České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Kvantová Weylova gravitace a její kosmologické implikace

Quantum Weyl gravity and its cosmological implications

Master's thesis

Author: **Bc. Jaroslav Kňap**

Supervisor: **Ing. Mast. Petr Jizba, PhD.**

Academic year: 2018/2019

Acknowledgement:

I would like to thank my supervisor Ing. Mast. Petr Jizba, PhD. for suggesting the topic of this research task and express my gratitude for his guidance and patience. I also wish to thank to Prof. dr. Ana Achúcarro for her guidance during my stay at Leiden University. Finally I would also like to thank Markéta Kofroňová for providing me support when I needed it the most.

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS19/183/OHK4/3T/14.

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Název práce:

Kvantová Weylova gravitace a její implikace na kosmologii

Autor: Bc. Jaroslav Kňap

Obor: Matematické inženýrství

Zaměření: Matematická fyzika

Druh práce: Diplomová práce

Vedoucí práce: Ing. Mast. Petr Jizba, PhD., Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, ČVUT v Praze

Abstrakt: Weylova teorie gravitace je unikátní konformně invariantní teorie gravitace ve čtyř dimensionálním časoprostoru. Zájem o tuto teorii roste díky její poruchové renormalizovatelnosti a nedávném úspěchu při vysvětlení galaktických rotačních křivek. Tato práce začíná přehledem základních konceptů v konvenční kosmologii, s důrazem na koncepty vztahující se k inflaci. Po teoretické motivaci konceptu inflace představuje některé základní důsledky tohoto konceptu a možné fyzikální mechanismy. Následně se zaměřujeme na diskuzi inflačního scénaře ve Weylově teorii, konkrétně ve fázi zlomené škálové symetrie. V poslední části pak začínáme výpočet jednosmyčkového efektivního potenciálu pro skalární pole v De Sitterově pozadí.

Klíčová slova: De Sitterův prostor, inflace, konformní symetrie, kvantová teorie gravitace, Weylova gravitace

Title:

Quantum Weyl gravity and its cosmological implications

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Abstract: Weyl's theory of gravity is a unique conformally invariant theory of gravity in four dimensional spacetime. The interest in this theory is growing, due to its power-counting renormalizability and its recent success in explaining galaxy rotation curves. This work begins with overview of basic concepts in conventional cosmology, with emphasis on concepts relating to inflation. After theoretical motivation of the concept of inflation, we introduce some basic consequences of this concept a its possible physical mechanism. We follow up by discussing inflationary scenarios in Weyl's theory of gravity, specifically in the phase of broken scale symmetry. In the final part we start a calculation of one-loop effective potential for scalar field in De Sitter background.

Key words: conformal symmetry, De Sitter space, inflation, quantum theory of gravity, Weyl gravity

Contents

Introduction

The success of ΛCDM model of cosmology in explaining the various observation is almost unreasonable. The model has 6 free parameters and using those and some theoretical assumptions it explains many observations over large range of length scales and over a 10Gyr of cosmic time. However, there are still limitations, as many ingredients of this model are mysterious from a viewpoint of fundamental physics. First, in this framework inflation must be introduced by hand, by postulating new scalar field (inflaton) with appropriate potential. There are also issues with General Relativity itself, such as non-renormalizability in the perturbative sense.

These issues lead us from two different directions to the same conclusion, that it is reasonable to consider modified theories of gravity. It has been demonstrated that one-loop effective action generates terms of R^2 in gravitational action, and so it is natural to consider modified action, such as Starobinsky action. Actions with quadratic term can also generate inflationary potentials which are in good agreement with observations.

A good candidate for modified theory of gravity is Weyl theory of gravity. It is a unique conformally invariant action in four dimensions, which leads to its power-counting renormalizability which makes it attractive from theoretical standpoint. Its success in explaining of rotation curves of galaxies ([4], [5]) only adds to its attractiveness. It can also be considered as an appropriate candidate for high-energy extension of General relativity, as after spontaneous breaking of scale symmetry it generates Starobinsky action.

Scope of the thesis

The goal of this work is familiarization with cosmology, with focus on inflationary cosmology, and application of this knowledge to discussion of inflationary scenario in Weyl gravity.

The first chapter introduces the usual FLRW cosmological metric. We than continue with basic discussion of the Standard Big Bang model and we point out some of its weaknesses. This naturally leads to introduction of concept of inflation, which allows to resolve those weaknesses. We then discuss potential physical mechanisms which could lead to inflationary phenomenology, and elaborate on some models in more detail. The chapter ends with discussion of primordially perturbations and mention of bounds on inflationary models from observations.

The second chapter, discussing Weyl gravity, starts with concept of classical Weyl gravity, its primary features and possibility of transforming its action to simpler form. We then explore inflationary scenario in Weyl gravity in phase of spontaneously broken scaling symmetry and compare results with most recent constraints on inflationary models from Planck 2018. In this exploration we use concepts from the first chapter and from appendices. We close this chapter with attempt to calculate one-loop effective potential for scalar field appearing due to fluctuations of metric field in De Sitter background. To this end we use the methods of $\zeta-$function regularization in curved background.

Appendices cover some of the tools used in this work. They discuss two transformations used in physics, namely Hubbard-Stratonovich transform which is a transformation on level of path integrals used in simplification of quadratic interactions, albeit at a cost of introducing new scalar field. Second is the transformation relating so-called Einstein and Jordan frames. These are two different formulations of gravitational theories related by conformal transformation and scalar field redefinitions. Question of their physical equivalence is also mentioned. Penultimate appendix discuss distinction between scale symmetry and conformal symmetry, and possibility of enhancement of scale symmetry to conformal symmetry in quantum field theories. Focus of the final appendix is $\zeta-$function renormalization in curved background and expansion of heat kernel.

Chapter 1

Inflationary cosmology

In this chapter we will introduce inflation in cosmology. To this end we will start by quickly reviewing homogeneous universe, specifically we will derive FLRW metric from general assumptions and follow by investigating its properties. After this brief overview, we will discuss conventional Big Bang model and demonstrate that it leads to problems with initial conditions, such as flatness problem and horizon problem. We follow by introducing the inflation mechanism as a possible remedy for the aforementioned problems. We then investigate the consequences of inflation and discuss possible physical mechanism driving this phenomenon. We discuss some inflationary models in more details. Finally we close this chapter with discussion of observable consequences of the model and of bounds that can be derived from observations. Readers seeking more details are referred to the following literature: [6], [7] for Big Bang Cosmology, and [8], [9] for more details on inflation.

1.1 FLRW space

In the field of cosmology, we are interested in the large-scale structure of the universe and its evolution. From astronomical observations it appears that the universe on the largest scales (i.e. super-galaxy scales) is (spatially) both homogeneous and isotropic, to quite high degree of accuracy. This is usually stated as the *cosmological principle*: *'At any instant in time, the universe looks the same at all positions in space, and all directions in space at any point are equivalent'*. This is of course a simplification, but it is a simplification which works remarkably well for modelling large-scale behaviour of universe as we shall see.

Before we can continue our discussion we must make the term 'instant in time' more precise, as there are no global inertial frames with respect to which we could define absolute time. We do this be talking not about instants of time, but about *three-dimensional space-like surfaces*. this means that we foliate the spacetime with non-intersecting 3D space-like surfaces, labelled by parameter t. This foliations, or slicing, can be done in multiple ways, so there is no preferred time. We also introduce idealised *fundamental observers*, who have no motion relative to the cosmological fluid (which consists of 'smeared-out' motion of the matter in the universe). For these observers we assume *Weyl's postulate*, which states that these observers provide a threading of the spacetime, i.e. through each non-singular spacetime point passes a unique worldline. With these assumptions it is possible to construct hypersurfaces of $t = const.$ such that 4−velocities of fundamental observers are orthogonal to the hypersurface.

The parameter t labelling the above constructed hypersurfaces can be taken as the proper time along the wordline of any fundamental observer, such parameter is then called *synchronous time coordinate*. We also introduce *comoving coordinates* $\left(x^1, x^2, x^3\right)$ which are spatial coordinates which remain constant along any wordline. In such coordinates the line element necessarily takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1.1}$$

where g_{ij} are generally functions of all four coordinates.

Now we will use the observation that the universe on large-scales looks isotropic and homogeneous. This leads us to the following general form of the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + S^2(t)\, h_{ij}\left(x^1, x^2, x^3\right)\mathrm{d}x^i\mathrm{d}x^j. \tag{1.2}$$

We require the 3−spaces to be homogeneous and isotropic, this leads us to consider *maximally symmetric 3-spaces*. These are spaces with the same number of symmetries as a Euclidean space of the same dimension. It can be shown that in such spaces the curvature is specified by a single number K, independent of coordinates. It is clear that such spaces are both homogeneous and isotropic. The metric of such 3−spaces takes the form

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{1.3}$$

Combining the last two expressions, we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + S^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right]. \tag{1.4}$$

We can simplify by absorbing the arbitrariness of the magnitude of K into the radial coordinate and the factor S^2. Defining $k = K/|K|$ and $r \to |K|^{1/2}r$ we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right], \tag{1.5}$$

with

$$R(t) = \begin{cases} \dfrac{S(t)}{|K|^{1/2}} \\ S(t). \end{cases} \tag{1.6}$$

Finally we define the dimensionless scale factor $a(t)$ as

$$a(t) = \frac{R(t)}{R_0}, \tag{1.7}$$

where R_0 is cosmological scale at present epoch. This leads us to the final form of the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric for the spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right], \tag{1.8}$$

where k is either $+1, 0, -1$ for Σ (spatial slice of spacetime) positively curved, flat or negatively curved respectively and the spatial coordinates are still the comoving coordinates.

As stated the FLRW metric describes homogeneous and isotropic universe, and as we can see, this permits time-dependent scale factor. Physical distance are obtained by multiplying the comoving distance by the scale factor, i.e. $R = a(t)\,r$, this is called the *proper distance*. Downside of this distance is that it cannot be directly measured, so in practice we must use different distance measures such as *angular diameter distance* or *luminosity measure* to define distance of objects, and then convert these to proper distances.

From the equation 1.8 it is clear that for our ansatz the evolution of the isotropic homogeneous universe is determined by the single function $a(t)$, which is determined by the matter content of the universe through the Einstein equations. Another important quantity characterizing the FLRW universe is the expansion rate of the universe, defined as follows

$$H = \frac{\dot{a}}{a}, \tag{1.9}$$

where $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$. This expansion rate is also known as the *Hubble parameter*. From definition it is clear it has units of inverse time, and so can be used to define characteristic time scale $t_{char} \sim H^{-1}$ for age of the universe. Using the speed of light as a conversion constant, we can also obtain characteristic length scale for the size of observable universe. Sign of the Hubble parameter determines whether the universe expands or contracts, for positive H the universe in expanding and for negative it is collapsing.

Now we will turn our attention to some basic kinematics in FLRW space. We start by defining new coordinates τ and χ as follows

$$\tau = \int \frac{\mathrm{d}t}{a(t)}$$
$$r^2 = \Phi_k\left(\chi^2\right) = \begin{cases} \sinh^2\chi & k=-1 \\ \chi^2 & k=0 \\ \sin\chi^2 & k=+1 \end{cases},$$

and in these coordinates the FLRW-metric takes the following form

$$\mathrm{d}s^2 = a\left(\tau^2\right)\left[-\mathrm{d}\tau^2 + \left[\mathrm{d}\chi^2 + \Phi_k\left(\chi^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right]\right]. \tag{1.10}$$

The new time coordinate τ is called *conformal time*, since it allows us to make the scale factor in conformal prefactor common to all coordinates. Metric in these coordinates is well suited for study of kinematics of massless particles, we immediately see that radial movement of a massless particle must satisfy

$$\mathrm{d}s^2 = 0 = a\left(\tau^2\right)\left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2\right], \tag{1.11}$$

so in these coordinates radial movement of massless particle looks like a movement in flat 1+1 spacetime, and solving for χ we get

$$\chi(\tau) = \pm\tau + \text{const.}, \tag{1.12}$$

obtaining a light cone in these coordinates, which simplifies analysis of causal structure.

Next we will consider extent of regions accessible by light signals (or generally anything propagating at light speed). We assume to have an emitter at some comoving coordinate χ_1 and observer at $\chi = 0$. If the emitter emits a photon at time t_1 which is then received by the observer at time t, then the the only signals emitted at time t_1 which the observer has received by time t are from $\chi < \chi_1$. The comoving coordinate of the emitter is given by

$$\chi_1 = \int_{t_1}^{t} \frac{\mathrm{d}t'}{a(t')} = \int_{a(t_1)}^{a(t)} \frac{\mathrm{d}\ln a}{aH(a)}. \tag{1.13}$$

Clearly if the above integral diverges for $t_1 \to 0$ (where $t_0 = 0 \Leftrightarrow a(t_0) = 0$), then in principle it is possible to receive signals emitted at sufficiently early time in the universe from *any* comoving particle. However if the above integral converges, than there is a limit at a distance which can be observed at given time t, our observational ability is limited by the *particle horizon*. Its comoving coordinate is given by

$$\chi_P(t) = \int_0^{t} \frac{\mathrm{d}t'}{a(t')} = \int_0^{a(t)} \frac{\mathrm{d}\ln a}{aH(a)}. \tag{1.14}$$

It can be shown that any model with decelerating expansion up to a time t will have finite particle horizon at that time, this will cause issues later on in the traditional Big Bang model.

Particle horizon defines which events we can see at given time, on the other hand we can wonder if there are events which we will never be able to observe or influence. This will lead to the definition of the so-called *event horizon*. Taking the equation (1.13) we see that if the integral diverges in the limit $t \to \infty$ than any event will eventually become observable. If however the integral converges in the large t limit then we can observe only those events whose comoving coordinate χ satisfies

$$\chi < \chi_E = \int_{t}^{t_{max}} \frac{\mathrm{d}t'}{a(t')}, \tag{1.15}$$

where t_{max} is either infinity or $a(t_{max}) = 0$, $t_{max} \neq 0$. By symmetry of observer and emitter $\chi_E(t)$ is also the maximal distance a light signal sent by the observer can ever reach.

Final distance we will discuss is defined using the Hubble parameter H. It is clear that Hubble parameter has units of t^{-1}, and so H^{-1} has units of time, since we have set $c = 1$, we can define *Hubble distance*

$$d_H(t) = H^{-1}(t), \tag{1.16}$$

and also *comoving Hubble distance*

$$\chi_H(t) = (aH)^{-1}(t) \tag{1.17}$$

Unlike the previous two distances, this one does not depend on the entire history and future evolution of the scale factor $a(t)$, it is a purely locally (in time sense) defined quantity. The Hubble distance corresponds to the typical scale on which physical process can operate coherently at given time t. From the equation (1.13) we see that the evolution of comoving Hubble distance can be used when calculating particle and event horizons. This fact will be important when considering horizon and flatness problems in the traditional Big Bang cosmology.

As we have seen earlier, dynamics of the isotropic, homogeneous universe is characterized by the evolution of the scale factor $a(t)$, which is determined from the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{1.18}$$

As we are interested in cosmological application we will be interested in the stress-energy tensor of a fluid. Introducing a set of observes whose worldlines are tangent to the timelike velocity 4-vector

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \tag{1.19}$$

where τ is proper time of the observers, so that the 4-velocity is normalized $g_{\mu\nu}u^\mu u^\nu = -1$. We define metric of 3-dimensional spatial sections orthogonal to u^μ as

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \tag{1.20}$$

this can be used to project quantities to the observers' instantaneous rest space. Now we can write the stress-energy tensor of the imperfect fluid as follows

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\gamma_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \Sigma_{\mu\nu}, \tag{1.21}$$

where $\rho = T_{\mu\nu}u^\mu u^\nu$ is the matter-energy density, $p = \frac{1}{3}T_{\mu\nu}\gamma^{\mu\nu}$ is the isotropic pressure, $q_\mu = -\gamma_\mu{}^\alpha T_{\alpha\beta}u^\beta$ is the energy flux vector and finally $\Sigma_{\mu\nu} = \gamma_{[\mu}{}^\alpha \gamma_{\nu]}{}^\beta T_{\alpha\beta}$. As always () signifies symmetrization of the enclosed indices and [] signifies antisymmetrization of the indices. For a perfect fluid there exists a unique 4-velocity such that $q_\mu = 0$ and $\Sigma_{\mu\nu} = 0$, and so the stress-energy tensor takes the form

$$T^\mu{}_\nu = (p+\rho)\, u^\mu u_\nu - p\delta^\mu{}_\nu, \tag{1.22}$$

where u^μ is the 4−velocity of the fluid. In a comoving frame we can choose $u^\mu = (1,0,0,0)$ to obtain a diagonal form of the stress-energy tensor (for mixed indices). Now we want to solve Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ with the stress-energy tensor of perfect fluid and metric given by FLRW metric. This leads to the following set of equations

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{1.23}$$

$$\dot H + H^2 = \frac{\ddot a}{a} = -\frac{1}{6}(\rho + 3p), \tag{1.24}$$

the above equations are known as the *Friedmann equations*, they are two coupled, non-linear ordinary differential equations. A cursory look reveals that in the expanding universe (i.e. $\dot a > 0$) filled with classical matter (i.e. matter satisfying strong energy condition $\rho + 3p \geq 0$) we obtain as a result of the equations that $\ddot a < 0$, implying existence of singularity in the finite past. Of course this conclusion only holds, if we assume that both general relativity and Friedmann equations are applicable to arbitrarily high energies and that no exotic forms of matter become relevant at such high energies. Combining the Friedmann equations leads to the continuity equation for the perfect fluid

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + p) = 0,$$

which can be alternatively rewritten as

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a} = -3(1+w), \tag{1.25}$$

where $w = \frac{p}{\rho}$ is a state parameter. Solving for ρ we get

$$\rho \propto a^{-3(1+w)}, \tag{1.26}$$

which we can immediately use to solve the first Friedmann equation for a, yielding

$$a(t) \propto \begin{cases} t^{\frac{2}{3}(1+w)} & w \neq -1 \\ \mathrm{e}^{Ht} & w = -1 \end{cases} \tag{1.27}$$

Assuming now that $k = 0$ in FLRW metric, this leads to $a(t) \propto t^{2/3}$, $a(t) \propto t^{1/2}$ and $a(t) \propto \exp(Ht)$ for universe dominated by non-relativistic matter ($w = 0$), radiation or relativistic matter $\left(w = \frac{1}{3}\right)$ and a cosmological constant ($w = -1$), respectively. If there are more types of matter relevant for cosmological behaviour (as in our universe), we simply write

$$p = \sum_i p_i \qquad\qquad \rho = \sum_i \rho_i, \tag{1.28}$$

where the summation index i runs over the relevant matter types. In such case we also define the present ration of energy density

$$\Omega_i = \frac{\rho_0^i}{\rho_{crit}}, \tag{1.29}$$

where subscript 0 signifies that the value is understood to be taken at present, and critical density is defined as $\rho_{crit} = 3H^2$. Using this we obtain for the Friedmann equations

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} + \Omega_k a^{-2} \tag{1.30}$$

$$\frac{1}{a_0 H_0^2}\frac{\mathrm{d}^2 a_0}{\mathrm{d}t^2} = -\frac{1}{2}\sum_i \Omega_i (1 + 3w_i) \quad (at\, t_0 = 0) \tag{1.31}$$

where $\Omega_k = \frac{-k}{a_0^2 H_0^2}$. From the current observation we get for the various Ω

$$\begin{aligned} \Omega_k &\sim 0 \\ \Omega_b &= 0.04 \\ \Omega_{dm} &= 0.23 \\ \Omega_\Lambda &= 0.72, \end{aligned}$$

where Ω_b is ration of baryonic matter, Ω_{dm} is the ratio of dark matter, Ω_Λ is the ratio of dark energy and Ω_k is the ratio given by curvature. Now we turn our attention to the Big Bang model.

1.2 Big Bang model, its problems and solutions

In this section we will introduce the Big Bang model, discuss some of its shortcoming and sketch a possible solution of these with the use of inflation.

The idea of expanding universe was formulated in 1920s, first with Friedmann's solution of Einstein's equations in 1924, where he neglected cosmological constant and obtained an solution describing an expanding universe. Three years later, in 1927 Georges Lemaitre proposed that our universe is an expanding one, to explain the observed redshift of spiral nebulae. During his work, he derived the Hubble's law and re-derived Friedmann's equations. Also his work led to the prediction that the amount of redshift was not constant, but that there was a relation between the distance from the observer and the observed redshift. Finally, in 1929, Edwin Hubble provided comprehensive observational evidence supporting Lemaitre's theory. His experimental observations have discovered that, relative to the Earth, all other observed bodies are receding in every direction at velocities proportional to their distance from the Earth. This isotropic nature of the expansion serves as a proof that it is a *space* which is expanding,and it was this discovery which lead to the formulation of the Standard Big Bang model.

The formulation of Standard Big Bang model began in earnest with the work of G. Gamow and his colleagues R. A. Alpher and R. C. Herman in the 1940s. They had postulated that the early universe was extremely hot and dense, to account for the abundance of elements in the universe. In 1948, Alpher and Herman predicted that one consequence of their model is presence of relic background radiation with temperature of order of few K.

In the 1950s Standard Big Bang theory was competing with a Steady state model of the universe. However, due to the additional observational evidence, such as quasars and radio galaxies, and a greater predictive power of the Big Bang model (such as an explanation of the formation and abundance of hydrogen and helium), led to a widespread acceptance of the Big Bang model as the leading cosmological model.

In 1964 the discovery of CMB has confirmed a supremacy of Standard Big Bang model over the Steady state universe hypothesis. Ironically detailed study of the CMB has also revealed a weakness in the Big Bang model, as it was unable to explain some features such as homogeneity of CMB, without invoking fine-tuning of initial time parameters.

In 1981, Alan Guth has proposed a solution of the flatness and horizon problems, proposing that early in its history the universe underwent a period of accelerated expansion, that is the idea of inflation. The first slow-roll inflation model was published in 1982. After this, numerous models were proposed, with some motivated by an attempt at making contact with particle physics (where the inflaton field can be the GUT Higgs field).

Last large modification to the cosmological paradigm came in 1998, with the observation of accelerated expansion. This discovery lead to the formulation of the current cosmological framework, the ΛCDM model.

In accordance with observations, the universe in Standard Big Bang model is described by the FLRW metric, which we introduced in the last section. The cosmological equation of motion are derived from Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.32}$$

where the $\Lambda g_{\mu\nu}$ is interpreted as an effective energy-momentum tensor of the vacuum. Assuming that the energy-momentum tensor of the matter content takes form of perfect fluid we obtain Friedmann-Lemaitre equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{1.33}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{1}{6}(\rho + 3p) \,. \tag{1.34}$$

For the following analysis it is useful to define critical density ρ_c, which is such that $k = 0$ when $\Lambda = 0$

$$\rho_c = 3H^2. \tag{1.35}$$

We then define cosmological density parameter Ω_{tot} relative to the critical density as

$$\Omega_{tot} = \frac{\rho_{tot}}{\rho_c}, \tag{1.36}$$

and use this to rewrite the first of Friedmann's equations

$$1 - \Omega_{tot} = -\frac{k}{(aH)^2}. \tag{1.37}$$

From this it is clear that $\Omega_{tot} < 1$ for $k = -1$ and so the Universe is open, when $\Omega_{tot} = 1$ $k = 0$ and Universe is spatially flat and finally when $\Omega_{tot} > 1$ $k = +1$ and the Universe is closed. For more detailed analysis it is useful to separate the Ω_{tot} into individual contribution. Therefore we define present-day density parameters for relativistic particles (and radiation) Ω_r and pressureless matter Ω_m. We also define a quantity $\Omega_\Lambda = \Lambda/3H^2$, which in more general models is no longer constant and so we define present-day density of the vacuum Ω_v. With all these contributions the Friedmann equation becomes

$$\Omega_m + \Omega_r + \Omega_v - 1 = \frac{k}{(a_0 H_0)^2}, \tag{1.38}$$

where the subscript 0 signifies present-day values. Sometimes the quantity $-k/(a_0 H_0)^2$ is denoted as Ω_k, however this notation implies that the spatial curvature contributes to the energy density of the Universe, which is not correct and so this notation is misleading.

From the Friedmann's equations we can derive how the density for single component behaves, we define state parameter $w = p/\rho$ to simplify the expressions. For the general equation of state we can obtain from the continuity equation $\dot{\rho} = -3H(1 + w)\rho$ the following expression

$$\rho \propto a^{-3(1+w)}. \tag{1.39}$$

We note that provided $w > -1/3$ the curvature term k/a^2 is less singular at early times and so can be neglected. This also implies that curvature domination era occurs at late times. The values w for radiation, matter and cosmological constant are

$$\begin{aligned} w_{MD} &= 0 \\ w_{RD} &= 1/3 \\ w_\Lambda &= -1. \end{aligned}$$

We can also plug the expression (1.39) into the Friedmann equations (for $w \neq -1$) to obtain a relation for scale factor

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{1.40}$$

For the case $w = -1$ we obtain from the Friedmann's equations exponential expansion

$$a(t) \propto e^{\sqrt{\Lambda/3}t}. \tag{1.41}$$

From the expression for the density it is clear that radiation and matter are dominant contribution to the energy tensor at sufficiently early times.

In the earliest hot and dense universe it is reasonable to assume that the matter can be described as a fluid of radiation/relativistic particles, for which $w = 1/3$. In this case, we obtain for energy density $\rho \propto a^{-4}$, where the additional factor of a^{-1} is due to cosmological redshift causing decrease in energy. For the scale factor and Hubble parameter we obtain

$$a(t) \propto t^{1/2}, \qquad H(t) = \frac{1}{2t}. \tag{1.42}$$

At later times, the non-relativistic matter starts to dominate the energy density. It has the equation of state of pressureless gas, so $w = 0$, and the density behaves as $\rho \propto a^{-3}$. The scale factor and Hubble parameter behave as follows

$$a(t) \propto t^{2/3}, \qquad H(t) = \frac{2}{3t}. \tag{1.43}$$

Finally, if the dominant contribution is from the vacuum energy V_0, it would act as cosmological constant with equation of state $w = -1$. The density would then be constant $\rho \propto a^0$, and scale factor would behave exponentially

$$a(t) \propto e^{\sqrt{\Lambda/3}t}, \qquad H = \sqrt{\Lambda/3}. \tag{1.44}$$

In the Standard Hot Big Bang model we assume that the early universe was dominated by radiation and relativistic matter, so the scale factor behaves as $a(t) \propto t^{1/2}$ for t sufficiently close to 0. This however means that the integral (1.14) defining particle horizon is convergent and so the particle horizon must be finite. This will lead to problems, as we will see from following example:

Take two objects that now have a proper distance of 10^9 light years from each other. Age of this universe is about $\sim 1.4 \times 10^10$ years, so there has been enough time for these objects to have exchanged about 14 light signals. We might assume, that since at earlier times the scale factor a was smaller and thus everything was closer together, the causal contact would be better. However it turns out that in continually decelerating universe this actually worsens causal contact. For example, at recombination the ratio of scale factors was

$$\frac{a(t_{rec})}{a_0} \approx 10^{-3}, \tag{1.45}$$

and so the proper distance of the objects was 10^6 light years. However, if we assume that prior to recombination the universe was radiation dominated, we obtain for time at recombination

$$\left(\frac{t_{rec}}{t_0}\right)^{2/3} = \frac{a_{rec}}{a_0} = 10^{-3}, \tag{1.46}$$

from which we obtain $t_{rec} = 1.4 \times 10^{5.5}$. From this we can obtain the proper distance to particle horizon

$$2ct_{rec} = 2.8 \times 10^5.5. \tag{1.47}$$

Since the two objects at t_{rec} where separated by proper distance of 10^6 light years, and the particle horizon was $2.8 \times 10^{5.5}$ light years away, they could not have exchanged even a single signal.

From the above analysis we see that there might be some problems with Standard Big Bang model, and so now we turn our attention to some of these problems. Here we use the word problem slightly loosely since most of these problems concern fine-tuning of initial conditions, which is strictly speaking not a problem of the theory. However we do feel that there might be a more robust solution for the evolution of early universe which would allow us to bypass fine-tuning.

We wish to discuss a Cauchy problem of the universe, i.e. a solution of Einstein equation given some initial conditions. To specify initial conditions we must consider a spatial slice of constant time Σ_i, for our purposes we will neglect gauge-dependence of this choice. On this initial surface we define positions and velocities of all particles, and then we use laws of gravity and fluid dynamics to evolve system forward in time. If we wish to obtain a universe we now observe we run into two problems.

One, we assumed isotropy and homogeneity of the universe, however inhomogeneities are gravitationally unstable (i.e. they grow with time). Since observations of the cosmic microwave background show that in the past inhomogeneities were much smaller at the time of last-scattering than today. Due to the aforementioned instability of inhomogeneities they should have been even smaller before last-scattering. The question then arises, why was early universe so extremely smooth? This is even more surprising in the light of the fact that (as we shall soon see), early universe in our model should consists of a large number of causally disconnected regions of space, and in the Big Bang there is no mechanism to explain why they apparently show such similar physical conditions. This is the so-called *horizon problem*.

Another issue concerns the initial velocities. Cauchy problem requires us to specify the velocity of the fluid at every point of the initial 3-surface. However for universe to remain homogeneous at late time requires very precise values of the initial velocities. Too small initial velocities would quickly lead to recollapse of the universe, and too large values would expand very rapidly and quickly become nearly empty. This fine-tuning becomes even more startling when combined with horizon problem, as the fine-tuning must take place across causally separated regions of space. As velocities of particles give us their kinetic energy and difference of potential and kinetic energies defines the local curvature, this problem is usually called *flatness problem*.

1.2.1 Horizon problem

We will now look at the horizon problem in more detail. Recalling the definition of the comoving particle horizon (i.e. the causal horizon)

$$\chi_P \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^{a_0} \frac{\mathrm{d}a}{Ha^2} = \int_0^{a_0} \mathrm{d}\ln a \left(\frac{1}{aH}\right), \tag{1.48}$$

where we have expressed the comoving horizon using the integral of the comoving Hubble radius $(aH)^{-1}$, this quantity will be crucial in inflation. From previous section we can write for the comoving Hubble radius in the universe dominated by the fluid with state parameter w

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}. \tag{1.49}$$

The qualitative behaviour depends on the sign of the quantity $1 + 3w$, in conventional Big Bang it holds that $w \gtrsim 0$ and so comoving Hubble radius $(aH)^{-1}$ grows monotonically with time, and as a result so does the comoving horizon τ, i.e. the fraction of the universe in causal contact increases with time. Specifically, for the radiation-dominated and matter-dominated universes we have for the comoving horizon

$$\chi_P = \int_0^a \frac{\mathrm{d}a_0}{Ha^2} \propto \begin{cases} a & RD \\ a^{1/2} & MD \end{cases} \tag{1.50}$$

The monotonic growth of the comoving horizon with time implies that comoving scales currently entering the horizon must have been far outside of the horizon at CMB decoupling. However the near-homogeneity of the CMB implies that the universe was very homogeneous at last-scattering, even on scales that are *a priori* causally independent, as can be seen from the example mentioned in the last section. The question now arises, how could homogeneity on such scales appear?

1.2.2 Flatness problem

Before we propose an answer to this question that does not require fine-tuning we will first discuss the second problem, the flatness problem. In General Relativity, the spacetime is a dynamical quantity, curving in reaction to the matter content. This then begs the question, why is the universe well approximated by flat Minkowski space? If we consider the Friedmann equation

$$H^2 = \frac{1}{3}\rho(a) - \frac{k}{a^2}, \tag{1.51}$$

and rewrite it as follows

$$1 - \Omega(a) = \frac{-k}{(aH)^2}, \tag{1.52}$$

where

$$\Omega(a) \equiv \frac{\rho(a)}{\rho_{crit}(a)}, \qquad \rho_{crit} \equiv 3H(a)^2 .$$

From the equation 1.52 it is clear that since in the standard cosmology the comoving Hubble radius $(aH)^{-1}$ grows with time, the quantity $|\Omega - 1|$ must diverge with time. This means that the critical value $\Omega = 1$ is unstable fixed point. This means that the near-flatness $\Omega(a_0)$ we observe today requires extreme fine-tuning of Ω close to 1 in the early universe. Specifically one finds the following bounds on deviation from flatness

$$|\Omega(a_{BBN}) - 1| \leq O\left(10^{-16}\right) \tag{1.53}$$

$$|\Omega(a_{GUT}) - 1| \leq O\left(10^{-55}\right) \tag{1.54}$$

$$|\Omega(a_{Pl}) - 1| \leq O\left(10^{-61}\right), \tag{1.55}$$

where for the abbreviations we have: BBN signifies *Big Bang Nucleosynthesis*, GUT the *Grand Unification Theory* era, and Pl signifies the *Planck scale*. It is clear that fulfilling these constraints would require quite extreme fine-tuning, without any deeper theoretical reason for the initial conditions to be such.

1.2.3 Monopole problem

Final problem of the traditional Big Bang model we will mention here is the monopole problem. This problem is unrelated to the initial conditions and instead arises due to the fact that the Standard model of particle physics ha undergone numerous phase transitions as the universe developed. Specifically this is related to the symmetry breaking, such as electroweak transitions

$$SU(3) \otimes SU(2) \otimes U(1)_Y \to SU(3) \otimes U(1)_{em}. \tag{1.56}$$

These symmetry breaking phase transitions lead to emergence of topological defects, that is field configurations with non-zero energy where the field is not in vacuum state and which cannot be removed. Examples of such defects are monopoles, cosmic strings, domain walls and others.

The problem is as follows, assuming a semi-simple GUT group which is broken down to the SM, monopoles form. They are heavy and pointlike, this means they behave as cold matter $\rho \sim a^{-3}$. After lengthy calculations we will obtain that in each Hubble volume of the universe there should be a number of topological defects of order unity. Since topological defects are quite energetic objects which should have formed extremely early in the universe, we should be able to observe some evidence of their presence. However, we have so far found no evidence of topological defects, suggesting that the real density is of lower order. The monopole problem can thus be states as follows: 'Why is the observed density of topological defects much lower than theoretical predictions?'.

1.2.4 Summary of problems

Finally as stated previously the aforementioned issues can be resolved by a fine-tuning of the initial conditions of the Cauchy problem of the universe. In this sense they are strictly speaking not inconsistencies in the conventional Big Bang model. However the solution through fine-tuning of the initial conditions can be unsatisfactory, as it limits the predictive power of the theory (i.e. the observed flatness cannot be explained without assuming it in the initial conditions in the first place). A way to resolve the apparent issues without invoking fine-tuning will be the primary topic of the next section.

1.3 Inflation

In the previous section we have quickly reviewed the main features of the traditional Big Bang model, highlighted two of its main weaknesses, the horizon and flatness problems and also mentioned monopole problem. We also emphasized the role the comoving Hubble Radius $(aH)^{-1}$ plays in those problems. Two of those issues arise because in the conventional cosmology the comoving Hubble radius is strictly increasing. This naturally suggests a simple solution, change the behaviour of the comoving Hubble radius, specifically *invert* its behaviour in the early universe, i.e. make it *decrease* sufficiently early in the universe. This is the basic idea behind inflation.

The evolution of the comoving particle horizon is of critical importance in the idea of inflation, so we will look at it in a bit more detail. Recalling its definition as a logarithmic integral of the comoving Hubble radius

$$\chi_P(t) = \int_0^{a(t)} \mathrm{d}\ln a' \frac{1}{a' H(a')}, \tag{1.57}$$

we emphasize a distinction between the comoving Hubble radius $(aH)^{-1}(t)$ and the comoving horizon $\chi_P(t)$: particles separated by distances greater than $\chi_P(t)$ could never have communicated, whereas particles separated by distances greater than $(aH)^{-1}(t)$ cannot communicate now. This distinction is important, as it is possible that $\chi_P(t)$ is much larger now than $(aH)^{-1}(t)$, so particles that cannot communicate now were in causal contact in early universe. This could have happened if the comoving Hubble radius was much larger in the early universe that it is now. This implies two things, first that the largest contribution to $\chi_P(t)$ is from early universe and second that there must have been a period of decreasing comoving Hubble radius. In inflation H is approximately constant, and a grows exponentially, which results in the decrease in the comoving Hubble radius as we want.

A nice side effect of this is that the decrease of the comoving Hubble radius provides a way for quantum generation of cosmological perturbations. Quantum fluctuations are generated on subhorizon scales, but exit the horizon when the Hubble radius becomes smaller than comoving wavelength of the fluctuations. In physical coordinates this corresponds to the superluminal expansion, which stretches the perturbations to acausal distances. These than become classical superhorizon density perturbations which re-enter the horizon later on in the Big Bang evolution and gravitationally collapse to form the large-scale structures.

Armed with this knowledge of the how comoving horizon and the comoving Hubble radius evolve during inflation, we can resolve the apparent problems of the Big Bang model. First we look at the flatness problem, recalling the Friedmann equation 1.52 in non-flat universe

$$|1 - \Omega(a)| = \frac{1}{(aH)^2}, \tag{1.58}$$

we can see that if the comoving Hubble radius decreases, the universe is driven *towards* flatness, and so the solution $\Omega = 1$ becomes an attractor during the inflationary phase. As we can see the flatness problem is quickly and elegantly resolved.

Next we turn our attention to the horizon problem, how can inflation help us establish homogeneity all across the early universe? Quite simply, the large scales only now entering the present universe, were inside the comoving horizon before the inflation. This is possible thanks to a decreasing comoving Hubble radius. Since they were inside the horizon in the early universe, they were in causal contact and so the causal physics established the homogeneity across these scales.

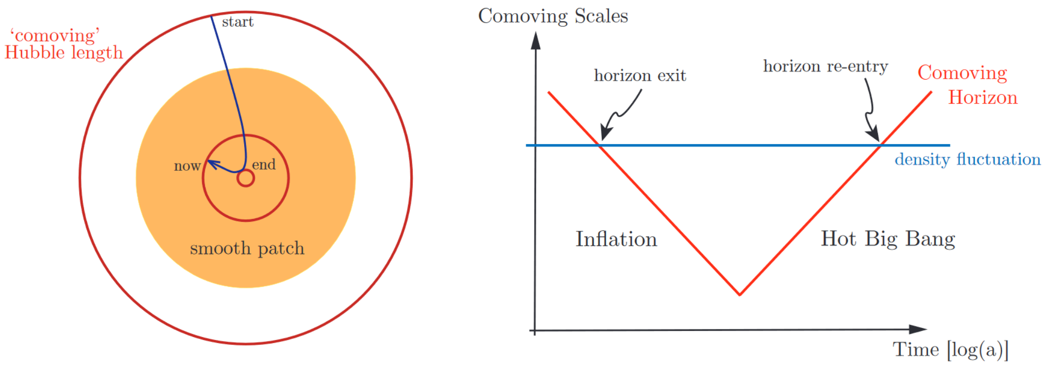

Figure 1.1: *Left:* Evolution of $(aH)^{-1}$, the comoving Hubble radius, in the inflationary universe. As seen, the comoving Hubble radius shrinks during the inflation, and as the inflation ends begins to expand. *Right:* The solution to the horizon problem. According to theory, all scales relevant to the cosmological observations today were larger than Hubble radius until a 10^{-5}. However at sufficiently early times, they were smaller than Hubble radius and thus in causal contact. Also the scales of cosmological interest only came back into the horizon in relatively recent time. From [9].

Finally the monopole problem is also resolved, as due to the accelerated expansion their density is diluted by approximately 10^{-50}, and so are rendered practically unobservable.

Now however a question appears, what conditions are related to inflation? To answer this, we look at the comoving Hubble radius. As we have seen the to resolve the previously mentioned problems of the Big Bang model, we assume that for some period of time the comoving Hubble radius was shrinking. How can we achieve this? From the Friedmann equations we can obtain relation of the acceleration and the pressure of the universe to the comoving Hubble radius

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0 \quad \Leftrightarrow \quad \rho + 3p < 0. \tag{1.59}$$

Thus we obtain three equivalent conditions for the inflation:

1. *Decreasing comoving Hubble radius*
   This is our fundamental definition of what inflation is, as it directly relates to the horizon and flatness problems.

2. *Accelerated expansion*
   If we simply take the derivative in the expression for the decreasing Hubble radius, we obtain

   $$-\frac{\ddot{a}}{(aH)^2} < 0, \tag{1.60}$$

   which immediately implies

   $$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0. \tag{1.61}$$

   This expression gives motivation for the often used definition of inflation as a period of accelerated expansion. The second time derivative of the scale parameter can be related to the Hubble parameter

   $$\frac{\ddot{a}}{a} = H^2\left(1 + \frac{\dot{H}}{H^2}\right). \tag{1.62}$$

   Acceleration then corresponds to the case when

   $$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\mathrm{d}\ln H}{\mathrm{d}N} < 1, \tag{1.63}$$

   where we have defined $\mathrm{d}N = H\mathrm{d}t = \mathrm{d}\ln a$, which measures the number of $e-$folds N of inflationary expansion. This means that ϵ is then a fractional change of the Hubble parameter per $e-$fold, and that during inflation this change is small.

3. *Negative pressure*
   The last obtained condition relates to the question of what kind of matter can source inflationary acceleration. We can infer that accelerated expansion requires

   $$p < -\frac{1}{3}\rho, \tag{1.64}$$

   that is, it requires negative pressure, which violates the strong energy condition. Ways such a situation can arise will be explored in subsequent sections of this work.

Before we end this section and start discussion possible physical mechanism of inflation, we will provide two figures, whose comparison should prove illuminating as to the way inflation resolves the problems. Both images use conformal spacetime diagrams, to emphasise causal relations in the spacetime.

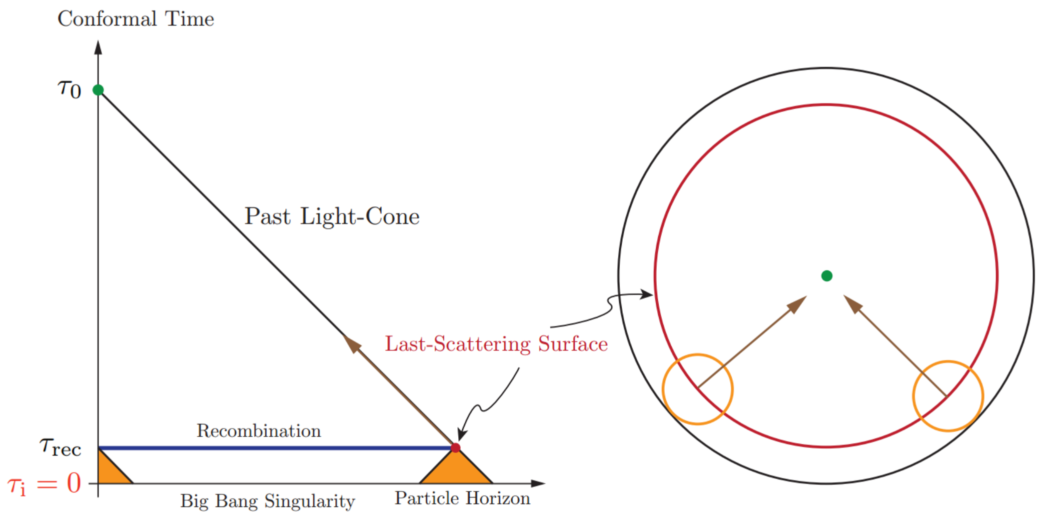

Figure 1.2: Conformal diagram of Standard Big Bang cosmology. We see that light cones from recombination era do not intersect, and so sufficiently distant areas were never in causal contact at recombination. This is due to the fact that there was insufficient amount of time between Big Bang singularity and recombination. It can shown that in this model, the CMB at recombination consists of 10^5 causally disconnected regions. From [9].

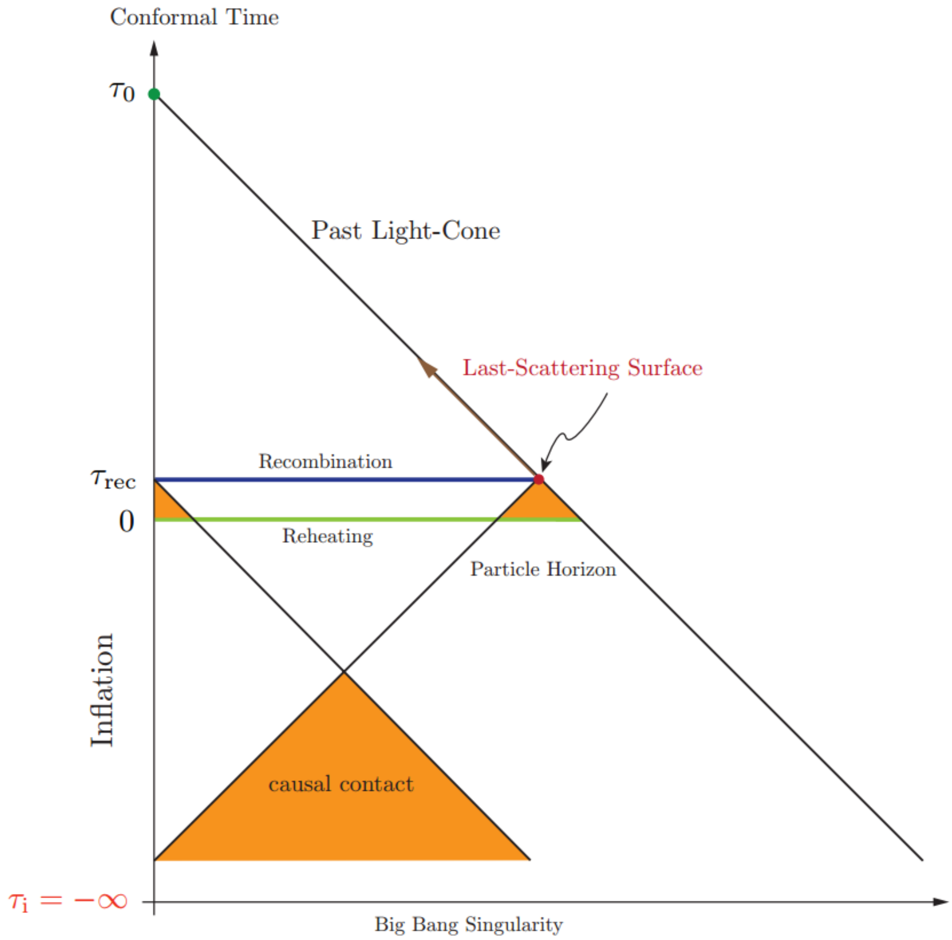

Figure 1.3: Conformal diagram of cosmology with inflation. We can see that inflation allows to extend conformal time to negative values, which in turn allows for intersection of light cones and so resolves horizon problem, if the inflation lasts sufficiently long time (50-60 e−folds). The apparent Big Bang at $\tau = 0$ is not a singularity, but a result of reheating at the end of inflation. From [9].

1.4 Physical mechanisms of inflation

So far we have merely pointed out some issues with the usual Big Bang cosmology and shown that a possible solution would require an accelerated growth of the scale factor $a(t)$. We have also noted that to obtain such an acceleration in Einstein gravity we would require a negative pressure of energy (or equivalently a nearly constant energy density). Under what physical conditions can this arise?

The most simple models of inflation involve a single scalar field ϕ, called the *inflation*, on which we will demonstrate basics of the inflationary physics. One of the reasons for assuming a scalar field (or fields) could be responsible for inflation is the fact that scalar field with non-zero values are generated by spontaneous symmetry breaking. If we assume a minimally coupled scalar field as a basis of inflation, we obtain action

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.65}$$

which is a sum of Einstein-Hilbert action and a an action for the scalar field ϕ with a canonical kinetic term. The potential term for the field ϕ describes its self-interactions. The equation of motion of the scalar field is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0, \tag{1.66}$$

and the stress-energy tensor is

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right). \tag{1.67}$$

If we assume that the metric field is the FLRW one and that the scalar field is homogeneous (i.e. $\phi(\boldsymbol{x}, t) = \phi(t)$), the stress-energy momentum tensor takes the form of the perfect fluid, with the pressure and density being

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.68}$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + v(\phi). \tag{1.69}$$

We immediately see that if the potential energy V dominates over the kinetic term, then the pressure becomes negative. From the equation of state

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}, \tag{1.70}$$

we see that the negative pressure corresponds to the $w_\phi < 0$ and the accelerated expansion to $w_\phi < -1/3$. The dynamics of the homogeneous scalar field and of the FLRW geometry is determined by two equations

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \tag{1.71}$$

$$H^2 - \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = 0, \tag{1.72}$$

from which we can see that for large values of the potential V, the scalar field will experience considerable Hubble friction from the $H\dot{\phi}$ term. After some manipulation, the acceleration equation for a universe dominated by a homogeneous scalar field takes the following form

$$\frac{\ddot{a}}{a} \equiv H^2(1-\epsilon) = -\frac{1}{6}\left(\rho_\phi + 3p_\rho\right), \tag{1.73}$$

where

$$\epsilon \equiv \frac{3}{2}\left(w_\phi + 1\right) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}. \tag{1.74}$$

The ϵ is the *Hubble slow-roll parameter*, which is related to the evolution of the Hubble parameter as $\epsilon = -\dot{H}/H^2$, as stated in the previous section. We remind that the accelerated expansion occurs if the $\epsilon < 1$. The de Sitter limit ($p_\phi \to \rho_\phi$) corresponds to $\epsilon \to 0$, and in this case the potential energy dominates over the kinetic energy, i.e. $V(\phi) \gg \dot{\phi}^2$. The accelerated expansion is only sustained until $\frac{1}{2}\dot{\phi}^2 \approx V(\phi)$, this means that in order for it to be sustained for sufficient period of time the second derivative of ϕ must be small enough

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|. \tag{1.75}$$

Under this condition, the term $\ddot{\phi}$ can be neglected in the Klein-Gordon equation, leading to approximate identity

$$3H\dot{\phi} \approxeq -V_{,\phi}. \tag{1.76}$$

The requirement $|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|$ translates to the requirement of the smallness of the second slow-roll parameter η

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \frac{1}{2\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}N}, \tag{1.77}$$

where $|\eta| < 1$ ensures that fractional change of η per $e-$fold is small.

The conditions on ϵ and $|\eta|$ are known as the called the slow-roll conditions and use of these is known as slow-roll approximation. The slow-roll conditions can be expressed as requirements on the shape of the scalar potential driving the inflation (i.e. the inflationary potential)

$$\epsilon_v \equiv \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \tag{1.78}$$

$$\eta_v \equiv M_{pl}^2\frac{V_{,\phi\phi}}{V}, \tag{1.79}$$

where we have reintroduced the Planck mass to make both ϵ_v and η_v explicitly dimensionless. These new parameters are called *potential slow-roll parameters*, in the slow-roll approximation they are related to the *Hubble slow-roll parameters* ϵ and η as follows

$$\epsilon \approx \epsilon_v, \qquad \eta \approx \eta_v - \epsilon_v. \tag{1.80}$$

The slow-roll conditions can now be stated as

$$\epsilon_v, |\eta_v| \ll 1, \tag{1.81}$$

which can be interpreted as conditions on the shape of inflationary potential. Specifically these conditions require the slope and curvature of the potential to be small. We can also identify $V^{,\phi\phi}$ with the effective mass of inflaton field, then the slow-roll condition on η requires that the effective mass of inflaton field must be small compared to Hubble scale.

Using the slow-roll conditions we obtain for the evolution of the dynamical quantities

$$H^2 \approx \frac{1}{3}V(\phi) \approx const. \tag{1.82}$$

$$\dot{\phi} \approx -\frac{V_{,\phi}}{3H}, \tag{1.83}$$

from which it follows that the resulting spacetime is approximately *de Sitter spacetime*

$$a(t) \approx \mathrm{e}^{Ht}. \tag{1.84}$$

Inflation will end when the slow-roll conditions are violated

$$\epsilon(\phi_{end}) = 1, \qquad \Longrightarrow \qquad \epsilon_v(\phi_{end}) \approx 1. \tag{1.85}$$

We can calculate the number of $e-$folds before the end of inflation as follows

$$N(\phi) \equiv \ln\frac{a_{end}}{a} = \int_t^{t_{end}} H\mathrm{d}t = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}}\mathrm{d}\phi \approx \int_{\phi_{end}}^{\phi} \frac{V}{V_{,\phi}}\mathrm{d}\phi. \tag{1.86}$$

This can also be expressed using the slow-roll parameters as

$$N(\phi) = \int_{\phi_{end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon}} \approx \int_{\phi_{end}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_v}}. \tag{1.87}$$

In order to resolve both the horizon and flatness problem we require the total number N of inflationary $e-$folds to satisfy $N > 40$

$$N_{total} = \ln\frac{a_{end}}{a_{start}} \gtrsim 40, \tag{1.88}$$

with precise value depending on the energy scale of inflation and on the details of the reheating process after inflation. From the observations we know that the fluctuations observed in the CMB are created at $N_{CMB} \approx 50-60$ $e-$folds before the end of the inflation (again, the precise value depends on the details of the inflationary and reheating processes). We thus have an integral constraint on the corresponding field value ϕ_{CMB}

$$\int_{\phi_{end}}^{\phi_{CMB}} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_v}} = N_{CMB} \approx 50-60. \tag{1.89}$$

Although the slow-roll inflation can lead to exponentially large universe, which is almost spatially flat and homogeneous, the energy density is still locked in the potential energy of the inflaton field. This energy must be converted to particles and thermalised to recover a hot Big Bang cosmology at the end of inflationary era. This phase is called *reheating*.

Close to minimum the scalar potential can be approximated by a quadratic function, the inflaton field at the end of inflation then begins oscillating around this minimum, and acts like pressureless matter

$$\frac{d\bar{\rho}_\phi}{dt} + \left(3H + \Gamma_\phi\right)\bar{\rho}_\phi = 0, \tag{1.90}$$

where the Γ_ϕ is a coupling parameter of the scalar field to other fields, heavily dependent on complex model-dependent physical process. The presence of the parameter Γ_ϕ leads to the decay of the inflaton energy, and increase in energy of the standard model degrees of freedom leading to the apparent hot Big Bang.

1.5 Models of inflation

In the previous discussion we did not consider potential microscopic origins of the inflationary field ϕ. It was simply an order parameter parametrizing the time-evolution of the inflationary energy density. There are many question left over, including the nature of the inflaton, shape of its potential and many others. The largest issue we encounter when trying to answer these question is the fact that inflation is believed to have taken place at an energy scale which is fat beyond our particle accelerators (as high as 10^{15} GeV). This means that any description of inflationary era requires a large extrapolation of the known laws of physics, and then comparing the predictions to the known experimental measures we have from the CMB.

Ultimately the definition of the inflationary model comes down to the specification of the inflationary action (relevant fields and their kinetic and potential terms) and its coupling to gravity (or potentially modification of gravity). The discussion in the previous section was in terms of a single-field slow-roll inflation, which is the simplest model with the action

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{1.91}$$

It is the shape of the inflationary potential $V(\phi)$ which determines the dynamics of the inflaton field from the of creation of CMB fluctuation to the end of inflation. The different possible choices of $V(\phi)$ can be classified by determining whether they allow the inflaton field to move small or large distances $\Delta\phi = \phi_{CMB} - \phi_{end}$, with this differences measured in Planck units.

In the case of the small-field inflation the field ϕ moves only over sub-Planckian distance, i.e. $\Delta\phi < M_{pl}$. The observational consequence of this is that the amplitude of the gravitational waves produced during inflation is too small to be detected. Potentials of this class usually arise due to *spontaneous symmetry breaking*, where the field moves from an unstable equilibrium towards a vacuum state (as in Higgs model). Example of this category is Higgs-like potential

$$V(\phi) = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^2\right]^2, \tag{1.92}$$

which we can generalize to the following expansion

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^p \right] + \cdots , \tag{1.93}$$

with dots representing higher-order terms which become relevant close to the end of the inflation and during the reheating process. One historically famous example is the Coleman-Weinberg potential, which arises from the radiatively-induced symmetry breaking in the electroweak and grand unified theories

$$V(\phi) = V_0 \left[\left(\frac{\phi}{\mu} \right)^4 \left(\ln \left(\frac{\phi}{\mu} - \frac{1}{4} \right) \right) + \frac{1}{4} \right]. \tag{1.94}$$

While the original values of the V_0 and μ arising from the $SU(5)$ theory are not compatible with experimental data, specifically with the small amplitude of the inflationary fluctuations, this model remains a useful from phenomenological perspective.

In the second class, the large-field inflation, the field starts at large field values and evolves towards a minimum at the origin $\phi = 0$. For the super-Planckian evolution, the produced gravitational waves should become observable in near future. The prototypical example of this class of inflationary potentials is the so-called *chaotic inflation*, where a single monomial dominates the potential

$$V(\phi) = \lambda_p \phi^p. \tag{1.95}$$

Here the slow-roll parameters are independent of the coupling constant λ_p, and when the field has super-Planckian field values $\left(\phi \gg M_{pl}\right)$ the slow-roll parameters are small. Another possible example, and a very elegant one at that, is the so-called *natural inflation*, with the following potential

$$V(\phi) = V_0 \left[\cos \left(\frac{\phi}{f} + 1 \right) \right]. \tag{1.96}$$

This often arises if the inflaton field is taken to be an axion, particle postulated to help resolve the strong CP problem in QCD. Depending on the parameter f this can be either large-field or a small-field type. However taken as a large-field model, with $2\pi f > M_{pl}$, it is very attractive as a shift symmetry of axions can be exploited to protect the potential from corrective terms even over large field ranges.

So far we have discussed only a single-field slow-roll inflation, however there are many other possibilities, as the inflation is not a single theory but more of a framework for a possible theory of the early universe. We can roughly categorize other possible inflationary actions in the following way:

1. *Multi-field inflation*
   The most obvious way to generalize the inflation is to allow more than one field to be dynamically relevant for the inflationary process, allowing for large freedom in the way models can be

designed. However due to this freedom the theory loses a lot of predictive power as parameters can be quite freely chosen.

2. *Modified gravity*
   Another obvious possibility is to modify the gravitational term of the action at high energies. However the simplest examples of this, the $f(R)$ theories can be transformed to the case of a minimally coupled scalar field with a potential $V(\phi)$ with usual Einstein-Hilbert gravity.

3. *Non-minimal coupling to gravity*
   All actions discussed so far have assumed a minimal coupling of the inflationary field to gravity, i.e. one without a direct coupling of the inflaton field and the metric field. We could easily create a theory with more general couplings, however using field redefinitions such theory could be rewritten as a minimally coupled one, effectively reducing this category to the previous cases, albeit one allowing a new perspective on models.

4. *Non-canonical kinetic term*
   Lastly, we have so far only assumed canonical kinetic term

$$K = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{1.97}$$

   and due to this inflation can only take place if the potential $V(\phi)$ is very flat. However, we can consider a possibility that a high-energy theory can have fields with non-canonical kinetic term

$$K = F(\phi, X), \tag{1.98}$$

   where $F(\phi, X)$ is a function of the inflaton field and its derivatives. In such a theory the inflation can be also driven by the kinetic term and as such occur even if the potential $V(\phi)$ is too steep for the inflation to occur with the canonical potential.

Of course the above list is not exhaustive of possible inflationary models, and it is possible to combine features from the different categories, e.g. an inflationary theory with a non-canonical kinetic term and multiple fields. In the remainder of this section we would like to focus on two models of inflation:

1. *Starobinsky model*, which is an example of an inflationary theory with modified gravity, specifically it is a $f(R)$ theory with the gravity action

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(M_{Pl}^2 R + \frac{R^2}{6M^2} \right), \tag{1.99}$$

   where $M_{Pl} = \sqrt{1/8\pi G}$ is the reduced Planck mass. This is the action Starobinsky originally used. More generally we refer to the inflationary model being Starobinsky (or Starobinsky-like) if the action of gravity has a linear and quadratic terms.

2. *Hybrid inflation*, in which the bulk of potential of inflationary field ϕ (or field multiplet) is generated by displacement from the vacuum of some so-called *waterfall field*. In this case the inflation ends not when the inflation condition is violated (although that can in principle happen too), but when the waterfall field is destabilized as the inflaton field moves through some critical value.

1.5.1 R^2 inflation

We start by looking at the R^2 inflation, also known as Starobinsky model after its discoverer. The action 1.99 includes two terms, the linear is the same as in ordinary General relativity and there is an additional quadratic term. It is the simplest example of the $f(R)$ model of gravity, which generally have an action of the form

$$S_f\left[g_{\mu\nu}\right] = \frac{M_{Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,f(R)\,, \tag{1.100}$$

where f is some function of the Ricci scalar R. The action can be rewritten in the following form

$$S_f\left[g_{\mu\nu},\chi\right] = \frac{M_{Pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\left[f'(\chi)(R-\chi)+f(\chi)\right], \tag{1.101}$$

where we have introduced a real scalar field χ. The two actions are equivalent, as if we choose $\xi = R$ we recover the original action, and the equations of motion of the field ξ lead to $\xi = R$ if $f'' \neq 0$. Next we Weyl transform the metric and redefine the scalar field as follows

$$\tilde{g}_{\mu\nu} \equiv \frac{\partial f}{\partial R}g_{\mu\nu} \tag{1.102}$$

$$\varphi \equiv \sqrt{\frac{3}{2}}M_{pl}\ln\frac{\partial f}{\partial R}, \tag{1.103}$$

resulting in the action

$$S_f\left[\tilde{g}_{\mu\nu},\varphi\right] = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{M_{pl}^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right], \tag{1.104}$$

with the following form of the potential

$$V(\varphi) = \frac{M_{pl}^2}{2}\frac{R\frac{\partial f}{\partial R} - f}{\left(\frac{\partial f}{\partial R}\right)^2} \tag{1.105}$$

The action 1.100 was in the so-called Jordan frame and the procedure performed above allowed us to switch to a new form of action 1.104 which is in the so-called Einstein frame. For more details about the uses and differences of the two frames we refer the reader to the appendix of this work.

In Starobinsky case, the action has the form 1.99 in Jordan frame and in Einstein frame has the form

$$S = \frac{1}{2}\int \mathrm{d}^4x\ \sqrt{-g}\left[M_{Pl}^2 R + \partial_\mu\phi\partial^\mu\phi - \frac{3}{2}M^2 M_{Pl}^4\left(1 - \mathrm{e}^{-\sqrt{2/3}\phi/M_{Pl}}\right)^2\right] \tag{1.106}$$

where ϕ is the inflaton field. The reason that Starobinsky model stands out is that it predicts a low value of scalar-to-tensor ration r

$$r \approx \frac{12}{N^2}, \tag{1.107}$$

where N is the number of $e-$folds during inflation, giving for $N \approx 50 - 60$ a value of around $r \approx 0.0048$, which is well within bounds established by combined analysis of BICEP-Keck-Array-Planck data which is $r \lesssim 0.1$, see Figure 1.4. The mass scale fixed by the anisotropy of the CMB at $M \approx 10^{-5}$.

Advantage of the Starobinsky model is that the inflaton field does not have to be introduced by hand, it naturally emerges from the higher-order gravitation term, which can be generated by a quantum effects even in General Relativity. This also means that inflation is in principle driven fully by curvature, through the R^2 term.

1.5.2 Hybrid inflation

Now we will look at the hybrid inflation in a bit more detail. In hybrid inflation there are two fields, the inflaton field ϕ and waterfall field ψ, with the bulk of the potential of the inflationary field being generated by a displacement of the waterfall field from the vacuum state.

In the traditional example of hybrid inflation, the potential takes the following form

$$V(\phi, \psi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{\lambda'}{4}\left(\phi^2 - \Delta^2\right)^2 + \frac{\lambda}{2}\phi^2\psi^2, \tag{1.108}$$

where λ, λ' are coupling constants and Δ is a constant. If the inflation proceeds along a valley given by $\phi = 0$, the potential reduces to an effective single field potential, and we effectively recover single field inflation. Such case is called *Valley hybrid inflation (VHI)*. However, it has been recently shown, that the hybrid potential can also support inflationary phase along a mixed valley-waterfall trajectories, this then leads to genuine two-field dynamics.

An example of this is model of hybrid inflation where the role of the waterfall field is played by a GUT Higgs field. Specifically the theory to be discussed is the UV extension of Higgs inflation with R^2-term, discussed in [10]. It has been recently noted that combination of Higgs inflation and R^2 inflation acts as the UV extension of Higgs inflation. The Lagrangian has the following form

$$\frac{\mathcal{L}}{\sqrt{-g}} = \left(\frac{M_{Pl}^2}{2} + \xi|\mathcal{H}|^2\right)R + \frac{M_{Pl}^2}{12M^2}R^2 - |D_\mu\mathcal{H}|^2 - \lambda|\mathcal{H}|^4. \tag{1.109}$$

After a switch to Einstein frame the R^2 term provides a new scalar field ϕ, called scalaron. In Einstein frame the Lagrangian becomes

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{M_{Pl}^2}{2}R - \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{2}{2}\mathrm{e}^{-\sqrt{2/3}\phi/M_{Pl}}|D_\mu\mathcal{H}|^2 - U\left(\phi,\mathcal{H}\right), \tag{1.110}$$

with

$$U\left(\phi,\mathcal{H}\right) = \lambda\mathrm{e}^{-2\sqrt{2/3}\phi/M_{Pl}}|\mathcal{H}|^4 + \frac{3}{4}M_{Pl}^2M^2\left[1-\left(1+\frac{2\xi}{M_{Pl}^2}|\mathcal{H}|^2\right)\mathrm{e}^{-\sqrt{2/3}\phi/M_{Pl}}\right]^2. \tag{1.111}$$

If the coupling constants λ is sufficiently large, the inflation dynamics become effectively single field, for some appropriate combination of fields ϕ and $\mathcal{H}$. However, if the mass M is sufficiently small, the dynamics are no longer well approximated by single field, and there is a genuine two-field dynamic.

1.6 Primordial perturbations

Inflation was originally introduced to resolve the flatness and horizon fine-tuning problems, however it was quickly realised that inflation also offers a mechanism to generate the inhomogeneities in initial conditions required for large-scale structure formation. In the following we assume only single-field dynamics for the inflation.

In classical inflationary cosmology driven by a scalar field, the inflaton field is uniform on hypersurfaces of constant time, i.e. $\phi = \phi_0\left(t\right)$. However, quantum fluctuations will inevitably lead to breakdown of spatial homogeneity, leading to the inhomogeneous field

$$\phi\left(t,x^i\right) = \phi_0\left(t\right) + \delta\phi\left(t,x^i\right). \tag{1.112}$$

This then leads us to consider inhomogeneous perturbation of the metric

$$\mathrm{d}s^2 = \left(1+2A\right)\mathrm{d}^2t - 2RB_i\mathrm{d}t\mathrm{d}x^i - R^2\left[\left(1+2C\right)\delta_{ij} + \partial_i\partial_jE + h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j, \tag{1.113}$$

where A, B, C, E are scalar perturbations, h_{ij} is trace-free, transverse (TT) tensor perturbation and R is dimensionful cosmological scale factor. The tensor perturbations are invariant under temporal gauge transformations, however both scalar metric and scalar field perturbations transform. However, one can construct gauge invariant combinations, such as

$$Q = \delta\phi - \frac{\dot{\phi}_0}{H}C, \tag{1.114}$$

which describes scalar field perturbations on spatially-flat hypersurfaces ($C = 0$). This can then be related to the curvature perturbations on hypersurfaces of uniform field ($\delta\phi = 0$) as follows

$$\mathcal{R} = C - \frac{H}{\dot{\phi}_0}\delta\phi = -\frac{H}{\dot{\phi}_0}Q. \tag{1.115}$$

In the case of slow-roll inflation $\rho \approxeq \rho(\phi)$ this coincides with curvature perturbation on the uniform-density hypersurfaces

$$\zeta = C - \frac{H}{\dot{\rho}_0}\delta\rho, \tag{1.116}$$

which then implies that the evolution of the inflation field couples scalar metric and scalar field perturbations.

The tensor metric perturbation are gauge-invariant and at first order decoupled from scalar perturbations. Because of this, they are free excitations and describe gravitational waves. Tensor mode with given wavevector $\boldsymbol{k}$ can be decomposed into two linearly-independent TT polarisation states,

$$h_{ij}(\boldsymbol{k}) = h_{\boldsymbol{k}} q_{ij} + \tilde{h}_{\boldsymbol{k}} \tilde{q}_{ij}. \tag{1.117}$$

We can plug this into linearised Einstein equations, which then yield evolution equation for the amplitude $h_{\boldsymbol{k}}$ (similarly for the amplitude $\tilde{h}_{\boldsymbol{k}}$)

$$\ddot{h}_{\boldsymbol{k}} + 3H\dot{h}_{\boldsymbol{k}} + \frac{k^2}{R^2}h_{\boldsymbol{k}} = 0, \tag{1.118}$$

which is the same evolution equation as for massless field in FLRW spacetime. Rewriting the above equation in terms of conformal time η and introducing the conformally rescaled field

$$u_{\boldsymbol{k}} = \frac{M_{Pl} R h_{\boldsymbol{k}}}{\sqrt{32\pi}} \tag{1.119}$$

we obtain

$$u''_{\boldsymbol{k}} + \left(k^2 - \frac{R''}{R}\right)u_{\boldsymbol{k}} = 0, \tag{1.120}$$

where $'$ denotes derivative with respect to conformal time. However, this is the equation of canonical scalar field in Minkowski spacetime with time-dependant mass.

During slow-roll we can approximate the time-dependent term

$$\frac{R''}{R} \approxeq (2 - \epsilon)R^2H^2, \tag{1.121}$$

this allows us to quantise the linearised metric fluctuations on sub-Hubble scales, $H^2 \ll k^2/R^2$, as the background expansion can be neglected.

We know that during inflationary expansion the comoving Hubble length decreases with time, this then means that all modes start inside the Hubble horizon and we can take initial field fluctuations to be in vacuum state on small scales or at early times

$$\langle u_{\boldsymbol{k}_1} u_{\boldsymbol{k}_2} \rangle = \frac{\mathrm{i}}{2} (2\pi)^3 \, \delta^{(3)} \left(\boldsymbol{k}_1 + \boldsymbol{k}_2 \right). \tag{1.122}$$

We can rewrite the above equation in terms of the amplitude of TT metric perturbations, yielding

$$\langle h_{\boldsymbol{k}_1} h_{\boldsymbol{k}_2} \rangle = \frac{1}{2} \frac{\mathcal{P}_T \left(k_1 \right)}{4\pi k_1^3} (2\pi)^3 \, \delta^{(3)} \left(\boldsymbol{k}_1 + \boldsymbol{k}_2 \right), \tag{1.123}$$

where the factor $1/2$ is due to the fact that two polarisation states contribute to the total tensor power spectrum

$$\mathcal{P}_T \left(k \right) = \frac{64\pi}{M_{Pl}^2} \left(\frac{k}{2\pi R} \right)^2. \tag{1.124}$$

Looking now at super-Hubble scales, $H^2 \gg k^2/R^2$, we obtain a growing mode solution $u_{\boldsymbol{k}} \propto R$ for Eq. (1.120). Returning to tensor amplitude $h_{\boldsymbol{k}}$, we find for this solution that $h_{\boldsymbol{k}} \to const.$, this implies that tensor modes are 'frozen-in' on super-Hubble scales, during and after inflation. We can now connect the initial vacuum fluctuations on sub-Hubble scales to the late-time power spectrum for tensor modes at Hubble exit during inflation, this gives us

$$\mathcal{P}_T \left(k \right) \simeq \frac{64\pi}{M_{Pl}^2} \left(\frac{H_*}{2\pi} \right)^2, \tag{1.125}$$

where $k = R_* H_*$. If we take de Sitter limit, $\epsilon \to 0$, the Hubble rate becomes time-independent and the tensor spectrum becomes scale-invariant on super-Hubble scales. However, slow-roll inflation leads to weak time dependence of H_* and this in turn leads to scale-dependency of spectrum on large scales, with *spectral tilt* given by

$$n_t = \frac{\mathrm{d} \ln \mathcal{P}_T}{\mathrm{d} \ln k} \approxeq -2\epsilon_*. \tag{1.126}$$

Now we would like to obtain a similar power spectrum for scalar density fluctuations, so that we can obtain tensor-to-scalar ratio. We start by observing that inflaton field fluctuation on spatially-flat hypersurfaces are coupled to scalar metric perturbations, however this can be eliminated with the use of Einstein constraints equations, after this elimination we obtain the following evolution equation

$$\ddot{Q}_{\boldsymbol{k}} + 3H\dot{Q}_{\boldsymbol{k}} + \left[\frac{k^2}{R^2} + V'' - \frac{8\pi}{3M_{Pl}^2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{R^3 \dot{\phi}^2}{H} \right) \right] Q_{\boldsymbol{k}}. \tag{1.127}$$

The effect of the fluctuations of gravity to the first order are represented by the terms which are proportional to M_{Pl}^{-2}, this means that in the constant background field limit this terms vanishes. As a result, in the slow-roll approximation this term is suppressed, however the effective mass term V'' is of the same order and so both must be included to correctly model deviations from exact de Sitter symmetry.

If we define a new field $v_{\boldsymbol{k}} = RQ_{\boldsymbol{k}}$, we can put the Eq. (1.127) into the canonical form

$$v_{\boldsymbol{k}}'' + \left(k^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}} = 0, \tag{1.128}$$

with z defined as

$$z = \frac{R\dot{phi}}{H} \qquad \frac{z''}{z} \cong (2 + 5\epsilon - 3\eta) R^2 H^2. \tag{1.129}$$

In the slow-roll approximation, the above approximate equality holds at leading order.

Now we proceed as in the case of tensor metric perturbations, we quantise the linearised field fluctuations $v_{\boldsymbol{k}}$ on sub-Hubble scales, $H^2 \ll k^2/R^2$, so that we can neglect the background expansion. We then impose

$$\langle v_{\boldsymbol{k}_1} v_{\boldsymbol{k}_2} \rangle = \frac{\mathrm{i}}{2} \delta^{(3)} (\boldsymbol{k}_1 + \boldsymbol{k}_2). \tag{1.130}$$

Again we rewrite this in terms of field perturbations, this corresponds to

$$\langle Q_{\boldsymbol{k}_1} Q_{\boldsymbol{k}_2} \rangle = \frac{\mathcal{P}_Q (k_1)}{4\pi k_1^3} (2\pi)^3 \delta^{(3)} (\boldsymbol{k}_1 + \boldsymbol{k}_2), \tag{1.131}$$

with the power spectrum of the field fluctuations on sub-Hubble scales being simply

$$\mathcal{P}_Q (k) = \left(\frac{k}{2\pi R}\right)^2, \tag{1.132}$$

If we now plug in the expression $k = R_* H_*$ at Hubble exit we obtain the classic result for vacuum fluctuations for massless field in de Sitter

$$\mathcal{P}_Q (k) \cong \left(\frac{H}{2\pi}\right)^2_*. \tag{1.133}$$

However, in practice there are corrections due to slow-roll, specifically due to small mass η and field evolution ϵ.

While the slow-roll corrections to field fluctuations are small on sub-Hubble scales, they can become significant as the field evolves on super-Hubble scales. This means it is more practical to instead work with curvature perturbation ζ defined in Eq. (1.116), which, for adiabatic density perturbations, remains constant on super-Hubble scales both during and after inflation. The curvature power spectrum is defines as

$$\mathcal{P}_\zeta (k) = \left[\left(\frac{H}{\dot{\phi}}\right)^2 \mathcal{P}_Q (k)\right]_* \cong \frac{4\pi}{M_{Pl}^2} \left[\frac{1}{\epsilon}\left(\frac{H}{2\pi}\right)^2\right]_*. \tag{1.134}$$

We note that during slow-roll inflation the scalar amplitude is boosted by the $1/\epsilon_*$ factor, this means that small scalar field fluctuations can lead to relatively large curvature perturbations on the hypersurfaces defined with respect to density, provided that the potential energy only weakly depends on the scalar field. In de Sitter limit, $\epsilon \to 0$, the potential energy becomes independent of the scalar field at first order and curvature perturbations on constant-density hypersurfaces becomes ill defined at first order.

Now we can define tensor-to-scalar ratio, be comparing the above expression with the primordial gravitational wave power spectrum

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \cong 16\epsilon_*. \tag{1.135}$$

Since both tensor tilt (1.126) and the tensor-to-scalar ratio are determined by the first slow-roll parameter at Hubble exit ϵ_*, we obtain an important consistency test for single field inflation

$$n_t = -\frac{r}{8}. \tag{1.136}$$

For small r this can be hard to verify, which would make tensor tilt n_t difficult to measure, however it can help rule out single-field slow-roll inflationary models if either r or n_t are too large.

Since slow-roll corrections lead to slow time-dependence of both H_* and ϵ_*, the scalar power spectrum gains a weak scale-dependence. It has proven easier to measure this *scalar tilt* , owing to relative largeness of scalar power spectrum. The scalar tilt is conventionally defined as $n_s - 1$

$$n_s - 1 \equiv \frac{\mathrm{d}\ln \mathcal{P}_\zeta}{\mathrm{d}\ln k} \cong -6\epsilon_* + 2\eta_*. \tag{1.137}$$

We can also define a running of this tilt at second-order in slow-roll parameters

$$\frac{\mathrm{d}n_s}{\mathrm{d}\ln k} \cong -8\epsilon_* \left(3\epsilon_* - 2\eta_*\right) - 2\xi_*^2, \tag{1.138}$$

where we introduce new second-order slow-roll parameter

$$\xi^2 = \frac{M_{Pl}^4}{64\pi^2} \frac{V_{,\phi} V_{,\phi\phi\phi}}{V^2}. \tag{1.139}$$

We can now ask, what are the bounds given by observations on the various quantities defined in this section, this will be subject of the next section.

1.7 Observational bounds on inflation

The observational fact of scale-dependence of the power spectrum means it is necessary to specify the Hubble-exit time $H_* = k/a_*$ and hence the comoving scale at which quantities are constrained. This can be quantified in terms of number of e-folds from the end of inflation

$$N_*(k) \cong 67 - \ln\left(\frac{k}{a_0 H_0}\right) + \frac{1}{4}\ln\left(\frac{V_*^2}{M_{Pl}^4 \rho_{end}}\right) + \frac{1}{12}\ln\left(\frac{\rho_{rh}}{\rho_{end}}\right) - \frac{1}{12}\ln g_*, \tag{1.140}$$

where $1/(a_0 H_0)$ is the present comoving Hubble length. The dependency on different models of reheating and on densities, expressed through ρ_{rh} term, leads to a range of possibilities for N_* for a fixed physical scale. This means that given inflation model will have a range of observable predictions.

The data on temperature and polarisation from Planck 2018 are consistent with a featureless smooth power spectrum over a range of comoving wavenumber, $0.005\text{Mpc}^{-1} \leq k \leq 0.2\text{Mpc}^{-1}$. The data measure the spectral index of scalar perturbations to be

$$n_s = 0.9649 \pm 0.0042, \tag{1.141}$$

at 68% confidence level, which corresponds to deviation from scale-invariance in the excess of the 8σ level. The 2018 Planck collaboration also finds no evidence of the scale dependence of the spectral index, either as running or as running of the running. This is in contrast with Planck 2015 data, which severely constrained the running but have not yet ruled it out. An analysis of the BICEP2/Keck Array, Planck and other data puts an upper bound on the tensor-to-scalar ratio at $k = 0.05\text{Mpc}^{-1}$

$$r < 0.064, \tag{1.142}$$

at 95% confidence level.

The observational bound mentioned above can be converted into bounds on the slow-roll parameters and therefore on the potential during slow-roll inflation. If we set the higher than second-order slow-roll parameters to zero, we obtain from the Planck collaboration the following bounds on the potential slow-roll parameters

$$\epsilon_V < 0.0097 \tag{1.143}$$

$$\eta_V = -0.010^{+0.007}_{-0.011} \tag{1.144}$$

$$\xi^2_V = 0.0035^{+0.0078}_{-0.0072}. \tag{1.145}$$

The figure (1.4), compares the observational CMB constraints on the spectral tilt n_s and the tensor-to-scalar ratio r.

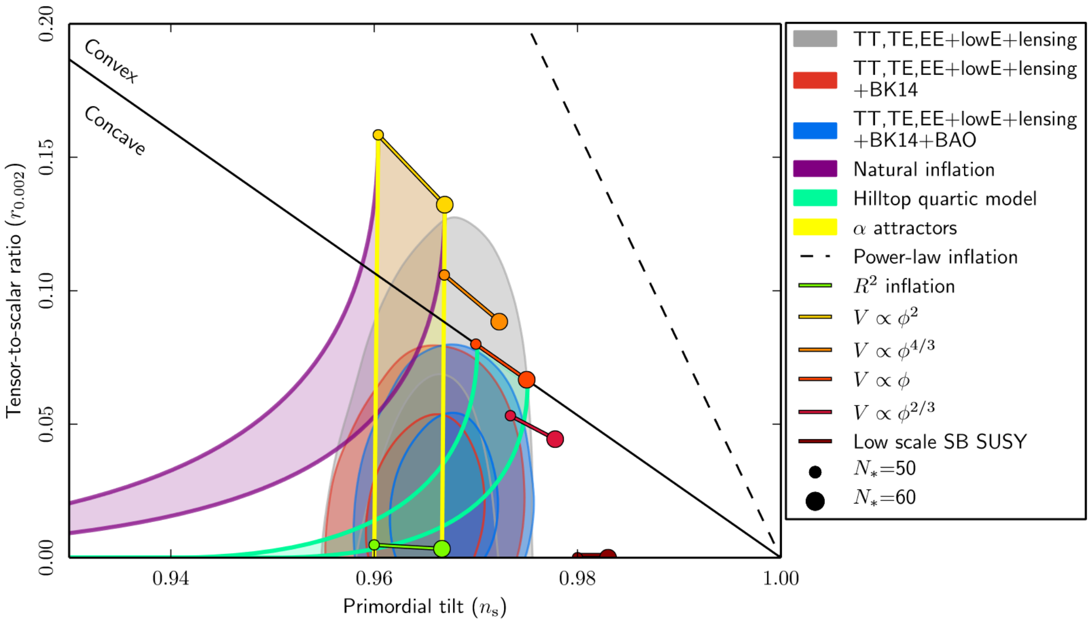

Figure 1.4: Marginalized joint 68% and 95% CL regions for n_s and r at $k = 0.002\text{Mpc}^{-1}$ from *Planck* alone and in combination with BK14 or BK14 plus BAO data, compared to the theoretical predictions of some inflationary models. Note that the marginalized joint regions assume no running in spectral index. From [11].

Generally the data from Planck rules out power-law inflation and hybrid models with $n_s > 1$. The most recent data from Planck 2018 in combination with other data sources such as BK14 strongly disfavour monomial models $V(\phi) \propto \phi^p$ with $p \geq 2$ and low-scale SUSY models. The models which provide good fit to Planck 2018 and BK14 data are those with exponential tails, such as R^2 Starobinsky model or D-brane inflation. More generally we can say that the concave potentials are strongly favoured compared to the convex ones, and de Sitter-like quasi-exponential expansion provides a better fit to data. More details about recent constraints on inflation please refer to [11].

Chapter 2

Weyl gravity

This chapter will be focused on the formulation of one-loop effective potential of Weyl theory of gravity in an arbitrary classical background. First we will quickly review properties and main features of Weyl' gravity, then we will move on to the expansion of its action to the second order in variations around some background metric. The chapter is ended with calculation of one-loop effective potential for a De Sitter background metric, using the method of $\zeta-$function regularization.

2.1 Weyl theory of gravity

Weyl theory of gravity is quite old alternative theory of gravity, first proposed by Hermann Weyl in 1918. Unlike general relativity, Weyl gravity has additional symmetries it is invariant under *Weyl transformation*

$$g_{\mu\nu}(x) \to \mathrm{e}^{2\omega(x)} g_{\mu\nu}(x) \,. \tag{2.1}$$

It can be also demonstrated to be invariant under the action of the conformal group, leading to the alternative name, conformal gravity.

The action principle of conformal gravity is

$$S = -\frac{1}{8\alpha_c^2} \int \mathrm{d}^4x \sqrt{-g}\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}, \tag{2.2}$$

where α_c is a dimensionless coupling constant and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, defined as the totally traceless part of Riemann tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{6} R g_{\mu[\rho} g_{\sigma]\nu}. \tag{2.3}$$

We also note that we are using metric signature $(-,+,+,+)$.

The action 2.2 is the unique conformal action in four dimensional spacetime. The action can be rewritten using only Riemann tensor and its contractions as

$$S = -\frac{1}{8\alpha_c^2} \int d^4x \sqrt{-g} \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \right], \tag{2.4}$$

this is a result of definition of Weyl tensor in terms of Riemann tensor and its contractions. Even further simplification can be achived with the use of Gauss-Bonnet invariant

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{2.5}$$

which contributes only a total divergnce. Substracting this term we obtain final form of the action

$$S = -\frac{1}{4\alpha_c^2} \int d^4x \sqrt{-g} \left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \right). \tag{2.6}$$

We need to stress that the equation (2.6) holds only if the dimension is four, otherwise the Gauss-Bonnet invariant is not a total divergence and so does not contribute only as a surface term. Because of this, in the following we will only use regularization and renormalization schemes in fixed dimension.

The reason we are interested in this theory of gravity is the idea, that at sufficiently high energies, physics should approach conformal symmetry. This in turn implies that at the beginning stages of the universe, confomal symmetry would be appropriate. Of course, in its current form the universe does not posses conformal symmetry, so it has to be broken in some fashion, such as by spontaneous symmetry breaking.

2.1.1 Hubbard-Stratonovich transform of the generating functional

Hubbard-Stratonovich transform is an integral transform based on the Gaussian integrals. It can be used to transform quadratic exponents to linear ones, at a cost of introducing a new variable which then has to be integrated out. More details about this transformation can be found in Appendix A.

The reason we will apply Hubbard-Stratonovich to our action is to linearise part of the R^2 term, and in doing so we will obtain a term in our action which looks identical to the *Starobinsky action* which has the following form

$$A_{Starobinsky} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \xi^2 R^2 \right), \tag{2.7}$$

where $\kappa = \sqrt{8\pi G} = M_{Pl}^{-1}$. The Starobinsky action is one of the possible descriptions of the process of inflation. It assumes that Einstein action is modified by additional quadratic term R^2, this term would dominate when curvature is large (i.e. inflation era), but for space which is close to flat (as current observations suggest about current state of our universe) the action would be dominated by the linear term. Because of this, it is quite an attractive feature of the conformal gravity that it can make contact with Starobinsky theory. In all the following equations we silently assume that the action is in the exponent, otherwise the transformations would not make sense.

Our starting point will be the Weyl action with Gauss-Bonnet term subtracted

$$A = -\frac{1}{4\alpha_c^2} \int d^4x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) = A_{R^2} + A_{R_{\mu\nu} R^{\mu\nu}}, \tag{2.8}$$

where

$$A_R = \frac{1}{12\alpha_c^2} \int d^4x \sqrt{-g} R^2 \quad \text{and} \quad A_{R_{\mu\nu} R^{\mu\nu}} = -\frac{1}{4\alpha_c^2} \int d^4x \sqrt{-g} R_{\mu\nu} R^{\mu\nu}. \tag{2.9}$$

Applying the Hubbard-Stratonovich transform on the $\exp[-A_{R^2}]$ we obtain

$$\exp[-A_{R^2}] = \exp\left[-\frac{1}{12\alpha_c^2} \int d^4x \sqrt{-g} R^2 \right] = \int \mathcal{D}\lambda \exp\left[\int d^4x \sqrt{-g} \left(3\alpha_c^2 \lambda^2 - R\lambda \right) \right], \tag{2.10}$$

where λ is an auxiliary scalar field to be integrated out. However we wish for our action to contain part resembling Starobinsky action, so we need part of the R^2 term to survive. To do this we first decompose A_{R^2} into two parts using identity $1 = \cosh^2\theta - \sinh^2\theta$

$$A = -\frac{1}{4\alpha_c^2} \int d^4x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - C^2 \frac{1}{3} R^2 \right) - \frac{S^2}{12\alpha_c^2} \int d^4x \sqrt{-g} R^2, \tag{2.11}$$

where θ is an arbitrary mixing angle. We have also denoted $\sinh\theta = S$ and $\cosh\theta = C$ to shorten the equations since θ is a constant angle chosen later using a *principle of minimal sensitivity*. As a next step we apply the Hubbard-Stratonovich transform only to the part proportional to S^2

$$A\left[g_{\mu\nu}, \lambda \right] = \int d^4x \sqrt{-g} \left[-\frac{1}{4\alpha_c^2} R_{\mu\nu} R^{\mu\nu} + \frac{C^2}{12\alpha_c^2} R^2 + \frac{3\alpha_c^2}{4S^2} \lambda^2 - \frac{1}{2} R\lambda \right] \tag{2.12}$$

where we also rescaled the auxiliary scalar field $\lambda \to -\lambda/2S^2$, which lacks a bare kinetic term. The kinetic term for the scalar field λ is dynamically generated by quantum fluctuations of the metric tensor $g_{\mu\nu}$. Finally we again rescale the scalar field $\lambda \to \lambda/\kappa^2$ to ease the comparison with the Starobinsky action, this final form then reads

$$A = \int d^4x \sqrt{-g} \left[-\frac{1}{4\alpha_c^2} R_{\mu\nu} R^{\mu\nu} + \frac{C^2}{12\alpha_c^2} R^2 + \frac{3\alpha_c^2}{4S^2\kappa^4} \lambda^2 - \frac{1}{2\kappa^2} R\lambda \right]. \tag{2.13}$$

We note that the scalar field λ can be separated into a background field $\langle\lambda\rangle$ which corresponds to the VEV of the field, and into fluctuations $\delta\lambda$ around this background. These fluctuations have to be considered to keep the theory fully equivalent to the original action (2.8), however neglecting the fluctuations can still provide a useful approximation. Since we wish to recover a long-range behaviour of Starobinsky's model, we need to show that in the there exists a set of parameters in the model's parameter space such that $\langle\lambda\rangle = 1$. This is the topic of the next section and will be done through effective potential for the field λ due to the one-loop corrections of the metric.

2.2 Inflation in Weyl gravity

As we have seen we can, through a use of Hubbard-Stratonovich transform, induce a term in action proportional to R at the cost of introducing a new scalar field λ. This field can be separated into its VEV $\langle\lambda\rangle = \bar{\lambda}$ and fluctuations around it $\delta\lambda$. Neglecting the fluctuations of λ field we can write the action

$$A = \int d^4x \sqrt{-g}\left[-\frac{1}{4\alpha_c^2}R_{\mu\nu}R^{\mu\nu} + \frac{C^2}{12\alpha_c^2}R^2 + \frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2 - \frac{1}{2\kappa^2}R\bar{\lambda}\right], \tag{2.14}$$

where $C = \cosh\theta$ and $S = \sinh\theta$ as previously. The only difference between (2.14) and (2.13) is that the action (2.14) is for the VEV only.

From the form of action we see that phenomenologically correct long-range behaviour is secured if $\bar{\lambda} = 1$. To see if this is possible we can calculate one-loop effective potential for the $\bar{\lambda}$ induced by metric fluctuations.

2.2.1 Effective potential in Minkowski background

In Minkowski background this results in the following form of effective potential [2]

$$V_{eff} = -\frac{9\alpha_c^4\bar{\lambda}^2}{16\pi^2\kappa^4\left(4S^2+1\right)^2}\left[\ln\frac{6\alpha_c^2\bar{\lambda}}{\left(1+4S^2\right)\kappa^2\mu^2} - \frac{3}{2}\right] + \frac{3\alpha_c^4\bar{\lambda}^2}{8\pi^2\kappa^4}\left[\ln\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2} - \frac{3}{2}\right] - \frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2, \tag{2.15}$$

with μ the subtraction point. It also admits trivial solution of $\bar{\lambda} = 0$

The effective potential (2.15) was obtained through dimensional regularization, which could be problematic as some of the transformation (such as subtraction of Gauss-Bonnet invariant) can be done only in four dimensions. However, the same form of the potential was also obtained for fixed dimension through zeta-function regularization in the authors research project, so we can state this seems to be correct form of the effective potential.

We determine the value of λ corresponding to the VEV by vanishing of $\frac{\partial V_{eff}}{\partial\bar{\lambda}}$. This then yields minimal V_{eff} for

$$\bar{\lambda}(S) = \frac{\kappa^2\mu^2}{2\alpha_c^2}\exp\left[\frac{3\alpha_c^2S^2\ln\left(\frac{3}{4S^2+1}\right) + 4\pi^2\left(4S^2+1\right)^2}{\alpha_c^2S^2\left(32S^4+16S^2-1\right)}\right]. \tag{2.16}$$

With this solution, it holds for $S^2 > \left(\sqrt{6}-2\right)/8$ that $V_{eff} < 0$ independent of the values α_c and κ. In light of this solution we see that the trivial solution represents a local maximum and so is unstable, for the given range of S^2.

The value $\bar{\lambda}$ in (2.16) is still dependent on the mixing angle θ, however the full theory is independent on this mixing angle. This dependence is due to the fact that we truncated the perturbation series after a finite loop order (specifically one). To reduce this effect we use the *principle of minimal sensitivity*. This is a procedure known from renormalization-group calculus, used in $\delta-$perturbation expansion.

As a result, the one-loop level value of θ is determined by the vanishing of $\frac{\mathrm{d}V_{eff}}{\mathrm{d}(S^2)}$,

$$\frac{\mathrm{d}V_{eff}}{\mathrm{d}\left(S^2\right)} = \frac{\partial\bar{\lambda}}{\partial\left(S^2\right)}\frac{\partial V_{eff}}{\partial\bar{\lambda}} + \frac{\partial V_{eff}}{\partial\left(S^2\right)} = 0. \tag{2.17}$$

As we are interested in the potential at minimum for $\bar{\lambda}$, the derivative with respect to this field vanishes, so the condition for θ becomes

$$\frac{\partial V_{eff}}{\partial\left(S^2\right)} = 0. \tag{2.18}$$

Writing this out, we obtain the following equation

$$\frac{128S^6 + 96S^4 + 36S^2 - 1}{S^4\left(32S^4 + 16S^2 - 1\right)} = \frac{12\alpha_c^2 \ln\left(\frac{4S^2+1}{3}\right)}{\pi^2\left(32S^4 + 16S^2 - 1\right)}. \tag{2.19}$$

This equation admits two branches of real solutions

$$S^2 = 0.0259237 - 0.0000197\alpha_c^2 + O\left(\alpha_c^4\right), S^2 \sim \frac{\xi}{\kappa} \sim 10^5.$$

The first solution does not give table $\bar{\lambda}$ as the effective potential becomes positive. The other solution is maximal allowed value within the range of validity of one-loop approximation, and leads to stable solution for $\bar{\lambda}$. We use to obtain final solution for one-loop VEV $\bar{\lambda}$

$$\bar{\lambda} = \frac{\kappa^2\mu^2}{2\alpha_c^2}\mathrm{e}^{1+2\pi^2/\alpha_c^2 S^2} \sim \frac{\kappa^2\mu^2}{2\alpha_c^2}\mathrm{e}^{1+2\pi^2\kappa^2/\alpha_c^2\xi^2} \tag{2.20}$$

to order $O\left(1/S^4\right)$. From the form of the solution, we see that for any value of dimensionless coupling α_c we can choose the renormalization mass scale μ such that $\bar{\lambda} = 1$, which ensures phenomenologically correct long-range behaviour.

We emphasize that the above analysis is done in flat background and from phenomenological standpoint it would be more useful to perform this analysis in for example De Sitter background, which is attempted in other sections of this work.

2.2.2 Conformal action in broken phase

We have seen that at least in flat background (and possibly backgrounds which only weakly deviate from flatness) we can select parameters of the theory such that the R term has correct sign, securing correct long-range behaviour. If we also additionally assume that in the phase of broken symmetry the cosmologically relevant metric is FLRW metric, then we obtain additional condition on curvature tensors due to conformal flatness of FLRW metric

$$\int \mathrm{d}^4x\,\sqrt{-g}\,3R_{\mu\nu}R^{\mu\nu} = \int \mathrm{d}^4x\,\sqrt{-g}\,R^2, \tag{2.21}$$

which holds modulo topological terms. If we combine this condition with the expanded action (2.14) and with solution for VEV (2.20) we obtain low-energy limit for conformal action in the broken phase

$$A_{broken} = -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\left(R - \xi^2 R^2 - 2\Lambda'\right), \tag{2.22}$$

where

$$\xi^2 = \frac{\kappa^2 S^2}{6\alpha_c^2}, \qquad \Lambda' = \frac{3\alpha_c^2}{4S^2\kappa^2}. \tag{2.23}$$

This action nicely corresponds to the Starobinsky action (1.99), however there is also induced cosmological constant of purely geometric origin. It is also interesting that this cosmological constant has the opposite sign compared to the usual matter induced cosmological constant.

In the unbroken phase the scalar field λ cannot be in the on-shell spectrum, due to local conformal symmetry. However, in the broken phase we expect it to appear via radiatively induced gradient term of the field λ. To the lowest order, the induced kinetic term takes the form

$$\frac{1}{2\kappa^2\lambda}\partial_\mu\lambda\partial^\mu\lambda, \tag{2.24}$$

which is positive, so in the broken phase we expect λ to become a propagating scalar mode. The detailed calculation can be found in [2]. We note that in non-flat background the only modification to the inflationary action in the broken phase would appear in the kinetic term of the field λ, and so if the background was for example maximally symmetric space with $|\Lambda| \ll 1$, the kinetic term from flat background would provide good approximation.

It is also important to note that in the broken phase, the kinetic energy of the scalar field is positive and the effective potential is bounded from below. This then implies that broken one-loop conformal gravity does not have ghost states.

2.2.3 Inflation in broken phase

In the previous sections we have seen that in the broken phase conformal gravity in FLRW background reduces to the following action

$$A_{broken,\lambda} = -\frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left[\lambda R - \xi^2 R^2 - \frac{1}{\lambda}\partial_\mu\lambda\partial^\mu\lambda - 2\Lambda'\lambda^2\right], \tag{2.25}$$

where we have also included dynamics of the field λ. The action includes Starobinsky term $\xi^2 R^2$ with additional scalar degree of freedom. We would also like to point out that the scalar scalar field is coupled in a similar way to Brans-Dicke gravity. We use Hubbard-Stratonovich transform to introduce new field ϕ and rid ourselves of the R^2 term, obtaining

$$A_{broken,J} = -\frac{1}{\kappa^2}\int d^4x\,\sqrt{-g}\left[\frac{\lambda + 2\xi\phi}{2}R + \frac{\phi^2}{2} - \frac{1}{2\lambda}\partial_\mu\lambda\partial^\mu\lambda - \Lambda'\lambda^2\right]. \tag{2.26}$$

Both scalar field are non-minimally coupled to gravity, so the action is in Jordan frame. Switching to Einstein frame we obtain

$$A_{broken,E} = -\frac{1}{\kappa^2}\int \sqrt{-g}\left[\frac{R}{2} - \frac{3\xi^2\left(\partial_\mu\phi\right)^2}{(\lambda+2\xi\phi)^2} - \frac{3\xi\left(\partial_\mu\lambda\right)(\partial^\mu\phi)}{(\lambda+2\xi\phi)^2} - \frac{\left(\partial_\mu\lambda\right)^2}{2\bar{\lambda}\,(\lambda+2\xi\phi)} \right.$$
$$\left. - \frac{3\left(\partial_\mu\lambda\right)^2}{4\,(\lambda+2\xi\phi)^2} + \frac{\phi^2}{2\,(\lambda+2\xi\phi)^2} - \frac{\Lambda'\lambda^2}{(\lambda+2\xi\phi)^2}\right., \tag{2.27}$$

where we have rescaled the metric

$$g_{\mu\nu} \to (\lambda+2\xi\phi)^{-1}\, g_{\mu\nu}. \tag{2.28}$$

This rescaling is valid only if $\lambda + 2\xi\phi > 0$, otherwise we would not preserved sign of the metric. The action (2.27) can be diagonalised by passing to field $(\lambda,\, \psi)$, with the field ψ obtained from redefinition

$$\phi = \frac{\exp\left(\sqrt{2/3}|\psi| - \lambda\right)}{2\xi}. \tag{2.29}$$

The diagonalized action then reads

$$A_{\lambda,\psi,E} = -\frac{1}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\left(\partial_\mu\psi\right)^2 - \mathrm{e}^{-\sqrt{2/3}|\psi|}\frac{\left(\partial_\mu\lambda\right)^2}{2\bar{\lambda}} + \frac{1}{8\xi^2}\left(1-2\lambda\mathrm{e}^{-\sqrt{2/3}|\psi|}\right)\right]. \tag{2.30}$$

We note that for large values of dimensionless field ψ (i.e. ψ/κ is large compared to Planck scale), the coefficient at the gradient term for the field λ is very small, which leads to strong oscillations of the λ field. If the symmetry breaking happened before inflation then, after a short period of severe oscillations, the oscillations are dampened at $\psi/\kappa \lesssim 10m_{Pl}$ and λ field settles at its effective potential minimum, $\bar{\lambda} = 1$.

From the action (2.30) we can read off inflationary potential

$$V(\lambda,\psi) = \frac{1}{8\xi^2}\left(1-2\lambda\mathrm{e}^{-\sqrt{2/3}|\psi|}\right). \tag{2.31}$$

It is clear that $V(\lambda,\psi) \leq 1/8\xi^2 \ll M_{Pl}$, which is a necesarry condition for inflation. We also note that for the field values $\psi/\kappa \sim 10M_{Pl}$ the potential is sufficently flat to produce slow-roll inflation dynamics, with the field ψ acting as the inflaton. The λ field is static during inflation and plays the role of spectator field. From the shape of inflationary potential we can also conclude that this represents hybrid inflation, in the sense that were $\lambda = 0$ the potential would be constant, and so displacement of λ from 0 generates the potential.

As the field λ is not dynamic during inflation we use the potential slow-roll parameters for single-field inflation, which are

$$\epsilon_V = \frac{M_{Pl}}{2}\left(\frac{\partial_\psi V\left(\bar{\lambda},\psi\right)}{V\left(\bar{\lambda},\psi\right)}\right) \tag{2.32}$$

$$\eta_V = M_{Pl}^2\frac{\partial^2_\psi V\left(\bar{\lambda},\psi\right)}{V\left(\bar{\lambda},\psi\right)}, \tag{2.33}$$

with the field ψ evaluated at the end of inflation. The parameters can then be used to compute both scalar tilt n_s and tensor-to-scalar ratio r

$$n_s = 1 - 6\epsilon_V + 2\eta_V \tag{2.34}$$

$$r = 16\epsilon_V. \tag{2.35}$$

It is useful to calculate N, the number of e−folds left to the end of inflation

$$N = -\kappa^2\int_{\psi_f}^{\psi'} \mathrm{d}\psi\frac{V\left(\bar{\lambda},\psi\right)}{\partial_\psi V\left(\bar{\lambda},\psi\right)} \approx \frac{3}{4\bar{\lambda}}\mathrm{e}^{\sqrt{2/3}|\psi_f|}, \tag{2.36}$$

with ψ_f is the value of inflaton at the end of inflation, i.e. when $\mathrm{e}^{-\sqrt{2/3}|\psi_f|} \sim 1$. Using this relation for N we can write the scalar tilt and tensor-to-scalar ratio to first relevant order as

$$n_s \approx 1 - \frac{2}{N} = 0.96 \div 0.96667 \tag{2.37}$$

$$r \approx \frac{12}{N^2} = 0.0033 \div 0.0048, \tag{2.38}$$

with the values of N estimated from CMB to be $N = 50 \div 60$. The current constraints on these values from Planck 2018 [11] are

$$r < 0.064, \qquad n_s = 0.9649 \pm 0.0042. \tag{2.39}$$

From this we see that the derived values are consistent with the Planck 2018 data, which is encouraging. This fact is not surprising, as at this level the described behaviour is the one of Starobinsky model, albeit with modified minimum of potential and with gravi-cosmological constant Λ'. We stress that the cosmological constant Λ' has its origin in the gravitational sector and enters with the opposite sign of the usual matter sector cosmological constant, and so can cancel at least part of it.

Concerning the end of inflation and reheating, we note that the inflation ends when the inflaton field ψ becomes small-valued and picks up kinetic energy, which leads to λ regaining its kinetic term. This can prove relevant for reheating, as from the action (2.30) we see that at small values of $|\psi|$ the dominant interaction channel is $|\psi|\left(\partial_\mu\lambda\right)^2$ which allows to transfer energy stored in the inflaton field through inflaton decay

$$\psi \to \lambda + \lambda, \tag{2.40}$$

so the model provides a possible reheating channel. The detailed analysis of this channel is a good subject for further work.

2.3 Effective potential in De Sitter background

In the previous section we have mentioned result for effective potential of the field λ in flat spacetime. However better approximation to the geometry of the very early universe is thought to be provided by De Sitter metric. It is therefore appropriate to calculate one-loop effective potential in this background and compare result with the result obtained for flat spacetime.

2.3.1 Expansion of Weyl action to 2nd order in variations

Since we want to calculate one-loop corrections to the effective action, we are interested in small perturbations of our field of interest, the metric field. This takes the form

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \alpha_c h_{\mu\nu}, \tag{2.41}$$

where $\bar{g}_{\mu\nu}$ is the full metric which we split into classical part $g_{\mu\nu}$ and quantum fields $h_{\mu\nu}$, and α_c is small dimensionless parameter in which we perform perturbation expansion. This simplification is justified if we assume that transition between conformal phase and Poincaré phase happens when gravitational variations are small and best described as small fluctuations over a fixed background. We will now expand the geometric objects of interest to at most second order in α_c and we will continue to use bar over symbols to denote full objects, i.e. those that take quantum fluctuations $h_{\mu\nu}$ into account. We start by calculating contravariant form of metric tensor $\bar{g}^{\mu\nu}$, we do this by demanding that

$$\bar{g}^{\mu\nu}\bar{g}_{\nu\rho} = \delta^{\mu}_{\rho}. \tag{2.42}$$

From the above equation it follows after a short calculation,

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \alpha_c h^{\mu\nu} + \alpha_c^2 h^{\mu\sigma} h_{\sigma}{}^{\nu}, \tag{2.43}$$

where we neglecting terms which are $O\left(\alpha_c^3\right)$ and higher. Next object of interest is the square root of determinant of full metric $\sqrt{-\bar{g}}$, this can be calculated using Taylor expansion and trace formula for determinant, to the final result (up to higher order terms)

$$\sqrt{-\bar{g}} = \sqrt{-g}\left(1 + \frac{1}{2}\alpha_c h + \frac{1}{8}\alpha_c^2 h'^2 - \frac{1}{4} h_{\alpha\beta} h^{\alpha\beta}\right), \tag{2.44}$$

where $h = g^{\mu\nu}h_{\mu\nu}$. Using the previous results wee find Ricci tensor

$$\begin{aligned}\bar{R}_{\mu\nu} &= R_{\mu\nu} + \frac{\alpha_c}{2}\left(\nabla_\sigma\nabla_\mu h^\sigma{}_\nu + \nabla_\sigma\nabla_\nu h^\sigma{}_\mu - \nabla_\mu\nabla_\mu h - \Box h_{\mu\nu}\right)\\ &+ \frac{\alpha_c^2}{2}h^{\sigma\tau}\left(\nabla_\nu\nabla_\mu h_{\sigma\tau} + \nabla_\sigma\nabla_\tau h_{\mu\nu} - \nabla_\sigma\nabla_\mu h_{\nu\tau} - \nabla_\sigma\nabla_\nu h_{\mu\tau}\right)\\ &+ \frac{\alpha_c^2}{2}\left(\nabla_\sigma h^{\sigma\tau}\right)\left(\nabla_\tau h_{\mu\nu} - \nabla_\mu h_{\nu\tau} - \nabla_\nu h_{\mu\tau}\right)\\ &+ \frac{\alpha_c^2}{2}\left[\frac{1}{2}\nabla_\mu h_{\sigma\tau}\nabla_\nu h^{\sigma\tau} + \nabla_\sigma h_\mu{}^\tau\nabla^\sigma h_{\nu\tau} - \nabla_\sigma h_\mu{}^\tau\nabla_\tau h_\nu{}^\sigma\right.\\ &+ \left.\frac{1}{2}\nabla_\sigma h\nabla_\mu h^\sigma{}_\nu + \frac{1}{2}\nabla_\sigma h\nabla_\nu h^\sigma{}_\mu - \frac{1}{2}\nabla_\sigma h\nabla^\sigma h_{\mu\nu}\right],\end{aligned} \tag{2.45}$$

where ∇ and $\Box$ are with respect to the classical background $g_{\mu\nu}$. Calculation of Ricci scalar can be done by tracing

$$\begin{aligned}\bar{R} &= R + \alpha_c\left(\nabla_\sigma\nabla_\tau h^{\sigma\tau} - \Box h - h^{\sigma\tau}R_{\sigma\tau}\right)\\ &+ \alpha_c^2\left(h^{\sigma\tau}\nabla_\sigma\nabla_\tau h + h^{\sigma\tau}\Box h_{\sigma\tau} - h^{\mu\sigma}h^\nu{}_\sigma R_{\mu\nu}\right.\\ &- h^{\mu\nu}\nabla_\sigma\nabla_\mu h^\sigma{}_\nu - h^{\mu\nu}\nabla_\mu\nabla_\sigma h^\sigma{}_\nu + \nabla_\sigma h^{\sigma\tau}\nabla_\tau h - \nabla_\sigma h^{\sigma\tau}\nabla_\lambda h^\lambda{}_\tau\\ &- \left.\frac{1}{2}\nabla_\sigma h_{\sigma\tau}\nabla^\tau h^{\sigma\lambda} + \frac{3}{4}\nabla_\lambda h_{\sigma\tau}\nabla^\lambda h^{\sigma\tau} - \frac{1}{4}\nabla_\sigma h\nabla^\sigma h\right).\end{aligned} \tag{2.46}$$

Now that we have expanded all quantities of interest, we wish to prepare the terms appearing in action. Since we have already simplified the action using a suitable surface term we will continue this trend and use integration by parts to simplify the resulting formulas, ignoring the appearing surface terms. Also in the following we will write only terms which are of order α_c^2, as those are terms which will be relevant for calculating one-loop effective action, we will denote this by upper index (2). First we will be interested in the term $\sqrt{-\bar{g}}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$, after a tedious calculation we obtain

$$\begin{aligned}\left[\sqrt{-\bar{g}}\bar{R}^{\mu\nu}\bar{R}_{\mu\nu}\right]^{(2)} &= \frac{\alpha_c^2}{2}\sqrt{-g}h_{\mu\nu}\left[\frac{1}{2}\delta^{\mu\nu,\rho\sigma}\Box^2 + \frac{1}{2}g^{\rho\sigma}g^{\mu\nu}\Box^2 - \Box\nabla^\mu\nabla^\rho g^{\nu\sigma} + \nabla^\sigma\nabla^\rho\nabla^\mu\nabla^\nu\right.\\ &- \nabla^\rho\nabla^\sigma\Box g^{\mu\nu} + \left(R^{\mu\rho\nu\sigma} + R^{\mu\rho}g^{\nu\sigma}\right)\Box\\ &+ \left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)R^{\lambda\tau}\nabla_\lambda\nabla_\tau - \frac{3}{2}R^{\rho\lambda}g^{\nu\sigma}\left(\nabla^\mu\nabla_\lambda + \nabla_\lambda\nabla^\mu\right)\\ &+ \frac{1}{2}R^{\mu\rho}\left(\nabla^\sigma\nabla^\nu + \nabla^\nu\nabla^\sigma\right) - R^{\rho\mu\sigma\lambda}\left(\nabla^\nu\nabla_\lambda + \nabla_\lambda\nabla^\nu\right)\\ &+ g^{\mu\nu}R^{\rho\lambda}\left(\nabla_\lambda\nabla^\sigma + \nabla^\sigma\nabla_\lambda\right) + \frac{5}{2}g^{\mu\rho}R^{\nu\lambda}R_\lambda{}^\sigma + 2R^\sigma{}_\lambda R^{\mu\rho\nu\lambda}\\ &+ R^{\rho\lambda\sigma\tau}R^\mu{}_\lambda{}^\nu{}_\tau - 2R^{\mu\lambda}R_\lambda{}^\nu g^{\rho\sigma} + \frac{1}{2}R^{\mu\rho}R^{\nu\sigma}\\ &- \left.\frac{1}{2}R_{\lambda\tau}R^{\lambda\tau}\left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)\right]h_{\rho\sigma}.\end{aligned} \tag{2.47}$$

Next we will calculate the second order of $\sqrt{-\bar{g}}\bar{R}^2$, obtaining

$$\begin{aligned}\left[\sqrt{-\bar{g}}\bar{R}^2\right]^{(2)} = \frac{\alpha_c^2}{2}\sqrt{-g}h_{\rho\sigma}\,[&2\nabla^\rho\nabla^\sigma\nabla^\mu\nabla^\nu - 2g^{\mu\nu}\nabla^\rho\nabla^\sigma\Box - 2\Box\nabla^\mu\nabla^\nu g^{\rho\sigma} \\ &+ 2g^{\mu\nu}g^{\rho\sigma}\Box^2 + 2R^{\rho\sigma}\left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right) \\ &+ R\left(g^{\rho\sigma}\nabla^\mu\nabla^\nu + g^{\mu\nu}\nabla^\rho\nabla^\sigma + \delta^{\mu\nu,\rho\sigma}\Box - g^{\mu\nu}g^{\rho\sigma}\Box - g^{\mu\rho}\nabla^\nu\nabla^\sigma - g^{\mu\rho}\nabla^\sigma\nabla^\nu\right) \\ &+ 2R^{\mu\nu}\left(g^{\rho\sigma}\Box - \nabla^\rho\nabla^\sigma\right) - \frac{1}{2}R^2\left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right) \\ &+ R\left(5R^{\mu\rho}g^{\nu\sigma} - R^{\mu\rho\nu\sigma} - R^{\mu\nu}g^{\rho\sigma}\right) + 2R^{\mu\nu}R^{\rho\sigma}]\,h_{\mu\nu}. \end{aligned} \tag{2.48}$$

And lastly we will calculate second order of $\sqrt{-\bar{g}}\bar{R}$,

$$\begin{aligned}\left[\sqrt{-\bar{g}}\bar{R}\right]^{(2)} = \frac{\alpha_c^2}{2}\sqrt{-g}h_{\rho\sigma}\Bigg[&\frac{1}{2}\delta^{\mu\nu,\rho\sigma}\Box - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\Box + g^{\rho\sigma}\nabla^\mu\nabla^\nu - g^{\nu\sigma}\nabla^\mu\nabla^\rho \\ &+ 2R^{\mu\rho}g^{\nu\sigma} - R^{\mu\nu}g^{\rho\sigma} - \frac{1}{2}R\left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)\Bigg]h_{\mu\nu}. \end{aligned} \tag{2.49}$$

In the above expression the symbol $\delta^{\mu\nu,\rho\sigma}$ denotes a Kronecker delta over the pairs of indices $(\mu\nu)$ and (ρ,σ).

2.3.2 De Sitter spacetime

Now we have all the second order variations prepared. As we are mainly interested in one-loop effective potential in inflationary era, and it is generally taken that during inflation spacetime was De Sitter-like, we can take the above expansion to be around De Sitter spacetime. In the following we will quickly overview its relevant features.

De Sitter spacetime is the Lorentzian analog of an 4-sphere, i.e. it is a maximally symmetric manifold of constant positive curvature. Interestingly, its symmetry group, the de Sitter group $O(1,4)$ is related to the Poincaré group trough the process of group contraction. Specifically, Poincaré group can be considered as an 'infinite radius' limit of the de Sitter group.

The most simple definition of de Sitter space is as a submanifold of Minkowski space $\mathbb{R}^{1,4}$, where it can be considered as a one-sheeted hyperboloid with the equation

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = \rho^2, \tag{2.50}$$

where ρ is a non-zero constant with dimension of length, it can be considered as a 'radius' of de Sitter space. The induced metric is non-degenerate and of Lorentzian signature, having in the spherical coordinates the following form

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2}{\rho^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{\rho^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{2.51}$$

Using the metric we can calculate the curvature tensors, ultimately yielding the following

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{\rho^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \\ R_{\mu\nu} &= \frac{3}{\rho^2}g_{\mu\nu} \\ R &= \frac{12}{\rho^2}. \end{aligned}$$

From the value of Ricci tensor we see that de Sitter spacetime is a Einstein manifold, that is a manifold where Ricci tensor is proportional to the metric. There is also relation between cosmological constant Λ and parameter ρ, it is

$$\Lambda = \frac{3}{\rho^2}. \tag{2.52}$$

We can use this to rewrite the curvature tensors with the cosmological constant, as that is a physical parameter. The results of this substitution are

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{2.53}$$

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.54}$$

$$R = 4\Lambda. \tag{2.55}$$

We note that the above written expressions for curvature tensors are valid in any maximally symmetric space. Positive Λ corresponds to space positive curvature, i.e. De Sitter space, and is the case we are interested in. Negative Λ then corresponds to space of negative curvature, Anti-De Sitter space. Finally $\Lambda = 0$ simply corresponds to the flat case.

We can now plug the expression for the curvature tensors into the formulas for the second variations we have obtained in the last section, this results in the following expressions:

$$\begin{aligned} \left[\sqrt{-\bar{g}}\bar{R}^{\mu\nu}\bar{R}_{\mu\nu}\right]^{(2)} = \frac{\alpha_c^2}{2}\sqrt{-g}\,h_{\mu\nu}\Bigg[&\frac{\Box^2}{2}\left(\delta^{\mu\nu,\rho\sigma} + g^{\rho\sigma}g^{\mu\nu}\right) - \Box\nabla^\mu\nabla^\rho g^{\nu\sigma} + \nabla^\sigma\nabla^\rho\nabla^\mu\nabla^\nu - \nabla^\rho\nabla^\sigma\Box g^{\mu\nu} \\ &+ \frac{\Lambda}{3}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\rho\nu} + 3g^{\mu\rho}g^{\nu\sigma}\right)\Box + \left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)\Lambda g^{\lambda\tau}\nabla_\lambda\nabla_\tau \\ &- \frac{3}{2}\Lambda g^{\rho\lambda}g^{\nu\sigma}\left(\nabla^\mu\nabla_\lambda + \nabla_\lambda\nabla^\mu\right) + \frac{\Lambda}{2}g^{\mu\rho}\left(\nabla^\sigma\nabla^\nu + \nabla^\nu\nabla^\sigma\right) + \Lambda g^{\mu\nu}g^{\rho\lambda}\left(\nabla_\lambda\nabla^\sigma + \nabla^\sigma\nabla_\lambda\right) \\ &- \frac{\Lambda}{3}\left(g^{\rho\sigma}g^{\mu\lambda} - g^{\rho\lambda}g^{\mu\sigma}\right)\left(\nabla^\nu\nabla_\lambda + \nabla_\lambda\nabla^\nu\right) + \frac{5}{2}\Lambda^2 g^{\mu\rho}g^{\nu\lambda}\delta^\sigma_\lambda + \frac{2}{3}\Lambda^2\delta^\sigma_\lambda\left(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\rho\nu}\right) \\ &+ \frac{\Lambda^2}{9}\left(g^{\rho\sigma}g^{\lambda\tau} - g^{\rho\tau}g^{\lambda\sigma}\right)\left(g^{\mu\nu}g_{\lambda\tau} - \delta^\mu_\tau\delta^\nu_\lambda\right) - 2\Lambda^2 g^{\mu\lambda}\delta^\nu_\lambda g^{\rho\sigma} + \frac{\Lambda^2}{2}g^{\mu\rho}g^{\nu\sigma} \\ &- \frac{\Lambda^2}{2}g_{\lambda\tau}g^{\lambda\tau}\left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)\Bigg]h_{\rho\sigma}, \end{aligned} \tag{2.56}$$

$$
\begin{aligned}
\left[\sqrt{-\bar{g}}\bar{R}^2\right]^{(2)} = \frac{\alpha_c^2}{2}\sqrt{-g}\,h_{\rho\sigma}\Big[&2\nabla^\rho\nabla^\sigma\nabla^\mu\nabla^\nu - 2g^{\mu\nu}\nabla^\rho\nabla^\sigma\Box - 2\Box\nabla^\mu\nabla^\nu g^{\rho\sigma} + 2g^{\mu\nu}g^{\rho\sigma}\Box^2 \\
&+ 2\Lambda g^{\rho\sigma}\left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right) + 2\Lambda g^{\mu\nu}\left(g^{\rho\sigma}\Box - \nabla^\rho\nabla^\sigma\right) \\
&+ 4\Lambda\left(g^{\rho\sigma}\nabla^\mu\nabla^\nu + g^{\mu\nu}\nabla^\rho\nabla^\sigma + \delta^{\mu\nu,\rho\sigma}\Box - g^{\mu\nu}g^{\rho\sigma}\Box - g^{\mu\rho}\nabla^\nu\nabla^\sigma - g^{\mu\rho}\nabla^\sigma\nabla^\nu\right) \\
&- 8\Lambda^2\left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right) + \frac{4}{3}\Lambda^2\left(15g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu} - 3g^{\mu\nu}g^{\rho\sigma}\right) \\
&+ 2\Lambda^2 g^{\mu\nu}g^{\rho\sigma}\Big]h_{\mu\nu},
\end{aligned} \tag{2.57}
$$

$$
\begin{aligned}
\left[\sqrt{-\bar{g}}\bar{R}\right]^{(2)} = \frac{\alpha_c^2}{2}\sqrt{-g}\,h_{\rho\sigma}\Big[&\left(\delta^{\mu\nu,\rho\sigma} - g^{\mu\nu}g^{\rho\sigma}\right)\frac{\Box}{2} + g^{\rho\sigma}\nabla^\mu\nabla^\nu - g^{\nu\sigma}\nabla^\mu\nabla^\rho + 2\Lambda g^{\mu\rho}g^{\nu\sigma} \\
&- \Lambda g^{\mu\nu}g^{\rho\sigma} - 2\Lambda\left(\delta^{\mu\nu,\rho\sigma} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)\Big]h_{\mu\nu}.
\end{aligned} \tag{2.58}
$$

The expressions above can be heavily simplified, by exploiting the symmetries of the tensors involved, by grouping the terms of the same order in Λ and finally by using partial integration and neglecting the boundary terms (as we are interested in the dynamics in the bulk). These manipulations result in the following expressions

$$
\begin{aligned}
\left[\sqrt{-\bar{g}}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\right]^{(2)} = \frac{\alpha_c^2}{4}\sqrt{-g}\Big[&h_{\mu\nu}\Box^2 h^{\mu\nu} + h_\mu{}^\mu\Box^2 h_\rho{}^\rho + h_{\mu\nu}{}^{;\mu}\Box h_\rho{}^{\nu;\rho} + h_{\mu\nu}{}^{;\mu\nu}h_{\rho\sigma}{}^{;\rho\sigma} - h_{\mu\nu}{}^{;\mu\nu}\Box h_\rho{}^\rho \\
&+ \frac{\Lambda}{3}\left(10h_{\mu\nu}\Box h^{\mu\nu} + 3h_\mu{}^\mu\Box h_\rho{}^\rho - 14h_{\mu\nu}{}^{;\mu}h_\sigma{}^{\nu;\sigma} - 12h_\mu^\mu h_{\rho\sigma}{}^{;\rho\sigma}\right) \\
&+ \frac{64}{9}\Lambda^2\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{4}h_\mu^\mu h_\rho^\rho\right)\Big],
\end{aligned} \tag{2.59}
$$

$$
\begin{aligned}
\left[\sqrt{-\bar{g}}\bar{R}^2\right]^{(2)} = \alpha_c^2\sqrt{-g}\Big[&h_{\mu\nu}{}^{;\mu\nu}h_{\rho\sigma}{}^{;\rho\sigma} - 2h_{\mu\nu}{}^{;\mu\nu}\Box h_\rho{}^\rho + h_\mu{}^\mu\Box^2 h_\rho{}^\rho \\
&+ 2\Lambda\left(h_\mu{}^\mu h_{\mu\nu}{}^{;\mu\nu} + 2h_{\mu\nu}{}^{;\mu}h_\rho{}^{\nu;\rho} + h_{\mu\nu}\Box h^{\mu\nu}\right) + 4\Lambda^2\left(h_{\mu\nu}h^{\mu\nu} + \frac{1}{4}h_\mu{}^\mu h_\rho{}^\rho\right)\Big],
\end{aligned} \tag{2.60}
$$

$$
\left[\sqrt{-\bar{g}}\bar{R}\right]^{(2)} = \frac{\alpha_c^2}{2}\sqrt{-g}\left[\frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{2}h_\mu{}^\mu\Box h_\rho{}^\rho + h_\mu{}^\mu h_{\rho\sigma}{}^{;\rho\sigma} + h_{\mu\nu}{}^{;\mu}h_\rho{}^{\nu;\rho} - \frac{4}{3}\Lambda\left(h_{\mu\nu}h^{\mu\nu} - \frac{1}{4}h_\mu{}^\mu h_\rho{}^\rho\right)\right]. \tag{2.61}
$$

We can immediately see that the terms which are without the Λ or Λ^2 prefactor are the same as in the expression for these variations in the Minkowski background, as in the [2], which is satisfactory, as the limit $\Lambda \to 0$ of the de Sitter space is the Minkowski space, and so the second order expressions should in de Sitter space should go in this limit to the same expressions in the Minkowski space.

We now denote the traceless part of $h_{\mu\nu}$ as

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4} h_\mu{}^\mu, \tag{2.62}$$

and use this to simplify the above expressions. Specifically we use this to eliminate mixed terms as follows

$$\begin{aligned} h_{\mu\nu}{}^{;\mu\nu} h_\rho{}^\rho &= 2\bar{h}_{\mu\nu}{}^{;\mu} \bar{h}^{\rho\nu}{}_{;\rho} - 2h_{\mu\nu}{}^{;\mu} h^{\rho\nu}{}_{;\rho} + \frac{1}{8} h_\mu{}^\mu \Box h_\rho{}^\rho \\ h_{\mu\nu}{}^{;\mu\nu} \Box h_\rho{}^\rho &= 2\bar{h}_{\mu\nu}{}^{;\mu} (\Box - \Lambda) \bar{h}^{\rho\nu}{}_{;\rho} - 2h_{\mu\nu}{}^{;\mu} (\Box - \Lambda) h^{\rho\nu}{}_{;\rho} + \frac{1}{8} h_\mu{}^\mu \Box^2 h_\rho{}^\rho. \end{aligned}$$

Using this we obtain

$$\begin{aligned} \left[\sqrt{-\bar{g}} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} \right]^{(2)} = \frac{\alpha_c^2}{4} \sqrt{-g} \Bigg[& h_{\mu\nu} \left(\Box^2 + \frac{10}{3} \Lambda \Box + \frac{64}{9} \Lambda^2 \right) h^{\mu\nu} - h_\mu{}^\mu \left(\frac{7}{8} \Box^2 - \frac{1}{2} \Lambda \Box + \frac{16}{9} \Lambda^2 \right) h_\rho{}^\rho \\ & + h_{\mu\nu}{}^{;\mu} \left(3\Box g^{\nu\sigma} - \nabla^\nu \nabla^\sigma + \frac{7}{3} \Lambda g^{\nu\sigma} \right) h_{\rho\sigma}{}^{;\rho} - \bar{h}_{\mu\nu}{}^{;\mu} \left(2\Box g^{\nu\sigma} + 6\Lambda g^{\nu\sigma} \right) \bar{h}_{\rho\sigma}{}^{;\rho} \Bigg] \end{aligned} \tag{2.63}$$

$$\begin{aligned} \left[\sqrt{-\bar{g}} \bar{R}^2 \right]^{(2)} = \alpha_c^2 \sqrt{-g} \Bigg[& h_{\mu\nu} \left(2\Lambda \Box + 4\Lambda^2 \right) h^{\mu\nu} + h_\mu{}^\mu \left(\frac{3}{4} \Box^2 + \frac{\Lambda}{4} \Box + \Lambda^2 \right) h_\rho{}^\rho \\ & + h_{\mu\nu}{}^{;\mu} \left(4\Box g^{\nu\sigma} - \nabla^\nu \nabla^\sigma - 4\Lambda g^{\nu\sigma} \right) h_{\rho\sigma}{}^{;\rho} - \bar{h}_{\mu\nu}{}^{;\mu} \left(4\Box g^{\nu\sigma} - 8\Lambda g^{\nu\sigma} \right) \bar{h}_{\rho\sigma}{}^{;\rho} \Bigg] \end{aligned} \tag{2.64}$$

$$\left[\sqrt{-\bar{g}} \bar{R} \right]^{(2)} = \frac{\alpha_c^2}{2} \sqrt{-g} \left[h_{\mu\nu} \left(\frac{1}{2} \Box - \frac{4}{3} \Lambda \right) h^{\mu\nu} - h_\mu{}^\mu \left(\frac{1}{2} \Box - \frac{1}{3} \Lambda \right) h_\rho{}^\rho + 2\bar{h}_{\mu\nu}{}^{;\mu} g^{\nu\sigma} \bar{h}_{\rho\sigma}{}^{;\rho} - h_{\mu\nu}{}^{;\mu} g^{\nu\sigma} h_{\rho\sigma}{}^{;\rho} \right] \tag{2.65}$$

This concludes the preparatory part of our calculation and now we turn our attention how we will apply this in the transformed form of the action, to obtain effective potential for the auxiliary field λ.

2.3.3 Effective action of Weyl gravity

We formally define the Euclidean generating functional of the quantum theory to be

$$Z = \sum \int \mathcal{D} g_{\mu\nu} \mathrm{e}^{-A}, \tag{2.66}$$

where the sum is over all possible distinct topologies and integral is over all distinct metric field on the spacetime manifold, where metrics are distinct if they are not related by transformation from Diff × Weyl, i.e. the group of diffeomorphisms of spacetime plus Weyl transformation.

At this point we need to discuss the problem of the gravitational functional measure. This discussion will mostly follow that in [12]. In order to define an invariant volume element in function space we must first define a metric in the function space

$$G^{\mu\nu,\alpha\beta}\left(g\left(x\right)\right)=\frac{1}{2}\sqrt{g\left(x\right)}\left[g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}+\lambda g^{\mu\nu}G^{\alpha\beta}\right]\left(x\right), \tag{2.67}$$

where $\lambda \neq -2/d$. This is the so-called *DeWitt supermetric* in space of metric deformations. This allows to define a norm for metric deformations as

$$\|\delta g\|^2=\int \mathrm{d}^d x\, \delta g_{\mu\nu}\left(x\right) G^{\mu\nu,\alpha\beta}\left(g\left(x\right)\right) g_{\alpha\beta}\left(x\right). \tag{2.68}$$

Now we can formulate an expression for functional measure $\mathcal{D}g_{\mu\nu}$

$$\int \mathcal{D}g_{\mu\nu}=\int \prod_x \left[\det G\left(g\left(x\right)\right)\right]^{1/2} \prod_{\mu\geq\nu} \mathrm{d}g_{\mu\nu}\left(x\right). \tag{2.69}$$

The determinant of G can be evaluated giving, up-to a multiplicative constant,

$$\det G\left(g\left(x\right)\right)\propto\left(1+\frac{1}{2}d\lambda\right)\left[g\left(x\right)\right]^{(d-4)(d+1)/4}. \tag{2.70}$$

This expression explains the condition on λ, it is to prevent the determinant from vanishing identically. The functional measure in four dimensions becomes

$$\int \mathcal{D}g_{\mu\nu}=\int \prod_x \prod_{\mu\geq\nu} \mathrm{d}g_{\mu\nu}\left(x\right). \tag{2.71}$$

The problem with this construction is that supermetric is not necessarily unique, for example we could drop the $\sqrt{g\left(x\right)}$ factor, yielding

$$\tilde{G}^{\mu\nu,\alpha\beta}\left(g\left(x\right)\right)=\frac{1}{2}\left[g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}+\lambda g^{\mu\nu}G^{\alpha\beta}\right]\left(x\right) \tag{2.72}$$

this is called *Misner supermetric*. Functional measure in four dimensions would then take form

$$\int \mathcal{D}g_{\mu\nu}=\int \prod_x \left[g\left(x\right)\right]^{-5/2} \prod_{\mu\geq\nu} \mathrm{d}g_{\mu\nu}\left(x\right) \tag{2.73}$$

Both of the previous cases can be subsumed into a general case with a factor $\sqrt{g\left(x\right)}^{(1-\omega)}$

$$G_{\omega}^{\mu\nu,\alpha\beta}\left(g\left(x\right)\right)=\frac{1}{2}\sqrt{g\left(x\right)}^{(1-\omega)}\left[g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}+\lambda g^{\mu\nu}G^{\alpha\beta}\right]\left(x\right), \tag{2.74}$$

where $\omega = 1$ and $\omega = 0$ correspond to the Misner and DeWitt measures, respectively. This would lead to functional measure

$$\int \mathcal{D}g_{\mu\nu}=\int \prod_x \left[g\left(x\right)\right]^{\sigma/2} \prod_{\mu\geq\nu} \mathrm{d}g_{\mu\nu}\left(x\right), \tag{2.75}$$

with σ a function of ω. Since other field would also contribute $g^{\gamma/2}(x)$ term into measure, it is reasonable to work 2.75 with some constant σ. Because of this freedom in choosing a measure, it can be suspected that the $g^{\sigma/2}(x)$ does not play role in physical properties of the theory and may be an irrelevant parameter. If that is the case then from practical point of view is the DeWitt choice the simplest one. Either way, equation 2.75 with some suitable regularization describes general choice of measure in functional integral of gravity.

In our case we evaluate the functional integral in a formal saddle-point approximation, so we do not have to consider details of the measure. We will also neglect the sum over topologies, as even formally it is not clear over what subset it should be taken, and we are also interested in only small perturbations, and topology changing effect are most definitely not "small perturbations". We also switch to Euclidean regime, then generating functional takes form

$$Z = \int \mathcal{D}h_{\mu\nu} \mathrm{e}^{-A[h_{\mu\nu}]}, \tag{2.76}$$

where $A\left[h_{\mu\nu}\right]$ is Euclidean action. The reason for moving to Euclidean space is that it makes the path integrals better behaved, we just have to remember the relation of Minkowski effective potential and Euclidean effective potential

$$V^M_{eff} = V^E_{eff}. \tag{2.77}$$

Next, we fix gauges and perform transformations outlined in the previous sections, this yields

$$Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\lambda\ \delta\left[\chi - \zeta\right]\delta\left[\chi^\nu - \zeta^\nu\right]\det\left(\mathcal{M}_{FP}\right)\det\left(\mathcal{N}_{FP}\right)\mathrm{e}^{-A[h_{\mu\nu},\lambda]}, \tag{2.78}$$

where λ is auxiliary field introduced in the Hubbard-Stratonovich transformation, the delta functions in the integrand are gauge fixing terms

$$\chi^\nu \equiv \partial_\mu h^{\mu\nu} = \zeta^\nu(x) \quad \wedge \quad \chi \equiv \bar{h} = \zeta(x) \tag{2.79}$$

first being the coordinate gauge and the second a conformal gauge. The functions $\zeta(x)$ and $\zeta^\nu(x)$ are arbitrary functions of x. The determinants $\det(\mathcal{M}_{FP})$ and $\det(\mathcal{N}_{FP})$ are Faddeev-Poppov terms for coordinate and conformal gauges respectively. The gauges themselves have form

$$(\mathcal{M}_{FP})_{\mu\nu} = -\Box_{\mu\nu} - \partial_\mu\partial_\nu \quad \wedge \quad \mathcal{N}_{FP} = (D-1)\,\delta^{(D)}(x-y). \tag{2.80}$$

Since the $\mathcal{N}_{FP}$ is simply a multiple of unit operator its determinant is not physically relevant and can be included into a normalization constant. To deal with the gauge fixing delta functions we use 't Hooft's averaging trick

$$\delta\left[\chi - \zeta\right] \to \int \mathcal{D}\zeta\, \mathrm{e}^{-\int \zeta\mathcal{H}\zeta}\delta\left[\chi - \zeta\right] = \mathrm{e}^{-\int \chi\mathcal{H}\chi}\left(\det\mathcal{H}\right)^{1/2}, \tag{2.81}$$

where $\mathcal{H}$ is an arbitrary symmetric operator. This also gives us a tool to cancel some terms in the action, by careful choice of the operator $\mathcal{H}$.

With the use of the expanded action (2.13) and simplified expressions for second order terms (2.63), (2.64) and (2.65) we can now write the action as follows

$$S^{(2)} = \int \mathrm{d}^4x\,\sqrt{-g}\left[h_{\mu\nu}Ah^{\mu\nu} + h_{\mu}{}^{\mu}Bh_{\rho}{}^{\rho} + h_{\mu\nu}{}^{;\mu}C^{\nu\sigma}h_{\rho\sigma}{}^{;\rho} + \bar{h}_{\mu\nu}{}^{;\mu}D^{\nu\sigma}\bar{h}_{\rho\sigma}{}^{;\rho}\right], \tag{2.82}$$

where

$$A = -\frac{1}{16}\Box^2 + \left[\left(\frac{C^2}{12} - \frac{5}{24}\right)\Lambda - \frac{\alpha_c^2\bar{\lambda}}{8\kappa^2}\right]\Box + \left[\left(\frac{C^2}{3} - \frac{4}{9}\right)\Lambda^2 + \frac{\alpha_c^2\bar{\lambda}}{3\kappa^2}\Lambda - \frac{3\alpha_c^2\bar{\lambda}^2}{16S^2\kappa^4}\right], \tag{2.83}$$

$$B = \left(\frac{7}{128} + \frac{C^2}{48}\right)\Box^2 + \left[\left(\frac{C^2}{48} - \frac{1}{32}\right)\Lambda + \frac{\alpha_c^2\bar{\lambda}}{8\kappa^2}\right]\Box + \left[\left(\frac{C^2}{12} - \frac{1}{9}\right)\Lambda^2 - \frac{\alpha_c^2\bar{\lambda}}{12\kappa^2}\Lambda - \frac{3\alpha_c^2\bar{\lambda}^2}{32S^2\kappa^4}\right], \tag{2.84}$$

$$C^{\nu\sigma} = \left(\frac{C^2}{3} - \frac{3}{16}\right)\Box g^{\nu\sigma} + \left(\frac{1}{16} - \frac{C^2}{12}\right)\nabla^\nu\nabla^\sigma + \left[\frac{\alpha_c^2\bar{\lambda}}{4\kappa^2} - \left(\frac{C^2}{3} + \frac{7}{48}\right)\Lambda\right]g^{\nu\sigma}, \tag{2.85}$$

$$D^{\nu\sigma} = \left(\frac{1}{8} - \frac{C^2}{3}\right)\Box g^{\nu\sigma} + \left[\left(\frac{3}{8} + \frac{2C^2}{3}\right)\Lambda - \frac{\alpha_c^2\bar{\lambda}}{2\kappa^2}\right]g^{\nu\sigma}, \tag{2.86}$$

where we have also replaced the field λ with its vacuum expectation value $\bar{\lambda}$. We can employ the gauge conditions to eliminate two quadratic expression in favour of determinants, this yields

$$Z = N\det\mathcal{M}_{FP}\det\mathcal{N}_{FP}\,(\det B)^{1/2}\,(\det C^{\nu\sigma})^{4/2}\int\mathcal{D}h_{\mu\nu}\exp\left[-\int\mathrm{d}^4x\,\sqrt{-g}\left(h_{\mu\nu}Ah^{\mu\nu} + \bar{h}_{\mu\nu}{}^{;\mu}D^{\nu\sigma}\bar{h}_{\rho\sigma}{}^{;\rho}\right)\right] \tag{2.87}$$

where the additional power of determinant $\det C^{\nu\sigma}$ is due to the fact that it acts on a four-vector. The determinants of $C^{\nu\sigma}$ and B where brought out of the exponential through t'Hooft's averaging trick. Further calculation is complicated by the fact that in the exponential we have two quadratic terms which are not independent. So far we have been unable to rewrite the exponent in diagonal form with independent terms. The equation (2.87) will provide a starting point for future work.

Appendix A

Hubbard-Stratonovich transformation

The Hubbard-Stratonovich transformation originally arose from work on mean field theory with infinite interaction range. It allows to transform interacting field theory to a field theory of non-interacting degrees of freedom, at a cost of introducing new fluctuating field.

We will start by describing a useful identity in a system with a finite degrees of freedom, then we will formally generalize it to the case of fields and define the transformation.

The starting point will be the following identity

$$\exp\left[\frac{\beta^2}{4\alpha}\right] = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x \exp\left[-\alpha x^2 + \beta x\right], \tag{A.1}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. This is a simple identity for a Gaussian integral, which can be easily generalized to many variables using the results of previous section, as follows

$$\exp\left[\frac{1}{2}\boldsymbol{b}\mathbf{A}^{-1}\boldsymbol{b}\right] = \sqrt{\frac{\det(\mathbf{A})}{(2\pi)^n}} \int_{-\infty}^{+\infty} \prod_i \mathrm{d}x_i \exp\left[-\frac{1}{2}\sum_i x_i A_{ij} x_j + b_i x_i\right], \tag{A.2}$$

where $\mathbf{A}$ is an $N \times N$ real symmetric matrix, $\boldsymbol{b}$ and $\boldsymbol{x}$ are N component vectors. If the matrix A^{-1} is a diagonal matrix with positive entries it can be interpreted as a mass matrix of some system and therefore a Lagrangian, the identity then tells us that we can reinterpret this free system as an interacting system with some new auxiliary variables. We can also go in the other direction by integrating out the interacting field to a system of free variables.

Using the results of the previous section formal extension to infinite degrees of freedom is straightforward

$$\exp\left[\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, J(x)\, A^{-1}(x,y)\, J(y)\right] = \frac{1}{Z[0]}\int \mathcal{D}\phi\, \exp\Big[\int \mathrm{d}x\mathrm{d}y \left(-\frac{1}{2}\phi(x)\, A(x,y)\, \phi(y)\right) \int \mathrm{d}x\, J(x)\, \phi(x)\Big], \tag{A.3}$$

or, using the simplified notation established in the previous section

$$\exp\left[\frac{1}{2}J_xA^{-1}_{xy}J_y\right] = \frac{1}{Z\left[0\right]}\int\mathcal{D}\phi\exp\left[-\frac{1}{2}\phi_xA_{xy}\phi_y + J_xA_x\right]. \tag{A.4}$$

Now if we take $A = \lambda\mathbb{I}$, with $0 \neq \lambda \in \mathbb{C}$ then we can interpret left-hand side as a quadratic self-interaction of the field J, which can then be expressed using a functional integral over a new auxiliary field with which it interacts. We note that the new auxiliary field ϕ is of the same type as the field on the left hand side, i.e. both the spacetime indices and internal indices are the same.

As to some example of Hubbard-Stratonovich transformation, it is interesting that Legendre transformation can be interpreted as such. For simplicity consider the following action

$$A\left[q_i, \dot{q}_i\right] = \int \mathrm{d}t\,\frac{1}{2}\dot{q}_iM_{ij}\dot{q}_j - V\left(q\right), \tag{A.5}$$

where M is mass matrix. We now consider Hubbard-Stratonovich transformation of this object

$$\exp\left[\int\mathrm{d}t\,\frac{1}{2}\dot{q}_iM_{ij}\dot{q}_j - V\left(q\right)\right] = \int\mathcal{D}p\,\exp\left[-\int\mathrm{d}t\,\frac{1}{2}p_iM^{-1}_{ij}p_j + V\left(q\right) - p_i\dot{q}_i\right]. \tag{A.6}$$

On the level of exponents this is

$$\frac{1}{2}\dot{q}_iM_{ij}\dot{q}_j - V\left(q\right) = -\frac{1}{2}p_iM^{-1}_{ij}p_j - V\left(q\right) + p_i\dot{q}_i, \tag{A.7}$$

which when written in terms of Lagrangian and Hamiltonian is

$$L\left(q_i, \dot{q}_i\right) = -H\left(q_i, p_i\right) + p_i\dot{q}_i, \tag{A.8}$$

This is exactly the Legendre transformation relating Lagrangian and Hamiltonian.

Appendix B

Jordan and Einstein frames

Scalar-tensor theories of gravity (such as Brans-Dicke theory), have two possible formulations, either in so-called Jordan frame or in Einstein frame. These two frames are related by a conformal transformation and by a redefinition of the gravitational scalar field present in the theory. The possibility of different formulations also exists in Kaluza-Klein theories and in higher derivative theories of gravity. The problem of the two possible formulations is whether they are equivalent or not, and if not which is the physical formulation. This problem is a matter of lively debates, not yet settled. In the following we will describe the two frames and whether they are problematic from theoretical stand point. For more details look refer to [13].

The usual formulation of scalar-tensor theories of gravity is in the Jordan frame, in which the action takes the form

$$S = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left[f(\phi)\,R - \frac{\omega(\phi)}{\phi}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \Lambda(\phi)\right] + \int d^4x\,\sqrt{-g}\mathcal{L}_{matter}, \tag{B.1}$$

where $\mathcal{L}_{matter}$ is the Lagrangian density of the matter content of the theory, and the couplings $f(\phi)$ and $\omega(\phi)$ are regular functions of the scalar field ϕ. In the following discussion we will limit ourselves to the Brans-Dicke theory, where the ω and Λ are assumed constant. Neglecting the matter part of the theory (which is not relevant for our current interest), assuming $\Lambda = 0$ we obtain the field equations in the following form

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{\omega}{2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) \\ \Box\phi + \frac{\phi R}{2\omega} &= 0. \end{aligned}$$

Despite the fact that this is the form usually presented in textbooks, there is one issue with the Jordan frame, and that is the fact that the kinetic term of the scalar field ϕ is not positive definite. This means that there is no stable ground state in the theory, and the system decays towards lower and lower energy states without bound.

However since the original paper on the Brans-Dicke theory, it is known that there is another formulation of the theory. After the conformal transformation

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \phi g_{\mu\nu}, \tag{B.2}$$

and the scalar field redefinition

$$\phi \longrightarrow \tilde{\phi} = \int \frac{(2\omega + 3)^{1/2}}{\phi} \mathrm{d}\phi, \tag{B.3}$$

where $\omega > -3/2$, the theory is recast into the Einstein frame (occasionally also referred to as Pauli frame). In this frame the theory becomes the theory of the Einstein gravity and a non self-interacting scalar field

$$S = \int \mathrm{d}^4 x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} \right], \tag{B.4}$$

and the field equations are simply the usual Einstein equations with the scalar field as the source and usual scalar field equation (i.e. Klein-Gordon equation)

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = 8\pi \left(\tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}^\alpha \tilde{\phi} \tilde{\nabla}_\alpha \tilde{\nabla} \right)$$

$$\tilde{\Box} \tilde{\phi} = 0.$$

If we had included also the $\mathcal{L}_{matter}$ term of the ordinary matter, it would appear in the transformed action with a multiplicative factor $\exp\left(-\alpha\tilde{\phi}\right)$. This anomalous coupling would then lead to a violation of the equivalence principle in the Einstein frame.

As we can see both frames have their own issues, however it can be seen that they (at least in some scenarios) are not physically equivalent. The Jordan frame formulation of the scalar-tensor theory seems to be physically unviable, as the energy density of the scalar gravitational field is not bound from below (which violates weak energy condition), leading to instability of the system. It has to be noted that in principle quantum system *can* poses states of negative energy density, so it is possible that more complete analysis taking into account quantum behaviour could provide justification for using this frame. On the other hand, the Einstein frame formulation is free from this problem, however when we include the Lagrangian density of ordinary matter we obtain anomalous coupling, which leads to violation of the equivalence principle. However it has to be noted that this violation is sufficiently small as to be compatible with current experimental bounds on the equivalence principle.

Appendix C

Scale transformation, Conformal transformation and Weyl transformation

In literature we encounter terms such as scale transformation, conformal transformation, or Weyl transformation, and related invariance terms (scale invariance, etc.), and often these terms are used interchangeably. However, mathematically there are distinctions. In this section we will describe these notions and both mathematical and physical relations between them, for more details please refer to [14].

We start by defining the transformations. Scale transformations are simply dilations of spacetime coordinates

$$x_\mu \to \lambda x_\mu, \tag{C.1}$$

i.e. these transformations rescale all coordinates by some amount, and so preserve ratios of length. Conformal transformations are a group of coordinate transformations

$$x_\mu \to x'_\mu = F_\mu (x), \tag{C.2}$$

which leave metric invariant up to a conformal factor

$$g_{\mu\nu}(x) = \Omega(x')\, g'_{\rho\sigma}(x')\, \frac{\partial F^\rho}{\partial x^\mu}\frac{\partial F^\rho}{\partial x^\mu}. \tag{C.3}$$

From this is clear that conformal transformations preserve *angles* between curves. Finally, Weyl transformations are pointwise rescalings of the metric and fields

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu} = \mathrm{e}^{2\omega(x)} g_{\mu\nu}(x), \tag{C.4}$$

$$\phi(x) \to \tilde{\phi}(x) = \mathrm{e}^{-\Delta\omega(x)} \phi(x), \tag{C.5}$$

where $\omega(x)$ is arbitrary scalar function of spacetime and Δ is conformal weight of field ϕ.

We quickly return to scale transformations. Under the scale transformation $x_\mu \to \lambda x_\mu$ metric transforms as

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} = \mathrm{e}^{2\omega} g_{\mu\nu}, \tag{C.6}$$

where the second equality follows from the fact that λ must be non-zero real number. We can take the formula (C.6) as the definition of scale transformations, with scaling weights of fields determined canonically from kinetic term. From this it is clear that Weyl transformations can be interpreted as gauged scale transformations.

From the above definitions it is clear that there is following relation for invariance under given transformation

$$Weyl\ invariance \implies Conformal\ invariance \implies Scale\ invariance. \tag{C.7}$$

This follows from the fact that conformal transformations can be considered as a subgroup of Weyl transformations that leave metric invariant up to a diffeomorphism. The relation of scale transformations and conformal transformations is better seen from structure of the spacetime symmetry group. Both scale and conformal transformations are coordinate transformations, so it is natural to also consider Poincaré algebra, which is taken to be the fundamental symmetry of our spacetime. It has the following commutation relations

$$\begin{aligned} \mathrm{i}\,[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\ \mathrm{i}\,[P^{\mu}, J^{\rho\sigma}] &= \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho} \\ [P^{\mu}, P^{\nu}] &= 0. \end{aligned} \tag{C.8}$$

For massless scale invariant theory we can enhance Poincaré algebra by adding dilation generator D

$$\begin{aligned} [D, P^{\mu}] &= -\mathrm{i}P^{\mu} \\ [D, J^{\mu\nu}] &= 0. \end{aligned} \tag{C.9}$$

Under certain assumptions (for more details look in [14]) it can be shown that maximally enhanced bosonic symmetry of the spacetime for massless particles can be obtained by adding the special conformal transformations K^μ, with the following commutation relations

$$\begin{aligned} [K^{\mu}, D] &= -\mathrm{i}K^{\mu} \\ [K^{\mu}, P^{\nu}] &= -2\mathrm{i}\eta^{\mu\nu} D + 2\mathrm{i}J^{\mu\nu} \\ [K^{\mu}, K^{\nu}] &= 0 \\ [K^{\mu}, J^{\rho\sigma}] &= \mathrm{i}\eta^{\mu\sigma} K^{\rho} - \mathrm{i}\eta^{\mu\rho} K^{\sigma}. \end{aligned} \tag{C.10}$$

From the commutation relations we see that the group of conformal transformations is 15 dimensional, whereas scale enhanced Poincaré group is 11 dimensional. Also from the closure of commutation relations above it is clear that conformal invariance demands scale invariance, but the inverse is not true, so in theory we can have scale invariance without conformal invariance.

From the above relation one might come to the conclusion that conformal invariance implies scale invariance and that is it. However, in quantum field theories we often find that scale invariance is enhanced to conformal invariance, i.e. presence of scale invariance implies conformal invariance.

For study of this enhancement it is useful to use energy-momentum tensor $T_{\mu\nu}$. Since we assume Poincaré invariance, we know from Noether's theorem that theory will posses conserved energy-momentum tensor $\partial^\mu T_{\mu\nu} = 0$ which can be chosen to be symmetric due to Lorentz invariance, $T_{\mu\nu} = T_{\nu\mu}$. Invariance under scale transformations requires that

$$T^\mu{}_\mu = \partial^\mu J_\mu, \tag{C.11}$$

where J_μ is called local virial current. This allows to construct a conserved scale current

$$D_\mu = x^\rho T_{\mu\rho} - J_\mu. \tag{C.12}$$

Invariance under special conformal transformations requires that energy-momentum tensor is traceless

$$T^\mu{}_\mu = 0. \tag{C.13}$$

For conformal invariance it is also important that the energy-momentum tensor is not unique, it can be improved without spoiling conservation law as follows

$$T_{\mu\nu} \to T_{\mu\nu} + \partial^\rho B_{\mu\nu\rho}, \qquad B_{\mu\nu\rho} = -B_{\rho\nu\mu}. \tag{C.14}$$

This has the following consequence, if we suppose that the energy-momentum tensor is given by

$$\begin{aligned} T^\mu{}_\nu &= \partial^\mu \partial_\nu L \quad (d = 2) \\ T^\mu{}_\nu &= \partial^\mu \partial^\rho L_{\nu\rho} \quad (d \geq 3)\,, \end{aligned} \tag{C.15}$$

so that trace is

$$\begin{aligned} T^\mu{}_\mu &= \partial^\mu \partial_\mu L \quad (d = 2) \\ T^\mu{}_\mu &= \partial^\mu \partial^\nu L_{\mu\nu} \quad (d \geq 3)\,, \end{aligned} \tag{C.16}$$

where L, $L_{\mu\nu}$ are certain local operators. It can be then shown that energy-momentum tensor $T_{\mu\nu}$ can be enhanced to energy-momentum tensor $\Theta_{\mu\nu}$ which is traceless, which is precisely the condition for conformally invariant theory.

From the above discussion it is clear that the improvement of scale invariance to conformal invariance is possible if virial current is a divergence of some local operator

$$J_\mu = \partial_\mu L \quad (d=2)$$
$$J_\mu = \partial^\rho L_{\mu\rho} \quad (d \geq 3)\,. \tag{C.17}$$

In $d = 2$ dimension there is a proof that scale invariance is enhanced to conformal invariance under following assumptions

- unitarity
- Poincaré invariance
- unbroken scale invariance
- discrete spectrum of scaling dimension
- existence of scale current,

this is knows as Zamolodchikov-Polchinski theorem. It is conjectured that this enhancement occurs also in higher dimension under the same assumptions.

C.1 Curved background

The above discussion was done implicitly assuming flat background on which the field theory is formulated. It turns out that most natural way to consider curved backgrounds is to study quantum behaviour of the trace of energy-momentum tensor. To this end, it is useful to formulate generating functional of renormalized operators, the Schwinger functional. Its formal path integral expression is

$$\mathrm{e}^{-W[g^I(x)]} = \int \mathcal{D}X \mathrm{e}^{-S_0[X] - \int \mathrm{d}^d x \lambda^I(x) O_I(x)}, \tag{C.18}$$

where λ_I are sources for operators O_I. We are interested in the energy-momentum tensor, which is sourced by the background metric tensor. This then leads us to consider quantum field theories on curved backgrounds. An important property of Schwinger functional is the Schwinger action principle

$$\langle O_I(x) \rangle = \frac{\delta W}{\delta \lambda^I(x)}. \tag{C.19}$$

We can define the energy-momentum tensor of matter sector as

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{C.20}$$

where $S = \int \mathrm{d}^d x \mathcal{L}$ is the matter action. From the diffeomorphism invariance of action we immediately get conservation law $D^\mu T_{\mu\nu} = 0$. If we consider scale invariance (which can be written as constant Weyl invariance, $g_{\mu\nu} \to \mathrm{e}^{2\bar{\omega}} g_{\mu\nu}$), then we obtain the following relation for trace of energy-momentum tensor

$$T^{\mu}{}_{\mu} = D^{\mu} J_{\mu}, \tag{C.21}$$

where $\delta\mathcal{L} = -\bar{\omega} D^{\mu} J_{\mu}$. This can be considered as the origin of virial current. Naturally we can now consider a scenario where action is invariant under Weyl transformations, from which it follows that the action density must also be Weyl invariant, and so we have

$$T^{\mu}{}_{\mu} = 0. \tag{C.22}$$

The above analysis did not include quantum effects, so now we turn our attention to them. For the expectation value of the energy-momentum tensor we obtain

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{g}} \frac{\delta W \left[g_{\mu\nu} \right]}{\delta g^{\mu\nu}}. \tag{C.23}$$

In conformal field theories we would assume that this would be zero, however in curved backgrounds this is not so due to *Weyl anomaly.* In two dimensions this takes the following form

$$\langle T_{\mu\nu} \rangle = \frac{1}{2\pi} \frac{c}{12} R, \tag{C.24}$$

where c is the so-called central charge. It appears due to the fact that in two dimension conformal algebra is enhanced to the infinite dimensional Virasoro algebra, and c is the center of this algebra. Since the anomaly is proportional to the scalar curvature, the theory is still conformally invariant in flat space.

In four dimensions the Weyl anomaly has more complicated form. The most general possible form is as follows

$$\langle T_{\mu\nu} \rangle = cW^2 - aE + BR^2 + \tilde{b} \Box R + d\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta\rho\sigma}, \tag{C.25}$$

where

$$W = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \qquad E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2,$$

are the Weyl tensor and Gauss-Bonnet term respectively. Some of the terms can eliminated by counter terms and with the use of various consistency conditions. For free field theories the Weyl anomaly can be computed from the one-loop determinant, trough Schwinger-DeWitt computation. Specifically we consider heat kernel of differential operator of Laplace type (e.g. $\Delta = \Box + U$)

$$K_{\Delta}(x, x; \tau) = \sum_{n} \tau^{n-d/2} \int \mathrm{d}^d x \sqrt{|g|}\, e_n (x|\Delta). \tag{C.26}$$

The Schwinger-DeWitt heat kernel computes one-loop logarithmic divergence which gives gravitational beta functions, we can relate this to Weyl anomaly as

$$\langle T^{\mu}{}_{\mu} \rangle = e_{d/2}\,(x|\Delta) \tag{C.27}$$

For example for a scalar field in four dimension we have

$$\Delta = -\Box + \xi R, \tag{C.28}$$

and for the DWSG coefficient

$$e_2\,(x|\Delta) = \frac{1}{16\pi^2}\left[\frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 - \frac{1}{180}R_{\mu\nu}^2 + \frac{1}{180}R_{\mu\nu\rho\sigma}^2\right]. \tag{C.29}$$

It is clear that for conformally coupled scalar field (i.e. $\xi = 1/6$) the coefficient in front of R^2 is zero, and the $\Box R$ can be eliminated by a counter term.

Appendix D

$\zeta-$function regularization in flat and curved background

In quantum field theory, one often encounters integrals or other quantities which are divergent. To deal with some of these infinities we need some regularization schemes, such as dimensional regularization, Pauli-Villars regularization or $\zeta-$function regularization. We will focus on $\zeta-$function regularization, since it can be quite naturally applied to the problem of calculating determinants of operators. We will start by a quick overview of regularization methods used for evaluating determinants of operators, then we will define spectral $\zeta-$function and finish by using heat kernel method to find suitable integral representation of spectral $\zeta-$function.

D.1 Spectral $\zeta-$function

We will start by a definition of the most famous $\zeta-$function, one with which the reader will most likely by familiar with, the Riemann $\zeta-$function. The Riemann $\zeta-$function, a function of complex variable s, is defined as

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1. \tag{D.1}$$

However the sum on the left side is holomorphic on the the half plane $\Re s$ so we can analytically continue it to the $\mathbb{C}$ to obtain meromorphic function on whole of $\mathbb{C}$. Since analytic continuation is unique, if we prove that some function $g(s)$ equals the Riemann $\zeta-$function on some subdomain of $\mathbb{C}$, then it is Riemann $\zeta-$function everywhere on $\mathbb{C}$. This enables us to find different representations of Riemann $\zeta-$function which might be more suitable for work in different contexts.

There are various generalizations of the Riemann $\zeta-$function, we will be interested in the so-called *spectral $\zeta-$functions* (also knows as $\zeta-$function of an operator). These work by replacing n in the denominator by λ_n, an eigenvalue of some operator O, giving

$$\zeta_O(s) = \sum_n^{\infty} \frac{1}{\lambda_n^s} \equiv \mathrm{tr}\left(O^{-s}\right), \tag{D.2}$$

where the trace is both over continuous and over discrete indices of the operator. This can be used to regularize (or simply directly define) determinant of an operator O. Derivation of this is simple, first we start by differentiating with respect to s

$$\frac{\mathrm{d}\zeta_O(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\sum_n^{\infty} \exp\left(-s\ln\lambda_n\right) = -\sum_n^{\infty}\exp\left(-s\ln\lambda_n\right)\ln\lambda_n, \tag{D.3}$$

then, setting $s = 0$ and using the basic property of logarithm we obtain

$$-\zeta_O'(0) = \sum_n^{\infty} \ln\lambda_n = \ln\prod_n^{\infty} = \ln\det O. \tag{D.4}$$

As a result we can define determinant of the operator O simply as

$$\det O \equiv \exp\left[-\zeta_O'(0)\right]. \tag{D.5}$$

This is the key formula used in ζ−function regularization. While it might seem we are done, we still need a way to calculate spectrum of the operator O to formulate the spectral ζ−function, this is obviously non-trivial issue. Luckily this problem can be bypassed by calculating *heat kernel*, which is the topic of the next section.

D.2 The Heat Kernel of Spectral ζ−Function

As stated previously, to calculate spectral ζ−function we need spectrum of the operator, which can be quite difficult. In this section we present an alternative formula for the spectral ζ−function, which will avoid this problem. To start with, we will define the exponent of the operator O

$$K(\tau) \equiv \exp\left[-\tau O\right] = \sum_n^{\infty} \exp\left[-\tau\lambda_n\right] |\psi_n\rangle\langle\psi_n|. \tag{D.6}$$

Next we will construct an integral representation of spectral ζ−function, a starting point will be integral representation of the gamma function

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-\tilde{\tau}}\tilde{\tau}^{s-1}\mathrm{d}\tilde{\tau} = k^s\int_0^{\infty} \mathrm{e}^{-k\tau}\tau^{s-1}\mathrm{d}\tau, \tag{D.7}$$

where in we have substituted $\tilde{\tau} = k\tau$ and $k \in \mathbb{R}\setminus\{0\}$. We can use this as a formula for k^{-s}

$$\frac{1}{k^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-k\tau} \tau^{s-1} d\tau. \tag{D.8}$$

The number k is arbitrary (except for the fact it has to be different from zero), so we can choose it to be some element of the spectrum of O. If $\sigma(0) \subset \mathbb{R}\setminus\{0\}$ then we write

$$\zeta_O(s) = \sum_n^\infty \frac{1}{\lambda_n^s} = \frac{1}{\Gamma(s)} \sum_n^\infty \int_0^\infty e^{-\lambda_n \tau} \tau^{s-1} d\tau, \tag{D.9}$$

and interchanging integration and summation we get for the spectral $\zeta-$function

$$\zeta_O(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_n^\infty e^{-\lambda_n \tau} \tau^{s-1} d\tau = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{tr} K(\tau) \tau^{s-1} d\tau. \tag{D.10}$$

As we can see, we have obtained a new representation of spectral $\zeta-$function, one which does not require knowledge of the spectrum of the operator O, provided we can calculate $\mathrm{tr}K(s)$.

The method of calculating the trace is reasonably straightforward, from the definition we know that $K(s)$ satisfies the following equation

$$\frac{\partial K(s)}{\partial s} = -OK(s), \tag{D.11}$$

and trace can be calculated in the x-representation as

$$\mathrm{tr}K(s) = \int dx K(x, x, s), \tag{D.12}$$

where $K(x, x', s) = \langle x| K(s) |x'\rangle$. If we now solve equation D.11 in the x-representation, we can calculate the trace right away. We can now call the operator $K(s)$ the *heat kernel*, since the equation it satisfies has a form of the heat equation.

All the discussion done above was in flat spacetime, however more physically interesting cases involve curved background. This will be the topic of the next section.

D.3 Heat Kernel in curved background

The formal manipulation to obtain determinant through spectral zeta function still holds in curved spacetime, as it is just mathematical manipulation, so we still have

$$\det O = \exp\left[-\frac{d}{ds}\zeta_O(s)|_{s=0}\right]$$
$$\zeta_O(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Ttr} K(\tau) \tau^{s-1} d\tau.$$

We are now interested in curved manifolds, this means the inner product must be modified accordingly. The natural inner product on $C^{\infty}(M)$ is

$$(\phi, \psi) = \mu^d \int_M \mathrm{d}^d x \sqrt{-g}\, \phi(x)\, \psi(x), \tag{D.13}$$

where μ is a constant with dimension of mass. Spectral decomposition of the operator O is

$$O\phi_n = \lambda_n \phi_n, \tag{D.14}$$

where the eigenfunctions ϕ_n are dimensionless and orthonormal with respect to the natural inner product. We also assume that the eigenfunctions ϕ_n form a basis in the space of functions on M. Using the decomposition of O we can write the spectral decomposition of the heat kernel of O as

$$K_O(x, y; t) = \sum_n \phi_n(x)\, \phi_n(y)\, \mathrm{e}^{-\lambda_n t}. \tag{D.15}$$

The trace of the heat kernel is then

$$\mathrm{Tr} K_O(t) = \int \mathrm{d}^d x \sqrt{-g}\, K_O(x, x; t) = \sum_n \mathrm{e}^{-\lambda_n t}. \tag{D.16}$$

As we have seen in the previous section, in the flat space the heat kernel can be easily calculated using the Fourier transform, which then also allows us to easily compute the trace of heat kernel. However, in general curved manifold we have no such tools available.

Every manifold locally looks like Euclidean (or Minkowski) space, so in the limit $t \to 0$ the trace of heat kernel must reduce to the same form as in the flat space. The deviations from the flat space solution must then be proportional to the deviations of the metric from flatness. This can be measured by curvature invariants. This reasoning leads us to the short-time asymptotic expansion for the heat kernel for operator O of order $2r$

$$\mathrm{Tr}\, K_O(t) \approx \frac{1}{(4\pi)^{d/2}} \sum_n E_n(O)\, t^{\frac{n-d}{2r}}, \quad t \to +0, \tag{D.17}$$

with $E_n(O)$ are integrated *heat kernel coefficients* $e_n(x|O)$,

$$E_n(O) = \int \mathrm{d}^d x \sqrt{-g}\, e_n(x|O). \tag{D.18}$$

The $e_n(x|O)$ are known as *heat kernel coefficients* also known as DeWitt-Seeley-Gilkey (DWSG) coefficients.

The operators can be either minimal or non-minimal. The minimal operators have the leading term is power of Laplace operator, non-minimal operators than have some more complicated expression as the leading term. We are interested particularly in the minimal fourth order operator, which has the form

$$\Box^2 + V^{\mu\nu}\nabla_\mu\nabla_\nu + N^\mu\nabla_\mu \ + X, \tag{D.19}$$

where $V^{\mu\nu}$, N^μ and X are a priori some matrix valued tensor fields. From [15] we know that the first two non-trivial coefficients are

$$e_2(x) = \frac{\Gamma\left(\frac{d-2}{4}\right)}{2\Gamma\left(\frac{d+2}{2}\right)}\left(\frac{1}{6}R + \frac{1}{2n}V\right), \tag{D.20}$$

$$\begin{aligned} e_4(x) = {} & \frac{\Gamma\left(\frac{d+4}{4}\right)}{2\Gamma\left(\frac{d+2}{2}\right)}\Bigg[(d-2)\left(\frac{1}{90}R^2_{\mu\nu\rho\sigma} - \frac{1}{90}R^2_{\mu\nu} + \frac{1}{36}R^2 + \frac{1}{15}\Box R + \frac{1}{6}W^2_{\alpha\beta}\right), \\ & + \frac{d+4}{6(d+2)}\Box V - \frac{2}{3}\frac{d+1}{d+2}\nabla^\alpha\nabla^\beta V_{(\alpha\beta)} + \frac{1}{4(d+2)}V^2 + \frac{1}{2(d+2)}V^{(\alpha\beta)}V_{(\alpha\beta)} \\ & + \frac{1}{6}VR - \frac{1}{3}V^{(\alpha\beta)}R_{\alpha\beta} - V^{[\alpha\beta]}W_{\alpha\beta} + \nabla^\alpha N_\alpha - 2X\Bigg], \end{aligned} \tag{D.21}$$

where $V = V^\alpha_\alpha$, $V^{(\alpha\beta)} = \frac{1}{2}\left(V^{\alpha\beta} + V^{\beta\alpha}\right)$, $V^{[\alpha\beta]} = \frac{1}{2}\left(V^{\alpha\beta} - V^{\beta\alpha}\right)$ and $W_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu + \left[w_\mu, w_\nu\right]$ is the bundle curvature. We now consider a particular case with $V^{\mu\nu} = Ag^{\mu\nu}$, $N^\mu = 0$, and we set A and X to be constants. We are also in De Sitter background in four dimensions, with Λ its cosmological constant. In this scenario the coefficients are after simplification

$$e_2(x) = \sqrt{\pi}\left[\frac{1}{6}\Lambda + \frac{1}{8}A\right] \tag{D.22}$$

$$e_4(x) = \frac{1}{4}\left[\frac{116}{135}\Lambda^2 + A^2 + \frac{4}{3}A\Lambda - 2X\right]. \tag{D.23}$$

We immediately notice on the left hand side we have constants, so that we can integrate to obtain

$$E_2 = \sqrt{\pi}\left[\frac{1}{6}\Lambda + \frac{1}{8}A\right]\Omega_4 \tag{D.24}$$

$$E_4 = \frac{1}{4}\left[\frac{116}{135}\Lambda^2 + A^2 + \frac{4}{3}A\Lambda - 2X\right]\Omega_4, \tag{D.25}$$

where $\Omega_4 = \int d^4x\sqrt{|g|}$. The terms contributing to divergences are those for which $n \leq d$ is in four dimensions we are interested only in coefficients $n = 0$, $n = 2$ and $n = 4$. We are also interested in non-minimal second order operators of the following form

$$-g^{\mu\nu}\Box + a\nabla^\mu\nabla^\nu + X^{\mu\nu}. \tag{D.26}$$

From [16] we know that for this operator the first two non-trivial coefficients in four dimensions are

$$e_{0\mu\nu} = g_{\mu\nu}\left[1 + \frac{a\,(2-a)}{4\,(1-a)^2}\right], \tag{D.27}$$

$$\begin{aligned} e_{2\mu\nu} = {} & g_{\mu\nu}R\frac{1}{6}\left(1 + \frac{a\,(2-a)}{4\,(1-a)^2}\right) - R_{\mu\nu}\frac{a\,(3a-4)}{12\,(1-a)^2} - W_{\mu\nu}\frac{a}{2\,(1-a)} \\ & - X_{(\mu\nu)}\left(1 + \frac{a\,(6-5a)}{12\,(1-a)^2}\right) - X_{[\mu\nu]}\left(1 + \frac{a}{2\,(1-a)}\right) - g_{\mu\nu}X^{\lambda}_{\lambda}\frac{a^2}{24\,(1-a)^2} \end{aligned} \tag{D.28}$$

In De Sitter background and with $X^{\mu\nu} = Bg^{\mu\nu}$ this reduces to

$$e_{0\mu\nu} = g_{\mu\nu}\left[1 + \frac{a\,(2-a)}{4\,(1-a)^2}\right], \tag{D.29}$$

$$e_{2\mu\nu} = g_{\mu\nu}\left[\Lambda\left(\frac{4}{6} + \frac{a\,(8-5a)}{12\,(1-a)^2}\right) - B\left(1 + \frac{a\,(2-a)}{4\,(1-a)^2}\right)\right]. \tag{D.30}$$

Again these are constants so we obtain for integrated values

$$E_{0\mu\nu} = g_{\mu\nu}\left[1 + \frac{a\,(2-a)}{4\,(1-a)^2}\right]\Omega_4, \tag{D.31}$$

$$E_{2\mu\nu} = g_{\mu\nu}\left[\Lambda\left(\frac{4}{6} + \frac{a\,(8-5a)}{12\,(1-a)^2}\right) - B\left(1 + \frac{a\,(2-a)}{4\,(1-a)^2}\right)\right]\Omega_4. \tag{D.32}$$

Conclusion

In this work we have familiarized ourselves with inflationary cosmology and observational bounds on inflationary models. We have also learned about $\zeta-$function regularization in curved spacetime, and related DeWitt-Seeley-Gilkey expansion.

We then quickly overviewed basic features of Weyl gravity and proceeded with exploration of inflationary model in the phase of broken scale symmetry. We found that the calculated values tensor-to-scalar ratio r and scalar tilt n_s are consistent with data from Planck 2018 observation, which makes the work promising. We also noted that due to the form of action we have a possible reheating (or preheating) channel in the form of inflaton decay $\psi \to \lambda + \lambda$. The study of details of this mechanism is a good topic for future work.

We followed by starting a work on calculating one-loop effective potential for VEV of field λ in De Sitter background. We have found relevant operators for this problem, however due to two remaining terms in the path integral which are not independent we were unable to finish the calculation. The form of partition function we have obtained is however makes it amendable to attempt simplification and so provides a good starting point for future work.

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