Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

A Survey on pp-wave Spacetimes

Analýza prostoročasů typu pp-vlny

Research task

Author: **Bc. Filip Garaj**

Supervisor: **prof. RNDr. Ladislav Hlavatý, DrSc.**

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- Zadání práce -

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V Praze dne 12. května, 2018 Bc. Filip Garaj

Název práce:

Analýza prostoročasů typu pp-vlny

Autor: Bc. Filip Garaj

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Vedoucí práce: prof. RNDr. Ladislav Hlavatý, DrSc., FJFI ČVUT, katedra fyziky

Abstrakt: Prvních několik oddílů tohoto textu se věnuje předběžným výsledkům jako jsou bivektory a Petrovova klasifikace. Tyto jsou potom užitečné pro účely poslední kapitoly, která si klade za cíl studovat některé typy prostoročasů. V této poslední kapitole představíme tři objekty (pp-vlna, zobecněná pp-vlna a rovinná vlna) a ukážeme, že jsou úzce spjaté. Podobnost tkví jak v explicitní formě zápisu v určitých vhodně zvolených souřadnicích, tak v algebraických/geometrických vlastnostech. V této práci je také zahrnuta sekce o tzv. Penroseových limitách, které v určitém smyslu přiřazují libovolnému metrickému tensoru pp-vlnu.

Klíčová slova: pp-vlna, zobecněná pp-vlna, bivektor, Petrovova klasifikace, Brinkmannovy souřadnice, Rosenovy souřadnice

Title:

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Abstract: We start our treatise by compiling various preliminary results on bivectors and Petrov classification, which both are useful for understanding the final chapter which is devoted to study various space-times. In the final chapter we introduce the three objects (pp-wave, generalized pp-wave and a plane wave) and show that they are closely related either by their explicit form in certain suitable coordinate systems or by their algebraic/geometric properties. In this text, there is also included a section on the so-called Penrose limits, which in some sense assign to a given metric tensor a pp-wave.

Key words: pp-wave, generalized pp-wave, plane wave, bivector, Petrov classification, Brinkmann coordinates, Rosen coordinates

Contents

Introduction

The aim of this work is to consistently and rigorously study definitions and properties of space-time metric tensors that are usually encountered under the names such as pp-wave, plane wave and generalized pp-wave. Since the use of these terms is anything but consistent throughout the available literature, one of the main goals is to find the relationship between the various definitions. We shall undergo the necessary heuristics later in order to clarify these, but here we offer the terminology for the three objects that we are concerned with the most, so that the possible reader is able to compare these definitions with what they are familiar with.

Firstly a *plane-wave spacetime* is a space-time manifold equipped with Lorentzian metric and there exists a global covariantly constant null vector field.

Secondly a $(d+2)$-dimensional space-time equipped with metric of a form

$$ds^2 = A_{ab}(u)x^a x^b du^2 + 2dudv + d\vec{x}^2 \qquad a, b = 1, \ldots, d$$

is called the *pp-wave in Brinkmann coordinates.* The "pp" that appears in the term pp-wave is an abbreviation for *plane fronted waves with parallel rays.* We discuss this further in chapter 4. There exist another coordinates, called the Rosen coordinates, that the pp-wave can be expressed in. In the same chapter we construct a transformation that allows us to transfer between the two coordinate systems.

Finally a $(d+2)$-dimensional space-time equipped with a metric of a form

$$ds^2 = 2H(u, x^a)du^2 + 2dudv + d\vec{x}^2 \qquad a = 1, \ldots, d$$

is called the *generalized pp-wave in Brinkmann coordinates.*

All three definitions above are stated again later with a useful context and they are also investigated in relationship with each other. This relationship can be summarized in a diagram

pp-wave	$\subset$	generalized pp-wave	$\subset$	plane wave

This means that every pp-wave is also a generalized pp-wave and that every generalized pp-wave is also a plane wave. Again we shall establish this later more formally.

There is also a somewhat less common definition of a generalized pp-wave and that is via the existence of a bivector (an antisymmetric tensor of second order) of certain properties. We adopted this definition because it is fully equivalent to our definition of a generalized pp-wave in Brinkmann coordinates and because (see section 4.2) it offers another interesting angle of view on this topic. In fact for a reader interested in properties of pp-waves only, the first three chapters may seem a little redundant, however the theory developed there is important for understanding the nuances in definitions of generalized pp-waves[1] and their relationship with the plane wave metric in order to comprehend fully what a pp-wave is.

[1]Also the author of this text has first encountered the generalized pp-waves in [2]. Then it seemed crucial to understand the geometry of bivectors and the Petrov classification.

Chapter 1

Geometric preliminaries

1.1 Differential geometry and conventions

Let (M, g) be an n-dimensional differentiable manifold. Here we specify our notation conventions for various geometrical objects derived from the metric tensor g. We use the standard notation for coordinate bases of tangent bundle resp. cotangent budle given as ∂_μ resp. dx^μ for coordinates $\{x^\mu\}$.

Remark 1.1.1. In this whole text we adopt the so-called *Einstein summation convention*, so that when the same indices appear, they are summed over, unless specified otherwise.

- Christoffel symbols for the Levi-Civita connection are

$$\Gamma^a_{bc} = \frac{1}{2} g^{am} (\partial_b g_{cm} + \partial_c g_{mb} - \partial_m g_{bc}) \qquad a, b, c, m = 1, \dots, n \tag{1.1}$$

- The Riemann tensor is

$$R^k{}_{lij} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il} \qquad i, j, k, l, m = 1, \dots, n \tag{1.2}$$

- The Ricci tensor is

$$\mathcal{R}_{lj} := \delta^i_k R^k{}_{lij} = R^i{}_{lij} \tag{1.3}$$

- The Ricci scalar is

$$\varrho := g^{lj} \mathcal{R}_{lj} = \mathcal{R}^j_j \tag{1.4}$$

The Weyl tensor

$$W_{lijk} := R_{lijk} + \frac{1}{n-2} \left[(\mathcal{R}_{ij} g_{kl} - \mathcal{R}_{ik} g_{jl} + \mathcal{R}_{lk} g_{ij} - \mathcal{R}_{lj} g_{ik}) + \frac{\varrho}{(n-1)} (g_{lj} g_{ki} - g_{lk} g_{ji}) \right] \tag{1.5}$$

The Weyl tensor satisfies the first Bianchi identity:

$$W_{abcd} + W_{adbc} + W_{acdb} = 0 \qquad \forall a, b, c, d = 1, \dots, n \tag{1.6}$$

and satisfies the following symmetries (simmilar to the Riemann tensor):

$$W_{lijk} = -W_{iljk} = -W_{likj} \qquad W_{lijk} = W_{jkli} \qquad \forall i, j, k, l = 1, \dots, n \tag{1.7}$$

Furthermore the Weyl tensor is trace-less, i.e.

$$W^l{}_{ilk} = 0 \qquad \forall i, k = 1, \dots, n \tag{1.8}$$

The totally antisymmetric Levi-Civita symbol is defined as

$$\epsilon_{l_1 l_2 \dots l_m} := \begin{cases} +1 & \text{if } (l_1 l_2 ... l_m) \text{ is an even permutation of } (12...m) \\ -1 & \text{if } (l_1 l_2 ... l_m) \text{ is an odd permutation of } (12...m) \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

We define a totally antisymmetric Levi-Civita tensor by putting

$$\varepsilon_{l_1 l_2 \dots l_m} := \sqrt{|\det g|}\ \epsilon_{l_1 l_2 \dots l_m} \tag{1.10}$$

Here, using the Levi-Civita tensor, we could generally define a global Hodge star operation, but since we shall need only its action on certain special spaces, we shall specify the less general definition when needed. Also we shall use the Hodge star only at a certain point $p \in M$, so that our manifold even need not be equipped with orientation. [1]

Remark 1.1.2. For an arbitrary 1-form ω the so-called Ricci identity is in effect:

$$\nabla_k \nabla_j \omega_i - \nabla_j \nabla_k \omega_i = R^l{}_{ijk} \omega_l \qquad \forall i, j, k \tag{1.11}$$

1.2 Minkowski geometry

In this section we provide useful definitions and theorems regarding the 4-dimensional Minkowski geometry. Although these facts are somewhat trivial, one might find some properties of Minkowski geometry rather nonintuitive, especially those matters that have to do with (pseudo-)orthogonality (with respect to a Lorentzian metric defined below). The knowledge contained in this paragraph shall prove fruitful for classifying bivectors (an object defined in the following chapter).

Definition 1.2.1. *Minkowski space M* is a 4-dimensional real vector space equipped with the so-called *Lorentz* non-degenerate bilinear form η, which is defined by

$$\eta(U, V) := \eta_{\mu\nu} U^\mu V^\nu \qquad \mu, \nu \in \{0, 1, 2, 3\}$$

where a basis can be chosen so that $\eta_{\mu\nu}$ are components of a diagonal matrix $\mathrm{diag}(-1, 1, 1, 1)$ and $U, V \in M$ have components U^μ and V^μ, respectively.

Definition 1.2.2. *Minkowski space-time* is a 4-dimensional vector space M regarded as a 4-dimensional differentiable manifold equipped with the global smooth Lorentz metric η whose components in the natural global chart (coordinate neighborhood is the whole vector space and the chart is the canonical isomorphism of 4-dimensional vector space to $\mathbb{R}^4$) on M are $\{\eta_{\mu\nu}\}_{\mu,\nu}$.

Remark 1.2.3. The tangent space at any point p of Minkowski space-time is Minkowski space when taken with the "inner product" $\eta|_p$. Since these two concepts are so closely related, we have used M to denote both.

Definition 1.2.4. A non-zero member V of Minkowski space is called, respectively, *spacelike*, *timelike* or *null* according as $\eta(V, V)$ is positive, negative or zero.
We denote $\mathcal{S}, \mathcal{T}$ and $\mathcal{N}$, respectively, the subsets of $\mathbb{R}^4$ consisting of all spacelike, timelike and null vectors.

Remark 1.2.5. We have obtained a disjoint decomposition (classification) of Minkowski space in the form

$$\mathbb{R}^4 = \mathcal{S} \cup \mathcal{T} \cup \mathcal{N} \cup \{0\}$$

where 0 is the zero vector of M.

Definition 1.2.6. A 1-dimensional subspace (henceforth called a 1-*space* or a *direction*) of Minkowski space is called *spacelike*, *timelike* or *null*, respectively, if it is spanned by a spacelike, timelike or null vector. The collections of such subspaces are denoted by $\mathcal{S}_1, \mathcal{T}_1$ and $\mathcal{N}_1$.

Definition 1.2.7. Let us define:

- Two vectors X, Y in Minkowski space are called *orthogonal*[1] if $\eta(X, Y) = 0$
- A vector V in Minkowski space is called a *unit* vector if $|\eta(V, V)| = 1$.
- Four linearly independent vectors in Minkowski space are reffered to as *tetrad*.
- A basis $\{E_j\}_{j=1}^4$ of Minkowski space consisting of four mutually orthogonal unit vectors is called an *orthonormal basis* (or *tetrad*).
- A basis $\{F_j\}_{j=1}^4$ of Minkowski space is called a *(real) null basis* (or *tetrad*) if the only non-vanishing "inner products" between the members of $\{F_j\}_{j=1}^4$ are

  $$\eta(F_3, F_3) = \eta(F_4, F_4) = \eta(F_1, F_2) = 1$$

- A set of four vector fields $\{L, N, M, \overline{M}\}$, where L and N are real, M and $\overline{M}$ are complex conjugates ($M := M_1 + iM_2$, $\overline{M} := M_1 - iM_2$ for M_1, M_2 real) and the only non-zero "inner products"

  $$\eta(L, N) = \eta(M, \overline{M}) = 1$$

  is called *Newman-Penrose* (or *complex null*) tetrad.

Proposition 1.2.8. [7] Let us consider Minkowski space. Then

[1] In the following text we shall use the term orthogonality exclusively, although pseudo-orthogonality would be more accurate. This prefix "pseudo" arises from η not being positive definite (as it is expected from a regular well-behaved inner product) and it actually accounts for most of the differences from the standard Euclidean geometry.

1. No two linearly independent null vectors are orthogonal. With the exception of λX, where λ is a scalar, every vector orthogonal to a given null vector X is spacelike.

2. No two timelike vectors are orthogonal. Any vector, which is orthogonal to a timelike vector, must necessarily be spacelike.

3. A vector, which is orthogonal to a spacelike vector, can be spacelike, timelike or null.

Proof. In the $(1+3)$-notation, where $\{X^\mu\} = (X^0, \vec{X})$:

1. Let X be a given null vector and Y some other vector orthogonal to X. Therefore we have conditions[2]:

$$\eta(X,X) = 0 \Leftrightarrow |X^0| = \|\vec{X}\| \qquad \text{and} \qquad \eta(X,Y) = 0 \Leftrightarrow X^0Y^0 = \vec{X}\cdot\vec{Y}$$

We can further compute, using the standard Cauchy-Schwarz inequality

$$|Y^0| = \frac{|X^0|\,|Y^0|}{|X^0|} = \frac{|X^0Y^0|}{|X^0|} = \frac{|\vec{X}\cdot\vec{Y}|}{|X^0|} \leq \frac{\|\vec{X}\|\|\vec{Y}\|}{|X^0|}$$

Now since X is null, we have $\|\vec{X}\| = |X^0|$, so that $|Y^0| \leq \|\vec{Y}\|$ and hence Y is either spacelike or null. When Y is null, the equations $\eta(X,X) = \eta(Y,Y) = 0$ and $\eta(X,Y) = 0$ imply

$$\vec{X}\cdot\vec{Y} = X^0Y^0 = \pm\|\vec{X}\|\|\vec{Y}\|$$

Comparing this with the standard formula for the Euclidean inner product $\vec{X}\cdot\vec{Y} = \|\vec{X}\|\|\vec{Y}\|\cos\vartheta$, where $\vartheta \in [0,\pi]$ is the angle between vectors $\vec{X},\vec{Y}$, we obtain the fact, that the angle between the space parts of X and Y is either 0 or π. Ergo $\vec{Y} = \lambda\vec{X}$, where $\lambda \in \mathbb{R}$ and using this in the orthogonality condition, we get

$$X^0Y^0 = \vec{X}\cdot\vec{Y} = \lambda(\vec{X}\cdot\vec{X}) = \lambda(X^0)^2 \qquad \Longrightarrow \qquad Y^0 = \lambda X^0$$

Hence $Y = \lambda X$ if Y is null and the proof is now complete.

2. Since we are again to discuss vectors orthogonal to a given vector, we may use the condition $X^0Y^0 = \vec{X}\cdot\vec{Y}$ and the Cauchy-Schwarz inequality similarly to the previous case, to obtain

$$|Y^0| \leq \frac{\|\vec{X}\|\|\vec{Y}\|}{|X^0|}$$

Here X is timelike, hence $|X^0| > \|\vec{X}\|$ which along with the previous condition gives

$$|Y^0| < \|\vec{Y}\|$$

and thus the vector Y has to be spacelike. This proves the second assertion. The first assertion, that no two timelike vectors are orthogonal, then also follows.

[2]In the following, the dot "$\cdot$" between space parts of two four-vectors denotes the standard Euclidean inner product $\vec{X}\cdot\vec{Y} := \sum_{j=1}^{3} X^jY^j$ and the double line $\left\|\vec{X}\right\| := \sqrt{\vec{X}\cdot\vec{X}}$ denotes the norm induced by this inner product.

3. For spacelike vector there arise no such restrictions (as in the other cases above), when we combine the required conditions.

□

Lemma 1.2.9. [7] Any linearly independent tetrad of mutually orthogonal vectors must necesarilly consist of three spacelike vectors and one timelike vector.

Remark 1.2.10. Apparently one member of $\{E_j\}_{j=1}^4$ is necessarily timelike and the other three are spacelike. Similarly for $\{F_j\}_{j=1}^4$ we know, that F_1 and F_2 are null and F_3 and F_4 are unit spacelike vectors.

Lemma 1.2.11. Let us consider Minkowski space. Then

1. Given any two timelike vectors T_1, T_2 there exist a spacelike (or zero vector in case T_1, T_2 are linearly dependent) vector S for which $\eta(T_2, S) = 0$ and a real number ζ such that T_1 and T_2 are bound by the following relation

$$T_1 = \zeta T_2 + S$$

2. Timelike vectors T_1, T_2 satisfy the "*Reversed Cauchy-Schwarz inequality*":

$$[\eta(T_1, T_2)]^2 \geq \eta(T_1, T_1)\eta(T_2, T_2) \tag{1.12}$$

where the equality holds if and only if T_1, T_2 are linearly dependent.

3. It is possible to obtain a null vector as a linear combination of two timelike vectors.

Proof. Let us prove:

1. In order to find such ζ we are to calculate

$$\eta(T_2, T_1) = \eta(T_2, \zeta T_2 + S) = \zeta \underbrace{\eta(T_2, T_2)}_{\neq 0} + \underbrace{\eta(T_2, S)}_{=0} \qquad \Longrightarrow \qquad \zeta = \frac{\eta(T_2, T_1)}{\eta(T_2, T_2)}$$

The vector S is obviously zero or spacelike, since the zero vector is orthogonal to every member of Minkowski space and the only non-zero vectors, that are orthogonal to a certain timelike vector, are spacelike.

2. Given two timelike vectors T_1, T_2 we are to use the previous point of this lemma, that has been already proven. We have $\zeta \in \mathbb{R}$ and S spacelike or zero (in case of linear dependency of T_1, T_2) such that $\eta(T_2, S) = 0$ and $T_1 = \zeta T_2 + S$. Obviously

$$\eta(T_1, T_1) = \zeta^2 \eta(T_2, T_2) + \eta(S, S)$$

Now making use of that, let us calculate

$$\begin{aligned}
[\eta(T_1, T_2)]^2 &= [\zeta\eta(T_2, T_2) + \eta(S, T_2)]^2 = \zeta^2[\eta(T_2, T_2)]^2 \\
&= [\eta(T_1, T_1) - \eta(S, S)]\eta(T_2, T_2) = \eta(T_2, T_2)\eta(T_1, T_1) \underbrace{-\eta(S, S)\eta(T_2, T_2)}_{\geq 0} \geq \\
&\geq \eta(T_2, T_2)\eta(T_1, T_1)
\end{aligned}$$

Here apparently the inequality is strict in case of S being a spacelike vector and the equality holds in case of S being the zero vector of Minkowski space. Thus the equality coincides with T_1 and T_2 being linearly dependent of each other.

3. We construct a non-trivial linear combination out of $M, B \in \mathcal{T}$

$$\alpha M + \beta B \qquad \alpha, \beta \in \mathbb{R}, \alpha, \beta \neq 0$$

We are to look for possible restrictions imposed on the coefficients α, β and enquire whether these restrictions allow for a null vector to exist as the aforementioned linear combination. The linear combination can be rewritten in a trivial way (after dividing by β), for it represents a member of the same direction

$$\lambda M + B \qquad \lambda \in \mathbb{R}$$

In order for this linear combination to be a null vector, the following condition must be satisfied.

$$0 = \eta(\lambda M + B, \lambda M + B) = \lambda^2 \eta(M, M) + 2\lambda \eta(M, B) + \eta(B, B)$$

We have obtained a quadratic equation in terms of λ. A real solution of this equation exists if and only if its discriminant is nonnegative. In our case

$$[2\eta(M, B)]^2 - 4\eta(M, M)\eta(B, B) \geq 0$$

After dividing by four and rearranging, this becomes exactly the "*Reversed Cauchy-Schwarz inequality*", that has been already proven.

□

Definition 1.2.12. A 2-dimensional subspace of Minkowski space is called

- *spacelike* if it contains no null vectors
- *null* if all null vectors contained in it are confined to a single null direction (i.e. are linearly dependent)
- *timelike* if all null vectors contained in it are confined to two distinct directions

Collections of these subspaces are denoted $\mathcal{S}_2, \mathcal{N}_2, \mathcal{T}_2$ respectively.

Theorem 1.2.13. The classification of 2-dimensional subspaces introduced in the previous definition is exhaustive.

Proposition 1.2.14. For the classification of 2-spaces, we have

1. Every non-zero vector in any member of $\mathcal{S}_2$ is spacelike.
2. Any member of $\mathcal{T}_2$ contains spacelike, timelike and null vectors.

3. Any member of $\mathcal{N}_2$ contains a unique null direction and all other vectors in it are spacelike and orthogonal to any non-zero vector in this null direction.

Proof. Let us proceed:

1. We discuss the possible combinations for two basis vectors of an arbitrary spacelike 2-space. Firstly, two timelike vectors are prohibited by the previous lemma 1.2.11. Secondly a combination of a timelike vector T and a spacelike vector S is prohibited because the quadratic equation in λ

$$0 = \eta(T + \lambda S, T + \lambda S) = \underbrace{\eta(T,T)}_{<0} + 2\lambda\eta(T,S) + \lambda^2 \underbrace{\eta(S,S)}_{>0}$$

   has always a positive discriminant, ergo two real solutions, which give rise to a null vector.

   Finally the remaining case is of two spanning spacelike vectors S_1, S_2. We search for possible conditions on λ to satisfy, in order for general linear combination $S_1 + \lambda S_2$ not to be a null vector. Apparently:

$$0 \neq \eta(S_1 + \lambda S_2, S_1 + \lambda S_2) = \eta(S_1, S_1) + 2\lambda\eta(S_1, S_2) + \lambda^2\eta(S_2, S_2)$$

   This is a quadratic equation in λ and for it not to have a real solution, the discriminant needs to be negative:

$$[\eta(S_1, S_2)]^2 - \eta(S_1, S_1)\eta(S_2, S_2) < 0 \tag{1.13}$$

   This resembles in form the Cauchy-Schwarz inequality (except that the inequality sign is strict). Thus all pairs of orthogonal spacelike vectors are acceptable for spanning the spacelike 2-space and by construction their linear combination can amount only to another spacelike vector. The same holds true for pairs of non-orthogonal spacelike vectors, only they need to satisfy the Cauchy-Schwarz inequality above[3].

2. We will show that it is possible to obtain spacelike or timelike vector as a linear combination of two linearly independent null vectors N_1, N_2. That amounts to solving the following inequality for $\lambda \in \mathbb{R}$

$$0 \overset{?}{\lessgtr} \eta(N_1 + \lambda N_2, N_1 + \lambda N_2) = \underbrace{\eta(N_1, N_1)}_{=0} + 2\lambda\eta(N_1, N_2) + \lambda^2 \underbrace{\eta(N_2, N_2)}_{=0}$$

   Here $\eta(N_1, N_2)$ is never zero, since we demanded N_1, N_2 to be linearly independent. The given inequality is clearly satisfied, no matter the sign of $\eta(N_1, N_2)$; it can be compensated for by the choice of $\lambda \in \mathbb{R}$.

[3]It is possible to obtain a null vector as a linear combination of two non-orthogonal spacelike vectors. As an example we consider $U = (1, 0, 0, 2)^T$, $V = (2, 0, 0, 3)^T$, where $U - V$ is obviously null. The vectors U, V violate the Cauchy-Schwarz inequality (1.13).

3. Follows immediately from the proposition 1.2.8 and the fact, that by definition, one of the spanning vectors is null.

□

Definition 1.2.15. The unique null direction of any $V \in \mathcal{N}_2$ is called the *principal null direction of* V. The two null directions in any memeber of $\mathcal{T}_2$ are reffered to as its *principal null directions.*

Lemma 1.2.16. There exist at most two mutually orthogonal, linearly independent spacelike vectors which are both orthogonal to a given null vector.

Proof. Any three mutually orthogonal, linearly independent spacelike vectors together with a null vector to which are all three of them orthogonal form a tetrad. Such tetrad is prohibited by lemma 1.2.9. □

Theorem 1.2.17. If $V \in \mathcal{N}_2$ with a principal null direction N, then $V^\perp \in \mathcal{N}_2$ and its principal null direction is also N, so that

$$V \cap V^\perp = \{N\}$$

Proof. Let $V \in \mathcal{N}_2$ be a 2-space spanned by two vectors - null N and spacelike S, which are according to point 3. of proposition 1.2.14 orthogonal. Now the orthogonal complement $V^\perp$ is the set

$$V^\perp = \{X \in M | \eta(X, Y) = 0, \forall Y \in V\}$$

Apparently the vector N itself is a memeber of $V^\perp$ as it is orthogonal to itself and to the vector S as well. Now since N is also a member of the original 2-space V, all other vectors in $V^\perp$ must be orthogonal to it. Hence $V^\perp$ cannot contain timelike vectors and any other null vector is linearly dependent to vector N. Furthermore, from lemma 1.2.16, there can only be two mutually orthogonal spacelike vectors orthogonal to a given null vector (in this case N). Ergo, $V^\perp$ is a 2-space generated by N and one other orthogonal spacelike vector. The assertion follows. □

Theorem 1.2.18. If $W \in \mathcal{T}_2$, then $W^\perp \in \mathcal{S}_2$ and the spanning vectors of W and $W^\perp$ together form a basis for Minkowski space.

Proof. The 2-space W is spanned by two linearly independent (non-orthogonal) null vectors N_1, N_2. Its orthogonal complement can contain only spacelike and null vectors by 1.2.8. However the inclusion of null vectors is prohibited, by the fact, that a member of an orthogonal complement has to be orthogonal (linearly dependent) to both, N_1 and N_2 which is impossible by our assumption of them being linearly independent. Hence $W^\perp$ is spanned by two (see lemma 1.2.16) spacelike vectors. □

1.3 Space-time, Einstein equations

Above we have developed a language that is natural for special theory of relativity, however the real physical world consists of accelerating and gravitating objects which are described by general theory of relativity for which the description by Minkowski space-time fits only locally.

Definition 1.3.1. A *space-time manifold* is a pair (M, g) where M is an n-dimensional differentiable, connected, Hausdorff manifold and g a smooth metric on M with Lorentz signature[4].

Remark 1.3.2. Sometimes we will use particular space-time dimension $n = 4$ which is characteristic for general theory of relativity. This theory is also characterized by imposing the *Einstein field equations*[5]:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\varrho g_{\mu\nu} = \frac{8\pi\varkappa}{c^4}T_{\mu\nu} \qquad \mu,\nu = 1,\ldots,4 \tag{1.14}$$

Here $\varkappa$ is Newton's gravitational constant, c is the speed of light in vacuum and $T_{\mu\nu}$ is the usual stress-energy tensor. We shall specify when we are using the dimension $n = 4$ and/or the Einstein equations.

1.4 Tensor symmetries

In this section we present some results concerning algebraical behavior of tensors with respect to action of the Hodge star on them. The set of tensors in question is taken to be satisfying the set of symmetries that the Riemann and the Weyl tensors satisfy, so that these results do apply to them. We shall not give any proofs in this section, for this topic is well-covered by other publications and the proofs usually follow from the straightforward calculation of appropriate contractions of the Levi-Civita tensors. All the results presented below are valid for a 4-dimensional space-time manifold, for we shall only need this special case in the following chapter on bivector analysis.

Definition 1.4.1. We define $\mathscr{W}|_p$ to be the set of $(0,4)$ tensors at $p \in M$, that satisfy the following algebraic symmmetries ($T \in \mathscr{W}|_p$):

$$T_{\alpha\beta\gamma\delta} = -T_{\beta\alpha\gamma\delta} = -T_{\alpha\beta\delta\gamma} \tag{1.15}$$

$$T_{\alpha\beta\gamma\delta} = T_{\gamma\delta\alpha\beta} \tag{1.16}$$

$$T_{\alpha\beta\gamma\delta} + T_{\alpha\delta\beta\gamma} + T_{\alpha\gamma\delta\beta} = 0 \tag{1.17}$$

[4]In this text, we shall use the *Lorentz signature* $(-1, \underbrace{1,\ldots,1}_{(n-1)\text{ times}})$ exclusively.

[5]In these equations there is sometimes and additional term $+\Lambda g_{\mu\nu}$ on the left-hand side, where Λ denotes the so-called cosmological constant. We set $\Lambda = 0$ throughout the whole text.

Definition 1.4.2. For every $T \in \mathscr{W}|_p$ we define the *left (Hodge) dual* $(\star T)$ and the *right (Hodge) dual* $(T\star)$ as follows[6]:

$$(\star T)_{\alpha\beta\gamma\delta} := \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}T^{\mu\nu}{}_{\gamma\delta} \qquad (T\star)_{\alpha\beta\gamma\delta} := \frac{1}{2}T_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\gamma\delta\mu\nu}$$

Lemma 1.4.3. For $T \in \mathscr{W}|_p$ and the left and the right dual, we have

$$(\star(\star T))_{\alpha\beta\gamma\delta} = -T_{\alpha\beta\gamma\delta} \qquad ((T\star)\star)_{\alpha\beta\gamma\delta} = -T_{\alpha\beta\gamma\delta}$$

Theorem 1.4.4. For $T \in \mathscr{W}|_p$ we have

$$\begin{aligned}(\star T\star)_{\alpha\beta\gamma\delta} = -T_{\alpha\beta\gamma\delta} &+ g_{\alpha\gamma}\left(T_{\delta\beta} - \frac{1}{4}Tg_{\delta\beta}\right) - g_{\alpha\delta}\left(T_{\gamma\beta} - \frac{1}{4}Tg_{\gamma\beta}\right) \\ &+ g_{\beta\delta}\left(T_{\alpha\gamma} - \frac{1}{4}Tg_{\alpha\gamma}\right) - g_{\beta\gamma}\left(T_{\alpha\delta} - \frac{1}{4}Tg_{\alpha\delta}\right)\end{aligned} \tag{1.18}$$

where $T_{\alpha\beta} := T^{\xi}{}_{\alpha\xi\beta}$ and $T := T^{\alpha}{}_{\alpha}$.

Corollary 1.4.5. For $T \in \mathscr{W}|_p$ the following statements are equivalent:

1. $(\star T\star)_{\alpha\beta\gamma\delta} = -T_{\alpha\beta\gamma\delta}$
2. $(\star T)_{\alpha\beta\gamma\delta} = (T\star)_{\alpha\beta\gamma\delta}$
3. $T_{\alpha\beta} - \frac{1}{4}Tg_{\alpha\beta} = 0 \qquad \forall\alpha,\beta$

Lemma 1.4.6. Weyl tensor W is a member of $\mathscr{W}|_p$, hence all the results of this section do apply to it.

Lemma 1.4.7. For the left and the right dual of Weyl tensor we have

$$(\star W)_{\alpha\beta\gamma\delta} = (W\star)_{\alpha\beta\gamma\delta} \tag{1.19}$$

Proof. The statement is obvious once we consider the equivalence of points 2. and 3. in corollary 1.4.5 and the trace-less property of Weyl tensor (equation (1.8)). □

Definition 1.4.8. We define *complex self-dual Weyl tensor at a point $p \in M$*

$$\hat{W} := W + i(\star W)$$

Remark 1.4.9. Tensor $\hat{W}$ is self-dual in a sense that $(\star\hat{W}) = -i\hat{W}$, indeed (using lemma 1.4.3) we have:

$$i(\star\hat{W}) = i(\star W) + (-i^2)(-W) = W + i(\star W) = \hat{W} \quad \Leftrightarrow \quad (\star\hat{W}) = -i\hat{W}$$

Later (in the next chapter on bivectors) we shall understand why this property is of interest.

Complex and real parts of self-dual Weyl tensor $\hat{W}$ satisfy the required symmetries, thus all the results of this section do apply to $\hat{W}$ as well.

[6] Because our work is confined to an arbitrary, yet fixed point $p \in M$, we need not to introduce orientation on the space-time manifold for the dual operations to work.

Chapter 2

Bivectors

In this chapter, we directly apply many of the results of the preceding chapter. The geometrical properties of vectors and especially the classification of 2-dimensional subspaces contained in there facilitates greatly the classification (and understanding) of bivectors. Motivation for including a study of bivectors in this text is the fact that the geometry of bivectors is closely related to a certain approach to description of generalized pp-waves (see chapter 4). Specifically, the theory concerning null bivectors shall be made good use of later in this text.

2.1 Definitions

Definition 2.1.1. Let (M, g) be a space-time. A $(2, 0)$ skew-symmetric tensor F (at $p \in M$) with components $F^{\alpha\beta}$ is called a *bivector* (at p).

Remark 2.1.2. The set $\mathscr{B}|_p$ of all bivectors at point p is isomorphic to a 6-dimensional real vector space.

Remark 2.1.3. We use the Hodge dual on $F \in \mathscr{B}|_p$ to obtain a dual $(0, 2)$ tensor of F:

$$(\star F)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$$

Here we list some identities for this duality operation without proofs, as these present a more or less straightforward excercise in tensor algebra:

Proposition 2.1.4. Let $F, H \in \mathscr{B}|_p$ then

$$(\star\star F)_{\mu\nu} := (\star(\star F))_{\mu\nu} = -F_{\mu\nu}$$

$$F_{\mu\nu}F^{\mu\nu} = -(\star F)_{\mu\nu}(\star F)^{\mu\nu} \tag{2.1}$$

$$H^{\mu\nu}F_{\nu\lambda} - (\star F)^{\mu\nu}(\star H)_{\nu\lambda} = \frac{1}{2}F^{\varrho\sigma}H_{\varrho\sigma}\delta^{\mu}_{\lambda} \tag{2.2}$$

Remark 2.1.5. According to theorem A.1.1 the rank of the corresponding matrix $F^{\alpha\beta}$ for $F \neq 0$ is equal to two or four.

Definition 2.1.6. Bivector F is called *simple* if the rank of $F^{\alpha\beta}$ is 2 and *non-simple* if the rank of $F^{\alpha\beta}$ is 4.

Theorem 2.1.7. The following statements are equivalent for a non-zero $F \in \mathscr{B}|_p$

1. F is simple
2. There exist independent $X, Y \in T_pM$ such that
$$F^{\mu\nu} = X^\mu Y^\nu - X^\nu Y^\mu$$
3. There exists $\omega \in T_p^*M$, $\omega \neq 0$ such that
$$F^{\mu\nu}\omega_\nu = 0$$

Proof. We prove this in a series of implications:

- (1. $\implies$ 2.) Here $F_{\mu\nu}$ is represented by a (4×4) skew-symmetric matrix whose rank is equal to two. According to linear algebra it is possible to find a basis such that the matrix of $F_{\mu\nu}$ in this basis has two rows and two columns consisting entirely of zeros. We are to use the outer product (see A.1.3) to find the two vectors. Using
$$\begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} U^1 \\ U^2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} T^1 & T^2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} T^1 \\ T^2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} U^1 & U^2 & 0 & 0 \end{pmatrix}$$
where b is a non-zero real number for $F_{\mu\nu}$ to be of rank 2, we find
$$b = U^1T^2 - T^1U^2 \qquad \text{and} \qquad -b = U^2T^1 - T^2U^1$$
This can be rewritten, so that
$$U^1 = \frac{T^1U^2 + b}{T^2} \qquad \text{and} \qquad U^2 = \frac{T^2U^1 + b}{T^1}$$
We choose T^1 and T^2 so that they are non-zero and therefore fully determine the vector T. That leaves a system of two linear equations for two unknown components of the vector U which obviously has a solution. In order to verify that U and T are linearly independent, we assume the opposite. Let
$$U^1 = \lambda T^1 \qquad \text{and} \qquad U^2 = \lambda T^2 \qquad \lambda \in \mathbb{R}$$
then from the previous expression of U^1, we have
$$\lambda T^1 = \frac{T^1(\lambda T^2) + b}{T^2} \qquad \implies \qquad \frac{b}{T^2} = 0$$
which is impossible, for we have chosen b to be non-zero. Formally, we have the existence of linearly independent $U, T \in T_pM$, such that
$$F^{\mu\nu} = U^\mu T^\nu - T^\mu U^\nu$$

- (2. $\implies$ 3.) Let us consider the 2-space spanned by the two vectors X and Y. We choose an arbitrary vector P orthogonal to this 2-space, so that

$$g_{\mu\nu}P^\mu X^\nu = g_{\mu\nu}P^\mu Y^\nu = 0$$

Now we calculate

$$F^{\alpha\beta}g_{\beta\lambda}P^\lambda = X^\alpha \underbrace{Y^\beta g_{\beta\lambda}P^\lambda}_{=0} - Y^\alpha \underbrace{X^\beta g_{\beta\lambda}P^\lambda}_{=0} = 0$$

We define $\omega_\nu := g_{\nu\lambda}P^\lambda$, which clearly exists.

- (3. $\implies$ 1.) The existence of a non-zero $\omega \in T^*_p M$ such that $F^{\alpha\beta}\omega_\beta = 0$ implies that the rank of $F^{\alpha\beta}$ is strictly less than four. Because the rank of $F^{\alpha\beta}$ has to be even number (see A.1.1), it is necessarily a simple bivector.

□

Definition 2.1.8. The 2-dimensional subspace of T_pM spanned by the two vectors corresponding to a simple bivector via previous theorem is called the *blade of a bivector.*

Definition 2.1.9. A bivector at a point p is called *spacelike, null* or *timelike* according as the corresponding blade of the bivector is spacelike, null, or timelike 2-space at p, respectively.

Corollary 2.1.10. Bivector $F \in \mathscr{B}|_p$ is simple if and only if $(\star F)$ is simple and then their blades are orthogonal complements of each other.

Proof. We use the equivalence in theorem 2.1.7, to prove both directions:

- ($\implies$) We assume the existence of $X, Y \in T_pM$ such that

$$F^{\gamma\delta} = X^\gamma Y^\delta - Y^\gamma X^\delta$$

Let us calculate

$$(\star F)_{\alpha\beta}Y^\beta = \varepsilon_{\alpha\beta\gamma\delta}(X^\gamma Y^\delta - Y^\gamma X^\delta)Y^\beta = \varepsilon_{\alpha\beta\gamma\delta}X^\gamma Y^\delta Y^\beta - \varepsilon_{\alpha\beta\gamma\delta}X^\delta Y^\gamma Y^\beta = 0$$

where we have used the fact that ε is a totally antisymmetric tensor and $Y^\mu Y^\nu$ is symmetric in the two indices. We found a non-zero 1-form $\omega_\gamma := Y^\beta g_{\gamma\beta}$ so that $(\star F)^{\alpha\beta}\omega_\beta = 0$. Here we have chosen vector Y to perform our calculations, but we might as well have chosen vector X with the same result.

- ($\impliedby$) Here we can reverse the preceding argument; we have the existence of two vectors X, Y so that $(\star F)^{\mu\nu} = X^\mu Y^\nu - X^\nu Y^\mu$. Now, we calculate:

$$F_{\mu\nu}Y^\nu = -(\star(\star F))_{\mu\nu}Y^\nu = -\varepsilon_{\mu\nu\alpha\beta}(\star F)^{\alpha\beta}Y^\nu = 0$$

Using the dual pairing again yields the sought after 1-form.

□

Lemma 2.1.11. Bivector $F \in \mathscr{B}|_p$ is simple if and only if (F is non-vanishing and)

$$(\star F)_{\mu\nu} F^{\nu\lambda} = 0$$

Proof. We prove two directions of this equivalence relation:

- ($\Longrightarrow$) Let bivector F be simple. From corollary 2.1.10 we have that this is equivalent to $(\star F)$ being simple, thus by theorem 2.1.7 there are two corresponding blades. Further (again by 2.1.7) these are orthogonal, i.e.

  $$\{X, Y\} \perp \{U, V\}$$

  Therefore, it is possible to express:

  $$(\star F)_{\mu\nu} F^{\nu\lambda} = (U_\mu V_\nu - V_\mu U_\nu)(X^\nu Y^\lambda - Y^\nu X^\lambda) = 0$$

- ($\Longleftarrow$) Let $(\star F)_{\mu\nu} F^{\nu\lambda} = 0$ hold. Then because F is non-vanishing (then also $(\star F)$ is non-vanishing), there exists a non-zero $K \in T_pM$ such that $(\star F)_{\mu\nu} K^\mu \neq 0$. We found a non-zero member of the covector space $\omega_\nu := (\star F)_{\mu\nu} K^\mu$, for which

  $$F^{\nu\lambda} \omega_\nu = \underbrace{(\star F)_{\mu\nu} F^{\nu\lambda}}_{=0} K^\mu = 0$$

  The necessary and sufficient condition 3. in theorem 2.1.7 has been satisfied, ergo F is simple.

□

Corollary 2.1.12. Bivector $F \in \mathscr{B}|_p$ is simple if and only if (F is non-vanishing and)

$$(\star F)_{\mu\nu} F^{\mu\nu} = 0$$

Proof. This is a trivial corollary to the previous lemma if we take equality (2.2) into account, where $H := (\star F)$. □

Remark 2.1.13. For any simple bivector, when written in the form 2. of the previous theorem 2.1.7 the vectors X, Y may be chosen to be orthogonal without the loss of generality.

Theorem 2.1.14. The bivector $F \in \mathscr{B}|_p$ is null if and only if

$$F_{\alpha\beta} F^{\alpha\beta} = (\star F)_{\alpha\beta} F^{\alpha\beta} = 0$$

Proof. We prove two directions of this equivalence relation in two implications:

- ($\Longrightarrow$) Let F be a null bivector; then it is necessarily simple and satisfies the condition $(\star F)_{\alpha\beta} F^{\alpha\beta} = 0$ via theorem 2.1.7. Bivector F being null, there is the corresponding blade spanned by a null vector X and a spacelike vector Y, that are orthogonal to each other, and also $F^{\alpha\beta} = X^\alpha Y^\beta - X^\beta Y^\alpha$. Thus

  $$F_{\alpha\beta} F^{\alpha\beta} = 2 X_\alpha X^\alpha Y_\beta Y^\beta - 2(X_\alpha Y^\alpha)^2$$

  Where

  $$X_\alpha X^\alpha = 0, X_\alpha Y^\alpha = 0 \qquad \Longrightarrow \qquad F_{\alpha\beta} F^{\alpha\beta} = 0$$

- ($\Longleftarrow$) Let us assume that $F_{\alpha\beta}F^{\alpha\beta} = (\star F)_{\alpha\beta}F^{\alpha\beta} = 0$. The second equality is equivalent to F being simple. We shall prove that it is a null bivector. Because F is simple, $(\star F)$ is also simple and both can be written in the form:

$$F^{\alpha\beta} = X^\alpha Y^\beta - X^\beta Y^\alpha, \qquad (\star F)^{\alpha\beta} = \tilde{X}^\alpha \tilde{Y}^\beta - \tilde{X}^\beta \tilde{Y}^\alpha,$$

where $X^\alpha Y_\alpha = 0$ and $\tilde{X}^\alpha \tilde{Y}_\alpha = 0$. Using the equality (2.1) and the orthogonality in algebraic steps analogous to those in the previous implication in the assumed condition $F_{\alpha\beta}F^{\alpha\beta} = 0$ we arrive at

$$(X_\alpha X^\alpha)(Y_\beta Y^\beta) = (\tilde{X}_\alpha \tilde{X}^\alpha)(\tilde{Y}_\beta \tilde{Y}^\beta) = 0$$

From this exactly one of the vectors in each pair has to be null (because X is orthogonal to Y, they cannot be simultaneously linearly independent and orthogonal and the same holds for $\tilde{X}, \tilde{Y}$ - see the first statement in proposition 1.2.8), without the loss of generality let us choose X and $\tilde{X}$ to be these null vectors. Because blades of F and $(\star F)$ are orthogonal to each other (corollary 2.1.10) we have that (without the loss of generality) X and $\tilde{X}$ are orthogonal null vectors. Because of that, they have to be members of the same direction, morover, Y and $\tilde{Y}$ both have to be spacelike vectors orthogonal to this null direction while simultaneously being orthogonal to each other. For F and $(\star F)$ we have found the corresponding 2-spaces (blades) with exactly one null direction in each (the same in both). Bivectors F and $(\star F)$ are then null by definition.

□

Proposition 2.1.15. The dual of a timelike bivector is spacelike and vice versa. The dual of a null bivector is null.

Proof. Follows immediately from theorems 1.2.17, 1.2.18 and corollary 2.1.10. □

2.2 Complex bivectors

Definition 2.2.1. We define $\mathscr{CB}|_p := \mathscr{B}|_p \oplus \mathscr{B}|_p$ where[1]

$$\mathscr{CB}|_p \ni (F_1, F_2) =: F_1 + iF_2 \qquad F_1, F_2 \in \mathscr{B}|_p$$

Furthemore we define a complex structure[2] on $\mathscr{CB}|_p$:

$$f : \mathscr{CB}|_p \to \mathscr{CB}|_p : F_1 + iF_2 \mapsto -F_2 + iF_1$$

By identifying f with the multiplication by the imaginary unit i, we have introduced a *complex vector space of bivectors at* $p \in M$.

[1]The direct sum of two copies of the same vector space naturally has an even dimension. Below it can be seen, that the idea of complexification is to start with a direct sum of two copies of a given (real) vector space and then proceed to extend the field from $\mathbb{R}$ to $\mathbb{C}$. For this extension, the even dimension of the direct sum vector space is important, for it is divided by two in this process.

[2]A complex structure on a vector space $W := V \oplus V$ is a map $f : W \to W : (v_1, v_2) \mapsto (-v_2, v_1)$ for $v_1, v_2 \in V$.

Remark 2.2.2. Obviously

$$\dim \mathscr{C}\mathscr{B}|_p = \dim \mathscr{B}|_p + \dim \mathscr{B}|_p = 12$$

as a real vector space. By introducing a complex structure, $\mathscr{C}\mathscr{B}|_p$ becomes a vector space over $\mathbb{C}$ and its dimension is now $\dim \mathscr{C}\mathscr{B}|_p = 6$.

Definition 2.2.3. By $\mathscr{S}^+|_p$ and $\mathscr{S}^-|_p$ we denote the subspaces of $\mathscr{C}\mathscr{B}|_p$ with elements

- $F \in \mathscr{S}^+|_p \quad \Leftrightarrow \quad (\star F) = -iF$
- $F \in \mathscr{S}^-|_p \quad \Leftrightarrow \quad (\star F) = iF$

respectively.

Remark 2.2.4. Let us consider complex bivectors of a form $F = A + iB$, where $A, B \in \mathscr{B}|_p$. Conditions for members of $\mathscr{S}^+|_p$ and $\mathscr{S}^-|_p$ from definition 2.2.3 then become respectively

- $(\star F) = (\star A) + \underbrace{(\star(iB))}_{=-i(\star B)} \stackrel{!}{=} -iF = (-i)(A + iB) = B - iA \quad \Leftrightarrow \quad A = (\star B)$
- $(\star F) = (\star A) + \underbrace{(\star(iB))}_{=-i(\star B)} \stackrel{!}{=} iF = i(A + iB) = -B + iA \quad \Leftrightarrow \quad A = -(\star B)$

Therefore the memebers of $\mathscr{S}^+|_p$ are exactly those memebers of $\mathscr{C}\mathscr{B}|_p$ of the form $F + i(\star F)$ and those in $\mathscr{S}^-|_p$ of the form $F - i(\star F)$ for a real bivector F.

Definition 2.2.5. We shall call the memebers of $\mathscr{S}^+|_p$ *self-dual bivectors* and the members of $\mathscr{S}^-|_p$ *anti self-dual bivectors*. We shall denote them by

$$\mathscr{S}^+|_p \ni \hat{H} = H + i(\star H) \qquad \mathscr{S}^-|_p \ni \tilde{H} = H - i(\star H)$$

for a real bivector H.

Remark 2.2.6. Any complex bivector $H \in \mathscr{C}\mathscr{B}|_p$ can be written as a sum of a self-dual and an anti self-dual bivector in a unique way as follows:

$$H = \frac{1}{2}\underbrace{(H + i(\star H))}_{\in \mathscr{S}^+|_p} + \frac{1}{2}\underbrace{(H - i(\star H))}_{\in \mathscr{S}^-|_p}$$

Thus $\mathscr{C}\mathscr{B}|_p$ is the direct sum vector space

$$\mathscr{C}\mathscr{B}|_p = \mathscr{S}^+|_p \oplus \mathscr{S}^-|_p$$

Definition 2.2.7. Let us define a tensor we shall refer to as *bivector metric*:

$$G_{\alpha\beta\gamma\delta} := \frac{1}{2}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}) \qquad \alpha, \beta, \gamma, \delta \in \{0, 1, 2, 3\}$$

Remark 2.2.8. The above definition works for any dimension n of the space-time; using it, we obtain the following decomposition of the Riemann curvature tensor (via the definition of the Weyl tensor (1.5))

$$R_{lijk} = W_{lijk} - \frac{1}{(n-2)} E_{lijk} - \frac{2\varrho}{(n-2)(n-1)} G_{lijk} \tag{2.3}$$

where

$$E_{lijk} := \mathcal{R}_{ij} g_{kl} - \mathcal{R}_{ik} g_{jl} + \mathcal{R}_{lk} g_{ij} - \mathcal{R}_{lj} g_{ik} \tag{2.4}$$

Remark 2.2.9. We have (for $F \in \mathscr{B}|_p$)

$$G_{\alpha\beta\gamma\delta} F^{\gamma\delta} = \frac{1}{2}(F_{\alpha\beta} - F_{\beta\alpha}) = F_{\alpha\beta}$$

and

$$G_{\alpha\beta\gamma\delta} F^{\alpha\beta} H^{\gamma\delta} = \frac{1}{2}(F_{\gamma\delta} H^{\gamma\delta} - F_{\delta\gamma} H^{\gamma\delta}) = F_{\gamma\delta} H^{\gamma\delta}$$

We see that G acts as a metric on the space of $\mathscr{B}|_p$:

$$G(F,H) = G(H,F) := F_{\alpha\beta} H^{\alpha\beta} \qquad \forall F, H \in \mathscr{B}|_p$$

Proposition 2.2.10. A bivector $F \in \mathscr{B}|_p$ is simple if and only if $G(F, (\star F)) = 0$.

Proof. This is quite obvious once we take the 6th equivalence in theorem 2.1.7 into consideration. □

Theorem 2.2.11. For a simple $F \in \mathscr{B}|_p$ the number $G(F,F)$ is positive, negative or zero according as F is spacelike, timelike or null.

Proof. For null bivectors this assertion follows straight from theorem 2.1.14. For spacelike or timelike bivector F we have the two blade-spanning vectors X, Y, so that

$$G(F,F) = F_{\alpha\beta} F^{\alpha\beta} = 2\eta(X,X)\eta(Y,Y) - 2[\eta(X,Y)]^2$$

If the bivector F is spacelike, then X, Y are both spacelike and they may be spanning the blade of F if they satisfy the Cauchy-Schwarz inequality (1.13) which amounts to $G(F,F) > 0$. On the other hand if F is timelike, then X, Y may be taken to be both timelike and hence by the *Reversed Cauchy-Schwarz* inequality in lemma 1.2.11 $G(F,F) < 0$ (they form a basis, so they are necessarily linearly independent). □

Remark 2.2.12. In the following discussion we shall informally extend the notion of bivector metric as defined in 2.2.7 (see also remark 2.2.9) to complex bivectors by putting:

$$G(H,F) := H_{\alpha\beta} F^{\alpha\beta} \qquad H, F \in \mathscr{C}\mathscr{B}|_p$$

Theorem 2.2.13. Let $\hat{F} \in \mathscr{S}^+|_p$ be a self-dual bivector represented via a real bivector $F \in \mathscr{B}|_p$ as $\hat{F} = F + i(\star F)$. Then

1. Bivector F is simple if and only if $G(\hat{F}, \hat{F}) \in \mathbb{R}$.

2. Bivector F is null if and only if $G(\hat{F}, \hat{F}) = 0$.

Proof. We have

$$G(\hat{F}, \hat{F}) = \hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta} = F_{\alpha\beta}F^{\alpha\beta} + 2iF_{\alpha\beta}(\star F)^{\alpha\beta} - (\star F)_{\alpha\beta}(\star F)^{\alpha\beta} \tag{2.5}$$

The last term can be rewritten due to equality (2.1), so that

$$G(\hat{F}, \hat{F}) = 2F_{\alpha\beta}F^{\alpha\beta} + 2iF_{\alpha\beta}(\star F)^{\alpha\beta}$$

Firstly $F_{\alpha\beta}(\star F)^{\alpha\beta} = 0$ is equivalent to F being a simple bivector due to 2.1.7, ergo it is equivalent to $G(\hat{F}, \hat{F}) = 2F_{\alpha\beta}F^{\alpha\beta} \in \mathbb{R}$. Secondly $F_{\alpha\beta}F^{\alpha\beta} = F_{\alpha\beta}(\star F)^{\alpha\beta} = 0$ is equivalent to F being null due to 2.1.14, thus it is equivalent to $G(\hat{F}, \hat{F}) = 0$. $\square$

Definition 2.2.14. A complex self-dual bivector $\hat{F} \in \mathscr{S}^+|_p$ represented via a real bivector $F \in \mathscr{B}|_p$ as $\hat{F} = F + i(\star F)$ is called *null* if F is null.

Chapter 3

Petrov classification

Petrov classification is not a central part of this thesis, however it is instrumental to derive the basics in order to understand results of this theory that we shall later apply. Here we have again the wonderful book [1] to thank for the general outline; we added some proofs.

3.1 Eigenbivectors

The Einstein field equtions (1.14) and the decomposition formula (2.3) show that the Weyl tensor is the only remaining part in the source-free regions of the space-time (where $T_{\mu\nu} \equiv 0$). The Weyl tensor may therefore be considered the quantity most characteristic of the gravitational field, which a pp-wave attempts to be model of (see section 4.1). A classification of Weyl tensor thus contributes to a classification of space-time metrics in question. [3]

Definition 3.1.1. A bivector $F \in \mathscr{C}\mathscr{B}|_p$ is called an *eigenbivector of* W (respectively, *of* $\hat{W}$) if the first (respectively, the second) following equation holds.

$$W_{\alpha\beta\gamma\delta}F^{\gamma\delta} = \lambda F_{\alpha\beta} \quad \lambda \in \mathbb{C} \tag{3.1}$$

$$\hat{W}_{\alpha\beta\gamma\delta}F^{\gamma\delta} = \mu F_{\alpha\beta} \quad \mu \in \mathbb{C} \tag{3.2}$$

Then λ (respectively, μ) denotes the associated *eigenvalue*.

Remark 3.1.2. In the following text WF shall be understood as an abbreviation for the summation taking place in the previous definition. Defining condition for eigenbivectors can be then written followingly:

$$WF = \lambda F \quad \lambda \in \mathbb{C} \qquad , \qquad \hat{W}F = \mu F \quad \mu \in \mathbb{C}$$

Remark 3.1.3. In the light of a previous definition, the tensor W at $p \in M$ can be regarded as a linear operator $W : \mathscr{C}\mathscr{B}|_p \to \mathscr{C}\mathscr{B}|_p$ or $W : \mathbb{C}^6 \to \mathbb{C}^6$ (for bivector at a certain point in a chosen basis is determined by 6 complex numbers).

Lemma 3.1.4. We have

$$(\star(WH)) = (\star W)H = (W\star)H = W(\star H)$$

Proof. Using the definition of Hodge dual on bivectors

$$(\star(WH))^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}(WH)_{\gamma\delta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}W_{\gamma\delta\mu\nu}H^{\mu\nu}$$

Now recalling the definition of the left dual for the members of $\mathscr{W}|_p$ and the fact that $W \in \mathscr{W}|_p$, we can write

$$(\star(WH))^{\alpha\beta} = \underbrace{\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}W_{\gamma\delta\mu\nu}}_{=(\star W)^{\alpha\beta}{}_{\mu\nu}}H^{\mu\nu} = g^{\alpha\varkappa}g^{\beta\iota}(\star W)_{\varkappa\iota\mu\nu}H^{\mu\nu}$$

This establishes the first equality of the lemma that we are to prove. Because of the identity (1.19) this can be rewritten yet again

$$(\star(WH))^{\alpha\beta} = g^{\alpha\varkappa}g^{\beta\iota}(W\star)_{\varkappa\iota\mu\nu}H^{\mu\nu}$$

Here the second equality has been proven. Now using the definition of the right dual for members of $\mathscr{W}|_p$ (definition 1.4.2) and the definition of the Hodge dual for bivectors respectively, we obtain

$$(\star(WH))^{\alpha\beta} = g^{\alpha\varkappa}g^{\beta\iota}\frac{1}{2}W_{\varkappa\iota}{}^{\varrho\sigma}\underbrace{\varepsilon_{\mu\nu\varrho\sigma}}_{=\varepsilon_{\varrho\sigma\mu\nu}}H^{\mu\nu} = g^{\alpha\varkappa}g^{\beta\iota}W_{\varkappa\iota}{}^{\varrho\sigma}(\star H)_{\varrho\sigma} = W^{\alpha\beta}{}_{\varrho\sigma}(\star H)^{\varrho\sigma}$$

The proof is now complete. ☐

Corollary 3.1.5. For every $H \in \mathscr{S}^+|_p$ the image WH also belongs to $\mathscr{S}^+|_p$.

Proof. For a member $H \in \mathscr{S}^+|_p$, equality $(\star H) = iH$ holds and the previous lemma yields

$$(\star(WH)) = W(\star H) = iWH \qquad \Longrightarrow \qquad WH \in \mathscr{S}^+|_p$$

☐

Remark 3.1.6. Let us denote by $WF = B \in \mathscr{B}|_p$ the image of an arbitrary real bivector F by W. For the corresponding subspaces of self-dual and anti self-dual complex bivectors, we have

$$W\underbrace{(F + i(\star F))}_{\in\mathscr{S}^+|_p} = \underbrace{WF}_{=B} + i\underbrace{W(\star F)}_{=(\star(WF))} = B + i(\star B) \in \mathscr{S}^+|_p$$

$$W\underbrace{(F - i(\star F))}_{\in\mathscr{S}^-|_p} = \underbrace{WF}_{=B} - i\underbrace{W(\star F)}_{=(\star(WF))} = B - i(\star B) \in \mathscr{S}^-|_p$$

where we have employed the identity from lemma 3.1.4.

Remark 3.1.7. Due to the fact, that $\mathscr{CB}|_p$ is a direct sum $\mathscr{CB}|_p = \mathscr{S}^+|_p \oplus \mathscr{S}^-|_p$, W is completely determined by its restrictions (linear maps) to the corresponding subspaces

$$W^+ := W|_{\mathscr{S}^+|_p} : \mathscr{S}^+|_p \to \mathscr{S}^+|_p$$

$$W^- := W|_{\mathscr{S}^-|_p} : \mathscr{S}^-|_p \to \mathscr{S}^-|_p$$

Definition 3.1.8. We define the *conjugation operator* $k : \mathscr{CB}|_p \to \mathscr{CB}|_p$ by putting

$$k(H) = \overline{H} := H_1 - iH_2 \qquad \underbrace{H_1}_{\in \mathscr{B}|_p} + i \underbrace{H_2}_{\in \mathscr{B}|_p} = H \in \mathscr{CB}|_p$$

Remark 3.1.9. Using equalities and notation from remark 3.1.6 we are to verify that $W^- = k \circ W^+ \circ k$. We have for all $H = F - i(\star F) \in \mathscr{S}^-|_p$

$$k \circ W^+ \circ kH = k \circ W^+ \circ k(F - i(\star F)) = k \circ W^+ \underbrace{(F + i(\star F))}_{\in \mathscr{S}^+|_p} =$$

$$= k(B + i(\star B)) = B - i(\star B) = W^-(F - i(\star F)) = W^- H$$

Because for the conjugation operator (a bijection) we have the trivial equality

$$k = k^{-1}$$

the relation between W^+ and W^- obtained in the previous remark is expressible as

$$W^- = k \circ W^+ \circ k^{-1}$$

This implies that W^+, W^- are similar (via the similarity transformation k) and as a result of linear algebra, their matrices in Jordan basis are of the same form (including degeneracies and with eigenvalues differing only by conjugation). Morover Jordan form of W is just the common Jordan form of W^+ and W^- "repeated" in an obvious way. This common Jordan form will be taken as the *algebraic type* of W. [1]

Proposition 3.1.10. Complex self-dual Weyl tensor $\hat{W}$ at $p \in M$ regarded as a linear map $\hat{W} : \mathscr{CB}|_p \to \mathscr{CB}|_p$ has its nontrivial range contained in $\mathscr{S}^+|_p$ and maps $\mathscr{S}^-|_p$ to zero bivectors.

Proof. First let us consider $H \in \mathscr{S}^+|_p$. Then

$$\hat{W}H = (W + i(\star W))H = WH + iW \underbrace{(\star H)}_{=-iH} = WH + WH = 2WH \in \mathscr{S}^+|_p$$

where we have made use of lemma 3.1.4 in the second equality. Now we consider $H \in \mathscr{S}^-|_p$ and proceed similarly, computing

$$\hat{W}H = (W + i(\star W))H = WH + iW \underbrace{(\star H)}_{=iH} = WH - WH = 0$$

□

Corollary 3.1.11. The Jordan form of $\hat{W}$ is determined by the Jordan form of its restriction to $\mathscr{S}^+|_p$. This Jordan form of $\hat{W}$ is (modulo the factor 2) identical to that of W^+ and hence to the algebraic type of W defined above.[1]

Remark 3.1.12. The problem of classifying W on a 6-dimensional complex vector space of bivectors $\mathscr{C}\mathscr{B}|_p$ has been transformed to the eigenvalue problem of classifying the restriction of $\hat{W}$ on the 3-dimensional complex vector space $\mathscr{S}^+|_p$.

3.2 Further problem transformation

Definition 3.2.1. We define the object $\mathbb{W}_{AB}$ related to Weyl tensor $W_{\alpha\beta\gamma\delta}$ via index notation convention. Here A, B are the so-called *bivector indices* taking value in $\{1, 2, 3, 4, 5, 6\}$ with one-to-one correspondence to the usual space-time indices defined as follows:

$$\begin{array}{ll} [23] \leftrightarrow 1 & [10] \leftrightarrow 4 \\ [31] \leftrightarrow 2 & [20] \leftrightarrow 5 \\ [12] \leftrightarrow 3 & [30] \leftrightarrow 6 \end{array} \tag{NC}$$

We refer to this convention as to *bivector notation convention* and to the object $\mathbb{W}_{AB}$ as to 6×6 *Weyl tensor.*

Proposition 3.2.2. The convention (NC) is well defined, i.e. the correspondence

$$[\mu\nu] \leftrightarrow A \qquad \text{where } \mu, \nu \in \{0, 1, 2, 3\}, A \in \{1, 2, \ldots, 6\}$$

is indeed one-to-one in terms of $\mathbb{W}_{AB}$ encompassing exactly all the information contained in Weyl tensor at $p \in M$.

Proof. This follows immediately from the symmetries satisfied by Weyl tensor:

$$W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma}$$

□

Remark 3.2.3. From the proof of the previous proposition it is quite obvious that the bivector notation convention can be generalized to any member of $\mathscr{W}|_p$.

Theorem 3.2.4. The 6×6 Weyl tensor is a symmetric matrix. Furthemore we have the following block structure

$$(\mathbb{W}_{AB}) = \begin{pmatrix} \mathbb{M} & \mathbb{N}^T \\ \mathbb{N} & \mathbb{P} \end{pmatrix} \tag{3.3}$$

where $\mathbb{M}, \mathbb{N}$ and $\mathbb{P}$ are 3×3 real matrices with $\mathbb{M}$ and $\mathbb{P}$ symmetric.

[1]See remark 3.1.9.

Proof. The symmetry of $\mathbb{W}_{AB}$ matrix in bivector indices $A, B \in \{1, 2, \dots, 6\}$ follows from the symmetry of Weyl tensor (1.7)

$$W_{\alpha\beta\gamma\delta} = W_{\gamma\delta\alpha\beta}$$

Using that in explicitly writing out the block structure, we have

$$(\mathbb{W}_{AB}) = \left(\begin{array}{ccc|ccc} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16} \\ w_{12} & w_{22} & w_{23} & w_{24} & w_{25} & w_{26} \\ w_{13} & w_{23} & w_{33} & w_{34} & w_{35} & w_{36} \\ \hline w_{14} & w_{24} & w_{34} & w_{44} & w_{45} & w_{46} \\ w_{15} & w_{25} & w_{35} & w_{45} & w_{55} & w_{56} \\ w_{16} & w_{26} & w_{36} & w_{46} & w_{56} & w_{66} \end{array}\right) =: \left(\begin{array}{c|c} \mathbb{M} & \mathbb{K} \\ \hline \mathbb{N} & \mathbb{P} \end{array}\right)$$

Apparently $\mathbb{K} = \mathbb{N}^T$ and $\mathbb{M}$ and $\mathbb{P}$ are symmetric matrices. □

Theorem 3.2.5. The block structure (3.3) of the previous theorem can be further simplified, so that finally

$$(\mathbb{W}_{AB}) = \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{N} & -\mathbb{M} \end{pmatrix} \tag{3.4}$$

where $\mathbb{M}$ and $\mathbb{N}$ are the same 3×3 real matrices (as in the previous theorem), furthemore they are symmetric and trace-free.

Proof. The discussion presented here was confined to an arbitrary point $p \in M$. At this point an orthonormal frame with respect to the metric $g|_p$ shall be introduced so that $(g_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1) = (\eta_{\alpha\beta})$. The trace-free property of Weyl tensor in this frame takes form of:

$$0 = W^{\xi}{}_{\alpha\xi\beta} = \eta^{\xi\nu} W_{\xi\alpha\nu\beta} = -W_{0\alpha0\beta} + W_{1\alpha1\beta} + W_{2\alpha2\beta} + W_{3\alpha3\beta}$$

There are 16 equations in terms of Weyl tensor for possible combinations of space-time indices that can be equivalently rewritten in terms of 6×6 Weyl tensor $\mathbb{W}$. In order:

$n.$	$\alpha\beta$	$W_{\alpha\beta\gamma\delta}$	$\mathbb{W}_{\mathbb{AB}}$
1.	00	$-W_{0000} + W_{1010} + W_{2020} + W_{3030} = 0$	$w_{44} + w_{55} + w_{66} = 0$
2.	01	$-W_{0001} + W_{1011} + W_{2021} + W_{3031} = 0$	$-w_{53} + w_{62} = 0$
3.	02	$-W_{0002} + W_{1012} + W_{2022} + W_{3032} = 0$	$w_{43} - w_{61} = 0$
4.	03	$-W_{0003} + W_{1013} + W_{2023} + W_{3033} = 0$	$-w_{42} + w_{51} = 0$
5.	10	$-W_{0100} + W_{1110} + W_{2120} + W_{3130} = 0$	$-w_{35} + w_{26} = 0$
6.	11	$-W_{0101} + W_{1111} + W_{2121} + W_{3131} = 0$	$-w_{44} + w_{33} + w_{22} = 0$
7.	12	$-W_{0102} + W_{1112} + W_{2122} + W_{3132} = 0$	$-w_{45} - w_{21} = 0$
8.	13	$-W_{0103} + W_{1113} + W_{2123} + W_{3133} = 0$	$-w_{46} - w_{31} = 0$
9.	20	$-W_{0200} + W_{1210} + W_{2220} + W_{3230} = 0$	$w_{34} - w_{16} = 0$
10.	21	$-W_{0201} + W_{1211} + W_{2221} + W_{3231} = 0$	$-w_{54} - w_{12} = 0$
11.	22	$-W_{0202} + W_{1212} + W_{2222} + W_{3232} = 0$	$-w_{55} + w_{33} + w_{11} = 0$
12.	23	$-W_{0203} + W_{1213} + W_{2223} + W_{3233} = 0$	$-w_{56} - w_{32} = 0$
13.	30	$-W_{0300} + W_{1310} + W_{2320} + W_{3330} = 0$	$-w_{24} + w_{15} = 0$
14.	31	$-W_{0301} + W_{1311} + W_{2321} + W_{3331} = 0$	$-w_{64} - w_{13} = 0$
15.	32	$-W_{0302} + W_{1312} + W_{2322} + W_{3332} = 0$	$-w_{65} - w_{23} = 0$
16.	33	$-W_{0303} + W_{1313} + W_{2323} + W_{3333} = 0$	$-w_{66} + w_{22} + w_{11} = 0$

Pairs of equations 2.&5., 3.&9. and 4.&13. for the elements of the matrix $\mathbb{W}_{AB}$ are dependent and they imply that the matrix $\mathbb{N}$ from the decomposition (3.3) is symmetric, i.e. $\mathbb{N} = \mathbb{N}^T$. Equation 1. implies that $\mathbb{P}$ in this decomposition is trace-free. Summing equations $1., 6., 11.$ and 16. and dividing by 2 yields

$$w_{11} + w_{22} + w_{33} = 0$$

which means that the matrix $\mathbb{M}$ is trace-free. Pairs of equations 7.&10., 8.&14. and 12.&15. are dependent and they imply (together with the fact that both $\mathbb{P}$ and $\mathbb{M}$ are trace-free) the relation $\mathbb{P} = -\mathbb{M}$ in the matrix decomposition (3.3). Finally the matrix $\mathbb{N}$ is trace-free as well, for the expression

$$w_{14} + w_{25} + w_{36}$$

is equivalent through the bivector indices convention to an expression

$$W_{0123} + W_{0231} + W_{0321}$$

This expression certainly vanishes due to the first Bianchi identity that is satisfied by Weyl tensor (see (1.6)). The proof is now complete. □

Remark 3.2.6. Using the orthonormal frame as in the proof of the previous theorem ($(g_{\alpha\beta}) = (\eta_{\alpha\beta})$) we can also rewrite the bivector metric G (see definition 2.2.7) in the above introduced 6×6 (bivector) notation with the convention (NC) so that

$$(\mathbb{G}_{AB}) = \frac{1}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \qquad \text{where} \qquad \mathbb{I} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Furthermore the Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\delta}$ in 6×6 notation is (in the orthonormal frame $\sqrt{|\det g|} = 1$)

$$(\mathbb{E}_{AB}) = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

By the same token an arbitrary bivector $F \in \mathscr{C}\mathscr{B}|_p$ is via (NC) expressed as

$$(F^A) = \begin{pmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \\ F^6 \end{pmatrix}$$

Further, we shall assume, that the bivector F is represented by these six components. Also because

$$(\mathbb{G}^{AB}) = 2\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

we have

$$F_A = \mathbb{G}_{AB}F^B \quad , \quad F^A = \mathbb{G}^{AB}F_B \qquad \text{and} \qquad (\star F)_A = \frac{1}{2}\mathbb{E}_{AB}F^B$$

Using this together and thinking of F_A as being composed of two triplets of complex numbers $\vec{A},\vec{B} \in \mathbb{C}^3$, so that $F_A = (\vec{A},\vec{B})^T$, we have $(\star F)_A = \frac{1}{2}\mathbb{E}_{AB}\mathbb{G}^{BC}F_C$ which written in the matrix form becomes:

$$\frac{1}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \times 2 \begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} -\vec{B} \\ \vec{A} \end{pmatrix}$$

Recalling the definition of self-dual bivectors 2.2.5, 2.2.3, we have that a complex bivector is self-dual if and only if $(F_A) = (\vec{A}, i\vec{A})^T$.

Remark 3.2.7. Using the tools from the previous remark, we shall calculate the explicit form of the complex 6×6 self-dual Weyl tensor from (3.4). Apparently:

$$\hat{\mathbb{W}}_{AB} = \mathbb{W}_{AB} + i(\star\mathbb{W})_{AB} = \mathbb{W}_{AB} + \frac{1}{2}i\mathbb{E}_{AC}\mathbb{G}^{CD}\mathbb{W}_{DB}$$

Or in the matrix form

$$(\hat{\mathbb{W}}_{AB}) = \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{N} & -\mathbb{M} \end{pmatrix} + \frac{1}{2}i\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \times 2 \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{N} & -\mathbb{M} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{N} & -\mathbb{M} \end{pmatrix} + i\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} \mathbb{M} & -\mathbb{N} \\ \mathbb{N} & \mathbb{M} \end{pmatrix} = \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{N} & -\mathbb{M} \end{pmatrix} + i\begin{pmatrix} -\mathbb{N} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbb{M} - i\mathbb{N} & \mathbb{N} + i\mathbb{M} \\ \mathbb{N} + i\mathbb{M} & -\mathbb{M} + i\mathbb{N} \end{pmatrix}$$

Now we define a complex matrix $\mathbb{Q} := \mathbb{M} - i\mathbb{N}$ and obtain a simple matrix form of the 6×6 complex self-dual Weyl tensor:

$$(\hat{\mathbb{W}}_{AB}) = \begin{pmatrix} \mathbb{Q} & i\mathbb{Q} \\ i\mathbb{Q} & -\mathbb{Q} \end{pmatrix} \tag{3.5}$$

Here we should note, that the matrix $\mathbb{Q}$, being a sum of trace-less matrices $\mathbb{M},\mathbb{N}$, is also trace-less.

From 3.1.10 we know, that $\hat{W}$ maps the space $\mathscr{S}^+|_p$ to itself and on $\mathscr{S}^-|_p$ it acts trivially. That result is verified in bivector notation for $\hat{F}_A = (\vec{D}, i\vec{D})$ as follows:

$$(\hat{\mathbb{W}})\mathbb{G}^{BC}\hat{F}_C = 2\begin{pmatrix} \mathbb{Q} & i\mathbb{Q} \\ i\mathbb{Q} & -\mathbb{Q} \end{pmatrix}\begin{pmatrix} \vec{A} \\ -i\vec{A} \end{pmatrix} = 2\begin{pmatrix} 2\mathbb{Q}\vec{A} \\ 2i\mathbb{Q}\vec{A} \end{pmatrix} = 4\begin{pmatrix} \mathbb{Q}\vec{A} \\ i\mathbb{Q}\vec{A} \end{pmatrix} \in \mathscr{S}^+|_p$$

It follows that any eigenbivector of $\hat{\mathbb{W}}$ with non-zero eigenvalue is necessarily self-dual and that, irrespective of the eigenvalue, $\hat{F}$ is a self-dual eigenbivector of $\hat{\mathbb{W}}$ if and only if the complex triplet $\vec{D}$ associated with $\hat{F}$ is an eigenvector of $\mathbb{Q}$. The original problem, given by equation (3.1) has now been transferred to the essentially equivalent and much simpler one of determining the eigenvector-eigenvalue structure of the symmetric trace-free complex matrix $\mathbb{Q}$. [1]

Definition 3.2.8. We classify space-times according to their Weyl tensor eigenbivector structure by equivalently classifying the corresponding complex trace-less matrices $\mathbb{Q}$ by their Segre characteristic. Let us assign the *Petrov type* to the matrix $\mathbb{Q}$ or equivalently to the Weyl tensor

Segre characteristic	matrix $\mathbb{Q}$	additional condition	Petrov type
$\{111\}$	$\begin{pmatrix} \alpha & 0 & \\ & \beta & 0 \\ & & \gamma \end{pmatrix}$	distinct $\alpha+\beta+\gamma=0$	**I**
$\{1(11)\}$	$\begin{pmatrix} -2\lambda & 0 & \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}$	$\lambda \neq 0$	**D**
$\{21\}$	$\begin{pmatrix} \lambda & 1 & \\ & \lambda & 0 \\ & & -2\lambda \end{pmatrix}$	$\lambda \neq 0$	**II**
$\{(21)\}$	$\begin{pmatrix} \alpha & 1 & \\ & \alpha & 0 \\ & & \alpha \end{pmatrix}$	$\alpha = 0$	**N**
$\{3\}$	$\begin{pmatrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{pmatrix}$	$\alpha = 0$	**III**
	zero matrix	$W\|_p \equiv 0$	**O**

The types $\mathbf{D}, \mathbf{II}, \mathbf{N}, \mathbf{III}$ and $\mathbf{O}$ are called *algebraically special* and $\mathbf{I}$ is reffered to as *algebraically general.*

Chapter 4

pp-wave space-times

In this chapter we shall talk about three types of metric tensor, called a plane wave, a pp-wave and a generalized pp-wave. All these metrics are closely related, yet somewhat different. Therefore it does not come as a surprise that the terminology and notation is rather inconsistent throughout the literature. One of our goals is to put all these various definitions into context and standardize the terminology. Our notation is more or less consistent with [4],[3],[2]. It is no coincidence, for these texts inspired the following the most.

4.1 Heuristics and definitions

We are to model weak gravitational fields, which represent the case when there is a source of such field, however the scales on which we inspect the field are large enough, so that the field is perceived to exhibit approximately pseudo-flat (Minkowski) geometry. It is a reasonable assumption that the linearized theory of general relativity accurately describes the physics that are "far" from the source of the inspected field. We shall consider a space-time metric

$$g_{\mu\nu} := \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ shall be treated as a small perturbation (at most linear orders occuring in all formulae) of the Minkowski metric $\eta_{\mu\nu}$. The inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and these yield the so-called linearized Einstein equations

$$\partial_\nu \partial_\alpha h^\alpha_\mu + \partial_\mu \partial_\alpha h^\alpha_\nu - \partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\mu \partial_\nu h^\alpha_\alpha - \eta_{\mu\nu}(\partial_\alpha \partial_\beta h^{\alpha\beta} - \partial_\alpha \partial^\alpha h^\beta_\beta) = \frac{8\pi\varkappa}{c^4} T_{\mu\nu}$$

A typical solution is ([4])

$$ds^2 = -dt^2 + dz^2 + (\delta_{ij} + h_{ij}(z-t))dy^i dy^j \qquad i, j \in \{2, 3\}$$

This metric is characterized by a wave traveling in the z direction that distorts it (from a flat metric) only in transverse directions. The underlying gravitational wave (whose phenomenological model the metric above is) is thus said to be transversally polarized. In terms of light-cone coordinates

$$U := (z - t) \qquad V := \frac{1}{2}(z + t)$$

this can be further transformed into

$$ds^2 = 2dUdV + (\delta_{ij} + h_{ij}(U))dy^i dy^j \qquad i, j \in \{2, 3\} \tag{4.1}$$

because $2dUdV = (dz - dt)(dz + dt) = dz^2 - dt^2$. A simple generalization is at hand.

Definition 4.1.1. A space-time equipped with metric of a form

$$ds^2 = 2dUdV + g_{ij}(U)dy^i dy^j \tag{4.2}$$

is called the *pp-wave in Rosen cordinates*[1]. To denote this metric we shall use $\mathcal{F}_{\mu\nu}$.

Remark 4.1.2. In the previous definition, we have not specified the range of summation over i, j indices. That is because from the previous heuristics it should amount to $2, 3$ out of $\{0, 1, 2, 3\}$ as in the example above, arising from the four-dimensional general theory of relativity. However the simple form (4.2) can be easily generalized to higher dimensions purely by extending the range of summation to an arbitrary $d \in \mathbb{N}$.

A significant property of the metric (4.2) above is:

Proposition 4.1.3. Let $(M, \mathcal{F})$ be a space-time equipped with the the metric (4.2). Then there exists a global (nowhere vanishing) covariantly constant null vector field.

Proof. An obvious candidate is the coordinate vector field ∂_V:

$$ds^2(\partial_V, \partial_V) = 2\underbrace{dU(\partial_V)}_{=0}\underbrace{dV(\partial_V)}_{=1} + g_{ij}(U)\underbrace{dy^i(\partial_V)}_{=0}\underbrace{dy^j(\partial_V)}_{=0} = 0$$

Also symbolically

$$\nabla_\mu \partial_V = \Gamma^\alpha_{V\mu}\partial_\alpha = \frac{1}{2}\mathcal{F}^{\alpha\varkappa}(\partial_V \mathcal{F}_{\varkappa\mu} + \partial_\mu \mathcal{F}_{V\varkappa} - \partial_\varkappa \mathcal{F}_{V\mu}) = 0$$

because in (4.2) no components depend on V and $\mathcal{F}_{VU} = 1$ with other V-components being zero. □

From this we can generalize by defining a space-time that is characterized only by the existence of global covariantly constant null vector.

Definition 4.1.4. A *plane wave space-time* is a space-time manifold equipped with a Lorentzian metric and there exists a global covariantly constant null vector field.

The term plane wave is chosen here, in analogy with other physical phenomena, where describing a radiation far from the source, we model it by infinite parallel wavefronts determined by a perpendicular vector. Here, again, we have specified neither the dimension of the space-time manifold nor the signature of the Lorentzian metric. Generally we consider a $d + 2$ dimensional manifold and the metric has signature

$$(-1, \underbrace{1, \ldots, 1}_{(d+1)\text{-times}})$$

for some $d \in \mathbb{N}$. Results that are relevant for general theory of relativity are easily retained by setting $d = 2$.

[1] The term "pp-wave" shall be elucidated in the following text.

Theorem 4.1.5. For a plane wave space-time, there exist coordinates $\{u, v, x^a\}$ such that the Lorentzian metric has form of

$$ds^2 = 2dudv + K(u, x^c)du^2 + 2A_a(u, x^c)dx^a du + g_{ab}(u, x^c)dx^a dx^b \tag{4.3}$$

for $a, b, c = 1, \ldots, 2d$.

Proof. Let G be the Lorentzian metric of the plane wave space-time and ∇ the Levi-Civita connection with respect to this metric. Let X be the globally defined covariantly constant null vector field of the metric F, so that $\nabla_\mu X^\nu = 0$. This implies

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \tag{4.4}$$

$$\nabla_\mu X_\nu - \nabla_\nu X_\mu = 0 \tag{4.5}$$

The first equation is a Killing equation, which equivalently says, that X is a Killing vector field. Vector field X is by our assumption nowhere zero, therefore, without the loss of generality, we can assume, that

$$X = \partial_v$$

for some coordinate function v, since this simply means that we are using a parameter along the integral curves of X as a coordinate v. We can write

$$X^\mu = (\partial_v)^\mu = \delta^\mu_v \qquad \Longrightarrow \qquad X_\lambda := G_{\lambda\mu}X^\mu = G_{\lambda\mu}\delta^\mu_v = G_{\lambda v}$$

From this, X being null, we have

$$0 = X_\mu X^\mu = G_{\mu v}\delta^\mu_v = G_{vv} \tag{4.6}$$

The Killing equation (4.4) implies, that

$$\partial_v G_{\mu\nu} = 0 \tag{4.7}$$

The equation (4.5) means that X is a "gradient" vector field, for

$$0 = \nabla_\mu X_\nu - \nabla_\nu X_\mu = \partial_\mu X_\nu + \Gamma^\alpha_{\mu\nu}X_\alpha - \partial_\nu X_\mu - \Gamma^\alpha_{\nu\mu}X_\alpha = \partial_\mu X_\nu - \partial_\nu X_\mu$$

The 1-form $X_\alpha = G_{\alpha\beta}X^\beta$ is then closed and by Poincaré lemma it is locally exact. Ergo there exists a neighborhood H and a smooth function $u \in C^\infty(H)$ such that $X = du$ as a 1-form. Hence the equality

$$G_{\mu v} = X_\mu = \partial_\mu u = \delta^u_\mu \qquad \Longrightarrow \qquad G_{vu} = G_{uv} = \delta^u_u = 1 \tag{4.8}$$

and also $G_{\mu v} = 0$ for $\mu \neq u$. From this and equations (4.6), (4.7), (4.8), we have the following form of metric via the change from general coordinates $\{x^\mu\}$ to $\{u, v, x^a\}$, where $a = 1, \ldots, d$:

$$\begin{aligned} ds^2 &= G_{\mu\nu}dx^\mu dx^\nu = \\ &= 2dudv + G_{uu}(u, x^c)du^2 + 2G_{au}(u, x^c)dx^a du + G_{ab}(u, x^c)dx^a dx^b = \\ &=: 2dudv + K(u, x^c)du^2 + 2A_a(u, x^c)dx^a du + G_{ab}(u, x^c)dx^a dx^b \end{aligned}$$

□

Here we have derived the most general form of a plane wave metric. Simplest model arises when we set $G_{ab} := \delta_{ab}$, $A_a(u, x^b) := 0$ and $K(u, x^c) := A_{ab}(u)x^a x^b$.

Definition 4.1.6. A space-time equipped with metric of a form

$$ds^2 = A_{ab}(u)x^a x^b du^2 + 2dudv + d\vec{x}^2 \qquad a, b = 1, \ldots, d \tag{4.9}$$

is called the *pp-wave in Brinkmann coordinates.* To denote this metric we shall use $\mathcal{G}_{\mu\nu}$.

Remark 4.1.7. Finally we can shed light on the term pp-wave. Actually pp is and abbreviation that stands for *plane-fronted waves with parallel rays.* "Plane-fronted" refers to the fact that the wave fronts $u =$ const. are planar (mathematically speaking restriction of the metric on the hypersurface $u =$ const. is flat) and "parallel rays" refers to the existence of a parallel null vector.

Remark 4.1.8. A pp-wave in Rosen coordinates

$$ds^2 = 2dUdV + g_{ij}(U)dy^i dy^j$$

is characterised by a single matrix-valued function of U, however two metrics with quite different $g = g(U)$ may well be isometric. In Brinkmann coordinates pp-wave metric is characterised by a single symmetric matrix-valued function $A_{ab}(u)$ of u. Generally there is very little redundancy in this description of pp-waves, i.e. there are very few residual coordinate transformations that leave the form of the metric invariant, and the metric is specified almost uniquely by $A_{ab}(u)$. [4]

4.2 Generalized pp-waves

In this section only, we shall restrict ourselves to 4-dimensional manifolds.

Definition 4.2.1. A *generalized pp-wave space-time* is a four-dimensional pseudo-Riemannian manifold equipped with the Levi-Civita connection and there exists a (real) global covariantly constant null bivector field. To denote the generalized pp-wave metric, we shall use $\tilde{\mathcal{G}}$.

Proposition 4.2.2. A space-time is a generalized pp-wave if and only if there exists a global covariantly constant complex (self-dual[2]) null bivector field.

Proof. For a complex (self-dual) null bivector $\hat{F}$ there is the corresponding real null bivector such that $\hat{F} = F + i(\star F)$. So either we can start with a real bivector and construct the complex self-dual bivector from it according to this formula or vice versa we start with a complex self-dual bivector and "decompose" it. The fact that the property of being covariantly constant is conserved via this correspondence is trivial, for the tensor (being constructed from the underlying metric tensor) $\varepsilon_{\alpha\beta\gamma\delta}$ commutes with the Levi-Civita connection. □

[2]Here we remind the reader that the complex null classification has been established only for self-dual bivectors.

The following is a very important theorem that first appeared in [3] and we cite it here because it sheds light on the relationship between various definitions of pp-wave like space-times that appear throughout the literature. We attempt to break down and to analyze the proof of this theorem in order to translate it into "language" of our definitions.

Theorem 4.2.3. [3][1] Space-time is a generalized pp-wave if and only if there exists a coordinate system $\{u, v, x, y\}$ (called the Brinkmann coordinates[3]) in which the metric tensor takes form of

$$ds^2 = 2H(u, x, y)du^2 + 2dudv + dx^2 + dy^2 \qquad v, x, y \in \mathbb{R},\ u \in [u_1, u_2]$$

This essential theorem connects the algebraic theory of bivectors to the metric tensors (hence the geometry as well) of a generalized pp-wave. The original proof is to be found in [3], however it uses a lot of implicit knowledge which we shall try to summarize in the following remark.

Remark 4.2.4. This remark uses the notation of [3] page 89. We start by summarizing several properties of the assumed covariantly constant null bivector field, let us denote it ω. Because this bivector is (real) null, we have locally the existence of its blade spanned by a spacelike vector t and a null vector l so that they are orthogonal to each other and also

$$\omega_{\alpha\beta} = l_\alpha t_\beta - l_\beta t_\alpha \tag{4.10}$$

We know that the blade of a dual bivector $(\star\omega)$ is orthogonal complement to that of ω and moreover they intersect in the null direction l; we shall denote the second spanning (necesarilly spacelike) vector by $\tilde{t}$. Thus

$$(\star\omega)_{\alpha\beta} = l_\alpha \tilde{t}_\beta - l_\beta \tilde{t}_\alpha \tag{4.11}$$

Here both spacelike vectors $t, \tilde{t}$ shall be normalized with respect to the Lorentzian metric so that from now on

$$t_\mu t^\mu = 1 \quad \text{and} \quad \tilde{t}_\mu \tilde{t}^\mu = 1 \tag{4.12}$$

Now we construct the so called complex null self-dual bivector as follows:

$$\hat{\omega}_{\alpha\beta} := \omega_{\alpha\beta} + i(\star\omega)_{\alpha\beta} = l_\alpha t_\beta - l_\beta t_\alpha + i l_\alpha \tilde{t}_\beta - i l_\beta \tilde{t}_\alpha = l_\alpha(\underbrace{t_\beta + i\tilde{t}_\beta}_{=:z_\beta}) - l_\beta(\underbrace{t_\alpha + i\tilde{t}_\alpha}_{=:z_\alpha}) \tag{4.13}$$

Ergo the bivector $\hat{\omega}$ is a complex bivector and the vector z is a complex vector for which the following relations hold:

$$z_\mu z^\mu = (t_\mu + i\tilde{t}_\mu)(t^\mu + i\tilde{t}^\mu) = 1 - 1 = 0 \tag{4.14}$$

$$l_\mu z^\mu = l_\mu(t^\mu + i\tilde{t}^\mu) = 0 + i0 = 0 = l_\mu \bar{z}^\mu \tag{4.15}$$

[3]Space-time equipped with a metric of the following form is reffered to as a *generalized pp-wave in Brinkmann coordinates.*

$$z_\mu \overline{z}^\mu = (t_\mu + i\tilde{t}_\mu)(t^\mu - i\tilde{t}^\mu) = 1 + 1 = 2 \tag{4.16}$$

The second property of assumed real bivector ω was the fact that it is covariantly constant, meaning that

$$\nabla_\nu \omega_{\alpha\beta} = 0 \qquad \forall \nu, \alpha, \beta \tag{4.17}$$

Because the Hodge star is constructed from the underlying metric tensor, it is effectively ignored by the Levi-Civita connection (or covariant differentiation), thus also the complex null self-dual bivector $\hat{\omega} := \omega + i(\star\omega)$ is covariantly constant

$$\nabla_\nu \hat{\omega}_{\alpha\beta} = 0 \qquad \forall \nu, \alpha, \beta \tag{4.18}$$

Starting from this property, we will show, that also the null vector l is covariantly constant. Let us first calculate:

$$\hat{\omega}_{\alpha\beta} \overline{\hat{\omega}}^{\beta\gamma} = (l_\alpha z_\beta - l_\beta z_\alpha)(l^\beta \overline{z}^\gamma - l^\gamma \overline{z}^\beta) = -2 l_\alpha l^\gamma \tag{4.19}$$

We have made use of (4.15) as well as of (4.16). Differentiating the equality (4.19) with respect to ∇_μ and using (4.18), we obtain:

$$0 = \nabla_\mu (\hat{\omega}_{\alpha\beta} \overline{\hat{\omega}}^{\beta\gamma}) = -2\nabla_\mu (l_\alpha l^\gamma) \quad \Longrightarrow \quad 0 = l_\beta \nabla_\mu l_\alpha + l_\alpha \nabla_\mu l_\beta \qquad \forall \alpha, \beta, \mu \tag{4.20}$$

Here we choose $\alpha = \beta$, however without the summation in effect, we obtain an equation $l_\alpha \nabla_\mu l_\alpha = 0$ for all α, μ. If given component l_α vanishes, then obviously $\nabla_\mu l_\alpha$ also vanishes. On the other hand, if $l_\alpha \neq 0$, we may divide our condition by it and arrive at $\nabla_\mu l_\alpha = 0$ for this particular α as well. All in all l is a covariantly constant null vector/1-form.

From this, we may deduce

$$0 = \nabla_\mu l_\alpha - \nabla_\alpha l_\mu = \partial_\mu l_\alpha - \Gamma^\nu_{\alpha\mu} l_\nu - \partial_\alpha l_\mu + \Gamma^\nu_{\mu\alpha} l_\nu = \partial_\alpha l_\mu - \partial_\mu l_\alpha \tag{4.21}$$

Here we have employed the symmetry of Christoffel symbols corresponding to the Levi-Civita connection. Now it is obvious that the 1-form l_α is closed and by Poincaré lemma, it is also locally exact, so that there exists a neighborhood V and a function $u \in C^\infty(V)$ such that

$$l = l_\alpha dx^\alpha = \partial_\alpha u dx^\alpha = du \tag{4.22}$$

We have constructed the coordinate function u used in theorem 4.2.3 the rest of the coordinates are constructed in [3].

Corollary 4.2.5. In a generalized pp-wave spacetime, there exists a global covariantly constant null vector field.

Proof. Using the previous theorem, we shall prove that the coordinate vector field ∂_v is covariantly constant and that it is nul. Firstly, for all $\mu \in \{u, v, x, y\}$

$$\nabla_\mu \partial_v = \Gamma^\lambda{}_{v\mu} \partial_\lambda = \frac{1}{2} \tilde{\mathcal{G}}^{\lambda\varkappa} (\partial_v \tilde{\mathcal{G}}_{\mu\varkappa} + \partial_\mu \tilde{\mathcal{G}}_{v\varkappa} - \partial_\varkappa \tilde{\mathcal{G}}_{v\mu}) \partial_\lambda$$

Here the Christoffel symbols are all zero since none of the coordinate expression functions are dependent on v and the coordinate function $\tilde{\mathcal{G}}_{vu} = 1$ does not depend on u, v, x, y. All in all we have that the coordinate vector field ∂_v is covariantly constant. Secondly, we have

$$ds^2(\partial_v, \partial_v) = H(u,x,y) \underbrace{du^2(\partial_v, \partial_v)}_{=0} + 2 \underbrace{du(\partial_v)}_{=0} \underbrace{dv(\partial_v)}_{=1} + \underbrace{dx^2(\partial_v, \partial_v)}_{=0} + \underbrace{dy^2(\partial_v, \partial_v)}_{=0} = 0$$

Hence ∂_v is the sought after covariantly constant null vector field. □

The proofs of the two following theorems are dependent on a considerable amount of previous knowledge, that is scattered across several other publications and that is beyond the scope of this text. We offer the first proof with proper citations in order to introduce the reader to its logical structure. The reader is redirected to [8] to learn more. All the mentioned exterior knowledge is to be found in the appendix B.

Theorem 4.2.6. [8][2] If in a non-flat space-time there exists a covariantly constant null vector field and the Petrov type is **N** or **O** then this space-time is a generalized pp-wave.

Proof. Let X be a covariantly constant null vector field. Then it satisfies the Ricci identity (1.11) so that

$$0 = \nabla_\nu \underbrace{\nabla_\mu X_\beta}_{=0} - \nabla_\mu \underbrace{\nabla_\nu X_\beta}_{=0} = R^\alpha{}_{\beta\mu\nu} X_\alpha \qquad \forall \mu, \nu, \beta \in \{0,1,2,3\}$$

Now, by lemma B.1.5 the Petrov type being **N** or **O** we have the vanishing of the Ricci scalar $\varrho = 0$. Furthermore X is a Debever-Penrose vector (see definition B.1.3). Then the Bel criterion B.1.4 yields $W^\alpha{}_{\beta\mu\nu} X_\alpha = 0$. Using the Riemann curvature tensor decomposition 2.3 we find that

$$\underbrace{R^\alpha{}_{\beta\mu\nu} X_\alpha}_{=0} = \underbrace{W^\alpha{}_{\beta\mu\nu} X_\alpha}_{=0} - \frac{1}{2} E^\alpha{}_{\beta\mu\nu} X_\alpha - \frac{1}{3} \underbrace{\varrho}_{=0} G^\alpha{}_{\beta\mu\nu} X_\alpha \qquad \Longrightarrow \qquad E^\alpha{}_{\beta\mu\nu} X_\alpha = 0$$

Here the lemma B.1.6 implies that there is $\lambda \in \mathbb{R}$ such that $\mathcal{R}_{\mu\nu} = \lambda X_\mu X_\nu$. Let $\{X^\alpha, Y^\alpha, N^\alpha, \overline{N}^\alpha\}$ be a Weyl canonical tetrad ([14]), and let us define

$$F_{\mu\nu} := X_\mu \overline{N}_\nu - X_\nu \overline{N}_\mu$$

Then by the same reasoning as in [3] the bivector F is recurrent (hence there is a 1-form ω such that $\nabla_\gamma F_{\mu\nu} = F_{\mu\nu}\omega_\gamma$) and has vanishing skew derivative ([12]):

$$\nabla_\beta \nabla_\gamma F_{\mu\nu} - \nabla_\gamma \nabla_\beta F_{\mu\nu} = 0 \qquad \forall \beta, \gamma, \mu, \nu$$

Combining these two facts, we can write:

$$0 = \nabla_\beta(F_{\mu\nu}\omega_\gamma) - \nabla_\gamma(F_{\mu\nu}\omega_\beta) = \underbrace{(\nabla_\beta F_{\mu\nu})}_{=F_{\mu\nu}\omega_\beta} \omega_\gamma + F_{\mu\nu}\nabla_\beta\omega_\gamma - \underbrace{(\nabla_\gamma F_{\mu\nu})}_{=F_{\mu\nu}\omega_\gamma} \omega_\beta - F_{\mu\nu}\nabla_\gamma\omega_\beta$$

And finally

$$\nabla_\beta \omega_\gamma = \nabla_\gamma \omega_\beta \quad \Longleftrightarrow \quad \partial_\beta \omega_\gamma = \partial_\gamma \omega_\beta \quad \forall \beta, \gamma$$

This is equivalent to ω being a closed 1-form. Thus, by Poincaré lemma, we have the existence of a coordinate neighborhood U and a smooth function defined on it so that $\omega = df$. By lemma B.1.7 there exists a covariantly constant bivector to which is F proportional, "conserving" the null property.

□

Theorem 4.2.7. [8][2] If in a space-time there exists a covariantly constant null vector field in some coordinate neighborhood U and the space-time is vacuum or the Ricci tensor is of Segre characteristic $\{(211)\}$ then this space-time is a generalized pp-wave.[4].

4.3 Geometric properties

In theorem 4.2.3 we have derived a certain simple form of a generalized pp-wave, that is equivalent to the original definition via bivector. This allows us to extend this definition into higher dimensions rather easily.

Definition 4.3.1. A space-time equipped with metric of a form

$$ds^2 = 2H(u, x^a)du^2 + 2dudv + d\vec{x}^2 \qquad a = 1, \ldots, d \tag{4.23}$$

is called the *generalized pp-wave in Brinkmann coordinates.*

From this definition whilst setting $2H(u, x^a) := A_{ab}(u)x^a x^b$ (in terms of definition 4.1.6) one does find that

Corollary 4.3.2. Every pp- wave is a generalized pp-wave.

In this section we shall discuss both, pp-waves and generalized pp-waves, in Brinkmann coordinates. We have already established the relationship between their expressions via these coordinates hence it suffices to derive the formulae solely for generalized pp-waves and then input $2H(u, x^a) := A_{ab}(u)x^a x^b$ to obtain similar expressions for pp-waves.

Theorem 4.3.3. The inverse metric tensor for the generalized pp-wave (4.23) is in Brinkmann coordinates represented via matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 1 & -2H(u,x^a) & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then (for generalized pp-wave in Brinkmann coordinates)

[4]This fact is easily reconcilable with the results of the next section, where we learn, that the Ricci tensor of a generalized pp-wave has only one non-vanishing component $\mathcal{R}_u^v$

1. the only nonzero Christoffel symbols are

$$\Gamma^{x^j}_{uu} = -\partial_{x^j} H(u, x^i) \qquad j = 1, \dots, d \tag{4.24}$$

2. the only nonzero components of the Riemann tensor are

$$R^{x^j}{}_{ux^k u} = -\partial_{x^k}\partial_{x^j} H(u, x^i) \qquad j, k = 1, \dots, d \tag{4.25}$$

3. the only nonzero component of the Ricci tensor is

$$\mathcal{R}_{uu} = -\sum_{j=1}^{d} \partial^2_{x^j} H(u, x^i) \tag{4.26}$$

Corollary 4.3.4. For a pp-wave in Brinkmann coordinates we have:

1. the only nonzero components of the Riemann tensor are

$$R^{x^j}_{ux^k u} = -A_{x^j x^k}(u) \qquad j, k = 1, \dots, d \tag{4.27}$$

2. the only nonzero component of the Ricci tensor is

$$\mathcal{R}_{uu} = -\sum_{j=1}^{d} A_{x^j x^j}(u) =: \mathrm{tr}A \tag{4.28}$$

4.4 Transition between Rosen and Brinkmann coordinates

In order to transform the non-flat transverse metric of a pp-wave in Rosen coordinates

$$ds^2 = 2dUdV + \underbrace{g_{ij}(U)dy^i dy^j}$$

to the flat transverse metric of a pp-wave in Brinkmann coordinates

$$ds^2 = A_{ab}(u)x^a x^b du^2 + 2dudv + d\vec{x}^2$$

we introduce a change of coordinates:

$$x^a = E^a{}_j(U) y^j \tag{4.29}$$

Here $E = E(U)$ is an orthonormal vielbein for the metric g in the sense that

$$g_{ij} = E^a{}_i E^b{}_j \delta_{ab} \tag{4.30}$$

Finding a differential of (4.29), we obtain (dot over E symbolizes a derivative with respect to U)

$$dx^a = \dot{E}^a{}_j y^j dU + E^a{}_j dy^j \qquad \Longrightarrow \qquad E^a{}_j dy^j = dx^a - \dot{E}^a{}_j y^j dU \tag{4.31}$$

Substituting (4.30) and this into the inspected non-flat transverse metric of a pp-wave in Rosen coordinates, we have

$$g_{ij}dy^i dy^j = E^a{}_i E^b{}_j \delta_{ab} dy^i dy^j = (dx^a - \dot{E}^a{}_i y^i dU)(dx^b - \dot{E}^b{}_j y^j dU)\delta_{ab}$$

Finally y can be expressed by inverse vielbein $E_a{}^c(U)E^a{}_j(U) = \delta^c_j$ as $y^j = E_a{}^j(U)x^a$ so that

$$g_{ij}dy^i dy^j = (dx^a - \dot{E}^a{}_i E_c{}^i x^c dU)(dx^b - \dot{E}^b{}_j E_d{}^j x^d dU)\delta_{ab}$$

Expanding, we arrive at

$$\begin{aligned} g_{ij}dy^i dy^j =& d\vec{x}^2 + \dot{E}^a{}_i E_c{}^i \dot{E}^b{}_j E_d{}^j x^c x^d \delta_{ab} dU^2 - \\ &- \underbrace{\dot{E}_{aj} E_d{}^j x^d dx^a dU}_{\alpha} - \underbrace{\dot{E}_{bi} E_a{}^i x^a dx^b dU}_{\beta} \end{aligned} \tag{4.32}$$

The form of the first two terms is the reason for undergoing this coordinate change, for they correspond to similar terms in the expression of pp-wave in Brinkmann coordinates. The last two terms are being underbraced and denoted with a greek letter for later convenience. On the other hand the last two terms shall be eliminated via shift

$$V \mapsto V + \frac{1}{2}\dot{E}_{ai} E_b{}^i x^a x^b \tag{4.33}$$

Now the differential of V shifts as

$$dV \mapsto dV + \frac{1}{2}(\ddot{E}_{ai} E_b{}^i x^a x^b dU + \dot{E}_{ai} \dot{E}_b{}^i x^a x^b dU + \underbrace{\dot{E}_{ai} E_b{}^i x^b dx^a}_{\alpha} + \underbrace{\dot{E}_{ai} E_b{}^i x^a dx^b}_{\beta}) \tag{4.34}$$

We substitute (4.32) and (4.34) into the pp-wave expressed in Rosen coordinates to obtain

$$ds^2 = 2dUdV + (\underbrace{\dot{E}^a{}_i E_c{}^i \dot{E}^b{}_j E_d{}^j \delta_{ab}}_{\eta} + \ddot{E}_{ci} E_d{}^i + \underbrace{\dot{E}_{ci} \dot{E}_d{}^i}_{\zeta}) x^c x^d dU^2 + d\vec{x}^2 \tag{4.35}$$

The terms underbraced with α have cancelled out and so do the terms underbraced with β if further "symmetry" condition is imposed:

$$\dot{E}_{bi} E_a{}^i \stackrel{!}{=} \dot{E}_{ai} E_b{}^i \tag{4.36}$$

Now we shall further rewrite the term underbraced with ζ in (4.35).

$$\dot{E}_{ci} \dot{E}_d{}^i = \dot{E}_{ci} \delta^i_j \dot{E}_d{}^j = \dot{E}_{ci} E_b{}^i E^b{}_j \dot{E}_d{}^j = \dot{E}_{bi} E_c{}^i E^b{}_j \dot{E}_d{}^j = \dot{E}^a{}_i E_c{}^i E^b{}_j \dot{E}_d{}^j \delta_{ab}$$

Here we have used inserting the identity, rewriting it as a product of a vielbein with its inverse, the symmetry condition (4.36) and a fact, that δ_{ab} is a flat metric, respectively. Into this we substitute the identity obtained as follows

$$0 = \frac{d}{dU}(\delta^b_d) = \frac{d}{dU}(E^b{}_j E_d{}^j) = \dot{E}^b{}_j E_d{}^j + E^b{}_j \dot{E}_d{}^j$$

So that finally:

$$\dot{E}_{ci}\dot{E}_d{}^i = -\dot{E}^a{}_i E_c{}^i \dot{E}^b{}_j E_d{}^j \delta_{ab}$$

and the term underbraced ζ subtracts with the term underbraced η in (4.35). All in all, we have the pp-wave in Brinkmann coordinates:

$$ds^2 = 2dUdV + \ddot{E}_{ci}E_d{}^i x^c x^d dU^2 + d\vec{x}^2$$

We summarize this:

Corollary 4.4.1. A pp-wave in Rosen coordinates $\{U, V, y^k\}$:

$$ds^2 = 2dUdV + g_{ij}(U)dy^i dy^j$$

can be transformed into Brinkmann coordinates with

$$\ddot{E}_{ci}(U)E_d{}^i(U) = A_{cd}(U) \qquad \Longleftrightarrow \qquad \ddot{E}_{ck}(U) = A_{cd}(U)E^d{}_k(U) \tag{4.37}$$

via the transformation

$$\begin{aligned} U &\mapsto U \\ V &\mapsto V + \frac{1}{2}\dot{E}_{ai}E_b{}^i x^a x^b \\ y^k &\mapsto E_a{}^k x^a \end{aligned} \tag{4.38}$$

for an arbitrary orthonormal vielbein

$$g_{ij} = E^a{}_i E^b{}_j \delta_{ab}$$

which satisfies the symmetry condition

$$\dot{E}_{bi}E_a{}^i = \dot{E}_{ai}E_b{}^i$$

Remark 4.4.2. Conversely, given a metric in Brinkmann coordinates with a certain $A = A(U)$, one can solve the equation (4.37) for the vielbein $E = E(U)$ and determine the non-flat transverse metric g_{ij} from it and thus obtain the pp-wave metric in Rosen coordinates.

4.5 Coordinates adapted to a choice of null geodesic

Let us consider a coordinate system $\{U, V, Y^k\}$ (for $k = 1, \ldots, d$, where $d \in \mathbb{N}$) and a metric expressed in this coordinate system as

$$ds^2 = 2dUdV + a(U, V, Y^k)dV^2 + 2b_j(U, V, Y^k)dVdY^j + g_{ij}(U, V, Y^k)dY^i dY^j$$

All the components of a, b_j, g_{ij} are generally depending on all of the coordinates $\{U, V, Y^k\}$. In this and in the following section, the metric of such form shall be denoted by G.

Remark 4.5.1. A special case of a metric of this form is a pp-wave in Rosen coordinates with $a, b_j = 0$ for all $j \in d$ and g_{ij} depending only on the coordinate U.

An important property of the metric above is that the nowhere vanishing coordinate vector field ∂_U is null. Further, we have:

Proposition 4.5.2. The vector field ∂_U is geodesic (with respect to G).

Proof. By definition ∂_U is geodesic if and only if

$$0 = \nabla_{\partial_U}\partial_U = \Gamma^{\mu}_{UU}\partial_\mu \qquad \Longleftrightarrow \qquad \Gamma^{\mu}_{UU} = 0 \quad \forall\mu$$

We have

$$\Gamma^{\mu}_{UU} = \frac{1}{2}G^{\mu\alpha}(\partial_U G_{\alpha U} + \partial_U G_{\alpha U} - \partial_\alpha \underbrace{G_{UU}}_{=0}) = G^{\mu\alpha}\partial_U G_{\alpha U} = G^{\mu V}\partial_U G_{VU}$$

Since $G_{VU} = 1$ is a constant it obviously does not depend on U and ergo the above condition for geodesic vector field is met. □

The coordinate U plays a role of the affine parameter for the integral curves of this geodesic vector field. The above metric defines a geodesic null congruence[5] so that in the region of validity of the coordinate system $\{U, V, Y^k\}$ there is a unique null geodesic passing through any point. The points in such coordinate neighborhood can be therefore parametrized by the affine parameter U and the transverse coordinates $\{V, Y^k\}$ labelling the geodesics.

The metric above (and the coordinates in which it has been expressed) had the significant property that it gave rise to the existence of a geodesic null congruence with certain properties. Now we shall be able to reverse this process and define:

Definition 4.5.3. Let γ be a null geodesic of a space-time with metric G. We say, that that $\{U, V, Y^k\}$ are *coordinates adapted to* γ (or simply *adapted coordinates* or *Penrose coordinates*) if the metric G expressed via these coordinates takes form of

$$ds^2_\gamma = 2dUdV + a(U, V, Y^k)dV^2 + 2b_j(U, V, Y^k)dV dY^j + g_{ij}(U, V, Y^k)dY^i dY^j \qquad (4.39)$$

and γ corresponds to the geodesic $V = Y^k = 0$ with U as the affine parameter.

Texts such as [4],[5] are concerned with proving that for a choice of metric tensor coordinates adapted to a certain null geodesic locally always exist.

4.6 Penrose limit

Let us consider a Lorentzian space-time with a metric G. We shall choose an arbitrary null geodesic γ and express the metric G in adapted coordinate system, where it takes form (4.39). Let us perform the asymetric scaling of coordinates

$$\{U, V, Y^k\} \mapsto \{U, \lambda^2 V, \lambda Y^k\} \qquad (4.40)$$

[5] Here the *congruence* means a set of integral curves defined by a nowhere vanishing vector field ∂_U.

(for a parameter λ), which leaves the coordinate/affine parameter U invariant. We denote this as

$$\{U, V, Y^k\} = \{u, \lambda^2 v, \lambda y^k\}$$

and thus obtain a one-parameter family of metrics $ds^2_\gamma \mapsto ds^2_{\gamma,\lambda}$ where $ds^2_{\gamma,\lambda}$ is the metric ds^2_γ expressed in the coordinates $\{u, v, y^k\}$:

$$ds^2_{\gamma,\lambda} = 2\lambda^2 dudv + \lambda^4 a(u, \lambda^2 v, \lambda y^k)dv^2 + 2\lambda^3 b_j(u, \lambda^2 v, \lambda y^k)dvdy^k + \lambda^2 g_{ij}(u, \lambda^2 v, \lambda y^k)dy^i dy^j$$

A metric $\lambda^{-2} ds^2_{\gamma,\lambda}$ that is conformal (with a constant scale factor λ^{-2}) to the metric $ds^2_{\gamma,\lambda}$ is expressed as

$$\lambda^{-2} ds^2_{\gamma,\lambda} = 2dudv + \lambda^2 a(u, \lambda^2 v, \lambda y^k)dv^2 + 2\lambda b_j(u, \lambda^2 v, \lambda y^k)dvdy^k + g_{ij}(u, \lambda^2 v, \lambda y^k)dy^i dy^j$$

Now taking the limit $\lambda \to 0$ results in a well-defined and non-degenerate metric

$$\begin{aligned} ds^2 : &= \lim_{\lambda \to 0} \lambda^{-2} ds^2_{\gamma,\lambda} = \\ &= 2dudv + 0 \cdot a(u, 0, 0)dv^2 + 2 \cdot 0 \cdot b_j(u, 0, 0)dvdy^k + \underbrace{g_{ij}(u, 0, 0)}_{=: g_{ij}(u)} dy^i dy^j = \\ &= 2dudv + g_{ij}(u)dy^i dy^j \end{aligned}$$

Here $g_{ij}(u) := g_{ij}(u, 0, 0)$ is the restriction of the original G to the null geodesic γ.

Remark 4.6.1. The absence of G_{Uj}- terms ($j = 1, \ldots, d$) in the initial metric, we originated from, expressed in an adapted coordinate system is crucial for this limit to exist. Terms corresponding to G_{Uj} would scale as λ^{-1} and thus result in a divergence.

This is the classical result (due to R. Penrose [6]) that starting from any space-time that admits the existence of a null geodesic, one can obtain via certain rescaling and limiting procedure a pp-wave in Rosen coordinates. (For definition see 4.2.)

Conclusion

The main goal of this text was to examine the various approaches to space-time metrics that are commonly reffered to as a pp-wave or a plane wave. We have explored the more heuristic point of view as well as the algebraic approach. For this algebraic, more rigorous approach, it was necessary to compile a lot of preliminary results which are included in the first three chapters. To the best of our knowledge the generalized pp-waves and the pp-waves have not appeared in the same text and have not been thoroughly compared. We aimed to include both in this work. After the above mentioned preliminaries were built and the definitions were clearly formulated, this turned out to be a fairly straightforward task. Primary benefiters of this work shall be readers new to the topic, for whom this text could save a lot of time, as even trying to understand what a pp-wave is, can be a rather tedious endeavor.

Appendix A

Linear algebra

A.1 Skew-symmetric matrices

Theorem A.1.1. The rank of a skew-symmetric matrix is even.

Proof. We shall prove this by mathematical induction on n for an $(n \times n)$ matrix $\mathbb{A}$.

1. Let $n = 1$. Then obviously $\mathbb{A}$ has only one component $\mathbb{A}_{11}$, which is necessarily zero by the skew-symmetry $\mathbb{A}_{11} = -\mathbb{A}_{11}$. Hence the rank of $\mathbb{A}$ is zero, which is an even number.

2. Let $\mathbb{A}$ be of an even rank. We have to show, that an $(n+1 \times n+1)$ skew symmetric matrix $\mathbb{B}$ defined by the following partition is of an even rank as well.

$$\mathbb{B} = \begin{pmatrix} \mathbb{A} & \vec{v} \\ -\vec{v}^T & 0 \end{pmatrix}$$

Here $\vec{v}$ is a column vector with n components. If the column vector $\vec{v}$ is linearly dependent of the columns of matrix $\mathbb{A}$ then by the skew-symmetry is $-\vec{v}^T$ linearly dependent of $-\mathbb{A}^T = \mathbb{A}$. Therefore neither $\vec{v}$, nor $-\vec{v}^T$ does contribute to the rank of $\mathbb{B}$ and we have

$$\operatorname{rank} \mathbb{B} = \operatorname{rank} \mathbb{A}$$

Should $\vec{v}$ be linearly independent of the columns of $\mathbb{A}$, we have by a similar inspection the linear independency of $-\vec{v}^T$ of the rows of $-\mathbb{A}^T = \mathbb{A}$. Both $\vec{v}$ and $-\vec{v}^T$ contribute to the rank of B and we obtain

$$\operatorname{rank} \mathbb{B} = \operatorname{rank} \mathbb{A} + 2$$

In both cases the rank of $\mathbb{B}$ is an even number ergo the proof by induction is now complete.

□

Definition A.1.2. The outer product of vectors $\vec{v}_1, \vec{v}_2$ is defined as

$$\vec{v}_1 \otimes \vec{v}_2 := \vec{v}_1 \vec{v}_2^T$$

where the expression on the right hand side is regarded as the usual matrix multiplication.

Remark A.1.3. Apparently if $\vec{v}_1$ is a column vector of n components and $\vec{v}_2$ is a vector of m components, then the result of their outer product is an $(n \times m)$ matrix. Similarly the result of an expression

$$\vec{u} \otimes \vec{v} - \vec{v} \otimes \vec{u}$$

where $\vec{u}$ and $\vec{v}$ are vectors of the same number of components, is a skew-symmetric matrix.

A.2 Segre characteristic

Let $A : V \to V$ be a linear map, where V is an n-dimensional vector space and $\mathbb{A}$ the corresponding matrix in the Jordan basis. From that we have the following block structure:

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_1 & & & \\ & \mathbb{A}_2 & & \\ & & \ddots & \\ & & & \mathbb{A}_r \end{pmatrix} \quad \text{where} \quad r = \#\{\lambda \in \mathbb{C} | \exists \vec{v} \in V, A\vec{v} = \lambda \vec{v}\}$$

and $\mathbb{A}_j \in \mathbb{R}^{m_j, m_j}$ for m_j being the associated multiplicity of the eigenvalue λ_j corresponding to the j-th block in $\mathbb{A}$. Furthemore, we have

$$\mathbb{A}_j = \begin{pmatrix} \mathbb{B}_{j1} & & & \\ & \mathbb{B}_{j2} & & \\ & & \ddots & \\ & & & \mathbb{B}_{jk(j)} \end{pmatrix}$$

where $\mathbb{B}_{jl}$ is a $p_{jl} \times p_{jl}$ matrix ($m_j = p_{j1} + \dots p_{jk(j)}$) whose diagonal entries are each equal to λ_j and whose superdiagonal entries are each equal to one and where $p_{j1} \geq \dots \geq p_{jk(j)}$. Once an ordering is established for the eigenvalues $\{\lambda_j\}_{j=1}^{r}$ then, with the above conventions, this so-called *Jordan canonical form* for A is uniquely determined.

Definition A.2.1. By symbol

$$\{(p_{11}, \dots, p_{1k(1)})(p_{21}, \dots, p_{2k(2)}) \dots (p_{r1}, \dots, p_{rk(r)})\}$$

we denote what is to be called the *Segre characteristic (Segre type, Segre symbol)* of a matrix $\mathbb{A}$ representing the linear map A. In writing out the Segre characteristic one does usually omit round brackets around a single digit.

Appendix B

Weyl tensor classification

B.1 Useful lemmas and theorems

Definition B.1.1. Let T be a nowhere zero (p, q) complex tensor in some neighborhood U. If on U exists a 1-form ω such that

$$\nabla_k T^{i_1 \dots i_p}_{j_1 \dots j_q} = T^{i_1 \dots i_p}_{j_1 \dots j_q} \omega_k$$

then the tensor is called *recurrent on* U and the 1-form is called the *recurrence* 1*-form*.

Definition B.1.2. If $\nabla_\mu \hat{W}_{\alpha\beta\gamma\delta} = 0$ in a contractable coordinate neighborhood U the Weyl tensor is called *conformally symmetric on* U.

Definition B.1.3. A null vector satisfying the equation

$$X_{[\mu} \hat{W}_{\alpha]\beta\gamma[\delta} X_{\nu]} X^\beta X^\gamma = 0 \qquad \forall \alpha, \beta, \mu, \nu$$

is called the *Debever-Penrose vector.*

Theorem B.1.4 (Bel criterion)**.** [8] Let W be a non-zero Weyl tensor at a point p. Then the Weyl tensor is Petrov type **N** if and only if there is a non-zero real vector X such that

$$W_{\alpha\beta\gamma\delta} X^\delta = 0 \qquad \forall \alpha, \beta, \gamma$$

The vector X is necessarily null and it is also Debever-Penrose vector.

Lemma B.1.5. [13][8] If a space-time admits a recurrent null vector field X then the Weyl tensor is algebraically special. Furthermore the Petrov type is **II** or **D** if and only if the Ricci scalar is non-zero; if the Weyl tensor is non-zero then X is a Debever-Penrose direction.

Lemma B.1.6. [8][15] Let $\varrho = 0$. If there is a non-zero vector X such that $E_{\alpha\beta\gamma\delta} X^\delta = 0$, then there exists $\lambda \in \mathbb{R}$ so that

$$\mathcal{R}_{\mu\nu} = \lambda X_\mu X_\nu$$

Lemma B.1.7. [2][8] Let T be a recurrent tensor on U and let ω be its recurrence 1-form. If there exists a function $f \in C^\infty(U)$ such that $\omega = df$, then there exists a covariantly constant tensor W to which is T proportional.

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