Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Physical Aspects of Conformal Gravity

Fyzikální aspekty konformní gravitace

Master's Thesis

Author: **Bc. Patrik Urban**
Supervisor: **Ing. MSc. Petr Jizba, Ph.D.**
Consultant: **Prof. Hagen Kleinert**
Academic year: 2017/2018

- Zadání práce -

- Zadání práce (zadní strana) -

Acknowledgment:

I would hereby like to thank to my supervisor, Ing. Petr Jizba Ph.D., for introducing me to the topic and for highly-valued guidance throughout the work on this difficult project.

Prohlášení:

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd...) uvedené v přiloženém seznamu.

Nemám závažný důvod proti použití tohoto školního díla ve smyslu § 60 Zákona č. 121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Praze dne Patrik Urban

Název práce:
Fyzikální aspekty konformní gravitace

Autor: Bc. Patrik Urban

Obor: Matematická fyzika
Druh práce: Diplomová práce

Vedoucí práce: Ing. Petr Jizba, PhD., Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Konzultant: Prof. Hagen Kleinert, Institute of Theoretical Physics, Freie Universitaet, Berlin, Germany

Abstrakt: Tato práce se věnuje studiu teoretického základu kvantové teorie pole, zejména konceptu efektivní akce, jednosmyčkového efektivního potenciálu a užití spektrální Zeta fukce při výpočtu funkcionálních determinantů. Veškeré poznatky jsou demonstrovány na ϕ^4 teorii, z čehož je čerpáno v pozdějších kapitolách. Tyto se věnují Weylově konformní teorii gravitace a jejímu kvantování. Jsou předkládány argumenty, proč právě tato teorie by měla být vhodnou náhradou Einsteinovy teorie v oblastech vysokých energií a Velkého Třesku. V této části je naším hlavním vodítkem článek P. Jizby, H. Kleinerta a F. Scardigliho [1], jehož výsledky pro jednosmyčkový efektivní potenciál se nám podaří potvrdit použitím jiné (pro gravitaci neekvivalentní metody) metody výpočtu pomocí Zeta funce regularizace. Dále jsou nalezeny parametry teorie takové, že Weylova gravitace přejde po dynamickém narušení konformní symetrie v gravitaci Starobinského. Na závěr je poskytnuta diskuse fenomenologických důsledků v kosmologii raného vesmíru.

Klíčová slova: kvantová gravitace, Weylova kvantová gravitace, kvantová teorie pole, efektivní akce, Zeta funkce regularizace

Title:
Physical Aspects of Conformal Gravity

Author: Bc. Patrik Urban

Abstract: In this work we peruse a theoretical background of standard QFT, namely the effective action and the one-loop effective potential and also the Zeta function regularization of functional determinants. The theoretical discussion is accompanied by illustrative calculations on the ϕ^4 theory, from which we benefit in later chapters. In these we discuss Weyl conformal theory of gravity and its quantum extensions. We put forward arguments showing this theory is phenomenologically suitable for the description of early universe cosmology. Our leading principle is the article by P. Jizba, H. Kleinert and S. Scardigli [1] and our goal is to recover same results, mainly for the one-loop effective potential. We find the one-loop effective potential to coincide with the form presented in Ref. [1] even when a different (and in gravity non-equivalent) regularization – namely zeta-function regularization – is employed. Further we find parameters of the theory so, that the Weyl conformal theory coincides with the Starobinsky gravity after the dynamical breakdown of conformal symmetry. Finally, we provide some discussion concerning potential phenomenological implications in the early universe cosmology.

Key words: Quantum gravity, Weyl quantum gravity, Quantum Field Theory, Effective action, Zeta function regularization

Contents

Chapter 1

Mathematical Preliminaries and Motivation

1.1 Why Quantum Gravity?

This work focuses on quantum gravity and it is therefore appropriate to start with a little bit of history and a discussion of the problems associated with the quantization of gravity. The gravitational interaction was probably the first interaction noticed by man. Nevertheless its subtleties eluded us until the beginning of the 20th century. The first description of gravitational pull that explained the movements of celestial objects and stated a precise mathematical law, however, was made by sir I. Newton (published in 1687). The law itself introduced a gravitational charge, that is numerically equal (by definition of the gravitational constant G) to the inertia of a body. This equality (or direct proportionality) was experimentally checked and rechecked over time, reaching precision of 10^{-13} as of today. Even though the Newton's gravitational law was and a great success, it has a very disturbing property – the instantaneous effect over any distance.

Until the electromagnetism was fully described by a set of four equations by J. C. Maxwell in 1861/62 there was no better foundations of physical principles than Galileo principle of relativity and Newton's second law of motion. However, with Maxwell's equations it became clear, that Newtonian theory based on Galilean invariance contradicts the consequences of electromagnetism, for the Maxwell's equations inevitably postulate an existence of a fundamental constant of the speed of light. It was clear then, that the Galileo transformation laws cannot be correct in general (assuming electromagnetism was more fundamental). This problem was resolved by A. Einstein in 1905 with his ground-breaking Principle of Special Relativity (or the Theory of Special Relativity (STR), however, it is more of a fundamental principle that should be obeyed by all other theories, than a theory by its own right). A. Einstein proposed to use a new transformation law, namely the Lorentz transformation, to make two inertial observers agree on a result of any experiment. This transformation has one parameter, the speed of light, that does not change and stays the same in all inertial reference frames, hence it respects the Maxwell's equations.

There was also the similarity between the Coulomb law of electric force and the Newton's law of gravitation. These are, however, different in one very important and fundamental fact – the Coulomb law is applied only in a static limit, whereas the Newton's law had no restrictions. By comparison with the Coulomb law, it was inferred then, that it should also apply just in a case of two masses with constant distance.

To extrapolate the fundamental principle of special relativity from inertial reference frames to any frames (thus incorporating accelerating frames) A. Einstein strained himself from 1905 to 1915 until he formulated the Theory of General Relativity (GR) in 1915. This theory is based on three fundamental postulates

1. *The Principle of Equivalence*

   Any accelerated frame is locally equivalent to that under the influence of a gravitational field and the inertia of the moving body is by definition equal to the gravitational charge/mass.

2. *The Principle of General Relativity*

   At every point in spacetime it is possible to operate in a Local (Cartesian) Inertial System (LIS) in which all laws of physics are equivalent to those governed by the Special Theory of Relativity.

3. *The Principle of General Covariance*

   The formulation of physical laws is independent of the choice of coordinates i.e. all coordinate frames are equal.

The theory of general relativity is together with quantum electrodynamic the most tested, experimentally verified and also most precise theory humankind has ever created. It predicted many new effects and answered some old questions, such as – the advance of the perihelion of Mercury, the gravitational red shift, the gravitational time dilatation, gravitational lensing, gravitational waves, frame dragging, black holes and much more. Its beauty lies in its geometrical interpretation of dynamical four dimensional background described by the metric tensor and shaped (curved and stretched) by the presence of matter, that serves as the arena for all physical reality around us. It has deepened our understanding and use of topology, differential geometry and of tensor calculus. It also introduced a new fundamental type of invariance of physical laws that now serves as a cornerstone for any general relativistic theory (including quantum theories of gravity) – the diffeomorphism invariance – which follows from the principle of general covariance.

The theory of general relativity was one of the biggest successes of science, however, after the birth of Quantum Mechanics it became clear, that these two theories cannot coexist in peace. The Quantum Mechanics was the solution to several seemingly unsolvable problems of the 20th century – the black body radiation (M. Planck), the photoeffect (A. Einstein) and the structure of the atom (A. Compton,E. Rutherford and others). It introduced the idea of a wave-particle duality of matter and a probabilistic character of the nature, forever discarding any chance of a fully deterministic theory of (not only) elementary particles. The main imperfection of the theory was, however, its purely non-relativistic regime. Meaningful relativistic extensions encountered surprising difficulties with one particle kinematics pointing to only one solution – the use of field theory with infinite degrees of freedom. And so the Quantum Field Theory was born thanks to P. Dirac, W. Pauli, V. Weisskopf and P. Jordan.

Quantum Field Theory is by construction a special relativistic theory that serves as a framework for the physics of elementary constituents of nature and their governing laws. The first "child" of the quantum field theory was Quantum Electrodynamics (QED) created by R. Feynman, F. Dyson, J. Schwinger, S. Tomonaga and many others. It fully covers the theory of electron and positron interaction intermediated by photons. The description of the nucleus and other elementary particles took a little longer than expected, because experimental physicists kept discovering dozens of new particles and it seemed there is no sense in that "zoo of particles".

The result is what we call today the Standard Model with the underlying theory – the Quantum Chromodynamics (QCD).

It seems that we can do almost miracles with only a few fundamental principles taken as a starting point of any physical theory. The principle of special relativity and the quantum theoretical postulates set the basic framework for all physics – hermitian operators on an Hilbert space, where we represent known physical invariances of nature. Quantum Mechanics was by itself successful only up to a certain degree of precision. The precision and applicability of the quantum theory deepened by imposing the invariance under the Lorentz transformation i.e. incorporating the principle of special relativity. It would seem obvious, that the next step in our understanding is merging the quantum theory with the principle/theory of general relativity, thus yielding so much wanted theory of everything (or at least theory of quantum gravity). After many decades of work, however, it seems that general relativity is absolutely incompatible with the quantum point of view.

Some problems are obvious – the theory of general relativity is formulated in a framework of four dimensional spacetime with space and time coordinates that are treated on equal footing. Einstein's equations put an equal sign between the curvature of the spacetime and the presence of energy/momentum in that very same spacetime. The theory is, by the principle of general covariance, background independent and can be vied as a description of the physical arena of nature. This arena is dynamical and the equations are non-linear in the sense, that the gravitational field itself carries a gravitational charge (i.e. energy), thus the field influences its own dynamics. This also means, that any form of energy gravitates, hence decreases its energy by creating a potential well of gravitational field around itself.

The quantum theory, on the other hand, proposes indeterministic behaviour and unpredictability of any value of dynamical degrees of freedom, giving birth to virtual pairs of particles and vacuum energy. The quantum field theories have space and time on the same footing (as a requirement of STR), nevertheless they distinguish time as the parameter of time evolution driven by the Hamiltonian. This by itself poses a great difficulty, for the Hamiltonian of general relativity (as of a fully constrained theory) is always zero (on-shell) and so there is nothing, that would generate "time evolution" in the usual sense. It would be preferable to view the metric tensor of general relativity as just another field, however the problem is the coordinate "space", in which this field should exist. An insurmountable problem represents the non-linearity of the gravitational field from the point of view of the quantum fluctuations. This is also related to the problem of the vacuum fluctuations – in general, quantum field theories add compensating terms to Lagrangians set the zero energy level to a convenient point, which physically corresponds to setting the energy of a vacuum to zero. The vacuum, however, is not "empty", but full of virtual particles and with interesting structure. The energy of the vacuum is not so much important in QFT, but since it would generate a non-trivial gravitational field, the real structure of the vacuum is a very important topic for quantum gravity.

We might question the need for the complete quantum theory based upon the principle of general relativity. For most of the purposes the theories we have are sufficient and it seems that quantum gravity would not bring anything profoundly new or important. However there are still problems, for which we have no definite answer. At the time of the birth of general relativity, there was only a little data concerning cosmology and/or data coming from the time near the beginning of the universe. Observations of clusters of galaxies and of stars in galaxies are pointing to the fact, that there must be additional matter contained in the universe, since the angular velocities of stars around the centres of galaxies (or galaxies around the centres of clusters of galaxies) do not agree with the theoretical predictions based on Keppler's laws. Because we cannot see this additional gravitating source, it does not interact electromagnetically or strongly. We call it the *Dark Matter* and the estimates based on the observations of the cosmic microwave background show, that it makes up about 25% of the known energy contents of the universe. We have also no reasonable explanation for the observed accelerated expansion of the universe which is apparently going on for 7 billion years already. It is either driven by an unknown form of energy responsible for negative pressure – the *Dark Energy* – which makes up about 70% of the energy content of the universe and is somehow related to the fabric of spacetime itself, since its density does not decrease with the expansion, or the theory of gravity we have now simply is not applicable at these scales.

Then there are problems involving high energies and Planck scale physics, where gravitational effects would play almost equally important role as quantum mechanics. That is because at the Planck scale, the Schwarzschild radius is comparable to the Compton's wavelength. Further we have no reliable and precise framework for the description of the beginning of the universe. The theory of horizons (cosmic or event) needs severe improvements, since the solutions given by the general relativity are classical and it is unthinkable to suppose there are no quantum effects like ripples propagating on the event horizon or the evaporization of a black hole caused by the event horizon separating tiny black holes as the consequence of the unpredictable high energy ripples. The Hawking radiation was the first effect that merges quantum mechanics with relativity, at least to some extent, leading however to the infamous information paradox. All these problems are pointing to the fact, that the theories we have are fundamentally incomplete and we should be seeking for unification of these fundamental physical theories and the principles they are based on.

In this thesis we will focus on a specific type of gravitational theory – Weyl conformal gravity – an alternative candidate, whose quantization shows great promise in its applicability to the description of the early universe. The goal of this thesis is to improve and clarify results obtained by P. Jizba, H. Kleinert and F. Scardigli in their article [1]. To that end we will first provide a discussion of needed tools namely the effective action and the Zeta function regularization. These will be then employed to the case of a quantum version of the Weyl gravity and we will present the key calculations leading to the confirmation of the results and their discussion.

1.2 Mathematical Background and Notation

In this section we recall some well known results based on calculations with Gaussian integrals, as they will be relevant for us in further physical applications. It seems appropriate not to include these into an appendix, since the below shown formulas will be seen later in the same form with a lot of physical meaning behind it thus the general properties might not be clear. We will also introduce notation which we will use throughout the text.

1.2.1 Gaussian Integrals and Their Generalizations

First we examine an integral of the Gaussian distribution

$$Z = \int \mathrm{d}^d x \, \exp\Big(-\frac{1}{2} x^T A x \Big) , \tag{1.1}$$

where $A \in \mathbb{R}^{d,d}$ is a symmetric matrix and the spectrum $\sigma(A) = \{a_i\}_{i=1}^{d}$ is positive (i.e. all $a_i > 0$). Under these assumptions the matrix is diagonalizable by an orthogonal matrix O, such that $x^T A x = x^T O^T O A O^T O x \equiv y^T D y$, where y is a new vector and D is a diagonal matrix having a_i as its elements. Since the matrix O is orthogonal, the Jacobian of the transformation is identity and we can decompose the Z into d independent 1-dimensional Gaussian integrals

$$Z = \prod_{i=1}^{d} \int \mathrm{d}y_i \, \mathrm{e}^{-\frac{1}{2}a_i y^2} = \prod_{i=1}^{d} \sqrt{\frac{2\pi}{a_i}} = \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \,. \tag{1.2}$$

Before we go further, we remark on the possibility of zero eigenvalue a_i of A. In that case, the integration over y_i cannot be performed and we are left with a divergent term coming from the integral of a constant over real numbers. In the context of quantum field theory, however, the integrand may contain further terms (for example a potential) and the integration may be performed and yield finite result. These zero eigenvalues correspond to the so-called *zero-modes* and play an important role for example in finding soliton solutions.

We will now generalize this result to

$$Z(b) = \int \mathrm{d}^d x \, \mathrm{e}^{-\frac{1}{2}x^T A x + b\,x} \,, \tag{1.3}$$

where $b \in \mathbb{R}^d$ and $b\,x$ is a standard scalar product. It is clear, that for a special choice of $b = 0$ we have $Z(0) = Z$. To calculate $Z(b)$, we find the minima of the exponent by solving $-A_{ij}x_j + b_i = 0$ to obtain new suitable variables for substitution (we are shifting to the point of the minima) $x_i \mapsto A_{ij}^{-1} b_j + y_i$ in order to rearrange the exponent and obtain

$$Z(b) = \int \mathrm{d}^d y \, \mathrm{e}^{-\frac{1}{2}y^T A y + \frac{1}{2} b^T A^{-1} b} = \mathrm{e}^{\frac{1}{2}b^T A^{-1} b} \int \mathrm{d}^d y \, \mathrm{e}^{-\frac{1}{2}y^T A y} = Z(0)\, \mathrm{e}^{\frac{1}{2}b^T A^{-1} b} \,. \tag{1.4}$$

Regarding the integrands as a distribution determined by the matrix A, we might be interested in computing expected (or mean) values of variables x_i denoted by

$$\langle x_i \dots x_j \rangle = \frac{1}{Z(0)} \int \mathrm{d}^d x \, x_i \dots x_j \, \mathrm{e}^{-\frac{1}{2}x^T A x} \,, \tag{1.5}$$

where the factor $1/Z(0)$ is there to normalize the expectation value (i.e. $\langle 1 \rangle = 1$). These are called by different names in different contexts. They are known as higher moments in statistics, correlation functions in statistical physics and green functions in quantum mechanics and field theory. Since these might become difficult to calculate, we use the so-called Feynman trick with a derivative in the form

$$\frac{\partial}{\partial b_k} Z(b) = \int \mathrm{d}^d x \, \frac{\partial}{\partial b_k} \mathrm{e}^{-\frac{1}{2}x^T A x + bx} = \int \mathrm{d}^d x \, x_k \, \mathrm{e}^{-\frac{1}{2}x^T A x + bx} \,. \tag{1.6}$$

Therefore we can easily write

$$\langle x_i \dots x_j \rangle = \frac{1}{Z(0)} \frac{\partial}{\partial b_i} \cdots \frac{\partial}{\partial b_j} \bigg|_{b=0} Z(b) = \frac{\partial}{\partial b_i} \cdots \frac{\partial}{\partial b_j} \bigg|_{b=0} \mathrm{e}^{\frac{1}{2}b^T A^{-1} b} \,, \tag{1.7}$$

since setting b equal to zero at the end of the calculation will replace $Z(b)$ with $Z(0) = Z$, which defines the Gaussian distribution. It is easy to see, that this trick extends to any function with a polynomial expansion series, thus we can write

$$\langle F(x) \rangle = \frac{1}{Z(0)} F\left[\frac{\partial}{\partial b}\right] \bigg|_{b=0} Z(b) = F\left[\frac{\partial}{\partial b}\right] \bigg|_{b=0} \mathrm{e}^{\frac{1}{2}b^T A^{-1} b} \,. \tag{1.8}$$

We shall also mention one form of the Wick expansion (in this context known as the cumulant expansion), which states, that the expectation value for odd number number of variables is zero and for even number we obtain

$$\langle x_{i_1} \dots x_{i_s} \rangle = \sum_{\substack{\text{all possible pairings} \\ p \text{ of } \{i_1 \dots i_s\}}} \left\langle x_{i_{p_1}} x_{i_{p_2}} \right\rangle \dots \left\langle x_{i_{p_{s-1}}} x_{i_{p_s}} \right\rangle , \tag{1.9}$$

This identity can, of course, be proven by induction, however, it can also be seen from the fact that $Z(b)$ is quadratic in b and that the exponential survives the derivative. Thus, only after we differentiate a term twice, there is a non-vanishing result (remember, that we set $b = 0$ at the end), yielding all possible pairs of x_i and x_j.

This expansion enables us to introduce a "connected" term in the Wick expansion. Imagine, we are to evaluate the following integral

$$I(\lambda) = \int \mathrm{d}^d x \; \mathrm{e}^{-\frac{1}{2}x^T A x - \lambda V(x)} , \tag{1.10}$$

where $V(x)$ is a polynomial in x. We might use (1.7) and the expansion of exp to obtain a relation

$$I(\lambda) = Z(0) \left\langle \mathrm{e}^{-\lambda V(x)} \right\rangle = Z(0) \sum_k \frac{(-\lambda)^k}{k!} \left\langle V(x)^k \right\rangle . \tag{1.11}$$

Here, we stumble upon terms $\langle V(x)^k \rangle$, for which the Wick expansion provides a new perspective – either the variables from each $V(x)$ are coupled only within themselves and form clusters, or they also couple with variables from another $V(x)$ in the product. Thus, for example, the connected clusters at the second order form a factorization of the form $\langle V(x)^2 \rangle = \langle V(x)^2 \rangle_{\mathrm{C}} + \langle V(x) \rangle_{\mathrm{C}}^2$. There exists a formula how the correlation functions (1.9) can be decomposed into a product of the cumulants. To make our work easier, we define a function, that includes only the connected terms

$$W(\lambda) \equiv \ln I(\lambda) = \ln Z(0) + \sum_{k=1} \frac{(-\lambda)^k}{k!} \left\langle V(x)^k \right\rangle_{\mathrm{C}} . \tag{1.12}$$

It is crucial to remark, that all these calculations can be generalized to an infinite number of dimensions ($d \to \infty$), which formally makes the index i continuous. The formulas then transform according to the following scheme (summation over repeating indices is always implied):

$$\begin{aligned} \sum_i &\to \int \mathrm{d}x_i \\ \frac{\mathrm{d}}{\mathrm{d}b_k} &\to \frac{\delta}{\delta b(x_k)} \\ \mathbf{a} \cdot \mathbf{b} = a_i \, b_i &\to \int \mathrm{d}x_i \, a(x_i) b(x_i) \\ a^T A b &\to \int \mathrm{d}x \, \mathrm{d}y \, a(x) A(x,y) b(y) \\ &\vdots \end{aligned} \tag{1.13}$$

Especially let us consider an inverse G of a finite-dimensional matrix A. In the continuous limit A becomes a differential operator – in particular, we are interested in cases, where A is a differential operator in one variable only, thus it is multiplied by a delta function

$$\begin{aligned} A_{ik}G_{kj} = \delta_{ij} \to \int \mathrm{d}x_k\, A(x_i,x_k)G(x_k,x_j) = \delta(x_i - x_j) \\ \int \mathrm{d}x_k\, A(x_i)\delta(x_i - x_k)G(x_k,x_j) = A(x_i)G(x_i,x_j) = \delta(x_i - x_j)\,, \end{aligned} \tag{1.14}$$

thus in this special case the inverse matrix is the Green function.

With this in mind, we will use the compact notation with indices “i”, “j”, etc., always keeping in mind, they might stand for a continuous “index” x_i and summation is replaced by integration. We will also sometimes use notation f_x instead of $f(x)$,to make it more clear, that the index is continuous.

1.2.2 Legendre Transform

In the following chapters will also make use of the Legendre transform and its properties, however, it seems that most textbooks of physics use this tool somewhat vaguely. We will try to be more careful with the terms and also remind us of the correct mathematical definition.

For a convex function $f : \mathbb{R} \mapsto \mathbb{R}$ we define a new function $g : \mathbb{R} \mapsto \mathbb{R}$, such that

$$g(y) = \sup_{x\in\mathbb{R}}\{xy - f(x)\} \equiv \max_{x\in\mathbb{R}} F(x,y)\,, \tag{1.15}$$

This transformation is know as Legendre–Fanchel transform and it becomes the usual Legendre transform in the case of f being a differentiable function. Under these conditions $F(x,y)$ is differentiable as well and we may search for the maximum by solving an equation $\frac{\partial F(x,y)}{\partial x} = 0$. Let x_0 be the solution, then

$$0 = \frac{\partial F(x_0,y)}{\partial x} = y - \frac{\partial f(x_0)}{\partial x} \Rightarrow \left.\frac{\partial f(x)}{\partial x}\right|_{x_0} = y\,. \tag{1.16}$$

It is clear, that this solution is parameterized by y so we should write $x_0(y)$. The Legendre transform of f is then a function

$$g(y) = x_0 y - f(x_0) = x_0(y)y - f(x_0(y))\,, \tag{1.17}$$

of only one variable y. With this, we obtain the familiar relations

$$\frac{\partial g(y)}{\partial y} = x_0, \qquad \frac{\partial g(y)}{\partial x_0} = y - \frac{\mathrm{d}f(x_0)}{\mathrm{d}x_0} = 0 \quad \text{and} \quad \frac{\partial f(x)}{\partial y} = \frac{\partial}{\partial y}\big(x_0 y - g(y)\big) = 0\,. \tag{1.18}$$

On the other hand, acting with a total differential operator on (1.17) instead of the partial one, one obtains

$$\frac{\mathrm{d}g(y)}{\mathrm{d}x_0} = y + x_0\frac{\mathrm{d}y(x_0)}{\mathrm{d}x_0} - \frac{\mathrm{d}f(x_0)}{\mathrm{d}x_0} = x_0\frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2} \quad \text{and} \quad \frac{\mathrm{d}f(x_0)}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\big(x_0 y - g(y)\big) = y\frac{\mathrm{d}^2 g(y)}{\mathrm{d}y^2}\,, \tag{1.19}$$

hence it is clear we must not forget to make distinction between the partial and the total derivatives and the fact that while doing a partial derivative, the implicitly defined relation $x_0(y)$ is being set as a constant.

We shall show one more identity, namely that the second derivatives of the functions f and g are inverse to each other. Taking a derivative of the defining relation for y in (1.16) w.r.t. x_0 and multiplying it it with the derivative of x_0 from (1.18) w.r.t. y, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x_0} = \frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2}, \quad \frac{\mathrm{d}x_0}{\mathrm{d}y} = \frac{\mathrm{d}^2 g(y)}{\mathrm{d}y^2} \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x_0}\frac{\mathrm{d}x_0}{\mathrm{d}y} = 1 = \frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 g(y)}{\mathrm{d}y^2}\,. \tag{1.20}$$

Now, that we have demonstrated the intricacies on the simple example, we will relax the notation of x_0 and use only x keeping in mind, that for total derivatives there is actually an implicit relation defined by (1.16).

Now we generalize the Legendre transform to higher dimensions – let $x \in \mathbb{R}^d$ and $f : \mathbb{R}^d \mapsto \mathbb{R}$ a convex and differentiable function (the convexity of the function may be ensured by the positivity of the Hessian matrix). We define a Legendre transform of f to be a function $g : \mathbb{R}^d \mapsto \mathbb{R}$ such that the following properties hold:

$$f(x) + g(y) = x_i y_i \quad \text{where} \quad y_i := \frac{\partial f(x)}{\partial x_i}\,. \tag{1.21}$$

Here we can again show (using the same tricks as in the one dimensional case), that the following relations hold

$$\frac{\partial g(y)}{\partial y_i} = x_i\,, \quad \frac{\partial f(x)}{\partial y_i} = 0 = \frac{\partial g(y)}{\partial x_i} \quad \text{but} \quad \frac{\mathrm{d}g(y)}{\mathrm{d}x_i} = \frac{\partial^2 f(x)}{\partial x_i \partial x_k} x_k\,, \quad \frac{\mathrm{d}f(x)}{\mathrm{d}y_i} = \frac{\partial^2 g(y)}{\partial y_i \partial y_k} y_k\,. \tag{1.22}$$

The analogue of (1.20) now states that the Hessian matrices of functions f and g are inverse to each other i.e.

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_k}\frac{\partial^2 g(y)}{\partial y_k \partial y_j} = \delta_{ij}\,. \tag{1.23}$$

As in the previous section, we can formally extend all these formulas to infinite dimension, with the same substitutions as in (1.13), however, we are no longer able to make distinction between the partial and the total derivatives, as the functional derivatives are directional by definition (they are the analogue of the partial derivatives). For this reason, we will use the symbols $\frac{\mathrm{d}}{\mathrm{d}a_i}$ and $\frac{\partial}{\partial a_i}$ also in the case of the continuous index in places, where there would be a possibility of confusion. We will, of course, use the full notation of the functional calculus, when appropriate.

Because we will be often working with objects several times differentiated w.r.t. to many variables, we will use a compact notation for these objects as

$$F^{(n)}_{i_1 \dots i_n} \equiv F^{(n)}_{i_1 \dots i_n}(x) \equiv \frac{\partial^n F(x)}{\partial x_{i_1} \dots \partial x_{i_n}} \longleftrightarrow \frac{\delta^n F[f(x)]}{\delta f(x_{i_1}) \dots \delta f(x_{i_n})}\,, \tag{1.24}$$

where the first definition states, that if we do not explicitly specify the argument, none special was substituted after the differentiation and the former argument is still in place. In this notation, the last formula considering the Legendre transformation (1.23) would read

$$f^{(2)}_{ik} g^{(2)}_{kj} = \delta_{ij}. \tag{1.25}$$

We also choose to work with a metric tensor respecting the Landau–Lifschitz convention $\eta = \mathrm{diag}(+,-,-,-)$ and in units where $c = 1 = \hbar$. We will, however, sometimes recover the Planck's constant for discussion of its physical meaning.

Chapter 2

Effective Action in Quantum Field Theories

This chapter will be devoted to a description of a path integral approach to quantum field theories, particularly to one of its useful tools – the Effective Action. We will try to show its relations to other physical quantities present in field theories and give some intuitive idea, what the effective action represents. We will see, that the name is not at all arbitrary – the effective action is in the semi-classical limit identical to the classical action and on the quantum level provides a systematic (at least in principle) way of calculating quantum corrections in the form of loop expansion. All theoretical framework will be accompanied by demonstrations on the ϕ^4 theory, and the goal of this chapter will by to introduce and work out an example of use of all tools needed for the calculation of the effective potential done in the Chapter 4. This chapter will be a summary based on textbooks [2, 3, 4, 5].

We would also like to remark on the fact, that there are several sign conventions and they are usually chosen to give nice formulas in specific textbooks. Thus, every textbook has its own specialities, which are then very hard to combine. The ambiguities in sing conventions include: overall sign of the Minkowskian action, sign of the propagator, sign of the source J, sign (or overall factor) of $W[J]$ etc. The inconsistencies of the results are sometimes very hard to overcome and we will stumble upon numerous discussions trying to make sense of it all.

2.1 Functional Integral Formulation of QFT

We will not go into any detail about the basic notions of path integrals in quantum mechanics or its generalizations to field theories. We are also acquainted with the problems of rigorous mathematical formulation of path integrals. Some of the problems might be addressed by performing all calculations in the so-called Euclidean regime, where the integral is well behaved. Here we rely on the paper by K. Osterwalder and R. Schrader [6], who proved the equivalence between the Minkowskian and the Euclidean regime in flat spacetimes.

We shall also relax strict distinction between the terms "path integral" and "functional integral" even though these should be differentiated, since in the field theory, we sum over field configurations and not paths in the sense of a trajectory of a particle. We also choose to work in a fixed dimension of four.

Let us first start by a formal definition of the Wick rotation, which replaces the Minkowskian functional integral with the Euclidean one. It is defined as a transformation of the coordinates $x^\mu \to \bar{x}^\mu$ so that $\bar{x}^0 = \mathrm{i}x^0$ and $\bar{x}^i = x^i$, which means we have rotated the time evolution from the real axis to the imaginary one. This transformation changes the sign in front of the kinetic term in the Lagrangian, effectively transforming the signature of the metric, which is the reason why we speak of a Euclidean regime. To see the change explicitly, let us first write a free field equations of motion for a (uncharged) scalar theory, which is given by the Klein–Gordon equation

$$\big(\partial_t^2 - \Delta + m^2\big)\phi(x) = \big(\Box + m^2\big)\phi(x) = 0\,. \tag{2.1}$$

A Lagrangian, which yields these equations of motion must be of the form (up to a sign, of course)

$$\mathcal{L} = \frac{1}{2}\int \mathrm{d}^3x\, \partial_\mu\phi\, \partial^\mu\phi - m^2\phi^2 = -\frac{1}{2}\int \mathrm{d}^3x\, \phi\big(\Box + m^2\big)\phi\,, \tag{2.2}$$

where we have performed an integration by parts and used the fact, that the total divergence part does not change the equations of motion. Upon creating action from this Lagrangian and performing the Wick rotation, we obtain

$$S[\phi] = -\int \mathrm{d}^4x\, \frac{1}{2}\phi(\Box + m^2)\phi \quad \rightarrow \quad \mathrm{i}S_{\mathrm{E}}[\phi] = \mathrm{i}\int \mathrm{d}^4x\, \frac{1}{2}\phi\big(-\bar{\partial}_\mu\bar{\partial}^\mu + m^2\big)\phi\,, \tag{2.3}$$

where $\bar{\partial}_\mu\bar{\partial}^\mu = \partial^2_{\bar{x}_0} + \bar{\Delta}$ and we now work with a positively definite metric. This transformation makes the weight factor in the path integral negative

$$\mathrm{e}^{\mathrm{i}S[\phi]} \quad \rightarrow \quad \mathrm{e}^{-S_{\mathrm{E}}[\phi]} \tag{2.4}$$

thus the path integral becomes well behaved. This chapter will be based on the Euclidean regime, drawing parallels with the general formalism introduced in the Chapter 1 and we shall return to the Minkowskian regime in Chapter 4 where we won't be able to make use of the Osterwalder–Schrader theorem. The relation between the Euclidean and the Minkowskian regimes will be presented as a summary in section 2.4.9

2.2 Generating Functional of the Full Green Functions

2.2.1 $Z_0[J]$ for a Free Theory, Wick Expansion, Basic Relations

To formulate one of the central objects of the path integral formulation of quantum physics, we define a Green function of the free field operator $\Box + m^2$, which we denote $\tilde{\Delta}$ and call the (free field) propagator. Because we want to use the Wick rotation, we must ensure, that the poles of the propagator are not in the 1st or 3rd quadrant of the complex plane, since that is where the rotation to imaginary time takes place. For that reason we shift the poles in such a way, that they lay in the 2nd and the 4th quadrant, which defines the Feynman propagator. Now, denoting the free field operator in the "x-representation" $(\Box + m^2)\delta(x-y) \equiv \tilde{K}(x,y)$, we can finally use the results we have prepared in the previous mathematical section and define

$$Z_{0\mathrm{M}}[\tilde{J}] = \int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left(-\frac{1}{2}\phi(x)(\Box + m^2)\phi + \tilde{J}(x)\phi(x)\right)} = \int \mathcal{D}\phi \exp\left\{\mathrm{i}\Big(-\frac{1}{2}\phi_x\tilde{K}_{xy}\phi_y + \tilde{J}_x\phi_x\Big)\right\}, \tag{2.5}$$

which very much resembles (1.3) except for the i in the exponent. The subscript 0 signifies we deal with the free theory and M that this is the Minkowskian regime. After performing the Wick rotation, we redefine the source $\tilde{J}(x)$ to $J(\bar{x})$ so, that the new exponent is equal to

$$Z_0[J] = \int \mathcal{D}\phi\, \mathrm{e}^{-S_{0\mathrm{E}}[\phi]+\int \mathrm{d}^4\bar{x} J(\bar{x})\phi(\bar{x})} \equiv \int \mathcal{D}\phi \exp\Big\{-\frac{1}{2}\phi_{\bar{x}} K_{\bar{x}\bar{y}} \phi_{\bar{y}} + J_{\bar{x}}\phi_{\bar{x}}\Big\} \equiv \int \mathcal{D}\phi\, \mathrm{e}^{-S_0[\phi,J]}\,, \quad (2.6)$$

where we have defined new Euclidean action $S_0[\phi, J] = S_{0\mathrm{E}}[\phi] - J_{\bar{x}}\phi_{\bar{x}}$ and we have dropped the subscript E to ease the notation. This redefinition of the source is comfortable, for now we need no compensating signs when searching for the expectation values. As we have already pointed out, the Wick rotation to imaginary time makes the integral bounded and well behaved. As a consequence we can now regard it as a Gaussian distribution defined by the infinite matrix $K_{\bar{x}\bar{y}}$ assigned to the Euclidean free field operator $-\bar{\partial}_\mu \bar{\partial}^\mu + m^2$. We will further ease the notation and leave out the bars above the Euclidean coordinates. To simplify the integral, we use formulas from chapter 1, namely (1.4) to obtain

$$Z_0[J] = Z_0[0] \exp\Big(\frac{1}{2} J_x \Delta_{xy} J_y\Big) = N\big(\det K\big)^{-1/2} \exp\Big(\frac{1}{2} J_x \Delta_{xy} J_y\Big)\,, \quad (2.7)$$

where Δ is now Euclidean free field Feynman propagator. This result can by nicely analytically continued in $\bar{x}_0$ back to real times which is what we expect and need for further calculations. The constant N contains for the factor $(2\pi)^{d/2}$ is seemingly infinite, but we can include it in the normalization of $Z_0[0]$[1]. We usually call the functional $Z_0[J]$ the *partition function* in statistical mechanics, but in this context, we use the term *generating functional of full n-point green functions.*

Le us not present some properties of the generating functional and give it a physical meaning. Firstly, from the general mathematical discussion in Chapter 1 we see it is possible to calculate expectation values of fields ϕ in terms of $Z[J]$ as

$$\langle \phi(x)\ldots\phi(y)\rangle_0 = \frac{\delta}{\delta J(x)} \cdots \frac{\delta}{\delta J(y)}\bigg|_{J=0} \exp\Big(\frac{1}{2} J_x \Delta_{xy} J_y\Big)\,. \quad (2.8)$$

These are called the *full n-point Green functions* or *full n-point correlation functions.* From the physical point of view, they correspond to the vacuum expectation values $\langle 0|T\big(\phi(x)\ldots\phi(y)\big)|0\rangle_0$ – this can be seen for example from the generalization of the path integral formulation of quantum mechanics. We can generalize this expectation value to

$$\langle 0_{\mathrm{out}}|T\big(\phi(x)\ldots\phi(y)\big)|0_{\mathrm{in}}\rangle_0^J = \frac{\delta}{\delta J(x)} \cdots \frac{\delta}{\delta J(y)} \exp\Big(\frac{1}{2} J_x \Delta_{xy} J_y\Big)\,, \quad (2.9)$$

which is a vacuum to vacuum matrix element of the operator $T(\phi(x)\ldots\phi(y))$. The superscript J points to the fact, that the source is active and it influences the system, changing the Hilbert spaces assigned to the system. Thus the ground states changes as well and they are no longer identical to each other, hence the notation $|0_{\mathrm{in}}\rangle$ and $|0_{\mathrm{out}}\rangle$. In this sense the generating functional $Z_0[J]$ itself corresponds to $\langle 0_{\mathrm{out}}|0_{\mathrm{in}}\rangle^J$ – a probability amplitude for the vacuum to evolve back into a vacuum under the influence of the sources. We call $\langle 0_{\mathrm{out}}|0_{\mathrm{in}}\rangle^J$ the *persistence of the vacuum.* From this we conclude, that $Z_0[0]$ corresponds to $\langle 0|0\rangle$ (no sources are present) which we would like to normalize to 1, hence it can be dropped from the formulas, making (2.7) go to

$$Z_0[J] = \exp\Big(\frac{1}{2} J_x \Delta_{xy} J_y\Big)\,. \quad (2.10)$$

[1] This is a usual trick, which will be drawn upon several times later in the text. The reasoning behind the legality of it is discussed below 2.19

Secondly, we shall point out that $Z_0[J]$ implicitly codes the dynamics of a free field theory, since it contains the propagator, which can be extracted as a second variation w.r.t. the sources and using the definition (2.8) we can also write it as

$$\left.\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y}\right|_{J=0} Z_0[J] = \Delta_{xy} = \langle\phi_x\phi_y\rangle_0 \ , \tag{2.11}$$

hence we see that by definition the propagator is equal to a two-point Green function $\langle\phi_x\phi_y\rangle_0$. It also means that the propagator is symmetrical in its indices. This new relation makes it possible for us to rephrase the Wick expansion for a free field theory in the language of the propagators as

$$\langle\phi_{i_1}\dots\phi_{i_s}\rangle_0 = \sum_{\substack{\text{all possible pairings}\\ p \text{ of } \{i_1\dots i_s\}}} \Delta_{i_{p_1}i_{p_2}}\dots\Delta_{i_{p_{s-1}}i_{p_s}} \ , \tag{2.12}$$

for s even, and 0 for s odd. At this point, we can introduce a diagrammatic representation of the objects we are dealing with. The representation in Feynman Diagrams makes it easier to manipulate with the formulas and equations (they will be soon enriched by nasty integrals) and more intuitive to grasp the quantum world. The propagator Δ_{ij} is usually represented as a line connecting two dots, as we see in Figure 2.1.

Figure 2.1: Diagrammatic representation of the free field propagator.

To give an example, what the free field Wick expansion looks like, we find $\langle\phi_i\phi_j\phi_k\phi_l\rangle_0$

$$\begin{aligned}\langle\phi_i\phi_j\phi_k\phi_l\rangle_0 &= \langle\phi_i\phi_j\rangle_0\langle\phi_k\phi_l\rangle_0 + \langle\phi_i\phi_k\rangle_0\langle\phi_j\phi_l\rangle_0 + \langle\phi_i\phi_l\rangle_0\langle\phi_j\phi_k\rangle_0 \\ &= \Delta_{ij}\Delta_{kl} + \Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk} \, ,\end{aligned} \tag{2.13}$$

and show its diagrammatic representation in Figure 2.2

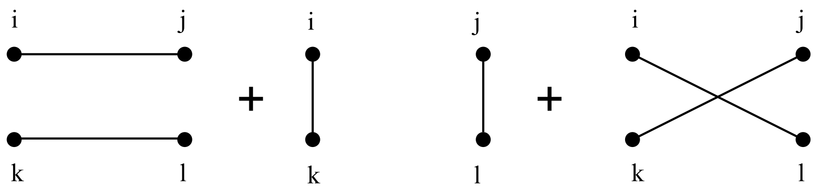


Figure 2.2: Diagrammatic representation of the 4-point Green function for the free field theory.

2.2.2 Interacting Theory, Wick Expansion, $Z[J]$ as a Generating Functional

To show another important property of the generating functional, we extend its definition to the interacting theory. Let us assume the Euclidean action is of the form

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\phi(x)K(x,y)\phi(y) + V_\mathrm{I}(\phi)\right) \equiv S_0[\phi] + \int \mathrm{d}^4x\, V_\mathrm{I}(\phi) \, , \tag{2.14}$$

with $V_\mathrm{I}(\phi)$ being the interacting potential and $S_0[\phi]$ the free field action. In order to be able to use our compact notation, we assume the interaction potential has a series expansion with coefficients $Y_{i_1\dots i_n}$ which stand for n-legged vertices (see fig. 2.3), i.e. we expand the interaction part of the action as

$$S_{\rm I}[\phi] \equiv \int \mathrm{d}^4x\, V_{\rm I}(\phi) \equiv \sum_{n=3} \frac{1}{n!} Y_{i_1\dots i_n}\phi_{i_1}\dots\phi_{i_n}\,. \tag{2.15}$$

It is clear, that there can be no term of power two, which would only rescale the mass of the field and also no linear term, since these are not physically interesting. For our example of the ϕ^4 theory, we have

$$S_{\rm I}[\phi] = \frac{1}{4!} Y_{ijkl}\phi_i\phi_j\phi_k\phi_l = \frac{g}{4!}\int \mathrm{d}^4x_i\,\phi(x_i)\,, \tag{2.16}$$

from where we conclude, that $Y_{ijkl} = g\,\delta_{ij}\delta_{ik}\delta_{il}$.

With this notation, the generating functional for the interacting theory is now defined as

$$Z[J] \equiv \int \mathcal{D}\phi\, \mathrm{e}^{-(\frac{1}{2}\phi_x K_{xy}\phi_y + S_{\rm I}[\phi] - J\phi)} = \int \mathcal{D}\phi\ \mathrm{e}^{-\int \mathrm{d}^4x V_{\rm I}(\phi)}\ \mathrm{e}^{-S_0[\phi,J]}\,. \tag{2.17}$$

Now we use the Feynman trick with the derivatives and obtain

$$Z[J] = \mathrm{e}^{-\int \mathrm{d}^4x V_{\rm I}\left(\frac{\delta}{\delta J}\right)} Z_0[J] = \mathrm{e}^{-\int \mathrm{d}^4x V_{\rm I}\left(\frac{\delta}{\delta J}\right)}\ \mathrm{e}^{\frac{1}{2}J_x\Delta_{xy}J_y}\,. \tag{2.18}$$

We shall now define expectation values of functions in this full, interacting theory by setting

$$\langle\phi(x)\dots\phi(y)\rangle = \frac{1}{Z[J]}\frac{\delta}{\delta J(x)}\cdots\frac{\delta}{\delta J(y)}\bigg|_{J=0} Z[J]\,, \tag{2.19}$$

where we put the normalization factor $1/Z[J]$ in order to cancel out the the vacuum divergences (we will expand on this topic later). It is also worth noting that any multiplicative numerical factor in front of $Z[J]$ is irrelevant to the expectation values (i.e. the physically relevant quantities) for if we chose $NZ[J]$ instead of $Z[J]$, the factor N would cancel itself out as can be seen from the definition.

The definition works also for functionals $F[\phi]$, such as in (2.18)

$$\begin{gathered}\langle F[\phi]\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi\, F[\phi]\,\mathrm{e}^{-S[\phi]} = \frac{1}{Z[J]} F\left[\frac{\delta}{\delta J}\right]\bigg|_{J=0}\int \mathcal{D}\phi\,\mathrm{e}^{-S[\phi,J]} = \frac{1}{Z[J]} F\left[\frac{\delta}{\delta J}\right]\bigg|_{J=0} Z[J]\,,\\ S[\phi,J] = S_0[\phi,J] + \int \mathrm{d}^4x\, V_{\rm I}(\phi)\,,\end{gathered} \tag{2.20}$$

We have already pointed out, that the for the free propagator the following property is very important

$$\Delta_{xy} = \langle\phi(x)\phi(y)\rangle_0 = Z^{(2)}_{0xy}[0]\,. \tag{2.21}$$

This notion of a propagator also holds in the interacting theory. Therefore now call $\langle\phi(x)\phi(y)\rangle$ the *full propagator* and we expect it to carry all information about a particle (field configuration) evolving from one configuration into another

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{Z[J]}\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y}\bigg|_{J=0} Z[J] = \left(\frac{1}{Z[J]}\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y} e^{-\int \mathrm{d}^4x V_{\rm I}\left(\frac{\delta}{\delta J}\right)} Z_0[J]\right)\bigg|_{J=0}\,. \tag{2.22}$$

Let us hint, how to find $Z[J]$ and the propagator for the ϕ^4 theory up to the second order of a coupling constant. We take $V_\mathrm{I}(\phi) = \frac{g}{4!}\phi^4(x)$ and expand exp in a Taylor series, assuming the coupling constant g is small, thus obtaining

$$Z[J] = \left[1 - \frac{g}{4!}\int \mathrm{d}^4x \left(\frac{\delta}{\delta J(x)}\right)^4 + \frac{1}{2}\left(\frac{g}{4!}\right)^2 \int \mathrm{d}^4x\,\mathrm{d}^4y \left(\frac{\delta}{\delta J(x)}\right)^4 \left(\frac{\delta}{\delta J(y)}\right)^4\right] \mathrm{e}^{\frac{1}{2}J_x\Delta_{xy}J_y}\,. \quad (2.23)$$

After computing the variations and setting $J=0$, one finds the generating functional of the full theory in the first order to be

$$Z[0] = Z_0[0]\Big(1 + \frac{g}{8}\Delta_{xx}\int \mathrm{d}^4x\Big)\,, \quad (2.24)$$

which is clearly divergent due to both the integral over the spacetime and $\Delta_{xx} = \Delta(0)$. This divergence accounts for the so-called *vacuum bubbles*, which are a manifestation of the fact, that there may be infinitely many interactions and interaction loops in an empty space (vacuum) that will be never detected in an external device and we must define the zero level of the energy. This is also the reason, why we put the normalization factor $1/Z[J]$ into the expectation value, since upon setting $J=0$ that term effectively eliminates all contributions of the vacuum bubbles.

The calculations are straightforward but lengthy and can be found in most textbooks on QFT. We will therefore not perform them here, but show an easy and systematic way, how to find a diagrammatic representation of the same thing in several different ways. The first possibility is to diagrammatically represent the equation (2.23). To that end we assign diagrams to J_i, $Y_{ij\ldots l}$ and the operator $\frac{\delta}{\delta J}$ as seen in figure 2.3

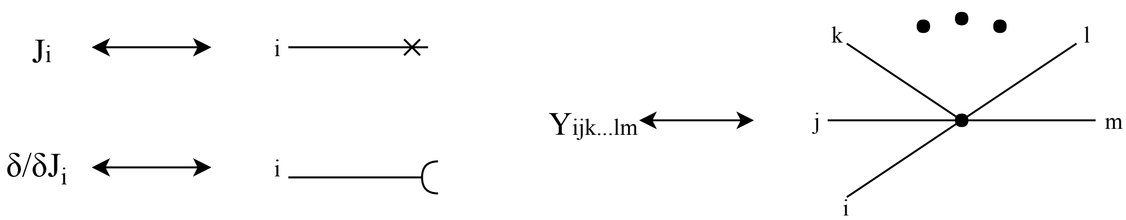


Figure 2.3: Diagrammatic representation of the source J_i and the operator $\frac{\delta}{\delta J}$ (left) and the interaction vertex $Y_{ijk\ldots lm}$ (right).

Figure 2.4 illustrates the expansion of $Z[J]$ and in the second line it shows all diagrams relevant for the full propagator $\langle\phi(x)\phi(y)\rangle$ to the second order of the coupling constant. This expansion is also called the Wick expansion. To obtain the full result we think of the sources J and the operators $\frac{\delta}{\delta J}$ as complementary and we join them together. This accounts for the action of the operator on the source. The operators $\frac{\delta}{\delta J}$ are attached to the interaction vertices as is prescribed by (2.23). For the full propagator we would then have to act on the resulting diagram twice more by $\frac{\delta}{\delta J}$ and then set $J=0$, therefore we leave two waiting legs with sources J_i on every end of the diagram – these will be acted on by the operators and the resulting diagrams will constitute the only non-zero contributions after setting $J=0$.

To arrive at the correct result, we must include some combinatorial factors to compensate for the factorials from the expansion. The first non-trivial term is easy – we have two ways of orienting $J_i\Delta_{ij}J_j$ so we multiply the term by 2, which yields the first result. The second term is a bit more complicated – we proceed systematically from the interaction vertex. Let us pick one leg of the vertex – there are 4 options to do that. To this chosen leg we can attach one of three

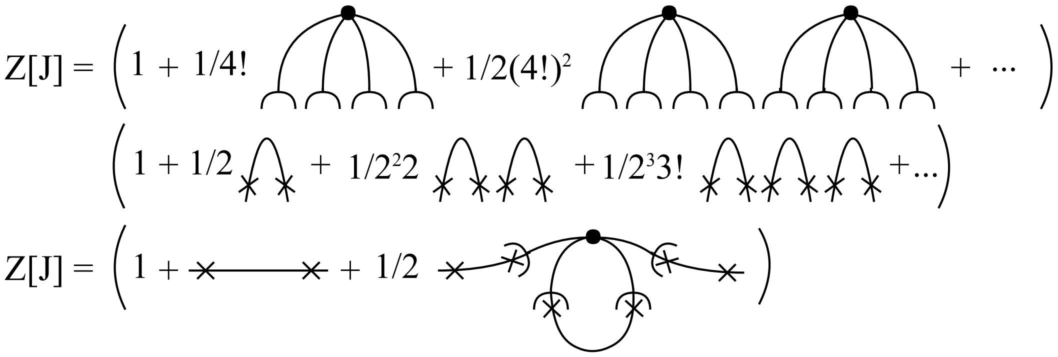


Figure 2.4: The diagramatic version of the Wick expansion of the generator $Z[J]$ and the specific choice of its solution for the full 2-point Green function.

$J_i\Delta_{ij}J_j$ in two ways (orientation) i.e. we get $4 \cdot 3 \cdot 2$ from the first step. Lets pick another leg (3 left) which can be joined with one of 2 remaining $J_i\Delta_{ij}J_j$, times 2 for the orientation – this step provides another $3 \cdot 2 \cdot 2$. Now we have last two legs on the vertex and one $J_i\Delta_{ij}J_j$ which will close a loop. That can happen only 2 different ways. Altogether, we have $4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 2$ options which exactly compensate for the factor $\frac{1}{4!}\frac{1}{3!}\frac{1}{2^3}$ to give the correct result $\frac{1}{2}$.

We have still not made clear, why $Z[J]$ is a generating functional of the full n-point Green functions. However, we have already said that the full n-point Green function is given by the expectation value $\langle\phi_{i_1}\dots\phi_{i_n}\rangle$, which is defined through the variation of $Z[J]$. Hence after expanding the functional around $J = 0$

$$Z[J] = \sum_{n=0} \frac{1}{n!} Z^{(n)}_{i_1\dots i_n}[0] J_{i_1}\dots J_{i_n}\,, \tag{2.25}$$

where $Z^{(n)}_{i_1\dots i_n}[0] = \frac{\delta}{\delta J_{i_1}}\dots\frac{\delta}{\delta J_{i_n}}\Big|_{J=0} Z[J] = Z[0]\,\langle\phi_{i_1}\dots\phi_{i_n}\rangle$, we find, that these objects are proportional to the n-point Green functions. This proportionality can be simplified by changing the normalization to $Z[0] = 1$. Since this can be always done, we will drop the normalization factor from now on.

In Figure 2.6 we show, how the generating functional can be represented as s Feynman diagram, for which we use also a diagrammatic representation of the n-point Green function illustrated in Figure 2.5

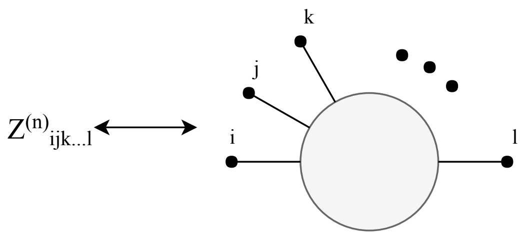


Figure 2.5: Diagrammatic representation of the full n-point Green function.

If we were to represent $\frac{\delta Z[J]}{\delta J_i}$ we would have similar expansion series as in Figure 2.6, but the right hand side would shift by one blob to the left (the coefficient would stay) and each blob would have one external leg with a dot at the end instead of a source. After removing the sources (setting $J = 0$), the only surviving term would be a blob having one pointing leg having a dot

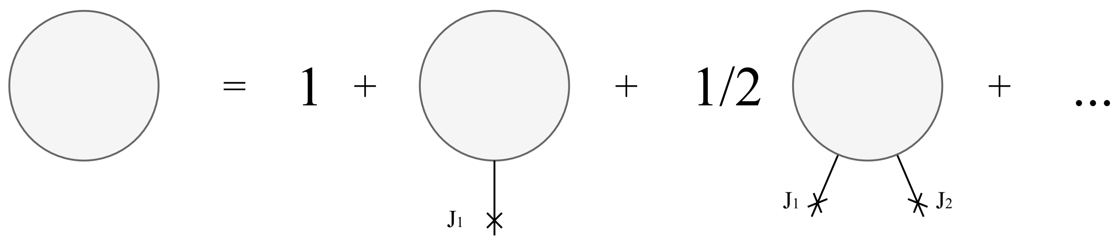


Figure 2.6: Diagrammatic representation of the generating functional of full Green functions $Z[J]$.

at its end, thus clearly respecting the definition in Figure 2.5. We also point out, that the full propagator is represented as a blob with two external legs and dots at the ends.

2.2.3 The Dyson–Schwinger Equation

We have one more thing to show considering the functional $Z[J]$ before we move on to other objects. We shall find an equation that would give full dynamics of the quantum system. Let us now consider transformations of the field ϕ such that the integral measure $\mathcal{D}\phi$ in invariant. Then, since the generating functional is independent of the field, we must have

$$\frac{\delta Z[J]}{\delta \phi_i} = 0 = -\int \mathcal{D}\phi \frac{\delta S[\phi, J]}{\delta \phi_i} \, \mathrm{e}^{-S[\phi,J]} \,. \tag{2.26}$$

This has a profound interpretation once we use (2.20) to rewrite it as

$$\left\langle \frac{\delta S[\phi, J]}{\delta \phi_i} \right\rangle^J = 0 \,, \tag{2.27}$$

which is nothing else than a statement, that the classical least action principle holds as a vacuum expectation value. Note here, that the expectation value is calculated with the source present, which is also respected by the equations of motion where the source is included. We may further rewrite it using the fact, that $S[\phi, J] = S[\phi] - J_x \phi_x$ and the definition of the expectation value as

$$\left(\frac{\delta S}{\delta \phi_i} \left[\frac{\mathrm{d}}{\mathrm{d}J} \right] - J_i \right) Z[J] = 0 \,, \tag{2.28}$$

obtaining a very important equation, which will accompany us throughout this chapter. This equation is called the Dyson–Schwinger (DS) equation and it provides full dynamics of the quantum system.

The Dyson-Swchinger equation has a nice diagrammatic representation (see 2.7), which we will now try to explain. The systematic approach to creating Feynman diagrams according to the Dyson–Swchinger equation [5] for a certain theory is as follows – a particle going into a blob will either survive and (freely propagate to another external leg) or interact once in all possible ways the theory allows. This algorithm is then iterated to obtain any order of precision. Every interaction vertex brings one power of the coupling constant, thus for small coupling constant, we can effectively discard high order corrections since they are negligible. Also, to compensate for the symmetry of the vertices, we add a factor $\frac{1}{(n-1)!}$ to all vertices with n legs.

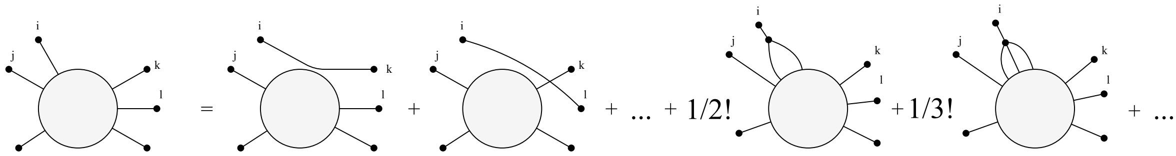


Figure 2.7: Dyson–Schwinger equation in terms of the Feynman diagrams.

Using this we can find the two point function of the ϕ^4 theory in the language of diagrams very easily – the "calculation" is presented in Figure 2.8. We clearly see, it confirms once already obtained result from the Wick expansion seen in Figure 2.4.

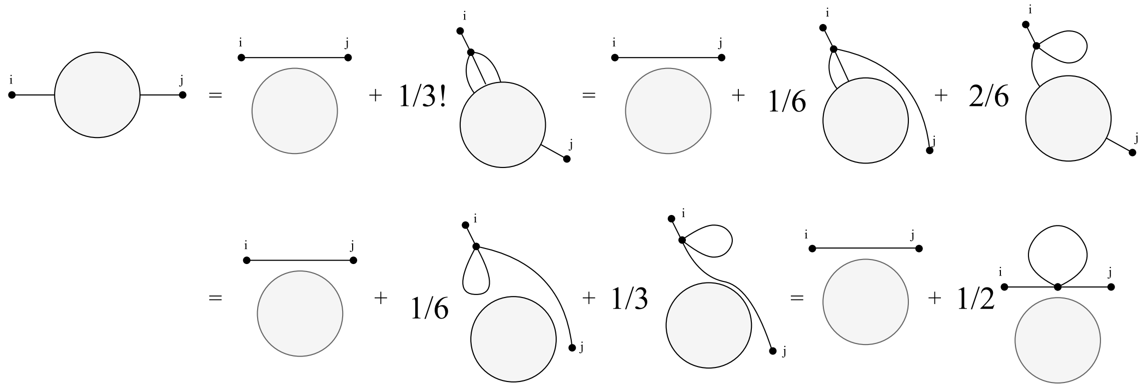


Figure 2.8: Dyson–Schwinger equation for the ϕ^4 theory.

We note there remains an unattached blob at each term, corresponding to the vacuum bubbles. These get removed simply by normalizing the expectation value – dividing by $Z[J]$ or setting $Z[0] = 1$. Note also that the propagator is ill behaved for the presence of the loop caused by $\Delta(0)$.

Identical result can be written in the mathematical language as seen in the equation (2.29)

$$
\begin{aligned}
&\langle \phi_i \phi_j \rangle = \Delta_{ij} + \frac{1}{2}\Delta_{ik} Y_{klmn} \Delta_{lm} \Delta_{nj} = \Delta(x_i - x_j) + \\
&+ \frac{1}{2}\int \mathrm{d}^4 x_k \mathrm{d}^4 x_l \mathrm{d}^4 x_m \mathrm{d}^4 x_n \mathrm{d}^4 x \, \Delta(x_i - x_k)\delta(x - x_k)\delta(x - x_l)\delta(x - x_m)\delta(x - x_n)\Delta(x_l - x_m)\Delta(x_n - x_j) \\
&= \Delta(x_i - x_j) + \frac{1}{2}\Delta(0)\int \mathrm{d}^4 x \, \Delta(x_i - x)\Delta(x - x_j)\,.
\end{aligned}
\tag{2.29}
$$

The result coincides with the one found using the Wick expansion, but we see here, that there indeed is a good reason to use a compact notation and the Feynman diagrams for its simplicity.

2.3 Generating Functional of the Connected Green Functions

In this section we introduce a new object, that was already mentioned in the first chapter. The whole point of this successive introduction of different generators is to simplify the classification of Feynman diagrams. That is why we now reduce the problem form the full Green functions to the connected ones. Later, we will also introduce basic building blocks of the connected ones.

2.3.1 Connectedness

We will now define a generator of connected Green functions. We remark here, that the sign convention for the definition is not unified. What is important, however, is the consequent definition of the effective action. This allows us to define it here with a plus sign, so that the formulas are not burdened by extra sign. Using the general formula from the first chapter, we can simply define it as (normalization $Z[0] = 1$ assumed)

$$W[J] = \ln Z[J] \equiv \sum_{n=1} \frac{1}{n!} W^{(n)}_{i_1 \ldots i_n}[0] J_{i_1} \ldots J_{i_n} \quad \Longleftrightarrow \quad Z[J] = \mathrm{e}^{W[J]}\,, \tag{2.30}$$

where $W^{(n)}_{i_1 \ldots i_n}[0] = \frac{\delta}{\delta J_{i_1}} \cdots \frac{\delta}{\delta J_{i_n}}\big|_{J=0} W[J]$ is the *connected n-point Green function.* If $Z[J]$ was a partition function of statistical physics, $W[J]$ would correspond (up to a sign) to the free energy of the system. That this object generates the connected Green functions can be illustrated the following way: we calculate the first few derivatives of $Z[J]$ and show it decomposes into a sum of one connected and several disconnected Green functions (Figure 2.9 illustrates this decomposition)

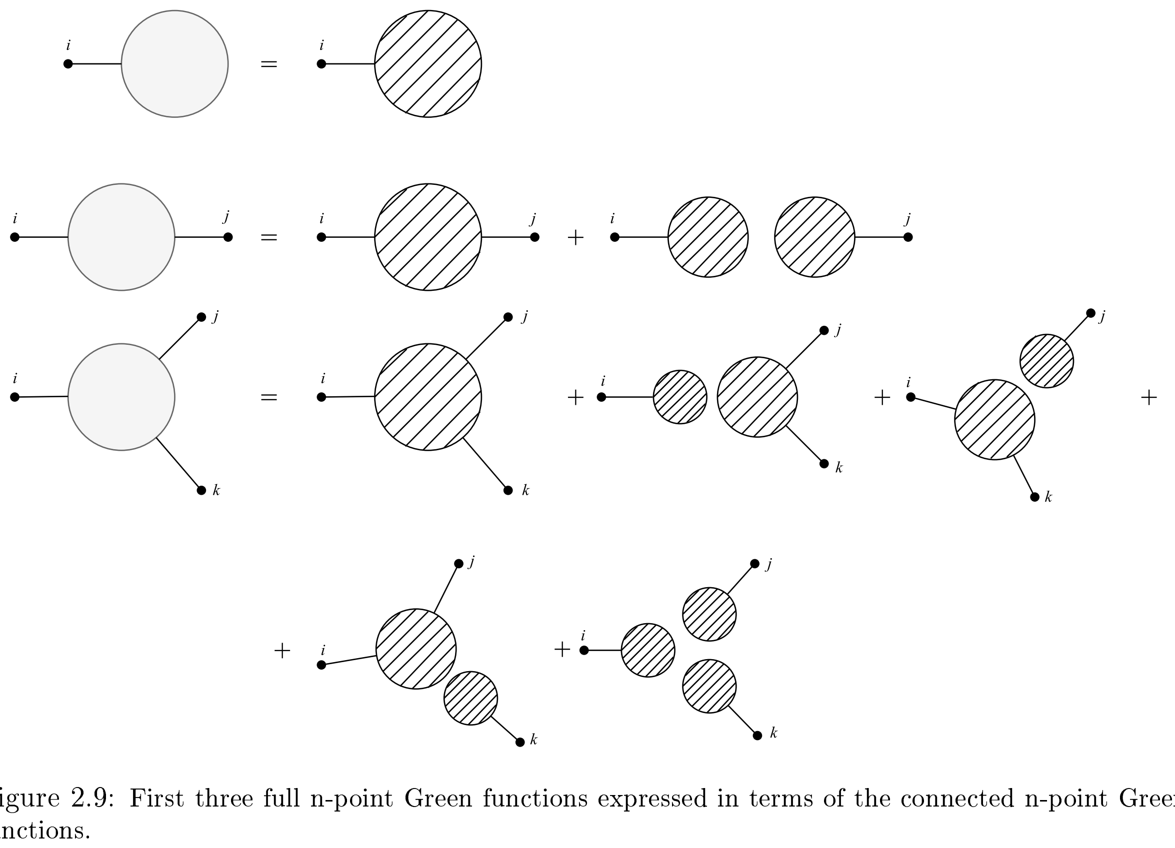


Figure 2.9: First three full n-point Green functions expressed in terms of the connected n-point Green functions.

$$
\begin{aligned}
Z^{(1)}_x[0] &= \frac{\delta Z[0]}{\delta J_x} = \frac{\delta}{\delta J_x}\Big|_{J=0} \mathrm{e}^{W[J]} = \frac{\delta W[0]}{\delta J_x} = W^{(1)}_x[0]\,, \\
Z^{(2)}_{xy}[0] &= \frac{\delta}{\delta J_x}\frac{\delta W[J]}{\delta J_y}\,\mathrm{e}^{W[J]}\Big|_{J=0} = W^{(2)}_{xy}[0] + W^{(1)}_x[0]W^{(1)}_y[0]\,, \\
Z^{(3)}_{xyz}[0] &= W^{(3)}_{xyz}[0] + W^{(1)}_x[0]W^{(2)}_{yz}[0] + W^{(1)}_y[0]W^{(2)}_{xz}[0] + W^{(1)}_z[0]W^{(2)}_{xy}[0] + W^{(1)}_x[0]W^{(1)}_y[0]W^{(1)}_z[0]\,, \\
&\vdots
\end{aligned}
\tag{2.31}
$$

There is yet another reasoning behind the definition of the functional $W[J]$ and it has roots in the language of diagrams. For every full one-point Green function we would like to separate the connected part attached to the external leg from the disconnected rest (which must again be the full generator $Z[J]$) – this is idea is illustrated in Figure 2.10. The mathematical equation

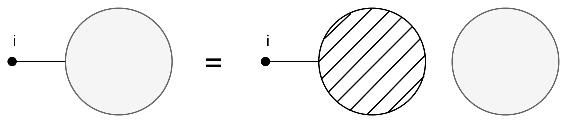


Figure 2.10: Alternative defining equation for the generator of the connected Green functions.

describing this relation reads

$$
\frac{\delta Z}{\delta J_x} = \frac{\delta W}{\delta J_x} Z[J]\,, \tag{2.32}
$$

and is solved by the $Z[J] = \mathrm{e}^{W[J]}$, thus the definition (2.30) can be seen in this more intuitive way.

Since we are not very much interested in disconnected diagrams (they correspond to unrelated processes and can be build from the connected ones) we define an expectation values using this new functional

$$
\langle \phi_x \dots \phi_y \rangle^J_{\mathrm{c}} \equiv W^{(n)}_{x\dots y}\,, \quad \text{where} \quad \langle \phi_x \dots \phi_y \rangle^{J=0}_{\mathrm{c}} \equiv \langle \phi_x \dots \phi_y \rangle_{\mathrm{c}}\,, \tag{2.33}
$$

and use it to search for all interesting physical quantities. In this sense, we can also define a connected propagator of the full theory to be the connected two-point Green function $\langle \phi_x \phi_y \rangle_{\mathrm{c}}$.

2.3.2 Dyson–Schwinger in Terms of Connected Green Functions

More interesting is the possibility to rewrite the Dyson–Schwinger equation in terms of the connected Green functions, which is clearly something we would like very much. To do so we first show a useful identity

$$
\frac{\delta}{\delta J}\big(Z[J]f[J]\big) = \frac{\delta W[J]}{\delta J} Z[J]f[J] + Z[J]\frac{\delta f[J]}{\delta J} = Z[J]\left(\frac{\delta W[J]}{\delta J} + \frac{\delta}{\delta J}\right) f[J]\,, \tag{2.34}
$$

thus after choosing the test function $f = 1$, we obtain

$$
\frac{\delta}{\delta J}\big(Z[J]\,\cdot\,\big) = Z[J]\left(\frac{\delta W[J]}{\delta J} + \frac{\delta}{\delta J}\right)\cdot\,, \tag{2.35}
$$

where the dot indicates it is an operator waiting for a function to act on. Let us now perform some gymnastics with the Dyson–Schwinger equation. We assume the function $\frac{\delta S[\phi]}{\delta \phi_i}$ has a polynomial expansion so that we can write

$$\frac{\mathrm{d}S}{\mathrm{d}\phi_i}\left[\frac{\delta}{\delta J}\right] Z[J] = \sum_{n=0} a_n \left(\frac{\delta}{\delta J}\right)^n Z[J]\,. \tag{2.36}$$

Next we insert an identity operator in an appropriate form between the operators $\frac{\delta}{\delta j}$ into each term in the expansion to get

$$\frac{\delta}{\delta J}\,\mathrm{e}^{W[J]}\,\mathrm{e}^{-W[J]}\frac{\delta}{\delta J}\dots\,\mathrm{e}^{-W[J]}\frac{\delta}{\delta J}\,\mathrm{e}^{W[J]}\,, \tag{2.37}$$

where we now use the identity (2.35) step by step from the right to get the factor $Z[J]$ on the left hand side of each operator $\frac{\delta}{\delta J_i}$, where it cancels with its counterpart. Thus we obtain an expansion in terms of $W[J]$

$$\frac{\delta S}{\delta \phi_i}\left[\frac{\delta}{\delta J}\right] Z[J] = Z[J]\sum_{n=0} a_n \left(\frac{\delta W[J]}{\delta J}+\frac{\delta}{\delta J}\right)^n = Z[J]\frac{\delta S}{\delta \phi_i}\left[\frac{\delta W[J]}{\delta J}+\frac{\delta}{\delta J}\right]. \tag{2.38}$$

Plugging this result back into the Dyson–Schwinger equation (2.28) and using the fact that the source J_i commutes with $Z[J]$, we can now divide the equation by $Z[J]$ to obtain

$$\frac{\delta S}{\delta \phi_i}\left[\frac{\delta W[J]}{\delta J}+\frac{\delta}{\delta J}\right] = J_i\,, \tag{2.39}$$

where we must not forget, it is still an operator acting on identity which we do not write explicitly.

Now we provide an example of the use of the Dyson–Schwinger equation in terms of the connected Green functions for the ϕ^4 theory. Since the Euclidean action is of the form $S[\phi] = \frac{1}{2}\phi_i\Delta^{-1}_{ij}\phi_j + \frac{1}{4!}Y_{ijkl}\phi_i\phi_j\phi_k\phi_l$ and we assume the interaction vertex is symmetrical in its indices, the DS equation yields

$$\begin{aligned} J_i &= \Delta^{-1}_{ij}\left(\frac{\delta W[J]}{\delta J_j}+\frac{\delta}{\delta J_j}\right) + \frac{1}{3!}Y_{ijkl}\left(\frac{\delta W[J]}{\delta J_j}+\frac{\delta}{\delta J_j}\right)\left(\frac{\delta W[J]}{\delta J_k}+\frac{\delta}{\delta J_k}\right)\left(\frac{\delta W[J]}{\delta J_l}+\frac{\delta}{\delta J_l}\right) \\ J_i &= \Delta^{-1}_{ij}W^{(1)}_j + \frac{1}{3!}Y_{ijkl}\left[W^{(1)}_jW^{(1)}_kW^{(1)}_l + 3W^{(1)}_jW^{(2)}_{kl} + W^{(3)}_{jkl}\right] \\ W^{(1)}_j &= \Delta_{ji}\left[J_i - \frac{1}{3!}Y_{ijkl}W^{(1)}_jW^{(1)}_kW^{(1)}_l - \frac{1}{2}Y_{ijkl}W^{(1)}_jW^{(2)}_{kl} - \frac{1}{3!}Y_{ijkl}W^{(3)}_{jkl}\right], \end{aligned} \tag{2.40}$$

where the minus signs arise because of the Euclidean regime where the action has a positive sign in front of the kinetic term (and due to the conventional choices we made in the text, as discussed in the first paragraph of this chapter, namely if we chose differently the sign of the propagator and of the source, we would have an exact result as in [5]). We did some straightforward reordering in the equation to obtain a formula for $W^{(1)}_j$ in the third line, which (after setting $J = 0$) corresponds to a one-point connected Green function. The right hand side provides a description of quantum dynamics based on the interaction potential. We see, that it can be iterated and further differentiated w.r.t. J to obtain higher order corrections. We also see from these diagrams, that setting the rules for creating Feynman diagrams is equivalent to writing a specific form of the action (the type of the interaction). We show diagrammatic representation of the equation (2.40) in Figure 2.11.

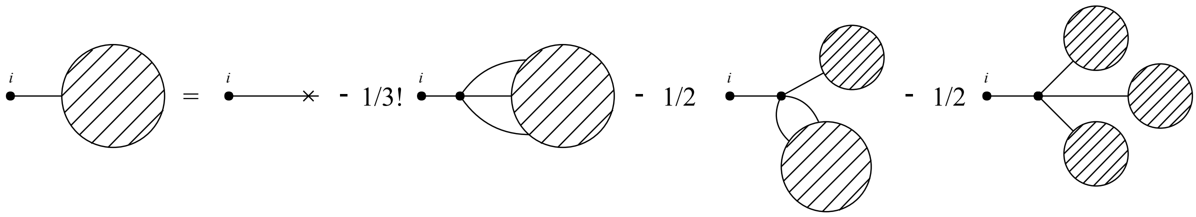


Figure 2.11: Dyson–Schwinger equation in terms of the connected Green functions for the ϕ^4 theory.

2.4 The Effective Action

Here we finally get to the main point of this chapter. We introduce an effective action, show some of its important properties as well as its connections to $W[J]$ and explain, why is it called the "effective action". Since there are two equivalent approaches to its introduction, we shall try to expand on each of them in a special subsection and then discuss the general properties.

2.4.1 Effective Action as a Generator of 1PI Green Functions

The first definition is based on the diagrammatic representation of field theory (see [5]). First we must define a *one-particle irreducible* (1PI) or *one-line irreducible* diagram – it is such a diagram, that cannot become disconnected by cutting one internal line. All other diagrams are one-particle reducible. Loops are examples and building blocks of 1PI of a diagrams (Green functions). It is clear that every Feynman diagram can be separated into 1PI parts joined by one internal line. Thus we would like to create a generating functional, which would systematically generate the 1PI Green functions. We propose the algorithm as follows:

1. Pick one external leg of a connected n-point function $W^{(1)}$
2. Follow it and pull out of the connected blob the 1PI part, that is connected to the rest of the diagram (represented by one or more $W^{(1)}$) by one leg(s)

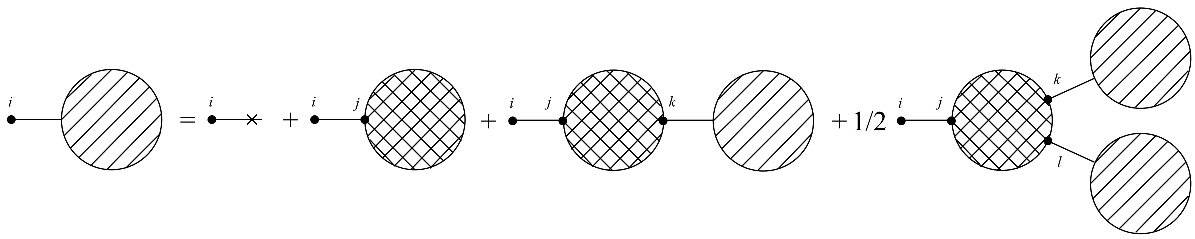


Figure 2.12: Diagrammatic definition of the generator of 1PI Green functions.

Figure 2.12 illustrates all possible outcomes of such a process and sums it all in an object that we denote by Γ. We have represented the n-point 1PI Green functions by cross-hatched blobs with n dots on its circumference. Upon rewriting the diagrammatic equation 2.12 into the language of mathematics, we get

$$W_i^{(1)} = \Delta_{ij}\left[J_j + \Gamma_j + \Pi_{jk}W_k^{(1)} + \frac{1}{2}\Gamma_{jkl}W_k^{(1)}W_l^{(1)} + \dots\right], \tag{2.41}$$

and with substitution $\frac{\delta W}{\delta J_i} \equiv \varphi_i$ and application of the inverse of the propagator we obtain

$$0 = J_j + \Gamma_j + \big(\Pi_{jk} - \Delta^{-1}_{jk}\big)\varphi_k + \frac{1}{2}\Gamma_{jkl}\varphi_k\varphi_l + \dots \,. \tag{2.42}$$

If we now collect all the 1PI Green functions into a generating functional

$$\Gamma[\varphi] = \sum_{n=1} \frac{1}{n!}\Gamma_{i_1 \dots i_n}\varphi_{i_1}\dots\varphi_{i_n}\,, \tag{2.43}$$

where we have denoted $\Gamma_{ij} = \Pi_{jk} - \Delta^{-1}_{jk}$ for reasons explained later, we can rewrite equation (2.42) as

$$0 = J_i + \frac{\delta\Gamma[\varphi]}{\delta\varphi_i}\,. \tag{2.44}$$

From this equation and from definition of φ_i it is clear that $W[J]$ and $\Gamma[\varphi]$ are related through the Legendre transformation. This will be our starting point for the next section.

2.4.2 Effective Action as a Legendre Transformation of $W[J]$

In this section we define the effective action by the Legendre transformation of $W[J]$, but we will use a different sign convention. We have already touched upon the problem of sings in the beginning of Section 2.3, where we defined $W[J]$ with a plus sign to obtain formulas without any extra signs. In Section 2.4.1 we introduced Γ using a specific notation to make it easier to compare it with the textbook [5]. Here, however, we must chose the sign so, that Γ reflects the properties we want it to have. Thus, as is done in [4], both W and Γ are defined with opposite signs. That is the reason, why we change the sign from Γ introduced in the previous section, as can be seen in the definition (2.46).

We have already mentioned, that $W[J]$ has the meaning of the free energy. Its Legendre transform Γ is usually known as the thermodynamic potential, however, in the context of QFT we call it the *effective action*. To perform the Legendre transform we need a new variable – that we have already defined in the previous section and the definition holds

$$\varphi_i \equiv \frac{\partial W}{\partial J_i}\,. \tag{2.45}$$

Here we have recovered the notation of partial and total derivatives to emphasise the difference. Its graphical representation is shown on the left hand side of equation in Figure 2.12. The Legendre transformation and its general consequences now read (see (1.22) in Chapter 1)

$$\begin{gathered} W[J] + \Gamma[\varphi] = J_i\varphi_i\,, \quad \frac{\partial\Gamma[\varphi]}{\partial\varphi_i} = J_i \quad \text{with} \quad \frac{\partial W[J]}{\partial\varphi_i} = 0 = \frac{\partial\Gamma[\varphi]}{\partial J_i}\,, \\ W^{(2)}_{ik}[J]\Gamma^{(2)}_{kj}[\varphi] = \delta_{ij}\,, \quad \frac{\mathrm{d}W[J]}{\mathrm{d}\varphi_i} = \Gamma^{(2)}_{ij}[\varphi]\varphi_j\,, \quad \text{and} \quad \frac{\mathrm{d}\Gamma[\varphi]}{\mathrm{d}J_i} = W^{(2)}_{ij}[\varphi]J_j\,. \end{gathered} \tag{2.46}$$

From the definition of the new field (2.45) we can immediately conclude its meaning. Upon setting $J = 0$ we get $\varphi_i = \langle\phi_i\rangle_{\mathrm{c}}$ – the expectation value of field ϕ when no sources are present. We can easily show that this quantity must be a constant

$$\langle\phi_i\rangle_{\mathrm{c}} = \langle\phi_i\rangle = \langle 0|\phi(x_i)|0\rangle = \langle 0|\,\mathrm{e}^{-\mathrm{i}Px}\phi(0)\,\mathrm{e}^{\mathrm{i}Px}|0\rangle = \phi(0)\,, \tag{2.47}$$

where we have used the generator of the spacetime translations P and the fact, that Poincaré invariance of the theory implies that the ground state is invariant under translations. Thus, when sources vanish, we have $\varphi = \phi(0) = \text{const.}$ We can use similar argument and show using a generator of $\mathbb{Z}_2$ parity $\mathcal{P}\phi(x)\mathcal{P}^\dagger = -\phi(x)$, that the expectation value is equal to minus itself (and hence to zero) in the case of a theory (action), that contains only even powers of fields. From this we conclude, that the expectation value is zero in both the free theory or and the ϕ^4 theory.

Further, the second relation in (2.46) states, that $\varphi = \phi(0)$ extremizes the effective action. This can be seen from the classical equations of motion

$$\frac{\delta S}{\delta \phi_i}[\phi] = J_i = \frac{\delta \Gamma}{\delta \varphi_i}[\langle \phi \rangle] = 0\,, \tag{2.48}$$

from where we conclude, that when ϕ is a solution of classical equations of motion without the presence of the source, then $\langle \phi \rangle$ is the solution of quantum equations of motion (which is the Dyson–Schwinger equation).

2.4.3 Effective Action of the Free Theory

Let us now concentrate on the free field theory – to distinguish the objects from those from the full theory, we add a subscript or superscript 0. Let us define φ_i^0 as in the case of the interacting theory

$$\begin{gathered} \varphi_i^0 = \frac{\partial W_0[J]}{\partial J_i} = \frac{1}{Z_0[J]}\frac{\partial Z_0[J]}{\partial J_i} = \frac{1}{Z_0[J]}\frac{\partial}{\partial J_i}\,\mathrm{e}^{\frac{1}{2}J_k \Delta_{kl} J_l} = \Delta_{ik} J_k \\ \Delta_{ji}^{-1}\varphi_i^0 = K_{ji}\varphi_i^0 = J_j\,, \end{gathered} \tag{2.49}$$

where we recall that K is the Euclidean operator appearing in the kinetic term of the action, i.e. $K(x,y) = (-\partial_\mu \partial^\mu + m^2)\delta(x-y)$. We can use the second relation to substitute for sources in the Legendre transformation to find the free field effective action Γ_0 by definition

$$\begin{aligned} \Gamma_0[\varphi^0] &= -W_0[J] + \varphi_i^0 J_i = -\frac{1}{2}J_k \Delta_{kl} J_l + \varphi_i^0 J_i = -\frac{1}{2}\varphi_j^0 K_{kj}\Delta_{jl}K_{lm}\varphi_m^0 + \varphi_i^0 K_{ik}\varphi_k^0 \\ &= \frac{1}{2}\varphi_i^0 K_{ik}\varphi_k^0 = S_{0E}[\varphi^0]\,. \end{aligned} \tag{2.50}$$

Now we see explicitly that there is indeed a good reason to call Γ the effective action since it seems to very similar to the classical action. Probably the strongest argument for the name *effective action* is, however, the Dyson–Schwinger equation in the language of 1PI Green functions, which will be our next task.

2.4.4 Dyson–Schwinger in Terms of the 1PI Green Functions

The last formulation of the DS equation we have obtained in (2.39) contained $W^{(1)}[J]$ and the operator $\frac{\mathrm{d}}{\mathrm{d}J}$. We can easily substitute the former, but to substitute the latter, we shall replace the (total) derivative w.r.t. J by a (total) derivative w.r.t. the new field variable φ. To do so we use the generalized chain rule for variations and the definition of the field φ

$$\frac{\mathrm{d}}{\mathrm{d}J_i} = \frac{\mathrm{d}\varphi_j}{\mathrm{d}J_i}\frac{\mathrm{d}}{\mathrm{d}\varphi_j} = \frac{\mathrm{d}}{\mathrm{d}J_i}\big(W_j^{(1)}[J]\big)\frac{\mathrm{d}}{\mathrm{d}\varphi_j} = W_{ij}^{(2)}[J]\frac{\mathrm{d}}{\mathrm{d}\varphi_j}\,. \tag{2.51}$$

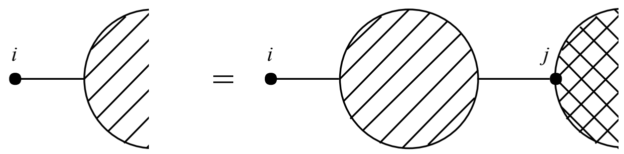


Figure 2.13: Diagrammatic representation of the relation (2.51).

This substitution has a nice diagrammatic counterpart shown in Figure 2.13, where the operators $\frac{\delta}{\delta J}$ and $\frac{\delta}{\delta\varphi}$ are being depicted as a leg attached to a half-blob, which symbolises the fact that the operators pull a leg out of the generating functional.

Replacing both terms in the Dyson–Schwinger equation, we obtain its new form

$$\frac{\delta S}{\delta \phi_i}\left[\varphi_j + W^{(2)}_{jk}[J]\frac{\mathrm{d}}{\mathrm{d}\varphi_k}\right] = J_i\,. \tag{2.52}$$

Further using the defining relation from the Legendre transformation (2.46) for the source and recovering[2] $\hbar$ we finally obtain a new form of the Dyson–Schwinger equation

$$\frac{\delta S}{\delta \phi_i}\left[\varphi_j + \hbar\, W^{(2)}_{jk}[J]\frac{\mathrm{d}}{\mathrm{d}\varphi_k}\right] = \frac{\delta\Gamma[\varphi]}{\delta\varphi_i}\,. \tag{2.53}$$

We immediately see that in the semi-classical limit as $\hbar \to 0$, the left hand side reduces to the classical equation of motion. Hence the dynamics described by the effective action is fully equivalent to the classical action in the semi-classical limit. This generalizes the result we have found in Section 2.4.2, which pointed to the fact, that they are equivalent for the free theory. We can also conclude that the effective action is identical as to the algebraical form (up to an additive constant). This conclusion will be discussed further in more detail and different circumstances. Because in the semi-classical limit the term with $\hbar$ vanishes and the field φ solves the classical equations of motion we sometimes call the field φ the *classical field.*

We must also conclude that the term containing $\hbar$ must be responsible for all the quantum corrections, since this equation gives the full quantum dynamics of the theory. The quantum corrections (involving virtual particles) are generated by $\frac{\mathrm{d}}{\mathrm{d}\varphi}$ which creates loops in the expansion. These could, of course, be seen already in previous result involving the DS equation and the graphical representations of the results, where the loops were generated by the operator $\frac{\mathrm{d}}{\mathrm{d}J}$.

Before we go further in the discussion of the general properties, we will show the DS equation in the terms of Γ for the ϕ^4 theory. The particular calculation is again straightforward, only one must not forget, that $\frac{\mathrm{d}\Gamma}{\mathrm{d}J} \neq 0 \neq \frac{\mathrm{d}W}{\mathrm{d}\varphi}$. With the use of relations from (2.46) the equation reads

$$\frac{\mathrm{d}\Gamma[\varphi]}{\mathrm{d}\varphi_i} = \Delta^{-1}_{ij}\varphi_j + \frac{1}{2}Y_{ijkl}\varphi_l W^{(2)}_{kl}[J] + \frac{1}{6}Y_{ijkl}W^{(2)}_{ja}[J]W^{(2)}_{kb}[J]W^{(2)}_{lc}[J]\Gamma^{(3)}_{abc}[\varphi]\,. \tag{2.54}$$

To find a suitable graphical representation, we set $J = 0$ to obtain only the first expansion term and we multiply the equation by Δ_{ki}. Thus one obtains a diagrammatical representation of the equation.

[2]The factor $\hbar$ would be present in the action $S[\phi, J]$ in front of the source J in the denominator, thus, the operator $\frac{\mathrm{d}}{\mathrm{d}J}$ would have to be defined with with a compensating factor $\hbar$ in the numerator

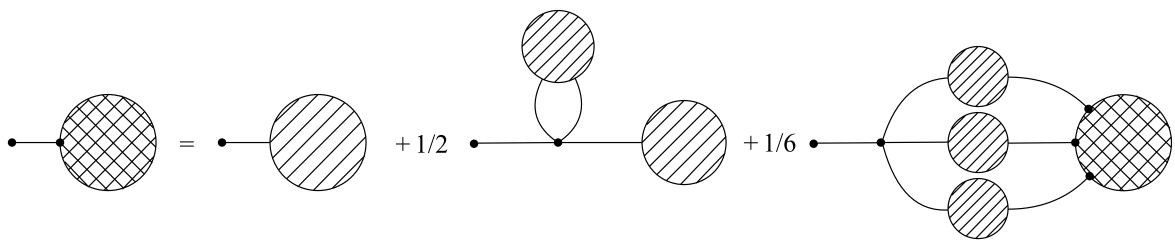


Figure 2.14: Diagrammatic representation of the Dyson–Schwinger equation in terms of the 1PI Green functions for the ϕ^4 theory.

2.4.5 Semi-Classical Limit – Tree Expansion

Let us return once more to the discussion of the semi-classical limit $\hbar \to 0$ of the Dyson–Schwinger equation. Specifically we examine if we truly recover classical dynamics. The DS equation must be then equivalent to the classical equation of motion

$$\frac{\delta S[\varphi]}{\delta \phi_i} = J_i \Longleftrightarrow \Delta^{-1}_{ij}\varphi_j + \frac{\delta S_{\mathrm{I}}[\varphi]}{\delta \phi_i} = J_i\,, \tag{2.55}$$

and after separating the classical solution φ we have

$$\varphi_i = \Delta_{ij}J_j - \Delta_{ij}\frac{\delta S_{\mathrm{I}}[\varphi]}{\delta \phi_j} = \Delta_{ij}J_j - \Delta_{ij}\frac{\delta S_{\mathrm{I}}}{\delta \phi_j}\left[\Delta_{ij}J_j - \Delta_{ij}\frac{\delta S_{\mathrm{I}}[\varphi]}{\delta \phi_j}\right] = \dots\,. \tag{2.56}$$

This iteratively solved series is called the *tree expansion* or *Born expansion* and from its diagrammatic representation (for the ϕ^4 theory only) presented in Figure 2.15 can be concluded, that it contains no loops. That is, however, exactly what we expect from a classical propagation.

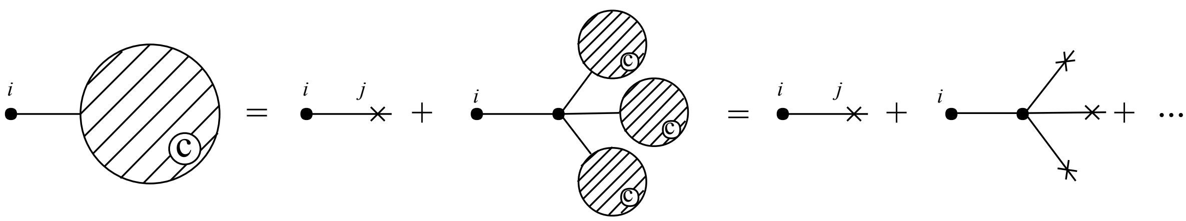


Figure 2.15: Diagrammatic representation of tree expansion for the ϕ^4 theory, demonstrating the semi-classical limit. The subscript c indicates we work with the solution of the classical equations of motion.

We should also point out a relation between the tree expansion and the definition of the generator of 1PI Green functions of the ϕ^4 theory seen in figure 2.12. If we replaced there the 1PI Green functions by a simple interaction vertex (where possible i.e. allowed by the theory), we would obtain the tree expansion, as depicted in figure 2.15. This corresponds to the fact, that in the semi-classical limit the 1PI green function shrinks to only one dot – the classical interaction vertex. This gives the effective action one more physical meaning – it generates the same form of a diagrammatic representation of the field dynamics as the classical action, but instead of bare interaction vertices, it contains the 1PI green functions. These are composed of the basic 1PI building blocks – loops. Therefore we can think of the effective action as of a functional which leads to the classical equation of motion, when we regard the 1PI Green functions (proper vertices) as of the *effective vertices* of the theory, which hide all quantum corrections in them.

2.4.6 The Amputated Green Functions

So far we have given no reason as to why the 1PI Green function is represented as a blob with dots on its circumference as opposed to any other blob we have seen so far. The first reason is the fact, that in the expansion (2.43) we see, it is connected to the fields φ_i, which already posses an external leg with a dot. However, there is one more reason related to the so-called *amputated green functions*. To illustrate it, we will need to introduce a new field variable $\xi(x) \equiv \varphi(x) - \langle\phi(x)\rangle$ which by definition vanishes, when the sources vanish. We also remark that from the discussion under the definition of φ it follows that $\xi(x) = \varphi(x)$ in case of both the free theory and the ϕ^4 theory. Next we take the formal expansion of $W[J]$ and differentiate it w.r.t. the source to get

$$\frac{\delta W[J]}{\delta J_i} = \sum_{n=1} \frac{1}{n!} W^{(n+1)}_{ij_1\ldots j_n}[0] J_{j_1} \ldots J_{j_n}, , \tag{2.57}$$

from which after subtracting the first term of the series corresponding to $\langle\varphi_j\rangle$ we obtain

$$\xi_j = \varphi_j - \langle\phi_j\rangle = W^{(2)}_{jj_1}[0] J_{j_1} + \frac{1}{2!} W^{(3)}_{jj_1j_2}[0] J_{j_1} J_{j_2} + \; . \tag{2.58}$$

Further we define inverse of the propagator of the full theory[3] as $S_{ik} W^{(2)}_{ki}[0] = \delta_{ij}$ which we will use to separate J from the relation above as

$$J_k = S_{kj}\xi_j - \sum_{n=2} \frac{1}{n!} S_{kj} W^{(n+1)}_{jj_1\ldots j_n}[0] J_{j_1} \ldots J_{j_n} \; . \tag{2.59}$$

This equation still reflects the fact, that setting $J = 0$ makes also $\xi = 0$ and the whole series vanishes. Next we define the so-called *amputated n-point Green function* as

$$W^{(n)}_{\text{amp.}\, i_1\ldots i_n}[0] = W^{(n)}_{j_1\ldots j_n}[0] S_{i_1 j_1} \ldots S_{i_n j_n} \; . \tag{2.60}$$

We call it amputated, since the inverses of the propagators cancel the usual external legs of the n-point functions. This amputated Green function is depicted as a blob with dots on its circumference, to which a propagator can be attached.

The equation (2.59) can be solved iteratively, yielding the first terms in the following form (in this particular formula, we will drop the argument [0] to make it shorter)

$$\begin{aligned}
J_k &= S_{kj}\xi_j - \frac{1}{2!} S_{kj} W^{(3)}_{jj_1j_2} J_{j_1} J_{j_2} - \frac{1}{3!} S_{kj} W^{(4)}_{jj_1j_2j_3} J_{j_1} J_{j_2} J_{j_3} = \\
&= S_{kj}\xi_j - \frac{1}{2} S_{kj} W^{(3)}_{jj_1j_2} S_{j_1n_1}\xi_{n_1} S_{j_2n_2}\xi_{n_2} + \frac{2}{2^2} S_{kj} W^{(3)}_{jj_1j_2} S_{j_1n_1}\xi_{n_1} S_{j_2m_1} W^{(3)}_{m_1m_2m_3} J_{m_2} J_{m_3} - \\
&\quad - \frac{1}{8} S_{kj} W^{(3)}_{jj_1j_2} S_{j_1m_1} W^{(3)}_{m_1m_2m_3} J_{m_2} J_{m_3} S_{j_2o_1} W^{(3)}_{o_1o_2o_3} J_{o_2} J_{o_3} - \\
&\quad - \frac{1}{6} S_{kj} W^{(4)}_{jj_1j_2j_3} J_{j_1} J_{j_2} J_{j_3} + \cdots = \\
&= S_{kj}\xi_j - \frac{1}{2} W^{(3)}_{\text{amp.}\, kn_1n_2}\xi_{n_1}\xi_{n_2} + \frac{1}{2} W^{(3)}_{\text{amp.}\, kn_1m_1}\xi_{n_1} W^{(2)}_{m_1n} S_{nm} W^{(3)}_{mm_2m_3} S_{m_2a_1}\xi_{a_1} S_{m_3a_2}\xi_{a_2} - \\
&\quad - \frac{1}{6} S_{kj} W^{(4)}_{jj_1j_2j_3} S_{j_1i}\xi_i S_{j_2l}\xi_l S_{j_3m}\xi_m + \ldots
\end{aligned} \tag{2.61}$$

[3]Note here, that by the general definition of the Legendre transform, $S_{ij} = \Gamma^{(2)}_{ij}[0]$.

$$
\begin{aligned}
J_k = S_{kj}\xi_j - \frac{1}{2}W^{(3)}_{\text{amp.}\,kn_1n_2}\xi_{n_1}\xi_{n_2} + \frac{1}{2}W^{(3)}_{\text{amp.}\,kn_1m_1}W^{(2)}_{m_1n}W^{(3)}_{\text{amp.}\,na_1a_2}\xi_{a_1}\xi_{a_2} - \\
- \frac{1}{6}W^{(4)}_{\text{amp.}\,kilm}\xi_i\xi_k\xi_l + \dots = \\
= S_{kl}\xi_l - \frac{1}{2}W^{(3)}_{\text{amp.}\,klm}\xi_l\xi_m - \frac{1}{2}\xi_l\xi_m\xi_n\Big(W^{(4)}_{\text{amp.}\,klmn} - W^{(3)}_{\text{amp.}\,kna}W^{(2)}_{ab}W^{(3)}_{\text{amp.}\,blm}\Big)\,.
\end{aligned}
\tag{2.62}
$$

To get it related to the effective action, let us recall, that it also has a formal expansion series, but now we choose to expand it around the expectation value $\langle\phi\rangle$ (which is equal to zero for the ϕ^4 theory). After taking the variation of the expansion w.r.t. to φ_k one gets

$$
\frac{\delta\Gamma}{\delta\varphi_k} = \sum_{n=0}\frac{1}{n!}\Gamma^{(n+1)}_{ki_1\dots i_n}[\langle\phi\rangle]\xi_{i_1}\dots\xi_{i_n}\,, \tag{2.63}
$$

but since $J_k = \frac{\delta\Gamma}{\delta\varphi_k}$ by definition, we can compare the two expansions as polynomials in ξ to find $\Gamma^{(n)}$ to be equal in the lowest orders to

$$
\begin{aligned}
&\Gamma^{(1)}[\langle\phi\rangle] = 0\,, \quad \Gamma^{(2)}_{ij}[\langle\phi\rangle] = S_{ij} = \Big(W^{(2)}_{ij}\Big)^{-1}\,, \\
&\Gamma^{(3)}_{ijk}[\langle\phi\rangle] = -W^{(3)}_{\text{amp.}\,ijk}[0]\,, \\
&\Gamma^{(4)}_{ijkl}[\langle\phi\rangle] = -W^{(4)}_{\text{amp.}\,ijkl}[0] + W^{(3)}_{\text{amp.}\,ika}[0]W^{(2)}_{ab}[0]W^{(3)}_{\text{amp.}\,bkl}\,.
\end{aligned}
\tag{2.64}
$$

These relations deserve an explanation. Here, we refer to what was already reasoned out – the algebraic form of the effective action is identical to the classical one. The terms $\Gamma^{(n)}$ then correspond to the coefficients in front of the n-th power of a field in the action. Thus we see, that there is no linear term in the action, the quadratic term is generated by the inverse of the full (effective) propagator (recall that the classical action is of the form $\frac{1}{2}\phi_x\Delta^{-1}_{xy}\phi_y + \dots$ hence the full propagator composes of the free one and some terms from the interaction) and the first non-trivial term comes from the (possible) ϕ^3 theory. The term coming from the ϕ^4 theory, however, has more contributions – we see, that the relative sign of the two terms tells us, that from all connected (amputated) Green functions with 4 external channels we must subtract those, that are reducible, constructed with the use of the ϕ^3 interaction vertices. The results are shown also diagrammaticaly in Figure 2.16.

2.4.7 The Full Propagator as a Series of 1PI Green Functions

There is one more interesting property of the effective action to show, before we move on to its applications. It gives us a systematic way of generating two-point functions of the full theory (i.e. the full propagator) in terms of a series of diagrams composed of 1PI parts. To that end recall, that Γ is very similar to the classical action as to the algebraical form (it is identical for the free field theory, see (2.50)) We assume then that we can write the effective action in the following form

$$
\Gamma = \varphi_i K_{ij}\varphi_j + \Gamma_{\text{I}}\,, \tag{2.65}
$$

where K is the inverse of the free field propagator and Γ_{I} describes the interaction. The second variation gives

$$
\Gamma^{(2)} = K + \Gamma^{(2)}_{\text{I}} \equiv K + \Sigma\,. \tag{2.66}
$$

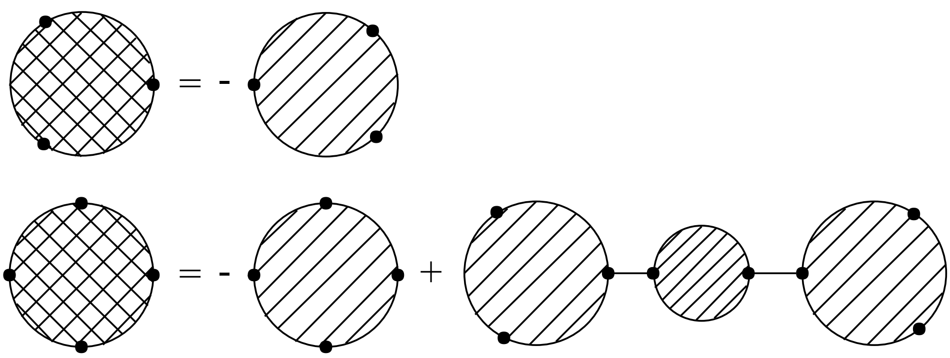


Figure 2.16: Diagrammatic representation of results from (2.64), relating the 1PI Green functions to the amputated ones.

Here Σ stands for all quantum corrections to $K = \Delta^{-1}$ coming from the interaction. We recall, that in equation (2.42) on page 32 we defined Γ_{ij} as $\Pi_{ij} - \Delta_{ij}$. Here we see the reason, why did so (meanwhile we also changed the sign of the action so there is some inconsistency here). Now we use a property of the Legendre transform

$$W^{(2)}\Gamma^{(2)} = 1 = W^{(2)}\big(K + \Sigma\big)\,, \tag{2.67}$$

from where we obtain

$$\begin{aligned} W^{(2)} &= \frac{1}{K+\Sigma} = \frac{1}{K(1+\Delta\Sigma)} \simeq K^{-1}\big(1 - \Delta\Sigma + \Delta\Sigma\Delta\Sigma + \dots\big) \\ W^{(2)} &= \Delta - \Delta\Sigma\Delta + \Delta\Sigma\Delta\Sigma\Delta - \dots\,. \end{aligned} \tag{2.68}$$

Identical result may be obtained another way – we multiply equation (2.67) by a propagator, separate $W^{(2)}$ on one side and then solve iteratively

$$W^{(2)} = \Delta - W^{(2)}\Sigma\Delta = \Delta - (\Delta - W^{(2)}\Sigma\Delta)\Sigma\Delta = \dots\,. \tag{2.69}$$

This equation enables us to calculate the full propagator approximatively in terms of the 1PI Green functions. We can also represent it diagrammatically, as shows figure 2.17.

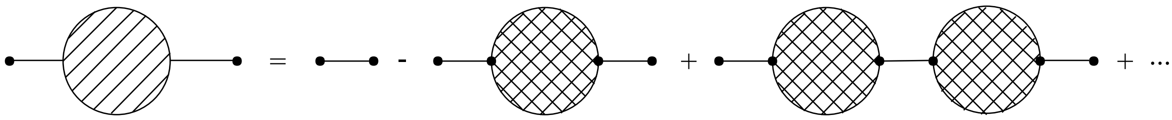


Figure 2.17: Diagrammatic representation of the series in (2.67)

This relation has a profound meaning. It tells us, that the full propagator is in the first approximation identical to the free propagator and that every higher order of approximation introduces at least one loop, since those are the basic building blocks of the 1PI Green functions. This result is confirmed by what we found for the ϕ^4 theory in Figures 2.8 and 2.4. We will show in the next section, that an expansion into a number of loops is best done in the formalism of the effective action.

2.4.8 Loop Expansion, One-Loop Effective Action, Effective Potential

We have discussed above the semi-classical limit of the DS equation which yielded the tree expansion. What we have not discussed about this solution is its relation to the path integral point of view. So far, we have not used path integrals in our calculations at all. As is written in [5], path integral is not as much of an integral in the sense of a continuous summation, but more of a tool, which transforms differentiation to multiplication, just the same way the Fourier transform does

$$\frac{\delta Z[J]}{\delta J_i} = \int \mathcal{D}\phi\, \phi_i\, \mathrm{e}^{-S[\phi,J]}\,. \tag{2.70}$$

Until now we have worked mostly in the world of differentiation but now we shall start using the path integrals. Let us try to evaluate the path integral using the saddle point approximation up to the first order. The integral is in this order of approximation dominated by one term, corresponding to the minima of the exponent, i.e. by the solution of classical equations of motion. Therefore we obtain

$$Z[J] \doteq \mathrm{e}^{-S[\phi^{\mathrm{c}},J]}\,, \quad \text{where} \quad \frac{\delta S[\phi^{\mathrm{c}}]}{\delta \phi_i} = J_i\,. \tag{2.71}$$

We already know, what this solution generates – that is exactly the Born expansion as seen in Section 2.4.5 and $\phi^{\mathrm{c}} = \varphi$. Here, however, we have given it a new physical meaning – that of a first approximation to the evaluation of the path integral.

We have also shown (see (2.53) and (2.50)) that on this level of approximation, the effective action is identical to the classical one, i.e. $\Gamma[\phi^{\mathrm{c}}] = S[\phi^{\mathrm{c}}]$, which will be useful later on. Let us now expand on the first quantum corrections coming from the saddle point approximation. In that case we take into account one more term

$$S[\phi] - \phi_i J_i \doteq S[\phi^{\mathrm{c}}] - \phi_i^{\mathrm{c}} J_i + \frac{1}{2}(\phi_i - \phi_i^{\mathrm{c}}) S^{(2)}_{ij}[\phi^{\mathrm{c}}](\phi_i - \phi_i^{\mathrm{c}})\,, \tag{2.72}$$

which we put into the definition of $Z[J]$ and since the third term is quadratic in the fields and the integral measure $\mathcal{D}\phi$ is invariant under translation, we obtain with the help of the general results from Chapter 1 the generating functional in the following form

$$Z[J] = N\, \mathrm{e}^{-S[\phi^{\mathrm{c}}] + \phi_i^{\mathrm{c}} J_i} \Big[\det\Big(S^{(2)}_{ij}[\phi^{\mathrm{c}}]\Big)\Big]^{-1/2}\,. \tag{2.73}$$

The normalization factor N can be forgotten, for as was already argued, it has no influence on the results of physically relevant quantities. We would like to be able to interpret the role of the determinant and possibly give it also a diagrammatical representation. To achieve that, we use a well known formula

$$\det A = \prod_i a_i = \mathrm{e}^{\ln(\prod_i a_i)} = \mathrm{e}^{\sum_i \ln a_i} = \mathrm{e}^{\mathrm{Tr}\ln A}\,, \tag{2.74}$$

with a_i being the eigenvalues of the matrix (operator) A. Using this identity, denoting $S^{(2)}_{ij}[\phi^{\mathrm{c}}] \equiv \Delta^{-1}_{ij} + \gamma_{ij}[\phi^{\mathrm{c}}] = \Delta^{-1}_{il}\big(\delta_{lj} + \Delta_{lk}\gamma_{kj}[\phi^{\mathrm{c}}]\big)$ and the fact that $\det AB = \det A \det B$ we obtain

$$Z[J] = \exp\Big[-S[\phi^{\mathrm{c}}] - \frac{1}{2}\,\mathrm{Tr}\ln(1 + \Delta\gamma[\phi^{\mathrm{c}}]) + \phi_i^{\mathrm{c}} J_i\Big]\sqrt{\det\Delta}\,, \tag{2.75}$$

and since $\sqrt{\det \Delta}$ is coming from $Z_0[J]$ for the free field theory without sources, we can simply set it equal to 1 as a part of the normalization. From the definition of the effective action (2.46) on page 32 and the definition of $W[J]$ follows that

$$\mathrm{e}^{-\Gamma[\phi]+\phi_i J_i} = Z[J] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi,J]} \,, \tag{2.76}$$

so we find, that the effective action is in this order of approximation equal to

$$\Gamma[\phi^{\mathrm{c}}] = S[\phi^{\mathrm{c}}] + \frac{1}{2} \operatorname{Tr} \ln(1 + \Delta\gamma[\phi^{\mathrm{c}}]) \equiv S[\phi^{\mathrm{c}}] + \Gamma^1[\phi^{\mathrm{c}}] \,. \tag{2.77}$$

Let us now examine the first quantum correction to the effective action $\Gamma^1[\phi^{\mathrm{c}}]$ in more detail. We expand the logarithm into a Taylor series and shift the sign to the propagator, which would correspond to defining the propagator with an opposite sing

$$\begin{aligned}\Gamma^1[\phi^{\mathrm{c}}] &= \frac{1}{2} \operatorname{Tr}((-\Delta_{ik})\gamma_{kj}[\phi^{\mathrm{c}}]) + \frac{1}{4} \operatorname{Tr}((-\Delta_{ik})\gamma_{kj}[\phi^{\mathrm{c}}](-\Delta_{jl})\gamma_{lm}[\phi^{\mathrm{c}}]) + \dots \\ \Gamma^1[\phi^{\mathrm{c}}] &= \frac{1}{2}(-\Delta_{ik})\gamma_{ki}[\phi^{\mathrm{c}}] + \frac{1}{4}(-\Delta_{ik})\gamma_{kj}[\phi^{\mathrm{c}}](-\Delta_{jl})\gamma_{li}[\phi^{\mathrm{c}}] + \dots \,.\end{aligned} \tag{2.78}$$

Before we show a diagram, we explain some of the terms here. First $\gamma_{ij}[\phi^{\mathrm{c}}] = S^{(2)}_{\mathrm{I}\,ij}[\phi^{\mathrm{c}}]$ contains all interaction vertices, that have two empty legs i and j waiting to be attached to something and the rest of the legs are attached to the classical Green function ϕ_c, represented as in the tree expansion in Figure 2.15. Hence we illustrate it as in Figure 2.18.

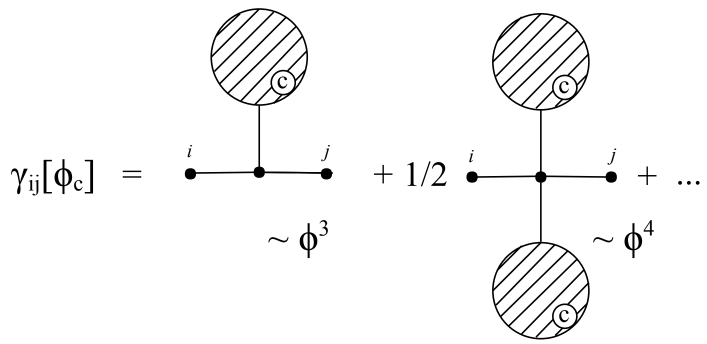


Figure 2.18: Diagrammatic representation of the interaction terms $\gamma_{ij}[\phi^{\mathrm{c}}]$ coming from the ϕ^3 and ϕ^4 interaction

Joining these together by propagators in the power series as determined by equation (2.78) we obtain (for all possible interactions) a diagram shown in Figure 2.19.

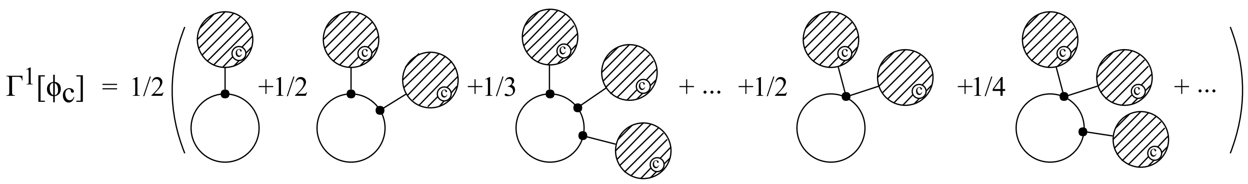


Figure 2.19: Diagrammatic representation of the one-loop expansion (2.78).

Here we can use the tree expansion of the classical field ϕ^c from Figure 2.15 and its generalization to other interacting potentials to find the loop expansion in another form, from which we conclude that the effective action satisfies the classical equations of motion of the tree structure, but it has effective vertices (loops) instead of the simple interaction vertices as in the classical theory. Figure 2.20 reflects this conclusion in full generality of interaction

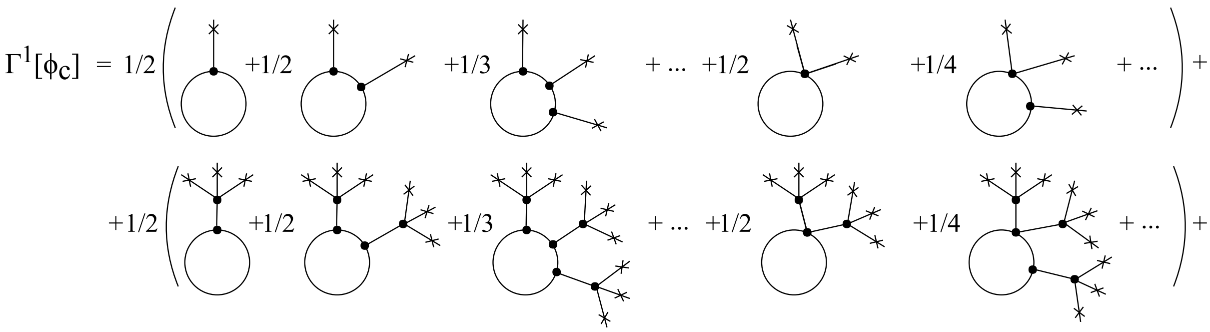


Figure 2.20: Diagrammatic representation of the one-loop expansion (2.78) including the tree expansion.

The last thing we mention about the effective action is the *effective potential.* We have already used the similarity with the classical action as to the form. Thus we expect, that the (Euclidean) effective action will be separable into kinetic and a potential part

$$\Gamma[\varphi] = \int \mathrm{d}^4x \, \frac{1}{2} A\big(\varphi(x)\big) \, \partial_\mu \varphi(x) \, \partial^\mu \varphi(x) + V_{\text{eff}}\big(\varphi(x)\big) + \dots \,, \tag{2.79}$$

with higher derivative terms neglected. Then when we need to explore static field configurations $\varphi(x) = \varphi$, the kinetic term vanishes and all we are left with is

$$\Gamma[\varphi] = \int \mathrm{d}^4x \, V_{\text{eff}}(\varphi) = V_{\text{eff}}(\varphi)\Omega_4 \,, \tag{2.80}$$

with Ω_4 being the four-volume of the Euclidean spacetime. Investigating these field configurations corresponds to searching for the stable configurations in the presence of the effective potential (which contains the quantum corrections). It is possible, that the effective potential has a different minimal value (e.g. different vacuum) than the classical potential. This has the effect, that the theory might (for example) break its symmetry as is the case of the Coleman–Weinberg mechanism or, as we will see in Chapter 3, the case of the Weyl conformal gravity. Since this symmetry breakdown occurs naturally only as a consequence of quantum corrections it is called the *dynamical breakdown of symmetry.*

The effective potential is useful also for finding "dressed" coupling constant and mass, also called effective, as (here for the case of the ϕ^4 theory)

$$m_{\text{eff}} \equiv \frac{\mathrm{d}^2 V(\varphi)}{\mathrm{d}\varphi^2} \,, \quad \text{and} \quad g_{\text{eff}} \equiv \frac{\mathrm{d}^4 V(M)}{\mathrm{d}\varphi^4} \,. \tag{2.81}$$

2.4.9 Euclidean and Minkowskian Regime

Here we present the overview of how to go from the Euclidean to the Minkowskian regime and definition of all the object discussed in the previous sections in the Minkowskian regime. We also recover $\hbar$ to see introduced quantities in full context.

First we recall our notation – the Wick rotation is performed by a change of coordinates $x \mapsto \bar{x}$ so that $\bar{x}^0 = \mathrm{i}x^0$ and $\bar{x}^i = x^i$. The classical action is of the following form

$$S[\phi] = -\frac{1}{2}\phi_x K_{xy}\phi_y - S_{\mathrm{I}}[\phi]\,, \quad \text{with} \quad K_{xy} = \partial_t^2 - \boldsymbol{\nabla}^2 + m^2\,. \tag{2.82}$$

Adding the source we have

$$S[\phi, J] = S[\phi] - J_x\phi_x\,. \tag{2.83}$$

The Euclidean action can be found as

$$S_{\mathrm{E}}[\phi(\bar{x}), J(\bar{x})] = -\mathrm{i}S[\phi(x), J(x)]\big|_{x^0=-\mathrm{i}\bar{x}^0} \quad \Longleftrightarrow \quad S[\phi(x), J(x)] = \mathrm{i}S_{\mathrm{E}}[\phi(\bar{x}), J(\bar{x})]\big|_{x^0=\mathrm{i}\bar{x}^0}\,, \tag{2.84}$$

which yields

$$S_{\mathrm{E}}[\phi(\bar{x}), J(\bar{x})] = \frac{1}{2}\phi_{\bar{x}}K_{\mathrm{E}\,\bar{x}\bar{y}}\phi_{\bar{y}} + S_{\mathrm{I}}[\phi(\bar{x})] - J_{\bar{x}}\phi_{\bar{x}}\,, \quad \text{with} \quad K_{\mathrm{E}\,\bar{x}\bar{y}} = -(\bar{\partial}_t^2 + \bar{\boldsymbol{\nabla}}^2) + m^2\,. \tag{2.85}$$

With this, we have by definition

$$Z[J] = \int \mathcal{D}\phi\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[\phi(x),J(x)]} = \int \mathcal{D}\phi\ \mathrm{e}^{-\frac{1}{\hbar}S_{\mathrm{E}}[\phi(\bar{x}),J(\bar{x})]}\,. \tag{2.86}$$

We must also change the form of the Feynman trick with $\frac{\delta}{\delta J_i}$ to $\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J_i}$ and $\hbar\frac{\delta}{\delta J_i}$ in the Minkowskian and Euclidean regime, respectively. Further, by definition of $W[J]$ we have

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}W[J]} = Z[J] = \mathrm{e}^{\frac{1}{\hbar}W_{\mathrm{E}}[J]}\,, \tag{2.87}$$

from where we conclude, that

$$W_{\mathrm{E}}[J(\bar{x})] = \mathrm{i}W[J(x)]\big|_{x^0=-\mathrm{i}\bar{x}^0} \quad \Longleftrightarrow \quad W[J(x)] = -\mathrm{i}W_{\mathrm{E}}[J(\bar{x})]\big|_{x^0=\mathrm{i}\bar{x}^0}\,, \tag{2.88}$$

and since $W[J] + \Gamma[\varphi] = \varphi_i J_i$ with $\varphi_i = \frac{\delta W[J]}{\delta J_i}$ we have also

$$\Gamma_{\mathrm{E}}[\varphi(\bar{x})] = -\mathrm{i}\Gamma[\varphi(x)]\big|_{x^0=-\mathrm{i}\bar{x}^0} \quad \Longleftrightarrow \quad \Gamma[\varphi(x)] = \mathrm{i}\Gamma_{\mathrm{E}}[\varphi(\bar{x})]\big|_{x^0=\mathrm{i}\bar{x}^0}\,. \tag{2.89}$$

We see the transformation rule coincides with the one for the classical action, as is of the effective action required. These relations conclude this chapter.

Chapter 3

Zeta Function Regularization

We have already stumbled upon an interesting object in the previous chapters – the determinant of a differential operator. We know what a determinant is for a finite-dimensional matrix, but taking a formal limit in the dimension to infinity, the matrix becomes a (differential) operator and the usual definition of the determinant through the sum of all permutations of matrix elements somewhat fails. We can use an equivalent definition such as that through a product of the eigenvalues, but we would be extremely lucky, if it converged. To find a finite and meaningful result (i.e. to regularize the determinant) we use a few tricks. We will expand on some of them in the following paragraph.

3.1 Brief Overview of Regularization Methods

Here we discuss some well known methods for calculating determinants of differential operators to obtain meaningful physical results.

- First thing we can do is to retreat back to the finite-dimensional case and discretize the operator. That means that to an operator O we assign finite-dimensional matrix O_n such, that in the limit as $n \to \infty$ we get $O_n \to O$. This can be done by substituting derivatives with finite differences from which we construct a matrix acting on a discretized vector $\phi(x_n)$. We calculate the determinant of such a matrix, which will depend on the dimension n through which we go back by taking the limit $n \to \infty$.

- It is sometimes possible in the discretized case to find a recurrence relation for the determinant by applying the determinant expansion by minors. After taking the limit $n \to \infty$, the determinant may be found as a solution of a differential equation specific to each operator.

- Another option is the so-called dimensional regularization. For that we apply the formula (2.74) relating the determinant with trace

$$\det O = \mathrm{e}^{\operatorname{tr} \ln O}\,, \tag{3.1}$$

where we calculate the trace in a conveniently chosen space, for example

$$\operatorname{tr} \ln O = \int \mathrm{d}^4 x \ln O(x,x)\,. \tag{3.2}$$

These integrals are, however, usually divergent, which is bypassed by analytically continuing the dimension of the integral measure to real numbers

$$\operatorname{tr}\ln O = \lim_{\epsilon\to 0} \eta^{2\epsilon} \int \mathrm{d}^{4-2\epsilon}x \ln O(x,x)\,. \tag{3.3}$$

Here we must introduce a new constant η, which carries non-trivial dimension identical to that of the integral measure $\mathrm{d}x$ to compensate for the change of the integral measure.

This regularization was chosen to calculate the effective potential in [1], however the applicability of this method is in this particular case questionable. The problem arises form the fact, that in their article, they also use a global topological invariant (a consequence of Gauss–Bonnet theorem), which holds only in a fixed dimension of four. The aim of this diploma thesis will be to confirm their result using the so-called zeta function regularization, which will be the topic of the following section.

Since it will be related to our case in Chapter 3, we also remark upon the so-called *dimensional transmutation*. This occures, when the regulating scale η does not vanish completely after all calculations of the integral are done and the limit $\epsilon \to 0$ applied. It was first described by Coleman and Weinberg [7], after whom the mechanism was also named. We will see that similar effect happens also in the case of the quantum Weyl gravity.

There exist many other methods, however, we will mention only one more – the use of the spectral zeta function.

3.2 The Spectral Zeta Function

It is probably appropriate to begin with the definition of the Riemmann zeta function, which enables us to make such funny statements as "sum of all natural numbers is $-\frac{1}{12}$" or at least give it a better sense. The zeta function is originally defined as

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \quad \operatorname{Re} s > 1\,, \tag{3.4}$$

nevertheless the function is meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C}\setminus\{1\}$ thus it can be analytically continued to the whole complex plane and the continuation is unique. Due to this property, we can search for other equivalent formulations of the same function and if we prove it equals to the zeta function on some subdomain in $\mathbb{C}$, we know from the uniqueness of the continuation that it indeed is the zeta function. Thus, we have other representations, such as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{\mathrm{e}^s - 1}\,\mathrm{d}x\ . \tag{3.5}$$

The most important property is the so-called functional equation, which enables us to reflect the domain of the ζ function to the second half of the complex plane

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\,. \tag{3.6}$$

The effect of the domain extension from $\operatorname{Re} s > 1$ to $\operatorname{Re} s < -1$ is to make sense of otherwise non-sensical formulas such as

$$\sum_{n=1}^{\infty} n \sim \zeta(-1) = -\frac{1}{12}\,, \quad \sum_{n=1}^{\infty} 1 \sim \zeta(0) = -\frac{1}{2}\,, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sim \zeta\Big(\frac{1}{2}\Big) = -1.46035\ldots\,, \tag{3.7}$$

and some of these even find its use in physics (e.g. the black body radiation) or the string theory.

There are many generalizations of the zeta function (see for example [8]) but we will be interested only in the so-called *spectral zeta function.* This is assigned to every operator A with eigenvalues a_n as

$$\zeta_A(s) \equiv \mathrm{Tr}\big(A^{-s}\big) \equiv \sum_{n}^{\infty} \frac{1}{a_n^s}\,. \tag{3.8}$$

Using this spectral zeta function, we are able to regularize (or even define) the determinant of an operator A. To show that, we first observe that

$$\frac{\mathrm{d}\zeta_A(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\sum_{n}^{\infty} \mathrm{e}^{-s\ln a_n} = -\sum_{n}^{\infty} \mathrm{e}^{-s\ln a_n} \ln a_n\,, \tag{3.9}$$

from where after setting $s = 0$ and employing the property of logarithm we obtain

$$-\zeta_A'(0) = \sum_{n}^{\infty} \ln a_n = \ln \prod_{n}^{\infty} a_n = \ln\det A\,. \tag{3.10}$$

Hence we can define a determinant of an operator through the spectral zeta function as

$$\det A \equiv \mathrm{e}^{-\zeta_A'(0)}\,, \tag{3.11}$$

which is the key equality to the zeta function regularization.

It seem all nice and clear, but apparently, we still have to solve the eigenproblem to find the spectral zeta function which is no improvement at all. To bypass the problem of finding the spectrum we use the so-called *heat kernel* which we describe in the following section.

3.3 The Heat Kernel for the Spectral zeta Function

Here we aim at finding an alternative formula for the spectral zeta function to the one using eigenvalues, since they are usually impossible to find. To that end let us define exponent of the operator A by

$$K(\tau) \equiv \mathrm{e}^{-\tau A} = \sum_n \mathrm{e}^{-\tau a_n} \left|\psi_n\right\rangle \left\langle\psi_n\right|\,, \quad \tau > 0\,. \tag{3.12}$$

By definition clearly $K(0) = 1$ and we can discard the one-dimensional projections $\left|\psi_n\right\rangle \left\langle\psi_n\right|$ by taking trace. The result is then a function of τ

$$\mathrm{Tr}\,K(\tau) = \sum_n \mathrm{e}^{-\tau a_n}\,. \tag{3.13}$$

It is unclear how the operator K or its trace might help us at the moment. But let us now devise an analogue of the integral representation of the Riemann zeta function from (3.5) also for the spectral one. We start with the integral representation of the gamma function

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-\tau}\tau^{s-1}\,\mathrm{d}\tau = \Big|\tau = k\tau\Big| = k^s \int_0^\infty \mathrm{e}^{-k\tau}\tau^{s-1}\,\mathrm{d}\tau \ , \tag{3.14}$$

where k is a real number. From here, we isolate k^{-s} as

$$\frac{1}{k^s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{e}^{-k\tau}\tau^{s-1}\,\mathrm{d}\tau \ . \tag{3.15}$$

Now since the number k is arbitrary, we can choose it from the spectra of A. Moreover, if we sum over all such choices to get

$$\sum_n \frac{1}{a_n^s} = \frac{1}{\Gamma(s)} \sum_n \int_0^\infty \mathrm{e}^{-a_n\tau}\tau^{s-1}\,\mathrm{d}\tau = \frac{1}{\Gamma(s)} \int_0^\infty \left(\sum_n \mathrm{e}^{-a_n\tau}\right)\tau^{s-1}\,\mathrm{d}\tau \ . \tag{3.16}$$

We have now obtained a very important formula

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr} K(\tau)\tau^{s-1}\,\mathrm{d}\tau \ . \tag{3.17}$$

This is a new way to calculate the spectral zeta function, without any use of the spectrum of A. The problem therefore transforms to that of finding $\operatorname{Tr} K(\tau)$. This can be done the following way – by definition, we know

$$\frac{\partial K(\tau)}{\partial \tau} = -AK(\tau)\,, \tag{3.18}$$

and since the trace can be easily calculated as

$$\operatorname{Tr} K(\tau) = \int \mathrm{d}x\, K(x,x,\tau)\,, \tag{3.19}$$

we are interested in solving an equation for $K(x,x',\tau)$ in the following form

$$\frac{\partial}{\partial \tau}K(x,x',\tau) = -\int \mathrm{d}x'' \ \big\langle x\big|A\big|x''\big\rangle\, K(x'',x',\tau)\,, \quad \text{where} \quad K(x,x',\tau) \equiv \big\langle x\big|K(\tau)\big|x'\big\rangle \ . \tag{3.20}$$

We must also supply the initial conditions for this differential equation which is $K(x,x',0) = \delta(x-x')$. We call the operator $K(\tau)$ the *heat kernel* since the differential equation (3.18) or (3.20) has a form of the heat equation – this becomes more apparent, when A is a differential operator in one variable only, thus $\langle x|A|x''\rangle = D(x)\delta(x-x'')$ and the integral on the right hand side simplifies, hence we obtain an equation of the form

$$\frac{\partial}{\partial \tau}K(x,x',\tau) = -D(x)K(x,x',\tau)\,. \tag{3.21}$$

The algorithm to find the determinant is now as follows:

1. Find a solution of the heat equation (3.20) for the heat kernel $K(x,x',\tau)$ and compute its trace

2. Compute the integral $\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr} K(\tau)\tau^{s-1}\,\mathrm{d}\tau$

3. Find the determinant as $\det A = \mathrm{e}^{-\zeta_A'(0)}$

We will now demonstrate the process on the ϕ^4 theory.

3.4 One-Loop Effective Potential for the ϕ^4 Theory

To find the one-loop effective action, we go back a bit to Section 2.4.8 and recall, that the first non-trivial quantum contribution to the effective action comes from the determinant of the second variation of the classical action. The Euclidean action reads

$$S[\phi] = \int \mathrm{d}^4\bar{x}\, \frac{1}{2}\phi(\bar{x})\big(- \bar{\partial}_x{}^2 + m^2\big)\phi(\bar{x}) + \frac{g}{4!}\phi^4(\bar{x})\,. \tag{3.22}$$

Hence, we are to calculate the determinant of the operator (see (2.73))

$$S^{(2)}[\phi] = -\bar{\partial}_x{}^2 + m^2 + \frac{g}{2}\phi^2(\bar{x})\,, \tag{3.23}$$

where ϕ now denotes the classical solution, which is in the order $O(\hbar)$ identical with φ used in the context of the effective action.

Let us follow the steps outlined in the previous section. First we solve the equation

$$\Big(- \bar{\partial}_x{}^2 + m^2 + \frac{g}{2}\phi^2(\bar{x})\Big)K(\bar{x},\bar{y},\tau) = -\frac{\partial K(\bar{x},\bar{y},\tau)}{\partial \tau}\,, \quad \text{with} \quad K(\bar{x},\bar{y},0) = \delta^{(4)}(\bar{x}-\bar{y})\,. \tag{3.24}$$

The problem can be simplified by separating the partial differential operator into two and first solve

$$\bar{\partial}_x{}^2 K_0(\bar{x},\bar{y},\tau) = \frac{\partial K_0(\bar{x},\bar{y},\tau)}{\partial \tau}\,, \quad \text{with} \quad K_0(\bar{x},\bar{y},0) = \delta^{(4)}(\bar{x}-\bar{y})\,. \tag{3.25}$$

To solve this equation we perform the Fourier transform from $\bar{x} \mapsto \bar{k}$ so that we now have an equation

$$\frac{\partial}{\partial \tau}\tilde{K}_0(\bar{k},\bar{y},\tau) = -\bar{k}^2\tilde{K}_0(\bar{k},\bar{y},\tau)\,, \tag{3.26}$$

for which it is easy to write the solution

$$\tilde{K}_0(\bar{k},\bar{y},\tau) = C(y)\,\mathrm{e}^{-\tau\bar{k}^2}\,. \tag{3.27}$$

This must be now Fourier-transformed back (we also include the initial condition setting $C(y) = 1$ and adding y into the exponent in the integrand to yield the integral representation of the delta function in the case of $\tau = 0$)

$$K_0(\bar{x},\bar{y},\tau) = \int \frac{\mathrm{d}^4\bar{k}}{(2\pi)^4}\,\mathrm{e}^{-\tau\bar{k}^2}\,\mathrm{e}^{-\mathrm{i}\bar{k}_\mu(\bar{x}^\mu-\bar{y}^\mu)} = \frac{1}{(2\pi)^4}\sqrt{\frac{\pi}{\tau}}^4\,\mathrm{e}^{-\frac{(\bar{x}-\bar{y})^2}{4\tau}} = \frac{1}{16\pi^2\tau^2}\,\mathrm{e}^{-\frac{(\bar{x}-\bar{y})^2}{4\tau}}\,. \tag{3.28}$$

The solution to the equation (3.24) can now be obtained from K_0 and K_p satisfying

$$\Big(m^2 + \frac{g}{2}\phi^2(\bar{x})\Big)K_p(\bar{x},\bar{y},\tau) = -\frac{\partial}{\partial \tau}K_p(\bar{x},\bar{y},\tau)\,, \tag{3.29}$$

as $K(\bar{x},\bar{y},\tau) = K_0(\bar{x},\bar{y},\tau)K_p(\bar{x},\bar{y},\tau)$ since in our special case of calculating the effective potential, we set $\phi(\bar{x}) \equiv \phi = \text{const}$ in the equation (3.29) as we have discussed in the previous chapter.

As a consequence of this choice, $K_p(\tau)$ is not a function of any coordinate and we can use

$$\begin{aligned}
&\Big(- \bar{\partial_x}^2 + m^2 + \frac{g}{2}\phi^2\Big) K_0(\bar{x},\bar{y},\tau)K_p(\tau) = \\
&\qquad = -K_p(\tau)\bar{\partial_x}^2 K_0(\bar{x},\bar{y},\tau) + K_0(\bar{x},\bar{y},\tau)\big(m^2 + \frac{g}{2}\phi^2\big)K_p(\bar{x},\bar{y},\tau) = \\
&\qquad = -K_p(\tau)\frac{\partial}{\partial\tau}K_0(\bar{x},\bar{y},\tau) - K_0(\bar{x},\bar{y},\tau)\frac{\partial}{\partial\tau}K_p(\tau) = -\frac{\partial}{\partial\tau}\big(K_0(\bar{x},\bar{y},\tau)K_p(\tau)\big)\,.
\end{aligned} \tag{3.30}$$

Fortunately, calculating $K_p(\tau)$ is easy and we find, that the full heat kernel can be written as

$$K(\bar{x},\bar{y},\tau) = \frac{1}{16\pi^2\tau^2}\,\mathrm{e}^{-\frac{(\bar{x}-\bar{y})^2}{4\tau}}\,\mathrm{e}^{-\tau(m^2+\frac{g}{2}\phi^2)}\,. \tag{3.31}$$

At this point it is necessary to make some adjustments. Since the exponent should be dimensionless, we rescale it by $\tau \to \tau/\mu^2$, where we have introduced a dimensionfull parameter μ, $[\mu] = \mathrm{kg} = \mathrm{m}^{-1}$ to compensate for the dimension of the $m^2 + \frac{g}{2}\phi^2$. Rewriting the result in the dimensionless form and tracing this function we get

$$\mathrm{Tr}\,K(\bar{x},\bar{y},\tau) = \int \mathrm{d}^4\bar{x}\,K(\bar{x},\bar{x},\tau) = \frac{\mu^4}{16\pi^2\tau^2}\,\mathrm{e}^{-\tau\mu^{-2}(m^2+\frac{g}{2}\phi^2)}\int \mathrm{d}^4\bar{x}\ , \tag{3.32}$$

and with the use of notation introduced in the previous chapter we will write the integral over the Euclidean spacetieme as Ω_4. We also note, that the trace of the heat kernel is independent of coordinates, hence its form will not change upon Wick rotation. This fact will be used later in the calculations of determinants in Section 4.6

We move on to the second step – calculating of the integral

$$\int_0^\infty \frac{\mu^4}{16\pi^2\tau^2}\,\mathrm{e}^{-\tau\mu^{-2}(m^2+\frac{g}{2}\phi^2)}\Omega_4\tau^{s-1}\,\mathrm{d}\tau = \frac{\mu^4}{16\pi^2}\Omega_4\int_0^\infty \mathrm{e}^{-\tau\mu^{-2}(m^2+\frac{g}{2}\phi^2)}\tau^{s-3}\,\mathrm{d}\tau\ . \tag{3.33}$$

We can now make a substitution $\tau\mu^{-2}(m^2 + \frac{g}{2}\phi^2) \to t$ to obtain a nice form of the Gamma function

$$\frac{\mu^4}{16\pi^2}\left[\frac{\mu^2}{m^2+\frac{g}{2}\phi^2}\right]^{s-2}\Omega_4\int_0^\infty \mathrm{e}^{-t}t^{s-3}\,\mathrm{d}t = \frac{\mu^4}{16\pi^2}\left[\frac{\mu^2}{m^2+\frac{g}{2}\phi^2}\right]^{s-2}\Omega_4\Gamma(s-2)\,. \tag{3.34}$$

Substituting this result back to the formula for the spectral zeta function one obtains

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\,K(\tau)\tau^{s-1}\,\mathrm{d}\tau = \frac{\Gamma(s-2)}{\Gamma(s)}\frac{\mu^4}{16\pi^2}\left[\frac{\mu^2}{m^2+\frac{g}{2}\phi^2}\right]^{s-2}\Omega_4\,, \tag{3.35}$$

where we use the fact, that the Gamma function is a generalization of the factorial $\Gamma(s-2)/\Gamma(s) = 1/(s-1)(s-2)$. Hence we write

$$\zeta_A(s) = \left(\frac{1}{s-2} - \frac{1}{s-1}\right)\frac{\mu^4}{16\pi^2}\left[\frac{\mu^2}{m^2+\frac{g}{2}\phi^2}\right]^{s-2}\Omega_4\,. \tag{3.36}$$

This thus concludes step two.

The last step is to find $\det\Big(-\bar{\partial_x}^2 + m^2 + \frac{g}{2}\phi^2\Big) = \zeta'(0)$ which is a simple calculus

$$\zeta'_{(-\bar{\partial_x}^2+m^2+\frac{g}{2}\phi^2)}(0) = \frac{\mu^4}{16\pi^2}\Omega_4\Big(\frac{m^2+\frac{g}{2}\varphi^2}{\mu^2}\Big)^2\Big[\frac{3}{4}+\frac{1}{2}\ln\Big(\frac{\mu^2}{m^2+\frac{g}{2}\varphi^2}\Big)\Big]. \tag{3.37}$$

Here again we stress the fact that there is no explicit coordinate in the form of $\zeta'(0)$. This is to our great benefit, for the result would be identical if we calculated it in the Minkowskian regime. Thus, we will be able to use this exact result in Chapter 4, where all calculations are done in Minkowskian regime.

Let us now find the Euclidean effective potential (see (2.80)) as

$$\Gamma[\varphi] = V_{\text{eff}}[\varphi]\Omega_4 = \Big(V[\varphi]+V^1[\varphi]\Big)\Omega_4\,, \tag{3.38}$$

where V is the classical potential and V^1 is the one-loop contribution. Further by definition (see (2.77))

$$\mathrm{e}^{-\Gamma[\phi]+J_i\phi_i} = \mathrm{e}^{-S[\phi]-\Gamma^1[\phi]+J_i\phi_i} = \mathrm{e}^{-S[\phi]+\phi_iJ_i}\Big[\det\Big(S^{(2)}_{ij}[\phi]\Big)\Big]^{-1/2}, \tag{3.39}$$

we obtain a relation for the one-loop effective potential to be

$$\begin{aligned}\mathrm{e}^{-\Gamma^1[\phi]} = \mathrm{e}^{-V^1[\phi]\Omega_4} = \Big[\det\Big(S^{(2)}_{ij}[\phi]\Big)\Big]^{-1/2} \quad\Longleftrightarrow\quad V^1[\phi]\int \mathrm{d}^4\bar{x} = \frac{1}{2}\ln\det\Big(S^{(2)}_{ij}[\phi]\Big)\\ V^1[\phi]\Omega_4 = -\frac{1}{2}\zeta'_{-\bar{\partial_x}^2+m^2+\frac{g}{2}\phi^2}(0)\,,\end{aligned} \tag{3.40}$$

and from here we see, that the volume integral Ω_4 will cancel on both sides of the equation.

The complete potential (the classical + the one-loop correction) reads (see c.f. [3])

$$V(\phi) = \frac{1}{2}m^2\phi^2+\frac{g}{4!}\phi^4+\frac{1}{64\pi^2}\Big(m^2+\frac{g}{2}\varphi^2\Big)^2\Big[-\frac{3}{2}+\ln\Big(\frac{m^2+\frac{g}{2}\varphi^2}{\mu^2}\Big)\Big]. \tag{3.41}$$

Having derived the form of the effective potential, we are now able to calculate the effective mass of the field and also its effective coupling constants, as proposed in Section 2.4.8.

$$m_{\text{eff}} = \frac{\partial^2 V(\phi)[0]}{\partial\phi^2} = m^2+\frac{m^2 g}{128\pi^2}\Big[-5+2\ln\Big(\frac{m^2}{\mu^2}\Big)\Big]. \tag{3.42}$$

We see, that the mass of the field changes, but also that we have and arbitrary parameter μ of the theory still in present. It is, however, possible to expressed it with respect to the coupling constant, which makes the potential a parameter of m and g and their scaling relation.

Chapter 4

Weyl Conformal Theory of Gravity and Its Quantization

This chapter will be devoted to the study of Weyl conformal theory of gravity and its quantum extensions. It is an example of a fourth-order theory with an action quadratic in curvature, containing all possible terms from R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in such a specific combination that the resulting action is conformally invariant. As a part of a general study of Riemann curvature tensor we stumble upon Ricci decomposition, where we find a traceless Weyl tensor which is conformally invariant. The action must be therefore composed of the contraction of this Weyl tensor. We will first present arguments, why conformal theory should be the right extension of GR and a good basis for a theory of quantum gravity. We will then show, how the general action can be simplified and perform its expansion into a linearized theory. Our next focus will be on the discussion of quantum conformal gravity with its implications found by P. Jizba, H. Kleinert and F. Scardigli in [1].

We note, that from this chapter on, we are working in the Minkowskian regime again. The reason for that is, that in curved spacetime, the Osterwalder–Schrader theorem does not hold and the Euclidean and the Minkowskian regimes are, in general, not equivalent. The question then arises – which of these regimes is the correct one to use? There are physicists (among them was for example S. Hawking), who postulate the correct form of the gravity to be Euclidean, nevertheless we will avoid this discussion and work in the Minkowskian regime assuming it is closer to physics.

4.1 Physical Motivation

First we pose a question as to why should the conformal symmetry be important? Generally, we use symmetries in physics to simplify problems – nowadays we use the symmetry of the Lorentz group as a starting point of our theories, requiring that the laws of physics are invariant under the action of the Lorentz (Poincaré) group. Assuming the system is invariant under space and time translations enables us to make physical predictions about far away places and both future and past times. Rotational invariance, on the other hand, guarantees isotropy and independence of direction and invariance under boosts tells us the laws of nature are independent of the speed of any observer. However, we have huge problems making predictions about different (energy) scales[1]. All perturbation approaches always assume the perturbation to be small i.e. the energy of the system must not change too dramatically. The Standard Model can predict to about a range of 1 TeV and our largest experimental apparatuses probe the energy scales of around 14 TeV, but we are still blind at greater energy scales, such as those needed by the Grand Unification Theories (GUT) 10^{13} TeV or the early stages of the universe 10^{30} TeV.

This would be elegantly resolved by the conformal invariance, whose physical interpretation is that the system must behave identically on all scales. This would be the missing symmetry, which would enable us to easily predict the behaviour of the system on any energy scale. However, except for some rare examples, the nature is not scale invariant. We must therefore assume the symmetry breaks during the evolution of our universe, giving birth to scale $\sim$ mass. We will show results confirming precisely this hypothesis.

One of the examples of systems that are scale invariant (are of fractal character) are systems undergoing a phase transformation. It is known, that some information about the system gets lost during a phase transformation (thus the process virtually violates unitarity). The conformal theory of gravity can therefore be a good candidate for the description of the first moments of existence of our universe. Whether it was the hot/cold Big Bang or the aftermath of Big Crunch, either way the process can be thought of as a phase transformation and conformal theory is appropriate for its description.

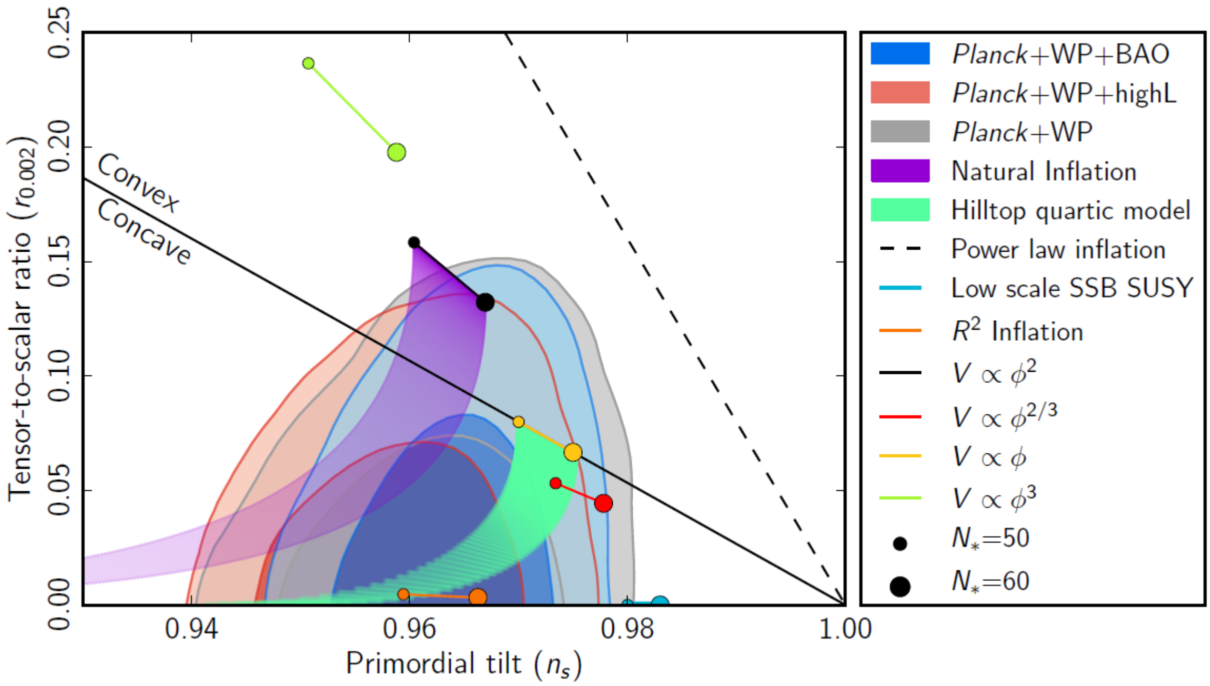
This early universe argument is strongly reinforced by cosmological observations, finding the most probable inflationary scenario to be curvature-driven thus best described by the Starobinsky gravity. It is assumed that our universe underwent an era of extreme expansion – *the inflationary era* – which lasted about 10^{-36}s and during which the size of the universe grew approximately $e^{40\div50}$-times. From the quantum mechanical veiwpoint, it is important to assume, that all fields that were present during the wild early moments of our universe must have fluctuated wildly. The inflation, however, caused an (almost perfect) smoothing out of all these fluctuations as we infer from the Cosmic Microwave Background (CMB) data. This is the basis for our present Standard Model of Cosmology.

From our phenomenological observations we are able to make some guesses about what the structure of the universe must have been in order for our universe to be as it is now. Right now, we are desperately waiting for gravitational waves coming from the inflationary era to be detected. What we have at our disposal at the moment are observations made by Planck and BICEP of the CMB, from which we are able to infer for example the Ω-parameters for energy content of the universe and the ratio of energy contained in tensorial or scalar modes of fluctuations present in the early universe matter. The tensor-to-scalar ratio of tensorial modes (the metric tensor) and of scalar modes (the assumed inflaton field, temperature etc.) is today at value $r < 0.11$ [9, 10]. In figure 4.1a and 4.1b, we see, that this ratio is theoretically best reached by the Starobinsky model, which takes for a source of the inflaton field (the field assumingly responsible for the inflation) higher orders of the curvature tensor – R^2. This is extremely important for conformal gravity because Starobinsky model arises in the conformal gravity as a low energy limit after the breakdown of conformal symmetry [1].

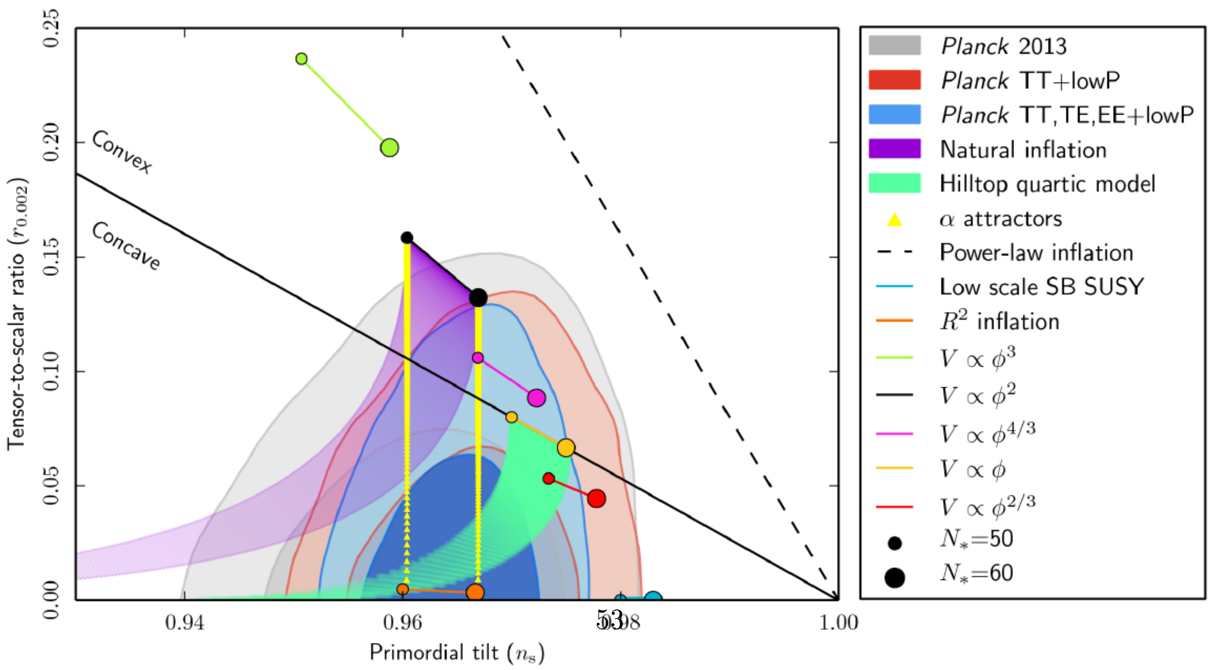
[1]Scale in this context is either length, mass or time, since these are the three units, that can be all set as $[x] = [t] = [m]^{-1} = \mathrm{L}$ by the choice of the speed of light c and the Planck constant $\hbar$.

Figure 4.1: Joint data results from experiments Planck, Wilkinson Microwave Anisotropy Probe (WP) and Baryon Acoustic Oscillation (BAO). Two shades of each colour represent 68% and 95% confidence level within the chosen experimental data. The data show the tensor-to-scalar ratio and its relation with the scalar spectral index n_s, which says how much the scalar fluctuations changes with scale. Since the observation shows that the fluctuations were not uniform on all scales, we must choose a pivotal scale $k_* = 0.002\,\mathrm{MPc}^{-1}$ to obtain the data. Theoretical predictions of inflationary models are plotted as color segments. N_* denotes the e-folding number, which is a parameter of the inflationary models. It is clear that the data favour the R^2 inflation model i.e. the Starobinsky model.

(a) Data from the 2013 data analysis, taken from [9]



(b) Data from the 2015 data analysis, taken from [10]



4.2 Mathematical Background

Let us define a *Weyl transformation* as

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)\,. \tag{4.1}$$

The Weyl transformation (or scaling transformation) is one of conformal subgroups, generated by a *conformal group.* In 4 dimensions, the conformal group has 15 infinitesimal generators – 6 Lorentz generators, 4 translations, 4 conformal boosts and 1 scaling. All except the Weyl transformation are already included in general diffeomorphism invariance of general relativity since these transformation act only as a change of coordinates. Thus, working with a generally relativistic theory it is sufficient to add only the Weyl transformation to obtain a conformally invariant theory.

All conformal transformations leave angles invariant, which can be easily seen from the definition of the cosine of an angle θ between vectors X^μ and Y^ν

$$\cos(\theta) = \frac{g_{\mu\nu} X^\mu Y^\nu}{\sqrt{g_{\mu\nu} X^\mu X^\nu g_{\alpha\beta} Y^\alpha Y^\beta}}\,. \tag{4.2}$$

Next we put forward some useful mathematical result that will be needed in the following sections. Let us begin by writing the action for the theory

$$S = -\frac{8}{\alpha_c^2} \int \mathrm{d}^4 x \sqrt{-g}\, C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}\,, \tag{4.3}$$

where α_c is a small dimensionless coupling constant and $C_{\alpha\beta\gamma\delta}$ is the so-called Weyl tensor defined as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}(g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha}) + \frac{1}{6} R g_{\alpha[\gamma} g_{\delta]\beta}\,, \tag{4.4}$$

It is clear that the action (4.3) is conformally invariant since $C^\alpha_{\beta\gamma\delta}$ is and we can write the integrand as

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = C^\alpha_{\beta\gamma\delta} g_{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} C^\mu_{\nu\rho\sigma} \to \Omega^{-6+2} C^\alpha_{\beta\gamma\delta} g_{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} C^\mu_{\nu\rho\sigma}\,, \tag{4.5}$$

where the term Ω^{-4} cancels out with the term from the transformation rule for the determinant $g = \det g_{\mu\nu} \to \Omega^8 g$. Using the definition of the Weyl tensor, known definitions for contractions of the Riemann tensor and $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ we rewrite the integrand of the action as

$$
\begin{aligned}
&C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \\
&= \frac{1}{4}\big(g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha}\big)\big(g^{\alpha\gamma}R^{\delta\beta} - g^{\alpha\delta}R^{\gamma\beta} - g^{\beta\gamma}R^{\delta\alpha} + g^{\beta\delta}R^{\gamma\alpha}\big) + \\
&+ R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \frac{1}{36}R^2\big(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}\big)\big(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta}\big) - \\
&- R_{\alpha\beta\gamma\delta}\big(g^{\alpha\gamma}R^{\delta\beta} - g^{\alpha\delta}R^{\gamma\beta} - g^{\beta\gamma}R^{\delta\alpha} + g^{\beta\delta}R^{\gamma\alpha}\big) + R_{\alpha\beta\gamma\delta}\frac{1}{3}R\big(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta}\big) - \\
&- \frac{1}{6}\big(g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha}\big)R\big(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta}\big) = \\
&= R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \frac{1}{4}4\big(4R_{\alpha\beta}R^{\alpha\beta} - R_{\alpha\beta}R^{\alpha\beta} - R_{\alpha\beta}R^{\alpha\beta} + R^2\big) + \frac{1}{36}R^2\big(4\cdot 4 + 4\cdot 4 - 4 - 4\big) - \\
&- 4R_{\alpha\beta}R^{\alpha\beta} + \frac{4}{6}R^2 - \frac{4}{6}R(4R - R) = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R^{\alpha\beta} + R^2\big(1 + \frac{24}{36} + \frac{4}{6} - \frac{12}{6}\big)
\end{aligned} \tag{4.6}
$$

so we can equivalently write the action (4.3) using the Riemann tensor, Ricci tensor and the Ricci scalar as

$$
S = -\frac{8}{\alpha_c^2}\int \mathrm{d}^4x\,\sqrt{-g}\,\Big[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2\Big]\,. \tag{4.7}
$$

The action can be further simplified with the use of the Gauss–Bonnet invariant

$$
\mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,, \tag{4.8}
$$

since the term $\sqrt{-g}\,\mathcal{G}$ contributes only a total divergence term – the Gauss–Bonnet theorem. It is important to remark, that the above Gauss–Bonnet theorem holds only in a fixed dimension of four. Using the theorem, it is possible to subtract the $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ term from the action and get an equivalent action in the form

$$
S = -\frac{1}{4\alpha_c^2}\int \mathrm{d}^4x\,\sqrt{-g}\,\Big(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2\Big)\,. \tag{4.9}
$$

The action in this form retains the conformal invariance and also exhibit the general diffeomorphism invariance. Variating this action w.r.t. the metric yields the so-called Bach's equations

$$
\frac{1}{2}g^{\alpha\beta}\big(R^{\gamma\delta}R_{\gamma\delta} - \frac{1}{3}R^2\big) - \nabla^2\big(R^{\alpha\beta} - \frac{1}{6}Rg^{\alpha\beta}\big) + R^{\alpha\gamma;\beta}{}_{;\gamma} - \frac{2}{3}R^{;\alpha\beta} = 0\,. \tag{4.10}
$$

Now we see again that the equations are of fourth order in the metric. It has been shown [12], that the ensuing linearized equations have six plane wave solutions corresponding to six propagating physical degrees of freedom – massless spin-2 graviton, massless spin-1 vector boson, identifiable with photon and a massless spin-2 ghost particle. This will also be indicated later in our search for the effective potential.

4.3 The Linearization of Weyl Gravity

To linearize the theory, we take $g_{\mu\nu} = \eta_{\mu\nu} + \alpha_c h_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ being the metric of a flat spacetime, α_c the small coupling constant from the Weyl action and $h_{\mu\nu}$ a disturbance.

It is not possible to assume anything about the $h_{\mu\nu}$ since it later appears in the path integral measure, hence it must run over all possible field configurations.

We now find contribution to the action to the lowest order of α_c. Let us start by the linearizing the connection

$$\Gamma_{\alpha\beta\gamma} = \alpha_c \frac{1}{2}\big(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\big) = \alpha_c \frac{1}{2}\big(h_{\alpha\beta,\gamma} + h_{\alpha\gamma,\beta} - h_{\beta\gamma,\alpha}\big) + O(\alpha_c^2)\,. \tag{4.11}$$

Next we recall the definition of the Riemann tensor and substitute result for the connection to find

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \Gamma_{\alpha\beta\delta,\,\gamma} - \Gamma_{\alpha\beta\gamma,\,\delta} + \Gamma_{\alpha\gamma\sigma}\Gamma^{\sigma}_{\beta\delta} - \Gamma_{\alpha\delta\sigma}\Gamma^{\sigma}_{\beta\gamma} = \\ &= \alpha_c \frac{1}{2}\big(h_{\alpha\beta,\delta} + h_{\alpha\delta,\beta} - h_{\beta\delta,\alpha}\big)_{,\,\gamma} - \alpha_c \frac{1}{2}\big(h_{\alpha\beta,\gamma} + h_{\alpha\gamma,\beta} - h_{\beta\gamma,\alpha}\big)_{,\,\delta} + O(\alpha_c^2) = \\ &= \alpha_c \frac{1}{2}\big(h_{\alpha\delta,\,\beta\gamma} + h_{\beta\gamma,\,\alpha\delta} - h_{\beta\delta,\,\alpha\gamma} - h_{\alpha\gamma,\,\beta\delta}\big)\,. \end{aligned} \tag{4.12}$$

From here we find by contractions, that the Ricci tensor is equal to

$$R_{\mu\nu} = \alpha_c \frac{1}{2}\Big(h^{\alpha}{}_{\mu,\,\nu\alpha} + h^{\alpha}{}_{\nu,\,\mu\alpha} - \Box h_{\mu\nu} - h^{\alpha}{}_{\alpha,\,\mu\nu}\Big)\,, \tag{4.13}$$

where $\Box = \partial_\alpha\,\partial^\alpha$. Tracing again, we get the Ricci scalar

$$R = \alpha_c\big(h^{\alpha\beta}{}_{,\,\alpha\beta} - \Box h^{\alpha}{}_{\alpha}\big)\,. \tag{4.14}$$

Since all curvature terms are squared on the action, it is clear, they will be at least of second order in α_c. We shall now find all the terms from the action (4.9) in the second order of α_c to have everything ready for further calculations. We begin wit the square of the Ricci tensor

$$\begin{aligned} 4R_{\mu\nu}R^{\mu\nu}\alpha_c^{-2} &= \Big(h^{\alpha}{}_{\mu,\,\nu\alpha} + h^{\alpha}{}_{\nu,\,\mu\alpha} - \Box h_{\mu\nu} - h^{\alpha}{}_{\alpha,\,\mu\nu}\Big)\Big(h^{\alpha\mu\,\nu}{}_{,\,\alpha} + h^{\alpha\nu\,\mu}{}_{,\,\alpha} - \Box h^{\mu\nu} - h^{\alpha}{}_{\alpha,}{}^{\mu\nu}\Big) = \\ &= \overbrace{h^{\alpha}{}_{\mu,\,\nu\alpha}h^{\beta\mu\,\nu}{}_{,\,\beta}}^{\text{A}} + \overbrace{h^{\alpha}{}_{\nu,\,\mu\alpha}h^{\beta\nu\,\mu}{}_{,\,\beta}}^{\text{B}} + \overbrace{\Box h_{\mu\nu}\Box h^{\mu\nu}}^{\text{C}} + \overbrace{h^{\alpha}{}_{\alpha,\,\mu\nu}h^{\beta}{}_{\beta,}{}^{\mu\nu}}^{\text{D}} + \\ &+ 2\Big[\overbrace{h^{\alpha}{}_{\mu,\,\nu\alpha}h^{\beta\nu\,\mu}{}_{,\,\beta}}^{①} - \overbrace{h^{\alpha}{}_{\mu,\,\nu\alpha}\Box h^{\mu\nu}}^{②} - \overbrace{h^{\alpha}{}_{\mu,\,\nu\alpha}h^{\beta}{}_{\beta,}{}^{\mu\nu}}^{③} - \overbrace{h^{\alpha}{}_{\nu,\,\mu\alpha}\Box h^{\mu\nu}}^{④} - \overbrace{h^{\alpha}{}_{\nu,\,\mu\alpha}h^{\beta}{}_{\beta,}{}^{\mu\nu}}^{⑤} + \overbrace{\Box h_{\mu\nu}h^{\beta}{}_{\beta,}{}^{\mu\nu}}^{⑥}\Big]\,. \end{aligned} \tag{4.15}$$

As we are searching for terms that will be put back into the action, we are not interested in total derivatives. As a consequence, we can shift the derivatives in the products from one term to the other compensating by an extra minus sign for every shift and forgetting the total derivative terms. Thus, for example

$$h^{\alpha}{}_{\mu,\,\nu\alpha}h^{\beta\mu\,\nu}{}_{,\,\beta} \simeq -\,\partial_\alpha h^{\alpha}{}_{\mu}\Box\,\partial_\beta h^{\beta\mu}\,, \tag{4.16}$$

where the symbol $\simeq$ reminds us, that the total derivatives are not taken into account. We rearrange the terms one by one, obtaining relations

$$\begin{gathered} \text{A} \simeq \text{B} \simeq -\,\partial_\alpha h^{\alpha}{}_{\mu}\Box\,\partial_\beta h^{\beta\mu}\,, \quad \text{C} \simeq h_{\mu\nu}\Box^2 h^{\mu\nu}\,, \quad \text{D} \simeq h^{\alpha}{}_{\alpha}\Box^2 h^{\beta}{}_{\beta} \\ ① \simeq h^{\alpha\nu}{}_{,\alpha\nu}h^{\beta\mu}{}_{,\beta\mu}\,, \quad ② \simeq ④ \simeq -\text{A}\,, \quad ③ \simeq ⑥ \simeq ⑤\,. \end{gathered} \tag{4.17}$$

Using these results, the square of the Ricci tensor can be written as

$$R_{\mu\nu}R^{\mu\nu} = \frac{\alpha_c^2}{4}\Big[2\,\partial_\alpha h^{\alpha\mu}\Box\eta_{\mu\nu}\,\partial_\beta h^{\beta\nu} + h_{\mu\nu}\Box^2 h^{\mu\nu} + h^\alpha{}_\alpha\Box^2 h^\beta{}_\beta + 2h^{\alpha\nu}{}_{,\alpha\nu}h^{\beta\mu}{}_{,\beta\mu} - 2h^{\alpha\nu}{}_{,\alpha\nu}\Box h^\beta{}_\beta\Big]\,. \tag{4.18}$$

The square of the Ricci scalar is easy to obtain straight from (4.14)

$$R^2 = \alpha_c^2\big(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h^\alpha{}_\alpha\big)^2\,. \tag{4.19}$$

We must not forget to expand the term $\sqrt{-g} = \sqrt{-\det(\eta_{\mu\nu}+\alpha_c h_{\mu\nu})}$ into a series in α_c. To that end we use $\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta g$ and the definition of g

$$\det(\eta_{\mu\nu}+\alpha_c h_{\mu\nu}) = \det(\eta_{\mu\alpha})\det\Big(\delta^\alpha_\nu + \alpha_c\eta^{\alpha\beta}h_{\beta\nu}\Big) \equiv \eta\det\Big(\delta^\alpha_\nu + \alpha_c\eta^{\alpha\beta}h_{\beta\nu}\Big)\,. \tag{4.20}$$

Now we compute the determinant with the use of the trace-log formula

$$\det\Big(\delta^\alpha_\nu + \alpha_c\eta^{\alpha\beta}h_{\beta\nu}\Big) = \exp\mathrm{Tr}\ln\Big(1+\alpha_c\eta^{\alpha\beta}h_{\beta\nu}\Big) \doteq \exp\mathrm{Tr}\Big(\alpha_c\eta^{\alpha\beta}h_{\beta\nu} - \frac{1}{2}\alpha_c^2\eta^{\alpha\beta}h_{\beta\gamma}\eta^{\gamma\delta}h_{\delta\nu}\Big) =$$
$$= \exp\big(\alpha_c h^\alpha{}_\alpha - \frac{1}{2}\alpha_c^2 h^{\alpha\beta}h_{\alpha\beta}\big) \doteq 1 + \alpha_c h^\alpha{}_\alpha - \frac{1}{2}\alpha_c^2 h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{4}\alpha_c^2 h^\alpha{}_\alpha h^\beta{}_\beta\,, \tag{4.21}$$

and now by the definition of variation

$$\det(\eta_{\mu\nu}+\alpha_c h_{\mu\nu}) = \eta + \delta\eta = \eta + \eta\Big(\alpha_c h^\alpha{}_\alpha - \frac{1}{2}\alpha_c^2 h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{4}\alpha_c^2 h^\alpha{}_\alpha h^\beta{}_\beta\Big)\,. \tag{4.22}$$

Using the fact, that $\eta = \det\eta_{\mu\nu} = -1$ we find the variation of $\sqrt{-g}$ to be

$$\sqrt{-g} = 1 + \frac{\alpha_c}{2}h^\alpha{}_\alpha + \frac{a_c^2}{8}\Big(h^\alpha{}_\alpha h^\beta{}_\beta - 2h^{\alpha\beta}h_{\alpha\beta}\Big)\,. \tag{4.23}$$

4.4 Hubbard–Stratonovich Transformation, Linearized Action

This section will be devoted to recovering a specific form of action used by P. Jizba and his colleges in their article [1]. To be able to make an easy comparison of the results, we will follow their notation. It is important to remark, that since we are interested in the partition function Z and since any physically relevant quantity derived from Z is insensitive to the normalization of Z, we will, in the course of our computations, usually forget all extra numerical factors.

We start this section by writing the Weyl action again

$$A = -\frac{1}{4\alpha_c^2}\int \mathrm{d}^4x\,\sqrt{-g}\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big)\,. \tag{4.24}$$

The overall goal of the rest of this chapter will be to show, that the quantum Weyl gravity dynamically breaks its conformal invariance and turns morphs a the Starobinsky gravity after the breakdown. The Starobinsky gravity is described by the action

$$A_{\mathrm{St}} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\Big(R - \xi^2 R^2\Big)\,, \tag{4.25}$$

where $\kappa^2 = 8\pi G = 8\pi/m_{\text{Planck}}^2 \doteq 5 \cdot 10^{-16}\text{kg}^2$, G being the Newton's gravitational constant and ξ is a small constant for which the observations of the CMB give constraints $\xi/\kappa \sim 10^5$ [9, 10]. We will take several steps to change the form of the action (4.24) to show it indeed transforms into the Starobinsky gravity.

The first part of this section will shortly introduce the Hubbard–Stratonovich transformation. We start from the one-dimensional integral identity

$$\mathrm{e}^{-ax^2} = \sqrt{\frac{1}{4\pi a}} \int_{-\infty}^{\infty} \mathrm{d}y \; \mathrm{e}^{-\frac{y^2}{4a} - \mathrm{i}xy} \,, \tag{4.26}$$

which can be generalized analogously to the first chapter to

$$\mathrm{e}^{-x^T A x} = \left(\frac{1}{4\pi}\right)^{d/2} \big(\det A\big)^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}^d y \; \mathrm{e}^{-\frac{1}{4} y^T A^{-1} y - \mathrm{i} x^T y} \,, \tag{4.27}$$

and sending the dimension to infinity, we formally obtain

$$\exp\left[-\int \mathrm{d}x\,\mathrm{d}y\, f(x) A(x,y) f(y)\right] = N \int \mathcal{D}g\, \mathrm{e}^{-\frac{1}{4}\int \mathrm{d}x \mathrm{d}y g(x) A^{-1}(x,y) g(y) - \mathrm{i}\int \mathrm{d}x f(x) g(x)} \,. \tag{4.28}$$

In our notation, we thus write

$$\mathrm{e}^{-f_x A_{xy} f_y} = \int \mathcal{D}g\, \mathrm{e}^{-\frac{1}{4} g_x A^{-1}_{xy} g_y - \mathrm{i} f_x g_x} \,. \tag{4.29}$$

This so-called Hubbard–Stratonovich (HS) transformations enables us to substitute a quadratic term f^2 in the exponent for a linear term at the price of introducing a new variable g. With this transformation also arises a new (divergent) constant N, but we include it in the measure $\mathcal{D}g$. We will use the HS transformation on the R^2 part of the action to obtain a new (dimensionfull) scalar field λ and to reduce R^2 to R. Specifically, we transform

$$\mathrm{e}^{\mathrm{i}A_{R^2}} = \exp\Big(\frac{\mathrm{i}}{12\alpha_c^2} \int \mathrm{d}^4x \sqrt{-g} R^2\Big) = \int \mathcal{D}\lambda \exp\Big[-\mathrm{i} \int \mathrm{d}^4x \sqrt{-g} \Big(3\alpha_c^2 \lambda^2 + R\lambda\Big)\Big] \,, \tag{4.30}$$

and to obtain the same result as in [1], we rescale the field $\lambda \to \lambda/2$

$$\mathrm{e}^{\mathrm{i}A_{R^2}} = \int \mathcal{D}\lambda \exp\Big[-\mathrm{i} \int \mathrm{d}^4x \sqrt{-g} \Big(\frac{3\alpha_c^2}{4} \lambda^2 + \frac{R}{2}\lambda\Big)\Big] \,. \tag{4.31}$$

Here, the dimensions of λ are the same as those of R, namely $[\lambda] = \mathrm{L}^{-2}$ in order to have a dimensionless action.

To reproduce the complete action as in [1], we further use a trivial identity $1 = \cosh^2\theta - \sinh^2\theta$ on the R^2 term

$$A = -\frac{1}{4\alpha_c^2} \int \mathrm{d}^4x \sqrt{-g} R_{\mu\nu} R^{\mu\nu} + \big(\cosh^2\theta - \sinh^2\theta\big) \frac{1}{12\alpha_c^2} \int \mathrm{d}^4x \sqrt{-g} R^2 \,, \tag{4.32}$$

and apply the Hubbard–Stratonovich transformation only to the part with $\sinh\theta$ (which effectively means we rescale $\alpha_c^2 \to -\alpha_c^2/\sinh^2\theta$). After denoting $\sinh\theta \equiv S$ and $\cosh\theta \equiv C$ for brevity we get

$$A[g_{\mu\nu},\lambda] = \int \mathrm{d}^4x\,\sqrt{-g}\Big[-\frac{1}{4\alpha_c^2}R_{\mu\nu}R^{\mu\nu} + \frac{C^2}{12\alpha_c^2}R^2 + \frac{3\alpha_c^2}{4S^2}\lambda^2 - \frac{R}{2}\lambda\Big]. \tag{4.33}$$

We see now, that the trivial identity $1 = S^2 - C^2$ provided us with additional term R^2 in the action, which is crucial if we want to obtain the Starobinsky model. To make to comparison even easier, we rescale the field $\lambda \to \lambda/\kappa^2$, so that we can easily identify the terms present the action with the Starobinsky model. Due to the rescaling, the field λ became dimensionless and we can now write the final form of the action with two dynamical fields $g_{\mu\nu}$ and λ

$$A = \int \mathrm{d}^4x\,\sqrt{-g}\Big[-\frac{1}{4\alpha_c^2}R_{\mu\nu}R^{\mu\nu} + \frac{C^2}{12\alpha_c^2}R^2 + \frac{3\alpha_c^2}{4S^2\kappa^4}\lambda^2 - \frac{1}{2\kappa^2}R\lambda\Big]. \tag{4.34}$$

By comparison with the Starobinsky action (4.25) we see, that $\frac{C^2}{12\alpha_c^2}$ corresponds to $\xi^2/2\kappa^2$ and that the long-range behaviour of the Weyl gravity will coincide with the Starobinsky model, when $\lambda = 1$. That this is possible will be our goal to show.

For future calculations is interesting to discuss the magnitude of each term present in the action. Since $\alpha_c \sim C\kappa/\xi \sim C10^{-5}$ is our small parameter of the expansion (thus surely $\alpha_c < 1$), we see, that the first term is of magnitude $C^{-2}10^{10} \div C^{-2}$, the second one 10^{10}, the third one $C^2S^{-2}\kappa^{-4}10^{-10}$ (which for $S > 1$ or $C > 1$ goes to $\kappa^{-4}10^{-10}$) and the last one is of magnitude κ^{-2}. Since $\kappa^2 \doteq 5\cdot 10^{-16}\mathrm{kg}^2$ we immediately see that the last two terms are of magnitude $\sim 10^{22}\mathrm{kg}^{-2}$ and $\sim 10^{17}\mathrm{kg}^{-2}$, thus they clearly dominate the action. It is therefore sufficient to expand the first two terms only up to the order α_c^0 as any higher terms would be further suppressed by the fact that α_c is small.

This suppression is, however, not so significant in the last term, where the expansion into higher orders of α_c is compensated by κ^{-2}, hence we will expand the last term up to the second order of α_c. We notice here, that the results prepared in the previous section do not provide $\sqrt{-g}R$ up to the second order. In this place we refer to [11] for more precise result

$$\begin{aligned}\sqrt{-g}R = \alpha_c\big(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h^{\alpha}{}_{\alpha}\big) + \alpha_c^2\Big[\frac{1}{2}h^{\gamma}{}_{\gamma}\big(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h^{\alpha}{}_{\alpha}\big) - h_{\alpha\gamma}\Big(2h_{\beta}{}^{\gamma,\beta\alpha} - h^{\beta}{}_{\beta}{}^{,\alpha\gamma} -\\ - \Box h^{\alpha\gamma}\Big) + \frac{1}{4}\Big(3h_{\beta\gamma,\alpha}h^{\beta\gamma,\alpha} - 2h_{\beta\gamma,\alpha}h^{\alpha\gamma,\beta} - 4h^{\alpha}{}_{\beta,\alpha}h^{\beta\gamma}{}_{,\gamma} + 4h^{\alpha}{}_{\beta,\alpha}h^{\gamma}{}_{\gamma}{}^{,\beta} - h^{\beta}{}_{\beta,\alpha}h^{\gamma}{}_{\gamma}{}^{,\alpha}\Big)\Big],\end{aligned} \tag{4.35}$$

After rearranging the derivatives and forgetting the total derivative terms, we can substitute in the action term by term the following expressions

$$\begin{aligned}\sqrt{-g}R_{\mu\nu}R^{\mu\nu} &\simeq \frac{\alpha_c^2}{4}\Big[2\,\partial_\alpha h^{\alpha\mu}\Box\eta_{\mu\nu}\,\partial_\beta h^{\beta\nu} + h_{\mu\nu}\Box^2h^{\mu\nu} + h^{\alpha}{}_{\alpha}\Box^2h^{\beta}{}_{\beta} + 2(h^{\alpha\nu}{}_{,\alpha\nu})^2 - 2h^{\alpha\nu}{}_{,\alpha\nu}\Box h^{\beta}{}_{\beta}\Big]\\ \sqrt{-g}R^2 &\simeq \alpha_c^2\big(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h^{\alpha}{}_{\alpha}\big)^2\\ \sqrt{-g}R &\simeq -\alpha_c\Box\bar{h} + \frac{\alpha_c^2}{2}\Big(-h^{\alpha}{}_{\alpha}\Box\bar{h} + h^{\alpha\mu}{}_{,\alpha}\eta_{\mu\nu}h^{\beta\nu}{}_{,\beta} + \frac{1}{2}h_{\alpha\beta}\Box h^{\alpha\beta} + \frac{1}{2}h^{\alpha}{}_{\alpha}\Box h^{\beta}{}_{\beta}\Big),\end{aligned} \tag{4.36}$$

where we have already used notation $\bar{h}$ introduced further down the text.

Before we substitute these terms into the action, we set $\lambda = \bar{\lambda} + \delta\lambda$ and use only the the expectation value of the field $\bar{\lambda}$. This corresponds to the choice of $\phi(x) = \phi$ from Section 3.4 for the calculation of the effective potential of the ϕ^4 theory. Another consequence of this choice is, that $\bar{\lambda}$ is no longer a dynamical field and therefore we do not integrate over it in the path integral.

It is apparent the terms the in the action are very distinct in specific ways, hence we will work with the action piece by piece. Firstly, the term proportional to λ^2 has no way to merge with the other terms and will survive in this form until the end. Secondly, the last term with $1/2\kappa^2$ will also survive separately, however, the first two terms will merge in some way. Let us separate the action as $A = A_1 + A_{\lambda^2} + A_{\kappa^{-2}}$ and focus on the A_1 part first

$$A_1 = \int \mathrm{d}^4x \left[-\frac{1}{8}\,\partial_\alpha h^{\alpha\mu} \Box\eta_{\mu\nu}\,\partial_\beta h^{\beta\nu} - \frac{1}{16} h_{\mu\nu}\Box^2 h^{\mu\nu} - \frac{1}{16} h^\alpha{}_\alpha \Box^2 h^\beta{}_\beta - \frac{1}{8}(h^{\alpha\nu}{}_{,\alpha\nu})^2 + \frac{1}{8} h^{\alpha\nu}{}_{,\alpha\nu}\Box h^\beta{}_\beta + \frac{C^2}{12}\left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h^\alpha{}_\alpha\right)^2 \right]. \tag{4.37}$$

To systematically simplify this part of the action, we will use the knowledge about the constraints. As the theory is conformally and diffeomorphically invariant, we must fix a gauge. The diffeomorphism constraint is put on functions $\chi^\nu \equiv \partial_\mu h^{\mu\nu}$ and the conformal constraint is put on $\chi \equiv \bar{h} = h^\alpha{}_\alpha - \partial_\mu \Box^{-1}\,\partial_\nu h^{\mu\nu}$ [13]. We further denote an operator $H_{\mu\nu} \equiv \frac{1}{2}\,\partial_\mu\,\partial_\nu - \Box\eta_{\mu\nu}$ to follow [1]. We will now try to identify these newly denoted objects in A_1.

First we notice, that $\Box\bar{h} = \Box h^\alpha{}_\alpha - \partial_\mu\,\partial_\nu h^{\mu\nu}$ is present in the last term in the second line. Moreover we rewrite the square of this term, forgetting total derivatives, as $\left(\Box\bar{h}\right)^2 \simeq \bar{h}\Box^2\bar{h}$.

The definition of $H_{\mu\nu}$ can be used in the first term to extract $\Box\eta_{\mu\nu} = \frac{1}{2}\,\partial_\mu\,\partial_\nu - H_{\mu\nu}$. After substituting this expression into the first term, we get $-\,\partial_\alpha h^{\alpha\mu}\Box\eta_{\mu\nu}\,\partial_\beta h^{\beta\mu} \simeq \partial_\alpha h^{\alpha\mu} H_{\mu\nu}\,\partial_\beta h^{\beta\mu} + \frac{1}{2} h^{\alpha\mu}{}_{,\alpha\mu} h^{\beta\nu}{}_{,\beta\nu}$. With the use of these identities, we find this part of the action to turn into

$$\begin{aligned} A_1 &= \int \mathrm{d}^4x \left[\frac{1}{8}\,\partial_\alpha h^{\alpha\mu} H_{\mu\nu}\,\partial_\beta h^{\beta\mu} + \frac{1}{16}\left(h^{\alpha\mu}{}_{,\alpha\mu}\right)^2 - \frac{1}{16} h_{\mu\nu}\Box^2 h^{\mu\nu} - \frac{1}{16} h^\alpha{}_\alpha\Box^2 h^\beta{}_\beta - \frac{1}{8}\left(h^{\alpha\mu}{}_{,\alpha\mu}\right)^2 + \right. \\ &\quad \left. + \frac{1}{8} h^{\alpha\nu}{}_{,\alpha\nu}\Box h^\beta{}_\beta + \frac{C^2}{12}\bar{h}\Box^2\bar{h}\right] = \\ &= \int \mathrm{d}^4x \left[\frac{1}{8}\,\partial_\alpha h^{\alpha\mu} H_{\mu\nu}\,\partial_\beta h^{\beta\mu} - \frac{1}{16}\left(h^{\alpha\mu}{}_{,\alpha\mu}\right)^2 - \frac{1}{16} h_{\mu\nu}\Box^2 h^{\mu\nu} - \frac{1}{16} h^\alpha{}_\alpha\Box^2 h^\beta{}_\beta + \right. \\ &\quad \left. + \frac{1}{8} h^{\alpha\nu}{}_{,\alpha\nu}\Box h^\beta{}_\beta + \frac{C^2}{12}\bar{h}\Box^2\bar{h}\right]. \end{aligned} \tag{4.38}$$

Next we notice, that the second and the fourth term are equal to $\left(h^{\alpha\mu}{}_{,\alpha\mu}\right)^2 + h^\alpha{}_\alpha\Box^2 h^\beta{}_\beta \simeq \left(\Box\bar{h}\right)^2 + 2h^{\alpha\mu}{}_{,\alpha\mu}\Box h^\beta{}_\beta \simeq \bar{h}\Box^2\bar{h} + 2h^{\alpha\mu}{}_{,\alpha\mu}\Box h^\beta{}_\beta$. Thus, we can simplify the action again

$$\begin{aligned} A_1 &= \int \mathrm{d}^4x \left[\frac{1}{8}\,\partial_\alpha h^{\alpha\mu} H_{\mu\nu}\,\partial_\beta h^{\beta\mu} - \frac{1}{16}\left(\bar{h}\Box^2\bar{h} + 2h^{\alpha\mu}{}_{,\alpha\mu}\Box h^\beta{}_\beta\right) - \frac{1}{16} h_{\mu\nu}\Box^2 h^{\mu\nu} \right. \\ &\quad \left. + \frac{1}{8} h^{\alpha\nu}{}_{,\alpha\nu}\Box h^\beta{}_\beta + \frac{C^2}{12}\bar{h}\Box^2\bar{h}\right] = \\ &= \int \mathrm{d}^4x \left[\frac{1}{8}\,\partial_\alpha h^{\alpha\mu} H_{\mu\nu}\,\partial_\beta h^{\beta\mu} + \bar{h}\Box^2\bar{h}\left(\frac{C^2}{12} - \frac{1}{16}\right) - \frac{1}{16} h_{\mu\nu}\Box^2 h^{\mu\nu}\right]. \end{aligned} \tag{4.39}$$

These terms are already present in the action written in [1] hence manipulations with A_1 are now complete.

We now turn our attention to the $A_{\kappa^{-2}}$ part of the action

$$
\begin{aligned}
A_{\kappa^{-2}} &= -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\,R\bar{\lambda} = \\
&= -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\left[-\alpha_c\bar{\lambda}\Box\bar{h} + \frac{\alpha_c^2}{2}\bar{\lambda}\Big(-h^{\alpha}{}_{\alpha}\Box\bar{h} + h^{\alpha\mu}{}_{,\alpha}\eta_{\mu\nu}h^{\beta\nu}{}_{,\beta} + \frac{1}{2}h_{\alpha\beta}\Box h^{\alpha\beta} + \frac{1}{2}h^{\alpha}{}_{\alpha}\Box h^{\beta}{}_{\beta}\Big)\right].
\end{aligned}
\tag{4.40}
$$

Comparison with the article tells us that the term linear in α_c is already in the correct form. There is, however, lot to be done with terms in multiplied by α_c^2. We transform the first term in the round brackets using $h^{\alpha}{}_{\alpha} = \bar{h} + \partial_\mu\Box^{-1}\,\partial_\nu h^{\mu\nu}$

$$
\begin{aligned}
-h^{\alpha}{}_{\alpha}\Box\bar{h} &= -\bar{h}\Box\bar{h} - \partial_\mu\Box^{-1}\,\partial_\nu h^{\mu\nu}\Box h^{\alpha}{}_{\alpha} + \partial_\mu\Box^{-1}\,\partial_\nu h^{\mu\nu}\Box\,\partial_\rho\Box^{-1}\,\partial_\sigma h^{\rho\sigma} \\
&\simeq -\bar{h}\Box\bar{h} - h^{\mu\nu}{}_{,\mu\nu}\Box h^{\alpha}{}_{\alpha} + h^{\mu\nu}{}_{,\mu\nu}\Box^{-1}h^{\rho\sigma}{}_{,\rho\sigma}\,.
\end{aligned}
\tag{4.41}
$$

and insert an identity $\Box^{-1}\Box$ to the second term to be able to use $\Box\eta_{\mu\nu} = \frac{1}{2}\,\partial_\mu\,\partial_\nu - H_{\mu\nu}$ again

$$
\begin{aligned}
h^{\alpha\mu}{}_{,\alpha}\Box^{-1}\Box\eta_{\mu\nu}h^{\beta\nu}{}_{,\beta} &= \frac{1}{2}h^{\alpha\mu}{}_{,\alpha}\Box^{-1}\,\partial_\mu\,\partial_\nu h^{\beta\nu}{}_{,\beta} - h^{\alpha\mu}{}_{,\alpha}\Box^{-1}H_{\mu\nu}h^{\beta\nu}{}_{,\beta} \simeq \\
&\simeq -\frac{1}{2}h^{\alpha\mu}{}_{,\alpha\mu}\Box^{-1}h^{\beta\nu}{}_{,\beta\nu} - h^{\alpha\mu}{}_{,\alpha}\Box^{-1}H_{\mu\nu}h^{\beta\nu}{}_{,\beta}\,.
\end{aligned}
\tag{4.42}
$$

With the use of these identities, we rewrite $A_{\kappa^{-2}}$ so that it now reads

$$
\begin{aligned}
A_{\kappa^{-2}} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\Big[&-\alpha_c\bar{\lambda}\Box\bar{h} - \frac{\alpha_c^2}{2}\bar{\lambda}\bar{h}\Box\bar{h} - \frac{\alpha_c^2}{2}\bar{\lambda}\,\partial_\alpha h^{\alpha\mu}\Box^{-1}H_{\mu\nu}\,\partial_\beta h^{\beta\mu} + \frac{\alpha_c^2}{4}\bar{\lambda}h_{\alpha\beta}\Box h^{\alpha\beta} - \\
&- \frac{\alpha_c^2}{2}\bar{\lambda}h^{\mu\nu}{}_{,\mu\nu}\Box h^{\alpha}{}_{\alpha} + \alpha_c^2\Big(\frac{1}{2} - \frac{1}{4}\Big)\bar{\lambda}h^{\mu\nu}{}_{,\mu\nu}\Box^{-1}h^{\rho\sigma}{}_{,\rho\sigma} + \frac{\alpha_c^2}{4}\bar{\lambda}h^{\alpha}{}_{\alpha}\Box h^{\beta}{}_{\beta}\Big]
\end{aligned}
\tag{4.43}
$$

Lastly, we notice that the whole second line can be transformed into only one term

$$
\frac{\alpha_c^2}{4}\bar{\lambda}\Big[h^{\alpha}{}_{\alpha}\Box h^{\beta}{}_{\beta} - 2h^{\mu\nu}{}_{,\mu\nu}\Box h^{\alpha}{}_{\alpha} + h^{\mu\nu}{}_{,\mu\nu}\Box^{-1}h^{\rho\sigma}{}_{,\rho\sigma}\Big] = \frac{\alpha_c^2}{4}\bar{\lambda}\bar{h}\Box\bar{h}\,. \tag{4.44}
$$

This term is already present in the action with coefficient $-1/2$, so we only add these two together to obtain $-1/4$. This was the last thing we needed to do since now we finally have identical result as compared with the article. Putting everything together, we obtain the action in the following form

$$
\begin{aligned}
A = &-\frac{1}{16}\int \mathrm{d}^4x\,h_{\mu\nu}\Box^2h^{\mu\nu} + \frac{1}{8}\int \mathrm{d}^4x\,\partial_\alpha h^{\alpha\mu}H_{\mu\nu}\,\partial_\beta h^{\beta\mu} + \Big(\frac{C^2}{12} - \frac{1}{16}\Big)\int \mathrm{d}^4x\,\bar{h}\Box^2\bar{h} - \\
&- \frac{1}{2\kappa^2}\int \mathrm{d}^4x\Big[-\alpha_c\bar{\lambda}\Box\bar{h} - \frac{\alpha_c^2}{4}\bar{\lambda}\bar{h}\Box\bar{h} - \frac{\alpha_c^2}{2}\bar{\lambda}\,\partial_\alpha h^{\alpha\mu}\Box^{-1}H_{\mu\nu}\,\partial_\beta h^{\beta\mu} + \frac{\alpha_c^2}{4}\bar{\lambda}h_{\alpha\beta}\Box h^{\alpha\beta}\Big] + \\
&+ \frac{3\alpha_c^2}{4S^2\kappa^4}\int \mathrm{d}^4x\,\bar{\lambda}^2\,.
\end{aligned}
\tag{4.45}
$$

We introduce notation as in [1] so that the lineaerized action can be put in a more compact form

$$\mathfrak{A} \equiv \frac{1}{16} + \frac{\alpha_c^2}{8\kappa^2}\bar{\lambda}\Box^{-1} \equiv -2\mathfrak{B}\,, \quad \mathfrak{C} = \Big(\frac{1}{16} - \frac{C^2}{12}\Big) - \frac{\alpha_c^2}{8\kappa^2}\bar{\lambda}\Box^{-1}\,. \tag{4.46}$$

With this notation it is possible to rewrite the action so that it is identical to that used by P. Jizba and his colleagues

$$A = \int \mathrm{d}^4x \Big[- h_{\mu\nu}\mathfrak{A}\Box^2 h^{\mu\nu} - \partial_\alpha h^{\alpha\mu}\mathfrak{B}H_{\mu\nu}\,\partial_\beta h^{\beta\mu} - \bar{h}\mathfrak{C}\Box^2\bar{h} + \frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Big]\,. \tag{4.47}$$

4.5 Quantization of Weyl Conformal Gravity

We formally define a quantum theory based on the Weyl gravity by

$$Z = \sum_i \int_{\Sigma_i} \mathcal{D}g_{\mu\nu}\, \mathrm{e}^{\mathrm{i}A}\,, \tag{4.48}$$

where Σ_i denote topologically distinct manifolds and where the measure $\mathcal{D}g_{\mu\nu}$ must be further treated by the Faddeev–Poppov method as the system has gauge symmetry Diff $\times$ Weyl(Σ_i). We must also include factors of $(-\det g_{\mu\nu}(x))^\omega$, where $\omega = -5/2$ in the Misner's convention or $\omega = (D-4)(D+1)/8$ in the De Witt's convention (D is the number of dimensions). There are also many other conventions, but here, for simplicity, we choose to drop this term (effectively we choose the De Witt's convention since we work in $D = 4$).

In our case there is no more integration over the scalar field λ as explained before and since we linearized the theory we substitute $\mathcal{D}g_{\mu\nu}$ by $\mathcal{D}h_{\mu\nu}$ which is now our only dynamical field. Hence, the partition function turns into

$$Z = \sum_i \int_{\Sigma_i} \mathcal{D}h_{\mu\nu}\, \mathcal{D}\lambda \delta[\chi - \zeta]\delta[\chi^\nu - \zeta^\nu] \det(\mathcal{M}_{\mathrm{FP}}) \det(\mathcal{N}_{\mathrm{FP}})\, \mathrm{e}^{\mathrm{i}A[h_{\mu\nu},\lambda]}\,. \tag{4.49}$$

The Faddeev–Poppov term for the coordinate gauge is known to be $(\mathcal{M}_{\mathrm{FP}})_{\mu\nu} = -\Box\eta_{\mu\nu} - \partial_\mu\,\partial_\nu$ [1] and $\mathcal{N}_{\mathrm{FP}} = (D-1)\delta^{(D)}(x-y)$ for the conformal gauge. As the operator $\delta(x-y)$ correspond to an infinite-dimensional unit operator it follows that the determinant $\det(\mathcal{N}_{\mathrm{FP}})$ is just a number, and as such will be included in the normalization. The functions $\delta[\cdot]$ play the role of constraints stemming from the gauge symmetry of the theory. We have already introduced the functions $\chi \equiv \bar{h}$ and $\chi^v \equiv \partial_\mu h^{\mu\nu}$ in the previous section. The constraint will be set by equating these to an arbitrary function ζ and ζ^ν, respectively – we will use this liberty of choice in a while to simplify all calculations.

The action that shall be used is given in (4.47). We use the property of exp to separate the terms of the action into a product of exponents, each of them playing a different role

$$\mathrm{e}^{\mathrm{i}A} = \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x h_{\mu\nu}\mathfrak{A}\Box^2 h^{\mu\nu}}\, \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x \chi^n \mathfrak{B}H_{\mu\nu}\chi^\mu}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \bar{h}\mathfrak{C}\Box^2\bar{h}}\, \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x \frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2}\,. \tag{4.50}$$

It is now possible to regard these as a product of integrals and thus calculate each term separately using the general theory of Gaussian integrals. For example we conclude that the first term yields

$$\int_{\Sigma_i} \mathcal{D}h_{\mu\nu}\, \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x h_{\mu\nu}\mathfrak{A}\Box^2 h^{\mu\nu}} = N \det\big(-\mathfrak{A}\Box^2\big)^{-1/2}\,. \tag{4.51}$$

Here, however, we must be careful about the dimension of the representation space of each operator, i.e. on how many fields it acts. Simply put, $h_{\mu\nu}$ represents 16 individual scalar fields cumulated into a tensor field by notation. Of course, not all of these fields are independent, as $h_{\mu\nu}$ is symmetric in its indices. Thus, we have 10 independent fields originating from $h_{\mu\nu}$, 4 from χ^ν and one from χ. This must be taken into account, hence we will denote the operators accordingly (e.g. $\mathfrak{A}\Box^2_{h_{\mu\nu}}$).

We must also deal with the constraints given by the delta functions. As will be seen shortly, these work to our advantage, due to the so-called t'Hooft averaging trick. The idea is as follows – since the function ζ (or ζ^ν) is arbitrary, why not average over a Gaussian distribution on the space of functions defined by an arbitrary symmetric operator O? In this way, the constraint turns into

$$\delta[\chi-\zeta] \to \sqrt{\det O}\int \mathcal{D}\zeta\, \mathrm{e}^{\mathrm{i}\zeta_x O_{xy}\zeta_y}\delta[\chi-\zeta] = \mathrm{e}^{\mathrm{i}\int \chi O \chi}(\det O)^{1/2}\,, \tag{4.52}$$

where the determinant arose from normalization. This trick gives us a way to cancel some of the exponential terms in Z by a convenient choice of O, which will create an exact counter-term. Since the constraints are on χ and χ^ν, we are able to cancel the two middle terms by choosing $O = \mathfrak{B}H_{\mu\nu}$ and $O = \mathfrak{C}\Box^2$ (they surely are symmetric), respectively.

After the cancellation of the two terms with the help of the t'Hooft trick and calculating the last Gaussian path integral, the partition function obtains the following form

$$\begin{aligned} Z &= N\det(\mathcal{M}_{\mathrm{FP}})\big(\det(\mathfrak{B}H_{\mu\nu})_{\chi^\nu}\big)^{\frac{1}{2}}\big(\det\big(\mathfrak{C}\Box^2\big)_{\bar{h}}\big)^{\frac{1}{2}}\sum_i\int_{\Sigma_i}\mathcal{D}h_{\mu\nu}\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x h_{\mu\nu}\mathfrak{A}\Box^2 h^{\mu\nu}}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\int \mathrm{d}^4x} = \\ &= \tilde{N}\det(\mathcal{M}_{\mathrm{FP}})\big(\det(\mathfrak{B}H_{\mu\nu})_{\chi^\nu}\big)^{1/2}\big(\det\big(\mathfrak{C}\Box^2\big)_{\bar{h}}\big)^{1/2}\Big[\det\Big(-\mathfrak{A}\Box^2_{h_{\mu\nu}}\Big)\Big]^{-1/2}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4}\,, \end{aligned} \tag{4.53}$$

where we have denoted Ω_4 the volume of the Minkowskian spacetime. Now we use the fact that $\det(AB) = \det A \det B$ and the discussion about the dimensions of the representation space of the operators (the constant N is unimportant and may change between two equality signs!), as well as the fact, that $\mathfrak{B} \sim \mathfrak{A}$ to write[2]

$$\begin{aligned} Z &= N(\det\mathcal{M}_{\mathrm{FP}})(\det\mathfrak{B}H_{\mu\nu})^{4/2}(\det\mathfrak{C}\Box^2)^{1/2}(\det -\mathfrak{A}\Box^2)^{-10/2}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4} = \\ &= N(\det\mathcal{M}_{\mathrm{FP}})(\det H_{\mu\nu})^2(\det\mathfrak{A})^{-5+2}(\det\mathfrak{C})^{1/2}(\det(-\Box))^{-10+1}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4}\,. \end{aligned} \tag{4.54}$$

Now we must simplify the determinant of $H_{\mu\nu}$ and of the Faddeev–Poppov operator $(\mathcal{M}_{\mathrm{FP}})_{\mu\nu} = -\Box\eta_{\mu\nu} - \partial_\mu\,\partial_\nu$. Let us start with the Faddeev–Poppov term. We use the properties of determinants again, which enables us to extract one d'Alambertian out of the determinant

$$\det(-\Box\delta^\mu{}_\nu - \partial^\mu\,\partial_\nu) = \det(-\Box)\det\big(\delta^\mu{}_\nu + \partial^\mu\Box^{-1}\,\partial_\nu\big)\,, \tag{4.55}$$

and the rest can be easily calculated with the help of the trace-log formula

[2]Here we are somewhat vague about the signs. We use the property of the determinant $\det(cA) = c^n\det A$ from finite dimension to forget all constant infinities. The question, however, arises, when we were to extract -1 this way from the determinant. Since this might be a very illegal operation, we will keep the signs in the determinants and, when situation arises, cancel them between themselves.

$$\det\big(\delta^\mu{}_\nu + \partial^\mu \Box^{-1}\,\partial_\nu\big) \equiv \exp \mathrm{Tr} \ln(1+D)^\mu_\nu = \exp \mathrm{Tr} \sum_k \frac{(-1)^{k+1}}{k} \big(D^k\big)^\mu_\nu = \\ = \exp \sum_k \frac{(-1)^{k+1}}{k} \big(D^k\big)^\mu_\mu = \exp \sum_k \frac{(-1)^{k+1}}{k}\,. \tag{4.56}$$

This is clearly a convergent sum, i.e. a finite number and as such will be included in the normalization and forgotten.

Let us now proceed by calculating the $(\det H_{\mu\nu})^2 \sim \det(H_{\mu\nu}H^{\nu\rho})$ in a very similar way

$$H_{\mu\nu}H^{\nu\rho} = \Big(\frac{1}{2}\,\partial_\mu\,\partial_\nu - \Box\eta_{\mu\nu}\Big)\Big(\frac{1}{2}\,\partial^\nu\,\partial^\rho - \Box\eta^{\nu\rho}\Big) = \frac{1}{4}\Box\,\partial_\mu\,\partial^\rho - \frac{2}{2}\Box\,\partial_\mu\,\partial^\rho + \Box^2\delta^\rho_\mu = \\ = \Box^2\big(\delta^\rho_\mu - \frac{3}{4}\,\partial_\mu\Box^{-1}\,\partial^\rho\big)\,, \tag{4.57}$$

thus again we can extract the square of thed'Alambertian out of the determinant and calculate the rest with the use of the trace-log formula

$$\det H_{\mu\nu}H^{\nu\rho} = \det\big(\Box^2\big) \exp \mathrm{Tr} \ln\bigg(1 - \frac{3}{4}D\bigg)^\mu_\nu = \det\big(\Box^2\big) \exp\left[\sum_k \frac{1}{k}\left(\frac{3}{4}\right)^k\right]. \tag{4.58}$$

We find that the sum converges again, hence will be forgotten as an unimportant number. Now that we have found the determinants, we must remember that Faddeev–Poppov operator carries two Lorentzian indices (i.e. it acts on a Lorentzian vectors) which has to be taken into account by raising the power of the determinant to 4. All in all, we obtain

$$(\det \mathcal{M}_{\mathrm{FP}})(\det H_{\mu\nu})^2 \sim \big(\det(-\Box)\big)^4 \det\big(\Box^2\big) = (\det \Box)^6\,. \tag{4.59}$$

We can now substitute this result into the expression for the partition function and see, that there are now only two non-trivial determinants left

$$Z = N(\det \mathfrak{A})^{-3}(\det \mathfrak{C})^{1/2}(\det(-\Box))^{-3}\, \mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4}\,. \tag{4.60}$$

Here we recall we have already discussed the number of degrees of freedom in the linearized theory. The fact that the free linearized theory contains six propagating degrees of freedom can be seen also here, after rewriting $(\det(-\Box))^{-3} = \big[(\det(-\Box))^{-1/2}\big]^6$. Traditionally, each term $(\det(-\Box))^{-1/2}$ corresponds to one of the degrees of freedom, hence we see there are six of them.

This argument might seem vague, as there are inverse d'Alambertians present in $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ (see (4.46)). From the definition of these operators we see, however, that each $\Box^{-1}$ is multiplied by α_c. This parameter will however be vanishingly small (or even zero) in the case of the linearized theory, therefore the operators $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ contribute in first approximation only by a constant.

The last thing left for us to do is to compute the determinants of the operators $\mathfrak{A}$, $\mathfrak{C}$. We do so in the following section.

4.6 Functional Determinants, The One-Loop Effective Potential

The goal now is to find out, whether the one-loop effective potential obtained from the partition function (4.60) coincides with the one obtained by the authors of [1]. In their article, they used dimensional regularization to calculate the remaining two determinants ($\det\mathfrak{A}$ and $\det\mathfrak{C}$), which is (as was already argued) inappropriate in this context, since we rely on the Gauss–Bonnet theorem valid only in a fixed dimension. To confirm and improve the validity of the results we employ in this thesis the zeta function regularization.

The calculation of determinants of $\mathfrak{A}$ and $\mathfrak{C}$ should be quite easy since they are analogous to what we have already computed in Section 3.4. Even though the calculations were then performed in the Euclidean regime, we have already emphasised the fact, that relevant results for the trace of the heat kernel and thus also for $\zeta'(0)$ are independent of coordinates, hence can be immediately used also in our calculations.

In Section 3.4 we calculated the determinant of the operator $\Box + m^2 + \frac{g}{2}\phi$, where $m^2 + \frac{g}{2}\phi$ is simply a constant. This operator has clearly the same structure as the ones we are trying to calculate. We begin with the operator $\mathfrak{A}$

$$\mathfrak{A} = \frac{1}{16} + \frac{\alpha_c^2\bar{\lambda}}{8\kappa^2}\Box^{-1} = \frac{1}{16}\Box^{-1}\left(\Box + \frac{2\alpha_c^2\bar{\lambda}}{\kappa^2}\right), \tag{4.61}$$

where the multiplicative constant is unimportant and with the help of the property of the determinant, we are now interested solely in

$$\det(\Box + A)\,, \quad \text{where} \quad A \equiv \frac{2\alpha_c^2\bar{\lambda}}{\kappa^2}\,. \tag{4.62}$$

Inspecting the physical dimension of the operator $\Box + A$, we find that both the d'Alambertian and the constant carry a dimension L^{-2}. We must therefore introduce a dimensionfull constant (also called regulator or regularization mass scale) μ, $[\mu] = \mathrm{L}$ and compute the determinant of $(\Box + A)/\mu^2$. We can now basically copy the solution from the ϕ^4 case, where the ζ function was equal to

$$\zeta_{(\Box+A)/\mu^2}(s) = \left(\frac{1}{s-2} - \frac{1}{s-1}\right)\frac{\mu^4}{16\pi^2}\left(\frac{\mu^2}{A}\right)^{s-2}\Omega_4\,, \tag{4.63}$$

which yields $\zeta'(0)$ after trivial differentiation as

$$\zeta'(0) = \frac{\mu^4}{32\pi^2}\Omega_4\left(\frac{2\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2}\right)^2\left[\frac{3}{2} - \ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2}\right)\right]. \tag{4.64}$$

From here we find the determinant to be equal to by definition (3.11)

$$\begin{aligned}\det\left(\frac{\Box}{\mu^2} + \frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right) &= \exp\left\{-\frac{\mu^4}{32\pi^2}\Omega_4\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right)^2\left[\frac{3}{2} - \ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right)\right]\right\}\\ &= \exp\left\{\frac{\alpha_c^4\bar{\lambda}^2}{8\pi^2\kappa^4}\Omega_4\left[\ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right) - \frac{3}{2}\right]\right\}.\end{aligned} \tag{4.65}$$

We proceed similarly also in the case of the operator $\mathfrak{C}$, where we use the fact, that

$$\left(\frac{1}{16}-\frac{C^2}{12}\right)=\frac{1}{4}\frac{3-4C^2}{12}=-\frac{4S^2+1}{48}\,, \tag{4.66}$$

and therefore the operator can be written as

$$\mathfrak{C}=-\frac{1}{48}\big(4S^2+1\big)\Box^{-1}\big(\Box+\tilde{C}\big)\,,\quad \text{where}\quad \tilde{C}\equiv\frac{6\alpha_c^2\bar{\lambda}}{\kappa^2(4S^2+1)}\,. \tag{4.67}$$

Since the functional form is identical to the previous operator, we trivially find the determinant to be equal to

$$\begin{aligned}\det\left(\frac{\Box}{\mu^2}+\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)&=\exp\left\{-\frac{\mu^4}{32\pi^2}\Omega_4\left(\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)^2\left[\frac{3}{2}-\ln\left(\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)\right]\right\}\\&=\exp\left\{\frac{9\alpha_c^4\bar{\lambda}^2}{8\pi^2\kappa^4(4S^2+1)}\Omega_4\left[\ln\left(\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)-\frac{3}{2}\right]\right\}.\end{aligned} \tag{4.68}$$

The two results (4.65) and (4.68), as well as the extra terms $\det\big(\Box^{-1}\big)$ will now help us in rewriting the partition function into a neat expression

$$\begin{aligned}Z&=N(\det\mathfrak{A})^{-3}(\det\mathfrak{C})^{1/2}(\det(-\Box))^{-3}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4}\\&\sim\Big[\det\Box^{-1}\det(\Box+A)\Big]^{-3}\Big[\det\big(-\Box^{-1}\big)\det(\Box+C)\Big]^{1/2}(\det(-\Box))^{-3}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4}\\&\sim\big(\det-\Box\big)^{+3-1/2-3}\,\mathrm{e}^{-\mathrm{i}\frac{3\alpha_c^4\bar{\lambda}^2}{8\pi^2\kappa^4}\Omega_4\left[\ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right)-\frac{3}{2}\right]}\,\mathrm{e}^{\mathrm{i}\frac{9\alpha_c^4\bar{\lambda}^2}{16\pi^2\kappa^4(4S^2+1)}\Omega_4\left[\ln\left(\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)-\frac{3}{2}\right]}\,\mathrm{e}^{\mathrm{i}\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\Omega_4}\\&=N\big(\det-\Box\big)^{-1/2}\,\mathrm{e}^{-\mathrm{i}\Omega_4\left\{\frac{3\alpha_c^4\bar{\lambda}^2}{8\pi^2\kappa^4}\left[\ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right)-\frac{3}{2}\right]-\frac{9\alpha_c^4\bar{\lambda}^2}{16\pi^2\kappa^4(4S^2+1)}\left[\ln\left(\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)-\frac{3}{2}\right]-\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\right\}}\,.\end{aligned} \tag{4.69}$$

We are now at the end of our calculations. We have found the partition function in such a convenient form, that the effective potential can be readily found from the definition $Z=\mathrm{e}^{-\mathrm{i}V_{\text{eff}}\Omega_4}$ to be equal to

$$V_{\text{eff}}=\frac{3\alpha_c^4\bar{\lambda}^2}{8\pi^2\kappa^4}\left[\ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right)-\frac{3}{2}\right]-\frac{9\alpha_c^4\bar{\lambda}^2}{16\pi^2\kappa^4(4S^2+1)}\left[\ln\left(\frac{6\alpha_c^2\bar{\lambda}}{\mu^2\kappa^2(4S^2+1)}\right)-\frac{3}{2}\right]-\frac{3\alpha_c^2}{4S^2\kappa^4}\bar{\lambda}^2\,. \tag{4.70}$$

This expression exactly duplicates the result obtained by P. Jizba, H. Kleinert and F. Scardigli, even though we have used different (and in gravity non-equivalent) regularization method. The last determinant $\big(\det-\Box\big)^{-1/2}$ can be thought of as a normalization. Usually, we would normalize by a determinant of the inverse propagator of free field theory. Since all fields present in our theory are by definition massless, the inverse propagator is simply $-\Box$. Hence, if we normalize the free theory to 1, we can simply drop this term.

4.7 The Emergence of Scale

Since we arrived at the same form of the effective potential as is in the article [1], we can proceed similarly. The next logical step is to find its minima of the potential w.r.t. $\bar{\lambda}$ which correspond to stable static field configurations of the field λ.

A calculation of the stationary configuration $\bar{\lambda}$ can be broken into several steps. First we obtain, after some algebra, that

$$\frac{\partial V_{\text{eff}}}{\partial \bar{\lambda}} = \frac{3\alpha_c^2 \bar{\lambda}}{8\pi^2 \kappa^2 S^2 \left(4S^2+1\right)^2} \Bigg[2\alpha_c^2 \left(4S^2+1\right)^2 S^2 \ln\left(\frac{2\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2}\right) - 3\alpha_c^2 S^2 \ln\left(\frac{6\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2 \left(4S^2+1\right)}\right) - \\ - 4\pi^2 \big(1+8S^2+16S^4\big) + \alpha_c^2 S^2 \big(1-16S^2-32S^4\big) \Bigg] , \tag{4.71}$$

which we put equal to zero to find the minimal value of the potential. One immediate solution is $\bar{\lambda} = 0$. To find other solutions, we concentrate on solving the equation

$$\begin{aligned} &2\alpha_c^2 \left(4S^2+1\right)^2 S^2 \ln\left(\frac{2\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2}\right) - 3\alpha_c^2 S^2 \ln\left(\frac{6\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2 \left(4S^2+1\right)}\right) = \\ &= 4\pi^2 \big(1+8S^2+16S^4\big) + \alpha_c^2 S^2 \big(-1+16S^2+32S^4\big) . \end{aligned} \tag{4.72}$$

We rewrite the left hand side of the equation as

$$\left[2\alpha_c^2 \left(4S^2+1\right)^2 S^2 - 3\alpha_c^2 S^2\right] \ln\left(\frac{2\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2}\right) - 3\alpha_c^2 S^2 \ln\left(\frac{3}{4S^2+1}\right) , \tag{4.73}$$

from where we separate the term with $\bar{\lambda}$

$$\ln\left(\frac{2\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2}\right) = \frac{3\alpha_c^2 S^2 \ln\left(\frac{3}{4S^2+1}\right) + 4\pi^2 \big(4S^2+1\big)^2 + \alpha_c^2 S^2 \big(32S^4+16S^2-1\big)}{\alpha_c^2 S^2 (32S^4+16S^2+2-3)} . \tag{4.74}$$

It is now easy to see the solution for $\bar{\lambda}$

$$\bar{\lambda}(S) = \frac{\mathrm{e}\kappa^2 \mu^2}{2\alpha_c^2} \exp\left\{ \frac{3\alpha_c^2 S^2 \ln\left(\frac{3}{4S^2+1}\right) + 4\pi^2 \big(4S^2+1\big)^2}{\alpha_c^2 S^2 (32S^4+16S^2-1)} \right\} . \tag{4.75}$$

Once we obtained have all solutions, we are interested in the values of the effective potential at the points of the minima. It is again easy to see, that for $\bar{\lambda} = 0$ the effective potential is $V_{\text{eff}} = 0$. For the non-trivial solutions, we find that for $S^2 > (\sqrt{6}-2)/8$ the potential $V_{\text{eff}} < 0$ for all α_c and κ. This can be seen, when we substitute the extremal point into the effective potential and rearrange the terms as follows

$$\frac{16\pi^2\kappa^4}{\alpha_c^4 \bar{\lambda}^2} V_{\text{eff}} = 6\left[\ln\left(\frac{2\alpha_c^2 \bar{\lambda}}{\kappa^2 \mu^2}\right) - \frac{3}{2}\right] - \frac{9}{(4S^2+1)^2}\left[\ln\left(\frac{6\alpha_c^2 \bar{\lambda}}{\mu^2 \kappa^2 (4S^2+1)}\right) - \frac{3}{2}\right] - \frac{12\pi^2}{\alpha_c^2 S^2} . \tag{4.76}$$

We turn our attention to the non-trivial terms. Clearly, the last term is always positive, therefore we do not have to consider it in our immediate discussion. We will reorganize the remaining terms in the following way

$$-\frac{3}{2}\left(6-\frac{9}{(4S^2+1)^2}\right)-\frac{9}{(4S^2+1)^2}\ln\left(\frac{3}{4S^2+1}\right)+\ln\left(\frac{2\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2}\right)\left(6-\frac{9}{4S^2+1}\right), \tag{4.77}$$

with the first bracket being equal to $3\big(32S^4+16S^2-1\big)/(4S^2+1)^2$ where the numerator can be rewritten as $32S^4+16S^2-1=2(4S^2+1)^2-3$. We denote the first term in (4.77) as $-3K/2$, K representing the first bracket.

The remaining terms include logarithms and so we are forces to substitute for $\bar{\lambda}$ the point of the minimal value of the potential, hoping the terms will cancel out

$$\begin{aligned}
&-\frac{9}{(4S^2+1)^2}\ln\left(\frac{3}{4S^2+1}\right)+\left(6-\frac{9}{(4S^2+1)^2}\right)\left[1+\frac{3\alpha_c^2S^2\ln\left(\frac{3}{4S^2+1}\right)+4\pi^2\big(1+4S^2\big)^2}{\alpha_c^2S^2(32S^4+16S^2-1)}\right]\\
&-\frac{9}{(4S^2+1)^2}\ln\left(\frac{3}{4S^2+1}\right)+3\frac{32S^4+16S^2-1}{(4S^2+1)^2}\left[1+\frac{3\alpha_c^2S^2\ln\left(\frac{3}{4S^2+1}\right)+4\pi^2\big(1+4S^2\big)^2}{\alpha_c^2S^2(32S^4+16S^2-1)}\right].
\end{aligned} \tag{4.78}$$

Here the term in front of ln e = 1 is equal to K, and therefore we just add them together – $-3/2+1=-1/2$. The remaining terms read

$$-\frac{9}{(4S^2+1)^2}\ln\left(\frac{3}{4S^2+1}\right)+\frac{9}{(4S^2+1)^2}\ln\left(\frac{3}{4S^2+1}\right)+\frac{12\pi^2}{\alpha_c^2S^2}, \tag{4.79}$$

which means that the logarithm indeed cancel out and we are left with a term, which will cancel the uninteresting term we left already at the beginning of the discussion. Hence, the only contribution to the effective potential at the point of the minima comes from $-K/2$

$$\frac{16\pi^2\kappa^4}{\alpha_c^4\bar{\lambda}^2}V_{\text{eff}}=-\frac{32S^4+16S^2-1}{2(4S^2+1)}\Longleftrightarrow V_{\text{eff}}=-\frac{\alpha_c^4\bar{\lambda}^2}{32\pi^2\kappa^4}\frac{32S^4+16S^2-1}{(4S^2+1)}. \tag{4.80}$$

It is now obvious, that whenever $32S^4+16S^2-1>0$, the effective potential is negative irrespective of the values of α_c or κ. The solutions to the inequality is, as we have already written $S^2>(\sqrt{6}-2)/8\doteq 0.056$.In this range of S^2 the solution $\bar{\lambda}=0$ corresponds to a local maximum, for at that point the effective potential reaches its highest value $V_{\text{eff}}=0$.

We shall now ensure that the value of $\bar{\lambda}$ no longer depends on the mixing angle θ introduced by $\sinh\theta$ since we have started with a theory that was independent of this parameter. So, although, the full theory (i.e. theory valid to all orders of the perturbative calculus) should be θ independent, this is generally not the case at any particular finite order. To deal with this situation, one can invoke the principle of minimal sensitivity [14] known from the renormalization-group calculus. The point of the principle is that, whenever a theory depends on an unphysical parameter (θ or S in our case), we must ensure that at every ordered the physically relevant quantities depend on the parameter in the weakest possible way. To fulfil this requirement we find the minima of the effective potential w.r.t. the unphysical parameter, in our case represented by S

$$0=\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}S^2}=\frac{\partial V_{\text{eff}}}{\partial S^2}+\frac{\partial V_{\text{eff}}}{\partial\bar{\lambda}}\frac{\partial\bar{\lambda}}{\partial S^2}=\frac{\partial V_{\text{eff}}}{\partial S^2}, \tag{4.81}$$

at the point of the minima of the effective potential w.r.t. $\bar{\lambda}$, hence $\frac{\partial V_{\text{eff}}}{\partial\bar{\lambda}}=0$. We find the derivative to be equal to

$$\frac{\partial V_{\text{eff}}}{\partial S^2} = \frac{9\alpha_c^4\bar{\lambda}^2}{4\pi^2\kappa^4(4S^2+1)^3} + \frac{3\alpha_c^2\bar{\lambda}^2}{4\kappa^4 S^4} + \frac{9\alpha_c^4\bar{\lambda}^2\left[\log\left(\frac{6\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2(4S^2+1)}\right) - \frac{3}{2}\right]}{2\pi^2\kappa^4(4S^2+1)^3}\,. \tag{4.82}$$

The minimal-sensitivity principle leads to following chain of identities

$$\begin{aligned} 0 &= 3\alpha_c^2 + \pi^2 S^{-4}(4S^2+1)^3 + 6\alpha_c^2\left[\ln\left(\frac{6\alpha_c^2\bar{\lambda}}{\kappa^2\mu^2(4S^2+1)}\right) - \frac{3}{2}\right] \\ &= \pi^2 S^{-4}(4S^2+1)^3 + 6\alpha_c^2\ln\left(\frac{3}{4S^2+1}\right) + \frac{18\alpha_c^2 S^2\ln\left(\frac{3}{4S^2+1}\right) + 24\pi^2\left(1+4S^2\right)^2}{S^2(32S^4+16S^2-1)}\,, \end{aligned} \tag{4.83}$$

where we used the solution (4.75) for $\bar{\lambda}$. Multiplying by the denominator from the last term we find that the coefficient in front of $\ln\left(\frac{3}{4S^2+1}\right)$ is equal to

$$6\alpha_c^2 S^2\left[(32S^4+16S^2-1)+3\right] = 12\alpha_c^2 S^2\left(4S^2+1\right)^2. \tag{4.84}$$

Hence the equation goes to

$$\frac{12\alpha_c^2 S^2}{\pi^2}\left(4S^2+1\right)^2\ln\left(\frac{4S^2+1}{3}\right) = (4S^2+1)^3 S^{-2}(32S^4+16S^2-1) + 24\left(1+4S^2\right)^2, \tag{4.85}$$

and since $\left(4S^2+1\right)^2 > 0$ we can safely divide by this term to obtain

$$\begin{aligned} \frac{12\alpha_c^2 S^4}{\pi^2}\ln\left(\frac{4S^2+1}{3}\right) &= (4S^2+1)(32S^4+16S^2-1) + 24S^2 \\ \frac{12\alpha_c^2 S^4}{\pi^2}\ln\left(\frac{4S^2+1}{3}\right) &= 128S^6 + 96S^4 + 36S^2 - 1\,. \end{aligned} \tag{4.86}$$

This equation has only one real solution [1] $S^2 = 0.0259237 - 0.0000197\alpha_c^2 + O(\alpha_c^4)$, which does not fall in the region of negative potential and therefore does not correspond to a stable solution. However, looking at the formula for $\partial V_{\text{eff}}/\partial S^2$ in (4.76), we immediately see it tends to go to zero in the limit $S \to \infty$. Since we are searching for the point, where the choice of θ (i.e. S) influences the effective potential in the least possible way, any large value of S^2 would obviously be our next best choice. Here, however, we must be more careful. The whole theory was build on the assumption, that α_c is a small constant, at most equal to 1 in order for the linearization $g_{\mu\nu} = \eta_{\mu\nu} + \alpha_c h_{\mu\nu}$ to work. Later, we recognized by comparison with the Starobinsky models in (4.25), that $C^2/\alpha_c^2 \sim \xi^2/\kappa^2$, hence for large values of C we can use $C \sim S \sim \alpha_c\xi/\kappa$ which is at most ξ/κ. Thus the largest value we may use for S, in order to for all the calculations hold true, is $\xi/\kappa \sim 10^5$ as currently estimated by [9] (as was already mentioned).

It is important to examine the behaviour of the solution (4.75) for the minimum of the effective potential in the limit as $S \to \infty$ or better at the point $S^2 = \xi^2/\kappa^2$. Due to the magnitude of S, we expand the exponent to the order $O(1/S^4)$ and drop the rest. In more detail, the exponent in (4.75) reads

$$\frac{3\alpha_c^2 S^2\ln\left(\frac{3}{4S^2+1}\right) + 4\pi^2\left(1+4S^2\right)^2}{\alpha_c^2 S^2(32S^4+16S^2-1)} = \frac{3\ln\left(\frac{3}{4S^2+1}\right)}{32S^4+16S^2-1} + \frac{4\pi^2(4S^2+1)^2}{\alpha_c^2 S^2(32S^4+16S^2-1)}\,. \tag{4.87}$$

It is clear from the Taylor series of $\ln(x) = (x-1) - (x-1)^2/2 + \ldots$, that the first term does not contribute to order $O(1/S^4)$. To find how contributes the second term, we divide the two polynomials

$$(32S^6 + 16S^2 - S^2) : (16S^4 + 8S^2 + 1) = 2S^2 - \frac{3S^2}{16S^4 + 8S^2 + 1} = 2S^2\Big(1 - \frac{3/2}{16S^4 + 8S^2 + 1}\Big)\,. \tag{4.88}$$

From here it is easy to see, that the inverse of this ratio can be expanded into a Taylor series

$$\begin{aligned}\frac{(4S^2+1)^2}{S^2(32S^4+16S^2-1)} &= \frac{1}{2S^2\Big(1 - \frac{3/2}{16S^4+8S^2+1}\Big)} \simeq \frac{1}{2S^2}\Big(1 + \frac{3/2}{16S^4 + 8S^2 + 1} + \ldots\Big) = \\ &= \frac{1}{2S^2} + O(1/S^4)\,,\end{aligned} \tag{4.89}$$

from where we conclude, that the solution for $\bar{\lambda}$ (see (4.75)) at the order $O(1/S^4)$ is equal to

$$\bar{\lambda} = \frac{\kappa^2\mu^2}{2\alpha_c^2}\exp\Big(1 + \frac{2\pi^2}{\alpha_c^2 S^2}\Big) \sim \frac{\kappa^2\mu^2}{2\alpha_c^2}\exp\Big(1 + \frac{2\pi^2\kappa^2}{\alpha_c^2\xi^2}\Big)\,. \tag{4.90}$$

This is an important result since it shows, that for any (relvant i.e. smaller than 1) initial value of the parameter of the theory α_c we are able to choose the renormalization mass scale μ so, that $\bar{\lambda} = 1$. We will further discuss the importance of this result in the following section.

4.8 Physical Interpretation and Discussion

Let us now summarize what we have achieved so far. We have shown that there exists a set of parameters of our theory which allows for the appearance of the Starobinsky gravity in the low-energy sector of the broken Weyl-gravity's phase. We recall that the action we used was

$$A_{\text{conf}} = \int \mathrm{d}^4x\,\sqrt{-g}\Big[-\frac{1}{4\alpha_c^2}R_{\mu\nu}R^{\mu\nu} + \frac{C^2}{12\alpha_c^2}R^2 + \frac{3\alpha_c^2}{4S^2\kappa^4}\,\lambda^2 - \frac{1}{2\kappa^2}R\lambda\Big]\,. \tag{4.91}$$

compared to the Starobinsky model

$$A_{\text{St}} = \int \mathrm{d}^4x\,\sqrt{-g}\Big(\frac{\xi^2}{2\kappa^2}R^2 - \frac{R}{2\kappa^2}\Big)\,. \tag{4.92}$$

The fact that it is possible to fit the vacuum expectation value of the Hubbard–Stratonovich field $\bar{\lambda} = 1$, means the theory easily transforms to the low curvature limit of Einstein's gravity. The factor in front of R^2 can also be fitted in a wide range of $\alpha_c \sim 1 \div 10^{-5}$ to $\xi^2/2\kappa^2$, thus obtaining Starobinsky model relevant in description of the inflationary era.

The scalar field λ deserves more detailed discussion. Firstly, λ was not present in the theory before. Someone might object, that the field λ is non-physical since it is not dynamical for it has no kinetic term in the action (it represents a scalar field with infinite mass). This problem can be resolved by the conformal symmetry of the theory we have started with. To explain the solution of the kinematics of λ, we recall, that λ arose from the transformation of the R^2 term in the action

$$\mathrm{e}^{\mathrm{i}A_{R^2}} = \exp\Big(\frac{\mathrm{i}}{12\alpha_c^2}\int \mathrm{d}^4x\,\sqrt{-g}R^2\Big) = \int \mathcal{D}\lambda \exp\Big[-\mathrm{i}\int \mathrm{d}^4x\,\sqrt{-g}\Big(\frac{3\alpha_c^2}{4}\lambda^2 + \frac{R}{2}\lambda\Big)\Big]\,. \quad (4.93)$$

It is clear, that the action in this form possesses no kinetic term for λ. However, the R^2 term of the action is conformally invariant by itself (under additional conditions). This can be seen from the transformation law for R under an infinitesimal conformal change $g_{\mu\nu} \to (1+\alpha(x))g_{\mu\nu}$. The Ricci scalar transforms as $R \to R(1-\alpha(x)) - 3\,\nabla^2\alpha(x)$ [15], where ∇_μ is the covariant derivative, hence under additional restriction $\nabla^2\alpha(x) = 0$, the term $R^2 \to R^2(1-2\alpha(x)+\ldots)$ exactly compensates with the transformation of the determinant $g \to (1+4\alpha(x))g$ and the R^2 term is global scale invariant.

We might now use this property to rescale the whole action $g_{\mu\nu} \to g_{\mu\nu}/|\lambda|$ which will generate a kinetic term [1] in the global scale invariant part of the action

$$-\int \mathrm{d}^4x\,\sqrt{-g}\Big(\frac{3\alpha_c^2}{4}\lambda^2 + \frac{R}{2}\lambda\Big) \longrightarrow \int \mathrm{d}^4x\,\sqrt{-g}\Big(-\frac{\lambda R}{2|\lambda|} + \frac{3}{4\lambda^2}\,\partial_\mu\lambda\,\partial^\mu\lambda - \frac{3\alpha_c^2}{4}\lambda^2\Big)\,, \quad (4.94)$$

The fact that the kinematics of λ is gauge dependent points to the fact, that λ is not a physical field at the time of introduction by the HS transformation. However, if we assume, that it obtains a kinetic term before the breakdown of the conformal symmetry, we would be left with a dynamical and physical scalar field.

We have found, that the effective potential has minima (vacuum) at the point $\bar\lambda = 1$ even though the classical theory did not. Since the new field $\bar\lambda$ is related to μ, as seen from

$$\bar\lambda = \frac{\kappa^2\mu^2}{2\alpha_c^2}\exp\left(1 + \frac{2\pi^2\kappa^2}{\alpha_c^2\xi^2}\right), \quad (4.95)$$

we see conclude, that it is a *dimensionally transmuted* parameter. This means that it depends on the renormalization mass scale even though it is a physically relevant quantity. Since this stable solution $\bar\lambda$ stems only from the fact that the effective potential has a new minimum, the symmetry breakdown is dynamical. This situation is similar to what happens in the Coleman–Weinberg mechanism, where a dimensionless coupling constant transmutes into a dimensionfull one due to one-loop quantum corrections [7].

We should also comment on the general properties of the other two terms present in the Weyl action, which are not in the Starobinsky action. Let us focus our attention to the $R_{\mu\nu}R^{\mu\nu}$ term. If we assume that the cosmologically relevant metric after the symmetry breakdown is the FLRW metric, we are able to make use of the conformal flatness of this metric to rewrite

$$\int \mathrm{d}^4x\,\sqrt{-g}\,3R_{\mu\nu}R^{\mu\nu} = \int \mathrm{d}^4x\,\sqrt{-g}\,R^2\,, \quad (4.96)$$

modulo topological term which is of no importance for us. If we use this identity and the solution for $\bar\lambda$ from (4.95) in (4.91), we obtain the action corresponding to the Weyl theory of gravity after the symmetry breakdown

$$\begin{aligned} A &= \int \mathrm{d}^4x\,\sqrt{-g}\Big[-\frac{1}{12\alpha_c^2}R^2 + \frac{C^2}{12\alpha_c^2}R^2 + \frac{3\alpha_c^2}{4S^2\kappa^4}\,\bar\lambda^2 - \frac{1}{2\kappa^2}R\bar\lambda\Big] \\ A &= \int \mathrm{d}^4x\,\sqrt{-g}\Big[\frac{S^2}{12\alpha_c^2}R^2 + \frac{3\alpha_c^2}{4S^2\kappa^4} - \frac{1}{2\kappa^2}R\Big]\,, \end{aligned} \quad (4.97)$$

which goes to

$$A = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\Big(R - \xi^2 R^2 - 2\Lambda\Big) \quad \text{where} \quad \xi^2 = \frac{S^2\kappa^2}{6\alpha_c^2}\,, \quad \Lambda = \frac{3\alpha_c^2}{4S^2\kappa^2}\,. \tag{4.98}$$

This action corresponds to a Starobinsky model with a cosmological term. It is worth noting, that our cosmological term is of geometric origin (hence it might be called *gravi-cosmological*) and has an opposite sing to what we are used to from a matter-induced cosmological term. This is not a problem, since the action (4.98) descries an empty space. Soon after the breakdown of the symmetry, matter appears and with it also the matter-induced cosmological constant, which corresponds to $\langle 0|\mathrm{tr}\,T_{\mu\nu}|0\rangle$ and has opposite sign to that of the gravi-cosmological one. This might be of great value since it gives us a way to compensate for the matter-induced term and obtain a very small overall cosmological term. It would conveniently address the problem of 120 orders of difference in the theoretically predicted value of the cosmological term from estimates based on observations.

Chapter 5

Conclusions

This work naturally splits into two parts; in the first part we presented some prerequisite theoretical material (functional integrals, effective action, zeta-function regularization) that will be needed in the second, i.e., core part of the thesis. This part of the thesis was based on Refs. [3, 5, 4, 2] and the principal aim was to give a full, self-contained and mathematically sound summary of the relevant mathematical techniques.

The second part of the Thesis discusses the Weyl (or conformal) gravity both from classical and quantum-theory point of view. We first put forward phenomenological reasons why the quantized Weyl gravity qualifies as a good candidate for the bona fide quantum gravity. Secondly, we discuss some algebraic and topological properties of the theory and ensuing simplifications which they inflict on the action functional. In the following sections we closely followed the paper of P. Jizba, H. Kleinert and F. Scardigli [1] with the explicit goal to reproduce (or refute) results obtained by the authors when new, physically more relevant regulating scheme is employed. In particular, our focus was on the zeta-renormalization scheme (fixed-dimension renormalization) rather than dimensional regularization. We have found that the zeta-function regularization leads to exactly the same form of the one-loop effective potential as found in [1], which is the key result of This thesis.

Our following discussion focused on proving that the quantized Weyl gravity dynamically breaks the scale symmetry via dimensional transmutation, yielding a fundamental scalar field — Hubbard–Stratonovich field. Non-zero vacuum expectation value of this field can be chosen so that in the low-energy broken phase regime the Weyl gravity morphs into a Starobinsky gravity with a gravi-cosmological constant. In view of recent PLANCK and BICEP II data which favour Starobinsky model of inflation, this is important and relevant conclusion since it shows that the Weyl gravity is a good and viable candidate for quantum gravity.

Bibliography

[1] P. Jizba, H. Kleinert and F. Scardigli, Eur. Phys. J. C **75**, 245 (2015).

[2] A. Das, *Field Theory - A Path integral Approach 2nd Ed.* (World Scientific Publishing, Singapore, 2006) .

[3] P. Rammond, *Field Theory: A Modern Primer 2nd Ed.* (Avalon Publishing, New York, 1997).

[4] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena 4th Ed.* (Claredon Press, Oxford, 2002).

[5] P. Cvitanović, *Field Theory - Lecture notes prepared by Ejnar Gyldenrekne* [http://chaosbook.org/FieldTheory/]

[6] Osterwalder, Konrad; Schrader, Robert. Axioms for Euclidean Green's functions. Comm. Math. Phys. 31 (1973), no. 2, 83–112.

[7] S. Coleman and E. Weinberg, Phys. Rev. D **8**, 1888 (1973).

[8] E. Elizalde, *Ten Physical Applications if Spectral Zeta Functions 2nd Ed.* (Springer, Heidelberg, 2012).

[9] Planck Collaboration, Astron. Astrophys. **571**, A22 (2014).

[10] Planck Collaboration, Astron. Astrophys. **594**, A20 (2016).

[11] I. Antoniadis, N.C. Tsamis, SLAC Publication, SLAC-PUB-3297 (1984), Phys. Lett. **144B**, 55 (1984).

[12] R.J. Riegert, Phys. Lett. A **105**, 105 (1984).

[13] E.S. Fradkin, A.A. Tseytlin, Phys. Rep. **119**, 233 (1985).

[14] P.M. Stevenson, Phys. Rev. D **23**, 2916 (1981).

[15] P. Urban, *Physical Aspects of Conformal Gravity*, Research Project, on request by email urbanpat@fjfi.cvut.cz