigroup we obtain a weak solution of evolution problem (1.8) for $\mathcal{A}$. Also we know that $U(t)$ is a continuous function from $[0, +\infty) \to \mathcal{H}$. Moreover if $U_0 \in \mathrm{Dom}(\mathcal{A})$ then $U(t) \in \mathrm{Dom}(\mathcal{A})$ and according to Theorem 1.2.6, $U(t)$ is unique and continuously differentiable on $[0, +\infty)$. Recall $U(t) = \begin{pmatrix} u \\ u_t \end{pmatrix}$ and $U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$. In particular given $U_0 \in \mathrm{Dom}(\mathcal{A})$ this means that there exists a unique solution $u(t, x)$ of (1.8) satisfying

$$u(t, \cdot) \in C^0([0, +\infty), \mathrm{Dom}(-\Delta)) \cap C^1([0, +\infty), H_0^1(\Omega)) \cap C^2([0, +\infty), L^2(\Omega)).$$

1.3.3 Time evolution

Our aim now is to show that under a sign-changing condition evolution system (1.8) with the operator $\mathcal{A}$ possesses an unstable solution for sufficiently large damping in some sense. In particular we parameterize the damping a by a positive multiplication constant α, i.e. $a = \alpha b$ and accordingly we denote the operator $\mathcal{A}$ with damping αb, the parameterized damped wave operator, by $\mathcal{A}_\alpha$, i.e.

$$\mathcal{A}_\alpha := \begin{pmatrix} 0 & I \\ \Delta & -\alpha b \end{pmatrix} \tag{1.49}$$

and of course the domain of $\mathcal{A}_\alpha$ remains unchanged. Henceforth we assume the sign-changing condition for the damping to be

$$\operatorname*{ess\,inf}_{x \in \Omega} b(x) < 0 \quad \text{and} \quad \operatorname*{ess\,sup}_{x \in \Omega} b(x) > 0.$$

To prove the instability we would need the existence of at least one positive point in the spectrum. It was proven by Krejčiřík and Freitas in [7, Theorem 2] that there exists $\alpha_0 > 0$ such that for $\alpha > \alpha_0$ there is at least one positive point in the spectrum of $\mathcal{A}_\alpha$. Denote this point by λ. If λ lies in the point spectrum then the evolution problem transforms to

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \lambda U(t), \quad U(0) = U_0$$

and thus $U(t) = U_0 e^{\lambda t}$. Since $\lambda > 0$ we see that there exist initial conditions U_0 for which the solution grows exponentially in time and thus is unstable.

On the other hand if λ lies in the essential spectrum we obtain the instability from a result by Solà-Morales in [18, Theorem 1]. It states that since $\mathcal{A}$ generates a C_0-semigroup evolution problem (1.8) has the so called global instability property if there is a positive point in the essential spectrum of $\mathcal{A}$. The global instability property means that for every residual subset of initial values in $\mathrm{Dom}(\mathcal{A})$ and for every initial value U_0 from such a set the corresponding semiorbit $\gamma^+(U_0) := \{U(t) : U(0) = U_0,\ t \in [0, +\infty)\}$ is unbounded. A residual set is the complement of the set formed by a countable union of nowhere dense sets (sets where the interior of the closure is an empty set). Moreover this implies that $\mathrm{Dom}(\mathcal{A})$ has no positively invariant bounded sets with points stable in the sense of Lyapunov [18].

Chapter 2

Schrödinger operator

Henceforth let Ω be an arbitrary domain in $\mathbb{R}^d$ as in the previous chapter. In this chapter we state the definition of the self-adjoint Laplace operator on $L^2(\Omega)$ with Dirichlet boundary conditions (the Dirichlet Laplacian) and afterwards we properly define the self-adjoint Schrödinger operator as a perturbation of the Dirichlet Laplacian for suitable class of potentials. Furthermore we state some of its spectral properties needed in the rest of the thesis.

2.1 Dirichlet Laplacian

The Dirichlet Laplacian on $L^2(\Omega)$ is defined as the Friedrichs extension of the minimal operator

$$\dot{\mathcal{T}}\psi := -\Delta\psi, \quad \mathrm{Dom}(\dot{\mathcal{T}}) := C_0^\infty(\Omega)$$

where $-\Delta\psi = -\sum_{i=1}^d \frac{\mathrm{d}^2\psi}{\mathrm{d}x_i^2}$ and the derivatives are meant in the weak sense. $C_0^\infty(\Omega)$ denotes the space of smooth functions with compact support in Ω. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ this operator is densely defined. Moreover for $\psi, \phi \in C_0^\infty(\Omega)$ using the integration by parts we get

$$(\psi, -\Delta\phi) = (\nabla\psi, \nabla\phi) = (-\Delta\psi, \phi) \tag{2.1}$$

where ∇ stands for the weak gradient. Hence $\dot{\mathcal{T}}$ is symmetric. Plugging $\phi := \psi$ into (2.1) we get $(\psi, -\Delta\psi) = (\nabla\psi, \nabla\psi) = \|\nabla\psi\|^2 \geq 0$ from which it follows that $\dot{\mathcal{T}}$ is a positive operator.

Moreover the quadratic form induced by $\dot{\mathcal{T}}$ is

$$\mathcal{Q}_{\dot{\mathcal{T}}}[\psi] := (\psi, \dot{\mathcal{T}}\psi) = \|\nabla\psi\|^2, \quad \mathrm{Dom}(\mathcal{Q}_{\dot{\mathcal{T}}}) := C_0^\infty(\Omega)$$

which is again densely defined, symmetric and positive (more specifically bounded from below with the bound equal to 0) using the same arguments as before. Since this form is induced by a positive and symmetric operator it is closable [4, Theorem 4.4.5] and its closure is

$$\mathcal{Q}_{\mathcal{T}}[\psi] := \|\nabla\psi\|^2, \quad \mathrm{Dom}(\mathcal{Q}_{\mathcal{T}}) := H_0^1(\Omega).$$

Indeed since $\mathcal{Q}_{\dot{\mathcal{T}}}$ is bounded from below by 0 it induces an inner product on $\mathrm{Dom}(\mathcal{Q}_{\dot{\mathcal{T}}})$ which is equal to the inner product defined on $H^1(\Omega)$ and moreover the closure of $\mathrm{Dom}(\mathcal{Q}_{\dot{\mathcal{T}}})$ in $H^1(\Omega)$ is by the definition the space $H_0^1(\Omega)$, see [20, Section 5.5].

Finally using the Representation theorem [20, Theorem 5.37] we get that there exists a self-adjoint and bounded from below operator $\mathcal{T}$ associated with $\mathcal{Q}_{\mathcal{T}}$ defined as

$$\mathcal{T}\psi := -\Delta\psi, \quad \mathrm{Dom}(\mathcal{T}) := \{\psi \in \mathrm{Dom}(\mathcal{Q}_{\mathcal{T}}) : \exists \phi \in L^2(\Omega),\ \forall \varphi \in \mathrm{Dom}(\mathcal{Q}_{\mathcal{T}}),\ \mathcal{Q}_{\mathcal{T}}(\varphi, \psi) = (\varphi, \phi)\}$$

where $\mathcal{Q}_\mathcal{T}(\cdot,\cdot)$ is the sesquilinear form determined uniquely by the quadratic form $\mathcal{Q}_\mathcal{T}[\cdot]$ via the polarization identity. In this particular case $\mathcal{Q}_\mathcal{T}(\psi,\phi) = (\nabla\psi, \nabla\phi)$. Looking at the domain of $\mathcal{T}$ we see that it can be rewritten using the definition of the weak Laplacian, i.e.

$$\mathcal{T}\psi := -\Delta\psi, \quad \mathrm{Dom}(\mathcal{T}) := \{\psi \in H_0^1(\Omega) : \Delta\psi \in L^2(\Omega)\}.$$

The operator $\mathcal{T}$ will be called as the Dirichlet Laplacian. Further specifications of the domain of $\mathcal{T}$ are possible however they require some regularity of Ω.

Remark 2.1.1 (Domain of $\mathcal{T}$)**.** *Here we state some examples of the domains of $\mathcal{T}$. This list is not complete, another examples are known.*

- *For arbitrary $\Omega \subset \mathbb{R}^d$ it holds*

$$\mathrm{Dom}(\mathcal{T}) = \left\{\psi \in H_0^1(\Omega) : \Delta\psi \in L^2(\Omega)\right\}.$$

- *For $\Omega \subset \mathbb{R}^d$ bounded and of class C^2 we have*

$$\mathrm{Dom}(\mathcal{T}) = H_0^1(\Omega) \cap H^2(\Omega)$$

which follows from the elliptic regularity theory.

- *For $\Omega = \mathbb{R}^d$ we get*

$$\mathrm{Dom}(\mathcal{T}) = H^2(\mathbb{R}^d)$$

since it can be shown that $H_0^1(\mathbb{R}^d) = H^1(\mathbb{R}^d)$.

Remark 2.1.2 (Spectrum of $\mathcal{T}$)**.** *We state some known facts about the spectrum of $\mathcal{T}$ depending on Ω which will be useful in the following.*

- *For arbitrary $\Omega \subset \mathbb{R}^d$ it holds that $\sigma(\mathcal{T}) \subset [0,+\infty)$ since $\mathcal{T}$ is a positive operator.*
- *For bounded $\Omega \subset \mathbb{R}^d$ we know that the spectrum is purely discrete, i.e. $\sigma(\mathcal{T}) = \sigma_{\mathrm{disc}}(\mathcal{T})$.*
- *For $\Omega = (0, L)$ (and also for every other bounded interval) we know the explicit formula for the spectrum*

$$\sigma(\mathcal{T}) = \left\{\left(\frac{n\pi}{L}\right)^2\right\}_{n=1}^{+\infty}.$$

- *For $\Omega = \mathbb{R}^d$ the spectrum is purely continuous and equal to the upper half-line, in particular*

$$\sigma(\mathcal{T}) = \sigma_{\mathrm{c}}(\mathcal{T}) = \sigma_{\mathrm{ess}}(\mathcal{T}) = [0,+\infty). \tag{2.2}$$

2.2 Schrödinger operator

Now we move on to the definition of the Schrödinger operator being a perturbation of the Dirichlet Laplacian. Henceforth let $V_0 \in L^\infty(\Omega,\mathbb{R})$ denote the potential. We define the multiplication operator $\mathcal{V}_0$ associated with V_0 as

$$\mathcal{V}_0\psi := V_0\psi, \quad \mathrm{Dom}(\mathcal{V}_0) := L^2(\Omega)$$

and with the quadratic form

$$\mathcal{Q}_{\mathcal{V}_0}[\psi] := (\psi, \mathcal{V}_0\psi) = \int_\Omega V_0|\psi|^2, \quad \mathrm{Dom}(\mathcal{Q}_{\mathcal{V}_0}) := L^2(\Omega).$$

For the norm of $\mathcal{V}_0$ we can easily obtain the following bound

$$\|\mathcal{V}_0\| = \sup_{\substack{\psi\in\mathrm{Dom}(\mathcal{V}_0)\\ \psi\neq 0}} \frac{\|\mathcal{V}_0\psi\|}{\|\psi\|} = \sup_{\substack{\psi\in L^2(\Omega)\\ \psi\neq 0}} \frac{\|V_0\psi\|}{\|\psi\|} \leq \sup_{\substack{\psi\in L^2(\Omega)\\ \psi\neq 0}} \frac{\|V_0\|_\infty\|\psi\|}{\|\psi\|} = \|V_0\|_\infty$$

hence $\mathcal{V}_0$ is a bounded operator (in fact it holds that $\|\mathcal{V}_0\| = \|V_0\|_\infty$). Now we state the perturbation theorem which will enable us to define the self-adjoint Schrödinger operator.

Theorem 2.2.1 (Kato-Rellich, [17, Theorem X.12]). *Let A be a self-adjoint operator on Hilbert space H bounded from below by M. Let B be a symmetric and A-bounded operator on H with the relative bound less than* 1*. Then $A + B$ is self-adjoint on* $\mathrm{Dom}(A)$ *and bounded from below by $M - \max\{\frac{b}{1-a}, a|M| + b\}$ where a, b are defined in* (2.3).

The property of A-boundedness from the theorem means that $\mathrm{Dom}(B) \supset \mathrm{Dom}(A)$ and that there exist $a, b \in \mathbb{R}$ such that

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\| \tag{2.3}$$

holds for all $\psi \in \mathrm{Dom}(A)$. The infimum of such a is called a relative bound. Since $\mathcal{V}_0$ is bounded, it is certainly $\mathcal{T}$-bounded with relative bound equal to 0. Thus using the Kato-Rellich theorem (in fact since V_0 is bounded this could be done more easily) we can define the self-adjoint and bounded from below Schrödinger operator $\mathcal{S} := \mathcal{T} + \mathcal{V}_0$. In particular

$$\mathcal{S}\psi := -\Delta\psi + V_0\psi, \quad \mathrm{Dom}(\mathcal{S}) := \{\psi \in H_0^1(\Omega) : \Delta\psi \in L^2(\Omega)\}$$

with the corresponding quadratic form

$$\mathcal{Q}_{\mathcal{S}}[\psi] := \int_\Omega |\nabla\psi|^2 + \int_\Omega V_0|\psi|^2, \quad \mathrm{Dom}(\mathcal{Q}_{\mathcal{S}}) := H_0^1(\Omega).$$

This operator is bounded from below by $-\|V_0\|_\infty$. The Theorem 2.2.1 allows us to define the self-adjoint Schrödinger operator for larger class of the potentials however this is not needed in the thesis.

According to [4, Section 4.5] we define the numbers

$$\lambda_n := \inf_{\substack{L_n\subset\mathrm{Dom}(\mathcal{S})\\ \dim L_n = n}} \sup_{\substack{\psi\in L_n\\ \psi\neq 0}} \frac{(\psi, \mathcal{S}\psi)}{\|\psi\|^2} = \inf_{\substack{L_n\subset\mathrm{Dom}(\mathcal{Q}_{\mathcal{S}})\\ \dim L_n = n}} \sup_{\substack{\psi\in L_n\\ \psi\neq 0}} \frac{\mathcal{Q}_{\mathcal{S}}[\psi]}{\|\psi\|^2} \tag{2.4}$$

for $n \in \mathbb{N}$. It holds that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\infty$ with $\lambda_\infty := \lim_{n\to+\infty}\lambda_n = \inf\sigma_{\mathrm{ess}}(\mathcal{S})$ and with the convention $\lambda_\infty := +\infty$ if $\sigma_{\mathrm{ess}}(\mathcal{S}) = \emptyset$. These numbers represent either a discrete eigenvalue of $\mathcal{S}$ (which is below the essential spectrum) or the threshold of $\sigma_{\mathrm{ess}}(\mathcal{S})$. It also holds that $\lambda_1 = \inf\sigma(\mathcal{S})$.

Now we will state selected spectral properties of the operator $\mathcal{S}$ which will be needed in the rest of the thesis. First we will state a well-known result in the case $\Omega = \mathbb{R}^d$. The proof of this statement is made by showing that under some assumptions $\mathcal{V}_0$ is relatively compact with respect to $\mathcal{T}$ which ensures that the essential spectrum of the Schrödinger operator $\mathcal{S}$ with potential $\mathcal{V}_0$ is equal to the essential spectrum of the operator without potential (2.2).

Definition 2.2.2 (Relative compactness)**.** *Let A be a self-adjoint operator on a Hilbert space H. Let C be an operator on H such that $\mathrm{Dom}(C) \supset \mathrm{Dom}(A)$. Then C is relatively compact with respect to A if and only if $CR_A(\lambda)$ is compact for some $\lambda \in \rho(A)$.*

If $CR_A(\lambda)$ is compact for some $\lambda \in \rho(A)$ then it is compact for all points in $\rho(A)$. This follows easily from the first resolvent identity [17, Theorem VI.6]

$$R_A(\lambda_1) - R_A(\lambda_2) = (\lambda_1 - \lambda_2)R_A(\lambda_1)R_A(\lambda_2), \quad \lambda_1, \lambda_2 \in \rho(A).$$

Moreover every compact operator is relatively compact with respect to a self-adjoint operator whose domain is included in the domain of the compact operator.

Theorem 2.2.3 (Weyl, [17, Corrolary XIII.4.2])**.** *Let A be a self-adjoint operator on a Hilbert space H and let C be a relatively compact with respect to A. Then $A + C$ is a closed operator and $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A + C)$.*

We now prove the relative compactness of $\mathcal{V}_0$ provided that $V_0 \xrightarrow[|x|\to+\infty]{} 0$. First consider the operator $R_{\mathcal{T}}(\lambda)$ for some $\lambda \in \rho(\mathcal{T})$. For concreteness we choose $\lambda = -1$. Then using the properties of the Fourier transform F on $L^2(\mathbb{R}^d)$ we get

$$\mathcal{F}\psi \equiv R_{\mathcal{T}}(-1)\psi = F^{-1}\left(-\frac{1}{1+|p|^2}(F\psi)(p)\right)$$

where $|\cdot|$ now stands for the standard form on $\mathbb{R}^d$. Hence define the function

$$f : \mathbb{R}^d \to \mathbb{R} : p \mapsto -\frac{1}{1+|p|^2}$$

which is certainly bounded and tends to 0 as $|p| \to +\infty$. Denote the restrictions of the functions f and V_0 on the ball B_R in $\mathbb{R}^d$ with radius R centered at the origin extended by zero on the rest of $\mathbb{R}^d$ by f_R and V_R respectively. Moreover denote the associated multiplication operators by $\mathcal{F}_R$ and $\mathcal{V}_{0R}$ respectively. The functions f_R and V_R lie in $L^2(\mathbb{R}^d)$ thus

$$\mathcal{F}_R\psi(x) := F^{-1}\left((f_R(p)(F\psi)(p))\right)(x)$$

is a Hilbert-Schmidt operator which is known to be compact. Next we estimate

$$\|\mathcal{V}_0\mathcal{F} - \mathcal{V}_{0R}\mathcal{F}_R\| = \|\mathcal{V}_0(\mathcal{F} - \mathcal{F}_R) + (\mathcal{V}_0 - \mathcal{V}_{0R})\mathcal{F}_R\| \le \|V_0\|_\infty\|f - f_R\|_\infty + \|V_0 - V_R\|_\infty\|f_R\|_\infty \tag{2.5}$$

since

$$\|\mathcal{V}_0\| = \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|\mathcal{V}_0\psi\|}{\|\psi\|} \le \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|V_0\|_\infty\|\psi\|}{\|\psi\|} = \|V_0\|_\infty$$

as well as

$$\|\mathcal{V}_0 - \mathcal{V}_{0R}\| = \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|(\mathcal{V}_0 - \mathcal{V}_{0R})\psi\|}{\|\psi\|} \le \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|V_0 - V_R\|_\infty\|\psi\|}{\|\psi\|} = \|V_0 - V_R\|_\infty$$

and moreover

$$\|\mathcal{F}_R\| = \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|F^{-1}\left(f_R F\psi\right)\|}{\|\psi\|} = \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|f_R\|_\infty\|\psi\|}{\|\psi\|} = \|f_R\|_\infty$$

and

$$\|\mathcal{F}-\mathcal{F}_R\| = \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|F^{-1}\left((f-f_R)F\psi\right)\|}{\|\psi\|} = \sup_{\substack{\psi\in L^2(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\|f-f_R\|_\infty\|\psi\|}{\|\psi\|} = \|f-f_R\|_\infty$$

provided that the Fourier transform is a linear and unitary operator. From

$$V_0 \xrightarrow[|x|\to+\infty]{} 0 \quad \text{and} \quad f \xrightarrow[|p|\to+\infty]{} 0$$

it follows that

$$\|f-f_R\|_\infty \xrightarrow[R\to+\infty]{} 0 \quad \text{and} \quad \|V_0-V_R\| \xrightarrow[R\to+\infty]{} 0.$$

Thus from (2.5) we get

$$\|\mathcal{V}_0\mathcal{F}-\mathcal{V}_{0R}\mathcal{F}_R\| \xrightarrow[R\to+\infty]{} 0.$$

Since $\mathcal{V}_{0R}$ is a bounded operator $\mathcal{V}_{0R}\mathcal{F}_R$ is compact. Finally we see that $\mathcal{V}_0\mathcal{F}$ is a limit of compact operators hence a compact operator provided that the space of compact operators is closed in the space of bounded operators. Thus from Weyl theorem 2.2.3 we get that $\sigma_{\text{ess}}(\mathcal{S}) = \sigma_{\text{ess}}(\mathcal{T})$ which is in the case $\Omega = \mathbb{R}^d$ equal to $[0,+\infty)$, see (2.2). Thus we have just proven the following theorem.

Theorem 2.2.4. *Let $\Omega = \mathbb{R}^d$ and $V_0 \in L^\infty(\mathbb{R}^d)$. If $V_0 \xrightarrow[|x|\to+\infty]{} 0$ then $\sigma_{\text{ess}}(\mathcal{S}) = [0,+\infty)$.*

This theorem implies that under its assumptions $\sigma_{\text{disc}}(\mathcal{S}) \subset (-\infty,0)$ thus all the numbers λ_n are nonpositive. It follows that one needs V_0 to be strictly negative at least somewhere on $\mathbb{R}^d$ for $\mathcal{S}$ to have non-empty discrete spectrum (otherwise $\mathcal{S}$ would be a positive operator). Summarizing the assumptions on V_0 we have

$$\Omega = \mathbb{R}^d, \quad V_0 \in L^\infty(\mathbb{R}^d,\mathbb{R}) \quad \text{and} \quad V_0 \xrightarrow[|x|\to+\infty]{} 0. \tag{2.6}$$

We conclude with the following propositions. Some of them require weaker assumptions on the potential however this will not be needed in the thesis.

Proposition 2.2.5. *Assume* (2.8). *If λ_1 is an eigenvalue then it is non-degenerate and the corresponding eigenfunction can be chosen to be strictly positive. If $V \leq 0$ and λ_n defined in* (2.4) *is a discrete eigenvalue then it is strictly negative.*

Proof. The property that λ_n is strictly negative was already discussed in the previous paragraph. Let λ_1 be an eigenvalue. The positivity of its eigenfunction ψ_1 follows directly from the definition of λ_1 (2.4) which in the case $n=1$ simplifies into

$$\lambda_1 = \inf_{\substack{\psi\in H_0^1(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\mathcal{Q}_\mathcal{S}[\psi]}{\|\psi\|^2} = \frac{\displaystyle\int_{\mathbb{R}^d}|\nabla\psi_1|^2 + \int_{\mathbb{R}^d} V_0|\psi_1|^2}{\displaystyle\int_{\mathbb{R}^d}|\psi_1|^2}$$

and we see that if ψ_1 is an eigenvalue then $|\psi_1|$ is also an eigenvalue. The fact that $|\psi_1| > 0$ follows from the so called Harnack inequality. For more details see [9, Theorem 8.38]. From this it also follows that λ_1 is simple since its eigenfunction is either strictly positive or strictly negative and two such functions cannot be orthogonal. $\square$

Proposition 2.2.6. *Let $d \leq 2$ and let $V_0 \in L^1(\mathbb{R}^d)$. If $\int_{\mathbb{R}^d} V_0 < 0$ then $\inf \sigma(\mathcal{S}) < 0$.*

Proof. Let $d \leq 2$ and consider the test function

$$\psi_n(x) := \varphi_n(|x|), \quad \varphi_n(r) := \begin{cases} 1 & r < n \\ \frac{\log n^2 - \log r}{\log n^2 - \log n} & n < r < n^2 \\ 0 & r > n^2 \end{cases}.$$

direct computation shows that $\psi_n \in H_0^1(\mathbb{R}^d)$ for every $n \in \mathbb{N}$. Moreover it holds

$$\psi_n(x) \xrightarrow[n\to+\infty]{} 1, \quad \forall x \in \mathbb{R}^d \quad \text{and} \quad \|\nabla\psi_n\| \xrightarrow[n\to+\infty]{} 0.$$

Recall that for all $\psi \in H_0^1(\mathbb{R}^d)$ we have

$$\inf \sigma(\mathcal{S}) = \lambda_1 \leq \frac{\displaystyle\int_{\mathbb{R}^d} |\nabla\psi|^2 + \int_{\mathbb{R}^d} V_0|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2}.$$

Hence plugging ψ_n into this formula we get

$$\frac{\displaystyle\int_{\mathbb{R}^d} |\nabla\psi_n|^2 + \int_{\mathbb{R}^d} V_0|\psi_n|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi_n|^2} \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}^d} V_0 < 0$$

and thus $\inf \sigma(\mathcal{S})$ can be made negative by taking n sufficiently large. □

2.3 Schrödinger operator with energy dependent potential

In this section we will analyze the Schrödinger operator with energy dependent potential. The fact that it is energy-dependent will be seen at the end of this section. We parameterize the potential of the operator $\mathcal{S}$ by a multiplication constant. In particular we define $V_0 := \mu V$ for $\mu \in \mathbb{R}$ and denote the operator $\mathcal{S}$ with the potential $\mu\mathcal{V}$ by $\mathcal{S}_\mu := \mathcal{T} + \mu\mathcal{V}$ where $\mathcal{V}$ is the multiplication operator generated by V, i.e.

$$\mathcal{S}_\mu\psi := -\Delta\psi + \mu V\psi, \quad \mathrm{Dom}(\mathcal{S}_\mu) := \{\psi \in H_0^1(\Omega) : \Delta\psi \in L^2(\Omega)\} \tag{2.7}$$

with the associated quadratic form

$$\mathcal{Q}_{\mathcal{S}_\mu}[\psi] := \int_\Omega |\nabla\psi|^2 + \mu\int_\Omega V|\psi|^2, \quad \mathrm{Dom}(\mathcal{Q}_{\mathcal{S}_\mu}) := H_0^1(\Omega).$$

Moreover we denote numbers (2.4) corresponding to $\mathcal{S}_\mu$ by $\lambda_n(\mu)$. The assumptions on V_0 from the previous section now transform to formally same assumptions since multiplication by a constant does not affect them. Therefore we assume

$$\Omega = \mathbb{R}^d, \quad V \in L^\infty(\mathbb{R}^d, \mathbb{R}), \quad V \xrightarrow[|x|\to+\infty]{} 0 \quad \text{and} \quad \mu \in \mathbb{R}. \tag{2.8}$$

Now we state some results concerning $\lambda_n(\mu)$. Some of them again require weaker assumptions than (2.8) however this will not be needed in the thesis.

Proposition 2.3.1. *Assume* (2.8). *Then $\mu \mapsto \lambda_n(\mu)$ is Lipschitz continuous function on $\mathbb{R}$.*

Proof. Let $\mu_1,\ \mu_2 \in \mathbb{R}$. Then from definition (2.4) we have

$$\begin{aligned}
\lambda_n(\mu_1) &= \inf_{\substack{L_n \subset \mathrm{Dom}(\mathcal{S}) \\ \dim L_n = n}} \sup_{\substack{\psi \in L_n \\ \psi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^d} |\nabla \psi|^2 + \mu_1 \int_{\mathbb{R}^d} V|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2} \\
&= \inf_{\substack{L_n \subset \mathrm{Dom}(\mathcal{S}) \\ \dim L_n = n}} \sup_{\substack{\psi \in L_n \\ \psi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^d} |\nabla \psi|^2 + (\mu_1 - \mu_2) \int_{\mathbb{R}^d} V|\psi|^2 + \mu_2 \int_{\mathbb{R}^d} V|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2} \\
&\leq \lambda_n(\mu_2) + |\mu_1 - \mu_2| \|V\|_\infty .
\end{aligned}$$

Therefore interchanging the roles of μ_1 and μ_2 we obtain

$$|\lambda_n(\mu_1) - \lambda_n(\mu_2)| \leq |\mu_1 - \mu_2| \|V\|_\infty$$

which means that $\lambda_n(\mu)$ is Lipschitz continuous. □

Moreover there exists an asymptotic formula for $\lambda_n(\mu)$ as $\mu \to +\infty$ respectively $\mu \to -\infty$.

Proposition 2.3.2 ([7, Theorem 4])**.** *Assume* (2.8). *Then $\lambda_n(\mu)$ satisfies the uniform asymptotics (not depending on the parameter n)*

$$\lambda_n(\mu) = \begin{cases} V_{\min}\mu + o(\mu), & \mu \to +\infty \\ V_{\max}\mu + o(\mu), & \mu \to -\infty \end{cases}$$

where $V_{\min} = \operatorname{ess\,inf}_{x \in \mathbb{R}^d} V(x)$ and $V_{\max} = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} V(x)$.

As a next step we derive the formula for the first derivative of the function $\mu \mapsto \lambda_1(\mu)$. For this we need to show that $\mathcal{S}_\mu$ is an analytic family of type (A) which will ensure that $\lambda_1(\mu)$ and the corresponding eigenfunction are differentiable in μ.

Definition 2.3.3 (Analytic family of type (A))**.** *Let R be a connected domain in the complex plane and let $T(\beta)$ be a closed operator on Hilbert space H with non-empty resolvent set for every $\beta \in R$. Then $T(\beta)$ is an analytic family of type (A) if and only if*

1. $\mathrm{Dom}(T(\beta))$ *is independent of β*

2. $T(\beta)\psi$ *is a vector-valued analytic function of β for every $\psi \in \mathrm{Dom}(T(\beta))$.*

There exists a useful criterion which states that in the case where $T(\beta) = H + \beta W$, the family $T(\beta)$ of closed operators with non-empty resolvent set is an analytic family of type (A) if W is a bounded operator, see the Lemma and the corollary of its proof of Section XII.2 in [17]. Assuming that $\mathcal{V}$ is bounded, $\mathcal{S}_\mu,\ \mu \in \mathbb{R}$ satisfies the criterion and hence is an analytic family of type (A).

Now assume (2.8) and $V \leq 0, V \neq 0$, i.e. V is non-trivial. From Theorem XII.9 in [17] it follows that $\lambda_1(\mu)$ is an eigenvalue and an analytic function of the parameter μ provided that it is non-degenerate and $\inf \sigma(\mathcal{S}_\mu) < 0$. The non-degeneracy follows from Proposition 2.2.5. Moreover the fact that $\inf \sigma(\mathcal{S}_\mu) < 0$ holds for all V and $\mu > 0$ if $d \leq 2$, see Proposition 2.2.6.

For $d > 2$ we see from Proposition 2.3.2 that for sufficiently large $\mu > 0$ the number $\lambda_1(\mu)$ is strictly negative. Combining we have the analyticity of $\lambda_1(\mu)$ for $\mu > 0$ if $d \leq 2$ or $\mu > \mu_0 > 0$ if $d > 2$. The fact that the corresponding eigenvector is analytic in μ follows from [10, Section 7.3.2]. We can therefore compute the derivatives of $\lambda_1(\mu)$ and its eigenfunction. Note that we could have also assumed V positive and $\mu < 0$.

First we provide a formula for the first derivative of $\lambda_1(\mu)$. Let $\lambda_1(\mu)$ be an eigenvalue and $\psi = \psi(x, \mu) \in H^1(\mathbb{R}^d)$ its eigenfunction. From the Representation theorem we have that for all $\phi \in \mathrm{Dom}(\mathcal{Q}_{\mathcal{S}_\mu}) = H^1(\mathbb{R}^d)$

$$\mathcal{Q}_{\mathcal{S}_\mu}(\psi, \phi) = \int_{\mathbb{R}^d} \overline{\nabla\psi}\nabla\phi + \mu \int_{\mathbb{R}^d} V\overline{\psi}\phi = \lambda_1(\mu) \int_{\mathbb{R}^d} \overline{\psi}\phi = \lambda_1(\mu)(\psi, \phi). \tag{2.9}$$

We compute the derivative of this formula with respect to μ. For every $\phi \in H^1(\mathbb{R}^d)$ (different ϕ than before in general) we get

$$\int_{\mathbb{R}^d} \overline{\nabla\phi}\frac{\mathrm{d}\nabla\psi}{\mathrm{d}\mu} + \mu \int_{\mathbb{R}^d} V\overline{\phi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \int_{\mathbb{R}^d} V\overline{\phi}\psi = \frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu} \int_{\mathbb{R}^d} \overline{\phi}\psi + \lambda_1(\mu) \int_{\mathbb{R}^d} \overline{\phi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu}. \tag{2.10}$$

Both formulas hold for all $\phi \in H^1(\mathbb{R}^d)$ and thus we are able to substitute $\phi = \frac{\mathrm{d}\psi}{\mathrm{d}\mu}$ in (2.9) and $\phi = \psi$ in (2.10). Indeed $\frac{\mathrm{d}\psi}{\mathrm{d}\mu} \in H_0^1(\mathbb{R}^d)$ since the analyticity of $\mathcal{S}_\mu\psi$ means that there exists functions ψ^n such that for every $\mu > 0$ and $\epsilon > 0$

$$\psi = \psi(\mu) = \psi^0 + (\mu - \epsilon)\psi^1 + \frac{(\mu - \epsilon)^2}{2}\psi^2 + \dots.$$

Thus taking $\mu = \epsilon$ we get that $\psi^0 \in H_0^1(\mathbb{R}^d)$. Moreover

$$\frac{\mathrm{d}\psi}{\mathrm{d}\mu} = \psi^1 = \lim_{\mu \to \epsilon} \frac{\psi(\mu) - \psi^0}{\mu - \epsilon} \quad \text{in } L^2(\mathbb{R}^d)$$

and since $H_0^1(\mathbb{R}^d)$ is a closed subspace of $L^2(\mathbb{R}^d)$ we obtain that $\frac{\mathrm{d}\psi}{\mathrm{d}\mu} \in H_0^1(\mathbb{R}^d)$.

By substituting we obtain

$$\int_{\mathbb{R}^d} \overline{\nabla\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}\nabla\psi + \mu \int_{\mathbb{R}^d} V\overline{\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}\psi = \lambda_1(\mu) \int_{\mathbb{R}^d} \overline{\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}\psi \tag{2.11}$$

and

$$\int_{\mathbb{R}^d} \overline{\nabla\psi}\frac{\mathrm{d}\nabla\psi}{\mathrm{d}\mu} + \mu \int_{\mathbb{R}^d} V\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \int_{\mathbb{R}^d} V\overline{\psi}\psi = \frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu} \int_{\mathbb{R}^d} \overline{\psi}\psi + \lambda_1(\mu) \int_{\mathbb{R}^d} \overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu}. \tag{2.12}$$

Now we subtract (2.11) from complex conjugate of (2.12) and get

$$\int_{\mathbb{R}^d} V\overline{\psi}\psi = \frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu} \int_{\mathbb{R}^d} \overline{\psi}\psi$$

from which it follows

$$\frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu} = \frac{\displaystyle\int_{\mathbb{R}^d} V|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2}$$

which proves the following statement.

Theorem 2.3.4. *Assume* (2.8) *and let $V \leq 0$ and non-trivial. Let $\lambda_1(\mu)$ be an eigenvalue with eigenfunction ψ for all $\mu \in [\mu_0, +\infty)$. Then the function $\mu \mapsto \lambda_1(\mu)$ is analytic on $(\mu_0, +\infty)$ and its first derivative is*

$$\frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu} = \frac{\displaystyle\int_{\mathbb{R}^d} V|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2}.$$

This statement implies that $\lambda_1(\mu)$ is decreasing. This can be shown easily for all $\mu \mapsto \lambda_n(\mu)$ using definition (2.4). Indeed let $\mu_1 > \mu_2 > 0$ then

$$\begin{aligned}\lambda_n(\mu_1) &= \inf_{\substack{L_n \subset \mathrm{Dom}(\mathcal{S}) \\ \dim L_n = n}} \sup_{\substack{\psi \in L_n \\ \psi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^d} |\nabla\psi|^2 + (\mu_1 - \mu_2)\int_{\mathbb{R}^d} V|\psi|^2 + \mu_2 \int_{\mathbb{R}^d} V|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2} \\ &= \lambda_n(\mu_2) + \inf_{\substack{L_n \subset \mathrm{Dom}(\mathcal{S}) \\ \dim L_n = n}} \sup_{\substack{\psi \in L_n \\ \psi \neq 0}} \frac{\displaystyle(\mu_1 - \mu_2)\int_{\mathbb{R}^d} V|\psi|^2}{\displaystyle\int_{\mathbb{R}^d} |\psi|^2}.\end{aligned}$$

Since the last term is negative whenever $V \leq 0$ we have $\lambda_n(\mu_1) - \lambda_n(\mu_2) \leq 0$ which implies that $\mu \mapsto \lambda_n(\mu)$ is decreasing.

Proposition 2.3.5. *Assume* (2.8) *and let $V \leq 0$ and non-trivial. The function $\mu \mapsto \lambda_n(\mu)$ is decreasing on $(0, +\infty)$ for all $n \in \mathbb{N}$.*

In the case of $\lambda_1(\mu)$ which is analytic on a suitable interval we are also able to compute the higher derivatives. We will now derive the formula for the second derivative. First we compute the derivative of (2.10), i.e. the second derivative of (2.9). For all $\phi \in H^1(\mathbb{R}^d)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \overline{\nabla\phi}\frac{\mathrm{d}^2\nabla\psi}{\mathrm{d}\mu^2} + 2\int_{\mathbb{R}^d} V\overline{\phi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \mu\int_{\mathbb{R}^d} V\overline{\phi}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2} \\ &= 2\frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu}\int_{\mathbb{R}^d} \overline{\phi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \lambda_1(\mu)\int_{\mathbb{R}^d} \overline{\phi}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2} + \frac{\mathrm{d}^2\lambda_1(\mu)}{\mathrm{d}\mu^2}\int_{\mathbb{R}^d} \overline{\phi}\psi.\end{aligned} \tag{2.13}$$

Now we substitute $\phi = \frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2}$ into (2.9) ,($\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2} \in H^1_0(\mathbb{R}^d)$ using the same arguments as before) and $\phi = \psi$ into (2.13) obtaining

$$\int_{\mathbb{R}^d} \overline{\nabla\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2}}\nabla\psi + \mu\int_{\mathbb{R}^d} V\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2}}\psi = \lambda_1(\mu)\int_{\mathbb{R}^d} \overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2}}\psi$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d} \overline{\nabla\psi}\frac{\mathrm{d}^2\nabla\psi}{\mathrm{d}\mu^2} + 2\int_{\mathbb{R}^d} V\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \mu\int_{\mathbb{R}^d} V\overline{\psi}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2} \\ &= 2\frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu}\int_{\mathbb{R}^d} \overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \lambda_1(\mu)\int_{\mathbb{R}^d} \overline{\psi}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mu^2} + \frac{\mathrm{d}^2\lambda_1(\mu)}{\mathrm{d}\mu^2}\int_{\mathbb{R}^d} \overline{\psi}\psi.\end{aligned}$$

Subtracting the complex conjugate of the former from the latter we obtain

$$2\int_{\mathbb{R}^d} V\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} = 2\frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu}\int_{\mathbb{R}^d} \overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} + \frac{\mathrm{d}^2\lambda_1(\mu)}{\mathrm{d}\mu^2}\int_{\mathbb{R}^d} \overline{\psi}\psi$$

which implies

$$\frac{\mathrm{d}^2\lambda_1(\mu)}{\mathrm{d}\mu^2} = 2\frac{\displaystyle\int_{\mathbb{R}^d} V\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} - \frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu}\int_{\mathbb{R}^d}\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}{\displaystyle\int_{\mathbb{R}^d}|\psi|^2}.$$

Now we substitute $\phi = \frac{\mathrm{d}\psi}{\mathrm{d}\mu}$ into (2.10) obtaining

$$\int_{\mathbb{R}^d} V\overline{\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}\psi - \frac{\mathrm{d}\lambda_1(\mu)}{\mathrm{d}\mu}\int_{\mathbb{R}^d}\overline{\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}\psi = -\int_{\mathbb{R}^d}\left|\nabla\frac{\mathrm{d}\psi}{\mathrm{d}\mu}\right|^2 - \mu\int_{\mathbb{R}^d} V\left|\frac{\mathrm{d}\psi}{\mathrm{d}\mu}\right|^2 + \lambda_1(\mu)\int_{\mathbb{R}^d}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\mu}\right|^2$$

which is negative from the definition of $\lambda_1(\mu)$. Indeed from (2.4) it holds

$$\lambda_1(\mu) = \inf_{\substack{\psi\in H_0^1(\mathbb{R}^d)\\ \psi\neq 0}} \frac{\mathcal{Q}_{\mathcal{S}_\mu}[\psi]}{\|\psi\|^2} \leq \frac{\displaystyle\int_{\mathbb{R}^d}\left|\nabla\frac{\mathrm{d}\psi}{\mathrm{d}\mu}\right|^2 + \mu\int_{\mathbb{R}^d} V\left|\frac{\mathrm{d}\psi}{\mathrm{d}\mu}\right|^2}{\displaystyle\int_{\mathbb{R}^d}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\mu}\right|^2}.$$

Recalling Proposition 2.3.4 we have just proven the following statement.

Theorem 2.3.6. *Assume* (2.8) *and let $V \leq 0$ and non-trivial. Let $\lambda_1(\mu)$ be an eigenvalue with eigenfunction ψ for all $\mu \in [\mu_0, +\infty)$. Then*

$$\frac{\mathrm{d}^2\lambda_1(\mu)}{\mathrm{d}\mu^2} = 2\frac{\displaystyle\int_{\mathbb{R}^d}|\psi|^2\int_{\mathbb{R}^d} V\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu} - \int_{\mathbb{R}^d} V|\psi|^2\int_{\mathbb{R}^d}\overline{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\mu}}{\displaystyle\left(\int_{\mathbb{R}^d}|\psi|^2\right)^2}.$$

Moreover $\frac{\mathrm{d}^2\lambda_1(\mu)}{\mathrm{d}\mu^2}$ is negative for all $\mu \in [\mu_0, +\infty)$ and therefore $\mu \mapsto \lambda_1(\mu)$ is concave.

Finally we provide an example of the family $\mathcal{S}_\mu$ where the potential is the so called finite rectangular well. Consider the Schrödinger operator denoted by $\mathcal{S}_\mu^a$, $\mu > 0$, $a < 0$ on $L^2(\mathbb{R})$ with the potential $\mathcal{V}_a$ generated by the function

$$V_a(x) = \begin{cases} 0, & x < a \\ a, & a < x < -a \\ 0, & x > -a. \end{cases}$$

Since $\mu\mathcal{V}_a$ suffices the assumptions of Theorem 2.2.4 the essential spectrum of $\mathcal{S}_\mu^a$ is $[0, \infty)$. We will now find the discrete spectrum. For this we compute the point spectrum thus we have to solve the equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \mu V_a\psi = \lambda\psi$$

for some $0 \neq \psi \in \mathrm{Dom}(\mathcal{S}_\mu^a) = H^2(\mathbb{R})$ and $\lambda \in \mathbb{R}$ (since $\mathcal{S}_\mu^a$ is self-adjoint) which involving the definition of V implies

$$\begin{aligned} -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} &= \lambda\psi, \quad x < a \\ -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \mu a\psi &= \lambda\psi, \quad a < x < -a \\ -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} &= \lambda\psi, \quad x > -a. \end{aligned}$$

The general solution of these equations is

$$\begin{aligned} \psi_1(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \quad x < a \\ \psi_2(x) &= C_3 e^{\sqrt{-(\lambda-\mu a)}x} + C_4 e^{-\sqrt{-(\lambda-\mu a)}x}, \quad a < x < -a \\ \psi_3(x) &= C_5 e^{\sqrt{-\lambda}x} + C_6 e^{-\sqrt{-\lambda}x}, \quad x > -a \end{aligned}$$

where we denoted the partial solutions by ψ_i. For ψ to lie in $L^2(\mathbb{R})$ we need $\lambda < 0$ (which implies there are no eigenvalues embedded in the essential spectrum) and also $C_2 = C_5 = 0$. Since $\mathcal{S}_\mu^a$ is bounded from below by $-\mu\|V_a\|_\infty = \mu a$ (which follows for example from 2.2.1) we have that $\inf\sigma(\mathcal{S}_\mu^a) \geq \mu a$ therefore we are looking for $\lambda \in (\mu a, 0)$.

Moreover since from the Sobolev embedding theorem it follows that $H^2(\mathbb{R}) \subset C^1(\mathbb{R})$ we need to assure that

$$\psi_1(a) = \psi_2(a), \quad \psi_2(-a) = \psi_3(-a), \quad \frac{\mathrm{d}\psi_1}{\mathrm{d}x}(a) = \frac{\mathrm{d}\psi_2}{\mathrm{d}x}(a) \quad \text{and} \quad \frac{\mathrm{d}\psi_2}{\mathrm{d}x}(-a) = \frac{\mathrm{d}\psi_3}{\mathrm{d}x}(-a).$$

Hence

$$\begin{aligned} C_1 e^{a\sqrt{-\lambda}} &= C_3 e^{ai\sqrt{\lambda-\mu a}} + C_4 e^{-ai\sqrt{\lambda-\mu a}} \\ C_3 e^{-ai\sqrt{\lambda-\mu a}} + C_4 e^{ai\sqrt{\lambda-\mu a}} &= C_6 e^{a\sqrt{-\lambda}} \\ C_1\sqrt{-\lambda}\, e^{a\sqrt{-\lambda}} &= C_3 i\sqrt{\lambda-\mu a}\, e^{ai\sqrt{\lambda-\mu a}} - C_4 i\sqrt{\lambda-\mu a}\, e^{-ai\sqrt{\lambda-\mu a}} \\ C_3 i\sqrt{\lambda-\mu a}\, e^{-ai\sqrt{\lambda-\mu a}} - C_4 i\sqrt{\lambda-\mu a}\, e^{ai\sqrt{\lambda-\mu a}} &= -C_6\sqrt{-\lambda}\, e^{a\sqrt{-\lambda}} \end{aligned}$$

which is a linear homogeneous system for the four constants C_1, C_3, C_4 and C_6. It has a non-trivial solution only if the corresponding determinant is zero, i.e.

$$\begin{vmatrix} e^{a\sqrt{-\lambda}} & -e^{ai\sqrt{\lambda-\mu a}} & -e^{-ai\sqrt{\lambda-\mu a}} & 0 \\ 0 & e^{-ai\sqrt{\lambda-\mu a}} & e^{ai\sqrt{\lambda-\mu a}} & -e^{a\sqrt{-\lambda}} \\ \sqrt{-\lambda}\, e^{a\sqrt{-\lambda}} & -i\sqrt{\lambda-\mu a}\, e^{ai\sqrt{\lambda-\mu a}} & i\sqrt{\lambda-\mu a}\, e^{-ai\sqrt{\lambda-\mu a}} & 0 \\ 0 & i\sqrt{\lambda-\mu a}\, e^{-ai\sqrt{\lambda-\mu a}} & -i\sqrt{\lambda-\mu a}\, e^{ai\sqrt{\lambda-\mu a}} & \sqrt{-\lambda}\, e^{a\sqrt{-\lambda}} \end{vmatrix} = 0.$$

Computing the determinant we get

$$2\sqrt{\lambda(-\lambda+\mu a)}\cos(2a\sqrt{\lambda-\mu a}) + (2\lambda-\mu a)\sin(2a\sqrt{\lambda-\mu a}) = 0.$$

Using the double-angle formulas for sine and cosine we obtain

$$\cos^2(a\sqrt{\lambda-\mu a}) - \sin^2(a\sqrt{\lambda-\mu a}) + \frac{2\lambda-\mu a}{\sqrt{\lambda(-\lambda+\mu a)}}\sin(a\sqrt{\lambda-\mu a})\cos(a\sqrt{\lambda-\mu a}) = 0.$$

which is equivalent to

$$\begin{aligned} &\cos^2(a\sqrt{\lambda-\mu a}) - \sin^2(a\sqrt{\lambda-\mu a}) \\ &+ \left(\sqrt{\frac{\lambda-\mu a}{-\lambda}} - \sqrt{\frac{-\lambda}{\lambda-\mu a}}\right)\sin(a\sqrt{\lambda-\mu a})\cos(a\sqrt{\lambda-\mu a}) = 0 \end{aligned}$$

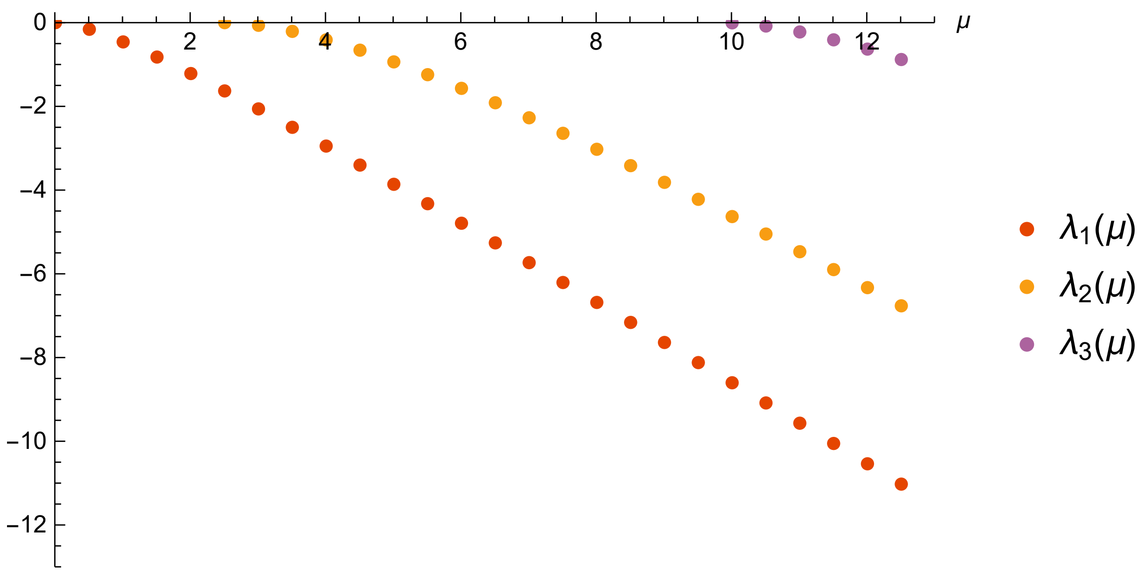

Figure 2.1: Plot of the eigenvalues of $\mathcal{S}_\mu^{-1}$ for $\mu \in (0, 13)$

and thus

$$\left(\cos(a\sqrt{\lambda-\mu a}) - \sqrt{\frac{-\lambda}{\lambda-\mu a}}\sin(a\sqrt{\lambda-\mu a})\right) \\ \times\left(\cos(a\sqrt{\lambda-\mu a}) + \sqrt{\frac{\lambda-\mu a}{-\lambda}}\sin(a\sqrt{\lambda-\mu a})\right) = 0.$$

Now we divide the equation by the term $\sin(a\sqrt{\lambda-\mu a})\cos(a\sqrt{\lambda-\mu a})$. This is possible for if $\sin(a\sqrt{\lambda-\mu a}) = 0$ then from the equation also $\cos(a\sqrt{\lambda-\mu a}) = 0$ and vice versa. This cannot happen at the same time. Hence

$$\left(\cot(a\sqrt{\lambda-\mu a}) - \sqrt{\frac{-\lambda}{\lambda-\mu a}}\right)\left(1 + \sqrt{\frac{\lambda-\mu a}{-\lambda}}\tan(a\sqrt{\lambda-\mu a})\right) = 0.$$

Employing the oddness of tangent and cotangent we get that the eigenvalues λ can be found as the solutions of the following two equations

$$\tan(-a\sqrt{\lambda-\mu a}) = \sqrt{\frac{-\lambda}{\lambda-\mu a}} \quad \text{and} \quad \cot(-a\sqrt{\lambda-\mu a}) = -\sqrt{\frac{-\lambda}{\lambda-\mu a}}.$$

The eigenvalues have finite multiplicity and no limit point therefore they coincide with the discrete spectrum and we would denote them by $\lambda_n(\mu)$ as before. On Figures 2.1 and 2.2 we provide the dependence of $\lambda_n(\mu)$ on the parameter μ for various values of a. It can be seen that in accordance with 2.3.5 and 2.3.2 the functions $\mu \mapsto \lambda_n(\mu)$ are decreasing and behave like linear functions as $\mu \to +\infty$. Moreover the first eigenvalue appears immediately as μ is non-zero. This corresponds with Proposition 2.2.5.

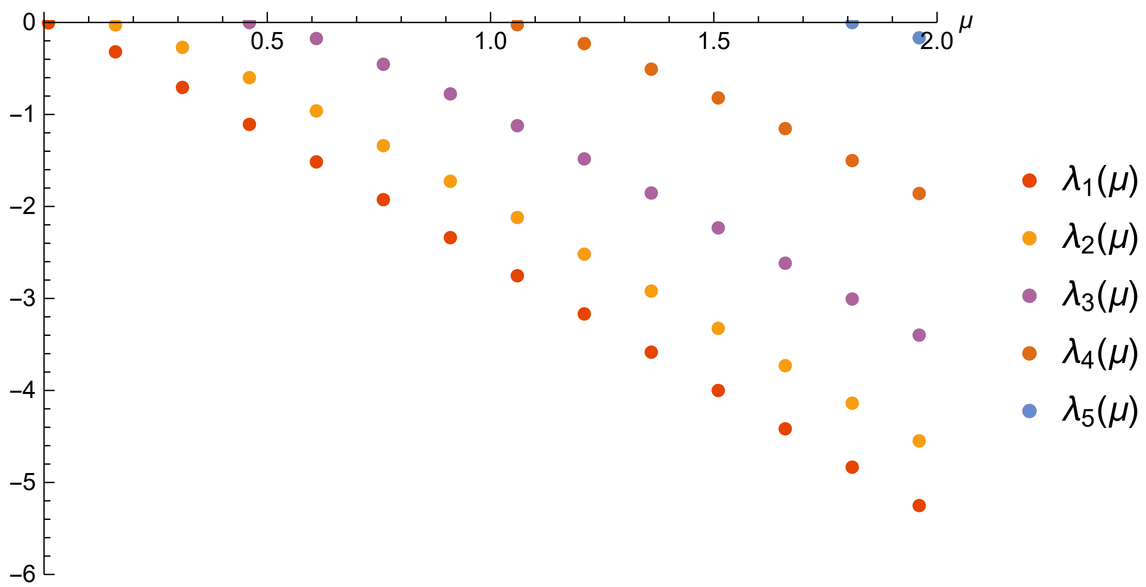

Figure 2.2: Plot of the eigenvalues of $\mathcal{S}_\mu^{-2.8}$ for $\mu \in (0, 2)$

2.4 Correspondence with damped wave operator

In this section we state a theorem of crucial importance for this thesis. It provides a connection between the spectrum of the self-adjoint Schödinger operator $\mathcal{S}_\mu$ with bounded potential $\mathcal{V}$ and the spectrum of the non-self-adjoint parameterized damped wave operator $\mathcal{A}_\alpha$ on $H_0^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ defined in (1.49) with bounded damping $b := V$. In particular

$$\mathcal{A}_\alpha = \begin{pmatrix} 0 & I \\ \Delta & -\alpha V \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}_\alpha) = H^2(\mathbb{R}^d) \times H_0^1(\mathbb{R}^d). \tag{2.15}$$

The discrete and essential spectrum of a non-self-adjoint operator were defined in Subsection 1.2.4.

Theorem 2.4.1 ([7, Lemma 2])**.** *Let $\mathcal{A}_\alpha$ be damped wave operator* (2.15)*. Then for $\mu \in \mathbb{R}$ and $\alpha > 0$ it holds*

1. $-\left(\frac{\mu}{\alpha}\right)^2 \in \sigma_{\mathrm{p}}(\mathcal{S}_\mu) \iff \frac{\mu}{\alpha} \in \sigma_{\mathrm{p}}(\mathcal{A}_\alpha)$
2. $-\left(\frac{\mu}{\alpha}\right)^2 \in \sigma_{\mathrm{ess}}(\mathcal{S}_\mu) \implies \frac{\mu}{\alpha} \in \sigma_{\mathrm{ess}}(\mathcal{A}_\alpha)$.

This theorem justifies the fact that we call $\mathcal{S}_\mu$ the energy dependent Schrödinger operator since μ is the eigenvalue (energy) of the non-parameterized damped wave operator $\mathcal{A}$. Next we have the following useful proposition.

Proposition 2.4.2. *Let $\mathcal{A}_\alpha$ be defined by* (2.15) *and assume that the damping V is nonpositive. Then for all $\mu \in \sigma_p(\mathcal{A}_\alpha)$ it holds $\Re\mu \geq 0$.*

Proof. Let $\mu \in \sigma_p(\mathcal{A}_\alpha)$. Then there exists $0 \neq \Psi \in \mathrm{Dom}(\mathcal{A}_\alpha)$ such that $\mathcal{A}_\alpha \Psi = \mu\Psi$. Denoting $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ we get

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 - \mu\alpha V \psi_2 = \mu\psi_2 \quad \text{and} \quad \psi_2 = \mu\psi_1$$

which together gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_1 - \mu\alpha V\psi_1 - \mu^2\psi_1 = 0.$$

Note that this equation can be viewed as a problem for a quadratic operator pencil, see [15]. Nevertheless multiplying by $\overline{\psi}_1$ and integrating over $\mathbb{R}^d$ implies

$$\int_{\mathbb{R}^d} |\nabla\psi_1|^2 + \mu\alpha\int_{\mathbb{R}^d} V|\psi_1|^2 + \mu^2\int_{\mathbb{R}^d} |\psi_1|^2 = 0$$

where we used the integration by parts. This is a quadratic equation for μ hence we can compute the solution

$$\mu_\pm = \frac{-\alpha\int_{\mathbb{R}^d} V|\psi_1|^2 \pm \sqrt{D}}{2\int_{\mathbb{R}^d}|\psi_1|^2}$$

where the discriminant D is

$$D = \alpha^2\left(\int_{\mathbb{R}^d} V|\psi_1|^2\right)^2 - 4\int_{\mathbb{R}^d}|\psi_1|^2\int_{\mathbb{R}^d}|\nabla\psi_1|^2 \begin{cases} = 0 \Rightarrow \mu_\pm \geq 0 \\ \geq 0 \Rightarrow \mu_\pm \geq 0 \\ \leq 0 \Rightarrow \Re\mu_\pm \geq 0. \end{cases}$$

□

Finally we can immediately see from Theorem 2.4.1 that since $\mathcal{S}_0 = \mathcal{T}$ and $0 \in \sigma_{\mathrm{ess}}(\mathcal{T})$ it holds that $0 \in \sigma_{\mathrm{ess}}(\mathcal{A}_\alpha)$. Moreover we know that $0 \in \sigma_{\mathrm{c}}(\mathcal{T})$ which implies that $0 \notin \sigma_{\mathrm{p}}(\mathcal{A}_\alpha)$.

Proposition 2.4.3. *Let $\mathcal{A}_\alpha$ be defined by* (2.15)*. Then $0 \notin \sigma_{\mathrm{p}}(\mathcal{A}_\alpha)$.*

Chapter 3

Bounds for eigenvalues of damped wave operator

In this chapter we derive numerous bounds for the eigenvalues of damped wave operator $\mathcal{A}_\alpha$ (2.15) using the correspondence between the spectrum of the damped wave operator and the Schrödinger operator $\mathcal{S}_\mu$ provided by Theorem 2.4.1. We use selected known bounds for the spectrum of the Schrödinger operator namely the Lieb-Thirring inequalities, the Buslaev-Faddeev-Zakharov trace formulae and the Birman-Schwinger principle.

Before proceeding recall the already obtained upper bound stated in Proposition 1.3.11 which follows directly from the fact that $\mathcal{A}$ generates a C_0-semigroup.

Next we consider the non-parameterized damped wave operator $\mathcal{A} \equiv \mathcal{A}_1$. We are able to obtain a simple bound for $\mu \in \sigma_p(\mathcal{A}),\ \mu \in \mathbb{R}$ using only Theorem 2.4.1 and the property of bounded from below self-adjoint operators which for $\mathcal{S}_\mu$ states that

$$\inf \sigma(\mathcal{S}_\mu) \geq -\|\mu V\|_\infty.$$

Assume (2.8). Then we know that $\sigma_{\mathrm{ess}}(\mathcal{S}_\mu) = [0, +\infty)$. All $\lambda_n(\mu)$ defined in (2.4) thus coincide with the strictly negative part of the point spectrum of $\mathcal{S}_\mu$ and we have $\lambda_n(\mu) \geq -|\mu|\|V\|_\infty$. Since from Proposition 2.4.3 we know that $\mu \neq 0$ we get from Theorem 2.4.1

$$\mu \in \sigma_{\mathrm{p}}(\mathcal{A}) \iff -\mu^2 \in \sigma_{\mathrm{p}}(\mathcal{S}_\mu) \iff \exists j \in \mathbb{N},\ \lambda_j(\mu) = -\mu^2 = -|\mu|^2$$

hence

$$-|\mu|^2 \geq -|\mu|\|V\|_\infty$$

and we conclude with the following statement.

Proposition 3.0.1. *Let $\mathcal{A}$ be damped wave operator* (2.15) *with $\alpha = 1$ and the damping V satisfying assumptions* (2.8). *Let $\mu \in \sigma_p(\mathcal{A}),\ \mu \in \mathbb{R}$. Then it holds that $|\mu| \leq \|V\|_\infty$.*

3.1 Lieb-Thirring inequalities

Consider a self-adjoint Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$ with real potential V and negative eigenvalues $\{\lambda_n : n \in \mathbb{N},\ n \leq N\}$ where the number of eigenvalues N can be finite or infinite. The Lieb-Thirring inequalities (see for example the summarizing paper [12]) provide an upper bound for the moments (sums of various powers) of the negative eigenvalues in terms of the integral of some power of the negative part of the potential $V_- := \frac{1}{2}(|V| - V)$. In particular it holds

$$\sum_{n=1}^{N} |\lambda_n|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \tag{3.1}$$

where $L_{\gamma,d} \in \mathbb{R}$ and the parameter γ can be chosen in the following way depending on the dimension d:

- $d = 1,\ \gamma \geq \frac{1}{2}$,
- $d = 2,\ \gamma > 0$,
- $d \geq 3,\ \gamma \geq 0$.

The values of γ satisfying this properties will be called the suitable values of γ. The sharp values of $L_{\gamma,d}$ (the less possible for which (3.1) holds) are known only in some cases, specifically

- $L_{\frac{1}{2},1} = 2L^{cl}_{\frac{1}{2},1} = \frac{1}{2}$
- $d \geq 1,\ \gamma \geq \frac{3}{2}$ then $L_{\gamma,d} = L^{cl}_{\gamma,d}$

where the so called classical constants $L^{cl}_{\gamma,d}$ arising from the Weyl's asymptotic formulae for the sum of negative eigenvalues $\lambda_n(\beta)$ of the Schrödinger operator $-\Delta + \beta V$ as $\beta \to +\infty$

$$\lim_{\beta \to +\infty} \beta^{-\gamma-\frac{d}{2}} \sum_{n=1}^{N} |\lambda_n(\beta)| = L^{cl}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}$$

satisfy the explicit formula

$$L^{cl}_{\gamma,d} = \frac{\Gamma(\gamma+1)}{2^d \pi^{\frac{d}{2}} \Gamma(\gamma + \frac{d}{2} + 1)}. \tag{3.2}$$

The remaining sharp constants are not known.

Assume (2.8). As was shown in the previous chapter the numbers $\lambda_n(\mu)$ coincide with the discrete and thus strictly negative part of the point spectrum of operator $\mathcal{S}_\mu$ (2.7) with the potential μ. Thus we have the Lieb-Thirring bound

$$\sum_{n=1}^{N_\mu} |\lambda_n(\mu)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} (\mu V)_-^{\gamma+\frac{d}{2}} \tag{3.3}$$

for all suitable γ. Here N_μ denotes the number of negative eigenvalues of $\mathcal{S}_\mu$. A trivial consequence is that for all n it holds

$$|\lambda_n(\mu)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} (\mu V)_-^{\gamma+\frac{d}{2}}. \tag{3.4}$$

Now we would like to use this bound to obtain a bound for some eigenvalues of damped wave operator $\mathcal{A}_\alpha$ (2.15). For this the parameterization of $\mathcal{A}_\alpha$ by α would not be necessary therefore we choose $\alpha = 1$ and denote this operator by $\mathcal{A}_1 \equiv \mathcal{A}$ as in the first chapter.

We will consider only the real part of the point spectrum of $\mathcal{A}$. Thus let $\mu \in \sigma_{\mathrm{p}}(\mathcal{A})$ and let $\mu \in \mathbb{R}$. We proceed as above. From Proposition 2.4.3 we know that $\mu \neq 0$. Using Theorem 2.4.1 we get

$$\mu \in \sigma_{\mathrm{p}}(\mathcal{A}) \iff -\mu^2 \in \sigma_{\mathrm{p}}(\mathcal{S}_\mu) \iff \exists j \in \mathbb{N},\ \lambda_j(\mu) = -\mu^2. \tag{3.5}$$

Next let $\mu > 0$. Using (3.4) we compute

$$\mu^{2\gamma} = |\lambda_j(\mu)|^\gamma \leq L_{\gamma,d}\, \mu^{\gamma+\frac{d}{2}} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}.$$

If on the other hand $\mu < 0$ we have

$$|\mu|^{2\gamma} = |\lambda_j(\mu)|^\gamma \leq L_{\gamma,d}\, |\mu|^{\gamma+\frac{d}{2}} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

where $V_+ := \frac{1}{2}(|V| + V)$. Choosing $\gamma = \frac{d}{2}$ which can be done for all $d \geq 1$ we obtain

$$\int_{\mathbb{R}^d} V_-^d \geq \frac{1}{L_{\frac{d}{2},d}}$$

in the case when $\mu > 0$ and

$$\int_{\mathbb{R}^d} V_+^d \geq \frac{1}{L_{\frac{d}{2},d}}$$

for $\mu < 0$. Since the potential can surely be chosen such that it satisfies our assumptions and is less than the right-hand side of the two formulas we see that the assumption that there exists some strictly positive or negative point in $\sigma_{\mathrm{p}}(\mathcal{A})$ was false in this case. Therefore we have just proven the following theorem.

Theorem 3.1.1. *Let $\mathcal{A}$ be non-parameterized damped wave operator* (2.15) *with the damping V which satisfies* (2.8). *If $V_- \in L^d(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} V_-^d < \frac{1}{L_{\frac{d}{2},d}}$$

then $\mathcal{A}$ has no positive eigenvalues. On the other hand if $V_+ \in L^d(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} V_+^d < \frac{1}{L_{\frac{d}{2},d}}$$

then $\mathcal{A}$ has no negative eigenvalues.

Remark 3.1.2 (Explicit bounds)**.** *Using the sharp values of $L_{\gamma,d}$ the explicit upper bounds for the integral of the potential V such that $\mathcal{A}$ has no positive respectively negative eigenvalues are known except for $d = 2$. Here W denotes V_- respectively V_+.*

- *For $d = 1$*

$$\int_{\mathbb{R}} W < 2.$$

- *For $d \geq 3$*

$$\int_{\mathbb{R}^d} W^d < \frac{2^d \pi^{\frac{d}{2}} \Gamma(d+1)}{\Gamma(\frac{d}{2}+1)}.$$

Equation (3.1) and (3.1) also provide a bound for μ for all suitable γ. Dividing them by $|\mu|^{\gamma+\frac{d}{2}}$ we conclude with the following theorem.

Theorem 3.1.3. *Let $\mathcal{A}$ be non-parameterized damped wave operator* (2.15) *with the damping V which satisfies* (2.8). *Let μ be its positive eigenvalue and $V_- \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$. Then*

$$\mu^{\gamma-\frac{d}{2}} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \tag{3.6}$$

on the other hand let μ be negative eigenvalue and $V_+ \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$. Then

$$|\mu|^{\gamma-\frac{d}{2}} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}} \tag{3.7}$$

for all suitable $\gamma \neq \frac{d}{2}$.

We see that for $d \geq 2$ this theorem provides a lower bound for the absolute value of the eigenvalue if $\gamma < \frac{d}{2}$, in particular

$$|\mu|^{\frac{d}{2}-\gamma} \geq \frac{1}{L_{\gamma,d} \displaystyle\int_{\mathbb{R}^d} W^{\gamma+\frac{d}{2}}}$$

and upper bound (3.6) respectively (3.7) if $\gamma > \frac{d}{2}$ where W denotes V_- respectively V_+. Nevertheless for $\gamma < \frac{d}{2}$ the sharp values of $L_{\gamma,d}$ are known only if $d \geq 4$.

The answer to the question which γ to choose to obtain the best bound depends on the size of μ and on the dimension d.

In the past few years there arose the so-called non-self-adjoint Lieb-Thirring inequalities which provide an upper bound for the sum of absolute values of the eigenvalues of Schrödinger operator with complex potential (which is non-self-adjoint) see [6]. These together with the generalized version of Theorem 2.4.1 for $\mu \in \mathbb{C}$ could provide another results for the spectrum of $\mathcal{A}$. This will be the aim of our next work.

3.2 Buslaev-Faddeev-Zakharov trace formulae

Consider again a self-adjoint Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R})$ with real potential V and negative eigenvalues $\{\lambda_n : n \in \mathbb{N},\ n \leq N\}$ where the number of eigenvalues N is finite. Then the Buslaev-Faddeev-Zakharov trace formulae [21] which follows from applying the inverse scattering method on the Korteweg-de Vries equation provides a lower bound for the sum of square roots of the eigenvalues. In particular

$$\sum_{n=1}^{N} |\lambda_n|^{\frac{1}{2}} \geq -\frac{1}{4} \int_{\mathbb{R}} V.$$

Assume (2.8). Consider operator $\mathcal{S}_\mu$ (2.7) on $L^2(\mathbb{R})$. Moreover assume $N_\mu < +\infty$, i.e. the number of negative eigenvalues is finite. Then the operator satisfies the above assumptions and we get

$$\sum_{n=1}^{N_\mu} |\lambda_n(\mu)|^{\frac{1}{2}} \geq -\frac{\mu}{4} \int_{\mathbb{R}} V \tag{3.8}$$

for numbers (2.4). Now we cannot use the little trick as before where we obtained (3.4) from (3.3). First we have to ensure that N_μ (the number of negative eigenvalues of $\mathcal{S}_\mu$) is exactly 1.

This will be achieved using the Bargmann bound [17, Problem 22] which provides an upper bound for the number of negative eigenvalues of $\mathcal{S}_\mu$

$$N_\mu \le 1 + \mu \int_{\mathbb{R}} |V(x)||x| \, \mathrm{d}x.$$

Moreover for $\mu > 0$ the assumption $\int_{\mathbb{R}} V < 0$ thanks to Proposition 2.2.6 implies that $\lambda_1(\mu)$ is an eigenvalue. Thus for

$$\mu < \left(\int_{\mathbb{R}} |V(x)||x| \, \mathrm{d}x \right)^{-1} \tag{3.9}$$

the number of negative eigenvalues N_μ is exactly 1. Consider non-parameterized (with $\alpha = 1$) damped wave operator $\mathcal{A}$ (2.15) on $H_0^1(\mathbb{R}) \times L^2(\mathbb{R})$. Let μ be its strictly positive eigenvalue and recall consequence (3.5) of Theorem 2.4.1. Hence for $\mu < \left(\int_{\mathbb{R}} |V(x)||x| \, \mathrm{d}x \right)^{-1}$ we get

$$|\mu| = |\lambda_1(\mu)|^{\frac{1}{2}} \ge -\frac{\mu}{4} \int_{\mathbb{R}} V \tag{3.10}$$

which implies

$$\int_{\mathbb{R}} V \ge -4.$$

On the other hand let $\mu < 0$. Then $\mathcal{S}_\mu$ has exactly one negative eigenvalue if $\int_{\mathbb{R}} V > 0$ and

$$|\mu| < \left(\int_{\mathbb{R}} |V(x)||x| \, \mathrm{d}x \right)^{-1} \tag{3.11}$$

holds and thus we again obtain (3.10) from which it now follows

$$\int_{\mathbb{R}} V \le 4.$$

Finally the theorem follows.

Theorem 3.2.1. *Assume* (2.8)*. Let $\mathcal{A}$ be damped wave operator* (2.15) *on $H_0^1(\mathbb{R}) \times L^2(\mathbb{R})$ with $\alpha = 1$ and $V \in L^1(\mathbb{R}, |x|\mathrm{d}x)$. Let μ be its real eigenvalue. If $\mu > 0$ and $\int_{\mathbb{R}} V < -4$ or $\mu < 0$ and $\int_{\mathbb{R}} V > 4$ then*

$$|\mu| \ge \left(\int_{\mathbb{R}} |V(x)||x| \, \mathrm{d}x \right)^{-1}.$$

Now recall parameterized damped wave operator $\mathcal{A}_\alpha$ (2.15) on $H_0^1(\mathbb{R}) \times L^2(\mathbb{R})$. Assuming $\int_{\mathbb{R}} V < 0$ and taking $\mu > 0$ such that (3.9) holds we have exactly one eigenvalue $\lambda_1(\mu)$ and it follows that

$$-\frac{\mu}{4} \int_{\mathbb{R}} V \le |\lambda_1(\mu)|^{\frac{1}{2}} \le \frac{\mu}{2} \int_{\mathbb{R}} V_-$$

from (3.4), (3.8) and the fact that $L_{\frac{1}{2},1} = \frac{1}{2}$. In the case of $\mathcal{A}_\alpha$ Theorem 2.4.1 now gives

$$-\left(\frac{\mu}{\alpha}\right)^2 \in \sigma_{\mathrm{p}}(\mathcal{S}_\mu) \Longleftrightarrow \frac{\mu}{\alpha} \in \sigma_{\mathrm{p}}(\mathcal{A}_\alpha)$$

which proves the following theorem.

Theorem 3.2.2. *Let $\mathcal{A}_\alpha$ be damped wave operator* (2.15) *on $H_0^1(\mathbb{R}) \times L^2(\mathbb{R})$ with the damping $V \in L^1(\mathbb{R}, |x|\mathrm{d}x)$ satisfying* (2.8) *and $\int_{\mathbb{R}} V < 0$. Then for any $\mu > 0$ such that* (3.9) *holds there exists exactly one α satisfying*

$$2\left(\int_{\mathbb{R}} V_-\right)^{-1} \leq \alpha \leq -4\left(\int_{\mathbb{R}} V\right)^{-1}$$

such that $\frac{\mu}{\alpha}$ is an eigenvalue of $\mathcal{A}_\alpha$.

This theorem gives us the existence of a positive eigenvalue of $\mathcal{A}_\alpha$ with specific damping. From the comments on time evolution in the first chapter we see that this implies that there exist initial conditions for which the solution of damped wave equation (1.1) which is generated by $\mathcal{A}_\alpha$ is unstable. Assuming $\mu < 0$ and $\int_{\mathbb{R}} V > 0$ leads us to analogous statement.

Theorem 3.2.3. *Let $\mathcal{A}_\alpha$ be damped wave operator* (2.15) *on $H_0^1(\mathbb{R}) \times L^2(\mathbb{R})$ with the damping $V \in L^1(\mathbb{R}, |x|\mathrm{d}x)$ satisfying* (2.8) *and $\int_{\mathbb{R}} V > 0$. Then for any $\mu < 0$ such that* (3.11) *holds there exists exactly one α satisfying*

$$2\left(\int_{\mathbb{R}} V_+\right)^{-1} \leq \alpha \leq 4\left(\int_{\mathbb{R}} V\right)^{-1}$$

such that $\frac{\mu}{\alpha}$ is an eigenvalue of $\mathcal{A}_\alpha$.

3.3 Birman-Schwinger principle

This section will use completely different techniques than the preceding parts. We will obtain some results for the spectrum of the damped wave operator on $H_0^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ with complex damping in general. This will be done by generalizing the Birman-Schwinger principle for the damped wave operator.

Thus consider a bounded complex-valued damping function $V \in L^\infty(\mathbb{R}^d, \mathbb{C})$ and define the corresponding damped wave operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & -V \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}) = H^2(\mathbb{R}^d) \times H_0^1(\mathbb{R}^d). \tag{3.12}$$

Since the perturbation by V is again bounded this operator is closed as in the real case.

Now we move on to the bound for the eigenvalues of $\mathcal{A}$. Let $\mu \in \sigma_{\mathrm{p}}(\mathcal{A})$, $\Re\mu \neq 0$. Then $\mathcal{A}\Psi = \mu\Psi$ for some $\Psi \in \mathrm{Dom}(\mathcal{A})$. Hence denoting $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ we have

$$\Delta\psi_1 - \mu V \psi_1 - \mu^2 \psi_1 = 0.$$

Moreover we write $V_{\frac{1}{2}} := \mathrm{sgn}(V)|V|^{\frac{1}{2}}$ where the complex signum function is defined as $\mathrm{sgn}(V) := e^{i\arg(V)}$. Thus $V = |V|^{\frac{1}{2}} V_{\frac{1}{2}}$ and the equation transforms to

$$(\mathcal{T} + \mu^2 I)\psi_1 = -\mu|V|^{\frac{1}{2}} V_{\frac{1}{2}} \psi_1 \tag{3.13}$$

where $\mathcal{T}$ is the Dirichlet Laplacian on $L^2(\mathbb{R}^d)$ defined in Section 2.1 and we use the same notation for the multiplicative operator and its generating function. Next denote $\phi := |V|^{\frac{1}{2}}\psi_1$. This function lies in $L^2(\mathbb{R}^d)$. Indeed recall that $\sigma(\mathcal{T}) = [0, +\infty)$. Therefore for any $\eta > 0$ we have that

$(\mathcal{T}+\eta I)^{-1}$ is a bounded bijection and thus there exists $\varphi \in L^2(\mathbb{R}^d)$ such that $(\mathcal{T}+\eta I)^{-1}\varphi = \psi_1$ and we have

$$\|\phi\| \le \||V|^{\frac{1}{2}}\psi_1\| \le \||V|^{\frac{1}{2}}(\mathcal{T}+\eta I)^{-1}\varphi\| \le \||V|^{\frac{1}{2}}(\mathcal{T}+\eta I)^{-1}\|\|\varphi\| < +\infty.$$

Therefore equation (3.13) implies

$$\mu|V|^{\frac{1}{2}}(\mathcal{T}+\mu^2 I)^{-1}V_{\frac{1}{2}}\phi = -\phi$$

provided that $\Re\mu \neq 0$. We are able to define the bounded Birman-Schwinger operator K_μ on $L^2(\mathbb{R}^d)$ (being a composition of three bounded operators) by

$$K_\mu\psi := \mu|V|^{\frac{1}{2}}(\mathcal{T}+\mu^2 I)^{-1}V_{\frac{1}{2}}\psi, \quad \mathrm{Dom}(K_\mu) := L^2(\mathbb{R}^d). \tag{3.14}$$

We immediately see that $-1 \in \sigma_{\mathrm{p}}(K_\mu)$. Thus we have just proven that for μ such that $\Re\mu \neq 0$ it holds $\mu \in \sigma_{\mathrm{p}}(\mathcal{A}) \Rightarrow -1 \in \sigma_{\mathrm{p}}(K_\mu)$.

Theorem 3.3.1 (Birman-Schwinger principle for damped wave operator)**.** *Let $\mathcal{A}$ be the damped wave operator with complex-valued bounded damping V and let K_μ be the Birman-Schwinger operator defined in* (3.14)*. For $\mu \in \mathbb{C}$, $\Re\mu \neq 0$ it holds*

$$\mu \in \sigma_{\mathrm{p}}(\mathcal{A}) \Rightarrow -1 \in \sigma_{\mathrm{p}}(K_\mu).$$

If some bounded operator T has the number -1 in its point spectrum with the corresponding eigenfunction ψ then certainly

$$\|T\| \ge \frac{\|T\psi\|}{\|\psi\|} = 1. \tag{3.15}$$

Also we have a formula for the integral kernel of K_μ

$$K_\mu(x,y) = \mu|V|^{\frac{1}{2}}(x)G_\mu(x,y)V_{\frac{1}{2}}(y)$$

where $G_\mu(x,y)$ is the integral kernel of the resolvent $(\mathcal{T}+\mu^2 I)^{-1}$. This is explicitly known for $d=1$ and $d=3$. First we focus on the case $d=1$. We have (see [19, Section 2.7.5])

$$G_\mu(x,y) = \frac{e^{-\mu|x-y|}}{2\mu}$$

and thus

$$K_\mu(x,y) = \mu|V|^{\frac{1}{2}}(x)\frac{e^{-\mu|x-y|}}{2\mu}V_{\frac{1}{2}}(y).$$

Using [17, Theorem VI.23] we can write for $V \in L^1(\mathbb{R})$

$$\|K_\mu\|^2 \le \|K_\mu\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}\times\mathbb{R}} |K_\mu(x,y)|^2 \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}\times\mathbb{R}} |\mu|^2|V(x)|\frac{\left|e^{-\mu|x-y|}\right|^2}{4|\mu|^2}|V(y)|\mathrm{d}x\mathrm{d}y$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. Further we assume $\Re\mu > 0$ to be able to estimate the exponential and we get

$$\int_{\mathbb{R}\times\mathbb{R}} |\mu|^2|V(x)|\frac{\left|e^{-\mu|x-y|}\right|^2}{4|\mu|^2}|V(y)|\mathrm{d}x\mathrm{d}y \le \int_{\mathbb{R}\times\mathbb{R}} |\mu|^2\frac{|V(x)||V(y)|}{4|\mu|^2}\mathrm{d}x\mathrm{d}y$$

since $|e^z| = e^{\Re z}$ for $z \in \mathbb{C}$. Using the Fubini theorem we arrive at

$$\|K_\mu\|^2 \leq \frac{1}{4}\|V\|_{L^1}^2$$

and employing (3.15) we conclude with the following statement.

Theorem 3.3.2. *Let $d = 1$ and $\mathcal{A}$ be the damped wave operator with complex-valued bounded damping $V \in L^1(\mathbb{R})$. If $\|V\|_{L^1} < 2$ then $\sigma_{\mathrm{p}}(\mathcal{A}) \subset \{\mu \in \mathbb{C} : \Re\mu \leq 0\}$.*

We see that this theorem is a generalization of the first statement of Theorem 3.1.1 for taking $V \leq 0$ we know that for $\mu \in \sigma_{\mathrm{p}}(\mathcal{A})$ we have $\Re\mu \geq 0$, see Proposition 2.4.2. This together with Proposition 2.4.3 implies the first statement of Theorem 3.1.1.

Now we analyze the case $d = 3$. The integral kernel of the resolvent $(\mathcal{T} + \mu^2 I)^{-1}$ is now

$$G_\mu(x, y) = \frac{e^{-\mu|x-y|}}{4\pi|x-y|}$$

and hence

$$K_\mu(x, y) = \mu|V|^{\frac{1}{2}}(x)\frac{e^{-\mu|x-y|}}{4\pi|x-y|}V_{\frac{1}{2}}(y).$$

Moreover we assume that $V \in R(\mathbb{R}^3)$ where $R(\mathbb{R}^3)$ is the Rollnik class which consists of all $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\|V\|_R^2 := \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y < +\infty.$$

As in the previous case we write

$$\|K_\mu\|^2 \leq \|K_\mu\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}^3\times\mathbb{R}^3} |\mu|^2|V(x)|\frac{\left|e^{-\mu|x-y|}\right|^2}{16\pi^2|x-y|^2}|V(y)|\mathrm{d}x\mathrm{d}y$$

and assuming $\Re\mu > 0$ we get

$$\|K_\mu\|^2 \leq \frac{|\mu|^2}{16\pi^2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y = \frac{|\mu|^2}{16\pi^2}\|V\|_R^2.$$

This proves the following theorem.

Theorem 3.3.3. *Let $d = 3$ and $\mathcal{A}$ be the damped wave operator with bounded complex-valued damping $V \in R(\mathbb{R}^3)$. Then*

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset \left\{\mu \in \mathbb{C} : \Re\mu \leq 0 \vee |\mu| \geq \frac{4\pi}{\|V\|_R}\right\}.$$

Consider now $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. From the sharp Hardy-Littlewood-Sobolev inequality [13, Theorem 4.3] we know that $L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow R(\mathbb{R}^3)$. In particular

$$\|V\|_R^2 \leq \sqrt[3]{4\pi^4}\|V\|_{L^{\frac{3}{2}}}^2$$

from which the following corollary follows.

Corollary 3.3.4. *Let $d = 3$ and $\mathcal{A}$ be the damped wave operator with bounded complex-valued damping $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Then*

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset \left\{\mu \in \mathbb{C} : \Re\mu \leq 0 \vee |\mu| \geq \frac{4\sqrt[3]{\pi}}{\sqrt[3]{2}\,\|V\|_{L^{\frac{3}{2}}}}\right\}.$$

Chapter 4

Finite rectangular well

In the final chapter we provide an explicitly computable example of the behavior of the point spectrum of the damped wave operator $\mathcal{A}$ on $H_0^1(\mathbb{R}) \times L^2(\mathbb{R})$ with the damping governed by the so called finite rectangular well. In particular let $a < 0,\ b > 0$ and let W denote the damping function

$$W(x) = \begin{cases} 0, & x < -b \\ a, & -b < x < b \\ 0, & x > b. \end{cases}$$

The damped wave operator with the potential W is denoted by $\mathcal{A}_W$, i.e.

$$\mathcal{A}_W = \begin{pmatrix} 0 & I \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} & -W \end{pmatrix}, \quad \mathrm{Dom}(\mathcal{A}) = H^2(\mathbb{R}) \times H_0^1(\mathbb{R}^d).$$

Let $\mu \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}_W$ then

$$\mathcal{A}_W \Psi = \mu \Psi$$

for some $0 \neq \Psi \in \mathrm{Dom}(\mathcal{A}_W)$ which means

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi - \mu W \psi - \mu^2 \psi = 0$$

where ψ denotes the first component of Ψ. Employing the definition of W we get

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi - \mu^2\psi = 0, \quad & x < -b \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi - \mu a \psi - \mu^2\psi = 0, \quad & -b < x < b \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi - \mu^2\psi = 0, \quad & x > b. \end{aligned}$$

The general solution of these equations is

$$\begin{aligned} \psi_1(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}, \quad & x < -b \\ \psi_2(x) = C_3 e^{\sqrt{\mu a + \mu^2}x} + C_4 e^{-\sqrt{\mu a + \mu^2}x}, \quad & -b < x < b \\ \psi_3(x) = C_5 e^{\mu x} + C_6 e^{-\mu x}, \quad & x > b \end{aligned}$$

where we denoted the partial solutions by ψ_i. The potential W satisfies assumptions (2.8) and thus Proposition 2.4.2 implies that $\Re\mu \geq 0$. Hence for ψ to lie in $L^2(\mathbb{R})$ we need $C_2 = C_5 = 0$. Moreover since the Sobolev embedding theorem implies that $H^2(\mathbb{R}) \subset C^1(\mathbb{R})$. This means we need to assure that

$$\psi_1(-b) = \psi_2(-b), \quad \psi_2(b) = \psi_3(b), \quad \frac{\mathrm{d}\psi_1}{\mathrm{d}x}(-b) = \frac{\mathrm{d}\psi_2}{\mathrm{d}x}(-b) \quad \text{and} \quad \frac{\mathrm{d}\psi_2}{\mathrm{d}x}(b) = \frac{\mathrm{d}\psi_3}{\mathrm{d}x}(b).$$

Therefore

$$\begin{aligned} C_1 e^{-\mu b} &= C_3 e^{-\sqrt{\mu a+\mu^2}b} + C_4 e^{\sqrt{\mu a+\mu^2}b} \\ C_3 e^{\sqrt{\mu a+\mu^2}b} + C_4 e^{-\sqrt{\mu a+\mu^2}b} &= C_6 e^{-\mu b} \\ C_1 \mu e^{-\mu b} &= C_3 \sqrt{\mu a+\mu^2}\, e^{-\sqrt{\mu a+\mu^2}b} - C_4 \sqrt{\mu a+\mu^2}\, e^{\sqrt{\mu a+\mu^2}b} \\ C_3 \sqrt{\mu a+\mu^2}\, e^{\sqrt{\mu a+\mu^2}b} - C_4 \sqrt{\mu a+\mu^2}\, e^{-\sqrt{\mu a+\mu^2}b} &= -C_6 \mu e^{-\mu b} \end{aligned}$$

which is a linear homogeneous equation for the four constants C_1, C_3, C_4 and C_6. Therefore it has a non-trivial solution only if the the corresponding determinant is equal to 0, in particular

$$\begin{vmatrix} e^{-\mu b} & -e^{-\sqrt{\mu a+\mu^2}b} & -e^{\sqrt{\mu a+\mu^2}b} & 0 \\ 0 & e^{\sqrt{\mu a+\mu^2}b} & e^{-\sqrt{\mu a+\mu^2}b} & -e^{-\mu b} \\ \mu e^{-\mu b} & -\sqrt{\mu a+\mu^2}\, e^{-\sqrt{\mu a+\mu^2}b} & \sqrt{\mu a+\mu^2}\, e^{\sqrt{\mu a+\mu^2}b} & 0 \\ 0 & \sqrt{\mu a+\mu^2}\, e^{\sqrt{\mu a+\mu^2}b} & -\sqrt{\mu a+\mu^2}\, e^{-\sqrt{\mu a+\mu^2}b} & \mu e^{-\mu b} \end{vmatrix} = 0.$$

Computing the determinant we arrive at

$$-2e^{-2b\mu}\mu \left(2\sqrt{\mu a+\mu^2}\cosh(2b\sqrt{\mu a+\mu^2}) + (a+2\mu)\sinh(2b\sqrt{\mu a+\mu^2})\right) = 0$$

which is equivalent to

$$2\sqrt{\mu a+\mu^2}\cosh(2b\sqrt{\mu a+\mu^2}) + (a+2\mu)\sinh(2b\sqrt{\mu a+\mu^2}) = 0.$$

In the special case when $\mu \in \mathbb{R}$ this reduces to

$$2\sqrt{-(\mu a+\mu^2)}\cos(2b\sqrt{-(\mu a+\mu^2)}) + (a+2\mu)\sin(2b\sqrt{-(\mu a+\mu^2)}) = 0$$

since $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$.

Hence the latter two equations determine the eigenvalues of $\mathcal{A}_W$. On Figures 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8 we can see their behavior with respect to different values of a when $b = 1$. We can see that the eigenvalues emerge from the imaginary axis and the deeper the well is the greater is their number.

Now we compare the bounds for the real eigenvalues obtained in the previous chapter. We analyze their behavior for different values of a with b fixed. In particular we employ the bound of Proposition 3.0.1 which for $\mathcal{A}_W$ has the form

$$\mu \leq \|W\|_\infty = -W_{\min}$$

where $W_{\min}$ denotes the essential infimum of W as before. Moreover we use the bound of Theorem 3.1.3 which states that

$$\mu \leq L_{\gamma,1}\int_{\mathbb{R}} |W|^{\gamma+\frac{1}{2}} = \left(\frac{\Gamma(\gamma+1)}{\pi^{\frac{1}{2}}\Gamma(\gamma+\frac{1}{2}+1)} b(-a)^{\gamma+\frac{1}{2}}\right)^{\frac{1}{\gamma-\frac{1}{2}}}$$

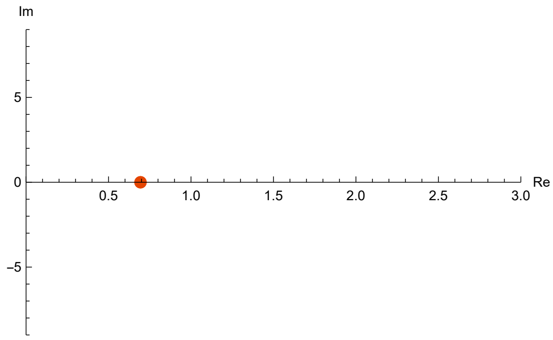

Figure 4.1: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -1.5$ and $b = 1$

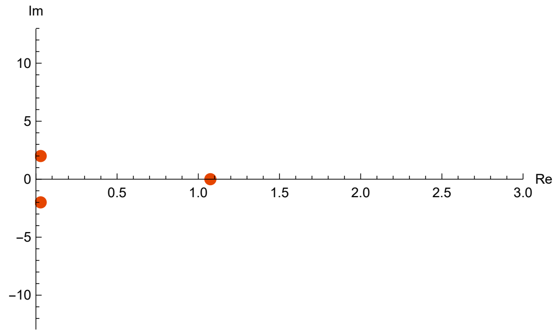

Figure 4.2: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -1.8$ and $b = 1$

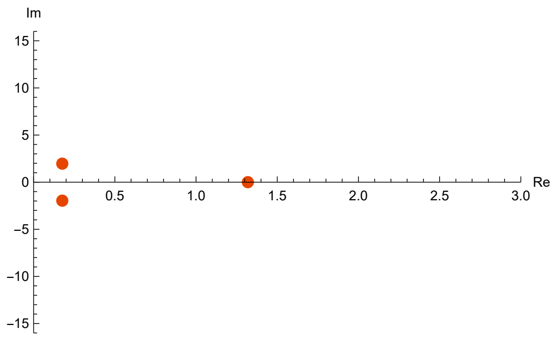

Figure 4.3: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -2$ and $b = 1$

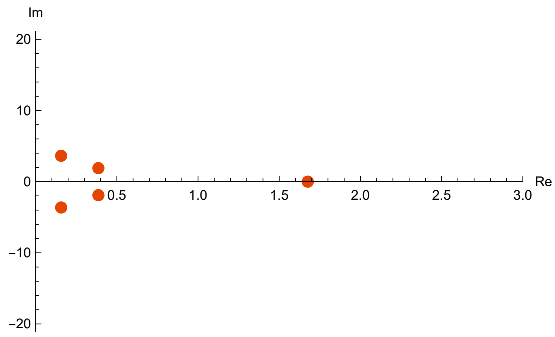

Figure 4.4: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -2.3$ and $b = 1$

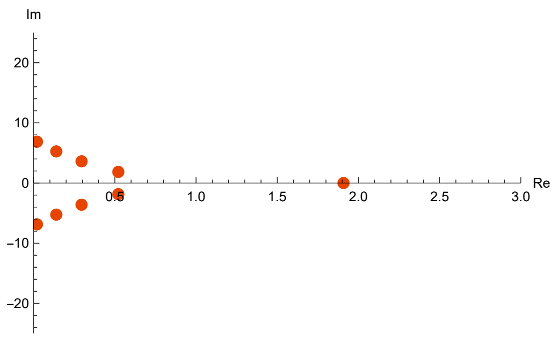

Figure 4.5: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -2.5$ and $b = 1$

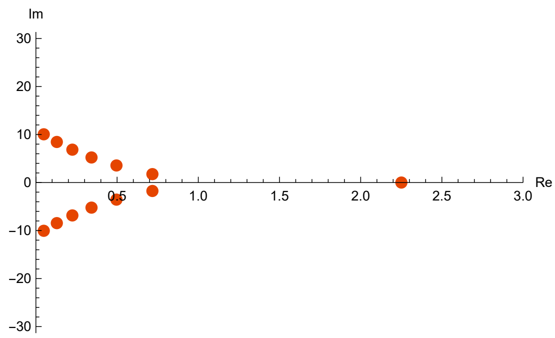

Figure 4.6: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -2.8$ and $b = 1$

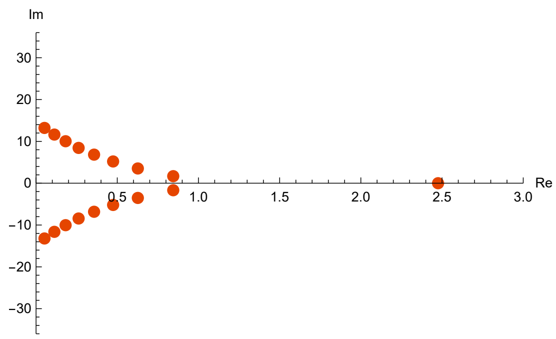

Figure 4.7: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -3$ and $b = 1$

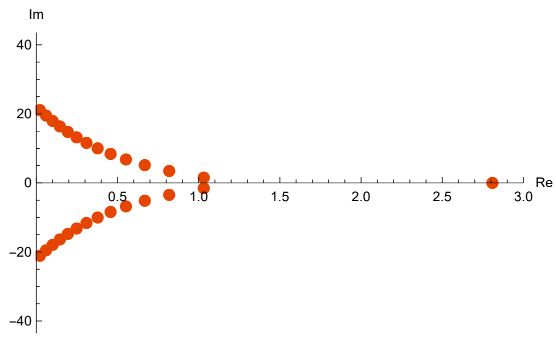

Figure 4.8: Plot of eigenvalues of $\mathcal{A}_W$ with $a = -3.3$ and $b = 1$

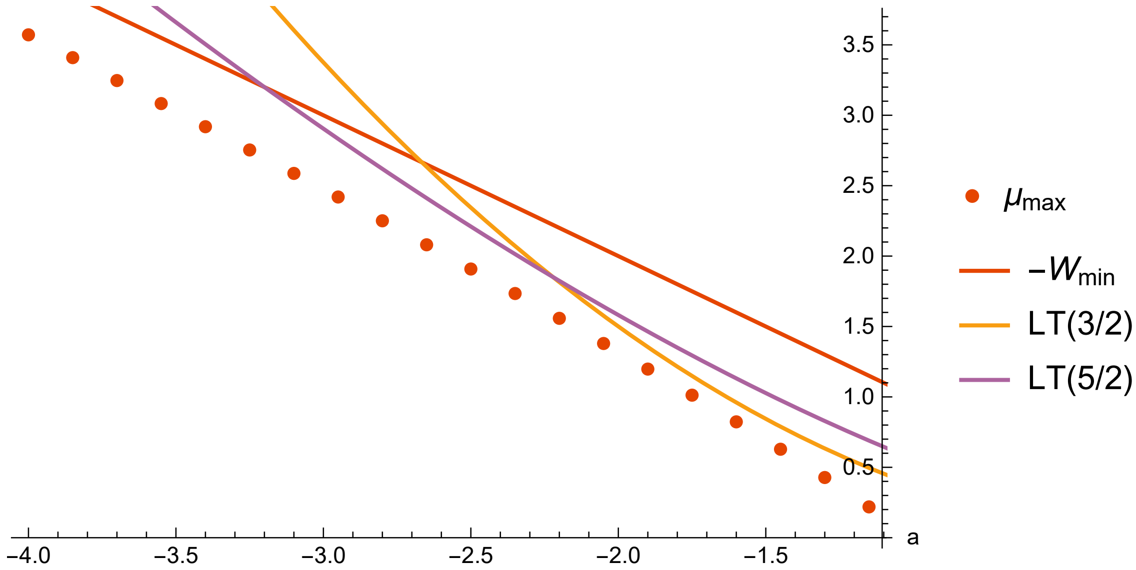

Figure 4.9: Plot of the bounds for eigenvalues of $\mathcal{A}_W$ with $b = 1$ and $a \in (-4, -1.1)$

for all $\gamma \geq \frac{3}{2}$ where we used explicit formula for $L_{\gamma,1}$ (3.2) since in this case $L_{\gamma,1} = L^{cl}_{\gamma,1}$. We plot these bounds for $\gamma = \frac{3}{2}$ and $\gamma = \frac{5}{2}$ denoted by $LT(3/2)$ and $LT(5/2)$ respectively. Since these bounds holds for every eigenvalue of $\mathcal{A}_W$ we compare them with the numerically computed greatest real eigenvalue denoted by $\mu_{\max}$. The results can be seen on Figures 4.9 and 4.10 for $b = 1$ and $b = 2$ respectively.

It can be seen that in both cases the bound $LT(3/2)$ is the best for the smallest values of $|a|$ for which $\mathcal{A}_W$ has a real eigenvalue. Then as $|a|$ grows the function $LT(3/2)$ is crossed by $LT(5/2)$ which thus starts to be the best bound. Finally for the largest values of $|a|$ the bound $-W_{\min}$ wins and provides the best estimate for any real eigenvalue μ. Also note that for small enough dampings there is no eigenvalue which is in agreement with Theorem 3.1.1.

Considering the case when a is fixed and b changes we obtain the behavior which can be seen on Figures 4.11 and 4.12 for $a = -1$ and $a = -2$ respectively. The quality of the bounds is the same as in the preceding, i.e. for the lowest values of b the bound $LT(3/2)$ is the best. Then with growing b the bound $LT(5/2)$ becomes better and with yet other growth the bound $-W_{\min}$ possesses the best information about μ.

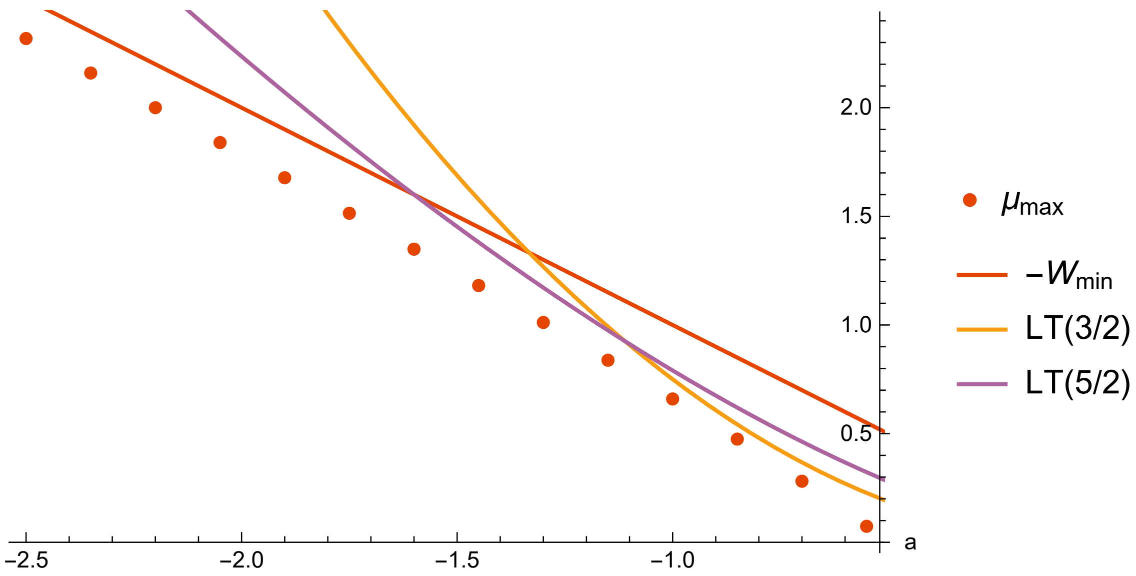

Figure 4.10: Plot of the bounds for eigenvalues of $\mathcal{A}_W$ with $b = 2$ and $a \in (-2.5, 0.5)$

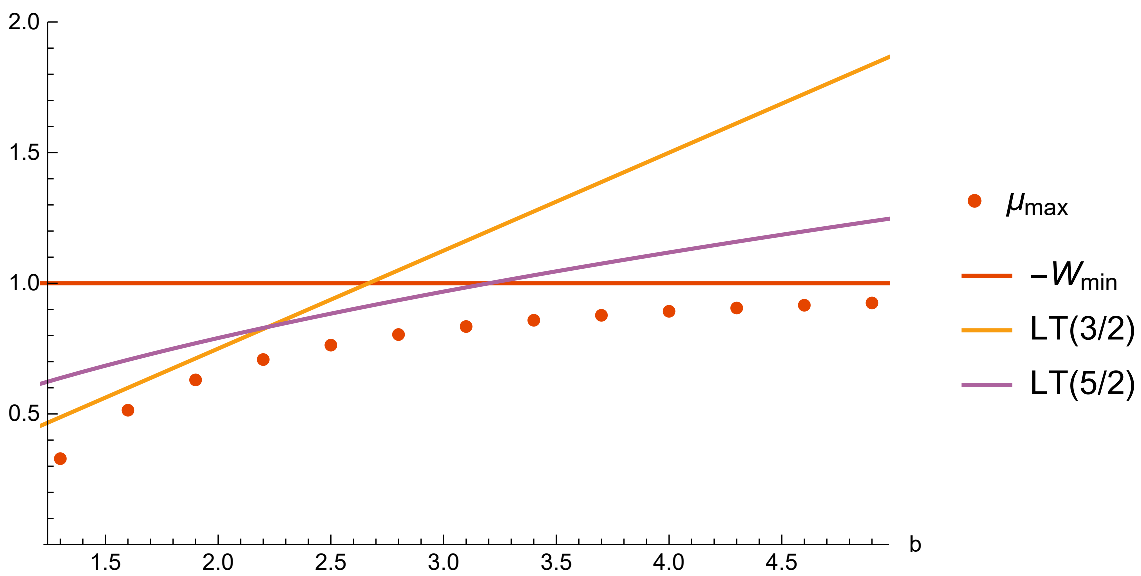

Figure 4.11: Plot of the bounds for eigenvalues of $\mathcal{A}_W$ with $a = -1$ and $b \in (1.2, 5)$

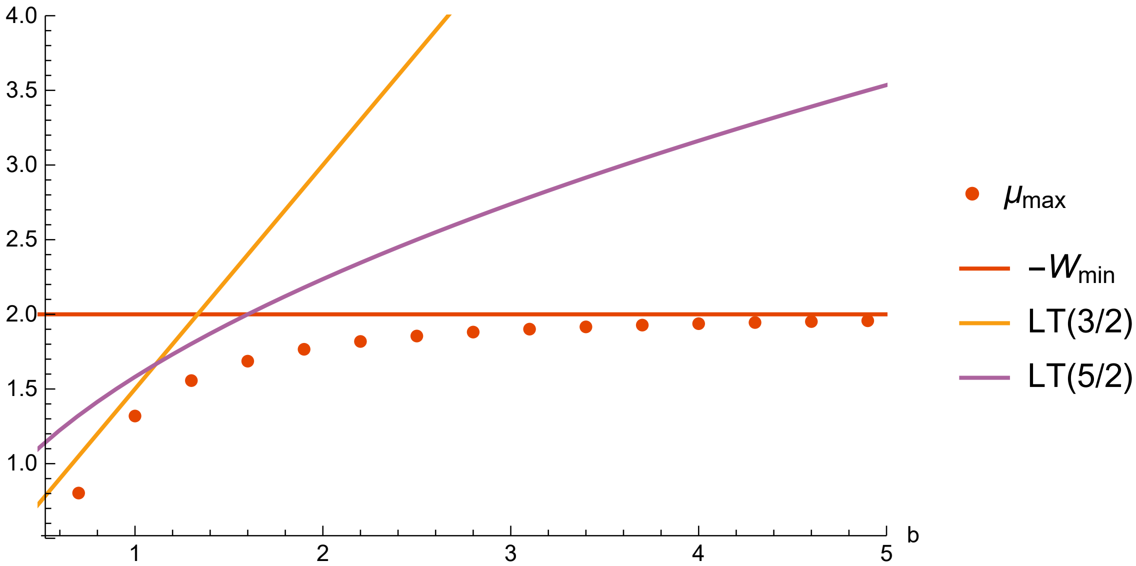

Figure 4.12: Plot of the bounds for eigenvalues of $\mathcal{A}_W$ with $a = -2$ and $b \in (0.5, 5)$

The last phenomena we are going to discuss is the sharpness of the bound of Theorem 3.1.1 respectively Theorem 3.3.2. The Birman-Schwinger principle can be used to obtain an analogous result for the Schrödinger operator which moreover states that the corresponding bound is sharp when the potential is the Dirac delta function. We would like to obtain such a result also for the damped wave operator at least from the numerical point of view. This will be done by parameterizing the delta function by an infinitely deep and narrow rectangular well whose L^1 norm converges to 2. More specifically we take the potential W and set

$$a = -h \quad \text{and} \quad b = \frac{1 + \frac{1}{e^h}}{h}$$

for some $h > 0$. On Figure 4.13 we can see the plot of the eigenvalue (which is unique and real) of $\mathcal{A}_W$ depending on the parameter h. We see that the larger is the parameter h the smaller is the eigenvalue and it is still present. Moreover for deep and narrow wells whose L^1 norm is slightly less than 2 the numerics show that there is no eigenvalue. This together with the fact that $\Re\mu \geq 0$ suggests that for the Dirac delta function the bound of Theorem 3.1.1 could be sharp. This of course has to be proven analytically which will be the goal of our next work.

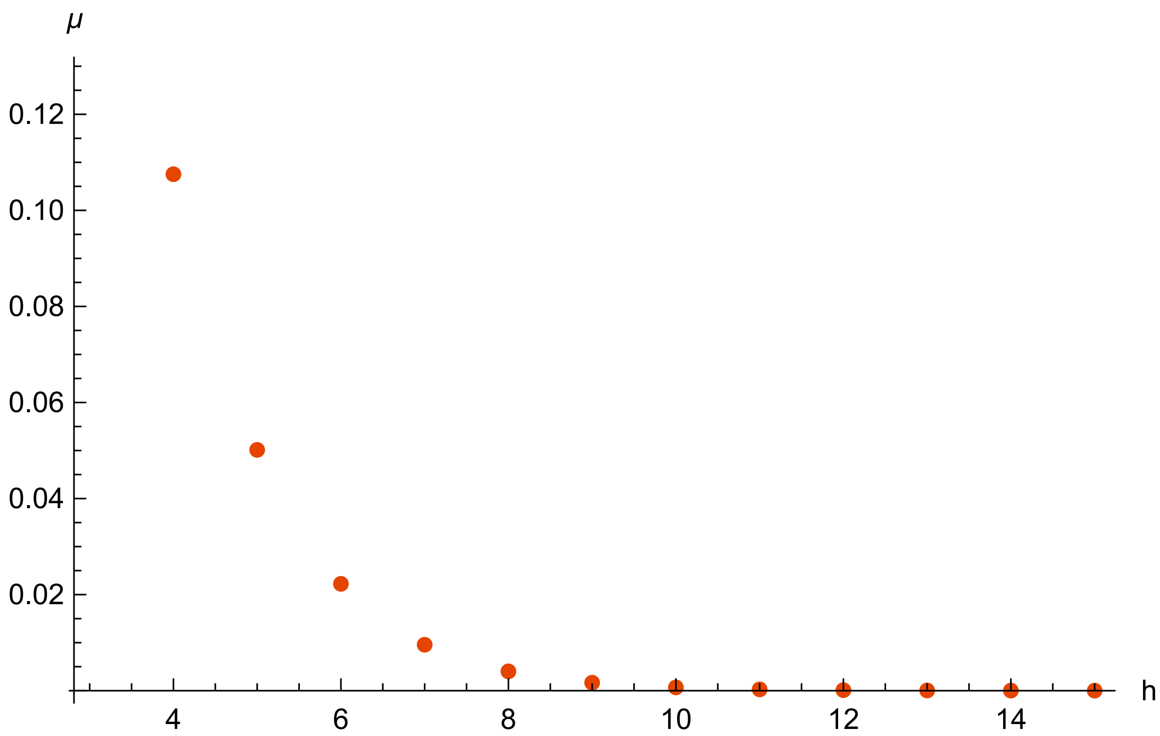

Figure 4.13: Plot of the eigenvalue of $\mathcal{A}_W$ depending on the parameter h

Conclusion

In this thesis we properly defined the damped wave operator as a generator of a C_0- semigroup which provided us with unique and regular solutions of the damped wave equation. We stated some known results on the time evolution and stability of these solutions. Afterwards we defined the Schrödinger operator with real potential as a bounded perturbation of the Dirichlet Laplacian and stated some of its spectral properties. We obtained the formulas for the first and second derivative of the first eigenvalue of this operator with respect to a multiplicative parameter which parameterizes its potential. In the main chapter of the thesis we obtained numerous bounds and criteria of existence of the eigenvalues of the damped operator using the correspondence between its spectrum and the spectrum of the Schrödinger operator and know results for the latter. We used namely the Lieb-Thirring inequalities, Buslaev-Faddeev-Zakharov trace formulae and the Birman-Schwinger principle. The last mentioned was used to establish the Birman-Schwinger principle for the damped wave operator even in the case of complex damping. Finally we demonstrated some of the obtained results in the case when the damping is the finite rectangular well.

In the future work we would like to focus on the connection between the Schrödinger operator with complex potential and the damped wave operator with complex damping and use the numerous number of results for the former to obtain some information about the latter.

Bibliography

[1] Adams, R. A., and J. J. F. Fournier. *Sobolev Spaces*. Academic Press, 2003.

[2] Cox, S., and E. Zuazua. "The Rate at Which Energy Decays in a Damped String." *Communications in Partial Differential Equations*, vol. 19, no. 1-2, 1994, pp. 213–243., doi:10.1080/03605309408821015.

[3] Curtain, R. F., and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer, 1995.

[4] Davies, E. B. *Spectral Theory and Differential Operators*. Cambridge Univ. Press, 1996.

[5] Evans, L. C. *Partial Differential Equations*. American Mathematical Society, 2010.

[6] Frank, R. L., A. Laptev, E. H. Lieb, and R. Seiringer. "Lieb–Thirring Inequalities for Schrödinger Operators with Complex-Valued Potentials." *Letters in Mathematical Physics*, vol. 77, no. 3, 2006, pp. 309–316., doi:10.1007/s11005-006-0095-1.

[7] Freitas, P., and Krejčiřık D. "Instability Results for the Damped Wave Equation in Unbounded Domains." *Journal of Differential Equations*, vol. 211, no. 1, 2005, pp. 168–186., doi:10.1016/j.jde.2004.06.006.

[8] Freitas, P., P. Siegl, and C. Tretter. "The Damped Wave Equation with Unbounded Damping" *Journal of Differential Equations*, vol. 264, no. 12, 2018, pp. 7023–7054., doi:10.1016/j.jde.2018.02.010.

[9] Gilbarg, D., and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, 1998.

[10] Kato, T. *Perturbation Theory for Linear Operators*. Springer, 1995.

[11] Ladyzhenskaya, O. A. *The Boundary Value Problems of Mathematical Physics*. Springer-Verlag, 1985.

[12] Laptev, A. "Spectral inequalities for Partial Differential Equations and their applications." *AMS/IP Studies in Advanced Mathematics*, vol. 51, 2012

[13] Lieb, E. H., and M. Loss. *Analysis*. American Mathematical Society, 2001.

[14] Liu, W. *Elementary Feedback Stabilization of the Linear Reaction-Convection-Diffusion Equation and the Wave Equation*. Springer Berlin Heidelberg, 2010.

[15] Markus, A. S. *Introduction to the Spectral Theory of Polynomial Operator Pencils*. American Mathematical Society, 1988.

[16] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, 1983.

[17] Reed, M., and B. Simon. *Methods of Modern Mathematical Physics*. Academic Press, 1975.

[18] Solà-Morales, J. "Global Instability and Essential Spectrum in Semilinear Evolution Equations." *Mathematische Annalen*, vol. 274, no. 1, 1986, pp. 125–131., doi:10.1007/bf01458020.

[19] Teschl, G. *Mathematical Methods in Quantum Mechanics: with Applications to Schroödinger Operators*. American Math. Soc., 2014.

[20] Weidmann, J. *Linear Operators in Hilbert Spaces*. Springer, 1980.

[21] Zakharov, V. E., and L. D. Faddeev. "Korteweg–De Vries Equation: A Completely Integrable Hamiltonian System." *Fifty Years of Mathematical Physics*, 2016, pp. 277–284., doi:10.1142/97898143409600023.