Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Waveguides with non-Hermitian boundary conditions

Vlnovody s nehermitovskými hraničními podmínkami

Bachelor Thesis

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Prohlášení:

Prohlašuji, že jsem svou bakalářskou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

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Abstrakt: Studujeme spektrum Laplaciánu v nezakřiveném rovinném pásku s nehermitovskými Robinovými hraničními podmínkami. Definujeme Robinův Laplacián coby m-sektoriální operátor pomocí teorie sektoriálních forem. Zabýváme se limitním případem křivého pásku s nekonečně malou šířkou. Dokazujeme, že za této situace nesamosdružený Robinův Laplacián konverguje k jednorozměrnému efektivnímu Hamiltoniánu ve slabém rezolventním smyslu.

Klíčová slova: kvantový vlnovod, Robinovy hraniční podmínky, spektrální teorie, sektoriální forma, m-sektoriální operátor, nesamosdružený operátor

Title:

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Abstract: We study the spectrum of the Laplacian in a straight planar strip subject to the non-Hermitian Robin boundary conditions. We define the Laplacian as an m-sectorial operator using the theory of sectorial forms. We deal with the limit situation of the curved strip with infinitely small width. We prove that in this case the non-selfadjoint Robin Laplacian converges to a one-dimensional effective Hamiltonian in the weak-resolvent sense.

Key words: quantum waveguide, Robin boundary conditions, spectral theory, sectorial form, m-sectorial operator, non-selfadjoint operator

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Chapter 1

Introduction

Recent progress in mesoscopic physics has allowed production of semiconductor structures which are small enough to exhibit quantum effects. Their size is comparable with that of atom so that they are called *nanostructures*. Considering the trend of decreasing size of components of electronics, it is expected that these nanostructures will play a key role in the future. Since the semiconductor materials are of crystallic structure, the motion of particle confined to such structure can be modeled by a free particle with an effective mass m^* constrained to a spatial region Ω. The particle can be associated with the Hamiltonian

$$H = -\frac{\hbar^2}{2m^*}\triangle$$

in the Hilbert space $L^2(\Omega)$, where $\hbar$ is the reduced Planck constant. Hereafter, we set $\frac{\hbar^2}{2m^*} = 1$. We refer to [5, 15] for more information about the physical background.

In this thesis we focus on a special category of nanostructures called *quantum waveguides*, which can be modeled by Ω being infinitely stretched tubular region in $\mathbb{R}^2$ or $\mathbb{R}^3$. The existing results for these structures is summarized in [7]. The fact that the particle is constrained to the waveguide can be modeled by imposing appropriate boundary conditions on the boundary of Ω. A natural choice are the Dirichlet boundary conditions

$$\psi = 0 \quad \text{on} \quad \partial\Omega$$

which represent that the wavefunction ψ associated with the particle is suppressed on the boundary due to a chemical potential. The simplest situation occurs if Ω is defined as a tubular neighbourhood of constant width of an infinite curve in $\mathbb{R}^2$. This model with Dirichlet boundary condition was studied in 1989 by P. Exner and P. Šeba [8]. They proved the existence of discrete spectrum of the Dirichlet Laplacian under assumption that the waveguide was asymptotically straight and sufficiently thin. This result was further extended in [9, 17]. The important feature of this model is that bending of the waveguide results in existence of discrete eigenvalues below the essential spectrum. Eigenfunctions associated with these eigenvalues are called bound states and it is known that they disturb the particle transport. This may be particularly problematic for applications where the particle is an electron carrying information in a nanowire. Even a tiny bending deformation of the wire may lead to a loss of data. For more information about the Dirichlet Laplacian in a curved strip we refer to the review paper [5].

In this thesis we are interested in the Robin-type boundary conditions which may be considered as a generalization of Dirichlet boundary conditions. Considering a infinite curved planar strip Ω with a unit outward normal vector n defined on its boundary, we define the Robin boundary conditions as

$$\frac{\partial \psi}{\partial n} + i\alpha\psi = 0 \text{ on } \partial\Omega, \tag{1.0.0}$$

where α is a real-valued function defined on $\partial\Omega$. Since the probability current does not vanish on $\partial\Omega$, the Robin boundary conditions can be used to model dissipative systems [10, 11].

We will be looking for the solutions of the problem

$$\begin{array}{ll} -\triangle\psi = \lambda\psi & \text{in } \Omega \\ \frac{\partial \psi}{\partial n} + i\alpha\psi = 0 & \text{on } \partial\Omega, \end{array}$$

This problem can be regarded as a spectral problem for an m-sectorial operator acting as Laplacian on functions defined in Ω and satisfying the Robin-type boundary conditions. However, this operator is not self-adjoint which is against the principles of quantum physics. On the other hand, this operator is $\mathcal{PT}$-symmetric. Here $\mathcal{P}$ denotes the spacial parity symmetry operator defined by

$$(\mathcal{P}\psi)(x) := \psi(-x),$$

and $\mathcal{T}$ represents the time-reversal operator acting as the complex conjugation

$$(\mathcal{T}\psi)(x) := \overline{\psi(x)}.$$

The interest in $\mathcal{PT}$-symmetric operators emerged with the recognition that many non-selfadjoint operators possess real spectra. It was proved in [2] that the operators of type $H = -\triangle + x^2(ix)^{\varepsilon}$, $\varepsilon \in \mathbb{R}_+$ acting in $L^2(\mathbb{R})$ have real, positive and discrete spectra. However, the $\mathcal{PT}$-symmetry is not a sufficient condition for a spectrum to be real.

Even though some non-selfadjoint operators have real spectra, they cannot be associated with observable in terms of classical quantum mechanics. Nonetheless, it was proved in [18] that for each irreducible set of quasi-Hermitean operator there exists a unique metric operator Θ. The non-selfadjoint operators can then be associated with observables in a Hilbert space with a scalar product defined by

$$(.,.)_\Theta := (.,\Theta .).$$

The metric operator for a Laplacian in $L^2((0,d))$ with Robin boundary conditions was found in [13].

Chapter 2

Preliminaries

In this chapter we state the definitions and theorems we shall use in subsequent chapters. Firstly, we introduce the notion of sectorial forms. Then we present few useful tools which are used for the study of the spectrum of an operator. In the last section we summarize basic properties of Sobolev spaces.

2.1 Sectorial forms

Quadratic forms are a convenient tool for studying Schrödinger operators, since they require less regularity on functions from their domain. Let us start with elementary definitions.

Definition 2.1.1. A map $t : \mathrm{Dom}(t) \times \mathrm{Dom}(t) \to \mathbb{C}$ is called a sesquilinear form in $\mathscr{H}$ if it is conjugate linear in the first argument and linear in the second. The function $t[u] := t(u, u)$ is called a quadratic form.

Contrary to linear operators, it is not difficult to find the adjoint form which is defined by $t^*(\psi, \varphi) := \overline{t(\varphi, \psi)}$, $\mathrm{Dom}(t^*) = \mathrm{Dom}(t)$. We say that a form is symmetric if $t(\psi, \varphi) = t^*(\psi, \varphi)$. Equipped with the adjoint form, we now can define its real and imaginary part as $\mathrm{Re}\, t := \frac{t+t^*}{2}$ and $\mathrm{Im}\, t := \frac{t-t^*}{2i}$. Note that neither $\mathrm{Re}\, t$ nor $\mathrm{Im}\, t$ are real-valued, however, it holds that $\mathrm{Re}\, t[\psi] = \mathrm{Re}(t[\psi])$, $\mathrm{Im}\, t[\psi] = \mathrm{Im}(t[\psi])$ and we can also write $t = \mathrm{Re}\, t + i\, \mathrm{Im}\, t$. The following notion is important for the definition of the sectorial form.

Definition 2.1.2. Let t be a sesquilinear form in $\mathscr{H}$. Its numerical range is defined by

$$\Theta(t) := \{t[\phi] \mid \phi \in \mathrm{Dom}(t), \|\phi\| = 1\}.$$

The numerical range of an operator T in $\mathscr{H}$ is defined by

$$\Theta(T) := \{(\phi, T\phi) \mid \phi \in \mathrm{Dom}(t), \|\phi\| = 1\}.$$

In general, a numerical range need not to be closed or open, nonetheless, it is always a convex subset of the complex plane. Now we are ready to define the sectorial form.

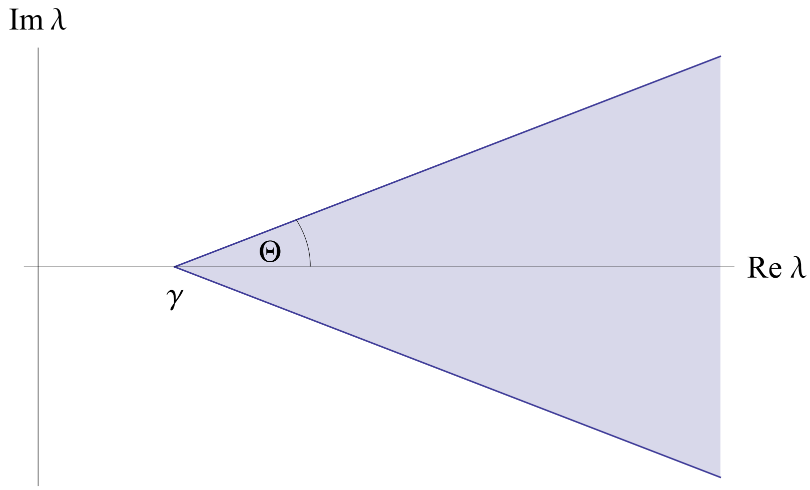

Figure 2.1: A sector with a vertex γ and a semi-angle θ.

Definition 2.1.3. A sesquilinear form t in $\mathscr{H}$ is called sectorial if its numerical range is a subset of a sector, *i.e.*

$$\Theta(h) \subset S_{\gamma,\theta} := \{\lambda \in \mathbb{C} \mid |arg(\lambda - \gamma)| \leq \theta\} \tag{2.1.0}$$

with a vertex $\gamma \in \mathbb{R}$ and a semi-angle θ such that $0 \leq \theta < \frac{\pi}{2}$.

Note that the parameters γ and θ are not uniquely determined by the form t. Indeed, a reduction of the semi-angle θ can be compensated by a reduction of the vertex γ. Every symmetric form is real-valued and if it is also bounded from below, then it is sectorial with $\gamma = 0$. Hence, the sectorial forms can be regarded as a generalization of symmetric forms bounded from below. In some cases it is convenient to use the perturbation theory to prove that a form is sectorial. For this purpose we define the notion of relative boundedness which specifies a relation between two forms.

Definition 2.1.4. Let t be a sectorial form in $\mathscr{H}$. A form t' in $\mathscr{H}$ is said to be relatively bounded with respect to t (or t-bounded), if $\mathrm{Dom}(t') \supset \mathrm{Dom}(t)$ and

$$|t'[u]| \leq a\|u\|^2 + b|t[u]|, \tag{2.1.0}$$

where $u \in \mathrm{Dom}(t)$ and a, b are nonnegative constants.

This property is useful in case that we are able to divide the examined form into the sum of a sectorial form and its relatively bounded perturbation.

Theorem 2.1.5. *([Theorem VI-1.33] [12]) Let t be a sectorial form and let t' be t-bounded with $b < 1$ in (2.1.4). Then $t + t'$ is sectorial. The form $t + t'$ is closed, if and only if t is closed.*

In case of linear opeartors, the notion of sectoriality becomes more complicated. Let us start with the following definitions.

Definition 2.1.6. A linear operator T in $\mathscr{H}$ is said to be accretive if $Re(\psi, T\psi) \geq 0$ for all $\phi \in \mathrm{Dom}(T)$, and quasi-accretive if $T + \alpha I$ is accretive for some $\alpha > 0$.

Definition 2.1.7. A closed linear operator T in $\mathscr{H}$ is said to be m-accretive if it satisfies

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\} \subset \rho(T),$$

$$\|(T-\lambda I)^{-1}\| \leq \frac{1}{|\operatorname{Re}\lambda|} \qquad \textit{for} \quad \operatorname{Re}\lambda < 0$$

If $T + \alpha I$ is m-accretive for some $\alpha > 0$, then T is said to be quasi-m-accretive.

If an operator is m-accretive it means that it is *maximal accretive* in the sense that it is accretive and there is no proper accretive extension.

Definition 2.1.8. A linear operator T in $\mathscr{H}$ is said to be sectorial if its numerical range lies in a sector defined by (2.1.3). We say that T is m-sectorial if it is sectorial and quasi-m-accretive

An important property of m-sectorial operators is that they are closed and densely defined. If a form t is bounded we can associate with it a bounded operator T so that $t(\psi, \varphi) = (\psi, T\varphi)$. This claim can be extended to densely defined, sectorial and closed form. In this case the associated operator is m-sectorial.

Theorem 2.1.9. *(The first representation theorem, [12, Theorem VI-2.1]) Let t be a densely defined, closed, sectorial sesquilinear form in $\mathscr{H}$. There exists an m-sectorial operator T such that*

i) *$Dom(T) \subset Dom(t)$ and*

$$t(u, v) = (u, Tv)$$

for every $u \in Dom(t)$ and $v \in Dom(T)$;

ii) *$Dom(T)$ is a core of t;*

iii) *if $v \in Dom(t)$, $w \in \mathscr{H}$ and*

$$t(u, v) = (w, v)$$

holds for every u belonging to a core of t, then $v \in Dom(T)$ and $Tv = w$.

The m-sectorial operator T is uniquely defined by the condition i

Furthermore, from the above theorem follows that there is a one-to-one correspondence between the set of all m-sectorial operators and the set of all densely defined, closed and sectorial sesquilinear forms.

2.2 Elements of spectral theory

First of all, we state some elementary definitions.

Definition 2.2.1. The linear operator H on a Hilbert space $\mathscr{H}$ is called *symmetric* if and only if

$$(H\psi, \varphi) = (\psi, H\varphi)$$

for all elements ψ, φ from the domain $\operatorname{Dom}(H)$.

Definition 2.2.2. Given a densely defined linear operator H on a Hilbert space $\mathscr{H}$, by the Riesz representation theorem there is a unique adjoint operator H^* defined by

$$(H\psi, \varphi) = (\psi, H^*\varphi), \quad \forall \psi \in \mathrm{Dom}(H), \varphi \in \mathrm{Dom}(H^*)$$

such that

$$\mathrm{Dom}(H^*) := \{\varphi \in \mathscr{H} \mid \exists \eta \in \mathscr{H}, \forall \psi \in \mathrm{Dom}(H), (H\psi, \varphi) = (\varphi, \eta)\}.$$

Definition 2.2.3. We say that an operator H is *self-adjoint* if and only if it is symmetric and $\mathrm{Dom}(H) = \mathrm{Dom}(H^*)$.

Definition 2.2.4. The operator H on a Hilbert space $\mathscr{H}$ is bounded from below if there exists a real number K such that

$$(\psi, H\psi) \geq K\|\psi\|^2 \quad \forall \psi \in \mathrm{Dom}(H).$$

We denote $H \geq K$.

Definition 2.2.5. We say that an operator H in a Hilbert space $\mathscr{H}$ is closed if and only if

$$\left.\begin{array}{r} \mathrm{Dom}(H) \ni \psi_n \xrightarrow{n\to\infty} \psi \in \mathscr{H} \\ H\psi_n \xrightarrow{n\to\infty} \phi \in \mathscr{H} \end{array}\right\} \implies (\psi \in \mathrm{Dom}(H) \wedge H\psi = \phi)\,.$$

Now we are able to define the spectrum $\sigma(H)$ of a closed operator H. We say that $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(H)$ if and only if the operator $H - \lambda I$ is not a bijection between $\mathrm{Dom}(H)$ and $\mathscr{H}$. This condition is met in the following situations:

1. The operator $H - \lambda I$ is not injective, *i.e.* there exists a non-zero vector $x \in \mathrm{Dom}(H)$ such that $Hx = \lambda x$. The number λ is then called the *eigenvalue* of H and the set of all eigenvalues forms the *point spectrum* of H denoted by $\sigma_p(H)$. The vector x corresponding to λ is called the *eigenvector* (if x is a function it is called an eigenfunction). The subspace $\ker(H - \lambda I)$ is the respective *eigenspace* of H and its dimension is the *multiplicity* of the eigenvalue λ.

2. The operator $H - \lambda I$ is not surjective but its range is dense in $\mathscr{H}$. Then we say that λ belongs to the *continuous spectrum* $\sigma_c(H)$.

3. The operator $H - \lambda I$ is not surjective and its range is not dense in $\mathscr{H}$. Then we say that λ belongs to the *residual spectrum* $\sigma_r(H)$.

The set $\varrho(H) := \mathbb{C} \setminus \sigma(H) = \{\lambda \in \mathbb{C} \mid (H - \lambda I)^{-1}\text{exists and is bounded}\}$ is called the *resolvent set* of H. The function R_H defined on $\varrho(H)$ by $R_H(\lambda) := (H - \lambda I)^{-1}$ is called the *resolvent* of H. The same name often refers to the operator $(H - \lambda I)^{-1}$. If the operator H is not closed then $\sigma(H) = \mathcal{C}$, thus it makes sense to study only the spectrum of closed operators. In case of self-adjoint operators the spectrum is a subset of real numbers ([4, Theorem 1.2.10]). By the closed-graph theorem [12, Section III-5.4], the pathological situation of $\lambda \in \sigma(H) \setminus \sigma_p(H)$ with $\mathrm{Ran}(H - \lambda I) = \mathscr{H}$ cannot occur, thus

$$\sigma(H) = \sigma_p(H) \cup \sigma_c(H) \cup \sigma_r(H)$$

and the unions are disjoint. However, for the purposes ot this thesis, it is more convenient to divide the spectrum in the following way.

Definition 2.2.6. Let H be a closed operator in a Hilbert space $\mathscr{H}$. Then we define the *discrete spectrum* $\sigma_{\text{disc}}(H)$ as a set of eigenvalues of finite multiplicity which are isolated point of the spectrum. The *essential spectrum* is consequently defined by

$$\sigma_{\text{ess}}(H) := \sigma(H) \setminus \sigma_{\text{disc}}(H).$$

We say that H has a *purely discrete spectrum* if $\sigma_{\text{ess}}(H) = \emptyset$.

The resolvent of a closed operator can be used to detemine if the operator has a purely discrete spectrum.

Theorem 2.2.7. *Let H be a closed operator. Suppose that there exists $\lambda_0 \in \varrho(H)$ such that $R_H(\lambda_0)$ is compact. Then R_H is compact for all $\lambda \in \varrho(H)$ and H has a purely discrete spectrum.*

Recalling the sectorial forms from previous section, we can use the perturbation theory to prove that an operator has a compact resolvent.

Theorem 2.2.8. *([12, Theorem VI-3.4]) Let s be a densely defined, closed sectorial form with $\operatorname{Re} s \geq 0$ and let S be the associated m-sectorial operator. Let $\tilde{t}$ be a s-bounded form satisfying (2.1.4) with $b < \frac{1}{2}$. Then $t = s + \tilde{t}$ is also sectorial and closed. Let T be the associated m-sectorial operator. Then the resolvents R_S and R_T exist. If S has a compact resolvent, the same is true of T.*

In case of self-adjoint operators we have the following useful criterion of *lambda* belonging to the spectrum.

Theorem 2.2.9. *(Weyl criterion) A number $\lambda \in \mathbb{R}$ lies in the spectrum of a self-adjoint operator H if and only if there exists a sequence of functions $\{\psi_n\}_{n\in\mathbb{N}}$, such that*

1. *$\psi_n \in Dom(H)$, $\forall n \in \mathbb{N}$*
2. *$\|\psi_n\| = 1$, $\forall n \in \mathbb{N}$*
3. *$\|H\psi_n - \lambda\psi_n\| \xrightarrow{n\to\infty} 0$.*

The following variation technique is useful for locating the discrete eigenvalues below the essential spectrum of a self-adjoint operator.

Theorem 2.2.10. *(Minimax principle, [4, Theorem 4.5.2]) Let H be a self-adjoint operator that is bounded from below on a Hilbert space $\mathscr{H}$ and let $\{\lambda_m\}_{m\in\mathbb{N}}$ be a non-decreasing sequence of real numbers defined by*

$$\lambda_m := \inf\left\{ \sup_{\psi\in\mathcal{P}} \frac{(\psi, H\psi)}{\|\psi\|^2} \,\middle|\, \mathcal{P} \subset Dom(H), \quad \dim\mathcal{P} = m \right\}.$$

Then one of the following cases occurs.

1. *σ_{ess} is empty if $\lambda_m \xrightarrow{m\to\infty} \infty$*
2. *There exists $a < \infty$ such that $\lambda_m < a$, $\forall m \in \mathbb{N}$ and $\lambda_m \xrightarrow{m\to\infty} a$. Then a is the smallest number of the essential spectrum and the part of the spectrum of H in $(-\infty, a)$ consists of the eigenvalues λ_m each repeated a number of times equal to its multiplicity.*
3. *There exists $a < \infty$ and $N < \infty$ such that $\lambda_N < a$ but $\lambda_m = a$ for all $m > N$. Then $a = \inf \sigma_{ess}$ and the part of the spectrum of H in $(-\infty, a)$ consists of the eigenvalues $\lambda_1, \ldots, \lambda_N$ each repeated a number of times equal to its multiplicity.*

2.3 Sobolev spaces

When dealing with partial differential equations, it might be difficult to find a solution among functions with derivatives understood in the classical sense. We remedy this problem by presenting the Sobolev spaces in which the derivatives are understood in a weak sense. In this section we give a short introduction into Sobolev spaces, more detailed information can be found in [1].

We avoid the general theory of Sobolev spaces and present the notions for the special case relevant to this thesis. Let us consider smooth functions in the interval $(-a, a)$. We define the *support* of a function f as the closure of the set of points $x \in (-a, a)$ such that $f(x) \neq 0$. We call *distribution* a linear functional mapping smooth functions with a compact support contained in $(-a, a)$ to complex numbers. Every locally integrable function f in $(-a, a)$ can be regarded as a distribution ϕ_f defined by

$$\phi_f(g) := \int_{-a}^{a} f(x)g(x)dx,$$

where g is an arbitrary smooth function with a compact support contained in $(-a, a)$. We define the *weak derivative* $D^\alpha \phi$ of a distribution ϕ with $\alpha \in \mathbb{N}$ as

$$(D^\alpha \phi)(g) := (-a)^\alpha \phi \left(\frac{d^\alpha g}{dx^\alpha} \right).$$

The weak derivative can be regarded as a generalization of the classical derivative, as some functions which are not differentiable in the classical sense have a weak derivative. For example the absolute value function is not differentiable in the classical sense but it is in the weak sense.

Now we can define the Sobolev space $W^{k,p}((-a, a))$ as the subset of functions in $L^p((-a, a))$ such that the weak derivative exists and belongs to $L^p((-a, a))$ for all $\alpha \leq k$, where $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$. The norm in this space is defined by

$$\|f\|_{W^{k,p}((-a,a))} := \sum_{\alpha \leq k} \|D^\alpha f\|_{L^p((-a,a))}. \tag{2.3.0}$$

We define the Sobolev space $W_0^{k,p}((-a, a))$ as the closure of $C_0^\infty((-a, a))$ with respect to the norm (2.3), where $C_0^\infty((-a, a))$ represents the set of all smooth functions with a compact support contained in $(-a, a)$.

Chapter 3

Straight Planar Strip

In the first section of this chapter we give proper definition of the Laplacian in a straight waveguide with Robin-type boundary conditions using the quadratic forms. In the second section we discuss the spectral properties of the Laplacian. Additionally, we mention Laplacians with Dirichlet and Neumann boundary conditions as special cases of the Robin Laplacian.

3.1 Definition of the Hamiltonian

Let us define the straight waveguide as the Cartesian product $\Omega_0 := \mathbb{R} \times (-a, a)$, where a is a positive number. We define here coordinates (s, t), where $s \in \mathbb{R}$ is the coordinate in the longitudinal direction and $t \in (-a, a)$ is the coordinate in the transversal direction. From now on we denote by "$_{,s}$" and "$_{,t}$" the partial derivatives with respect to s and t, respectively.

We are interested in the solutions ψ of the following stationary Schrödinger equation for a free particle with energy λ confined to the waveguide Ω_0 and satisfying the Robin-type boundary

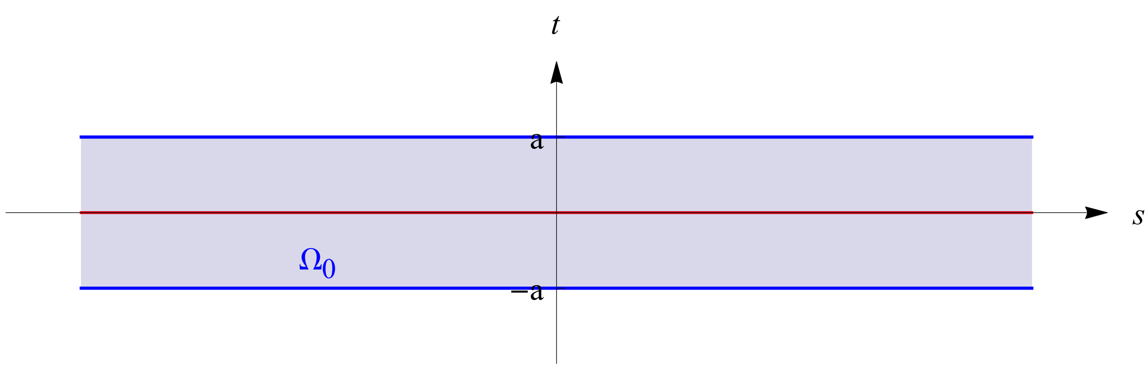

Figure 3.1: Straight waveguide Ω_0 of width $2a$.

conditions on $\partial\Omega_0$:

$$\begin{cases} -\triangle\psi = \lambda\psi & \text{in } \Omega_0 \\ \frac{\partial\psi}{\partial t} + i\alpha\psi = 0 & \text{on } \partial\Omega_0, \end{cases} \tag{3.1.0}$$

where α is a given real number.

The problem (3.1) can be regarded as a spectral problem for an m-sectorial operator in $L^2(\Omega_0)$ which we denote here by $-\triangle_\alpha^{\Omega_0}$. It turns out that it is correctly defined by

$$\begin{aligned} -\triangle_\alpha^{\Omega_0}\psi &:= -\triangle\psi \\ \mathrm{Dom}(-\triangle_\alpha^{\Omega_0}) &:= \left\{\psi \in W^{2,2}(\Omega_0) \;\middle|\; \frac{\partial\psi}{\partial t} + i\alpha\psi = 0 \;\text{ on } \partial\Omega_0\right\}. \end{aligned} \tag{3.1.1}$$

The derivations are understood in the weak sense and the Lapacian acts in the distributional sense.

The two-dimensional spectral problem (3.1) can be formally simplified using the separation of variables. Namely, we assume that every solution ψ can be rewritten as

$$\psi(s,t) = f(s)g(t).$$

Putting this ansatz to (3.1), we obtain

$$-\frac{f''(s)}{f(s)} - \frac{g''(t)}{g(t)} = \lambda \quad \text{in } \Omega_0.$$

This equation makes sense only for such $(s,t) \in \Omega_0$ that $f(s) \neq 0$ and $g(t) \neq 0$. The only functions f, g satisfying this formula are constant functions, thus we can write

$$\lambda + \frac{f''(s)}{f(s)} = -\frac{g''(t)}{g(t)} = C,$$

where C is a constant. Taking into account the Robin-type boundary conditions in (3.1), we obtain the following boundary conditions for g:

$$g'(\pm a) + i\alpha g(\pm a) = 0. \tag{3.1.1}$$

In conclusion, we transformed the two-dimensional problem (3.1) into two independent one-dimensional problems for the transversal operator $-\triangle_\alpha^{(-a,a)}$ and the longitudinal operator $-\triangle^{\mathbb{R}}$. In other words, the Hilbert space $L^2(\Omega_0)$ can be expressed in terms of tensor product as

$$L^2(\Omega_0) = L^2((-a,a)) \otimes L^2(\mathbb{R}) \tag{3.1.1}$$

and the operator $-\triangle_\alpha^{\Omega_0}$ can be consequently rewritten as

$$-\triangle_\alpha^{\Omega_0} = -\triangle_\alpha^{(-a,a)} \otimes I^{\mathbb{R}} + I^{(-a,a)} \otimes -\triangle^{\mathbb{R}}. \tag{3.1.1}$$

3.1.1 The transversal Hamiltonian

Firstly, let us address the proper definition of the transversal Hamiltonian $-\triangle_\alpha^{(-a,a)}$. It turns out that the correct definition is similarly to (3.1.1) given by

$$\begin{aligned} -\triangle_\alpha^{(-a,a)}\varphi &:= -\varphi'' \\ \mathrm{Dom}(-\triangle_\alpha^{(-a,a)}) &:= \left\{\varphi \in W^{2,2}((-a,a)) \;\middle|\; \varphi'(\pm a) + i\alpha\varphi(\pm a) = 0\right\}. \end{aligned} \tag{3.1.2}$$

Let us consider the quadratic form $\dot{h}_\alpha$ associated with the operator $-\triangle_\alpha^{(-a,a)}$. Using the integration by parts together with the boundary conditions (3.1), we get

$$\begin{aligned}\dot{h}_\alpha[\varphi] :=&(\varphi, -\triangle_\alpha^{(-a,a)}\varphi)_{L^2((-a,a))}\\ =&-\int_{-a}^{a}\overline{\varphi(t)}\varphi''(t)dt=\int_{-a}^{a}\overline{\varphi'(t)}\varphi'(t)dt-\left[\overline{\varphi(t)}\varphi'(t)\right]_{-a}^{a}\\ =&\|\varphi'\|^2+i\alpha|\varphi(a)|^2-i\alpha|\varphi(-a)|^2.\end{aligned} \tag{3.1.3}$$

In order to prove that the operator (3.1.2) is m-sectorial, let us start from the form which acts as $\dot{h}$. We define

$$\begin{aligned}h_\alpha[\varphi] &:= \|\varphi'\|^2+i\alpha|\varphi(a)|^2-i\alpha|\varphi(-a)|^2\\ \mathrm{Dom}(h_\alpha) &:= W^{1,2}(\mathbb{R}).\end{aligned} \tag{3.1.4}$$

The following lemma will help us prove that the form h is sectorial.

Lemma 3.1.1. *The inequality $|\varphi(\pm a)|^2\leq \frac{1}{2a}\|\varphi\|^2+2\|\varphi\|\|\varphi'\|$ holds for all $\varphi \in Dom(h_\alpha)$.*

Proof. We prove the inequality for the point a only, proof for $-a$ is analogous. Let us define an auxiliary function

$$\eta(x) = \begin{cases} 0 & x \in (-\infty, -a) \\ \frac{x+a}{2a} & x \in [-a, a] \\ 1 & x \in (a, \infty). \end{cases}$$

Using the properties of η combined with the Schwarz inequality, we obtain

$$\begin{aligned}|\varphi(a)|^2=&\int_{-a}^{a}\frac{d}{dt}\left(\eta(t|\varphi(t)|^2\right)dt\\ =&\int_{-a}^{a}\eta'(t)|\varphi(t)|^2dt+2\int_{-a}^{a}\eta(t)|\varphi(t)||\varphi'(t)|dt\\ \leq&\frac{1}{2a}\int_{-a}^{a}|\varphi(t)|^2dt+2\int_{-a}^{a}|\varphi(t)||\varphi'(t)|dt\\ \leq&\frac{1}{2a}\|\varphi\|^2+2\|\varphi\|\|\varphi'\|.\end{aligned}$$

□

In order to prove that the form h_α is sectorial, we regard it as a perturbation of the form associated with the Neumann Laplacian acting in $L^2((-a,a))$. Let us now divide the form h into two parts:

$$\begin{aligned}h_1[\varphi] &= \|\varphi'\|^2\\ h_2[\varphi] &= i\alpha|\varphi(a)|^2-i\alpha|\varphi(-a)|^2,\end{aligned} \tag{3.1.5}$$

where h_1 is the form associated with the Neumann Laplacian. Note that the domains of these forms are identical, *i.e.* $\mathrm{Dom}(h_1) = \mathrm{Dom}(h_2) = W^{1,2}((-a,a))$.

Theorem 3.1.2. *Let h_1 and h_2 be the forms defined by (3.1.5). Then the form h_2 is h_1-bounded and the constant b in (2.1.4) can be taken arbitrarily small.*

Proof. Using the Lemma 3.1.1, we can estimate the form h_2 by

$$\begin{aligned}|h_2[\varphi]| &\le |\alpha||\varphi(-a)|^2 + |\alpha||\varphi(a)|^2 \\ &\le |\alpha|\left(\frac{1}{a}\|\varphi\|^2 + 4\|\varphi\|\|\varphi'\|\right) \\ &\le |\alpha|\left(\frac{1}{a}\|\varphi\|^2 + \frac{4}{\varepsilon}\|\varphi\|^2 + \varepsilon\|\varphi'\|^2\right) \\ &= a\|\varphi\|^2 + b|h_1[\varphi]|,\end{aligned}$$

where $a = |\alpha|\left(\frac{1}{a} + \frac{4}{\varepsilon}\right)$ and $b = \varepsilon|\alpha|$. In the last inequality we used the Young inequality $2xy \le \frac{1}{\varepsilon}x^2 + \varepsilon y^2$ with $x = 2\|\varphi\|$ and $y = \|\varphi'\|$. This estimate holds for all $\varepsilon > 0$, thus the constant b can be taken arbitrarily small. Since the domains of the forms h_1 and h_2 are identical, we conclude that the form h_2 is h_1-bounded. □

Corollary 3.1.3. *The quadratic form h_α defined by (3.1.4) is densely defined, sectorial and closed.*

Proof. The domain $\mathrm{Dom}(h_\alpha) = W^{1,2}((-a,a))$ contains the set $C_0^\infty((-a,a))$ consisting of smooth functions with compact support. This set is dense in $L^2((-a,a))$ ([1, Theorem 2.19]), thus $\mathrm{Dom}(h)$ is dense in $L^2((-a,a))$ and the form h_α is densely defined.

The form h_1 associated with the Neumann Laplacian is sectorial and closed [6, Chapter IV]. According to the Theorem 3.1.2 the form h_2 is h_1-bounded with arbitrarily small constant b, therefore the form $h_\alpha = h_1 + h_2$ is by the Theorem 2.1.5 sectorial and closed. □

In the light of this corollary, we can use the First representation theorem 2.1.9 to associate the form h_α with an m-sectorial operator defined by

$$\begin{aligned} H_\alpha\varphi &= \eta \\ \mathrm{Dom}(H_\alpha) &= \left\{\varphi \in \mathrm{Dom}(h_\alpha) \;\middle|\; \begin{array}{l} \exists \eta \in L^2((-a,a)), \\ \forall \psi \in \mathrm{Dom}(h_\alpha), h_\alpha(\psi,\varphi) = (\psi,\eta) \end{array}\right\}. \end{aligned} \tag{3.1.6}$$

In fact, $\eta = -\varphi''$ and the Hamiltonian H_α equals to the operator $-\triangle_\alpha^{(-a,a)}$ defined by (3.1.2), as shown in the following theorem.

Theorem 3.1.4. *The Hamiltonian H_α defined by (3.1.6) equals to the operator $-\triangle_\alpha^{(-a,a)}$ defined by (3.1.2).*

Proof. If $\psi \in W^{2,2}$, there exists its second derivative and we can integrate $h_\alpha(\phi,\psi) = (\phi',\psi') - \left[\bar{\phi}\psi'\right]_{-a}^{a}$ by parts to get $h_\alpha(\phi,\psi) = (\phi,-\psi'')$. Thus $\eta := -\phi''$ and $H_\alpha \supset -\triangle_\alpha^{(-a,a)}$.

It remains to prove that $H_\alpha \subset -\triangle_\alpha^{(-a,a)}$. The course of the proof is inspired by [12, Example VI-2.16]. Let $\phi \in \mathrm{Dom}(h_\alpha)$, $\psi \in \mathrm{Dom}(H_\alpha)$ and $\eta \in L^2((-a,a))$ be the function defined by (3.1.6). Let ζ be the indefinite integral of η, then the relation $h_\alpha(\phi,\psi) = (\phi, T\psi) = (\phi,\eta)$ means

$$\int_{-a}^{a} \bar{\phi}'\psi' + i\alpha\overline{\phi(a)}\psi(a) - i\alpha\overline{\phi(-a)}\psi(-a) = \int_{-a}^{a}\bar{\phi}\eta = \int_{-a}^{a}\bar{\phi}\zeta' = \left[\bar{\phi}\zeta\right]_{-a}^{a} - \int_{-a}^{a}\bar{\phi}'\zeta.$$

This equation can be subsequently rewritten as

$$\int_{-a}^{a} \overline{\phi}'(\psi' + \zeta) + \overline{\phi(a)}(i\alpha\psi(a) - \zeta(a)) + \overline{\phi(-a)}(i\alpha\psi(-a) + \zeta(-a)) = 0 \tag{3.1.6}$$

which holds for every $\phi \in \text{Dom}(h_\alpha)$. However, in the next step we make a special choice of ϕ. For any $\phi' \in L^2((-a,a))$ satisfying $\int_{-a}^{a} \phi'(t)dt = 0$, the function $\phi(t) = \int_{-a}^{t} \phi(\tilde{t})d\tilde{t}$ lies in $\text{Dom}(h_\alpha)$ and $\phi(-a) = \phi(a) = 0$, so that $\psi' + \zeta$ is orthogonal to ϕ' by (3.1.4). The relation $\int_{-a}^{a} \phi'(t)dt = 0$ means that ϕ' is orthogonal to the constant function equal to 1, thus $\psi' + \zeta \in \left\{1^\perp\right\}^\perp = span\,\{1\}$, *i.e.*

$$\psi' + \zeta = C, \tag{3.1.6}$$

where C is a constant. Substituting this identity into (3.1.4), we get

$$\overline{\phi(a)}(i\alpha\psi(a) - \zeta(a) + C) + \overline{\phi(-a)}(i\alpha\psi(-a) + \zeta(-a) - C) = 0. \tag{3.1.6}$$

Since $\phi(a)$ and $\phi(-a)$ may attain any complex number when ϕ varies over $\text{Dom}(\dot{h}_\alpha)$, their coefficients in (3.1.4) must vanish. Together with (3.1.4) we therefore obtain

$$C = \zeta(\pm a) - i\alpha\psi(\pm a) = \zeta(\pm a) + \psi'(\pm a)$$

which means that every $\psi \in \text{Dom}(H_\alpha)$ satisfies the Robin boundary conditions

$$\psi'(\pm a) + i\alpha\psi(\pm a) = 0. \tag{3.1.6}$$

Furthermore, the derivation of (3.1.4) yields

$$-\psi'' = \zeta' = \eta \in L^2((-a,a)),$$

thus $\psi \in W^{2,2}((-a,a))$ and we conclude that $\text{Dom}(H_\alpha) \subset \text{Dom}(-\triangle_\alpha^{(-a,a)})$. Since $\eta = H_\alpha\psi = -\psi''$, we have proved that $H_\alpha \subset -\triangle_\alpha^{(-a,a)}$. □

The operator $-\triangle_\alpha^{(-a,a)}$ is not self-adjoint, nevertheless there exists a simple prescription for the adjoint operator.

Proposition 3.1.5. *Let $-\triangle_\alpha^{(-a,a)}$ be the operator defined by (3.1.2). Then its adjoint operator is defined by*

$$\left(-\triangle_\alpha^{(-a,a)}\right)^* = -\triangle_{-\alpha}^{(-a,a)}\,.$$

Proof. The operator $-\triangle_\alpha^{(-a,a)}$ is associated with the sesquilinear form

$$h_\alpha(\psi,\varphi) := \int_{-a}^{a} \overline{\psi'(t)}\varphi'(t)dt + i\alpha\overline{\psi(a)}\varphi(a) - i\alpha\overline{\psi(-a)}\varphi(-a).$$

It is convenient to work with sesquilinear forms, since it is easy to find the adjoint forms. Indeed, the adjoint form of h_α is given by

$$h_\alpha^*(\psi,\varphi) := \overline{h_\alpha(\varphi,\psi)} := \int_{-a}^{a} \overline{\psi'(t)}\varphi'(t)dt - i\alpha\overline{\psi(a)}\varphi(a) + i\alpha\overline{\psi(-a)}\varphi(-a).$$

This form is associated with the adjoint operator and since we have proved the Theorem 3.1.4 for all $\alpha \in \mathbb{R}$, it is also associated with $-\triangle_{-\alpha}^{(-a,a)}$. □

3.1.2 The longitudinal Hamiltonian

We will define the free Hamiltonian $-\triangle^{\mathbb{R}}$ via Friedrichs extension. Let us define the minimal Hamiltonian

$$\begin{aligned} \dot{H}\psi &:= -\triangle\psi \\ \operatorname{Dom}(\dot{H}) &:= C_0^\infty(\mathbb{R}), \end{aligned} \tag{3.1.7}$$

where $C_0^\infty(\mathbb{R})$ consists of smooth functions on $\mathbb{R}$ with compact support. This operator is associated with the quadratic form

$$\begin{aligned} \dot{h}[\psi] &= (\psi, \dot{H}\psi) = -\int_{-\infty}^{\infty} \bar{\psi}\psi'' = \int_{-\infty}^{\infty} \overline{\psi'}\psi' = \|\psi'\|^2 \\ \operatorname{Dom}(\dot{h}) &= \operatorname{Dom}(\dot{H}) = C_0^\infty(\mathbb{R}) \end{aligned} \tag{3.1.8}$$

Since the free Hamiltonian $\dot{H}$ is symmetric and bounded from below, its associated form $\dot{h}$ is closable ([12, Corollary VI-1.28]). The domain of the closure of $\dot{h}$ is given by the closure of $C_0^\infty(\mathbb{R})$ with respect to the topology induced by $\dot{h}$ which coincides with the norm (2.3), therefore it is by definition equal to $W_0^{1,2}(\mathbb{R})$. Additionally, in our case of the real axis we have $W_0^{1,2}(\mathbb{R}) = W^{1,2}(\mathbb{R})$. The closure of $\dot{h}$ is then defined by

$$\begin{aligned} h[\psi] &= \|\psi'\|^2 \\ \operatorname{Dom}(h) &= W^{1,2}(\mathbb{R}). \end{aligned} \tag{3.1.9}$$

Using the First representation theorem 2.1.9, we can associate with h an m-sectorial operator H given by

$$\begin{aligned} H\psi &:= \eta \\ \operatorname{Dom}(H) &:= \left\{\psi \in \operatorname{Dom}(h) \;\middle|\; \exists \eta \in L^2(\mathbb{R}), \forall \phi \in \operatorname{Dom}(h), h(\psi,\phi) = (\psi,\eta)\right\}. \end{aligned}$$

By the same argumentation as in the proof of the Theorem 3.1.4, we conclude that the Hamiltonian H equals to the operator $-\triangle^{\mathbb{R}}$ defined by

$$\begin{aligned} -\triangle^{\mathbb{R}}\psi &: -\triangle\psi \\ \operatorname{Dom}(-\triangle^{\mathbb{R}}) &:= W^{2,2}(\mathbb{R}). \end{aligned} \tag{3.1.10}$$

3.2 Spectral analysis

In this section we find the spectrum of the operator $-\triangle_\alpha^{\Omega_0}$ defined by (3.1.1). We divide this task into the study of the spectrum of the transversal operator $-\triangle_\alpha^{(-a,a)}$ and the longitudinal operator $-\triangle^{\mathbb{R}}$. In the transversal case we also mention the limit situations of Neumann and Dirichlet boundary conditions. Altogether, we will consider the following three boundary conditions on the interval $(-a, a)$:

$$\begin{array}{ll} \text{Neumann} & \phi'(\pm a) = 0 \\ \text{Dirichlet} & \phi(\pm a) = 0 \\ \text{Robin} & \phi'(\pm a) + i\alpha\phi(\pm a) = 0. \end{array} \tag{3.2.0}$$

Note that the Dirichlet boundary conditions are the limit case of Robin boundary conditions as α tends to infinity while the Neumann case can be obtained by setting $\alpha = 0$.

3.2.1 The spectrum of the transversal Hamiltonian

Let us start with the study of the point spectrum of the Hamiltonian $-\triangle_{\alpha}^{(-a,a)}$ which means to solve the spectral problem

$$\begin{cases} -\psi'' = \lambda\psi \\ \psi'(\pm a) + i\alpha\psi(\pm a) = 0. \end{cases} \tag{3.2.0}$$

The general solution of the above differential equation is

$$\psi(t) = A\sin(\sqrt{\lambda}t) + B\cos(\sqrt{\lambda}t), \tag{3.2.0}$$

where A and B are complex constants. This general solution is common to all the boundary conditions and we can obtain the point spectrum by imposing each of the boundary conditions in (3.2). In the case of Robin boundary conditions we get the following system of equations for A and B:

$$\begin{aligned} A\sqrt{\lambda}\cos(\sqrt{\lambda}a) + i\alpha A\sin(\sqrt{\lambda}a) - B\sqrt{\lambda}\sin(\sqrt{\lambda}a) + i\alpha B\cos(\sqrt{\lambda}a) = 0 \\ A\sqrt{\lambda}\cos(\sqrt{\lambda}a) - i\alpha A\sin(\sqrt{\lambda}a) + B\sqrt{\lambda}\sin(\sqrt{\lambda}a) + i\alpha B\cos(\sqrt{\lambda}a) = 0. \end{aligned}$$

We are interested only in non-trivial solutions, *i.e.* such that $A \neq 0$ and $B \neq 0$. In order to derive the condition for λ belonging to the point spectrum, we rewrite the system as a matrix equation

$$\mathbb{M}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.2.0}$$

with

$$\mathbb{M} = \begin{pmatrix} \sqrt{\lambda}\cos(\sqrt{\lambda}a) + i\alpha\sin(\sqrt{\lambda}a) & -\sqrt{\lambda}\sin(\sqrt{\lambda}a) + i\alpha\cos(\sqrt{\lambda}a) \\ \sqrt{\lambda}\cos(\sqrt{\lambda}a) - i\alpha\sin(\sqrt{\lambda}a) & \sqrt{\lambda}\sin(\sqrt{\lambda}a) + i\alpha\cos(\sqrt{\lambda}a) \end{pmatrix}.$$

Since the equation (3.2.1) is homogeneous, there exists a non-trivial solution if $\det\mathbb{M} = 0$. This condition yields

$$(\lambda - \alpha^2)\sin(2\sqrt{\lambda}a) = 0,$$

thus we conclude that the point spectrum is given by

$$\sigma_p\left(-\triangle_{\alpha}^{(-a,a)}\right) = \{\alpha^2\} \cup \left\{ \left(\frac{k\pi}{2a}\right)^2 \,\middle|\, k \in \mathbb{N} \right\}. \tag{3.2.0}$$

Zero is excluded from the spectrum because it leads to the eigenfunction $\psi(t) = B$ which meets the Robin boundary conditions only if $B = 0$. Hereafter we shall denote the eigenvalues by $\lambda_k = \left(\frac{k\pi}{2a}\right)^2$, with the convention that $\lambda_0 = \alpha^2$. The obtained eigenvalues now can be associated with the corresponding eigenfunctions by solving the system (3.2.1) with the given eigenvalues. For the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ the matrix $\mathbb{M}$ in (3.2.1) takes form

$$\mathbb{M} = \begin{pmatrix} \frac{k\pi}{2a}\cos(\frac{k\pi}{2}) + i\alpha\sin(\frac{k\pi}{2}) & -\frac{k\pi}{2a}\sin(\frac{k\pi}{2}) + i\alpha\cos(\frac{k\pi}{2}) \\ \frac{k\pi}{2a}\cos(\frac{k\pi}{2}) - i\alpha\sin(\frac{k\pi}{2}) & \frac{k\pi}{2a}\sin(\frac{k\pi}{2}) + i\alpha\cos(\frac{k\pi}{2}) \end{pmatrix}.$$

The corresponding eigenfunctions then differ for odd and even eigenvalues λ_k:

$$\psi_k(t) = \begin{cases} B\left(-\frac{k\pi i}{2\alpha a}\sin(\frac{k\pi}{2a}t) + \cos(\frac{k\pi}{2a}t)\right) & \text{for odd } k \\ B\left(-\frac{2\alpha a i}{k\pi}\sin(\frac{k\pi}{2a}t) + \cos(\frac{k\pi}{2a}t)\right) & \text{for even } k, \end{cases}$$

where B is a complex constant. Similarly, for the eigenvalue $\lambda_0 = \alpha^2$ we get

$$\psi_0(t) = B\left(\cos(\alpha t)\right) - i\sin(\alpha t)) = Be^{-i\alpha t}.$$

The normalization condition $\|\psi_k\|= 1$ yields

$$\begin{aligned} 1 &= |B|^2 a\left(1 - \left(\tfrac{k\pi}{2\alpha a}\right)^2\right) && \text{for odd } k \\ 1 &= |B|^2 a\left(1 - \left(\tfrac{2\alpha a}{k\pi}\right)^2\right) && \text{for even } k, k \geq 2 \\ 1 &= |B|^2 \tfrac{\sin(2\alpha a)}{\alpha} && \text{for } k = 0. \end{aligned}$$

Thus, we conclude that the normalized eigenfunctions corresponding to the eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$ are given by

$$\psi_k(t) = \begin{cases} \sqrt{\frac{\alpha}{\sin(2\alpha a)}}e^{-i\alpha t} & \text{for } k = 0 \\ \frac{2\alpha a}{\sqrt{a((2\alpha a)^2-(k\pi)^2}}\left(-\frac{k\pi i}{2\alpha a}\sin(\frac{k\pi}{2a}t) + \cos(\frac{k\pi}{2a}t)\right) & \text{for odd } k \\ \frac{k\pi}{\sqrt{a((k\pi)^2-(2\alpha a)^2}}\left(-\frac{k\pi i}{2\alpha a}\sin(\frac{k\pi}{2a}t) + \cos(\frac{k\pi}{2a}t)\right) & \text{for even } k, k \geq 2. \end{cases}$$

In case of the Neumann boundary conditions the matrix $\mathbb{M}$ in (3.2.1) takes form

$$\mathbb{M} = \begin{pmatrix} \sqrt{\lambda}\cos(\sqrt{\lambda}a) & -\sqrt{\lambda}\sin(\sqrt{\lambda}a) \\ \sqrt{\lambda}\cos(\sqrt{\lambda}a) & \sqrt{\lambda}\sin(\sqrt{\lambda}a) \end{pmatrix}.$$

The condition $\det\mathbb{M} = 0$ then means

$$\lambda\sin(2\sqrt{\lambda}a) = 0$$

and the point spectrum is given by

$$\sigma_p\left(-\triangle_N^{(-a,a)}\right) = \left\{\left(\frac{k\pi}{2a}\right)^2 \,\middle|\, k \in \mathbb{N}\cup\{0\}\right\}.$$

In this case zero is an acceptable solution because it leads to $\psi(t) = B$ which meets the Neumann boundary condition. The corresponding eigenfunctins are obtained by solving the system (3.2.1) with $\lambda \in \sigma_p\left(-\triangle_N^{(-a,a)}\right)$:

$$\psi_k(t) = \begin{cases} \sqrt{\frac{1}{2a}} & \text{for } k = 0 \\ \sqrt{\frac{1}{a}}\sin(\frac{k\pi}{2a}t) & \text{for odd } k \\ \sqrt{\frac{1}{a}}\cos(\frac{k\pi}{2a}t) & \text{for even } k, k \geq 2. \end{cases}$$

Finally, the Dirichlet boundary conditions lead to the system with

$$\mathbb{M} = \begin{pmatrix} \sin(\sqrt{\lambda}a) & \cos(\sqrt{\lambda}a) \\ -\sin(\sqrt{\lambda}a) & \cos(\sqrt{\lambda}a) \end{pmatrix}.$$

The eigenvalues are restricted by

$$\sin(2\sqrt{\lambda}a) = 0,$$

thus the point spectrum is given by

$$\sigma_p\left(-\triangle_D^{(-a,a)}\right) = \left\{ \left(\frac{k\pi}{2a}\right)^2 \,\middle|\, k \in \mathbb{N} \right\}.$$

Similarly to the Robin case zero leads to a trivial solution and is excluded from the point spectrum. The corresponding eigenfunctions are consequently given by:

$$\psi_k(t) = \begin{cases} \sqrt{\frac{1}{a}}\cos(\frac{k\pi}{2a}t) & \text{for odd } k \\ \sqrt{\frac{1}{a}}\sin(\frac{k\pi}{2a}t) & \text{for even } k. \end{cases}$$

Since the resolvents associated with the Laplacians with Neumann and Dirichlet boundary conditions are compact ([4, Theorems 7.2.2, 6.2.3]), the corresponding spectra are purely discrete, thus we can write

$$\begin{aligned} \sigma\left(-\triangle_N^{(-a,a)}\right) &= \sigma_{\text{disc}}\left(-\triangle_N^{(-a,a)}\right) = \left\{ \left(\frac{k\pi}{2a}\right)^2 \,\middle|\, k \in \mathbb{N} \cup \{0\} \right\}, \\ \sigma\left(-\triangle_D^{(-a,a)}\right) &= \sigma_{\text{disc}}\left(-\triangle_D^{(-a,a)}\right) = \left\{ \left(\frac{k\pi}{2a}\right)^2 \,\middle|\, k \in \mathbb{N} \right\}. \end{aligned} \tag{3.2.1}$$

In the following proposition we show that the same holds for Robin Laplacian.

Proposition 3.2.1. *Let $-\triangle_\alpha^{(-a,a)}$ be the operator defined by (3.1.2). Then it has a compact resolvent and its spectrum is given by*

$$\sigma\left(-\triangle_\alpha^{(-a,a)}\right) = \sigma_{disc}\left(-\triangle_\alpha^{(-a,a)}\right) = \{\alpha^2\} \cup \left\{ \left(\frac{k\pi}{2a}\right)^2 \,\middle|\, k \in \mathbb{N} \right\}.$$

Proof. Let us consider the quadratic form h_α defined by (3.1.4) and associated with the operator $-\triangle_\alpha^{(-a,a)}$. It can be divided into the sum of h_1 and h_2 as in (3.1.5). From the Theorem 3.1.2 follows that h_2 is h_1-bounded with relative bound b arbitrarily small. The form h_1 is positive and associated with the Neumann Laplacian which has a compact resolvent, thus by Theorem 2.2.8 the Robin Laplacian $-\triangle_\alpha^{(-a,a)}$ has a compact resolvent and purely discrete spectrum. □

Note that although the operator $-\triangle_\alpha^{(-a,a)}$ is not self-adjoint, its spectrum is real.

3.2.2 The spectrum of the longitudinal Hamiltonian

Contrary to the transversal operator $-\triangle_\alpha^{(-a,a)}$, the spectrum of $-\triangle^{\mathbb{R}}$ is purely essential which we prove in the following proposition.

Proposition 3.2.2. *Let $-\triangle^{\mathbb{R}}$ be the operator defined by (3.1.10). Then its spectrum is given by*

$$\sigma\left(-\triangle^{\mathbb{R}}\right) = \sigma_{ess}\left(-\triangle^{\mathbb{R}}\right) = [0, \infty).$$

Proof. The operator $-\triangle^{\mathbb{R}}$ is associated with the quadratic form defined by (3.1.9). This form is non-negative, thus from the minimax principle follows $\sigma\left(-\triangle^{\mathbb{R}}\right) \subset [0,\infty)$.

To prove the opposite inclusion, we use the Weyl criterion 2.2.9. More precisely, for every $\lambda \in [0,\infty)$ we define

$$\varphi_n(s) := \frac{1}{\sqrt{n}}\varphi_1(\frac{s}{n}),$$

where $\varphi_1 \in C_0^\infty(\mathbb{R})$ and $\|\varphi_1\|= 1$. All elements of this sequence are normalized to 1 in $L^2(\mathbb{R})$ and their derivatives satisfy

$$\begin{aligned} \|\varphi_n'\|^2 &= \frac{1}{n^2}\|\varphi_1'\|^2 \\ \|\varphi_n''\|^2 &= \frac{1}{n^4}\|\varphi_1''\|^2. \end{aligned} \tag{3.2.2}$$

Now we can define the sequence

$$\psi_n(s) := \varphi_n(s)e^{i\sqrt{\lambda}s}$$

for which we will prove that satisfies the hypothesis of the Weyl criterion. All its members are normalized to 1 in $L^2(\mathbb{R})$ and lie in $\mathrm{Dom}(-\triangle^{\mathbb{R}})$. Moreover, using the identities (3.2.2), we obtain

$$\begin{aligned} \|(-\triangle^{\mathbb{R}}-\lambda)\psi_n(s)\|^2 &= \|(-\varphi_n''(s) - 2i\sqrt{\lambda}\varphi_n'(s)e^{i\sqrt{\lambda}s}\|^2 \\ &\leq \|\varphi_n''(s)\|^2+4\lambda\|\varphi_n'(s)\|^2 \xrightarrow{n\to\infty} 0. \end{aligned}$$

In conclusion, for every $\lambda \in [0,\infty)$ the sequence $\{\psi_n\}_{n=1}^{\infty}$ satisfies the hypothesis of the Weyl criterion, thus $\sigma\left(-\triangle^{\mathbb{R}}\right) \supset [0,\infty)$ and $\sigma\left(-\triangle^{\mathbb{R}}\right) = [0,\infty)$. Since the interval contains no isolated points, we have $\sigma\left(-\triangle^{\mathbb{R}}\right) = \sigma_{\mathrm{ess}}\left(-\triangle^{\mathbb{R}}\right)$. □

3.2.3 The spectrum of the Hamiltonian of a straight strip

We conclude this chapter by showing that the spectrum of $-\triangle_\alpha^{\Omega_0}$ is given by the sum of the spectra of the transversal and longitudinal Hamiltonians.

Theorem 3.2.3. *Let $-\triangle_\alpha^{\Omega_0}$ be the Hamiltonian defined by (3.1.1). Then its spectrum is given by*

$$\sigma\left(-\triangle_\alpha^{\Omega_0}\right) = \sigma_{ess}\left(-\triangle_\alpha^{\Omega_0}\right) = [\lambda_{min},\infty),$$

where $\lambda_{min} := \min\left\{\alpha^2, \left(\frac{\pi}{2a}\right)^2\right\}$.

Proof. The Hamiltonian $-\triangle_\alpha^{\Omega_0}$ can be expressed in terms of tensor product by (3.1). If $-\triangle_\alpha^{(-a,a)}$ and $-\triangle^{\mathbb{R}}$ generate bounded holomorphic semigroups on the Hilbert spaces $L^2((-a,a))$ and $L^2(\mathbb{R})$, respectively, then by [16, Theorem XIII.35] the spectrum of $-\triangle_\alpha^{\Omega_0}$ equals to the sum

$$\sigma\left(-\triangle_\alpha^{\Omega_0}\right) = \sigma\left(-\triangle_\alpha^{(-a,a)}\right) + \sigma\left(-\triangle^{\mathbb{R}}\right) = [\lambda_{\min},\infty).$$

In chapter 3 we proved that the Hamiltonians $-\triangle_\alpha^{(-a,a)}$ and $-\triangle^{\mathbb{R}}$ are m-sectorial with a vertex 0 and all such operators generate bounded holomorphic semigroups by [12, Theorem IX-1.24]. □

Chapter 4

Narrow Curved Waveguide

4.1 The Geometry

Let Γ be an infinite planar curve parametrised by its arc-length, *i.e.* a C^2-smooth map $\Gamma : \mathbb{R} \to \mathbb{R}^2 : s \mapsto (\Gamma_1(s), \Gamma_2(s)$ satisfying $|\dot{\Gamma}(s)|= 1$ for all $s \in \mathbb{R}$. We define a normal vector field $N := (-\dot{\Gamma}_2, \dot{\Gamma}_1)$ and a tangent vector field $T := (\dot{\Gamma}_1, \dot{\Gamma}_2)$. The couple (T, N) then forms a Frenet frame. The curvature of Γ is defined through the Frenet formulae

$$\begin{pmatrix} \dot{T}(s) \\ \dot{N}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix} \tag{4.1.0}$$

by $\kappa(s) = \det(\dot{\Gamma}(s), \ddot{\Gamma}(s))$, where κ is a continuous function of the arc-length parameter s. Let $\Omega_0 := \mathbb{R} \times (-1, 1)$ be a straight waveguide and let $\varepsilon > 0$. We define a curved waveguide of width 2ε by $\Omega_\varepsilon := \mathcal{L}_\varepsilon(\Omega_0)$, where

$$\mathcal{L}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2 : (s, t) \mapsto \Gamma(s) + \varepsilon t N(s). \tag{4.1.0}$$

We understand Ω_ε as an open connected subset of $\mathbb{R}^2$ with cartesian coordinates x, y. In fact, the image Ω_ε has a geometrical meaning of a non-self-intersecting strip only under assumption that $\mathcal{L}_\varepsilon \restriction \Omega_0$ is a diffeomorphism. By the inverse map theorem the sufficient condition is that the restriction $\mathcal{L}_\varepsilon \restriction \Omega_0$ is injective and the Jacobian of $\mathcal{L}_\varepsilon$ is non-zero on Ω_0. From the Frenet formulae (4.1) follows that $\ddot{\Gamma}(s) = \kappa(s)\binom{-\dot{\Gamma}_2(s)}{\dot{\Gamma}_1(s)}$ and thus the Jacobian of $\mathcal{L}_\varepsilon$ is given by

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \dot{\Gamma}_1 - \varepsilon t\ddot{\Gamma}_2 & \varepsilon N_1 \\ \dot{\Gamma}_2 + \varepsilon t\ddot{\Gamma}_1 & \varepsilon N_2 \end{vmatrix} = \begin{vmatrix} N_2\,(1 - \varepsilon t\kappa) & \varepsilon N_1 \\ -N_1\,(1 - \varepsilon t\kappa) & \varepsilon N_2 \end{vmatrix} = \varepsilon\,(1 - \varepsilon t\kappa)\,.$$

Therefore the condition that the Jacobian of $\mathcal{L}_\varepsilon$ is non-zero on Ω_0 is met if $\varepsilon t\kappa(s) < 1$ for all $t \in (-1, 1)$ and $s \in \mathbb{R}$. Summing up our considerations, we will always assume:

Assumption 4.1.1. *i)* $\mathcal{L}_\varepsilon \restriction \Omega_0$ is injective

ii) $\kappa \in L^\infty(\mathbb{R})$

iii) $\varepsilon\|\kappa\|_\infty < 1$, where $\|\kappa\|_\infty := \sup_{s\in\mathbb{R}}|\kappa(s)|$

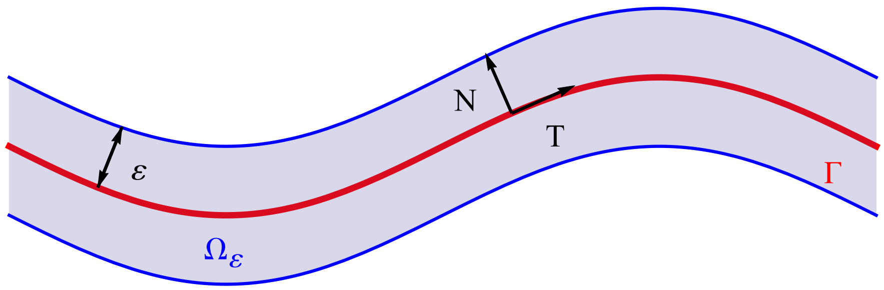

Figure 4.1: Curved strip Ω_ε defined via reference curve Γ.

From the third condition we derive the following useful estimates:

$$\forall s,t \in \Omega_0 \qquad 0 < C_- \leq 1 - \varepsilon t \kappa(s) \leq C_+ \quad \text{with} \quad C_\pm := 1 \pm \varepsilon \|\kappa\|_\infty. \tag{4.1.0}$$

4.2 The Hamiltonian

We are interested in the behaviour of the solutions $\psi \in L^2(\Omega_\varepsilon)$ to the boundary-value problem as $\varepsilon \to 0$

$$\begin{cases} -\triangle \psi = \lambda \psi & \text{in } \Omega_\varepsilon \\ \frac{\partial \psi}{\partial N} + i\alpha\psi = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4.2.0}$$

where $\alpha : \partial\Omega_\varepsilon \to \mathbb{R}$ is a given real-valued function. We assume $\alpha \in L^\infty(\partial\Omega_\varepsilon)$ and $\alpha \restriction \mathcal{L}_\varepsilon(\mathbb{R} \times \{-1\}) = \alpha \restriction \mathcal{L}_\varepsilon(\mathbb{R} \times \{1\})$ so that α can be identified with function $\alpha : \mathbb{R} \to \mathbb{R}$.

We understand (4.2) as the spectral problem for the m-sectorial operator with Robin-type boundary conditions in the Hilbert space $L^2(\Omega_\varepsilon)$

$$\begin{aligned} H_\varepsilon \psi &:= -\triangle \psi \\ \operatorname{Dom}(H\varepsilon) &:= \left\{ \psi \in W^{2,2} \,\middle|\, \frac{\partial \psi}{\partial N} \restriction \partial\Omega_\varepsilon + i\alpha\psi \restriction \partial\Omega_\varepsilon = 0 \right\}. \end{aligned} \tag{4.2.1}$$

The operator is associated with the form

$$\begin{aligned} Q_\varepsilon[\psi] &:= (\psi, H_\varepsilon \psi) = \int_{\Omega_\varepsilon} |\triangledown \psi|^2 \quad + \quad i \int_{\mathcal{L}(\mathbb{R} \times \{1\})} \alpha |\psi|^2 \quad - \quad i \int_{\mathcal{L}(\mathbb{R} \times \{-1\})} \alpha |\psi|^2 \\ \operatorname{Dom}(Q_\varepsilon) &:= W^{1,2}(\Omega_\varepsilon). \end{aligned} \tag{4.2.2}$$

Since the mapping $\mathcal{L}_\varepsilon$ is under Assumption 4.1.1 a global diffeomorphism between Ω_0 and Ω_ε it is natural to describe the Hamiltonian (4.2.1) in curvilinear coordinates (s,t) determined by the inverse of $\mathcal{L}_\varepsilon$. Additionally, it is convenient to rescale the space Ω_ε for the purpose of simplifying the scalar product. In the light of aforementioned considerations we introduce the following mapping:

$$U_\varepsilon : L^2(\Omega_\varepsilon, dxdy) \to \mathscr{H}_\varepsilon := L^2(\Omega_0, h_\varepsilon(s,t)dsdt) : \psi \mapsto \sqrt{\varepsilon}\psi \circ \mathcal{L}_\varepsilon := \varphi, \tag{4.2.2}$$

where $h_\varepsilon(s,t) := 1 - \varepsilon t\kappa$ is the Jacobian of $\mathcal{L}_\varepsilon$ up to the parameter ε. Since $\mathcal{L}_\varepsilon$ is a diffeomorphism and

$$(U_\varepsilon\psi, U_\varepsilon\varphi)_{\Omega_\varepsilon} = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \psi \circ \mathcal{L}_\varepsilon \varphi \circ \mathcal{L}_\varepsilon = \int_{\Omega_0} \psi\varphi h_\varepsilon = (\psi, \varphi)_{\mathscr{H}_\varepsilon},$$

it is clear that the mapping (4.2) is a unitary transformation. Hence the transformed Hamiltonian $\tilde{H}_\varepsilon := U_\varepsilon H_\varepsilon U_\varepsilon^{-1}$ is unitarily equivalent with the former Hamiltonian H_ε. In other words we replaced the simple operator H_ε on the complicated Hilbert space $L^2(\Omega_\varepsilon, dxdy)$ with the more complicated operator $\tilde{H}_\varepsilon$ on the simpler space $\mathscr{H}_\varepsilon$. The Hamiltonian $\tilde{H}_\varepsilon$ is associated with the form

$$\begin{aligned} \tilde{Q}_\varepsilon[\varphi] &:= Q_\varepsilon[U_\varepsilon^{-1}\varphi] \\ \mathrm{Dom}(\tilde{Q}_\varepsilon) &:= U_\varepsilon \mathrm{Dom}\,(Q_\varepsilon) = W^{1,2}\,(\Omega_0)\,. \end{aligned} \tag{4.2.3}$$

In particular, the gradient is transformed into curvilinear coordinates (s,t) as follows:

$$\begin{aligned} \forall\varphi \in \Omega_0 \quad |\bigtriangledown\varphi|^2 = |\partial_x\varphi|^2 + |\partial_y\varphi|^2 &= \left|\frac{\partial\psi}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial x}\right|^2 + \left|\frac{\partial\psi}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial y}\right|^2 \\ &= \left|\psi_{,s}\frac{N_2}{h} + \psi_{,t}N_1 h\right|^2 + \left|-\psi_{,s}\frac{N_1}{h} + \psi_{,t}N_2 h\right|^2 \\ &= \frac{1}{h^2}|\psi_{,s}|^2 + \frac{1}{\varepsilon^2}|\psi_{,t}|^2. \end{aligned}$$

Consequently the boundary part of quadratic form Q_ε is transformed via substitution in the curve integral and altogether the quadratic form $\tilde{Q}_\varepsilon$ in curvilinear coordinates is expressed as

$$\tilde{Q}_\varepsilon[\varphi] = \int\limits_{\Omega_0} \frac{|\varphi_{,s}|^2}{h_\varepsilon} + \frac{1}{\varepsilon^2}\int\limits_{\Omega_0} |\varphi_{,t}|^2 h_\varepsilon + \frac{1}{\varepsilon}\int\limits_{\mathbb{R}\times\{1\}} \alpha|\varphi|^2 h_\varepsilon - \frac{1}{\varepsilon}\int\limits_{\mathbb{R}\times\{-1\}} \alpha|\varphi|^2 h_\varepsilon. \tag{4.2.3}$$

The operator $\tilde{H}_\varepsilon$ then can be expressed as

$$\begin{aligned} \tilde{H}_\varepsilon &= -\frac{1}{h_\varepsilon}\partial_s\frac{1}{h_\varepsilon}\partial_s + \frac{1}{\varepsilon^2}\frac{1}{h_\varepsilon}\partial_t h_\varepsilon \partial_t \\ \mathrm{Dom}\left(\tilde{H}_\varepsilon\right) &= \left\{\varphi \in W^{2,2}(\Omega_0)\;\middle|\; \varphi_{,t} + i\varepsilon\alpha\varphi = 0 \quad \text{on} \quad \mathbb{R}\times\pm1\right\}. \end{aligned} \tag{4.2.4}$$

In the next step which is inspired by [14] we transform the Hamiltonian $\tilde{H}_\varepsilon$ into the operator which satisfies the usual Neumann boundary conditions on $\mathscr{H}_\varepsilon$. For this purpose we introduce another unitary transformation

$$V_\varepsilon : \mathscr{H}_\varepsilon \to \mathscr{H}_\varepsilon : \varphi \mapsto e^{i\varepsilon\alpha t}\varphi =: \phi \tag{4.2.4}$$

which leads to the unitarily equivalent operator $\hat{H}_\varepsilon = V_\varepsilon \tilde{H}_\varepsilon U_\varepsilon^{-1}$ associated with the form $\hat{Q}_\varepsilon[\phi] := \tilde{Q}_\varepsilon[V_\varepsilon^{-1}\phi]$, $\mathrm{Dom}(\hat{Q}_\varepsilon) := V_\varepsilon \mathrm{Dom}(\tilde{Q}_\varepsilon)$. However, this unitary transformation also yields additional condition on regularity of α and κ. Namely, we shall assume that $\alpha \in W^{2,\infty}$ and $\kappa \in W^{1,\infty}$ so that the Hamiltonian can be expressed as

$$\begin{aligned} \hat{H}_\varepsilon &= -\frac{1}{h_\varepsilon}\partial_s \frac{1}{h_\varepsilon}\partial_s + \frac{2i\dot{\alpha}\varepsilon t}{h_\varepsilon}\partial_s + V_1 + \frac{1}{\varepsilon^2}\frac{1}{h_\varepsilon}\partial_t h_\varepsilon \partial_t + \frac{2i\alpha}{\varepsilon}\partial_t + V_2 \\ \operatorname{Dom}(\hat{H}_\varepsilon) &= \left\{\phi \in W^{2,2}(\Omega_0) \mid \phi_{,t} = 0 \quad \text{on} \quad \mathbb{R} \times \pm 1\right\}, \end{aligned} \tag{4.2.5}$$

where V_1 and V_2 are potential terms given by

$$\begin{aligned} V_1 =& \frac{1}{h_\varepsilon^2}\left(\dot{\alpha}^2\varepsilon^2 t^2 - i\dot{\alpha}\varepsilon t \frac{h_{\varepsilon,s}}{h_\varepsilon} + i\ddot{\alpha}\varepsilon t\right) \\ V_2 =& \alpha^2 + \frac{i\alpha}{\varepsilon}\frac{h_{\varepsilon,t}}{h_\varepsilon} \end{aligned} \tag{4.2.6}$$

The associated form can then be rewritten as

$$\begin{aligned} \hat{Q}_\varepsilon[\phi] =& \int_{\Omega_0} \frac{|\phi_{,s}|^2}{h_\varepsilon} + \int_{\Omega_0} \frac{2i\dot{\alpha}\varepsilon t\bar{\phi}\phi_{,s}}{h_\varepsilon} + \int_{\Omega_0} V_1|\phi|^2 h_\varepsilon \\ &+ \frac{1}{\varepsilon^2}\int_{\Omega_0} |\phi_{,t}|^2 h_\varepsilon + \frac{1}{\varepsilon}\int_{\Omega_0} 2i\alpha\bar{\phi}\phi_{,t} h_\varepsilon + \int_{\Omega_0} V_2|\phi|^2 h_\varepsilon \\ \operatorname{Dom}(\hat{Q}_\varepsilon) =& W^{1,2}(\Omega_0). \end{aligned} \tag{4.2.7}$$

In conclusion, we replaced the m-sectorial operator H_ε satisfying Robin-type boundary conditions on $L^2(\Omega_\varepsilon, dxdy)$ with the operator $\hat{H}_\varepsilon$ satisfying Neumann boundary conditions on $\mathscr{H}_\varepsilon$. Since the operators are unitarily equivalent, they possess the same spectrum, thus we are going to study the operator $\hat{H}_\varepsilon$. Moreover, the unitary transform also preserves the m-sectoriality of the operator.

4.3 The limit

In this section we will show that the Hamiltonian $\hat{H}_\varepsilon$ weakly converges to an operator H_{eff} as the width of the strip tends to zero. It shall be later explained, how the limit is understood.

Since the Hamiltonian $\hat{H}_\varepsilon$ is m-sectorial, there exists $K \in \mathbb{R}$ such that $-K \in \varrho(\hat{H}_\varepsilon)$. Hence the resolvent is a bounded operator on $\mathscr{H}_\varepsilon$, *i.e.* $(\hat{H}_\varepsilon + K)^{-1} \in \mathscr{B}(\mathscr{H}_\varepsilon)$. For any function $f \in \mathscr{H}_\varepsilon$ we set $\phi_\varepsilon := (\hat{H}_\varepsilon + K)^{-1} f$ so that ϕ_ε satisfies the resolvent equation

$$\left(\hat{H}_\varepsilon + K\right)\phi_\varepsilon = f \tag{4.3.0}$$

which can be equivalently rewritten as

$$\forall \tilde{\varphi} \in W^{1,2}(\Omega_0) \qquad \hat{Q}_\varepsilon(\tilde{\varphi}, \phi_\varepsilon) + K(\tilde{\varphi}, \phi_\varepsilon)_\varepsilon = (\tilde{\varphi}, f)_\varepsilon, \tag{4.3.0}$$

where $(.,.)_\varepsilon$ denotes the scalar product in $\mathscr{H}_\varepsilon$. Specifically, for the choice $\tilde{\varphi} := \phi_\varepsilon$ we have

$$\hat{Q}_\varepsilon[\phi_\varepsilon] + K\|\phi_\varepsilon\|_\varepsilon^2 = (\phi_\varepsilon, f)_\varepsilon. \tag{4.3.0}$$

Following lemmas will be helpful in estimating the function ϕ_ε. Let us first introduce a convention of denoting constants.

Remark 4.3.1. *Henceforth we shall denote by C a generic constant depending on $\|\kappa\|_{C^1(\overline{\mathbb{R}})}$ and $\|\alpha\|_{C^2(\overline{\mathbb{R}})}$ but independent of ε. The constant may change from line to line.*

Lemma 4.3.2. *Let $\alpha \in W^{2,\infty}$, $\kappa \in W^{1,\infty}$ and $\hat{H}_\varepsilon$ be the Hamiltonian defined by (4.2.1). Than there exist constants $\varepsilon_0 = \varepsilon_0\left(\|\kappa\|_\infty\right)$ and*
$C = C\left(\|\kappa\|_\infty, \|\dot{\kappa}\|_\infty, \|\alpha\|_\infty, \|\dot{\alpha}\|_\infty, \|\ddot{\alpha}\|_\infty, \varepsilon_0\right)$ such that $\forall \varepsilon < \varepsilon_0$:

$$|V_1(s,t)| \le C\varepsilon,$$
$$|V_2(s,t)| \le C.$$

Proof. The bounds (4.1) yield the estimate $\frac{1}{h_\varepsilon} \le \frac{1}{1-\varepsilon\|\kappa\|_\infty}$. Furthermore, since $\kappa \in W^{1,\infty}$ the partial derivatives in potential terms V_1 and V_2 can be estimated by

$$|h_{\varepsilon,s}| = |u\varepsilon\dot{\kappa}| \le \varepsilon_0\|\dot{\kappa}\|_\infty$$
$$|h_{\varepsilon,t}| = |\varepsilon\kappa| \le \varepsilon_0\|\kappa\|_\infty.$$

In the light of the above inequalities we get the estimate of the first potential term:

$$\begin{aligned}
|V_1(s,t)| &= \left|\frac{1}{h_\varepsilon^2}\left(\dot{\alpha}^2\varepsilon^2 t^2 - i\dot{\alpha}\varepsilon t\frac{h_{\varepsilon,s}}{h_\varepsilon} + i\ddot{\alpha}\varepsilon t\right)\right| \\
&\le \frac{1}{h_\varepsilon^2}\left(\left|(\dot{\alpha}^2\varepsilon^2 t^2\right| + \left|\dot{\alpha}\varepsilon t\frac{h_{\varepsilon,s}}{h_\varepsilon}\right| + |\ddot{\alpha}\varepsilon t|\right) \\
&\le \frac{1}{\left(1-\varepsilon_0\|\dot{\kappa}\|_\infty\right)^2}\left(\|\dot{\alpha}\|_\infty^2\varepsilon_0^2 + \|\dot{\alpha}\|_\infty\varepsilon_0\frac{\varepsilon_0\|\dot{\kappa}\|_\infty}{1-\varepsilon_0\|\dot{\kappa}\|_\infty} + \|\ddot{\alpha}\|_\infty\varepsilon_0\right) \\
&\le C\varepsilon.
\end{aligned}$$

The last inequality arose from the assumption that $\alpha \in W^{2,\infty}$, $\kappa \in W^{1,\infty}$ and from the fact that if $\varepsilon_0 < 1$ then $\varepsilon_0^2 < \varepsilon_0$. We deal with the estimate of the second potential term in the same fashion:

$$\begin{aligned}
|V_2| &= \left|\alpha^2 + \frac{i\alpha}{\varepsilon}\frac{h_{\varepsilon,t}}{h_\varepsilon}\right| \le \alpha^2 + \left|\frac{\alpha}{\varepsilon}\frac{h_{\varepsilon,t}}{h_\varepsilon}\right| \\
&\le \|\alpha\|_\infty^2 + \frac{\|\alpha\|_\infty}{\varepsilon_0}\frac{\varepsilon_0\|\kappa\|_\infty}{1-\varepsilon_0\|\kappa\|_\infty} \le C.
\end{aligned}$$

□

Lemma 4.3.3. *For every function $\phi \in \mathscr{H}_\varepsilon$ and constant $\delta > 0$ the following inequalities hold:*

$$\left|\int\limits_{\Omega_0}\frac{2i\dot{\alpha}\varepsilon\overline{\phi}\phi_{,s}}{h_\varepsilon}\right| \le \delta\int\limits_{\Omega_0}\frac{|\phi_{,s}|^2}{h_\varepsilon} + \frac{1}{4\delta}\int\limits_{\Omega_0}\frac{4|\dot{\alpha}|^2\varepsilon^2}{h_\varepsilon}|\phi|^2 \tag{4.3.0}$$

$$\left|\frac{1}{\varepsilon}\int\limits_{\Omega_0}2i\alpha\overline{\phi}\phi_{,t}h_\varepsilon\right| \le \delta\frac{1}{\varepsilon^2}\int\limits_{\Omega_0}|\phi_{,t}|^2h_\varepsilon^2 + \frac{1}{4\delta}\int\limits_{\Omega_0}4\alpha^2|\phi|^2h_\varepsilon. \tag{4.3.0}$$

Proof. Using the Schwarz inequality we get:

$$\left|\int\limits_{\Omega_0} \frac{2i\dot{\alpha}\varepsilon\overline{\phi}\phi_{,s}}{h_\varepsilon}\right| \leq 2\sqrt{\int\limits_{\Omega_0} \frac{|\phi_{,s}|^2}{h_\varepsilon}}\sqrt{\int\limits_{\Omega_0} \frac{|\dot{\alpha}|^2\varepsilon^2}{h_\varepsilon}|\phi|^2}$$
$$\leq \delta \int\limits_{\Omega_0} \frac{|\phi_{,s}|^2}{h_\varepsilon} + \frac{1}{\delta}\int\limits_{\Omega_0} \frac{|\dot{\alpha}|^2\varepsilon^2}{h_\varepsilon}|\phi|^2,$$

where the second inequality follows from $2ab \leq a^2+b^2$ with $a = \sqrt{\delta \int\limits_{\Omega_0} \frac{|\phi_{,s}|^2}{h_\varepsilon}}$ and $b = \sqrt{\frac{1}{\delta}\int\limits_{\Omega_0} \frac{|\dot{\alpha}|^2\varepsilon^2}{h_\varepsilon}|\phi|^2}$. Similarly, using the same arguments we would get the inequality (4.3.3). □

From now on, we shall denote $\mathscr{H}_0 := L^2(\Omega_0, dsdt)$. Note that the spaces $\mathscr{H}_0$ and $\mathscr{H}_\varepsilon$ coincide as sets but they differ in topology. Therefore we can regard any element of $\mathscr{H}_0$ as an element of $\mathscr{H}_\varepsilon$, and vice versa. The norm in $\mathscr{H}_0$ is defined by $\|\psi\|_0^2 = \int\limits_{\Omega_0} |\psi(s,t)|^2 dsdt$ and using the bounds (4.1), we have

$$0 < C_-\|.\|_0^2 \leq \|.\|_\varepsilon^2 \leq C_+\|.\|_0^2, \tag{4.3.0}$$

where $C_\pm := 1 \pm \varepsilon\|\kappa\|_\infty$.

Now we are prepared to prove the following crucial estimates.

Theorem 4.3.4. *Let f be an arbitrary function from $\mathscr{H}_\varepsilon$ and ϕ_ε be the function satisfying the resolvent equation (4.3). Than the following inequalities hold:*

$$\begin{aligned} \|\phi_\varepsilon\|_0 &\leq C\|f\|_0 \\ \|\phi_{\varepsilon,t}\|_0 &\leq C\varepsilon\|f\|_0 \\ \|\phi_{\varepsilon,s}\|_0 &\leq C\|f\|_0. \end{aligned} \tag{4.3.0}$$

Proof. Using the previously stated Lemmata 4.3.2 and 4.3.3 with special choice $\delta = \frac{1}{2}$ we can estimate the equality (4.3) from below as follows:

$$\begin{aligned} \hat{Q}_\varepsilon[\phi_\varepsilon] + K\|\phi_\varepsilon\|_\varepsilon^2 = &\int\limits_{\Omega_0} \frac{|\phi_{\varepsilon,s}|^2}{h_\varepsilon} + \int\limits_{\Omega_0} \frac{2i\dot{\alpha}\varepsilon t\overline{\phi_\varepsilon}\phi_{\varepsilon,s}}{h_\varepsilon} + \int\limits_{\Omega_0} V_1|\phi_\varepsilon|^2 h_\varepsilon \\ &+ \frac{1}{\varepsilon^2}\int\limits_{\Omega_0} |\phi_{\varepsilon,t}|^2 h_\varepsilon + \frac{1}{\varepsilon}\int\limits_{\Omega_0} 2i\alpha\overline{\phi_\varepsilon}\phi_{\varepsilon,t}h_\varepsilon + \int\limits_{\Omega_0} V_2|\phi_\varepsilon|^2 h_\varepsilon + K\int\limits_{\Omega_0} |\phi_\varepsilon|^2 h_\varepsilon \\ \geq &\int\limits_{\Omega_0} \frac{|\phi_{\varepsilon,s}|^2}{h_\varepsilon} - \frac{1}{2}\int\limits_{\Omega_0} \frac{|\phi_{\varepsilon,s}|^2}{h_\varepsilon} - 2\int\limits_{\Omega_0} \frac{|\dot{\alpha}|^2\varepsilon^2}{h_\varepsilon}|\phi_\varepsilon|^2 - \varepsilon C\int\limits_{\Omega_0} |\phi_\varepsilon|^2 \\ &+ \frac{1}{\varepsilon^2}\int\limits_{\Omega_0} |\phi_{\varepsilon,t}|^2 h_\varepsilon - \frac{1}{2\varepsilon^2}\int\limits_{\Omega_0} |\phi_{\varepsilon,t}|^2 h_\varepsilon^2 - 2\int\limits_{\Omega_0} \alpha^2|\phi_\varepsilon|^2 h_\varepsilon - C\int\limits_{\Omega_0} |\phi_\varepsilon|^2 h_\varepsilon \\ &+ K\int\limits_{\Omega_0} |\phi_\varepsilon|^2 h_\varepsilon \\ \geq &\frac{1}{2}\int\limits_{\Omega_0} \frac{|\phi_{\varepsilon,s}|^2}{h_\varepsilon} + \frac{1}{2\varepsilon^2}\int\limits_{\Omega_0} |\phi_{\varepsilon,t}|^2 h_\varepsilon + (K-C)\int\limits_{\Omega_0} |\phi_\varepsilon|^2 h_\varepsilon. \end{aligned}$$

From the Schwarz inequality immediately follows that (4.3) is bounded from above by $\|\phi_\varepsilon\|_\varepsilon \|f\|_\varepsilon$ which together with the above inequality yields

$$\frac{1}{2}\left\|\frac{\phi_{\varepsilon,s}}{h_\varepsilon}\right\|_\varepsilon + \frac{1}{2\varepsilon^2}\|\phi_{\varepsilon,t}\|_\varepsilon + (K-C)\|\phi_\varepsilon\|_\varepsilon^2 \leq \|\phi_\varepsilon\|_\varepsilon \|f\|_\varepsilon. \tag{4.3.0}$$

Under the assumption that K is large enough with respect to C, the left hand side is composed of three non-negative terms, and we therefore conclude

$$\begin{aligned}
\|\phi_\varepsilon\|_\varepsilon &\leq \frac{1}{K-C}\|f\|_\varepsilon \\
\|\phi_{\varepsilon,t}\|_\varepsilon^2 &\leq 2\varepsilon^2\|\phi_\varepsilon\|_\varepsilon\|f\|_\varepsilon \leq \frac{2\varepsilon^2}{K-C}\|f\|_\varepsilon \\
\|\phi_{\varepsilon,s}\|_\varepsilon^2 &\leq 2\|\phi_\varepsilon\|_\varepsilon\|f\|_\varepsilon \leq \frac{2}{K-C}\|f\|_\varepsilon.
\end{aligned}$$

Taking into account the norm inequalities (4.3), we consequently obtain the desired statement (4.3.4). □

Since the operators $\hat{H}_\varepsilon$ and H_{eff} act on different Hilbert spaces, we need to explain how the convergence of corresponding resolvent operators is understood. Inspired by [3] we decompose the Hilbert space $\mathscr{H}_0$ into a sum of two mutually orthogonal subspaces $\mathscr{H}_0^{\text{const}}$ and $\mathscr{H}_0^{\perp}$. Let us consider the Neumann Laplacian $-\triangle_N^{(-1,1)}$ acting in $L^2\left((-1,1)\right)$. Seeing that its eigenfunctions $\{\chi_n\}_{n\in\mathbb{N}}$ form the orthonormal basis of the space, we can express any function $\phi \in \mathscr{H}_0$ in terms of Fourier expansion

$$\forall s \in \mathbb{R}, t \in (-1,1) \qquad \phi(s,t) = \sum_{n=1}^{\infty} \varphi_n(s)\chi_n(t), \tag{4.3.0}$$

where $\varphi_n(s) := \int_{-1}^1 \overline{\chi_n(t)}\phi(s,t)dt$. Let us now define the subspace $\mathscr{H}_0^{\text{const}}$ as the set consisting of first terms of the Fourier expasion, *i.e.*

$$\mathscr{H}_0^{\text{const}} := \left\{\phi_0 \in \mathscr{H}_0 \;\middle|\; (\exists f \in \mathscr{H}_0)\left(\phi_0 = \varphi\chi_1 = (\chi_1, f)_{L^2((-1,1))}\,\chi_1\right)\right\}. \tag{4.3.0}$$

Since the first eigenvalue of $-\triangle_N^{(-1,1)}$ is equal to zero and its corresponding eigenfunction is $\chi_1 = \frac{1}{\sqrt{2}}$, we have $\varphi(s) := \frac{1}{\sqrt{2}}\int_{-1}^1 \phi(s,t)dt$ and $\phi_0(s) = \varphi(s)\chi_1(t) = \frac{1}{2}\int_{-1}^1 \phi(s,t)dt$. As a result, the subspace $\mathscr{H}_0^{\text{const}}$ can be interpreted as consisting of functions constant in the transversal variable t, thus it can be naturally identified with $L^2(\mathbb{R})$. More precisely, the identity mapping $I : \mathscr{H}_0^{\text{const}} \to L^2(\mathbb{R}) : \{\phi \mapsto \phi\}$ is isometric isomorphism between $\mathscr{H}_0^{\text{const}}$ and $L^2(\mathbb{R})$. Therefore, with abuse of notation, we may identify any operator on $\mathscr{H}_0^{\text{const}}$ with the one acting in $L^2(\mathbb{R})$, and vice versa. The corresponding projection is then given by

$$\left(\mathcal{P}\phi\right)(s) := \frac{1}{2}\int_{-1}^{1}\phi(s,t)dt. \tag{4.3.0}$$

By the projection theorem, we now can decompose the Hilbert space as

$$\mathscr{H}_0 = \mathscr{H}_0^{\text{const}} \oplus \mathscr{H}_0^{\perp} \tag{4.3.0}$$

The projection onto the subspace $\mathscr{H}_0^{\perp}$ is subsequently defined by $\mathcal{P}^{\perp} := I - \mathcal{P}$. In conclusion, by virtue of the decomposition (4.3), we can express any function $\phi \in \mathscr{H}_0$ as

$$\phi(s,t) = \phi_0(s) + \phi^{\perp}(s,t), \tag{4.3.0}$$

where $\phi_0 = \varphi\chi_1 \in \mathscr{H}_0^{\mathrm{const}}$ and $\phi^{\perp} \in \mathscr{H}_0^{\perp}$. Moreover the orthogonality of ϕ_0 and $\phi^{\perp}$ can be rewritten as

$$\int_{-1}^{1} \phi^{\perp}(s,t)dt = 0 \quad \text{for almost every } s \in \mathbb{R}. \tag{4.3.0}$$

Additionally, if $\phi \in W^{1,2}(\Omega_0)$, we can differentiate the previous identity to get

$$\int_{-1}^{1} \phi^{\perp}_{,s}(s,t)dt = 0 \quad \text{for almost every } s \in \mathbb{R}. \tag{4.3.0}$$

In the light of the Hilbert space decomposition, we now can rewrite the inequalities (4.3.4) for $\phi_\varepsilon(s,t) = \frac{1}{\sqrt{2}}\varphi_\varepsilon(s) + \phi_\varepsilon^{\perp}(s,t)$.

Theorem 4.3.5. *Let f be an arbitrary function from $\mathscr{H}_\varepsilon$ and ϕ_ε be a function satisfying the resolvent equation (4.3). Than the following inequalities hold:*

$$\begin{aligned} \|\varphi_\varepsilon\|_{W^{1,2}(\mathbb{R})} &\le C\|f\|_0 \\ \|\phi_\varepsilon^{\perp}\|_0 &\le C\|f\|_0 \\ \|\phi_{\varepsilon,s}^{\perp}\|_0 &\le C\|f\|_0 \\ \|\phi_{\varepsilon,t}^{\perp}\|_0 &\le C\varepsilon\|f\|_0, \end{aligned} \tag{4.3.1}$$

where $\|\varphi_\varepsilon\|^2_{W^{1,2}(\mathbb{R})} = \|\varphi_\varepsilon'\|^2_{L^2(\mathbb{R})} + \|\varphi_\varepsilon\|^2_{L^2(\mathbb{R})}$.

Proof. Using the Hilbert space decomposition we have

$$\begin{aligned} \|\phi_\varepsilon\|_0^2 &= \int\limits_{\mathbb{R}} |\varphi_\varepsilon(s)|^2 ds + \int\limits_{\Omega_0} |\phi_\varepsilon^{\perp}(s,t)|^2 ds dt + 2\,\mathrm{Re} \int\limits_{\mathbb{R}} ds \overline{\varphi_\varepsilon}(s) \int_{-1}^{1} dt \chi_1(t)\phi_\varepsilon^{\perp}(s,t) \\ &= \|\varphi_\varepsilon\|^2_{L^2(\mathbb{R})} + \|\phi_\varepsilon^{\perp}\|_0^2, \end{aligned} \tag{4.3.2}$$

where the third term on the right hand side vanishes due to (4.3). Similarly, using (4.3), we have

$$\begin{aligned} \|\phi_{\varepsilon,s}\|_0^2 &= \int\limits_{\mathbb{R}} |\dot{\varphi}_\varepsilon(s)|^2 ds + \int\limits_{\Omega_0} |\phi_{\varepsilon,s}^{\perp}(s,t)|^2 ds dt + 2\,\mathrm{Re} \int\limits_{\mathbb{R}} ds \dot{\overline{\varphi}}_\varepsilon(s) \int_{-1}^{1} dt \chi_1(t)\phi_{\varepsilon,s}^{\perp}(s,t) \\ &= \|\dot{\varphi}_\varepsilon\|^2_{L^2(\mathbb{R})} + \|\phi_{\varepsilon,s}^{\perp}\|_0^2. \end{aligned} \tag{4.3.3}$$

Since χ_1 is in fact the constant $\frac{1}{\sqrt{2}}$, the partial derivation with respect to t reduces to the perpendicular part and we can write

$$\|\phi_{\varepsilon,t}\|_0^2 = \|\phi_{\varepsilon,t}^{\perp}\|_0^2. \tag{4.3.3}$$

From the identities (4.3.2), (4.3.3) and the former inequalities (4.3.4) immediately follows

$$\begin{aligned} \|\varphi_\varepsilon\|^2_{W^{1,2}(\mathbb{R})} &= \|\phi_{\varepsilon,s}\|_0^2 + \|\phi_\varepsilon\|_0^2 - \|\phi_\varepsilon^{\perp}\|_0^2 - \|\phi_{\varepsilon,s}^{\perp}\|_0^2 \le \|\phi_{\varepsilon,s}\|_0^2 + \|\phi_\varepsilon\|_0^2 \\ &\le C\|f\|_0. \end{aligned}$$

Rest of the inequalities can be proved in a similar manner. □

In fact, this theorem yields that $\phi_\varepsilon^\perp$ is negligible as $\varepsilon \to 0$.

Corollary 4.3.6. *Under the assumptions in Theorem 4.3.5, the following inequality holds*

$$\|\phi_\varepsilon^\perp\| \leq \frac{2}{\pi} C\varepsilon \|f\|,$$

thus $\phi_\varepsilon^\perp$ converges to 0 as $\varepsilon \to 0$ as an element of $\mathscr{H}_0$.

Proof. Let us consider the Neumann Laplacian $-\triangle_N^{(-1,1)}$ acting in $L^2((-1,1))$. Using the variational characterisation ([6, Lemma XI-1.1]), we can estimate its second eigenvalue as

$$\lambda_2 = \left(\frac{\pi}{2}\right)^2 = \inf_{\substack{\psi \in W^{1,2}((-1,1)),\\ (\psi,\chi_1)=0}} \frac{\|\psi'\|^2}{\|\psi\|^2} \leq \frac{\|\psi'\|^2}{\|\psi\|^2},$$

where the inequality holds for every $\psi \in W^{1,2}((-1,1))$ satisfying $(\psi, \chi_1) = 0$. Employing the orthogonal decomposition of $\mathscr{H}_0$, we see that $\phi_\varepsilon^\perp$ is orthogonal to χ_1, thus we can write

$$\left(\frac{\pi}{2}\right)^2 \|\phi_\varepsilon^\perp\|^2 \leq \|\phi_{\varepsilon,t}^\perp\|^2.$$

Combining the above inequality with (4.3.1), we eventually obtain

$$\|\phi_\varepsilon^\perp\| \leq \frac{2}{\pi} C\varepsilon \|f\|.$$

□

Furthermore, the inequalities (4.3.1) indicate that the set $\{\varphi_\varepsilon \mid \varepsilon > 0\}$ is bounded in $W^{1,2}(\mathbb{R})$, therefore weakly precompact in $W^{1,2}(\mathbb{R})$, *i.e.* its closure is compact in weak topology. Hence every sequence in $\{\varphi_\varepsilon \mid \varepsilon > 0\}$ has a convergent subsequence. Let φ_0 denote a weak limit point, *i.e.* there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \xrightarrow[k\to\infty]{} 0$ and

$$\varphi_{\varepsilon_k} \xrightarrow[k\to\infty]{w} \varphi_0 \quad \text{in} \quad W^{1,2}(\mathbb{R}). \tag{4.3.3}$$

Now we would like to pass to the limit $k \to \infty$ in (4.3) with $\varepsilon = \varepsilon_k$. The following theorem shows the limit for test functions $\varphi \in C_0^\infty(\mathbb{R})$. However, since $C_0^\infty(\mathbb{R})$ is dense in $W^{1,2}(\mathbb{R})$, it can be consequently extended.

Theorem 4.3.7. *Let $\phi_{\varepsilon_k} \in \mathscr{H}_0$ be the function defined by (4.3) and let $\{\varphi_{\varepsilon_k}\}_{k=0}^\infty$ be a sequence of projections $\varphi_{\varepsilon_k} = \int_{-1}^1 \overline{\chi_n(t)}\phi_{\varepsilon_k}(s,t)dt$ with weak limit φ_0. Then for every $\varphi \in C_0^\infty(\mathbb{R})$ following identity holds:*

$$(\dot\varphi, \dot\varphi_0)_{L^2(\mathbb{R})} + \left(\varphi, [\alpha^2 - i\alpha\kappa]\varphi_0\right)_{L^2(\mathbb{R})} + K(\varphi,\varphi_0)_{L^2(\mathbb{R})} = (\varphi, \frac{1}{\chi_1}\mathcal{P}f)_{L^2(\mathbb{R})}.$$

Proof. Let $\tilde{g} \in \mathscr{H}_0^{\text{const}}$ be the test function in (4.3). More precisely, we choose $\tilde{g} \equiv g \in C_0^\infty(\mathbb{R})$ which is dense in $W^{1,2}(\mathbb{R})$. This function is independent of the transversal variable t and can be rewritten as $g(s) = \varphi(s)\chi_1$. The resolvent equation (4.3) with the test function g then takes form

$$\left(\frac{g_{,s}}{h_\varepsilon}, \frac{\phi_{\varepsilon_k,s}}{h_\varepsilon}\right)_{\varepsilon_k} + \left(\frac{g}{h_\varepsilon}, \frac{2i\dot{\alpha}\varepsilon_k t\phi_{\varepsilon_k,s}}{h_\varepsilon}\right)_{\varepsilon_k} + (g, V_1\phi_{\varepsilon_k})_{\varepsilon_k}$$
$$+\frac{1}{\varepsilon_k^2}(g_{,t}, \phi_{\varepsilon_k,t})_{\varepsilon_k} + \frac{1}{\varepsilon_k}(g, 2i\alpha\phi_{\varepsilon_k,t})_{\varepsilon_k} + (g, V_2\phi_{\varepsilon_k})_{\varepsilon_k} + K(g, \phi_{\varepsilon_k})_{\varepsilon_k} = (g, f)_{\varepsilon_k}$$

We will prove the theorem by showing the limit term by term.

- Firstly, we will use the Hilbert space decomposition (4.3) to split the first term as

$$\left(\frac{g_{,s}}{h_\varepsilon}, \frac{\phi_{\varepsilon_k,s}}{h_\varepsilon}\right)_{\varepsilon_k} = \int_{\Omega_0} \frac{\overline{g_{,s}}\phi^\perp_{\varepsilon_k,s}}{h_\varepsilon} + \int_{\Omega_0} \frac{\overline{g}_{,s}\dot{\varphi}_{\varepsilon_k}\chi_1}{h_\varepsilon}. \tag{4.3.3}$$

  Considering the estimates (4.1), we have $\frac{1}{h_\varepsilon} - 1 = \frac{\varepsilon_k t\kappa}{1-\varepsilon_k t\kappa} \leq C\varepsilon_k$. Therefore we can write

$$\left|\int_{\Omega_0} \frac{\overline{g_{,s}}\phi^\perp_{\varepsilon_k,s}}{h_\varepsilon}\right| = \left|\int_{\Omega_0} \overline{g_{,s}}\phi^\perp_{\varepsilon_k,s} + \int_{\Omega_0} \overline{g_{,s}}\phi^\perp_{\varepsilon_k,s}\left(\frac{1}{h_\varepsilon} - 1\right)\right|$$
$$\leq C\varepsilon_k\|g_{,s}\|\ \|\phi^\perp_{\varepsilon_k,s}\| \leq C\varepsilon_k\|g_{,s}\|\ \|f\|,$$

  where the first term vanishes due to (4.3) and the last inequality follows from (4.3.1). The first term in (4.3.7) therefore converges to zero as $k \to \infty$. The other one can be rewritten as

$$\int_{\Omega_0} \frac{\overline{g}_{,s}\dot{\varphi}_{\varepsilon_k}\chi_1}{h_\varepsilon} = \int_{\Omega_0} \overline{g}_{,s}\dot{\varphi}_{\varepsilon_k}\chi_1 + \int_{\Omega_0} \overline{g}_{,s}\dot{\varphi}_{\varepsilon_k}\chi_1\left(\frac{1}{h_\varepsilon} - 1\right).$$

  The first integral converges to $\int_{\Omega_0} \overline{g}_{,s}\dot{\varphi}_0\chi_1$ and since the integrand is independent of t, it can be rewritten as $\int_{\mathbb{R}} \dot{\overline{\varphi}}\dot{\varphi}_0 = (\dot{\varphi}, \dot{\varphi}_0)_{L^2(\mathbb{R})}$. The other integral can be estimated by $\int_{\Omega_0} \overline{g}_{,s}\dot{\varphi}_{\varepsilon_k}\chi_1\left(\frac{1}{h_\varepsilon} - 1\right) \leq C\varepsilon_k\chi_1\|g_{,s}\|\|\dot{\varphi}_{\varepsilon_k}\|$, thus it is negligible in the limit. Summing up, we have shown

$$\left(\frac{g_{,s}}{h_\varepsilon}, \frac{\phi_{\varepsilon_k,s}}{h_\varepsilon}\right)_{\varepsilon_k} \xrightarrow{k\to\infty} (\dot{\varphi}, \dot{\varphi}_0)_{L^2(\mathbb{R})}.$$

- Using (4.1), we can estimate $\frac{1}{h_\varepsilon} = \frac{1}{1-\varepsilon_k t\kappa} \leq C$. Together with (4.3.4) this estimate yields

$$\left|\left(\frac{g}{h_\varepsilon}, \frac{2i\dot{\alpha}\varepsilon_k t\phi_{\varepsilon_k,s}}{h_\varepsilon}\right)_{\varepsilon_k}\right| = \left|\int_{\Omega_0} \frac{2i\dot{\alpha}\varepsilon_k t\overline{g}\phi_{\varepsilon_k,s}}{h_\varepsilon}\right| \leq C\varepsilon_k\left|\int_{\Omega_0} \overline{g}\phi_{\varepsilon_k,s}\right|$$
$$\leq C\varepsilon_k\|g\|\ \|\phi_{\varepsilon_k,s}\| \leq C\varepsilon_k\|g\|\ \|f\|.$$

  Hence, the second term converges to zero as $k \to \infty$.

- The term including the first potential V_1 can be estimated as

$$|(g, V_1\phi_{\varepsilon_k})_{\varepsilon_k}| \leq C\varepsilon_k |(g, \phi_{\varepsilon_k})_{\varepsilon_k}| \leq C\varepsilon_k \|g\| \, \|\phi_{\varepsilon_k}\| \leq C\varepsilon_k \|g\| \, \|f\|,$$

where the first inequality follows from Lemma 4.3.2 and in the last one we used (4.3.4). As a result the first term is negligible in the limit.

- Since g is independent of t, we can immediately write

$$\frac{1}{\varepsilon_k^2}(g_{,t}, \phi_{\varepsilon_k,t})_{\varepsilon_k} = 0.$$

- The term including the second potential can be rewritten by the definition of V_2 as

$$(g, V_2\phi_{\varepsilon_k})_{\varepsilon_k} = \int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k} - i \int_{\Omega_0} \alpha\kappa\overline{g}\phi_{\varepsilon_k}. \tag{4.3.3}$$

Subsequently, we rewrite the first integral as

$$\int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k} h_\varepsilon = \int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k} + \int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k}(h_\varepsilon - 1).$$

Using the Hilbert space decomposition, we obtain

$$\int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k} h = \int_{\Omega_0} \alpha^2 \overline{g}\varphi_{\varepsilon_k}\chi_1 + \int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k}^{\perp} + \int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k}(h_\varepsilon - 1).$$

The first integral converges to $\int_{\Omega_0} \alpha^2 \overline{g}\varphi_0\chi_1$ and because its integrand is independent of t, it can be rewritten as $\left(\varphi, \alpha^2\dot{\varphi}_0\right)_{L^2(\mathbb{R})}$. The Corollary 4.3.6 states that $\phi_{\varepsilon_k}^{\perp} \xrightarrow{k\to\infty} 0$, thus the second integral vanishes for $k \to \infty$. The third term can be estimated by $\int_{\Omega_0} \alpha^2 \overline{g}\phi_{\varepsilon_k}(h_\varepsilon - 1) \leq C\varepsilon_k \|g\| \|\phi_{\varepsilon_k}\|$ and since ϕ_{ε_k} is bounded by (4.3.1), the integral converges to zero.
The second integral in (4.3.7) can be rewritten as

$$\int_{\Omega_0} \alpha\kappa\overline{g}\phi_{\varepsilon_k} = \int_{\Omega_0} \alpha\kappa\overline{g}\varphi_{\varepsilon_k}\chi_1 + \int_{\Omega_0} \alpha\kappa\overline{g}\phi_{\varepsilon_k}^{\perp}.$$

The first term converges to $(\varphi, \alpha\kappa\dot{\varphi}_0)_{L^2(\mathbb{R})}$ and the second term vanishes by Corollary 4.3.6. In conclusion, we have shown

$$(g, V_2\phi_{\varepsilon_k})_{\varepsilon_k} \xrightarrow{k\to\infty} \left(\varphi, [\alpha^2 - i\alpha\kappa]\varphi_0\right)_{L^2(\mathbb{R})}.$$

- By the same arguments as in the previous case, we see that

$$K\,(g, \phi_{\varepsilon_k})_{\varepsilon_k} \xrightarrow{k\to\infty} K(\varphi, \varphi_0)_{L^2(\mathbb{R})}.$$

- The term on the right hand side can be rewritten as

$$(g,f)_{\varepsilon_k} = \int_{\Omega_0} \bar{g}f + \int_{\Omega_0} \bar{g}f(h_\varepsilon - 1).$$

As in the previous cases, the second term converges to zero. The first integral is independent of ε_k, thus it does not vary in the limit. Since the function g is independent of t, it can be rewritten as $\int_{\mathbb{R}} \chi_1 \overline{\varphi(s)} \left(\int_{-1}^{1} f(s,t)dt \right) ds$. Recalling the definition of the projection $\mathcal{P}$ onto the subspace $\mathscr{H}_0^{\text{const}}$, we finally obtain

$$(g,f)_{\varepsilon_k} \xrightarrow{k\to\infty} (\varphi, \frac{1}{\chi_1}\mathcal{P}f)_{L^2(\mathbb{R})}.$$

- The last and the most difficult step is to show that the remaining term

$$\frac{1}{\varepsilon_k}\left(g, 2i\alpha\phi_{\varepsilon_k,t}\right)_{\varepsilon_k} = \frac{1}{\varepsilon_k}\int_{\Omega_0} 2i\alpha\bar{g}\phi_{\varepsilon_k,t}h_\varepsilon \xrightarrow{k\to\infty} 0. \tag{4.3.3}$$

To prove it, we return to the equation (4.3). We choose here $\tilde{g}(s) := \alpha(s)g(s)t$ as the test function and multiply the whole equation by ε to get

$$\varepsilon \int_{\Omega_0} \frac{\dot{\alpha}\overline{g}t\phi_{\varepsilon,s} + \alpha\dot{\overline{g}}t\phi_{\varepsilon,s}}{h_\varepsilon} + 2i\varepsilon^2 \int_{\Omega_0} \alpha\dot{\alpha}t^2\overline{g}\phi_{\varepsilon,s} + \varepsilon \int_{\Omega_0} V_1 h_\varepsilon \alpha t \overline{g}\phi_\varepsilon + \frac{1}{\varepsilon}\int_{\Omega_0} \alpha\overline{g}\phi_{\varepsilon,t}h_\varepsilon$$
$$+2i\int_{\Omega_0} \alpha^2 t\overline{g}\phi_{\varepsilon,t}h_\varepsilon + \varepsilon\int_{\Omega_0} V_2\alpha\overline{g}t\phi_\varepsilon + K\varepsilon\int_{\Omega_0} \alpha\overline{g}t\phi_\varepsilon h_\varepsilon = \varepsilon\int_{\Omega_0} \alpha\overline{g}tfh_\varepsilon.$$

By the same arguments as in the previous cases, we conclude that all the terms on both sides of the equation (4.3.7) except for $\frac{1}{\varepsilon}\int_{\Omega_0} \alpha\overline{g}\phi_{\varepsilon,t}h_\varepsilon$ tend to zero as $\varepsilon \to 0$, thus

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_0} \alpha\overline{g}\phi_{\varepsilon,t}h_\varepsilon = 0,$$

which is the desired limit (4.3.7) up to the constant $2i$.

Since we showed the limit for any $g(s) = \varphi(s)\chi_1 \in C_0^\infty(\mathbb{R})$, we can say, that the identity holds for every $\varphi \in C_0^\infty(\mathbb{R})$. □

Since $C_0^\infty(\mathbb{R})$ is dense in $W^{1,2}(\mathbb{R})$, the obtained identity extends to $\varphi \in W^{1,2}(\mathbb{R})$. Hence φ_0 is the solution of the one-dimensional equation

$$\left(-\frac{d^2}{ds^2} + \alpha^2(s) - i\alpha(s)\kappa(s) + K\right)\varphi_0(s) = f(s) \quad \text{in } \mathbb{R}.$$

This equation can be regarded as an action of the operator $H_{\text{eff}} + K$, where

$$H_{\text{eff}} := -\frac{d^2}{ds^2} + \alpha^2 - i\alpha\kappa \tag{4.3.3}$$

is the operator in $L^2(\mathbb{R})$ associated with the quadratic form

$$h_{\text{eff}}[\varphi] = \|\dot{\varphi}\|_{L(\mathbb{R})} + \big(\varphi, [\alpha^2 - i\alpha\kappa]\varphi\big)_{L^2(\mathbb{R})}$$
$$\text{Dom}(h_{\text{eff}}) = W^{1,2}(\mathbb{R}).$$

Since the same result is obtained for any limit point, we have proved

$$\varphi_\varepsilon \xrightarrow[k\to\infty]{w} \varphi_0, \quad \text{in } L^2(\mathbb{R});$$
$$\phi_\varepsilon^\perp \xrightarrow[k\to\infty]{w} 0 \quad \text{in } \mathscr{H}_0^\perp.$$

That is

$$\phi_\varepsilon \xrightarrow[\varepsilon\to 0]{w} \chi_1\varphi_0 + 0 \quad \text{in } \mathscr{H}_0$$

with respect to the decomposition $\mathscr{H}_0 = \mathscr{H}_0^{\text{const}} \oplus \mathscr{H}_0^\perp$. Recalling the definition of ϕ_ε, we can summarize this section in the following corollary.

Corollary 4.3.8. *Let $\hat{H}_\varepsilon$ be the operator defined by (4.2.5) and H_{eff} be the operator defined by (4.3), then $\hat{H}_\varepsilon$ converges to H_{eff} in the weak resolvent sense, i.e.*

$$\left(\hat{H}_\varepsilon + K\right)^{-1} f \xrightarrow[\varepsilon\to 0]{w} \left(H_{\mathit{eff}} + K\right)^{-1} (\mathcal{P}f) + 0 \quad \mathit{in}\ \mathscr{H}_0.$$

This result takes into account that the operators act in different Hilbert spaces.

Chapter 5

Conclusion

We were interested in spectral properties of a curved planar waveguide, subject to Robin boundary conditions. We defined the Robin Laplacian using the theory of sectorial forms. We found the spectrum of the Robin Laplacian in a straight planar strip and we mentioned the Dirichlet and Neumann Laplacians as the limit cases. As a main result, we proved that the non-selfadoint Robin Laplacian in a curved planar strip converges to the self-adjoint effective Hamiltonian H_{eff} in a weak-resolvent sense as the width of the strip tends to zero. The problem that the operators act in different Hilbert spaces was solved by the Hilbert space decomposition.

There are several direction in which this thesis can be extended. First of all, we expect that the convergence actually holds in strong or even norm-resolvent sense. Additionally, we embedded the strip in the space $\mathbb{R}^2$ but it is possible to define it on a Riemannian manifold using the Fermi coordinates. The strip then represents a thin layer and its spectrum depends not only on the curvature of the strip but also on the curvature of the manifold. Lastly, we can extend this result to higher dimensions.

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