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Diploma thesis

Metamaterials: Spectral-theoretic approach

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Supervisor: Mgr. David Krejčiřík, PhD., DSc.

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Diplomová práce

Metamateriály: spektrálně-teoretický přístup

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Title:
Metamaterials: Spectral-theoretic approach

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Specialization: Mathematical physics

Sort of project: Diploma thesis

Supervisor: Mgr. David Krejčiřík, Ph.D., DSc.

Abstract: This thesis is focused on metamaterial with simultaneously negative permittivity and permeability and their application in cloaking devices. There is made a historical review about inventing such substances both in practical and theoretical way. We summarize the main properties and applications for the metamaterials and then focus just on the cloaking. Some mathematical treatments to this invisibility effect are mentioned. One of them is concept of so called anomalous localised resonance described in a chapter devoted only this approach. Inspired by it we confirm that cloaking due to anomalous localised resonance does not occur for the three dimensional ball and extend this result for higher dimensions. Using operator theory we introduce an indefinite laplacian on a rectangle and in the symmetric radial geometry and prove that both these operators are essentially self-adjoint. This is done in two dimensional space, we provide the same result for higher dimensions too. We discovere that zero lies in the essential spectra of these operators. This means that there is an inverse operator but it does not exist for all functions on the right side of Poisson equation.

Key words: metamaterials, superlenses, cloaking, invisibility, anomalous localised resonance, localization index, essentially self-adjoint operators, Bessel functions, spectral theory

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Autor: Bc. Filip Hložek

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Abstrakt: Tato práce je zaměřena na metamateriály se současně zápornou permitivitou a permeabilitou a jejich aplikace v maskovacích zařízeních. Je zde dán historický přehled výzkumu těchto materiálů jak v teoretické tak praktické rovině. Shrneme zde hlavní vlastnosti a aplikace metamateriálů, z nichž se zaměříme hlavně na maskování. Zmíníme několik přístupů k tomuto jevu. Jedním z nich je koncept tzv. anomálně lokalizované rezonance, kterému je věnována celá jedna kapitola. Sledujíc tento koncept dokážeme, že k maskování pomocí anomální lokalizované rezonance nedochází na třírozměrné kouli, a rozšíříme tento výsledek do vyšších dimenzí. S použitím operátorové teorie zavedeme indefinitní laplacián na obdélníku a v symetrické radiální geometrii. Dokážeme, že oba tyto operátory jsou podstatně samosdružené a tento výsledek ve dvou dimenzích rozšíříme opět do dimenzí vyšších. Poté ukážeme, že nula leží v esenciálních spektrech těchto operátorů. To sice zaručuje existenci inverzního operátoru, ale ne pro všechny funkce na pravé straně Poissonovy rovnice.

Klíčová slova: metamateriály, superčočky, maskování, neviditelnost, anomální lokalizovaná rezonance, lokalizační index, podstatně samosdružené operátory, Besselovy funkce, spektrální teorie

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Contents

Chapter 1

Introduction

In the recent years there was a rapid development in the theory of metamaterials. These artificial materials have some extraordinary properties that arise from their very specific periodic structure designed by Pendry *et al.* [41] in 1999. They proposed that if we want to achieve the unnatural properties such as negative permittivity ϵ and permeability μ (typical for metamaterials) we need to put very small sub-units in the structure which make it possible.

It is known that permittivity and permeability represent an average response of the system to applied electromagnetic fields and in ordinary materials they are always positive. That is on a length scale much greater than the separation between atoms where all we need to know about material is described by ϵ and μ. However if the sub-units are very specifically designed (structure of the split rings) and much smaller than the wavelength of radiation then electric permittivity and magnetic permeability can be negative (see Figure 1.1).

The idea of material with negative ϵ and μ comes from the Russian physicist Victor Veselago that published in 1967 the first theoretical article [48] about metamaterials (however he called them the left-handed substances). Until then he studied so-called magnetic semiconductors in order to slow down electromagnetic waves (more information about Veselago's life and studies can be found for example in [44]). The wave velocity depends on the refractive index n by relation

$$v = \frac{c}{n} \tag{1.1}$$

where c is the speed of light. The refractive index is given as a square root of electrical permittivity ϵ and magnetic permeability μ

$$n = \sqrt{\epsilon\mu}. \tag{1.2}$$

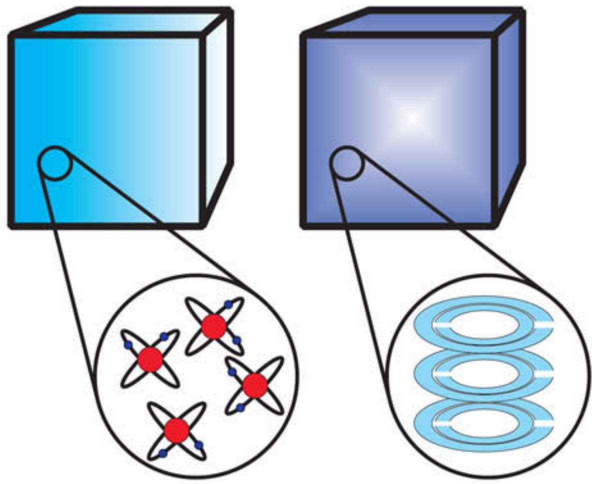

Figure 1.1: In the left there is a conventional material which derives its quantities ϵ and μ from the constituent atoms. However in the metamaterial (right) the role of the atoms is now played by small sub-units that consist of many atoms. The split ring structure of these units is the main reason for negative response of metamaterial in permittivity and permeability. [40]

To achieve slow velocities Veselago wanted to obtain higher values of n so he tried to increase both ϵ and μ in the magnetic semiconductor. But there is an issue that the high values of these quantities could not be realised simultaneously at any frequency and often one of the ϵ or μ became negative so the wave could not propagate (in such medium). Then he realised very important question: What would happen if both permittivity and permeability were simultaneously negative? Apparently the refractive index stays the same and real so it is obvious to ask whether there will be any difference against materials found in nature. Or whether such substances can even exist. Actually Veselago shows in his paper that these metamaterials do not contradict any fundamental laws of nature so they are in principle possible. He also presents here many extraordinary properties of such substances despite of the fact that they did not exist.

Other milestones in development of metamaterial are then the invention of material with negative permittivity [42] in 1996, three years later material with negative permeability [41] and finally the substance with simultaneously negative values of ϵ and μ was achieved in 2000 thanks to the work of Smith *et al.* [47].

The inspiration of this thesis comes from one of the most interesting application of metamaterial - the concept of invisibility, metamaterial cloaking. The first such cloaking device was developed in 2006 by Schurig *et al.* [45]. This cloak was able to be invisible for electromagnetic waves of microwave frequencies and its size was in the millimeter length scale.

There are many mathematical approaches to the theory of metamaterial cloaking. In Section 2.2 there are described some of them and the whole Chapter 3 is devoted to the concept of so called anomalous localised resonance. It is interesting that this description of cloaking has its origin also before the first manufacturing of metamaterial ([38] from 1994).

This thesis is organised as follows. In Chapter 2 we review the most important moments in the history of metamaterials, we mention here their properties and possible applications. The rest of that chapter is then dedicated to cloaking and different approaches describing it besides the anomalous localised resonance which is introduced in Chapter 3. Except the definition of this concept there are also the main forming results in this field divided into 3 sections each dedicated to the particular group of scientists. In the end of this chapter in Section 3.4 we present our own results about cloaking due to the anomalous localised resonance and whether it occurs on d-dimensional ball, $d \geq 3$. Inspired by the work of Behrndt and Krejčiřík [6] we introduce in Chapter 4 two indefinite Laplace operators in rectangular and rotational symmetrical setting and by the separation of variables we prove that these operators are essentially self-adjoint. Finally in Chapter 5 we briefly summarize so far found results about spectrum of the operator in rectangular setting and then we present our own calculation about spectrum of the operator in rotational setting. The main result is that 0 lies in the essential spectrum of both these operators.

Chapter 2

History of metamaterials

2.1 Physics of metamaterials

As stated in the introduction the first scientist investigating metamaterials (although only in theoretical way) was Victor Veselago. He introduced in [48] many extraordinary properties. For example that Poynting vector $\vec{S}$ and the wave vector $\vec{k}$ point in the opposite directions which results in reversed Doppler and Cerenkov effect and reversed Snell's law. Also the light pressure is then replaced by light attraction but the greatest attention is focused on the refraction of light in unusual way because of the negative refractive index n (see Figure 2.1). Now we con-

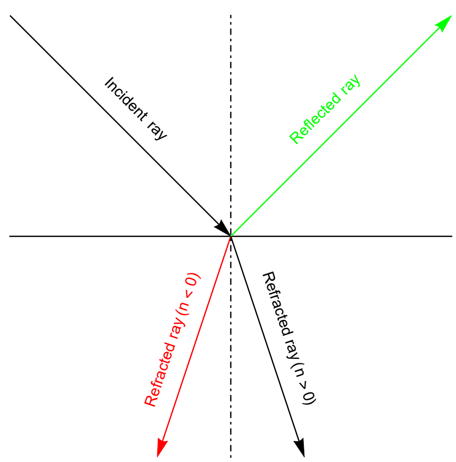

Figure 2.1: Propagation of a ray through the boundary between materials with positive (upper half plane) and negative refractive index (lower half plane).

centrate on the last property a little more. The reflected ray has always the same direction independently of the refractive index. But in a special case when permittivity and permeability of the metamaterial are exactly opposite to ϵ and μ of the surrounding medium then there is no reflected ray (refraction still occurs). This allows to design very interesting refraction system called Veselago's lens. It is a planar lens (but in fact it is not a lens in the usual sense, because it does not focus a bundle of rays coming from infinity at one point) and it is easy to see that if a radiation point is located at a distance shorter than a thickness of plate then such radiation is focused at a point (see Figure 2.2). It is also obvious that usual

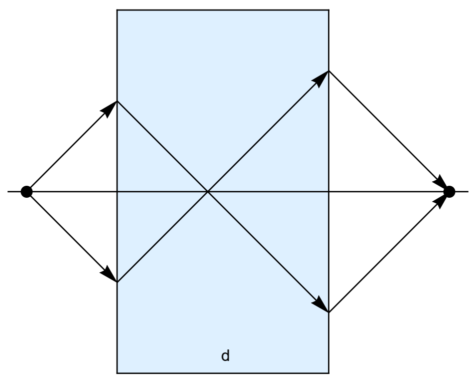

Figure 2.2: Propagation of two rays in Veselago's lens which is a metamaterial plate of thickness d

convex and concave lenses switch their role if they are made of metamaterial. It means that the rays passing through convex lens diverge and if they pass through the concave lens they converge.

The first negative permittivity material was achieved by Pendry *et al.* in 1996 (see [42]). It was a periodic structure of many thin infinite wires (see Figure 2.3) where permittivity is given by a relation

$$\epsilon_{\text{eff}}(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega} \tag{2.1}$$

where ω is a frequency of electromagnetic radiation, ω_p is an effective plasma frequency given by configuration of the wire field, i is an complex unit and γ is a coefficient of attenuation. From this it can be easily seen that for frequencies lower than plasma frequency the real part of permittivity is negative.

Pendry was also part of the team that created the first material with negative permeability. They stated in [41] (1999) that every material is composite in the

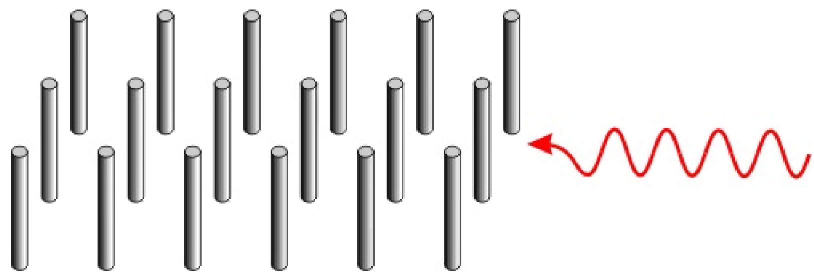

Figure 2.3: Periodic structure of thin infinite wires with incident wave on the right. [21]

sense that its smaller parts can be even atoms and molecules. And because permittivity and permeability only present a homogeneous view to the electromagnetic properties of a material, we can simply replace the atoms with some unit cells of characteristic dimensions which has to be much smaller than the wave length of the electromagnetic radiation. These cells are set in a periodic structure and their contents define the effective response of the system. The calculated dependence of the permeability is

$$\mu_{\text{eff}}(\omega) = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 - i\omega\Gamma} \tag{2.2}$$

where F, ω_0 and Γ are constants related to the geometry of the system. That periodic structure proposed by Pendry was field of so called split rings. Those are flat concentric disks separated from each other by a small distance and these rings are both divided on the exactly opposite sides (see Figure 2.4).

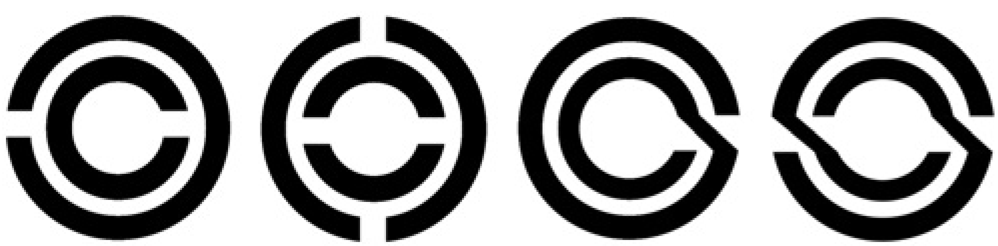

Figure 2.4: Various possibilities for split rings resonators. The setting on the left is the original proposed by Pendry. [21]

In [47] Smith *et al.* demonstrated a composite medium which combined both previous structures (see Figure 2.5). So after 33 years Viktor Veselago could see his theoretical ideas to become real.

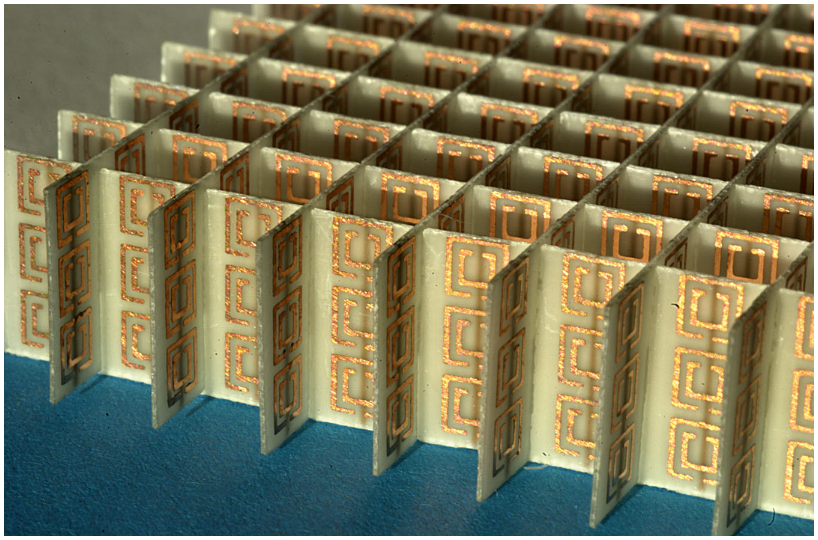

Figure 2.5: A split ring structure on copper circuit board with copper wires to give negative both permittivity and permeability [40].

Invention of such extraordinary and promising material caused huge interest of many scientists in this field. Many ways how to use metamaterials were suggested and plenty of them are still being investigated. Let us focus on some of them. In 2000, just after manufacturing the first metamaterial, Pendry [40] suggested that Veselago's lens on Figure 2.2 might act as a superlens. This means that such lens would provide a perfect image which is not possible for conventional lenses because their maximum resolution in the image can never be grater than

$$\Delta \approx \frac{2\pi}{k_{max}} = \frac{2\pi c}{\omega} = \lambda \tag{2.3}$$

where ω is a frequency of an infinitesimal dipole placed in front of the lens,λ is the wavelength of light and k_{max} is maximum value of the wave vector for propagating waves. Pendry proves here that the metamaterial amplify evanescent waves and so that both propagating and evanescent waves contribute to the resolution of the image.

This Pendry's proposal that the imaged object would have the same evanescent fields decaying exponentially away from it (as the real object) has been subject to controversy (see [34] or introduction in [33] for more details about this debate). Despite all the published articles dedicated to this problem there still has not been made mathematical proof for the superlensing until 2005 when Milton *et* *al.* accomplished it due to concept of anomalous localized resonance [33]. This approach will be discussed more precisely in Chapter 3).

Besides superlenses there are also other possible applications for metamaterials as for example metamaterial antennas [39], metamaterial absorber [25], [24], metamaterials sensors [20], terahertz detectors and of course metamaterial cloak which is the inspiration of this thesis.

2.2 Effect of invisibility

Manufacturing the first negatively refracting substance rapidly increased interest in the area of invisibility due to the metamaterial with negative refractive index n. In 2006 the first real metamaterial cloak for microwave frequencies was calculated [43] and created [45]. The approach used in [43] and also [46] is based on the fact that the metamaterials provide a freedom in their design so they can be used to control electromagnetic fields. There is used a coordinate transformation between orthogonal Cartesian mesh x, y, z and the distorted mesh $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ where u, v, w is the location of the new point with respect to the x, y, z axes (see Figure 2.6). Maxwell's equations have exactly the same form in any coordinate system but for the permittivity and permeability we must use their

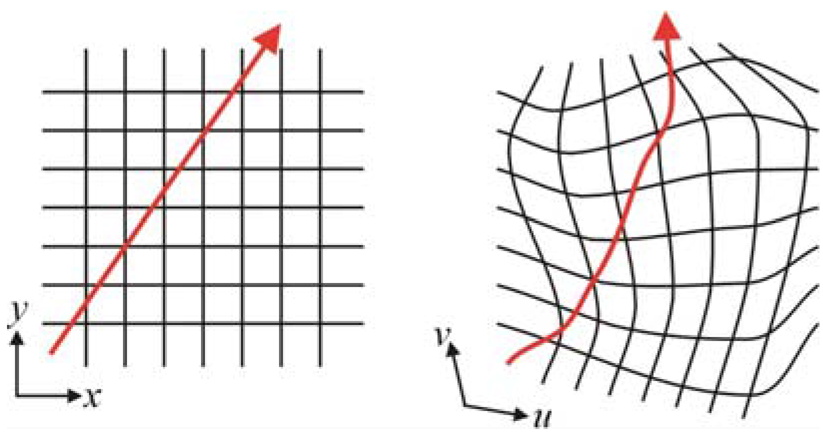

Figure 2.6: A field line in space with Cartesian coordinate system (left) and with the distorted system (right) [43]

values renormalized by the tensor transformation:

$$\epsilon'_j = \epsilon_j \frac{Q_u Q_v Q_w}{Q_j^2}, \qquad \mu'_j = \mu_j \frac{Q_u Q_v Q_w}{Q_j^2} \tag{2.4}$$

$$E'_j = Q_j E_j, \qquad H'_j = Q_j H_j \tag{2.5}$$

where $j \in \{u, v, w\}$ and

$$\begin{aligned} Q_u^2 &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ Q_v^2 &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\ Q_w^2 &= \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2 \end{aligned} \tag{2.6}$$

Purpose of that article was to hide an arbitrary object in space in a way that external observers would have no idea that there is any object hidden from them (which is apparently what one would expect from the invisibility effect). This is supposed to be achieved by metamaterial that would guide rays around the object and return them to their original trajectory (one can imagine water flowing around the stone in the river). Usually we think of the hiding object as a circle or sphere of radius R_c (core radius) and the cloaking region (filled with metamaterial) as an annulus $R_c < r < R_s$ (shell radius). For the object to be invisible we must find transformation that compress all fields in the region $r < R_s$ into the region $R_c < r < R_s$. The sketch of this transformation can be seen in Figure 2.7. However there are some issues to be solved. For example there is a singularity that can be seen if we consider a ray heading directly towards the centre of the circle (sphere). Rays near this singularity are bent very close around the inner circle and very tightly to each other. This implies that there must be very rapid changes in permittivity and permeability of the metamaterial. In practice big problem is also to achieve very small or very large values of ϵ' and μ'. Cloaking can still occur but will be imperfect. From the ray Figure 2.7 it can be easily seen that the cloaking is specific to a single frequency. Despite all these difficulties this theory gave birth to the first such metamaterial cloak working in the microwave frequencies [45].

Simultaneously with Pendry, Smith, Schurig *et al.* also Leonhardt was working on introduction of metamaterial cloak where invisibility should be again preserved by flowing electromagnetic fields around concealed object as if nothing was there. His tool for this theory was optical conformal mapping [27]. Such map

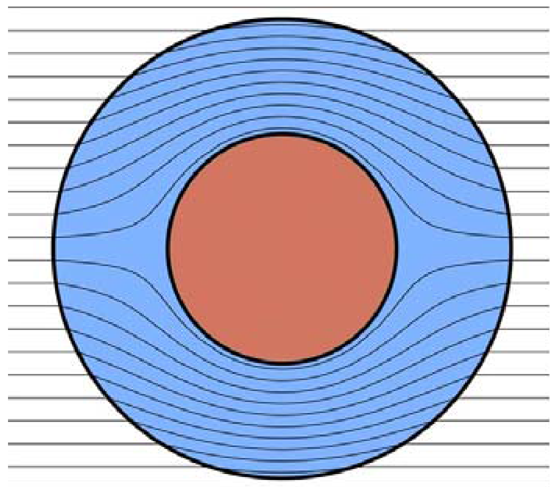

Figure 2.7: The electromagnetic field is made to avoid objects and flow around them and then returning undisturbed to its original trajectories. The red area represents the core and the blue area is the shell. [43]

preserves angles between the coordinate lines. Advantage of this method is that it is general so it can be used also for other forms of wave propagation like the sound waves. He considered here light propagating from infinity in a plane which is appropriate for complex valued functions and conformal mapping. It turns out that the main imperfections of invisibility are caused by reflections and time delays. But while reflections can be made exponentially small, the time delays are unavoidable. This delay was calculated by Leonhardt in [26].

The time delay of electromagnetic waves indicates that the general relativity could find some usage in this field of invisibility. In fact it is shown in [28] that it is not only usable for cloaking but that the general relativity unifies the whole theory behind controlling electromagnetic fields. Therefore when a desired function is given, we can calculate material properties for the device that turns this function into fact. This use of general relativity may be here quite surprising but actually it is not if we realize that the design concept of invisibility, superlenses and other applications is based on Fermat's principle that rays of light follow the shortest optical path in media. These paths are effective geodetics and general relativity provides tools for curved geometries. However the metamaterials can also find some use in the physics of gravitation because as shown in [28] they may be applied for laboratory analogues of artificial black holes.

Chapter 3

Anomalous localised resonance

This chapter is devoted to the most recent approach to metamaterial cloaking - the anomalous localised resonance (we will write shortly ALR or eventually CALR when we speak about cloaking due to anomalous localised resonance). There are many definitions of ALR. We take the one by Milton *et al.* stated for example in [33]: Inhomogeneous body exhibits ALR if as the loss goes to zero (or as the system of equations lose ellipticity) the field magnitude diverges to infinity throughout a specific region with sharp boundaries not defined by any discontinuities in the moduli, but converges to a smooth field outside that region.

Quasistatic approximation is considered here. It is quite useful since in this case the electric and magnetic problems decouple so we can think about metamaterials with only negative permittivity and positive permeability. Therefore we can use only electric part of Maxwell's equations

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{3.1}$$

$$\vec{\nabla} \times \vec{E} = 0, \tag{3.2}$$

where $\vec{D}$ is a displacement field related to the electric field $\vec{E}$ by relation $\vec{D} = \epsilon\vec{E}$ and ρ is an electric charge density. The second equation means that the electrostatic field is potential and so we can introduce potential V by relation $\vec{E} = -\nabla V$. We use this equation together with relation $\vec{D} = \epsilon\vec{E}$ in (3.1) to obtain

$$-\vec{\nabla} \cdot (\epsilon\vec{\nabla} V) = \rho. \tag{3.3}$$

Usually in most of the articles about ALR is the electric charge density considered zero so we have only

$$-\vec{\nabla} \cdot (\epsilon\vec{\nabla} V) = 0 \tag{3.4}$$

The sign changing coefficient in Maxwells equations can have negative consequences on the regularity of the solution and on the well-posedness of the problem. This was analysed for example by Bonnet-Ben Dhia et al., see for example [7], [8] and the references therein. Another area that mathematicians have to deal with is rigorous justification of the cloaking model. This was examined e.g. in [9], [11], [22], [23]. The techniques for homogenization are still being developed, however the problem is co complicated that they are set only for specific geometries like split-rings.

Equation (3.4) is the one to be dealt with but it has to be applied to some geometry. Usually there is a medium with permittivity ϵ_m in which there is placed a body containing a so called coated cylinder. Coated cylinder is made of cylindrical core and cylindrical shell surrounding the core and having permittivity ϵ_c and ϵ_s respectively. The coated cylinder is then characterized by these dielectric parameters and by the core radius r_c, the shell radius r_s and by the radius r_m around the cylinder centre where the medium has permittivity ϵ_m. Very important for the anomalous localized resonance is that the shell permittivity ϵ_s is a complex number with negative real part and a small non-negative imaginary part, we can choose it then for example as $\epsilon_s = -1 + i\delta$. Here δ is modelling losses in the material caused due to the electrical resistance of it (we know that electrical conductivity is calculated as $\sigma = \delta\omega$ where ω is a frequency of radiation). The crucial moment for ALR comes as δ tends to zero and the aim is to investigate what happens in this limit.

We introduce here some of the articles about ALR divided into several sections each concentrating on a specific approach to ALR. In the end there are summarized our own results contributing to this topic from our Research thesis. Before that let us

3.1 Milton, Nicorovici, McPhedran

Beginnings of anomalous localized resonance come even before the first metamaterial was manufactured. In 1994 Nicorovici, McPhedran and Milton analyzed the two-dimensional potential around a coated cylinder [38]. We briefly summarize it here.

We consider the coated cylinder described above. We recall that such medium is made of two concentric cylinders (core and shell) with permitivities ϵ_c, ϵ_s and radii r_c, r_s respectively and this medium is placed in a space with permittivity ϵ_m at least in some radius r_m around the cylinder centre. From the electrical

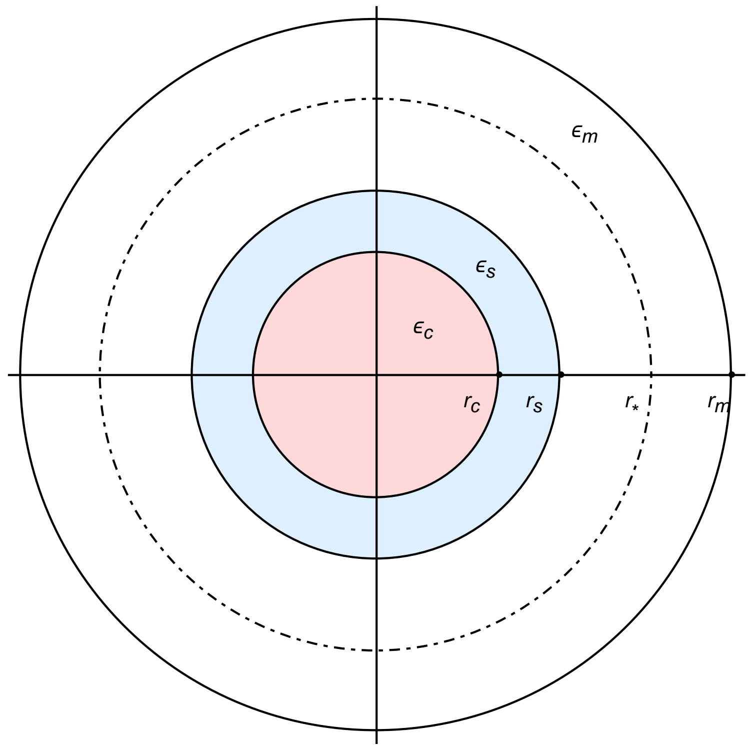

Figure 3.1: A coated cylinder in two dimensions with parameters $(\epsilon_c, r_c), (\epsilon_s, r_s), (\epsilon_m, r_m)$ describing core, shell and surrounding medium respectively. The radius $r_* = \sqrt{r_s^3/r_c}$ characterizes area in which ALR occurs.

part of Maxwell's equations we get the equation for complex potential V when charge density is zero (3.4). Since $V(z)$ is an analytic function of z we can find expansions for each region in our setting as

$$V_e(z) = A_0 + \sum_{l=1}^{+\infty} (A_l z^l + B_l z^{-l}) \qquad \text{for } r_s \leq r \leq r_m \tag{3.5}$$

$$V_s(z) = C_0 + \sum_{l=1}^{+\infty} (C_l z^l + D_l z^{-l}) \qquad \text{for } r_c \leq r \leq r_s \tag{3.6}$$

$$V_c(z) = E_0 + \sum_{l=1}^{+\infty} E_l z^l \qquad \text{for } r \leq r_c \tag{3.7}$$

Conditions of continuity for potential and for the normal component of the electrical displacement on the boundary (i.e. for r_s and r_c) allow us to express coefficients B_l, C_l, D_l, E_l in terms of the A_l

$$B_l = \frac{A_l}{\Delta}[\eta_{ms} + \eta_{sc}(\frac{r_c}{r_s})^{2l}]r_s^{2l}, \tag{3.8}$$

$$C_l = \frac{A_l}{\Delta}(1+\eta_{ms}), \tag{3.9}$$

$$D_l = \frac{A_l}{\Delta}\eta_{sc}(1+\eta_{ms})r_c^{2l}, \tag{3.10}$$

$$E_l = \frac{A_l}{\Delta}(1+\eta_{ms})(1+\eta_{sc}), \tag{3.11}$$

where $\Delta = 1 + \eta_{ms}\eta_{sc}(\frac{r_c}{r_s})^{2l}$ and parameters

$$\eta_{ms} = \frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_s}, \qquad \eta_{sc} = \frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}, \tag{3.12}$$

characterize the jumps in permittivities across the boundary between the external medium and the shell and between the shell and the core respectively.

The purpose of that article was to determine when the coated cylinder is equivalent to a solid cylinder. Apparently it occurs when the relation between A_l and B_l is exactly the same as for a solid cylinder because this relationship defines the response of a coated cylinder to an external field. It is stated there that there are six special cases that such equivalence occurs. But two of them are even more special than the others:

$$\epsilon_s + \epsilon_c = 0, \tag{3.13}$$

$$\epsilon_s + \epsilon_m = 0. \tag{3.14}$$

Using the first equation, we can calculate the limit

$$B_l = \lim_{\epsilon_s \to -\epsilon_c} \frac{\dfrac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_c} + \dfrac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}\left(\dfrac{r_c}{r_s}\right)^{2l}}{1 + \dfrac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_s}\dfrac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}\left(\dfrac{r_c}{r_s}\right)^{2l}} r_s^{2l} A_l = \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} r_s^{2l} A_l \tag{3.15}$$

This means that we can replace the coated cylinder by a solid cylinder with radius r_s and permittivity ϵ_c without changing the external potential V_e and therefore the core properties are extended up to the outer boundary of the shell. Limit for the second equation is calculated in a similar way

$$B_l = \lim_{\epsilon_s \to -\epsilon_m} \frac{\frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_c} + \frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}\left(\frac{r_c}{r_s}\right)^{2l}}{1 + \frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_s}\frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}\left(\frac{r_c}{r_s}\right)^{2l}} r_s^{2l} A_l = \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} a^{2l} A_l \qquad (3.16)$$

where $a = \frac{r_s^2}{r_c}$. This relation tells us that the core properties are now extended up even beyond the shell ($a = \frac{r_s^2}{r_c} > r_s$) without disturbing the potential outside the radius a. Thus equivalent solid cylinder has radius a and permittivity ϵ_c. If both equations (3.13) and (3.14) hold then the coated cylinder can be replaced by the medium material without altering the external field.

These extraordinary properties are in that article called partial resonance. For one of the two special cases we place a line dipole at point z_0 on the positive half of the x axis (see Figure 3.2). The magnitude of that dipole is chosen in a way that its potential is $\frac{1}{z-z_0}$ and may be expanded in z with coefficients $A_l = -(\frac{1}{z_0})^{l+1}$.

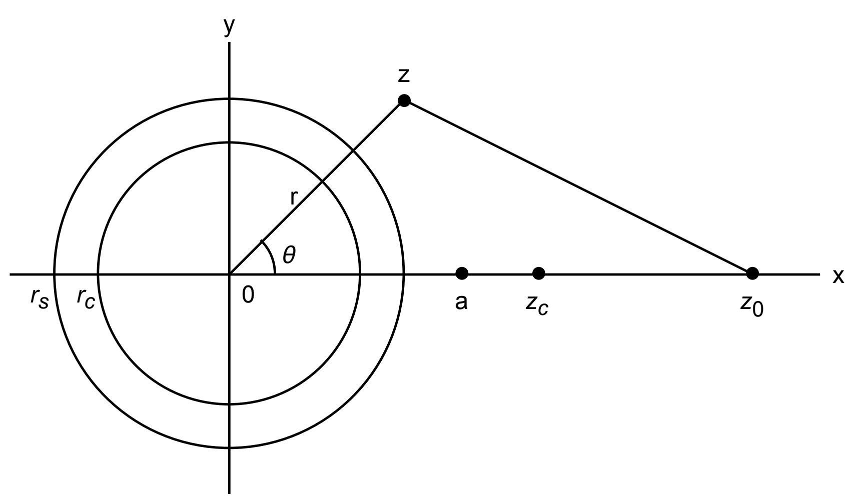

Figure 3.2: A coated cylinder in the electric field of a dipole placed at the point z_0 [38]

Then for the case $z_0 > z_c = \frac{a^2}{r_s} = \frac{r_s^3}{r_c^2}$

$$\tilde{V}_e(z) = \frac{1}{z - z_0} - \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} \frac{\frac{a^2}{z_0^2}}{z - \frac{a^2}{z_0}} \qquad \text{for } |z| \geq r_s, \quad (3.17)$$

$$\tilde{V}_s(z) = -\frac{2\epsilon_c}{\epsilon_m + \epsilon_c} \frac{1}{z_0} + \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} \frac{\frac{r_s^2}{a^2}}{z - z_0 \frac{r_s^2}{a^2}} - \frac{\frac{r_s^2}{z_0^2}}{z - \frac{r_s^2}{z_0}} \qquad \text{for } r_c \leq |z| \leq r_s, \quad (3.18)$$

$$\tilde{V}_e(z) = \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} \frac{1}{z_0} + \frac{2\epsilon_m}{\epsilon_m + \epsilon_c} \frac{\frac{r_s^2}{a^2}}{z - z_0 \frac{r_s^2}{a^2}} \qquad \text{for } 0 \leq |z| \leq r_c. \quad (3.19)$$

But if we take $z_0 < z_c$ then the ratio test shows that the series (3.5) does not converge for $r_s < r < \frac{a^2}{z_0}$ as well as the series (3.6) for $z_0 \frac{r_s^2}{a^2} < r < r_s$. There also appears two image dipoles (in later works they are called ghost sources) one in the medium at $\frac{a^2}{z_0} = r_{g_1}$ and one in the shell at $z_0 \frac{r_s^2}{a^2} = r_{g_2}$ where they produce unphysical singularities of V_e and V_s respectively. (A ghost source can appear not only in the medium and shell but also in the core as pointed out by authors later in [33]) An unpleasant consequence of this is that we have no physical solution of this problem when $\epsilon_s = -\epsilon_m$ and $z_0 < z_c$. This can be avoided if we add small imaginary part to the permittivity of the shell. In a physical way of speech the shell becomes lossy. Usually we take permittivity of the shell as

$$\epsilon_s = (-1 + i\delta)\epsilon_m \tag{3.20}$$

where $|\delta| \ll 1$. In this case the potential stays very close to the values given by (3.17) – (3.19) besides annulus between the two ghost sources $r_{g_1} \leq r \leq r_{g_2}$. The field outside this radius r_{g_2} converges to the field outside the equivalent solid cylinder. When we approach the ghost source from outside this radius, it looks like a true line source (in the limit when $\delta \to 0$). The last mentioned fact was discovered later in 2005 [33] as the first example of superlensing (see further in this section).

Of course with the existence of actual metamaterials this theory acquire more importance. In 2002 Milton wrote a book [30] about theory of composites where he mentioned in one small section this concept of localized resonance.

As mentioned in Section 2.1 Pendry's article [40] about superlenses started debate about whether such device is in principle possible to be created. Some numerical simulations and calculations were suggesting that it is not possible to achieve this effect of perfect lenses for any dispersive lossy lens. However many other people suggest that the slab lenses could work when their thickness d is much smaller than wavelength of radiation in free space λ_0. The first mathematically correct proof of superlensing was given in 2005 in [33]. There was again used a quasistatic limit and the principle of anomalous localized resonance (this full name was used here for the first time) since permittivity was considered $\epsilon = -1 + i\epsilon''$ where the imaginary part ϵ'' tends to zero.

Their result for a dipole line source being outside a coated cylinder is as follows: In the case when $\epsilon_m \neq \epsilon_c$ there is no ALR for $r_0 > r_{crit}$ in the limit $\delta \to 0$ but for smaller r_0 the potential becomes anomalously locally resonant in an annulus between two ghost sources one in the medium at r_{g_1} and the second one at r_{g_2} which can be in the shell (for r_0 between r_{crit} and a) or in the core (for r_0 between a and r_s), see Figure 3.3. In the case when $\epsilon_m = \epsilon_c$ there is no ALR for $r_0 > r_*$ in the limit $\delta \to 0$ but for r_0 between r_* and r_s the potential becomes anomalously locally resonant in two sometimes overlapping annuli. Outside the anomalously locally resonant regions the potential V converges to $\tilde{V}$ (for more detailed version of this theorem see [33]).

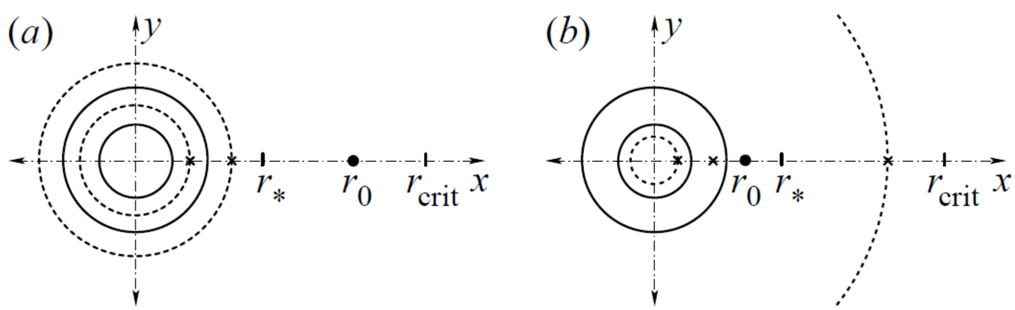

Figure 3.3: The location of ghost sources (marked by the crosses) depends on the position of r_0, namely when (a) r_0 is between r_* and r_{crit} or (b) when r_0 is between r_s and r_*. The solid circles represent the core and shell of the coated cylinder. Anomalously locally resonant region is marked by the dashed lines. [33]

They also considered another geometry than the one with the coated cylinder (now called cylindrical superlens) - the slab lens. They had $\epsilon_m = \epsilon_c = 1$ and let r_s, r_c and r_0 tend to infinity while $d = r_s - r_c$ and $d_0 = r_0 - r_s$ were kept fixed. They came to results that there is no resonance as $\delta \to 0$ for $d_0 > d$ and the potential converges to the one that satisfies the properties of a superlens. But for $d_0 < d$ there occurs ALR in two sometimes overlapping layers of thickness $2(d - d_0)$. From that (and private communication with Alexei Efros) they deduce that in a simple case when $\epsilon_s = -1 + i\epsilon_s''$ the loss scales as $|\epsilon_s''|^{2(d_0/d)-1}|\log \epsilon_s''|$ which goes to zero when $d_0 > \frac{d}{2}$ but diverges to infinity when $d_0 < \frac{d}{2}$. This divergence occurs because in this case the source lie in the ALR region and they have to do increasing amount of work against the locally resonant field here.

Another very important article by Milton and Nicorovici is from 2006 [31] about cloaking effects associated with anomalous localized resonance. They fluently continue in what they have done in [33] and use the results from there not for superlensing but for the invisibility effect. A quasistatic transverse magnetic (TM) field is considered where polarizable line with polarizability $\boldsymbol{\alpha}$ is placed. This TM field surrounds again a coated cylinder and the polarizable line is placed along $x = r_0 > r_s$, $y = 0$. Permittiivities of this setting are chosen to be $\epsilon_s \approx -\epsilon_m \approx -\epsilon_c$ where ϵ_m is assumed to be fixed, real and positive. ϵ_c should also remain fixed but can be complex with non-negative imaginary part. Further ϵ_s approaches $-\epsilon_m$ along the trajectory in the upper half of the complex plane as $\delta \to 0$, and this number δ is now taken from denotion

$$\frac{(\epsilon_s + \epsilon_c)(\epsilon_m + \epsilon_s)}{(\epsilon_s - \epsilon_c)(\epsilon_m - \epsilon_s)} = \delta e^{i\phi} \tag{3.21}$$

They introduced here two quantities: effective polarizability tenzor and effective source terms

$$\boldsymbol{\alpha}_* = [\boldsymbol{\alpha}^{-1} - c(\delta)\boldsymbol{I}]^{-1}, \quad \begin{pmatrix} k_*^e \\ -k_*^o \end{pmatrix} = [\boldsymbol{I} - c(\delta)\boldsymbol{\alpha}]^{-1} \begin{pmatrix} k^e \\ -k^o \end{pmatrix} \tag{3.22}$$

where $c(\delta)$ is exactly calculated in that article and it is shown that for $\delta \to 0$ this $c(\delta)$ tends to infinity. The quantities k^e and k^o are dipole moments of the polarizable line (k^e gives the amplitude of the dipole component with even symmetry about the x-axis and k^o the amplitude of the component with odd symmetry about the x-axis). If $|c(\delta)|$ is very large there are for the expressions (3.22) following estimates

$$\boldsymbol{\alpha}_* \approx \frac{-\boldsymbol{I}}{c(\delta)}, \quad \begin{pmatrix} k_*^e \\ -k_*^o \end{pmatrix} \approx \frac{-\boldsymbol{\alpha}^{-1}}{c(\delta)} \begin{pmatrix} k^e \\ -k^o \end{pmatrix} \tag{3.23}$$

Therefore both expression tend to zero as $\delta \to 0$ and that according to [31] explains why cloaking occurs.

The cloaking is shown even for the slab lens. In fact if we have again a lens of a thickness d then it is proved here that a polarizable line dipole located less than $\frac{d}{2}$ from the lens would be cloaked. This is due to the presence of a resonant field in front of the lens which was shown already in [33].

These three articles became a starting point of the anomalous localized resonance and inspired many other scientists in this area. Let us mention some other work by these three authors, for example [37] with some nice simulations of cloaking which clarify its physical mechanism, or [32] where the core radius r_c of the coated cylinder is supposed to be bigger than the shell radius r_s. They call such setting a folded geometry and give it a physical meaning by transforming it to an equivalent problem in unfolded geometry. Milton also worked together with Ammari, Ciraolo, Kang and Lee when they wrote [3], [4] and [2] to which Section 3.3 is dedicated.

3.2 Bouchitté, Schweizer

Another important moment for anomalous localized resonance comes with [10]. It is one of the first articles about ALR where the theory of operators was used. The operators investigated there are

$$\mathcal{L}^\eta = \nabla \cdot (a^\eta \nabla) \tag{3.24}$$

and interest of this article is in the solutions u^η of equation $\mathcal{L}^\eta u^\eta = 0$. The coefficient a^η is here for permittivity and it is defined as

$$a^\eta(x) = \begin{cases} -1 + i_0\eta & \text{for } x \in \Sigma \\ +1 & \text{for } x \in \mathbb{R}^2 \setminus \Sigma \end{cases} \tag{3.25}$$

Here $x \in \mathbb{R}^2$, $\eta > 0$, $i_0 \in \mathbb{C}$, $|i_0| = 1$ and $\operatorname{Im} i_0 > 0$. The set Σ denotes the ring $\Sigma = B_R(0) \setminus B_1(0)$, $R > 1$.

The theorems proved here simply say that there is the cloaking radius $R^* = \sqrt{\frac{R^3}{1}} = R^{3/2} > R$ (in [31] it was denoted as $r_\#$) which they are very closely related to. These are that if we choose a small ball $B_\epsilon(x_0)$ around a point $x_0 \in \mathbb{R}^2$ than we can get two results: First, if $x_0 \notin B_{R^*}(0)$ then a measurement of the whole setting shows there is no ring present (it is invisible). Second, if $x_0 \in B_{R^*}(0)$ then the measurement does not detect the ring but $B_\epsilon(x_0)$ either. For this theorems they introduce two numbers which express a measure for the visibility of the dipole inclusion

$$\mathcal{M}_q^\eta = \left(\int_{\partial B_q(0)} |\partial_n v^\eta|^2 \right)^{1/2} \tag{3.26}$$

and a measure of how much the true solution differs from the comparision solution

$$\mathcal{N}_q^\eta(f) = \left(\int_{\partial B_q(0)} |\partial_n u^\eta - \partial_n u^*|^2 \right)^{1/2} \tag{3.27}$$

The results for these quantities depend strongly on the position of x_0 or better whether x_0 lies in the B_{R^*} or not.

However without these coefficients or any Dirichlet-to-Neumann maps we can still simply study one limit which will indicate whether cloaking of the $B_\epsilon(x_0)$ occurs or not. (We concentrate on this because we use the same procedure in our calculations in Section 3.4.) In [10] they took the equation

$$\mathcal{L}^\eta u^\eta = 0 \tag{3.28}$$

whose solution in two dimensions (in the radial geometry stated above) can be found for fixed $k \in \mathbb{N}_0$ with the ansatz

$$u^\eta(x) = U(r)e^{ik\theta} \tag{3.29}$$

where $U : (0, \infty) \to \mathbb{C}$. It is usual to take $k \in \mathbb{Z}$ but the solution is symmetric in k with respect to 0 thus we do not need to write absolute values for every k. To find solution of (3.28) (which are piecewise harmonic functions) they made the following ansatz for complex numbers $a, b, \alpha, \beta \in \mathbb{C}$

$$U_k(r) = \begin{cases} r^k & \text{for } r \leq 1 \\ ar^k + br^{-k} & \text{for } 1 < r \leq R \\ \alpha r^k + \beta r^{-k} & \text{for } R < r \end{cases} \tag{3.30}$$

In the case when $r \leq 1$ is no complex coefficient because we require the solution to be bounded in $x = 0$ (the term r^{-k} has singularity in $r = 0$) and also we require normalizing to a unit monomial around 0. The four unkown coefficients a, b, α, β can be determined explicitly from conditions on continuity for u^η and $a^\eta \partial_r u^\eta$ in $r = 1$ and $r = R$. We then get four equations

$$1 = a + b \tag{3.31}$$

$$1 = Aa - Ab \tag{3.32}$$

$$aR^k + bR^{-k} = \alpha R^k + \beta R^{-k} \tag{3.33}$$

$$aAR^k + bAR^{-k} = \alpha R^k - \beta R^{-k} \tag{3.34}$$

Here they denoted $A = -1 + i_0\eta \in \mathbb{C}$. From these equations we express the wanted coefficients

$$a = \frac{A+1}{2A} \tag{3.35}$$

$$b = \frac{A-1}{2A} \tag{3.36}$$

$$\alpha = \frac{1}{4}R^{-k}\left[\frac{(1+A)^2}{A}R^k - \frac{(1-A)^2}{A}R^{-k}\right] \tag{3.37}$$

$$\beta = \frac{1}{4}R^{k}\left[\frac{1-A^2}{A}R^k - \frac{1-A^2}{A}R^{-k}\right] \tag{3.38}$$

Question is which term becomes dominant, αr^k or βr^{-k}. For this they introduced another very important number called localization index

$$P_k^\eta = \frac{\beta}{\alpha} \in \mathbb{C} \tag{3.39}$$

It is now straightforward to calculate this localization index when he have exact forms of coefficients α, β

$$P_k^\eta = R^{2k}\frac{(1-A^2)(R^k - R^{-k})}{(1+A)^2R^k - (1-A)^2R^{-k}} \tag{3.40}$$

To see the importance of cloaking radius R^* we need to create it somewhere in this expression. Also because we want to find out whether the term αr^k or βr^{-k} is dominant we take look at the number P_k^η / r^{2k} (we write this final result for all $k \in \mathbb{Z}$)

$$\frac{P_{|k|}^\eta}{r^{2|k|}} = \left(\frac{R^*}{r}\right)^{2|k|}\frac{(2i_0\eta - i_0^2\eta^2)(1 - R^{-2|k|})}{i_0^2\eta^2R^{|k|} - (2 - i_0\eta)^2R^{-|k|}} \tag{3.41}$$

The loss parameter is here η so we want to explore what happens if it tends to zero. In this case we get

$$\max_k \frac{|P_{|k|}^\eta|}{r^{2|k|}} \to 0 \quad \text{if } r > R^* \tag{3.42}$$

$$\max_k \frac{|P_{|k|}^\eta|}{r^{2|k|}} \to \infty \quad \text{if } r < R^* \tag{3.43}$$

The anomalous localized resonance is related to the fact that P_k^η can become very large. That is exactly what (3.43) says and then the truth is that there is the dominance of the term β over the term α in case when $r < R^*$.

This localization index can be translated into a Dirichlet-to-Neumann operator. We recall the definition from [10] for fixed $r > R$ and the boundary $\Gamma = \partial_n u^\eta|_\Gamma$

$$N^{r,\eta} : H^{1/2}(\Gamma, \mathbb{C}) \to H^{-1/2}(\Gamma, \mathbb{C}), u^\eta \mapsto \partial_n u^\eta|_\Gamma \tag{3.44}$$

Here u^η is the solution of equation (3.28) in $B_r(0)$ and $n(x)$ is the exterior normal to $B_r(0)$. We know that $(e^{ik\theta})_{k\in\mathbb{Z}}$ is a basis of both $H^{\pm 1/2}(\Gamma, \mathbb{C})$. Hence we can describe $N^{r,\eta}$ with its Fourier components

$$N^{r,\eta}(e^{ik\theta}) = N_k^{r,\eta} e^{ik\theta} \tag{3.45}$$

The solution u^η is known to us from (3.29) and (3.30) so for $r > R$ it has the form $u^\eta = c(\alpha r^{|k|} + \beta r^{-|k|})e^{ik\theta}$. Therefore we can find connection between the localization index P_k^η and the Dirichlet-to-Neumann operator $N^{r,\eta}$ by (3.45) and

$$N_k^{r,\eta} = \frac{\partial_r u^\eta}{u^\eta}|_{\partial B_r(0)} = \frac{|k|}{r} \frac{1 - P_k^\eta r^{-2|k|}}{1 + P_k^\eta r^{-2|k|}} \tag{3.46}$$

3.3 Ammari, Ciraolo, Kang, Lee and others

Now we will mention some more recent articles dedicated to anomalous localized resonance. From these we would like to point out especially work by Ammari, Ciraolo, Kang, Lee and Milton. In [3] they considered a bounded domain $\Omega \in \mathbb{R}^2$ and domain D whose closure is contained in Ω. With loss parameter δ they defined permittivity in $\mathbb{R}^2$ as

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus \bar{\Omega} \\ -1 + i\delta & \text{in } \Omega \setminus \bar{D} \\ 1 & \text{in } D \end{cases} \tag{3.47}$$

It is obvious that this geometry is more general than what we could see earlier in this thesis. But the setting with two concentric disks is also examined here as a special case of this more general one. Still we can think about D as a core with permittivity 1 surrounded by the shell $\Omega \setminus \bar{D}$ with permittivity $-1 + i\delta$.

The dielectric problem in $\mathbb{R}^2$ they dealt with was

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = \alpha f \tag{3.48}$$

Here αf is a source term where function f compactly supported in $\mathbb{R}^2$ satisfies physical condition of conservation of charge

$$\int_{\mathbb{R}^2} f dx = 0 \tag{3.49}$$

and V_δ fulfils the decay condition

$$\lim_{|x|\to\infty} V_\delta(x) = 0 \tag{3.50}$$

There aim was to find those functions f in such a way that (when $\alpha = 1$)

$$E_\delta = \int_{\Omega\setminus\bar{D}} \delta|\nabla V_\delta|^2 dx \to \infty \quad \text{as } \delta \to 0 \tag{3.51}$$

$$|V_\delta(x)| < C \quad \text{when } |x| > a \tag{3.52}$$

The quantity E_δ from (3.51) is proportional to the electromagnetic power dissipated into heat. The second equation (3.52) tells that V_δ remains bounded by some constant C outside certain radius a independent of δ. There is apparently unphysical situation because it results from (3.51) that amount of energy dissipated per unit time is infinite (in the limit $\delta \to 0$). Therefore we can choose $\alpha = 1/\sqrt{E_\delta}$ then the source αf produce the same power independent of α. The new associated solution of (3.48) is then $V_\delta/\sqrt{E_\delta}$ and it will approach zero outside the radius a. Such described case means that CALR occurs.

Now when the problem is stated let us mention the method used in [4] to deal with it. Their goal was to give a necessary and sufficient condition on the source term so the blow-up (3.51) takes place. For this they used techniques of layer potentials to reduce the dielectric problem to a singularly perturbed system of integral equations which is non-self-adjoint. The integral operators in this article are sometimes called Neumann-Poincaré operators (see [3] for details). A generalization of Calderón's identity is used to the non-self-adjoint system so they could express the solution in terms of the eigenfunctions of a self-adjoint compact operator.

Then they investigated well known case of an annulus (D and Ω are two concentric disks with radii r_i, r_e respectively) and found that there exists a cloaking radius $r_* = \sqrt{r_e^3 r_i^{-1}}$ such that any dipole sources placed in the annulus $B_{r_*} \setminus \bar{B}_e$ are cloaked.

In the following article [4] they considered the same problem a little more generally and defined the permittivity distribution as

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^d \setminus \bar{\Omega} \\ -\epsilon_s + i\delta & \text{in } \Omega \setminus \bar{D} \\ \epsilon_c & \text{in } D \end{cases} \tag{3.53}$$

And what more they did not restrict themselves only to two-dimensional problem but they also examined what happens in three dimensions for radially symmetric structure. Using the same procedure as in [3] they proved that CALR occurs in two dimensions only if $\epsilon_s = -1$ (assuming that $\epsilon_c = -1$). For other values of ϵ_s CALR does not occur. In three dimensions CALR does not occur whatever ϵ_s and ϵ_c are.

In fact this non-occurrence of CALR in three dimensions is related to the fact that ϵ_s is constant. It was discovered in [2] that if they use a shell with a specially designed anisotropic permittivity then the CALR occurs for the case when D and Ω are two concentric balls in $\mathbb{R}^3$ with radii r_i and r_e respectively and chosse r_0 in the way that $r_0 > r_e$. For a given loss parameter $\delta > 0$ they defined

$$\boldsymbol{\epsilon}_\delta(\boldsymbol{x}) = \begin{cases} \boldsymbol{I} & |\boldsymbol{x}| > r_e \\ (\epsilon_s + i\delta)a^{-1}(\boldsymbol{I} + \frac{b(b-2|\boldsymbol{x}|)}{|\boldsymbol{x}|^2}\hat{\boldsymbol{x}} \otimes \hat{\boldsymbol{x}}) & r_i < |\boldsymbol{x}| < r_e \\ \epsilon_c \sqrt{\frac{r_0}{r_i}}\boldsymbol{I} & |\boldsymbol{x}| < r_i \end{cases} \tag{3.54}$$

where $\boldsymbol{I}$ is the 3×3 identity matrix, ϵ_s, ϵ_c are constants, $\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ is unit vector in $\mathbb{R}^3$, $a = \frac{r_e - r_i}{r_0 - r_e} > 0$ and $b = (1+a)r_e$. Such $\boldsymbol{\epsilon}_d$ is anisotropic and variable in the shell and it is designed by unfolding a folded geometry.

There are still many interesting articles about cloaking via anomalous localized resonance but it is not possible to talk about all of them. Let us mention some of the most recent ones like for example [12] where they use the spectral analysis of the Neumann-Poincaré type operator on confocal ellipses to prove that CALR takes place in such setting. Another spectral analysis of Neumann-Poincaré operator is made in [5]. Resonance at eigenvalues and at the essential spectrum is investigated there. The resonance at eigenvalues of the Neumann-Poincaré operator is the plasmon resonance [16] and it is shown that the resonance at the essential spectrum is the anomalous localised resonance on ellipses (in $\mathbb{R}^2$) but does not occur on three dimensional balls. It is also shown that the resonance at the essential spectrum is weaker than at the eigenvalues. In [6] there is investigated an indefinite Laplacian on a rectangular domain in the plane. They found that such defined operator is self-adjoint and that its spectrum consists of nonzero eigenvalues of finite multiplicity and 0 is also in the spectrum but having infinite multiplicity (therefore belongs to essential part of the spectrum). Among articles dealing with this three dimensional problem we can mention for example [29], [35], [36] that all are very recent (2015).

3.4 ALR on the ball in three and higher dimensions

In our Research thesis [18] we proved that CALR does not occur on the spherical shell in three dimensions. This was proved however before in [4] but we used a different procedure inspired by [10] and their notion of localization index.

Let us write down again the equation (3.4)

$$\vec{\nabla} \cdot (\epsilon_\delta(x)\vec{\nabla}\Psi(x)) = 0 \tag{3.55}$$

with permittivity

$$\epsilon_\delta(x) = \begin{cases} +1, & \text{for } x \in B_1(0) \\ -1 + i\delta, & \text{for } x \in \Sigma \\ +1, & \text{for } x \in \mathbb{R}^d \setminus B_R(0) \end{cases} \tag{3.56}$$

where we denoted Σ as an annulus $B_R(0) \setminus B_1(0)$ for $R > 1$. d is here for the dimension of space and $\delta > 0$ as before. Since the permittivity (3.56) is constant in each of the three regions we are interested in the solution of Laplace equation

$$\Delta\Psi(x) = 0 \tag{3.57}$$

If we suppose that we can write solution of this equation in separated form as

$$\Psi(x) = \psi(r)\phi(\Omega) \tag{3.58}$$

where r denotes radial coordinate and Ω is for angular coordinates then we can write down boundary conditions for the solution as

$$\begin{aligned} \psi(1^-) &= \psi(1^+) \\ \psi(R^-) &= \psi(R^+) \\ \epsilon_\delta(1^-)\frac{d}{dr}\psi(1^-) &= \epsilon_\delta(1^+)\frac{d}{dr}\psi(1^+) \\ \epsilon_\delta(R^-)\frac{d}{dr}\psi(R^-) &= \epsilon_\delta(R^+)\frac{d}{dr}\psi(R^+) \end{aligned} \tag{3.59}$$

To get Laplace equation (3.57) in a radial form we first take a look at Laplace operator $-\Delta$ which can be written in d dimensions as

$$-\Delta = -\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\Delta_{S^{d-1}} \tag{3.60}$$

where $-\Delta_{S^{d-1}}$ is Laplace-Beltrami operator or spherical Laplace operator in $d-1$ dimensions [15]. Its spectrum is well known $\sigma(-\Delta_{S^{d-1}}) = \{l(d-2+l)\}_{l=0}^{\infty}$ and therefore we can write Laplace operator in a form

$$-\Delta = \bigoplus_{l=0}^{\infty}\bigoplus_{k=-l}^{l}\left(-\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) + \frac{l(d-2+l)}{r^2}\right) \tag{3.61}$$

Hence Laplace equation (3.57) in the radial form is

$$-\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\psi(r)\right) + \frac{l(d-2+l)}{r^2}\psi(r) = 0 \tag{3.62}$$

It is easy to check that functions $\psi_1(r) = r^l$ and $\psi_2(r) = r^{-(d-2+l)}$ are solutions to equation (3.62). In case of ψ_1 we have

$$\begin{aligned}
-\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\psi_1(r)\right) + r^{d-3}l(d-2+l)\psi_1(r) &= \\
= -\frac{d}{dr}\left(r^{d-1}lr^{l-1}\right) + r^{d-3}l(d-2+l)r^l &= \\
= -l(d-2+l)r^{d+3+l} + l(d-2+l)r^{d-3+l} &= 0
\end{aligned}$$

and in case of ψ_2 we get in the same way

$$\begin{aligned}
-\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\psi_2(r)\right) + r^{d-3}l(d-2+l)\psi_2(r) &= \\
= -\frac{d}{dr}\left(r^{d-1}(-d+2-l)r^{-d+1-l}\right) + r^{d-3}l(d-2+l)r^{-d+2-l} &= \\
= -l(d-2+l)r^{-l-1} + l(d-2+l)r^{-l-1} &= 0
\end{aligned}$$

Therefore we can express the solution of our equation (3.62) as a superposition of ψ_1 and ψ_2

$$\psi(r) = \sum_{l=0}^{+\infty}\left(A_l(r)r^l + B_l(r)r^{-l+2-d}\right) \tag{3.63}$$

Let us note here that these coefficients A_l, B_l are constants in each of the three regions $B_1(0), \Sigma$ and $\mathbb{R}^d \setminus B_R(0)$. For fixed $l \in \mathbb{N}_0$ we can denote one addend of the sum above as

$$\psi_l(r) = \begin{cases} r^l & \text{for } r \leq 1 \\ a_l r^l + b_l r^{-l+2-d} & \text{for } 1 < r \leq R \\ \alpha_l r^l + \beta_l r^{-l+2-d} & \text{for } R < r \end{cases} \tag{3.64}$$

We will be focused now on the case when $d = 3$. Using boundary conditions (3.59) we get four equations for four coefficients $a_l, b_l, \alpha_l, \beta_l$

$$\begin{aligned} 1 &= a_l + b_l \\ a_l R^l + b_l R^{-l+2-d} &= \alpha_l R^l + \beta_l R^{-l+2-d} \\ l &= A_\delta(a_l l + b_l(-l+2-d)) \\ A_\delta(a_l l R^{l-1} + b_l(-l+2-d)R^{-l+1-d}) &= \alpha_l l R^{l-1} + \beta_l(-l+2-d)R^{-l+1-d} \end{aligned} \tag{3.65}$$

here we denoted $A_\delta = -1 + i\delta$ similarly as in the [10]. From this system of equation it is possible to achieve the coefficients

$$\begin{aligned} a_l &= \frac{l + A_\delta\,(l-2+d)}{A_\delta\,(2l-2+d)} \\ b_l &= \frac{l\,(A_\delta - 1)}{A_\delta\,(2l-2+d)} \\ \alpha_l &= \frac{(A_\delta l + l - 2 + d)\,(l + A_\delta\,(l-2+d))\,R^{2l+d-2} - l\,(l-2+d)\,(A_\delta - 1)^2}{A_\delta\,(2l-2+d)^2\,R^{2l+d-2}} \\ \beta_l &= \frac{l\,(l + A_\delta\,(l-2+d))\,(A_\delta - 1)\,(1 - R^{2l+d-2})}{A_\delta(2l-2+d)^2} \end{aligned} \tag{3.66}$$

Following [10] we state now the theorem for CALR on spherical shell in $d \geq 3$ dimensions.

Theorem 3.1. *Suppose the spherical shell setting made of two concentric spheres with radii 1 and R in d dimensions, $d \geq 3$. Parameter ϵ_δ is for each area defined by (3.56). For all $l \in \mathbb{N}_0$ and every $\delta > 0$ the following limit with localization index $P_l^\delta = \beta_l/\alpha_l$ holds*

$$\lim_{\delta \to 0} \max_l \frac{|P_l^\delta|}{r^{2l+d-2}} = 0 \tag{3.67}$$

where α_l, β_l are the coefficients of functions $\psi_l(r)$ (3.64) calculated explicitly in (3.66).

Proof. We prove this theorem first for $d = 3$. To find maximum of $|P^l_\delta|/r^{2l+1}$ in l we need to examine the behaviour of such function (we denote it $g(l)$ for short). The explicit formula for this can be calculated

$$g(l) = \left(\frac{R}{r}\right)^{2l+1} \frac{l\sqrt{4+\delta^2}\sqrt{1+\delta^2(1+l)^2}(R^{2l+1}-1)}{\sqrt{\begin{array}{l} 16l^2+32l^3+16l^4+8l^2R^{2l+1}+R^{4l+2}+8lR^{2l+1}+8\delta^2l^2+ \\ 16\delta^2l^3+8\delta^2l^4+6\delta^2lR^{2l+1}+14\delta^2l^2R^{2l+1}+16\delta^2l^3R^{2l+1}+ \\ 8\delta^2l^4R^{2l+1}+\delta^2R^{4l+2}+2\delta^2lR^{4l+2}+2\delta^2l^2R^{4l+2}+\delta^4l^2+ \\ 2\delta^4l^3-\delta^4l^4+2\delta^4l^2R^{2l+1}-4\delta^4l^3R^{2l+1}+2\delta^4l^4R^{2l+1}+ \\ \delta^4l^2R^{4l+2}+2\delta^4l^3R^{4l+2}+\delta^4l^4R^{4l+2} \end{array}}} \tag{3.68}$$

Because of l in the numerator it is obvious that $g(0) = 0$ and now we want to see what happens in the infinity

$$\lim_{l\to+\infty} g(l) = \lim_{l\to+\infty}\left(\frac{R}{r}\right)^{2l+1} \frac{l^2R^{2l+1}}{l^2R^{2l+1}} \frac{\sqrt{4+\delta^2}\sqrt{\frac{1}{l^2}+\delta^2\left(\frac{1}{l}+1\right)^2}\left(1-\frac{1}{R^{2l+1}}\right)}{\sqrt{o(1)+\delta^4}} \tag{3.69}$$

This limit is zero since $r > R$. Because this function is smooth and positive on $(0, +\infty)$ and its values on borders are zero it must have a maximum somewhere in this interval. To find it we will make some estimates on function g.

First we want to enlarge the numerator in (3.68). For this we can estimate $\sqrt{4+\delta^2}\sqrt{1+\delta^2(1+l)^2}$ by $C(1+l)$ (where $C > 0$) and the last bracket easily as $R^{2l+1}-1 < R^{2l+1}$. To achieve maximum values of g we need to reduce its denominator. It is easy to see that the whole square root is greater than $\sqrt{R^{4l+2}(1+\delta^4l^4)}$ and that is obviously greater than $R^{2l+1}(1+\delta^2l^2)$. With denotation $\alpha = R/r$ we have so far

$$g(l) \le \alpha^{2l+1}\frac{lC(1+l)R^{2l+1}}{R^{2l+1}(1+\delta^2l^2)} = \alpha^{2l+1}\frac{lC(1+l)}{1+\delta^2l^2} \tag{3.70}$$

Crucial is now behaviour of function α^{2l+1}. For all l greater than some $l_0 > 0$ we can make an estimate that $\alpha^{2l+1} \le \frac{1}{l^2}$ and then

$$g(l) \le \frac{1}{l^2}\frac{lC(1+l)}{1+\delta^2l^2} \le C\frac{1}{1+\delta^2l^2} \le C\frac{1}{1+\delta^2l_0^2} \tag{3.71}$$

and therefore l_0 is the point in which function g is maximal and its maximum is written above on the right side of estimates (3.71). To include also points in the vicinity of 0 we can take interval $\langle 0, 1\rangle$ where we can estimate α^{2l+1} by constant value 1. Then we have

$$g(l) \leq \frac{lC(1+l)}{1+\delta^2 l^2} \leq \frac{2C}{1+\delta^2} \tag{3.72}$$

so in this case the function g is maximal for $l = 1$ and its maximal value is written above on the right side of estimates (3.72).

We can see that if δ tends to zero in (3.71) or (3.72) we get zero for the limit (3.67) which finishes the proof for $d = 3$.

In higher dimensions the situation is more or less the same. Calculation of the function $f(l) = |P_l^\delta|/r^{2l+d-2}$ in now a little more tedious but the importance is here in the exponent of the denominator which will be for $d > 3$ larger than in three dimensions. Therefore the limit (3.67) holds for all $d \geq 3$. □

We have seen in Section 3.2 that the anomalous localised resonance is related to the fact that P_l^δ can become very large in some area of space which meant the dominance of the β-term over the α-term. This was matched in two dimensions in the circle with radius $R^* = R^{3/2}$. But in the higher dimensions the opposite holds, α-term is dominant over the β-term and therefore no CALR occurs.

Chapter 4

Self-adjoint realisations of indefinite Laplace operator in separable systems

4.1 Rectangular symmetrical setting

Following [6] we consider a geometry which is sketched in Figure 4.1. There is a rectangle $\Omega = (-a/2, a/2) \times (0, b)$ made of two smaller rectangles $\Omega_+ = (0, a/2) \times (0, b)$, $\Omega_- = (-a/2, 0) \times (0, b)$ connected by a line $\mathcal{C} = \{0\} \times (0, b)$.

The operator studied here is defined as follows

$$Af = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix}, \tag{4.1}$$

$$\begin{aligned} \operatorname{dom} A = \Bigg\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : & f_\pm, \Delta f_\pm \in H^2(\Omega_\pm), \\ & f|_{\partial\Omega} = 0, f_+|_{\mathcal{C}} = f_-|_{\mathcal{C}}, \partial_{\mathbf{n}+} f_+|_{\mathcal{C}} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}} \Bigg\} \end{aligned} \tag{4.2}$$

There $f_\pm$ denote the restrictions of a function $f \in H^2(\Omega)$ onto $\Omega_\pm$, and $\partial_{\mathbf{n}\pm}$ are normal derivatives pointing outward of $\Omega_\pm$. Let us mention the boundary conditions in this domain more closely. The first one is condition for functions on the boundary of rectangle. It can be arbitrary so we choose zero for simplicity. The second one expresses that functions from the domain (4.2) must be continuous

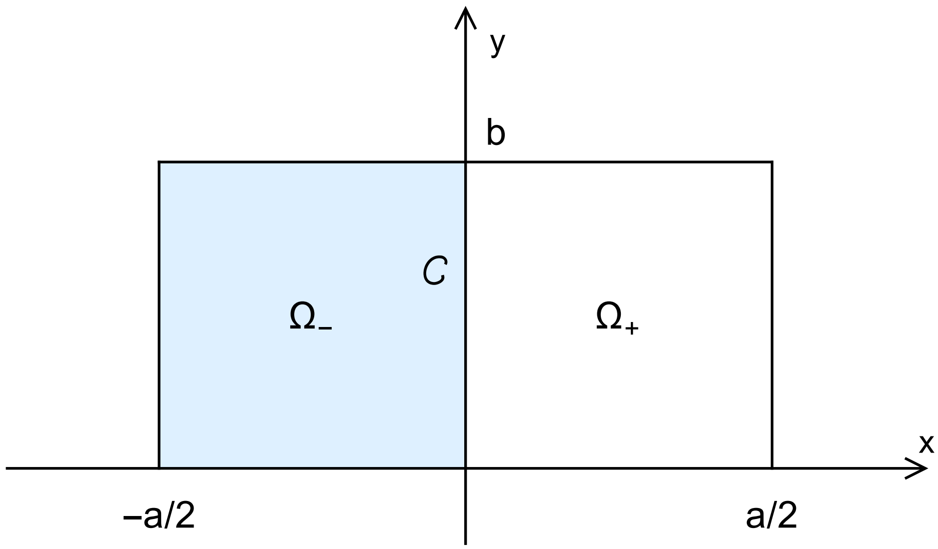

Figure 4.1: A rectangle $\Omega = (-a/2, a/2) \times (0, b)$ in $\mathbb{R}^2$. The left area is of the negative permittivity whereas the right one is of the positive permittivity.

on the transition between Ω_+ and Ω_-. The last condition may look a little odd. It means that the derivative of function from Ω_+ is in the opposite direction than from Ω_-. This condition arises from the sign change at interface $\mathcal{C}$ which is physically motivated by permittivity change between metamaterial and dielectricum.

Main result of this section is stated in the following theorem.

Theorem 4.1. *Operator A defined by relations (4.1), (4.2) is essentially self-adjoint.*

To prove this theorem we use Lemma A.6. Therefore we need to meet the assumptions that A is symmetric and its eigenfunctions form a complete orthonormal set. First we need to find eigenvalues and eigenfunctions of this operator.

Lemma 4.2. *Spectrum of A consists of roots of the equation*

$$\frac{\tanh\left(\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\right)}{\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}} = \frac{\tan\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\right)}{\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}} \tag{4.3}$$

for $\lambda \neq \pm\left(\frac{n\pi}{b}\right)^2$. These solutions can be arranged into the sequence $\{\lambda_{n,m}\}_{m\in\mathbb{Z}}$ for fixed $n \in \mathbb{N}$ in such a way that $\lambda_{n,0} = 0$ (zero is solution even for every $n \in \mathbb{N}$).

The eigenfunctions of A corresponding to $\lambda_{n,m}$ are then obtained by a separation of variables as $f_{n,m}(x,y) = \psi_{n,m\pm}(x)\chi_n(y)$ where $\chi_n(y) = \sqrt{\frac{2}{b}}\sin(n\pi\frac{y}{b})$ and

$$\psi_{n,m+}(x) = \mathcal{N}_{n,m}\sinh\sqrt{\lambda_n + \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\sin\sqrt{\lambda_n - \left(\frac{n\pi}{b}\right)^2}\left(\frac{a}{2} - x\right) \tag{4.4}$$

$$\psi_{n,m-}(x) = \mathcal{N}_{n,m}\sin\sqrt{\lambda_n - \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\sinh\sqrt{\lambda_n + \left(\frac{n\pi}{b}\right)^2}\left(\frac{a}{2} + x\right) \tag{4.5}$$

where domain of $\psi_{n,m+}(x)$ is $\left(0, \frac{a}{2}\right)$ and for $\psi_{n,m-}(x)$ it is $\left(-\frac{a}{2}, 0\right)$. The normalization constants $\mathcal{N}_{n,m}$ are here given by a relation

$$\begin{aligned}\frac{1}{\mathcal{N}_n^2} = \sin^2\sqrt{\lambda_n - \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\left(\frac{\sinh\sqrt{\lambda_n + \left(\frac{n\pi}{b}\right)^2}a}{4\sqrt{\lambda_n + \left(\frac{n\pi}{b}\right)^2}} - \frac{a}{4}\right) + \\ + \sinh^2\sqrt{\lambda_n + \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\left(\frac{a}{4} - \frac{\sin\sqrt{\lambda_n - \left(\frac{n\pi}{b}\right)^2}a}{4\sqrt{\lambda_n - \left(\frac{n\pi}{b}\right)^2}}\right)\end{aligned} \tag{4.6}$$

Proof. We need to find eigenvalues of operator A, i.e. to solve the equations

$$\mp\Delta f_\pm = \lambda f_\pm \tag{4.7}$$

in $\Omega_\pm$ respectively and then to consider the boundary conditions from domain (4.2). For this we use a separation of variables in the same way as in [6]. We decompose any eigenfunction $f \in H^2(\Omega)$ of A into the orthonormal Dirichlet basis $\{\chi_n\}_{n=1}^\infty$, $\chi_n(y) = \sqrt{\frac{2}{b}}\sin(n\pi\frac{y}{b})$.

$$f(x,y) = \sum_{n=1}^\infty \psi_n(x)\chi_n(y) \tag{4.8}$$

From (4.8) we have in fact two equations, one for $f_+(x,y)$ and the second for $f_-(x,y)$ with functions $\psi_{n+}(x)$ and $\psi_{n-}(x)$ on the right side respectively. Thus the equation (4.7) for eigenvalues of the operator A can be rewritten as

$$\mp\sum_{n=1}^\infty\left(\psi''_{n\pm}(x)\chi_n(y) - \psi_{n\pm}(x)\left(\frac{n\pi}{b}\right)^2\chi_n(y)\right) = \lambda\sum_{n=1}^\infty\left(\psi_{n\pm}(x)\chi_n(y)\right) \tag{4.9}$$

$\{\chi_n(y)\}_{n=1}^{\infty}$ forms a complete orthonormal set. Therefore if we multiply these two equations by $\chi_m(y)$ and integrate over y from 0 to b we get two differential equations for fixed n (relabel m and n) and only one variable x

$$-\psi''_{n+}(x) = \left(\lambda - \left(\frac{n\pi}{b}\right)^2\right)\psi_{n+}(x), \qquad x \in \left(0, \frac{a}{2}\right) \tag{4.10}$$

$$\psi''_{n-}(x) = \left(\lambda + \left(\frac{n\pi}{b}\right)^2\right)\psi_{n-}(x), \qquad x \in \left(-\frac{a}{2}, 0\right) \tag{4.11}$$

General solutions for such equations are

$$\psi_{n+} = c_{1+}\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\,x\right) + c_{2+}\cos\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\,x\right) \tag{4.12}$$

$$\psi_{n-} = c_{1-}\sinh\left(\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}\,x\right) + c_{2-}\cosh\left(\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}\,x\right) \tag{4.13}$$

To determine the coefficients $c_{j\pm}$, $j = 1, 2$ we need to use boundary interface conditions from (4.2). For one variable x they have simpler form

$$\psi_{n+}\left(\frac{a}{2}\right) = \psi_{n-}\left(-\frac{a}{2}\right) = 0, \quad \psi_{n+}(0) = \psi_{n-}(0), \quad \psi'_{n+}(0) = -\psi'_{n-}(0) \tag{4.14}$$

We denote the first coefficient c_{1+} as c and then the others are

$$c_{2+} = c_{2-}, \quad c_{1-} = -c\frac{\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}}{\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}}, \quad c_{2-} = -c\tan\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\,\frac{a}{2}\right) \tag{4.15}$$

Concurrently we determine the equation for eigenvalues $\lambda \neq \pm\left(\frac{n\pi}{b}\right)^2$

$$\frac{\tanh\left(\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}\,\frac{a}{2}\right)}{\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}} = \frac{\tan\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\,\frac{a}{2}\right)}{\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}} \tag{4.16}$$

so we can rewrite coefficient c_{1-}as

$$c_{1-} = -c\frac{\tan\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\,\frac{a}{2}\right)}{\tanh\left(\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}\,\frac{a}{2}\right)} \tag{4.17}$$

For short let us denote $\sqrt{\lambda_n \pm \left(\frac{n\pi}{b}\right)^2}$ as $\lambda_{n\pm}$ and for simplification of expressions (4.12) and (4.13) we write coefficient c as

$$c = -\mathcal{N}_n \sinh\left(\sqrt{\lambda + \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\right) \cos\left(\sqrt{\lambda - \left(\frac{n\pi}{b}\right)^2}\frac{a}{2}\right) \tag{4.18}$$

Then we obtain the eigenfunctions $\psi_{n,m+}$ and $\psi_{n,m-}$ in the form (4.4), (4.5). Now we determine the normalization constants $\mathcal{N}_{n,m}$

$$1 = \int_{-a/2}^{a/2} \psi_{n,m}^2(x)dx = \int_{-a/2}^{0} \psi_{n,m-}^2(x)dx + \int_{0}^{a/2} \psi_{n,m+}^2(x)dx$$

$$\begin{aligned}
\frac{1}{\mathcal{N}_{n,m}^2} =& \sin^2\lambda_-\frac{a}{2}\int_{-a/2}^{0}\sinh^2\lambda_+\left(\frac{a}{2}+x\right)dx + \\
&+ \sinh^2\lambda_+\frac{a}{2}\int_{0}^{a/2}\sin^2\lambda_-\left(\frac{a}{2}-x\right)dx = \\
=& \sin^2\lambda_-\frac{a}{2}\left[\frac{\sinh\lambda_+(a+2x)-\lambda_+(a+2x)}{4\lambda_+}\right]_{-a/2}^{0} + \\
&+ \sinh^2\lambda_+\frac{a}{2}\left[\frac{\sin\lambda_-(a-2x)-\lambda_-(a-2x)}{4\lambda_-}\right]_{0}^{a/2} = \\
=& \sin^2\lambda_-\frac{a}{2}\left(\frac{\sinh\lambda_+ a}{4\lambda_+}-\frac{a}{4}\right) + \sinh^2\lambda_+\frac{a}{2}\left(\frac{a}{4}-\frac{\sin\lambda_- a}{4\lambda_-}\right)
\end{aligned}$$

and therefore $\mathcal{N}_{n,m}$ satisfies (4.6). Let us note that this result is a generalization of the same one in [6] where the lengths of rectangle's sides are $a = 2$ and $b = 1$. □

Lemma 4.3. *Eigenfunctions $f_{n,m}$ of the operator A form an orthonormal set in $H^2(\Omega)$.*

Proof. Orthonormality of $\{\chi_n(y)\}$ is obvious and normalization for $\psi_{n,m}$ is obtained in the previous lemma. Therefore one has only to check the orthogonality of the eigenfunctions (4.4) and (4.5). It can be done directly but it would be cumbersome. So we better proceed in a different way. If there is a self-adjoint operator then its eigenfunctions must be orthogonal. That is why we want to prove for every $n \in \mathbb{N}$ the self-adjointness of the one-dimensional operator

$$(A_n\psi_n)(x) = \begin{cases} -\psi_{n+}''(x) + \left(\frac{n\pi}{b}\right)^2\psi_{n+}(x), & x \in \left(0, \frac{a}{2}\right) \\ \psi_{n-}''(x) - \left(\frac{n\pi}{b}\right)^2\psi_{n-}(x), & x \in \left(-\frac{a}{2}, 0\right) \end{cases} \tag{4.19}$$

$$\begin{aligned}\operatorname{dom} A_n = \Bigg\{ \psi_n = \begin{pmatrix}\psi_{n+}\\ \psi_{n-}\end{pmatrix} :& \psi_{n+}, \psi''_{n+} \in H^2((0, \frac{a}{2})),\\ & \psi_{n-}, \psi''_{n-} \in H^2((-\frac{a}{2}, 0)),\\ & \psi_{n+}(\frac{a}{2}) = \psi_{n-}(-\frac{a}{2}) = 0,\\ & \psi_{n+}(0) = \psi_{n-}(0), \psi'_{n+}(0) = -\psi'_{n-}(0) \Bigg\}\end{aligned} \tag{4.20}$$

This is basically the same operator as in our Bachelor thesis [17] thus we use the same method to prove the self-adjointness of this operator A_n.

First we need to show that this operator is symmetric. For all $\psi_n, \varphi_n \in \operatorname{Dom} A_n$ we have

$$\begin{aligned}(\psi_n, A_n\varphi_n) =& \int_{-a/2}^{0} \psi_{n-} \left(\varphi''_{n-} - \left(\frac{n\pi}{b}\right)^2 \varphi_{n-} \right) +\\ &+ \int_0^{a/2} \psi_{n+} \left(-\varphi''_{n+} + \left(\frac{n\pi}{b}\right)^2 \varphi_{n+} \right) =\\ &= - \int_{-a/2}^{0} \psi'_{n-}\varphi'_{n-} - \left(\frac{n\pi}{b}\right)^2 \int_{-a/2}^{0} \psi_{n-}\varphi_{n-} + \left[\psi_{n-}\varphi'_{n-}\right]_{-a/2}^{0} +\\ &+ \int_0^{a/2} \psi'_{n+}\varphi'_{n+} + \left(\frac{n\pi}{b}\right)^2 \int_0^{a/2} \psi_{n+}\varphi_{n+} - \left[\psi_{n+}\varphi'_{n+}\right]_0^{a/2} =\\ &= \int_{-a/2}^{0} \left(\psi''_{n-} - \left(\frac{n\pi}{2}\right)^2 \psi_{n-} \right) \varphi_{n-} + \left[\psi_{n-}\varphi'_{n-}\right]_{-a/2}^{0} - \left[\psi'_{n-}\varphi_{n-}\right]_{-a/2}^{0} +\\ &+ \int_0^{a/2} \left(-\psi''_{n+} + \left(\frac{n\pi}{2}\right)^2 \psi_{n+} \right) \varphi_{n+} - \left[\psi_{n+}\varphi'_{n+}\right]_0^{a/2} + \left[\psi'_{n+}\varphi_{n+}\right]_0^{a/2}\end{aligned}$$

The boundary terms gives zero because of the boundary conditions and so we get for all ψ_n, φ_n equality $(\psi_n, A_n\varphi_n) = (A_n\psi_n, \varphi_n)$ expressing that operator A_n is symmetric.

Now we need the domain of A_n and its adjoint operator to be the same. According to Definition A.4 we are interested in relation $(\varphi_n, A_n\psi_n) = (A_n^*\varphi_n, \psi_n)$ for $\psi_n \in \operatorname{Dom} A_n$ and $\varphi_n \in \operatorname{Dom} A_n^*$. But first we need to check whether the second derivative of φ_n truly exists. For this we consider a restriction of A_n

$$(\dot{A}_n\psi_n)(x) = \begin{cases} -\psi''_{n+}(x) + \left(\frac{n\pi}{b}\right)^2 \psi_{n+}(x), & x \in \left(0, \frac{a}{2}\right)\\ \psi''_{n-}(x) - \left(\frac{n\pi}{b}\right)^2 \psi_{n-}(x), & x \in \left(-\frac{a}{2}, 0\right)\end{cases} \tag{4.21}$$

$$\begin{aligned}\operatorname{dom}\dot{A}_n = \Bigg\{\psi_n = \begin{pmatrix}\psi_{n+}\\ \psi_{n-}\end{pmatrix} :& \psi_{n+}, \psi''_{n+} \in H^2((0,\frac{a}{2})),\\ & \psi_{n-}, \psi''_{n-} \in H^2((-\frac{a}{2},0)),\\ & \psi_{n+}(\frac{a}{2}) = \psi_{n-}(-\frac{a}{2}) = 0,\\ & \psi_{n+}(0) = \psi_{n-}(0) = \psi'_{n+}(0) = -\psi'_{n-}(0) = 0\Bigg\}\end{aligned} \tag{4.22}$$

Adjoint operator to this restriction is then

$$(\dot{A}_n^*\psi_n)(x) = \begin{cases} -\psi''_{n+}(x) + \left(\frac{n\pi}{b}\right)^2 \psi_{n+}(x), & x \in \left(0, \frac{a}{2}\right)\\ \psi''_{n-}(x) - \left(\frac{n\pi}{b}\right)^2 \psi_{n-}(x), & x \in \left(-\frac{a}{2}, 0\right)\end{cases} \tag{4.23}$$

$$\begin{aligned}\operatorname{dom}\dot{A}_n^* = \Bigg\{\psi_n = \begin{pmatrix}\psi_{n+}\\ \psi_{n-}\end{pmatrix} :& \psi_{n+}, \psi''_{n+} \in H^2((0,\frac{a}{2})),\\ & \psi_{n-}, \psi''_{n-} \in H^2((-\frac{a}{2},0)),\\ & \psi_{n+}(\frac{a}{2}) = \psi_{n-}(-\frac{a}{2}) = 0\Bigg\}\end{aligned} \tag{4.24}$$

Since $\dot{A}_n$ is a restriction of A_n we have $\dot{A}_n \subset A_n$ (in the sense that $\operatorname{Dom}\dot{A}_n \subset \operatorname{Dom} A_n$). For their adjointness it holds $A_n^* \subset \dot{A}_n^*$. Adding well known fact that adjoint operator is always the extension of the original one, i.e. $A_n \subset A_n^*$ we have all together that

$$\dot{A}_n \subset A_n \subset A_n^* \subset \dot{A}_n^* \tag{4.25}$$

and thus $\varphi''_n \in \operatorname{Dom} A_n^*$ exists since it exists in its superset $\operatorname{Dom}\dot{A}_n^*$.

Now we are able to use twice integration by parts in the scalar product

$$\begin{aligned}(\varphi_n, A_n\psi_n) = &\int_{-a/2}^{0} \varphi_{n-}\left(\psi''_{n-} - \left(\frac{n\pi}{b}\right)^2 \psi_{n-}\right) + \\ &+ \int_0^{a/1} \varphi_{n+}\left(-\psi''_{n+} + \left(\frac{n\pi}{b}\right)^2 \psi_{n+}\right) =\end{aligned}$$

$$
\begin{aligned}
&= \int_{-a/2}^{0} \left(\varphi_{n-}'' - \left(\frac{n\pi}{b} \right)^2 \varphi_{n-} \right) \psi_{n-} + \\
&\quad + \int_{0}^{a/1} \left(-\varphi_{n+}'' + \left(\frac{n\pi}{b} \right)^2 \varphi_{n+} \right) \psi_{n+} + \\
&\quad + \left[\varphi_{n-} \psi_{n-}' \right]_{-a/2}^{0} - \left[\varphi_{n+} \psi_{n+}' \right]_{0}^{a/2} - \left[\varphi_{n-}' \psi_{n-} \right]_{-a/2}^{0} + \left[\varphi_{n+}' \psi_{n+} \right]_{0}^{a/2}
\end{aligned}
$$

which we want to be equal to $(A_n^* \varphi_n, \psi_n)$. It is obvious that we need again the sum of boundary terms to be zero. We rewrite this sum with use of boundary conditions in $\mathrm{Dom}\, A_n$ for ψ_n and also use that $\mathrm{Dom}\, A_n^* \subset \mathrm{Dom}\, \dot{A}_n^*$ which implies conditions $\varphi_{n-}(-\frac{a}{2}) = \varphi_{n+}(\frac{a}{2}) = 0$

$$
(\varphi_{n-}(0^-) - \varphi_{n+}(0^+))\psi_{n-}'(0^-) - (\varphi_{n-}'(0^-) + \varphi_{n+}'(0^+))\psi_{n-}(0^-) = 0 \quad (4.26)
$$

From the definition of the adjoint operator this must be true for all $\psi_n \in \mathrm{Dom}\, A_n$. Therefore we choose some specific functions from the domain so this condition would be met.

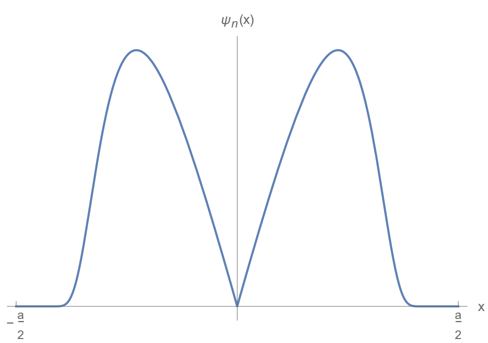

Figure 4.2: $\psi_n(x)$ vanishing on the boundary with nonzero derivatives at 0

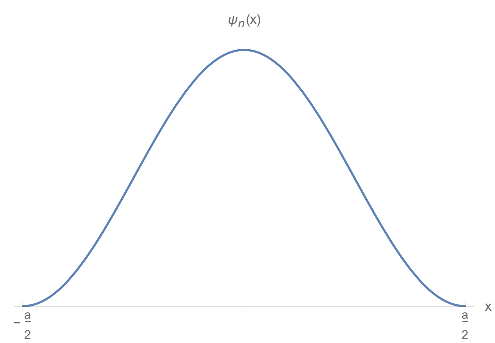

Figure 4.3: $\psi_n(x)$ with zero derivative at $x = 0$ and vanishing on the boundary

First we take $\psi'(-a/2) = \psi'(a/2) = 0$, thus these functions vanishes at the boundaries of interval $(-a/2, a/2)$. In the function on Figure 4.2 there are finite arbitrary nonzero derivatives from the left and right in zero with zero value at this point. Therefore (4.26) results now in the condition $\varphi_{n-}(0^-) = \varphi_{n+}(0^+)$. The situation with the second function on Figure 4.3 is opposite, it has nonzero value at zero but the derivative is here zero. Therefore we obtain $\varphi_{n-}'(0^-) = -\varphi_{n+}'(0^+)$.

These resulting relations together with the boundary conditions from superset $\mathrm{Dom}\, \dot{A}_n^*$ are clearly the same boundary conditions as for the original operator A_n.

It follows that domains of this operator and A_n^* are the same and therefore by definition operator A_n is self-adjoint.

Now we finally know that $\{\psi_{n,m}\}_{n=1}^{\infty}$ and $\{\chi_n\}_{n=1}^{\infty}$ are orthonormal sets in $H^2((-a/2, a/2))$ and $H^2((0, b))$ respectively. It is easy to see that then $f_{n,m}(x, y)$ also form an orthonormal set in $H^2(\Omega)$

$$\int_\Omega f_{n,m}(x, y) f_{\tilde{n},\tilde{m}}(x, y) dxdy = \int_{-a/2}^{a/2} \psi_{n,m}(x)\psi_{\tilde{n},\tilde{m}}(x)dx \int_0^b \chi_n(y)\chi_{\tilde{n}}(y)dy =$$
$$= \delta_{n\tilde{n}}\delta_{m\tilde{m}}$$

□

Lemma 4.4. *The functions $f_{n,m}$ with $n \in \mathbb{N}, m \in \mathbb{Z}$ form a complete orthonormal set in $H^2(\Omega)$.*

Proof. We will proceed as in [19] (V., Ex. 1.10). We need to show that if $(w, f_{n,m}) = 0$ for all n, m and any function w then $w = 0$. We define

$$w_{n,m}(y) = \int_{-a/2}^{a/2} w(x, y)\psi_{n,m}(x)dx \tag{4.27}$$

For such function we have by the Schwarz inequality that

$$|w_{n,m}(y)|^2 \le \int_{-a/2}^{a/2} |w(x, y)|^2 \, dx \int_{-a/2}^{a/2} |\psi_{n,m}(x)|^2 \, dx = \int_{-a/2}^{a/2} |w(x, y)|^2 \, dx \tag{4.28}$$

where the last equality is because of the fact that $\{\psi_{n,m}(x)\}$ form an orthonormal set. Also we have

$$\int_0^b |w_{n,m}(y)|^2 \, dy \le \int_\Omega |w(x, y)|^2 \, dxdy = \|w\|^2 \tag{4.29}$$

The inequalities (4.28) and (4.29) imply that $w_{n,m} \in H^2((0, b))$ thus we can write

$$(w, f_{n,m}) = (w, \psi_{n,m}\chi_n) = (w_{n,m}, \chi_n) \tag{4.30}$$

and therefore $(w, f_{n,m}) = 0$ for all n, m truly implies that $(w_{n,m}, \chi_n) = 0$ and hence $w_{n,m} = 0$ by the completeness of $\{\chi_n\}$. □

Proof of Theorem 4.1. "Symmetricity of A, all assumption of Lemma A.6 matched" Finally we have matched all assumptions of Lemma A.6 so we can apply it to our operator A since we have found the complete orthonormal set of eigenfunctions $f_{n,m}$ that all lie in $\operatorname{dom} A$. This proves that A is essentially self-adjoint operator. □

4.2 Rotationally symmetrical setting dielectricum-metamaterial-dielectricum

As we have seen in Chapter 3 usually a different geometry than the rectangular one from the previous section is considered when talking about metamaterial cloaking. And that the most interest is in the rotationally symmetrical setting such as the one sketched in Figure 4.4. There are three areas, one with negative

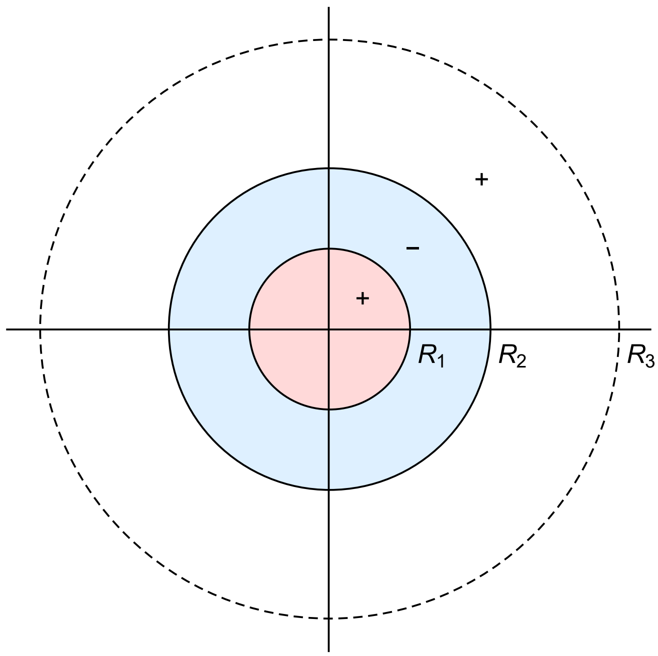

Figure 4.4: A sketch of a rotationally symmetrical setting. Signs $+$ and $-$ denote regions with positive and negative permittivity.

permittivity (metamaterial) and two with positive permittivities (dielectrics). For simplification we consider the same value of the positive permittivities. In the material it is supposed to be the same but with opposite sign. The reason for radius R_3 is that we would not find any eigenfunctions if we considered the whole space $\mathbb{R}^2$ rather than this bounded geometry. More practical explanation is that it is natural to consider that permittivity is constant only in some area around the cloaking device.

Since all balls here are centred at the origin we will write shortly B_r for a ball with radius r and centre 0. Conventionally we take $R_0 = 0$. Also we denote the three regions from the Figure 4.4 as $\mathcal{B}_1 = B_{R_1}, \mathcal{B}_2 = B_{R_2} \setminus B_{R_1}$ and $\mathcal{B}_3 = B_{R_3} \setminus B_{R_2}$. The corresponding operator for such geometry is then

$$Cf = \begin{pmatrix} -\Delta f_1 \\ \Delta f_2 \\ -\Delta f_3 \end{pmatrix}, \tag{4.31}$$

$$\begin{aligned} \operatorname{dom} C = \Bigg\{ f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad &: f_j, \Delta f_j \in H^2((R_{j-1}, R_j), r dr) \times H^2(S^1, d\varphi), \\ j = 1, 2, 3, \quad & f_3|_{\partial B_{R_3}} = 0, f_1|_{\partial B_{R_1}} = f_2|_{\partial B_{R_1}}, f_2|_{\partial B_{R_2}} = f_3|_{\partial B_{R_2}}, \\ & \partial_{\mathbf{n}+} f_1|_{\partial B_{R_1}} = \partial_{\mathbf{n}-} f_2|_{\partial B_{R_1}}, \partial_{\mathbf{n}+} f_2|_{\partial B_{R_2}} = \partial_{\mathbf{n}-} f_3|_{\partial B_{R_2}} \Bigg\} \end{aligned} \tag{4.32}$$

Opposite the rectangular case there is a change in the Hilbert space. We take here naturally polar coordinates and due to this fact the measure of the space is changed. Again $\partial_{\mathbf{n}\pm}$ are normal derivatives pointing outward of $\mathcal{B}_j$. The boundary conditions have the same character as before. Functions on the boundary must be zero (it can be chosen arbitrary so we take zero for simplification), they must be continuous on the interfaces between $\mathcal{B}_j$ and their derivatives on these borders must point in opposite directions.

The main result of this section is very similar to the previous one.

Theorem 4.5. *Operator C defined by relations (4.31), (4.32) is essentially self-adjoint.*

To prove it we proceed in the same way as in Section 4.1.

Lemma 4.6. *Spectrum of C consists of roots of the equation*

$$\frac{D}{C} = \frac{F}{E} \tag{4.33}$$

where we use shorter denotation for

$$A = \frac{J_n(-i\lambda_1)}{J_n(\lambda_1)} - i\frac{J_{n-1}(-i\lambda_1) - J_{n+1}(-i\lambda_1)}{J_{n-1}(\lambda_1) - J_{n+1}(\lambda_1)} \tag{4.34}$$

$$B = \frac{Y_n(-i\lambda_1)}{J_n(\lambda_1)} - i\frac{Y_{n-1}(-i\lambda_1) - Y_{n+1}(-i\lambda_1)}{J_{n-1}(\lambda_1) - J_{n+1}(\lambda_1)} \tag{4.35}$$

$$C = \frac{Y_n(\lambda_2)}{J_n(\lambda_2)} - \frac{Y_n(\lambda_3)}{J_n(\lambda_3)} \tag{4.36}$$

$$D = \frac{Y_n(-i\lambda_2)}{J_n(\lambda_2)} - \frac{B}{A}\frac{J_n(-i\lambda_2)}{J_n(\lambda_2)} \tag{4.37}$$

$$E = \frac{Y_{n-1}(\lambda_2) - Y_{n+1}(\lambda_2)}{J_{n-1}(\lambda_2) - J_{n+1}(\lambda_2)} - \frac{Y_n(\lambda_3)}{J_n(\lambda_3)} \tag{4.38}$$

$$F = \frac{i\left(Y_{n-1}(-i\lambda_2) - Y_{n+1}(-i\lambda_2) - \frac{B}{A}\left(J_{n-1}(-i\lambda_2) - J_{n+1}(-i\lambda_2)\right)\right)}{J_{n-1}(\lambda_2) - J_{n+1}(\lambda_2)} \tag{4.39}$$

Here λ_j *denotes product* $R_j\sqrt{\lambda}$ *for* $j = 1, 2, 3$. *The eigenfunctions of* C *are obtained by a separation of variables as*

$$f(r, \varphi) = \sum_{n=1}^{\infty} g_n(r)h_n(\varphi) \tag{4.40}$$

where $h_n(\varphi) = \frac{1}{\sqrt{\pi}}\sin(n\varphi)$ *and* $g_n(r)$ *has three different forms according to radius* r *as follows*

$$g_{1n}(r) = \mathcal{N}_n \frac{Y_n(-i\lambda_1) - \frac{B}{A}J_n(-i\lambda_1)}{J_n(\lambda_1)} J_n(\sqrt{\lambda}r), \qquad r \in (0, R_1) \tag{4.41}$$

$$g_{2n}(r) = \mathcal{N}_n \left(Y_n(-i\sqrt{\lambda}r) - \frac{B}{A}J_n(-i\sqrt{\lambda}r)\right), \qquad r \in (R_1, R_2) \tag{4.42}$$

$$g_{3n}(r) = \mathcal{N}_n \frac{D}{C}\left(Y_n(\sqrt{\lambda}r) - \frac{Y_n(\lambda_3)}{J_n(\lambda_3)}J_n(\sqrt{\lambda}r)\right), \qquad r \in (R_2, R_3) \tag{4.43}$$

The normalization constants $\mathcal{N}_n$ *can be calculated for a specific choice of diame-*

ters R_1, R_2, R_3 and positive integer n from relation

$$\frac{1}{|\mathcal{N}_n|^2} = \left|\frac{Y_n(-i\lambda_1) - \frac{B}{A}J_n(-i\lambda_1)}{J_n(\lambda_1)}\right|^2 \int_0^{R_1} \left|J_n(\sqrt{\lambda}r)\right|^2 rdr +$$
$$+ \int_{R_1}^{R_2} \left|(Y_n(-i\sqrt{\lambda}r)\right|^2 rdr - \left|\frac{B}{A}\right|^2 \int_{R_1}^{R_2} \left|J_n(-i\sqrt{\lambda}r)\right|^2 rdr +$$
$$+ \left|\frac{D}{C}\right|^2 \int_{R_2}^{R_3} \left|Y_n(\sqrt{\lambda}r)\right|^2 rdr - \left|\frac{DY_n(\lambda_3)}{CJ_n(\lambda_3)}\right|^2 \int_{R_2}^{R_3} \left|J_n(\sqrt{\lambda}r)\right|^2 rdr \tag{4.44}$$

Proof. The Laplace operator in polar coordinates is given as

$$-\Delta f(r,\varphi) = -\frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} \tag{4.45}$$

Therefore three equations for eigenvalues λ of the operator C are

$$\begin{aligned} -\sum_{n=1}^{\infty}\left(g''_{jn}(r) + \frac{1}{r}g'_{jn}(r) - \frac{n^2}{r^2}g_{jn}(r)\right)h_n(\varphi) &= \lambda\sum_{n=1}^{\infty}(g_{jn}(r)h_n(\varphi)), j = 1,3 \\ \sum_{n=1}^{\infty}\left(g''_{2n}(r) + \frac{1}{r}g'_{2n}(r) - \frac{n^2}{r^2}g_{2n}(r)\right)h_n(\varphi) &= \lambda\sum_{n=1}^{\infty}(g_{2n}(r)h_n(\varphi)) \end{aligned} \tag{4.46}$$

where $g_{jn}, j = 1,2,3$, denotes functions in the interval (R_{j-1}, R_j) respectively and $h_n(\varphi)$ is given as stated in the theorem. Notice that these are spherical harmonics that are usually denoted by Y. However, since we work mainly with Bessel functions, and Bessel functions of the second kind are also denoted by Y, we use h_n instead. $\{h_n(\varphi)\}_{n=1}^{\infty}$ form a complete orthonormal set so if we multiply these equations by $h_m(\varphi)$ and integrate over angle φ from 0 to 2π we get three differential equations for fixed n and one variable r

$$g''_{1n}(r) + \frac{g'_{1n}(r)}{r} + \left(\lambda - \frac{n^2}{r^2}\right)g_{1n}(r) = 0, \qquad r \in (0, R_1) \tag{4.47}$$

$$g''_{2n}(r) + \frac{g'_{2n}(r)}{r} - \left(\lambda + \frac{n^2}{r^2}\right)g_{2n}(r) = 0, \qquad r \in (R_1, R_2) \tag{4.48}$$

$$g''_{3n}(r) + \frac{g'_{3n}(r)}{r} + \left(\lambda - \frac{n^2}{r^2}\right)g_{3n}(r) = 0, \qquad r \in (R_2, R_3) \tag{4.49}$$

These are Sturm-Liouville equations and their solutions are given in terms of Bessel functions (see Appendix A.2) as

$$g_{1n}(r) = c_1 J_n(\sqrt{\lambda} r), \qquad r \in (0, R_1) \tag{4.50}$$
$$g_{2n}(r) = c_2 J_n(-i\sqrt{\lambda} r) + c_3 Y_n(-i\sqrt{\lambda} r), \qquad r \in (R_1, R_2) \tag{4.51}$$
$$g_{3n}(r) = c_4 J_n(\sqrt{\lambda} r) + c_5 Y_n(\sqrt{\lambda} r), \qquad r \in (R_2, R_3) \tag{4.52}$$

where J_n are Bessel functions of the first kind and Y_n are Bessel functions of the second kind. Because Y_n is singular at the origin we do not consider it as a part of solution g_{1n} since it is defined on $(0, R_1)$.

As before we need to determine the coefficients c_k, $k = 1, 2, 3, 4, 5$ with use of our boundary conditions which are for functions $g_{jn}(r), j = 1, 2, 3$ as follows

$$\begin{aligned} g_{3n}(R_3) = 0, \quad & g_{1n}(R_1) = g_{2n}(R_1), & g_{2n}(R_2) = g_{3n}(R_2), \\ & g'_{1n}(R_1) = -g'_{2n}(R_1), & g'_{2n}(R_2) = -g'_{3n}(R_2) \end{aligned} \tag{4.53}$$

With $\lambda_j = R_j\sqrt{\lambda}$ for $j = 1, 2, 3$ we get equations:

$$c_4 J_n(\lambda_3) + c_5 Y_n(\lambda_3) = 0 \tag{4.54}$$
$$c_1 J_n(\lambda_1) = c_2 J_n(-i\lambda_1) + c_3 Y_n(-i\lambda_1) \tag{4.55}$$
$$c_4 J_n(\lambda_2) + c_5 Y_n(\lambda_2) = c_2 J_n(-i\lambda_2) + c_3 Y_n(-i\lambda_2) \tag{4.56}$$
$$\begin{aligned} & c_1 \left(J_{n-1}(\lambda_1) - J_{n+1}(\lambda_1)\right) = \\ & = ic_2 \left(J_{n-1}(-i\lambda_1) - J_{n+1}(-i\lambda_1)\right) + ic_3 \left(Y_{n-1}(-i\lambda_1) - Y_{n+1}(-i\lambda_1)\right) \end{aligned} \tag{4.57}$$
$$\begin{aligned} & c_4 \left(J_{n-1}(\lambda_2) - J_{n+1}(\lambda_2)\right) + c_5 \left(Y_{n-1}(\lambda_2) - Y_{n+1}(\lambda_2)\right) = \\ & = ic_2 \left(J_{n-1}(-i\lambda_2) - J_{n+1}(-i\lambda_2)\right) + ic_3 \left(Y_{n-1}(-i\lambda_2) - Y_{n+1}(-i\lambda_2)\right) \end{aligned} \tag{4.58}$$

With notation (4.34) – (4.39) and if we take c_3 as a normalization factor $\mathcal{N}_n$ we have for our constants

$$c_1 = \mathcal{N}_n \frac{Y_n(-i\lambda_1) - \frac{B}{A} J_n(-i\lambda_1)}{J_n(\lambda_1)} \tag{4.59}$$
$$c_2 = -\mathcal{N}_n \frac{B}{A} \tag{4.60}$$
$$c_3 = \mathcal{N}_n \tag{4.61}$$
$$c_4 = -\mathcal{N}_n \frac{D}{C} \frac{Y_n(\lambda_3)}{J_n(\lambda_3)} \tag{4.62}$$
$$c_5 = \mathcal{N}_n \frac{D}{C} \tag{4.63}$$

and the equation for the eigenvalues is in this short notation really written as (4.33). Then the solutions of our Bessel equations are indeed of the form (4.41) – (4.43). Normalization factor $\mathcal{N}_n$ is to be determined from the equation

$$\begin{aligned} 1 = \int_0^{R_3} |g_n(r)|^2\, rdr = \int_0^{R_1} |g_{1n}(r)|^2\, rdr + \int_{R_1}^{R_2} |g_{2n}(r)|^2\, rdr + \\ + \int_{R_2}^{R_3} |g_{3n}(r)|^2\, rdr \end{aligned} \tag{4.64}$$

from which results relation (4.44). □

Lemma 4.7. *Eigenfunctions f_n of the operator C form an orthonormal set in $H^2(B_{R_3})$.*

Proof. Orthogonality of the eigenfunctions will be checked by the same method as in the previous section. We take this one-dimensional operator

$$(C_n g_n)(x) = \begin{cases} -g''_{1n}(r) - \frac{g'_{1n}(r)}{r} + \frac{n^2}{r^2} g_{1n}(r), & r \in (0, R_1) \\ g''_{2n}(r) + \frac{g'_{2n}(r)}{r} - \frac{n^2}{r^2} g_{2n}(r), & r \in (R_1, R_2) \\ -g''_{3n}(r) - \frac{g'_{3n}(r)}{r} + \frac{n^2}{r^2} g_{3n}(r), & r \in (R_2, R_3) \end{cases} \tag{4.65}$$

$$\begin{aligned} \operatorname{Dom} C_n = \Bigg\{ g_n = \begin{pmatrix} g_{1n} \\ g_{2n} \\ g_{3n} \end{pmatrix} : & g_{jn}, g''_{jn} \in H^2((R_{j-1}, R_j), rdr), j = 1,2,3 \\ & g_{1n}(R_1) = g_{2n}(R_1), g_{2n}(R_2) = g_{3n}(R_2), \\ & g'_{1n}(R_1) = -g'_{2n}(R_1), g'_{2n}(R_2) = -g'_{3n}(R_2), \\ & g_{3n}(R_3) = 0 \Bigg\} \end{aligned} \tag{4.66}$$

The next step is to prove that C_n is symmetric so for all $g_n, \tilde{g}_n \in \operatorname{Dom}(C_n)$

$$\begin{aligned} (g_n, C_n \tilde{g}_n) = \int_0^{R_1} g_{1n}(r) \left(-\tilde{g}''_{1n}(r) - \frac{\tilde{g}'_{1n}(r)}{r} + \frac{n^2}{r^2} \tilde{g}_{1n}(r) \right) rdr + \\ + \int_{R_1}^{R_2} g_{2n}(r) \left(\tilde{g}''_{2n}(r) + \frac{\tilde{g}'_{2n}(r)}{r} - \frac{n^2}{r^2} \tilde{g}_{2n}(r) \right) rdr + \\ + \int_{R_1}^{R_3} g_{3n}(r) \left(-\tilde{g}''_{3n}(r) - \frac{\tilde{g}'_{3n}(r)}{r} + \frac{n^2}{r^2} \tilde{g}_{3n}(r) \right) rdr \end{aligned}$$

We calculate for example the first integral (others result similarly). Using twice integration by parts we obtain

$$\int_0^{R_1}\left(-g''_{1n}\tilde{g}'_{1n}-\frac{g'_{1n}}{r}+\frac{n^2}{r^2}g_{1n}\right)\tilde{g}_{1n}rdr-[rg_{1n}\tilde{g}'_{1n}]_0^{R_1}+[rg'_{1n}\tilde{g}_{1n}]_0^{R_1}$$

and so for all terms together we get

$$\begin{aligned}(g_n, C_n\tilde{g}_n) = (C_n g_n, \tilde{g}_n) - [rg_{1n}\tilde{g}'_{1n}]_0^{R_1} + [rg'_{1n}\tilde{g}_{1n}]_0^{R_1} + [rg_{2n}\tilde{g}'_{2n}]_{R_1}^{R_2} - \\ - [rg'_{2n}\tilde{g}_{2n}]_{R_1}^{R_2} - [rg_{3n}\tilde{g}'_{3n}]_{R_2}^{R_3} + [rg'_{3n}\tilde{g}_{3n}]_{R_2}^{R_3}\end{aligned}$$

and since the boundary conditions provide here that the sum of boundary terms is zero we proved that C_n is a symmetric operator.

Its adjoint C_n^* is determined by the relation $(\tilde{g}_n, C_n g_n) = (C_n^*\tilde{g}_n, g_n)$ for any $g_n \in \mathrm{Dom}(C_n)$ and $\tilde{g}_n \in \mathrm{Dom}(C_n^*)$. Therefore we need to know whether there exists the second derivative of $\tilde{g}_n$. Repeating procedure from the previous section we take a restriction of C_n and its adjoint operator

$$(\dot{C}_n g_n)(r) = \begin{cases} -g''_{1n}(r) - \frac{g'_{1n}(r)}{r} + \frac{n^2}{r^2}g_{1n}(r), & r \in (0, R_1) \\ g''_{2n}(r) + \frac{g'_{2n}(r)}{r} - \frac{n^2}{r^2}g_{2n}(r), & r \in (R_1, R_2) \\ -g''_{3n}(r) - \frac{g'_{3n}(r)}{r} + \frac{n^2}{r^2}g_{3n}(r), & r \in (R_2, R_3) \end{cases} \tag{4.67}$$

$$\begin{aligned}\mathrm{dom}\,\dot{C}_n = \Bigg\{ g_n = \begin{pmatrix} g_{n1} \\ g_{n2} \\ g_{n3}\end{pmatrix} :& g_{jn}, g''_{jn} \in H^2((R_{j-1}, R_j), rdr), j = 1,2,3 \\ & g_{1n}(R_1) = g_{2n}(R_1) = g_{2n}(R_2) = g_{3n}(R_2) = 0, \\ & g'_{1n}(R_1) = g'_{2n}(R_1) = g'_{2n}(R_2) = g'_{3n}(R_2) = 0, \\ & g_{3n}(R_3) = 0 \Bigg\}\end{aligned} \tag{4.68}$$

$$(\dot{C}_n^* g_n)(r) = \begin{cases} -g''_{1n}(r) - \frac{g'_{1n}(r)}{r} + \frac{n^2}{r^2}g_{1n}(r), & r \in (0, R_1) \\ g''_{2n}(r) + \frac{g'_{2n}(r)}{r} - \frac{n^2}{r^2}g_{2n}(r), & r \in (R_1, R_2) \\ -g''_{3n}(r) - \frac{g'_{3n}(r)}{r} + \frac{n^2}{r^2}g_{3n}(r), & r \in (R_2, R_3) \end{cases} \tag{4.69}$$

$$\operatorname{dom}\dot{C}_n^* = \left\{ g_n = \begin{pmatrix} g_{n1} \\ g_{n2} \\ g_{n3} \end{pmatrix} : g_{jn}, g_{jn}'' \in H^2((R_{j-1}, R_j), rdr), j = 1,2,3 \right. \\ \left. g_{3n}(R_3) = 0 \right\} \tag{4.70}$$

From this we can show again that $\dot{C}_n \subset C_n \subset C_n^* \subset \dot{C}_n^*$ and therefore $\tilde{g}_n''$ truly exists. This enable us to do twice integration by parts in the expression

$$\begin{aligned}
(\tilde{g}_n, C_n g_n) = &\int_0^{R_1} \tilde{g}_{1n}(r) \left(-g_{1n}''(r) - \frac{g_{1n}'(r)}{r} + \frac{n^2}{r^2} g_{1n}(r) \right) rdr + \\
&+ \int_{R_1}^{R_2} \tilde{g}_{2n}(r) \left(g_{2n}''(r) + \frac{g_{2n}'(r)}{r} - \frac{n^2}{r^2} g_{2n}(r) \right) rdr + \\
&+ \int_{R_1}^{R_3} \tilde{g}_{3n}(r) \left(-g_{3n}''(r) - \frac{g_{3n}'(r)}{r} + \frac{n^2}{r^2} g_{3n}(r) \right) rdr = \\
&= \int_0^{R_1} \left(-\tilde{g}_{1n}''(r) - \frac{\tilde{g}_{1n}'(r)}{r} + \frac{n^2}{r^2} \tilde{g}_{1n}(r) \right) g_{1n}(r) rdr + \\
&+ \int_{R_1}^{R_2} \left(\tilde{g}_{2n}''(r) + \frac{\tilde{g}_{2n}'(r)}{r} - \frac{n^2}{r^2} \tilde{g}_{2n}(r) \right) g_{2n}(r) rdr + \\
&+ \int_{R_1}^{R_3} \left(-\tilde{g}_{3n}''(r) - \frac{\tilde{g}_{3n}'(r)}{r} + \frac{n^2}{r^2} \tilde{g}_{3n}(r) \right) g_{3n}(r) rdr - \\
&- [r\tilde{g}_{1n} g_{1n}']_0^{R_1} + [r\tilde{g}_{1n}' g_{1n}]_0^{R_1} + [r\tilde{g}_{2n} g_{2n}']_{R_1}^{R_2} - \\
&- [r\tilde{g}_{2n}' g_{2n}]_{R_1}^{R_2} - [r\tilde{g}_{3n} g_{3n}']_{R_2}^{R_3} + [r\tilde{g}_{3n}' g_{3n}]_{R_2}^{R_3}
\end{aligned}$$

which we need to be equal to $(C_n^* \tilde{g}_n, g_n)$ in the way that following sum must be zero

$$\begin{aligned}
&- [r\tilde{g}_{1n} g_{1n}']_0^{R_1} + [r\tilde{g}_{1n}' g_{1n}]_0^{R_1} + [r\tilde{g}_{2n} g_{2n}']_{R_1}^{R_2} - \\
&- [r\tilde{g}_{2n}' g_{2n}]_{R_1}^{R_2} - [r\tilde{g}_{3n} g_{3n}']_{R_2}^{R_3} + [r\tilde{g}_{3n}' g_{3n}]_{R_2}^{R_3} = 0
\end{aligned} \tag{4.71}$$

Now we take again some specific functions g_n from $\operatorname{Dom}(C_n)$ which help us to determine conditions on $\operatorname{Dom}(C_n^*)$. Moreover we have again boundary condition $g_{3n}(R_3) = 0$ since $\operatorname{Dom} C_n^* \subset \operatorname{Dom} \dot{C}_n^*$. These are sketched on Figures 4.5 and 4.6 where the nonzero terms are emphasized and which conditions it implies.

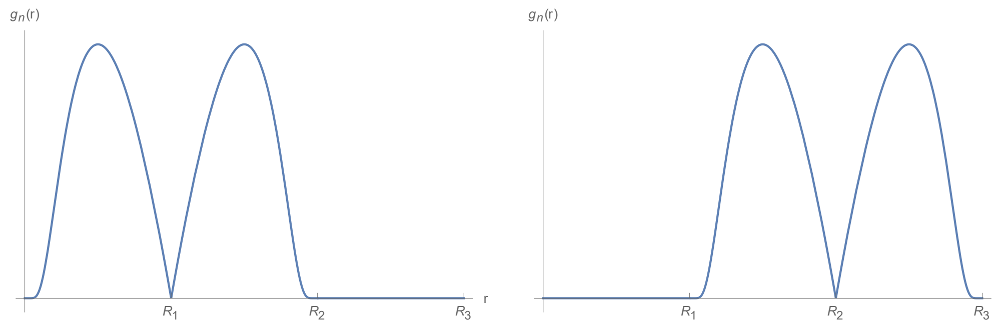

Figure 4.5: The functions with nonzero derivatives but zero values at points R_1 and R_2. On the left: $-R_1\tilde{g}_{1n}(R_1)g'_{1n}(R_1) - R_1\tilde{g}_{2n}(R_1)g'_{2n}(R_1) = 0$, $g'_{1n}(R_1) = -g'_{2n}(R_1) \Rightarrow \tilde{g}_{1n}(R_1) = \tilde{g}_{2n}(R_1)$. On the right: $R_2\tilde{g}_{2n}(R_2)g'_{2n}(R_2) + R_2\tilde{g}_{3n}(R_2)g'_{3n}(R_2) = 0, g'_{2n}(R_2) = -g'_{3n}(R_2) \Rightarrow \tilde{g}_{2n}(R_2) = \tilde{g}_{3n}(R_2)$

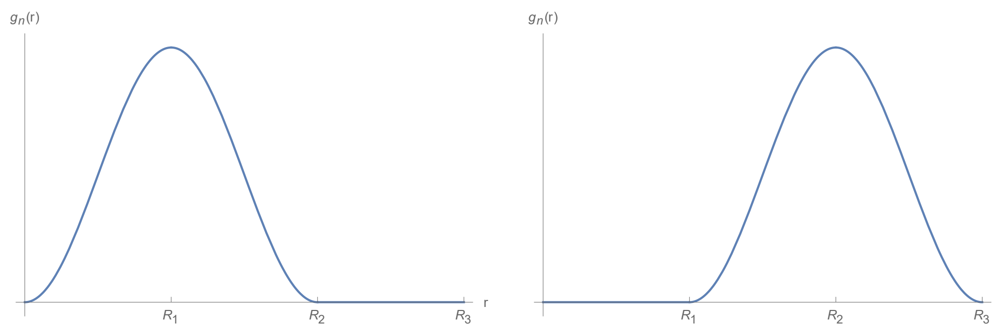

Figure 4.6: The functions with zero derivatives and nonzero values at points R_1 and R_2. On the left: $R_1\tilde{g}'_{1n}(R_1)g_{1n}(R_1) + R_1\tilde{g}'_{2n}(R_1)g_{2n}(R_1) = 0, g_{1n}(R_1) = g_{2n}(R_1) \Rightarrow \tilde{g}'_{1n}(R_1) = -\tilde{g}'_{2n}(R_1)$. On the right: $-R_2\tilde{g}'_{2n}(R_2)g_{2n}(R_2) - R_2\tilde{g}'_{3n}(R_2)g_{3n}(R_2) = 0, g_{2n}(R_2) = g_{3n}(R_2) \Rightarrow \tilde{g}'_{2n}(R_2) = -\tilde{g}'_{3n}(R_2)$.

From this the conditions on functions $\tilde{g}_n \in \operatorname{Dom} C_n^*$ are all together

$$\begin{aligned} \tilde{g}_{3n}(R_3) = 0, \quad & \tilde{g}_{1n}(R_1) = \tilde{g}_{2n}(R_1), && \tilde{g}_{2n}(R_2) = \tilde{g}_{3n}(R_2), \\ & \tilde{g}'_{1n}(R_1) = -\tilde{g}'_{2n}(R_1), && \tilde{g}'_{2n}(R_2) = -\tilde{g}'_{3n}(R_2) \end{aligned} \tag{4.72}$$

which are exactly the same conditions as on functions from $\operatorname{Dom} C_n$. Therefore we proved that $\operatorname{Dom} C_n = \operatorname{Dom} C_n^*$ and thus the operator C_n is self-adjoint and that is why functions $\{g_n\}_{n=1}^{\infty}$ form an orthonormal set in $H^2((0, R_3))$.

As a consequence of orthonormality of sets $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ it is obvious that also $\{f_n\}_{n=1}^{\infty}$ is orthonormal in $H^2(B_{R_3})$

$$\int_{B_{R_3}} f_n(r,\varphi) r dr d\varphi = \int_0^{R_3} g_n(r) g_{\tilde{n}}(r) r dr \int_0^{2\pi} h_n(\varphi) h_{\tilde{n}}(\varphi) d\varphi = \delta_{n\tilde{n}} \tag{4.73}$$

□

Lemma 4.8. *The functions f_n, $n \in \mathbb{N}$ form a complete orthonormal set in $H^2(B_{R_3})$.*

Proof. Now we repeat the procedure from [19] (V., Ex. 1.10) in the same way as in the previous section. The function $w_{n,m}$ is now defined

$$w_{n,m}(\varphi) = \int_0^{R_3} w(r,\varphi) \psi_{n,m}(r) r dr \tag{4.74}$$

By the Schwarz inequality and because $\{g_{n,m}(r)\}$ is an orthonormal set we have

$$|w_{n,m}(\varphi)|^2 \leq \int_0^{R_3} |w(r,\varphi)|^2 r dr \int_0^{R_3} |g_{n,m}(r)|^2 r dr = \int_0^{R_3} |w(r,\varphi)|^2 r dr \tag{4.75}$$

Another simple inequality is

$$\int_0^{2\pi} |w_{n,m}(\varphi)|^2 d\varphi \leq \int_{B_{R_3}} |w(r,\varphi)|^2 r dr d\varphi = \|w\|^2 \tag{4.76}$$

The inequalities (4.75) and (4.76) imply that $w_{n,m} \in H^2((0,2\pi))$ and therefore

$$(w, f_{n,m}) = (w, g_{n,m} h_n) = (w_{n,m}, h_n) \tag{4.77}$$

Thus $(w, f_{n,m}) = 0$ for all n, m implies that $(w_{n,m}, h_n) = 0$ and hence $w_{n,m} = 0$ by the completeness of $\{h_n\}$. □

Proof of Theorem 4.5. In this case of operator C we have matched again all assumptions of Lemma A.6 and so we have proved that the operator C is essentially self-adjoint. □

4.3 Higher dimensions

In this section we explore the previous two cases in general dimension d. First consider the rectangle in d dimensions. We have $\Omega_d = (-a/2, a/2) \times (0, a_2) \times (0, a_3) \times \cdots \times (0, a_d)$. This hyperrectangle is again divided on two: $\Omega_{d-} = (-a/2, 0) \times (0, a_2) \times \cdots \times (0, a_d)$ with negative permittivity and $\Omega_{d+} = (0, a/2) \times (0, a_2) \times \cdots \times (0, a_d)$ with positive one. The interface between them can be again denoted as $\mathcal{C}_d = \{0\} \times (0, a_2) \times \cdots \times (0, a_d)$. Than the operator is again similarly

$$A_d f = \begin{pmatrix} -\Delta f_+ \\ \Delta f_- \end{pmatrix} \tag{4.78}$$

$$\begin{aligned} \operatorname{dom} A_d = \Bigg\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : & f_\pm, \Delta f_\pm \in H^2(\Omega_{d\pm}), \\ & f|_{\partial\Omega_d} = 0, f_+|_{\mathcal{C}_d} = f_-|_{\mathcal{C}_d}, \partial_{\mathbf{n}+} f_+|_{\mathcal{C}_d} = \partial_{\mathbf{n}_-} f_-|_{\mathcal{C}_d} \Bigg\} \end{aligned} \tag{4.79}$$

The essential self-adjointness of operator A_d can be proved in the similar way as in lemmas 4.2 – 4.4. Therefore we solve the eigenvalue equation

$$\mp\Delta f_\pm = \lambda f_\pm \tag{4.80}$$

with the separation of variables, i.e. we decompose any eigenfunction $f \in H^2(\Omega_d)$ of A_d as

$$f(x, y_2, \ldots, y_d) = \sum_{n_2,\ldots,n_d}^{\infty} \psi_{n_2,\ldots,n_d}(x) \chi_{n_2}(y_2) \ldots \chi_{n_d}(y_d) \tag{4.81}$$

where $\{\chi_{n_2}\}_{n=1}^{\infty}, \ldots, \{\chi_{n_d}\}_{n=1}^{\infty}$ of the form $\chi_{n_j}(y_j) = \sqrt{\frac{2}{a_j}} \sin(n_j \pi \frac{y_j}{a_j})$ with $j = 2, \ldots, d$ are the complete ortonormal bases in their respective domain. The eigenvalue equation can be than rewritten as

$$\begin{aligned} \mp \sum_{n_2,\ldots,n_d}^{\infty} \left[\psi''_{n_2,\ldots,n_d\pm}(x) - \pi^2 \frac{N^2}{B^2} \psi_{n_2,\ldots,n_d\pm}(x) \right] \chi_{n_2}(y_2) \ldots \chi_{n_d}(y_d) = \\ = \lambda \sum_{n_2,\ldots,n_d}^{\infty} \psi_{n_2,\ldots,n_d\pm}(x) \chi_{n_2}(y_2) \ldots \chi_{n_d}(y_d) \end{aligned} \tag{4.82}$$

where we denoted

$$\frac{N^2}{B^2} = \sum_{j=2}^{d} \left(\frac{n_j}{a_j}\right)^2 \tag{4.83}$$

to emphasize the similarity with the two-dimensional case. However we have to keep in mind that the value of this sum depends on all indices $n_2, \ldots, n_d$ and not only on a single n as in the previous case. It will be useful to denote the multi-index $\nu_d = (n_2, \cdots, n_d)$. Now we use that $\{\chi_{n_2}\}_{n=1}^{\infty}, \ldots, \{\chi_{n_d}\}_{n=1}^{\infty}$ form complete orthonormal sets to achieve two one-dimensional equations

$$-\psi''_{\nu_d+}(x) = \left(\lambda - \left(\frac{N\pi}{B}\right)^2\right)\psi_{\nu_d+}(x), \qquad x \in \left(0, \frac{a}{2}\right) \tag{4.84}$$

$$\psi''_{\nu_d-}(x) = \left(\lambda + \left(\frac{N\pi}{B}\right)^2\right)\psi_{\nu_d-}(x), \qquad x \in \left(-\frac{a}{2}, 0\right) \tag{4.85}$$

It is obvious that we are now in the same situation as in the Section 4.1 since we have two differential equations of the same form as (4.10) and (4.11) with the difference in number of indices and that the coefficient n^2/b^2 was changed into N^2/B^2 defined as (4.83). Therefore it is no surprise that we get eigenfunctions of operator A_d as

$$\psi_{\nu_d,m+}(x) = \mathcal{N}_{\nu_d,m} \sinh\sqrt{\lambda_{\nu_d} + \left(\frac{N\pi}{B}\right)^2}\frac{a}{2} \sin\sqrt{\lambda_{\nu_d} - \left(\frac{N\pi}{B}\right)^2}\left(\frac{a}{2} - x\right) \tag{4.86}$$

$$\psi_{\nu_d,m-}(x) = \mathcal{N}_{\nu_d,m} \sin\sqrt{\lambda_{\nu_d} - \left(\frac{N\pi}{B}\right)^2}\frac{a}{2} \sinh\sqrt{\lambda_{\nu_d} + \left(\frac{N\pi}{B}\right)^2}\left(\frac{a}{2} + x\right) \tag{4.87}$$

with normalization constants $\mathcal{N}_{\nu_d,m}$ given as

$$\begin{aligned}\frac{1}{\mathcal{N}_{\nu_d}^2} = \sin^2\sqrt{\lambda_{\nu_d} - \left(\frac{N\pi}{B}\right)^2}\frac{a}{2}\left(\frac{\sinh\sqrt{\lambda_{\nu_d} + \left(\frac{N\pi}{B}\right)^2}a}{4\sqrt{\lambda_{\nu_d} + \left(\frac{N\pi}{B}\right)^2}} - \frac{a}{4}\right) + \\ + \sinh^2\sqrt{\lambda_{\nu_d} + \left(\frac{N\pi}{B}\right)^2}\frac{a}{2}\left(\frac{a}{4} - \frac{\sin\sqrt{\lambda_{\nu_d} - \left(\frac{N\pi}{B}\right)^2}a}{4\sqrt{\lambda_{\nu_d} - \left(\frac{N\pi}{B}\right)^2}}\right)\end{aligned} \tag{4.88}$$

The eigenvalue equation keeps the same form as well

$$\frac{\tanh\left(\sqrt{\lambda+\left(\frac{N\pi}{B}\right)^2}\frac{a}{2}\right)}{\sqrt{\lambda+\left(\frac{N\pi}{B}\right)^2}}=\frac{\tan\left(\sqrt{\lambda-\left(\frac{N\pi}{B}\right)^2}\frac{a}{2}\right)}{\sqrt{\lambda-\left(\frac{N\pi}{B}\right)^2}} \tag{4.89}$$

for $\lambda \neq \pm\left(\frac{N\pi}{b}\right)^2$. The next work would be only a repetition of our procedure in Section 4.1. Therefore let us conclude the result in the following theorem which is a simple generalization of Theorem 4.1 into general dimension d.

Theorem 4.9. *Operator A_d defined by relations (4.78), (4.79) is essentially self-adjoint.*

We make similar generalization for the radial case. The geometry is defined with three areas $\mathcal{B}_1 = B_{R_1}, \mathcal{B}_2 = B_{R_2} \setminus B_{R_1}$ and $\mathcal{B}_3 = B_{R_3} \setminus B_{R_2}$ where B_{R_j} for $j = 1, 2, 3$ are open hyperspherical balls in d dimensions with radii R_j. It is obvious that we use the hyperspherical coordinate system (r, Ω) where r represent the radial distance and Ω is a symbol for $d-1$ angular coordinates $\theta_1, \ldots, \theta_{d-1}$, $\theta_{d-1} \in< 0, 2\pi >$ and $\theta_k \in< 0, \pi >$ for $k = 1, \ldots, d-2$. In this scheme the considered operator is defined as

$$C_d f = \begin{pmatrix} -\Delta f_1 \\ \Delta f_2 \\ -\Delta f_3 \end{pmatrix}, \tag{4.90}$$

$$\begin{aligned}\operatorname{dom} C_d = \Bigg\{ f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad &: f_j, \Delta f_j \in H^2((R_{j-1}, R_j), r^{d-1}dr) \times H^2(S^{d-1}, d\Omega), \\ j = 1,2,3, \quad & f_3|_{\partial B_{R_3}} = 0, f_1|_{\partial B_{R_1}} = f_2|_{\partial B_{R_1}}, f_2|_{\partial B_{R_2}} = f_3|_{\partial B_{R_2}}, \\ & \partial_{\mathbf{n}+} f_1|_{\partial B_{R_1}} = \partial_{\mathbf{n}-} f_2|_{\partial B_{R_1}}, \partial_{\mathbf{n}+} f_2|_{\partial B_{R_2}} = \partial_{\mathbf{n}-} f_3|_{\partial B_{R_2}} \Bigg\}\end{aligned} \tag{4.91}$$

where $d\Omega = \sin^{d-2}(\theta_1)\sin^{d-3}(\theta_2)\ldots\sin(\theta_{d-2})\theta_{d-1}d\theta_1\ldots d\theta_{d-1}$. Once again differential equation $\Delta f = \lambda f$ is our concern with Laplace operator expressed in hyperspherical coordinates as

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^{d-1}} \tag{4.92}$$

where $\Delta_{S^{d-1}}$ is Laplace-Beltrami operator that we have met in Section 3.4. Suppose that the eigenfunctions f can be written by separation of variables as

$$f(r,\Omega)=\sum_{n=1}^{\infty} g_n(r)h_n(\Omega) \tag{4.93}$$

where $h_n(\Omega)$, $n\in\mathbb{N}$ are spherical harmonics (see e.g. [15]). Since $h_n(\Omega)$ are eigenfunctions of spherical Laplacian, i.e. $-\Delta_{S^{d-1}}h_n(\Omega)=n(n+d-2)h_n(\Omega)$, the eigenvalue equation $C_d f(r,\Omega)=\lambda f(r,\Omega)$ is of the form

$$\mp\sum_{n=1}^{\infty}\left(g''_{jn}(r)+\frac{1}{r}g'_{jn}(r)-\frac{n(n+d-2)}{r^2}g_{jn}(r)\right)h_n(\Omega)=\lambda\sum_{n=1}^{\infty}\left(g_{jn}(r)h_n(\Omega)\right) \tag{4.94}$$

where $g_{jn}(r)$ are functions in the interval (R_{j-1},R_j) respectively, $j=1,2,3$. The plus sign on the left side of the equation applies for $j=2$, otherwise there is the minus sign. Now we multiply these three equations by $h_m(\Omega)$ and integrate over Ω. Since $\{h_n(\Omega)\}_{n=1}^{\infty}$ is a complete orthonormal set, the equations (4.94) results in one-dimensonal

$$g''_{1n}(r)+\frac{g'_{1n}(r)}{r}+\left(\lambda-\frac{n(n+d-2)}{r^2}\right)g_{1n}(r)=0,\qquad r\in(0,R_1) \tag{4.95}$$

$$g''_{2n}(r)+\frac{g'_{2n}(r)}{r}-\left(\lambda+\frac{n(n+d-2)}{r^2}\right)g_{2n}(r)=0,\qquad r\in(R_1,R_2) \tag{4.96}$$

$$g''_{3n}(r)+\frac{g'_{3n}(r)}{r}+\left(\lambda-\frac{n(n+d-2)}{r^2}\right)g_{3n}(r)=0,\qquad r\in(R_2,R_3) \tag{4.97}$$

We see that these equations are of the same form as (4.47) – (4.49) with a change in coefficient at $g_{jn}(r)$, $j=1,2,3$. Therefore we denote $n(n+d-2)$ by N^2 just as we did in the rectangular case above. Using the same procedure as in Section 4.2 we get eigenfunctions of operator C_d

$$g_{1n}(r)=\mathcal{N}_n\frac{Y_N(-i\lambda_1)-\frac{B}{A}J_N(-i\lambda_1)}{J_N(\lambda_1)}J_N(\sqrt{\lambda}r),\qquad r\in(0,R_1) \tag{4.98}$$

$$g_{2n}(r)=\mathcal{N}_n\left(Y_N(-i\sqrt{\lambda}r)-\frac{B}{A}J_N(-i\sqrt{\lambda}r)\right),\qquad r\in(R_1,R_2) \tag{4.99}$$

$$g_{3n}(r)=\mathcal{N}_n\frac{D}{C}\left(Y_N(\sqrt{\lambda}r)-\frac{Y_N(\lambda_3)}{J_N(\lambda_3)}J_N(\sqrt{\lambda}r)\right),\qquad r\in(R_2,R_3) \tag{4.100}$$

where we have following relation for normalization constant $\mathcal{N}_n$

$$\frac{1}{|\mathcal{N}_n|^2} = \left|\frac{Y_N(-i\lambda_1) - \frac{B}{A}J_N(-i\lambda_1)}{J_N(\lambda_1)}\right|^2 \int_0^{R_1} \left|J_N(\sqrt{\lambda}r)\right|^2 rdr+$$
$$+ \int_{R_1}^{R_2} \left|(Y_N(-i\sqrt{\lambda}r)\right|^2 rdr - \left|\frac{B}{A}\right|^2 \int_{R_1}^{R_2} \left|J_N(-i\sqrt{\lambda}r)\right|^2 rdr+$$
$$+ \left|\frac{D}{C}\right|^2 \int_{R_2}^{R_3} \left|Y_N(\sqrt{\lambda}r)\right|^2 rdr - \left|\frac{DY_N(\lambda_3)}{CJ_N(\lambda_3)}\right|^2 \int_{R_2}^{R_3} \left|J_N(\sqrt{\lambda}r)\right|^2 rdr \tag{4.101}$$

and A, B, C, D, E, F are again denoted constants defined as (4.34) – (4.39) with n changed for $N = \sqrt{n(n+d-2)}$. Therefore we can again easily write down the eigenvalue equation

$$\frac{D}{C} = \frac{F}{E} \tag{4.102}$$

It is obvious that the rest of the procedure from Section 4.2 leads to the same result as Theorem 4.5 for general dimension d.

Theorem 4.10. *Operator C_d defined by relations (4.90), (4.91) is essentially self-adjoint.*

Chapter 5

Spectral analysis and perturbation theory

Now we would like to examine the spectra of the essentially self-adjoint operators A, C and their generalizations A_d, C_d from the previous chapter.

In case of the operator A on the rectangle $(-a/2, a/2) \times (0, b)$ we can simply summarize the results from [6] where authors considered such operator. They found that the spectrum of operator A consists of discrete eigenvalues $\lambda_{n,m}$ ($m \in \mathbb{Z}$, $n \in \mathbb{N}$) which for each fixed n form an increasing sequence of simple roots of equation (4.3). The eigenvalue 0 is of infinite multiplicity and therefore belongs to the essential spectrum of the rectangular operator A. As stated in that article such spectral properties are unexpected because for bounded rectangular domain one would think that the essential spectrum is empty. This peculiarity occurs here because of the extraordinary condition on the domain i.e. that the derivatives point in opposite directions at the interface between dielectricum and metamaterial. It is obvious that the same result holds also for spectrum $\sigma(A_d)$.

Further we focus on the spectrum of the operator C. Its generalization C_d will be discussed afterwards. We want to examine the behaviour of 0 as a point of $\sigma(C)$. It cannot be an eigenvalue (it is obvious if one tries to solve equations (4.47) – (4.49) for $\lambda = 0$ as we did in Section 3.4, also see [10]) but question is whether it could be contained in the essential spectrum which would be therefore nonempty just like in the rectangular case.

For this we modify the eigenvalue equation (4.33) for high order of n. We use relations (A.33), (A.34) for the Bessel functions, thus

$$J_n(z) = \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2n}\right)^n + o\left(\left(\frac{ez}{2}\right)^n \frac{1}{n^{n+\frac{1}{2}}}\right) \tag{5.1}$$

$$Y_n(z) = -\sqrt{\frac{2}{\pi n}} \left(\frac{2n}{ez}\right)^n + o\left(\left(\frac{2}{ez}\right)^n n^{n-\frac{1}{2}}\right) \tag{5.2}$$

$$\begin{aligned} J_{n-1}(z) - J_{n+1}(z) = & \frac{1}{\sqrt{2\pi}} \left(\frac{ez}{2}\right)^{n-1} \frac{1}{(n-1)^{n-\frac{1}{2}}} - \\ & - \frac{1}{\sqrt{2\pi}} \left(\frac{ez}{2}\right)^{n+1} \frac{1}{(n+1)^{n+\frac{3}{2}}} + o\left(\left(\frac{ez}{2}\right)^{n+1} \frac{1}{(n+1)^{n+\frac{3}{2}}}\right) \end{aligned} \tag{5.3}$$

$$\begin{aligned} Y_{n-1}(z) - Y_{n+1}(z) = & \sqrt{\frac{2}{\pi}} \left(\frac{2}{ez}\right)^{n+1} (n+1)^{n+\frac{1}{2}} - \\ & - \sqrt{\frac{2}{\pi}} \left(\frac{2}{ez}\right)^{n-1} (n-1)^{n-\frac{3}{2}} + o\left(\left(\frac{2}{ez}\right)^{n-1} (n-1)^{n-\frac{1}{2}}\right) \end{aligned} \tag{5.4}$$

for $n \to \infty$. The two last relations results immediately from (5.1), (5.2) and we write it here because they are very convenient when one modifies the eigenvalue equation. Using these formulae both sides of (4.33) are as follows

$$\begin{aligned} l.h.s. = & \frac{R_2^{2n}}{2\,(-i)^n\, R_1^{2n} \left[-1 + o\left(\frac{1}{n^2}\right)\right]} \left[1 - \frac{(n-1)^{n-\frac{1}{2}}(n+1)^{n+\frac{1}{2}}}{n^{2n}} + \right. \\ & \left. + \frac{e^2 R_1^2 \lambda}{4} \left(\frac{(n-1)^{2n-2}}{2^{2n}} - \frac{(n-1)^{n-\frac{1}{2}}}{(n+1)^{n+\frac{3}{2}}}\right) + o\left(\frac{1}{n^2}\right)\right] \end{aligned} \tag{5.5}$$

$$r.h.s. = \frac{R_2^{2n}}{2(-i)^n R_1^{2n}\left[\frac{(n+1)^{n+\frac{1}{2}}}{n^{2n}(n-1)^{n-\frac{1}{2}}} - \frac{e^2R_2^2\lambda}{4}\frac{1}{n^{2n}(n-1)} + o\left(\frac{(n-1)^{n-\frac{3}{2}}}{n^{2n}(n+1)^{n+\frac{3}{2}}}\right)\right]}\cdot$$
$$\cdot\left[\frac{(n+1)^{n+\frac{1}{2}}}{n^{2n}(n-1)^{n-\frac{1}{2}}} - \frac{1}{(n-1)^{2n-1}} + \frac{e^2\lambda}{4}\left[R_2^2\left(\frac{1}{n^{2n}(n+1)}-\right.\right.\right.$$
$$\left.-\frac{1}{(n-1)^{n-\frac{1}{2}}(n+1)^{n+\frac{3}{2}}}\right) + R_1^2\left(\frac{1}{n^{2n}(n-1)} + \left.\frac{1}{(n-1)^{n-\frac{1}{2}}(n+1)^{n+\frac{3}{2}}}\right)\right] +$$
$$\left.\frac{e^4R_1^2R_2^2\lambda^2}{16}\left[\frac{1}{(n+1)^{2n+3}} + \frac{(n-1)^{n-\frac{3}{2}}}{n^{2n}(n+1)^{n+\frac{3}{2}}}\right] + o\left(\frac{1}{n^{2n+3}}\right)\right] \tag{5.6}$$

If we put these results into the equation (4.33) we get a quadratic equation for λ (for $n \to \infty$)

$$0 = 2\frac{(n+1)^{n+\frac{1}{2}}}{(n-1)^{n-\frac{1}{2}}} - \frac{(n+1)^{2n+1}}{n^{2n}} - \frac{n^{2n}}{(n-1)^{2n-1}} -$$
$$-\lambda\frac{e^2}{4}\left[R_1^2\left(\frac{(n-1)^{n-\frac{3}{2}}(n+1)^{n+\frac{1}{2}}}{n^{2n}} - \frac{n^{2n}}{(n-1)^{n-\frac{1}{2}}(n+1)^{n+\frac{3}{2}}} - \frac{2}{n-1}\right) +\right.$$
$$\left.+R_2^2\left(\frac{(n-1)^{n-\frac{3}{2}}(n+1)^{n+\frac{1}{2}}}{n^{2n}} + \frac{n^{2n}}{(n-1)^{n-\frac{1}{2}}(n+1)^{n+\frac{3}{2}}} - \frac{2}{n-1}\right)\right] +$$
$$+\lambda^2\frac{e^4R_1^2R_2^2}{16}\left[2\frac{(n-1)^{n-\frac{3}{2}}}{(n+1)^{n+\frac{3}{2}}} + \frac{n^{2n}}{(n+1)^{2n+3}} - \frac{(n-1)^{2n-3}}{n^{2n}}\right] + o\left(\frac{1}{n^5}\right) \tag{5.7}$$

Here we make a series expansion of terms with n at infinity which reduces the equation to the form

$$0 = -\frac{e^2}{36n^5}+\lambda\frac{e^2}{4}\left[R_1^2\left(\frac{2}{n}+\frac{2}{n^3}+\frac{1}{3n^4}+\frac{2}{n^5}\right) + R_2^2\left(\frac{2}{n^2}+\frac{2}{n^4}+\frac{1}{3n^5}\right)\right] +$$
$$+\lambda^2\frac{e^2R_1^2R_2^2}{16}\left(\frac{2}{n^3}-\frac{4}{n^4}+\frac{5}{3n^5}\right) + o\left(\frac{1}{n^5}\right) \tag{5.8}$$

Finally we find two solutions $\lambda_\pm$ of this quadratic equation.

$$\lambda_\pm = -\frac{2}{R_1^2 R_2^2} \frac{6R_1^2 n^4 + 6R_2^2 n^3 + 6R_1^2 n^2 + (R_1^2 + 6R_2^2)n + 6R_1^2 + R_2^2}{(6n^2 - 12n + 5)} \pm$$
$$\pm \frac{2}{\sqrt{3} R_1^2 R_2^2 \, (6n^2 - 12n + 5)} \sqrt{\begin{aligned} &108R_1^4 n^8 + 216 R_1^2 R_2^2 n^7 + (216R_1^4 + 108R_2^4)n^6 + \\ &(36R_1^4 + 432R_1^2R_2^2)n^5 + (324R_1^4 + 216R_2^4 + \\ &+ 72R_1^2R_2^2)n^4 + (36R_2^4 + 432R_1^2R_2^2)n^3 + \\ &+ (219R_1^4 + 108R_2^4 + 78R_1^2R_2^2)n^2 + \\ &+ (36R_1^4 + 36R_2^4 + 210R_1^2R_2^2)n + 108R_1^4 + \\ &+ 3R_2^2 + 41R_2^2R_2^2 \end{aligned}} \tag{5.9}$$

Now it is easy to calculate the limits for $n \to \infty$.

$$\lim_{n\to\infty} \lambda_+ = 0 \qquad \lim_{n\to\infty} \lambda_- = -\infty \tag{5.10}$$

Therefore the left limit proves that 0 belongs to the essential spectrum of operator C.

Let us discuss here the case of general dimension d. We saw in Section 4.3 that the only difference in eigenvalue equation is in fact that we write $N^2 = n^2 + n(d-2)$ instead of n^2 (it is obvious that these are the same for $d = 2$). Therefore the calculation would look the same as in the two-dimensional case but with the difference that the above equations in the enlarged form (with explicitly written n, d) are much wider and actually there is nothing new in them. Thus we regretfully avoid writing these long expressions here and only state here the result that we have computed, i.e. $\lim\limits_{n\to\infty} \lambda_+ = 0$. We summarize these found result in the following theorem.

Theorem 5.1. *Consider essentially self-adjoint operator defined by relations* (4.31) *and* (4.32)*. The essential spectrum of both essentially self-adjoint operators C and C_d defined by relations* (4.31)*,* (4.32) *and* (4.90)*,* (4.91) *is non-empty and it holds that $0 \in \sigma_{\mathrm{ess}}(C)$.*

Chapter 6

Conclusions

In this thesis we made a quite thorough review about early development of metamaterials with simultaneously negative permittivity and permeability both in theoretical and practical way. We mentioned some important properties predicted by Veselago and also some possible applications in the future (some prototypes of metamaterial anntenas and metamaterial cloak already exist). The most interest is in the field of invisibility cloaks made of metamaterials. We presented here some most important mathematical approaches to the cloaking focused mainly on the concept of so called anomalous localised resonance. After introducing the leading scientists in this area we follow the work by Bouchitte and Schweizer and use it to prove that anomalous localised resonance does not occur on three (and more) dimensional balls. This result is not only a confirmation of the one showed in [4] but also a generalization for higher dimensions.

We also mentioned results from the work by Behrndt and Krejčiřík. They investigated the nonelliptic diferential expression $-\operatorname{div}\operatorname{sgn}\operatorname{grad}$ on a rectangular domain in two dimensions. In this thesis examination of similar operator is made for general rectangle with lengths a, b. As in [6] we used a separation of variables to prove that this operator is essentially self-adjoint. Then we repeated the same procedure with similar operator but in a symmetrically rotational setting dielectricum-metamaterial-dielectricum. These results were then generalized to higher dimensions d.

Since the spectrum of the rectangular operator is already known we only summarized that its eigenvalues are nonzero real numbers of finite multiplicity accumulating to $+\infty$ and $-\infty$ and that 0 is in its essential spectrum. We proved then the same result also for the radial operator. The fact that 0 lies in the essential spectrum means that there is an inverse operator to the one from the left side of

Poisson equation $-\operatorname{div}\operatorname{sgn}\operatorname{grad} u = f$ but this inversion is unbounded and so it does not exist for all functions f on the right side.

There are still many topics to investigate. We could use procedure from [14] to achieve better estimates in our calculations in Chapter 5. It would be also nice to prove the invisibility radius R^* with tools of functional analysis and therefore more understand this problem of cloaking in theory of linear operators.

Appendix A

Mathematical theory

A.1 Self-adjointness of unbounded linear operators

Let us review here some elementary definitions and theorems from the theory of linear operators. We are using these statements mostly from [13]. Since we are dealing only with unbounded operators in this thesis we focus here on them, i.e. $H : \mathrm{Dom}(H) \subset \mathscr{H} \to \mathscr{H}$ where $\mathscr{H}$ denotes considered Hilbert space.

Definition A.1. We call H densely defined operator if $\mathrm{Dom}(H)$ is dense in $\mathscr{H}$.

As the replacement for the boundedness we work with the closed operators. Spectral theory of unbounded operators would not be too interesting because such operators have the whole complex plane in the spectrum $\sigma(H) = \mathbb{C}$.

Definition A.2. Operator H is closed if and only if for all $\psi \in \mathscr{H}$ and for every sequence $\{\psi_n\} \subset \mathrm{Dom}(H)$ such that $\lim\limits_{n\to\infty} \psi_n = \psi, \lim\limits_{n\to\infty} H\psi_n = \phi$ one has $\psi \in \mathrm{Dom}(H)$ and $H\psi = \phi$.

The key property for unbounded operators it the one of self-adjointness. A part of it but much more elementary is the property of symmetry.

Definition A.3. We say that a densely defined operator H is symmetric if for all $\psi, \phi \in \mathrm{Dom}(H)$ we have $(H\psi, \phi) = (\psi, H\phi)$.

Here $(\cdot, \cdot)$ denotes a scalar product in the Hilbert space $\mathscr{H}$. It is usually quite simple to verify whether operator is symmetric or not. Much more demanding is to show that the operator is also self-adjoint. This property is demanded because otherwise one could not apply all the powerful tools of the spectral theory.

Definition A.4. Let A be a linear operator on a Hilbert space $\mathscr{H}$. Then the adjoint operator H^* is determined by the condition that $(\varphi, H\psi) = (H^*\varphi, \psi)$ uniquely for all $\psi \in \mathrm{Dom}(H)$ and $\varphi \in \mathrm{Dom}(H^*)$. The domain of H^* is defined to be the set $\mathrm{Dom}(H^*) = \{\phi \in \mathscr{H} | \exists \eta \in \mathscr{H}, \forall \psi \in \mathrm{Dom}(H), (\phi, H\psi) = (\eta, \psi), \eta = H^*\phi\}$

Definition A.5. Operator H is self-adjoint if H is symmetric and $\mathrm{Dom}(H) = \mathrm{Dom}(H^*)$. We say that H is essentially self-adjoint if it is symmetric and its closure is self-adjoint.

Apparently if H is symmetric then it is easy to see that the adjoint operator $H*$ is an extension of H. We write it as $H \subset H^*$ in the sense that this inclusion applies for the domains of operators H, H^*. Following lemma gives a method of proving essential self-adjointness however, as stated in [13], it is useful only for simple operators whose eigenvectors can be determined explicitly.

Lemma A.6. *Let H by a symmetric operator on $\mathscr{H}$, and let $\{\psi_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $\mathscr{H}$. If each ψ_n lies in $\mathrm{Dom}(H)$ and there exist $\lambda_n \in \mathbb{R}$ such that $H\psi_n = \lambda_n \psi_n$ for every n, then H is essentially self-adjoint. Moreover, the spectrum of $\bar{H}$ is the closure in $\mathbb{R}$ of the set of all λ_n.*

A.2 Bessel functions

Definition A.7. Differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0 \tag{A.1}$$

is called Bessel equation. Solutions are the Bessel functions of the first kind $J_{\pm\nu}(z)$, of the second kind $Y_\nu(z)$ (also called Weber's functions) and of the third kind $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ (Hankel functions).

Each Bessel function is a regular (holomorphic) function of z throughout the z-plane cut along the negative real axis, and for fixed $z \neq 0$ each in an entire (integral) function of ν. When $\nu = \pm n \in \mathbb{Z}$, $J_\nu(z)$ has no branch point and is an entire (integral) function of z.

Important features of the various solutions are as follows: $J_\nu(z)(\mathrm{Re}\,\nu \geq 0)$ is bounded as $z \to 0$ in any bounded range of $\arg z$. $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν.

$H_\nu^{(1)}(z)$ tends to zero as $|z| \to \infty$ in the sector $0 < \arg z < \pi$; $H_\nu^{(2)}(z)$ tends to zero as $|z| \to \infty$ in the sector $-\pi < \arg z < 0$. For all values of ν, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent.

Relation between $Y_\nu(z)$ and $J_\nu(z)$ is

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{A.2}$$

For ν an integer or zero (in this case we write n instead of ν) we must replace the right side of this equation by its limiting value. Further there holds for integer order

$$J_{-n}(z) = (-1)^n J_n(z) \tag{A.3}$$
$$Y_{-n}(z) = (-1)^n Y_n(z) \tag{A.4}$$

Also for Henkel functions we have relations

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) = \frac{i}{\cos(\nu\pi)}\left(e^{-\nu\pi i}J_\nu(z) - J_{-\nu}(z)\right) \tag{A.5}$$
$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) = \frac{i}{\cos(\nu\pi)}\left(J_{-\nu}(z) - e^{\nu\pi i}J_\nu(z)\right) \tag{A.6}$$
$$H_{-\nu}^{(1)}(z) = e^{\nu\pi i}H_\nu^{(1)}(z) \tag{A.7}$$
$$H_{-\nu}^{(2)}(z) = e^{-\nu\pi i}H_\nu^{(2)}(z) \tag{A.8}$$

Bessel functions of the first and second kind can be written as the ascending series

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)} \tag{A.9}$$

$$\begin{aligned} Y_n(z) = &-\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\frac{1}{2}z\right)J_n(z) - \\ &-\frac{\left(\frac{1}{2}z\right)^{n}}{\pi}\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(n+k+1)\right)\frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!} \end{aligned} \tag{A.10}$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is gamma function and $\psi(z) = \frac{d\ln\Gamma(z)}{dz}$ is psi or digamma function. For integer values of z it has simpler form $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$ where γ is Euler's constant.

Another look at Bessel functions can be as the solution to some special integrals. For example if $|\arg z| < \frac{1}{2}\pi$ then

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos\left(z\sin\theta - \nu\theta\right)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\sinh t - \nu t}dt \tag{A.11}$$

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin\left(z\sin\theta - \nu\theta\right)d\theta - \frac{1}{\pi}\int_0^\infty \left(e^{\nu t} + e^{-\nu t}\cos(\nu\pi)\right)e^{-z\sinh t}dt \tag{A.12}$$

In the case of $\nu = 0$ these integrals are only

$$J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta)d\theta = \frac{1}{\pi}\int_0^\pi \cos(z\cos\theta)d\theta \tag{A.13}$$

$$Y_0(z) = \frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi} \cos(z\cos\theta)\left(\gamma + \ln(2z\sin^2(\theta))\right)d\theta \tag{A.14}$$

If we denote all Bessel functions $J, Y, H^{(1)}, H^{(2)}$ as $\mathscr{C}$ we have for all of them following recurrence relations

$$\mathscr{C}_{\nu-1}(z) + \mathscr{C}_{\nu+1}(z) = \frac{2\nu}{z}\mathscr{C}_\nu(z) \tag{A.15}$$

$$\mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z) = 2\mathscr{C}'_\nu(z) \tag{A.16}$$

From these two relations we can deduce two useful expression for derivatives

$$\mathscr{C}'_\nu(z) = \mathscr{C}_{\nu-1}(z) - \frac{\nu}{z}\mathscr{C}_\nu(z) \tag{A.17}$$

$$\mathscr{C}'_\nu(z) = -\mathscr{C}_{\nu+1}(z) + \frac{\nu}{z}\mathscr{C}_\nu(z) \tag{A.18}$$

Since we know from (A.3) and (A.4) that $J_{-1}(z) = -J_1(z)$ and $Y_{-1}(z) = -Y_1(z)$ we have with use of (A.16) that

$$J'_0(z) = -J_1(z) \qquad Y'_0(z) = -Y_1(z) \tag{A.19}$$

Important are also formulae for derivatives of general order

$$\left(\frac{1}{z}\frac{d}{dz}\right)\left[z^\nu\mathscr{C}_\nu(z)\right] = z^{\nu-k}\mathscr{C}_{\nu-k}(z) \tag{A.20}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)\left[z^{-\nu}\mathscr{C}_\nu(z)\right] = (-1)^k z^{-\nu-k}\mathscr{C}_{\nu+k}(z) \tag{A.21}$$

where $k \in \mathbb{N} \cup \{0\}$. Therefore for k-th derivative of the Bessel function we have

$$\mathscr{C}_\nu^{(k)}(z) = \frac{1}{2^k}\left(\mathscr{C}_{\nu-k}(z) - \binom{k}{1}\mathscr{C}_{\nu-k+2}(z) + \binom{k}{2}\mathscr{C}_{\nu-k+4}(z) - \cdots + (-1)^k\mathscr{C}_{\nu+k}(z)\right) \tag{A.22}$$

Particularly for $k = 1$ we have relation (A.16)

$$\mathscr{C}'_\nu(z) = \frac{\mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z)}{2} \tag{A.23}$$

If $m \in \mathbb{Z}$ then we have

$$J_\nu\left(ze^{m\pi i}\right) = e^{m\pi\nu i}J_\nu(z) \tag{A.24}$$

$$Y_\nu(ze^{m\pi i}) = e^{-m\pi\nu i}Y_\nu(z) + 2i\sin(m\nu\pi)\cot(\nu\pi)J_\nu(z) \tag{A.25}$$

$$\sin(\nu\pi)H_\nu^{(1)}(ze^{m\pi i}) = -\sin\{(m-1)\nu\pi\}H_\nu^{(1)}(z) - e^{\nu\pi i}\sin(m\nu\pi)H_\nu^{(2)}(z) \tag{A.26}$$

$$\sin(\nu\pi)H_\nu^{(2)}(ze^{m\pi i}) = \sin\{(m+1)\nu\pi\}H_\nu^{(2)}(z) + e^{\nu\pi i}\sin(m\nu\pi)H_\nu^{(1)}(z) \tag{A.27}$$

We take a closer look on relation (A.24). For integer order of Bessel function the right exponential can be only ± 1. Therefore we can write the first relation in case of odd m (for even m it is trivial) as

$$J_n\left(-z)\right) = (-1)^n J_n(z) \tag{A.28}$$

This means that $J_n(z)$ is odd function for odd n and even function if the order is also even. Complex conjugation of Bessel functions of the first and second kind means conjugation of theirs arguments

$$\overline{J_\nu(z)} = J_\nu(\overline{z}) \qquad \overline{Y_\nu(z)} = Y_\nu(\overline{z}) \tag{A.29}$$

Henkel functions have the similar character but they switch under the complex conjugation

$$\overline{H_\nu^{(1)}(z)} = H_\nu^{(2)}(\overline{z}) \qquad \overline{H_\nu^{(2)}(z)} = H_\nu^{(1)}(\overline{z}) \tag{A.30}$$

Such switching in Henkel functions is also in relations

$$H_\nu^{(1)}(ze^{\pi i}) = -e^{-\nu\pi i}H\_nu^{(2)}(z) \tag{A.31}$$

$$H_\nu^{(2)}(ze^{-\pi i}) = -e^{\nu\pi i}H\_nu^{(1)}(z) \tag{A.32}$$

which are consequence of (A.26) and (A.27). For more recurrence relations see [1].

For purposes of this thesis we need also following asymptotic expansions of Bessel functions for large orders

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu \tag{A.33}$$

$$Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu} = -\sqrt{\frac{2}{\pi\nu}} \left(\frac{2\nu}{ez}\right)^\nu \tag{A.34}$$

where it is supposed that $\nu \to \infty$ through real positive values (the other variables are fixed).

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