Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Bounds on resonant frequencies of vibrational systems

Odhady na rezonanční frekvence vibrujících systémů

Bachelor's Degree Project

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- Zadání práce -

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Název práce:

Odhady na rezonanční frekvence vibrujících systémů

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Abstrakt: Laplacián je velmi významný operátor s mnoha aplikacemi. Nejprve ukážeme jeho důležitost v teorii hudby. Potom definujeme dirichletovský laplacián jako samosdružený operátor na omezených oblastech pomocí kvadratických forem a vyslovíme některé jeho spektrální vlastnosti. Aplikací pricipu mini-maxu a použitím scvrkávajících se nebo paralelních souřadnic obdržíme dva horní odhady na první vlastní hodnotu dirichletovského laplaciánu na jednoduše souvislých oblastech. Navíc vyslovíme vlastní horní odhad pro speciální ne jednoduše souvislé oblasti. Nakonec aplikujeme tyto odhady na speciální tvary oblastí, porovnáme je a následně ukážeme příklady chování našeho odhadu na speciálních ne jednoduše souvislých oblastech vytvořených z předtím uvedených oblastí.

Klíčová slova: dirichletovský laplacián, horní odhad na první vlastní hodnotu, omezené oblasti, spektrální geometrie, teorie hudby

Title:

Bounds on resonant frequencies of vibrational systems

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Abstract: The Laplacian is a very important operator with many applications. First we show its importance in the musical theory. Then we define the self-adjoint Dirichlet Laplacian on bounded domains using the quadratic forms and state some of his spectral properties. Applying the min-max principle and using the shrinking or parallel coordinates we obtain two upper bounds for the first eigenvalue of the Dirichlet Laplacian on simply-connected domains. Moreover we introduce our own upper bound for particular, not simply-connected domains. Finally we apply the obtained bounds to some special shapes of simply-connected domains, compare them and subsequently we show examples of behavior of our bound on the particular, not simply-connected domains created from the domains introduced before.

Key words: bounded domains, Dirichlet Laplacian, musical theory, spectral geometry, upper bounds for the first eigenvalue

Contents

Introduction

The spectrum of the self-adjoint Dirichlet Laplacian is interesting to study due to the huge amount of its applications and no less for the mathematics itself. The spectrum of the operator depends on the geometry of the domain in $\mathbb{R}^d$ on which it acts. These domains can be divided into two main groups, we have either unbounded or bounded domains. The unbounded domains according to the Glazman's classification are of three types, quasi-conical, quasi-cylindrical and quasi-bounded. For bounded domains it is known that the spectrum is purely discrete and for a few shapes (rectangular parallelepipeds, balls, tori) we even known the spectrum explicitly. This thesis studies the spectrum on bounded domains, more precisely we are mainly interested in the upper bounds for the first eigenvalue (non-trivial, sharp or in arbitrary dimension).

In the beginning of the thesis we study the application of the spectrum in music. Since the vibrations of the parts of the musical instruments which produce the sound can be modeled using the wave equation in which figures the Laplace operator (with the Dirichlet, Dirichlet-Neumann or Neumann boundary conditions), we show that there is a close relationship between the eigenvalues of the spectrum (not yet knowing that we have found the complete spectrum) and the frequencies of the sounding tones. Next we compute and compare the spectra of one-dimensional (string, air column) and two-dimensional vibrating objects (membrane of a drum) and show the huge importance of the spectrum of the string in the music. Finally we introduce a special kind of drum called a timpani which surprisingly has a spectrum very similar to the spectrum of the string, if other physical phenomena are taken into account.

In the second chapter we correctly define the self-adjoint Dirichlet Laplacian on bounded domains using the quadratic forms and we state some of its properties and the properties of its spectrum, which will be needed later. We also develop some spectral-analytic tools, such as the min-max principle, on which the subsequent chapter stands.

The third chapter is the main chapter of the thesis were we state some of the existing bounds for the first eigenvalue. We start with one lower bound, the Faber-Krahn inequality, first conjectured in the book [2]. Next from the monotonicity of the eigenvalues we can easily obtain the trivial upper bound. We then continue with the Pólya and Szegö's [5], respectively planar Payne and Weinberger's [17], upper bound for simply-connected domains which stand on the min-max principle and on the use of the shrinking, respectively parallel, coordinates. Moreover we introduce our own result, the generalization of the Pólya and Szegö's bound for particular, not simply-connected domains. In the end we state the Antunes and Freitas conjecture based on some numerical studies [21].

In the last chapter we first present some examples of simply-connected domains to which we apply the presented bounds and consequently we compare them. Finally we create the particular hole in these domains, apply our bound to them and show the dependence of the bound on the size of the domain and the hole.

Chapter 1

Musical motivation

1.1 Spectrum of 1D objects

Most musical instruments produce sound by the vibration of air column or string. Both these can be described by the one dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}.$$

The function $u = u(x,t)$ of the coordinate x and time t is the amplitude of vibration and c is a constant having a meaning of the phase velocity of the wave in the material or air. For our purposes, we can put this constant equal to 1 without loss of generality. Considering a string of length L with two fixed edges we have the Dirichlet boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0. \end{aligned} \tag{1.1}$$

Considering an air column, we have either the Neumann boundary conditions on both sides, i.e.,

$$\begin{aligned} \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} &= 0 \\ \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L} &= 0 \end{aligned} \tag{1.2}$$

or the Neumann boundary condition on one side and the Dirichlet boundary condition on the other. (An air column with the Dirichlet boundary condition on both sides is not possible because air which is blown into the instrument has to escape somewhere.)

1.1.1 Spectrum of string

First we solve the string boundary spectral problem obtaining the spectrum of the string

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{\partial^2 u(x,t)}{\partial t^2} \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

subject to the initial conditions.

Assuming $u(x,t) = X(x)T(t)$, we can separate the time and space part of the preceding equation. First we solve the space part

$$\frac{\partial^2 X(x)}{\partial x^2} = -\lambda X(x) \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} X(0) &= 0 \\ X(L) &= 0. \end{aligned} \tag{1.4}$$

We can also interpret this equation as an eigenvalue problem for the Laplace operator (in case of the string it consists only of the second partial derivative with respect to the coordinate x). We will see that not only for the string but also for other geometries in higher dimensions we can interpret the space part of the wave equation as an eigenvalue problem.

Now we will show that λ, eigenvalue of our problem defined in the previous paragraph, is positive. This will allow us to write the corresponding eigenvectors in the terms of sines and cosines. Let us take our equation, multiply it with its complex conjugate and integrate over the whole string

$$\int_0^L \frac{\partial^2 X(x)}{\partial x^2}\overline{X(x)}dx = -\lambda \int_0^L X(x)\overline{X(x)}dx.$$

Using the per partes method on the integral on the left hand side we obtain

$$\left[\frac{\partial X(x)}{\partial x}\overline{X(x)}\right]_0^L - \int_0^L \frac{\partial X(x)}{\partial x}\frac{\partial \overline{X(x)}}{\partial x}dx = -\lambda \int_0^L X(x)\overline{X(x)}dx$$

where the first term is zero because of the Dirichlet boundary conditions. Expressing λ we get

$$\lambda = \frac{\int_0^L \frac{\partial X(x)}{\partial x}\frac{\partial \overline{X(x)}}{\partial x}dx}{\int_0^L \left|X(x)\right|^2 dx} = \frac{\int_0^L \frac{\partial X(x)}{\partial x}\overline{\frac{\partial X(x)}{\partial x}}dx}{\int_0^L \left|X(x)\right|^2 dx} = \frac{\int_0^L \left|\frac{\partial X(x)}{\partial x}\right|^2 dx}{\int_0^L \left|X(x)\right|^2 dx}$$

which implies that λ is nonnegative.

Assuming $\lambda = 0$ we get

$$X(x) = C_1 x + C_2$$

as the general solution of the equation (1.3) and considering the boundary conditions (1.4) we obtain zero solution $X(x) = 0$, thus $\lambda = 0$ cannot be considered as the part of the spectrum, implying $\lambda > 0$.

Because λ is positive we can write the general solution of the equation (1.3), for example in the form

$$X(x) = C_3 \cos(\sqrt{\lambda}x) + C_4 \sin(\sqrt{\lambda}x).$$

Next we apply the Dirichlet conditions (1.4) to solve the boundary value problem

$$X(0) = 0 \Rightarrow C_3 = 0$$

$$X(L) = 0 \Rightarrow \lambda = \frac{k^2\pi^2}{L^2},\ k \in \mathbb{Z} - \{0\}.$$

Thus the spectrum of the string boundary value problem is

$$\sigma = \left\{ \frac{k^2\pi^2}{L^2} \middle| k \in \mathbb{Z} - \{0\} \right\} \tag{1.5}$$

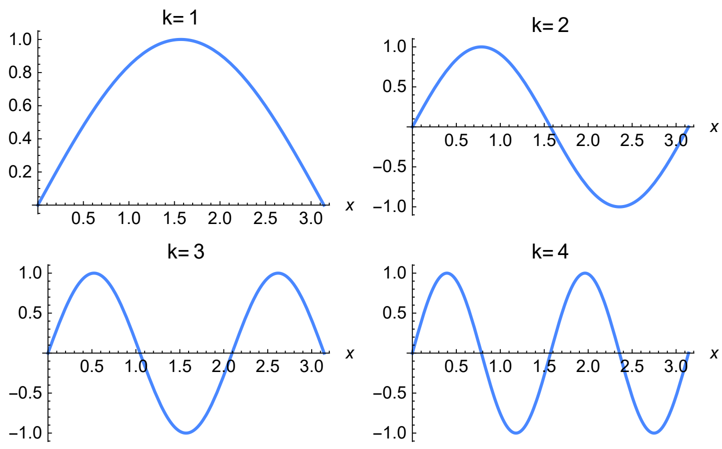


Figure 1.1: Plot of $u_k(x, 0)$ for $k \in \{1, 2, 3, 4\}$

and the corresponding eigenfunctions are

$$X(x) = C_4 \sin(\sqrt{\lambda} x).$$

While it is not our goal to find the complete solution, u, we will do it to find out the meaning of the eigenvalues (1.5).

The separated equation for time is

$$\frac{\partial^2 T(t)}{\partial t^2} = -\lambda T(t).$$

We already know that λ is positive, so we can write the solution in the form

$$T(t) = C \cos(\sqrt{\lambda} t + \phi). \tag{1.6}$$

The whole solution $u(x, t) = X(x)T(t)$ is then

$$u_k(x, t) = C_5 \sin(\sqrt{\lambda_k} x) \cos(\sqrt{\lambda_k} t + \phi), \quad \lambda_k \in \sigma$$

where $C_5 = C \cdot C_4$ and ϕ depend on the initial conditions. On Figure 1.1 the first four modes can be seen. From this equation we can see that $\sqrt{\lambda_k}$ plays the role of the angular frequency of the movement

$$\sqrt{\lambda_k} = \omega = 2\pi f$$

where f is the frequency. This implies that the frequencies of the modes of vibration of the string depend proportionally on k, $f \propto k$, $k \in \mathbb{Z} - \{0\}$. The mode with the lowest frequency ($k = 1$) is considered as the fundamental. Its frequency is $f = \frac{\pi}{L}$. All other modes' frequencies are a whole number multiples of this fundamental frequency.

1.1.2 Spectrum of air column with two open ends

Next we will look at the instruments where the sound is produced by vibrations of an air column with both ends open (for example a flute). An air column can be modeled by the wave equation for the acoustic pressure or for the amplitude as in the string case. For better compatibility we consider the wave equation for amplitude. A pressure node is equivalent to an amplitude antinode and vice versa. Thus the open ends on both sides which mean pressure nodes imply amplitude antinodes on both sides. We thus get the Neumann boundary conditions (1.2). So our goal is to solve the following boundary value problem

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{\partial^2 u(x,t)}{\partial t^2} \\ \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} &= 0 \\ \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L} &= 0 \end{aligned} \tag{1.7}$$

subject to the initial conditions. Again by assuming $u(x,t) = X(x)T(t)$ and separating time we obtain the following space problem

$$\begin{aligned} \frac{\partial^2 X(x)}{\partial x^2} &= -\lambda X(x) \\ \left.\frac{\partial X(x)}{\partial x}\right|_{x=0} &= 0 \\ \left.\frac{\partial X(x)}{\partial x}\right|_{x=L} &= 0. \end{aligned}$$

Analogically as in the string boundary value problem we can show that λ is nonnegative.

For $\lambda = 0$ we have the solution

$$X(x) = C_1 x + C_2 \tag{1.8}$$

and considering the Neumann boundary conditions (1.2) we get a constant eigenfunction $X(x) = C_2 \neq 0$.

For λ strictly positive we have the solution

$$X(x) = C_3 \cos(\sqrt{\lambda} x) + C_4 \sin(\sqrt{\lambda} x)$$

and applying (1.2) leads to

$$\begin{aligned} \left.\frac{\partial X(x)}{\partial x}\right|_{x=0} &= 0 \Rightarrow C_4 = 0 \\ \left.\frac{\partial X(x)}{\partial x}\right|_{x=L} &= 0 \Rightarrow \lambda = \frac{k^2\pi^2}{L^2},\ k \in \mathbb{Z} - \{0\}. \end{aligned}$$

Hence, we can write the spectrum

$$\sigma = \left\{ \frac{k^2\pi^2}{L^2} \middle| k \in \mathbb{Z} \right\} \tag{1.9}$$

with the corresponding eigenfunctions

$$X(x) = C_3 \cos(\sqrt{\lambda} x).$$

As we can see the only difference between this spectrum (1.9) and the previously obtained string boundary value problem spectrum (1.5) is the eigenvalue 0, which does not play role in the

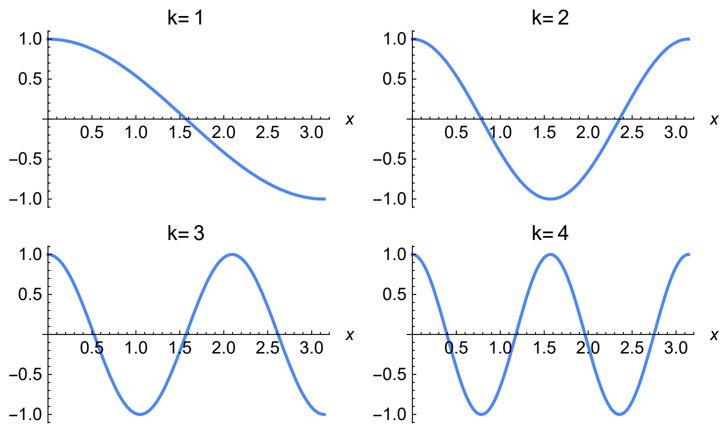


Figure 1.2: Plot of $u_k(x, 0)$ for $k \in \{1, 2, 3, 4\}$

sound because of its zero energy. So the mode frequencies are again the integer multiples of the fundamental frequency ($k = 1$).

The solution of time part of (1.7) is the same as in the string case, (1.6), so we can write the whole solution as

$$u_k(x,t) = C_5 \cos(\sqrt{\lambda_k} x) \cos(\sqrt{\lambda_k} t + \phi),\ \lambda_k \in \sigma$$

where $C_5 = C \cdot C_3$. On Figure 1.2 we can see the first four modes omitting the zero mode.

1.1.3 Spectrum of air column with one open end

An air column with both open and closed end is also a usual vibrating object in many musical instruments (for example a clarinet). We can model this again by the wave equation for the amplitude but now with the Dirichlet boundary condition on the closed end and the Neumann boundary condition on the open end. Let us assume that the end at the coordinate $x = 0$ is the closed one without loss of generality. Thus our boundary value problem is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$
$$u(0,t) = 0$$
$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L} = 0$$

subject to the initial conditions. Again by separating time we obtain the space problem

$$\frac{\partial^2 X(x)}{\partial x^2} = -\lambda X(x)$$
$$X(0) = 0$$
$$\left.\frac{\partial X(x)}{\partial x}\right|_{x=L} = 0$$

with the eigenvalue λ which is again nonnegative.

For $\lambda = 0$ we obtain the solution

$$X(x) = C_1 x + C_2$$

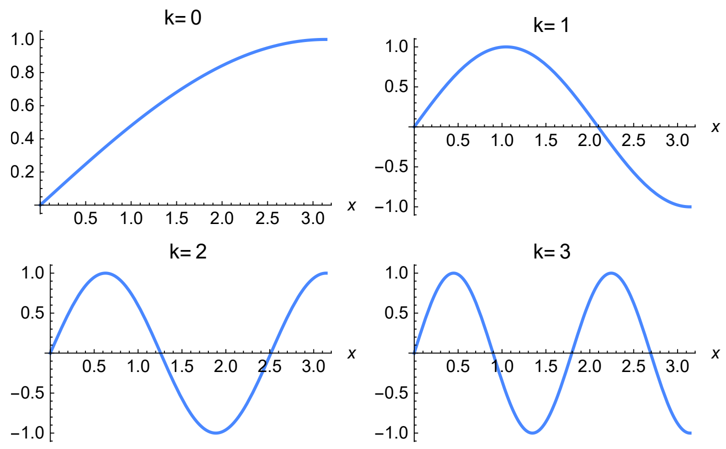


Figure 1.3: Plot of $u_k(x, 0)$ for $k \in \{0, 1, 2, 3\}$

which considering the boundary conditions leads to zero solution $X(x) = 0$ not being considered as an eigenfunction, thus $\lambda = 0$ is not an eigenvalue.

For $\lambda > 0$ the solution can be written in the same form as in the purely Neumann case

$$X(x) = C_3 \cos(\sqrt{\lambda} x) + C_4 \sin(\sqrt{\lambda} x)$$

and the boundary conditions imply

$$X(0) = 0 \Rightarrow C_3 = 0$$

$$\left.\frac{\partial X(x)}{\partial x}\right|_{x=L} = 0 \Rightarrow \lambda = \frac{\left(k + \frac{1}{2}\right)^2 \pi^2}{L^2},\ k \in \mathbb{Z}.$$

The spectrum is then

$$\sigma = \left\{ \frac{\left(k + \frac{1}{2}\right)^2 \pi^2}{L^2} \middle| k \in \mathbb{Z} \right\}$$

with the eigenfunctions

$$X(x) = C_4 \sin(\sqrt{\lambda} x).$$

On the first sight the corresponding frequencies are now not proportional to k, but if we look more carefully it can be seen that they are integer multiples of a frequency that is half the size of the frequency in the string case and every odd frequency (even the first one) is missing in the spectrum. As we will see later this is no problem for the sound. The important thing is that the spectrum still contains only the integer multiples of some frequency.

The whole solution obtained the same way as before using the time solution (1.6) is then

$$u_k(x, t) = C_5 \sin(\sqrt{\lambda} x) \cos(\sqrt{\lambda_k} t + \phi),\ \lambda_k \in \sigma$$

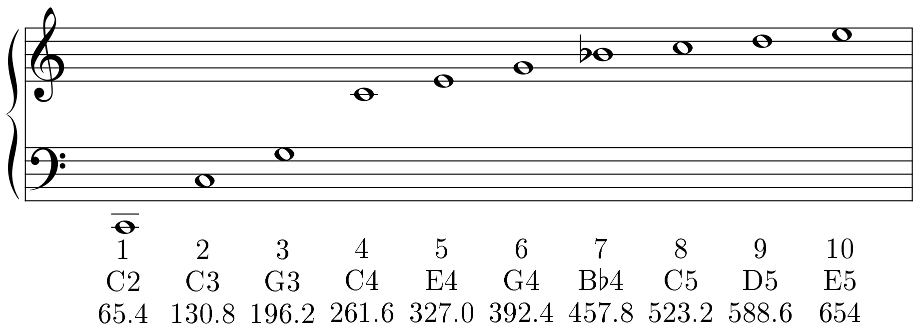
where $C_5 = C \cdot C_4$. On figure 1.3 we can see the corresponding first four modes.

1.1.4 Harmonic series

The fact that the previously obtained spectra contain frequencies that are integer multiples of some fundamental frequency is very important because almost all the sounds we hear and which can be considered as tones are produced by what can be modeled as a one-dimensional vibrating object (string, air column, bar, etc). The fundamental frequency with its integer multiples sounding above it is then the structure of tones our ear has usually been hearing since the ancient times. This implies our ear is being used to this structure, this spectrum, and it sounds nice to it. The frequencies which are included in it generate the so called harmonic series. This series is definitely the basis of the classical western music.

To show the importance of the harmonic series we will create one. We take 65.4 Hz (note C2) as the fundamental frequency. The integer multiples of the fundamental are then 130.8 Hz (note C3), 196.2 Hz (note G3), 261.6 Hz (note C4), 327.0 Hz (note E4), 392.4 Hz (note G4), 457.8 Hz (note B♭4) and so on. The names of the notes are only informative because an exact pitch associated with a specific note depends on the type of the tuning we choose.

Musical extract 1: Harmonic series. First line denotes the number of the tone, on the second line there are the names of the notes and the third line has the meaning of the frequency of the tones



We can see that the ratio of the frequencies of the first overtone and the fundamental is 2:1. In musical theory this ratio is called an octave. The ratio of the second and the first overtone is 3:2, called the perfect fifth. The following are 4:3 called the perfect fourth, 5:4 is the major third, 6:5 is the minor third and so on. These are the ratios (called intervals) defined by the harmonic series which our ear likes because it hears it in every tone. From these we can construct the so called just intonation, which contains strictly these intervals (ratios of small integers) and hence it is very consonant. However the music which can be produced in just intonation is very limited because it is almost impossible to maintain this small integer ratios between all the tones in every chord or within the succeeding tones. In the past there were many attempts to solve this problem, for example the Pythagorean tuning which uses only the perfect fifths to get all tones but leading to the Pythagorean comma which is another problem. The intense development of the western music in approximately last 500 years forced the formation of the so called equal temperament. It takes the interval of octave (2:1) and divides it into twelve parts of the same size equal to $\sqrt[12]{2}$. This means that no interval except the octave is an exact ratio of small integers. Hence all of them differ from the corresponding just interval which makes their sound slightly dissonant. However with equal temperament we can write as rich music as we want. We can change the keys during one piece (we can imagine this as changing the fundamental frequency of the harmonic series), we can use complicated chords (some of their notes do not have to be contained in the appropriate harmonic series) and so on. None of this is easily possible with just intonation. The classical western music uses, approximately from the times of J. S. Bach, mainly the equal temperament.

Although there are differences between intervals in just intonation and equal temperament, the harmonic series is still the basis of the western music. Not musically trained ear almost cannot notice that all the intervals (except octave) are dissonant. When performing music only on the instruments where the tone pitch can be continuously changed (for example violin or human voice) the performers often tend to play some intervals more just. Of course this is not possible when playing the instruments with fixed tone pitch (for example piano or organ).

In the next chapter we take a look on the spectrum of the instruments where the vibration is produced by some two-dimensional source (for example drums).

1.2 Spectrum of 2D domains

The musical instruments which have the vibrating object that can be modeled as one or more two-dimensional membranes are called the drums. The vast majority of this membranes has a round shape, so we will discuss only the spectrum of a circular membrane.

Useful model of vibrations of this membrane is the wave equation in the polar coordinates with the Dirichlet and cyclic boundary conditions

$$\begin{aligned} \Delta u(r,\phi,t) &= \frac{\partial^2 u(r,\phi,t)}{\partial t^2} \\ u(a,\phi,t) &= 0 \\ u(r,-\pi,t) &= u(r,\pi,t) \\ \frac{\partial u}{\partial \phi}(r,-\pi,t) &= \frac{\partial u}{\partial \phi}(r,\pi,t) \end{aligned} \tag{1.10}$$

subject to the initial conditions where a is the radius of the circle, $r \in (0,a)$, $\phi \in (-\pi,\pi)$ and

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

is the Laplace operator in the polar coordinates r and ϕ. We have again put the constant c equal to 1 without loss of generality. Now we will again use the method of the separation of the variables to separate the space part of the equation. Assuming the solution of type $u(r,\phi,t) = A(r,\phi)T(t)$ we obtain

$$\begin{aligned} \frac{\partial^2 A(r,\phi)}{\partial r^2} + \frac{1}{r}\frac{\partial A(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A(r,\phi)}{\partial \phi^2} &= -\lambda A(r,\phi) \\ A(a,\phi) &= 0 \\ A(r,-\pi) &= A(r,\pi) \\ \frac{\partial A}{\partial \phi}(r,-\pi) &= \frac{\partial A}{\partial \phi}(r,\pi) \end{aligned}$$

where λ is an eigenvalue. Our task is now to calculate all possible λ. By assuming $A(r,\phi) = R(r)\Phi(\phi)$ we can separate the radial and angular part

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + \lambda r^2 = -\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = \nu \tag{1.11}$$

$$R(a) = 0 \tag{1.12}$$

$$\Phi(-\pi) = \Phi(\pi) \tag{1.13}$$

$$\frac{d\Phi}{d\phi}(-\pi) = \frac{d\Phi}{d\phi}(\pi). \tag{1.14}$$

The solution of the angular part without considering the boundary conditions is

$$\Phi(\phi) = C_1 e^{i\sqrt{\nu}\phi} + C_2 e^{-i\sqrt{\nu}\phi}, \tag{1.15}$$

assuming $\nu \neq 0$.

Now we will apply the cyclic boundary conditions. For $\nu < 0$ we can rewrite the solution as

$$\Phi(\phi) = (C_1 + C_2)\cosh\sqrt{\nu}\phi + (C_2 - C_1)\sinh\sqrt{\nu}\phi.$$

From the condition (1.13) and considering the parity of the hyperbolic functions we obtain

$$-(C_2 - C_1)\sinh\sqrt{\nu}\pi = (C_2 - C_1)\sinh\sqrt{\nu}\pi.$$

This condition can be satisfied only when

$$\sinh\sqrt{\nu}\pi = 0 \vee C_2 - C_1 = 0.$$

The first equation is equal to $\nu = 0$, but we are now interested only in ν strictly negative. This implies that $C_2 - C_1 = 0$. From the condition (1.14) and again considering the parity we get

$$-(C_1 + C_2)\sinh\sqrt{\nu}\pi = (C_1 + C_2)\sinh\sqrt{\nu}\pi$$

which analogically leads to $C_1 + C_2 = 0$. Hence for ν strictly negative we get only a trivial solution for Φ.

For $\nu = 0$ we have different fundamental system than in the solution (1.15). We can write the solution for $\nu = 0$ as

$$\Phi(\phi) = C_3 + C_4\phi.$$

The boundary conditions imply $C_4 = 0$, leading to a constant solution.

For $\nu > 0$ we can again rewrite solution (1.15) as

$$\Phi(\phi) = i(C_1 - C_2)\sin\sqrt{\nu}\phi + (C_1 + C_2)\cos\sqrt{\nu}\phi.$$

Omitting the trivial solution and considering the parity of the trigonometric functions, the condition (1.13) and (1.14) imply

$$\begin{aligned}\sin\sqrt{\nu}\pi &= 0\\ \sqrt{\nu}\pi &= m\pi,\ m \in \mathbb{Z}\\ \nu &= m^2.\end{aligned}$$

Thus we may take $\nu = m^2, m \in \mathbb{Z}_0^+$ and the solution of the angular part considering boundary conditions as

$$\Phi_m(\phi) = i(C_1 - C_2)\sin m\phi + (C_1 + C_2)\cos m\phi.$$

Now we solve the radial part of equation (1.11). First we have to show that λ is positive. Let us denote our membrane by D and its boundary by ∂D. λ is an eigenvalue of the problem

$$\Delta A(r,\phi) = -\lambda A(r,\phi).$$

Analogically as in the one-dimensional case, we can multiply this equation by the complex conjugate of A and then integrate it over the whole membrane D

$$\int_D \bar{A}\,\Delta A dS = -\lambda\int_D \bar{A}AdS. \tag{1.16}$$

Now we can use the Divergence theorem ([22], Thm. 5.8) for the function $\bar{A}\Delta A$

$$\int_{\partial D}(\bar{A}\nabla A)\cdot\boldsymbol{n}dt = \int_D \bar{A}\Delta A dS + \int_D |\nabla A|^2 dS$$

where $\boldsymbol{n}$ is the outward unit normal vector to dS. Here we can substitute the first integral on the right hand side from the equation (1.16) and use the Dirichlet boundary conditions which imply

$$\int_{\partial D} (\bar{A}\nabla A)\cdot \boldsymbol{n} dt = 0$$

because $A(r, \phi) = 0$ on ∂D. We obtain an expression for λ

$$\lambda = \frac{\displaystyle\int_D |\nabla A|^2 dS}{\displaystyle\int_D |A|^2 dS}$$

from which we can see that $\lambda \geq 0$.

Now we can continue. After substituting for $\sqrt{\nu}$ and rearranging the equation into the form

$$r^2 \frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + R\Big(\big(\sqrt{\lambda}r\big)^2 - m^2\Big) = 0$$

which is possible because $\lambda \geq 0$, we obtain the Bessel equation of order m. Its solution with respect to m can be written as a linear combination of the Bessel functions of order m of the first and second kind

$$R_m(r) = C_5 J_m(\sqrt{\lambda}r) + C_6 Y_m(\sqrt{\lambda}r).$$

The Bessel functions can be defined using the power series

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}, \ \alpha \in \mathbb{C}$$

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}, \ \alpha \in \mathbb{C} - \mathbb{Z}$$

$$Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x), \ n \in \mathbb{Z}$$

where $\Gamma(x)$ is the gamma function defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

An important property of the Bessel functions of the second kind is

$$\lim_{x\to 0^+} Y_p(x) = -\infty.$$

For our purposes we need a finite displacement of the membrane

$$\lim_{x\to 0^+} R_m(r) \neq \pm\infty$$

which implies that $C_6 = 0$ and

$$R_m(r) = C_5 J_m(\sqrt{\lambda}r). \tag{1.17}$$

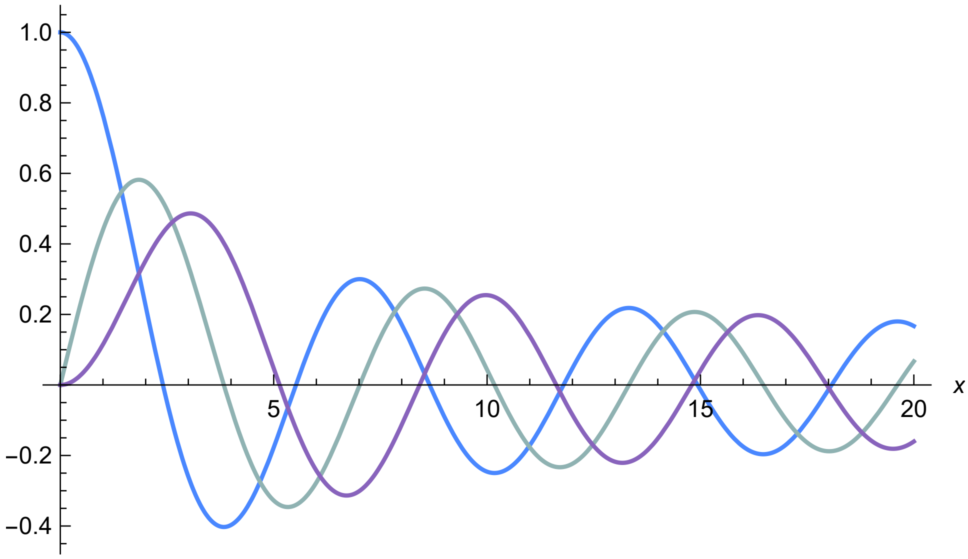


Figure 1.4: Plot of $J_m(x)$ for $m \in \{0, 1, 2\}$

Table 1.1: Table of first j_{mn} ordered by size

m	n	j_{mn}
0	1	2.405
1	1	3.832
2	1	5.136
0	2	5.520
3	1	6.380
1	2	7.016
4	1	7.588
2	2	8.417

Next we have to apply the Dirichlet boundary conditions (1.12)

$$R(a) = 0 \implies J_m(\sqrt{\lambda}a) = 0.$$

The Bessel function of the first kind has infinitely many zeros. The plot of J_m can be seen on Figure 1.4. We shall denote the *nth* zero of the Bessel function of the first kind of the order m as j_{mn}, $J_m(j_{mn}) = 0$. Table 1.2 shows the first j_{mn} ordered by size.

Thus we can write

$$\sqrt{\lambda}a = j_{mn}$$

from which we can finally obtain the spectrum

$$\lambda_{mn} = \left(\frac{j_{mn}}{a}\right)^2,\ m \in \mathbb{Z}_0^+, n \in \mathbb{N}. \tag{1.18}$$

The solution of the time part of the equation (1.10)

$$\frac{\partial^2 T(t)}{\partial t^2} + \lambda T(t) = 0$$

is the same as in the one-dimensional case

$$T(t) = C_7 \cos(\sqrt{\lambda}t + \phi).$$

Thus the whole solution can be written for example in the form

$$u_{mn}(r, \phi, t) = K J_m(\sqrt{\lambda_{mn}}r)\big(i(C_1 - C_2)\sin m\phi + (C_1 + C_2)\cos m\phi\big)\cos(\sqrt{\lambda_{mn}}t + \phi) \tag{1.19}$$

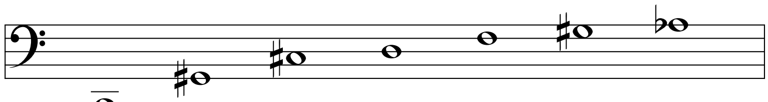
where $K = C_7 \cdot C_5$. On Figure 1.5, the ten first modes sorted in a non-decreasing order can be seen. From (1.19) we can see that $\sqrt{\lambda_{mn}}$ again plays the role of the angular frequency $\sqrt{\lambda_{mn}} = \omega = 2\pi f$.

We have obtained the frequencies of the modes of vibration

$$f_{mn} = \frac{1}{2\pi}\sqrt{\lambda_{mn}} = \frac{j_{mn}}{2\pi a}.$$

These frequencies are definitely not a whole number multiples of the fundamental frequency f_{01}. Now we would assume $a = 0.00585$ to attain the same fundamental frequency (65.4 Hz, note C2) as in the one-dimensional case so we could better compare them. (The radius of the membrane seems unrealistic because we dismissed the constant c at the beginning). First modes sounding above the fundamental have frequencies of 104.25 Hz (note G♯2), 139.72 Hz (note C♯3), 150.18 Hz (note D3), 173.58 Hz (note F3), 190.87 Hz (note G♯3) and 206.45 Hz (note A♭3). The names of the notes are again only informative, the differences between these frequencies and frequencies obtained using equal temperament for the same notes would be significant.

Musical extract 2: Spectrum of circular membrane



1	2	3	4	5	6	7
C2	G♯2	C♯3	D3	F3	G♯3	A♭3
65.4	104.25	139.72	150.18	173.58	190.87	206.45

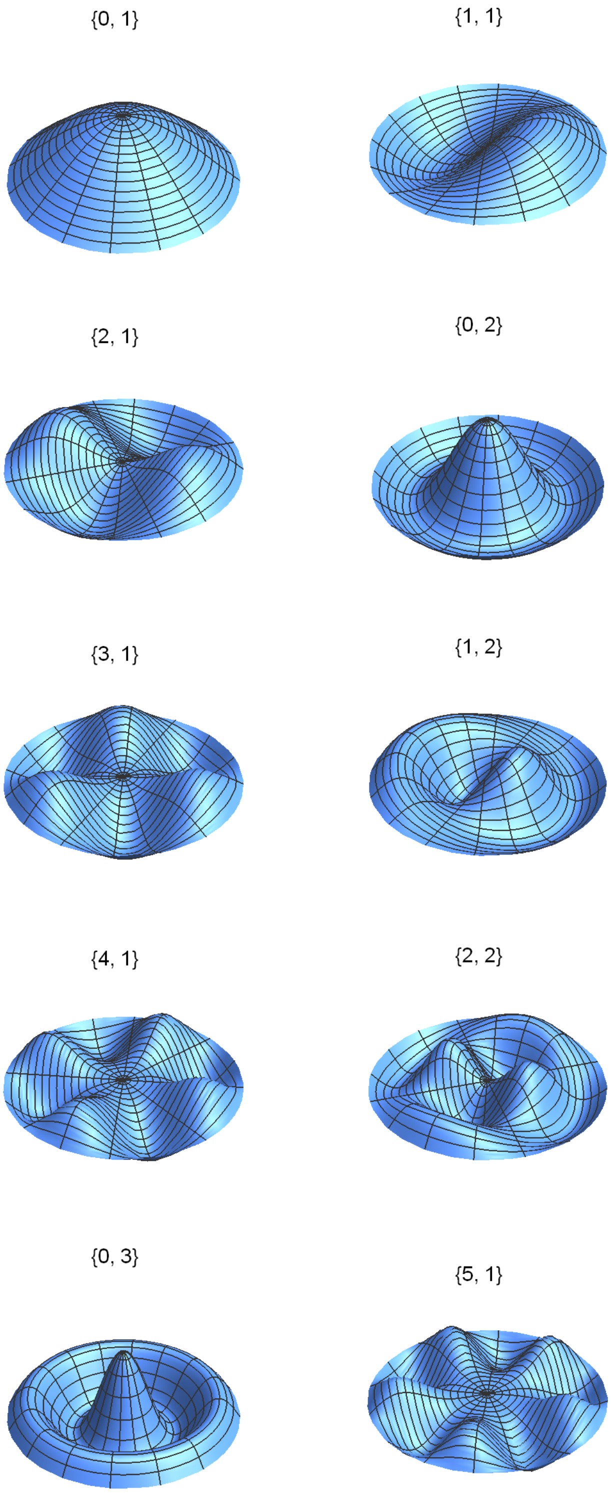


Figure 1.5: Plot of $u_{mn}(r, \phi, 0)$ for $(m, n) \in \{(0,1),(1,1),(2,1),(0,2),(3,1),(1,2),(4,1),(2,2),(0,3),(5,1)\}$

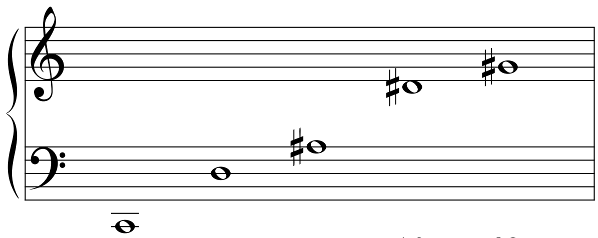
As we can see this series is very different from the previously obtained harmonic series. This is caused by the two-dimensionality of the membrane and it is the reason for why our ear does not perceive the sound of a drum as a tone but more as a noise. But there are some drums which sound musically. First of their sounding overtones are consistent with the harmonic series thus we can determine the drum's pitch and classify its sound as a tone. This can be accounted for other factors from which the most important are the sound radiation and considering the vibrations of air enclosed in the kettle and above the membrane. We will talk about this in the next chapter.

1.3 Sound radiation and air loading

In this section we discuss the phenomena that makes some drums with the kettle sound more musically. The most important drum with these properties is the timpani. It is being used in many different types of musical ensembles, for example in the classical symphonic orchestra or in marching bands.

The first thing we should consider is the place where the membrane is struck because this significantly affects the decay rate of the modes and thus the sounding spectrum. We shall denote the individual modes as in the previous section by the two indices m and n, where $m \in \mathbb{Z}_0^+$ means the number of the nodal diameters and $n \in \mathbb{N}$ has the value of the number of the nodal circles. If one would strike the timpani in the middle, only the modes with $m = 0$ would participate in the sound, because all the remaining have a node in the place of the stroke and thus cannot be excited by this way.

Musical extract 3: Spectrum of membrane when struck in middle



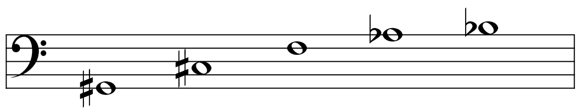
1	4	9	16	23
C2	D3	A♯3	D♯4	G♯4
65.4	150.18	235.43	320.80	406.21

If we now look at the corresponding tones we can see that for example the interval between the fundamental and first sounding overtone is approximately 1 : 2.296, something between major and augmented ninth which is a very dissonant interval. Hence this is not the best way to strike the timpani. In the book The Theory of Sound [2], Lord Rayleigh showed that striking the membrane approximately one quarter of its radius from the edge almost does not excite the first mode, thus the first sounding mode is the second one in the spectrum ($m = 1, n = 1$). This way of striking also causes the modes with $n = 1$ to be mainly represented in the spectrum (their amplitudes are the highest). As we will show later, these modes have small decay rates in contrast with the modes with $m = 0$ which decay very rapidly and thus their sound can be considered more as a thump. More importantly the frequencies of the slowly decaying modes can be shifted to be considered as a part of some harmonic series. Therefore from this point we would assume the membrane has been struck approximately one quarter of its radius from the edge.

Now we will look at the sound radiation of the individual modes (see [3]). The mode ($m = 0, n = 1$) acts as a monopole source which radiates sound very effectively and thus has very high decay rate and considering the place of hit allows us to omit this mode in the sounding spectrum. The second mode ($m = 1, n = 1$) acts as a dipole source which radiates sound less effectively than the monopole source and so it decays more slowly. The third mode ($m = 2, n = 1$) acts as a quadrupole source to which it takes even more time to decay than to the second mode. The fourth mode can be considered as something between the monopole and dipole source and its decay time is something between the first and the second mode. However it can be shown that this mode does not play a big role in the sounding spectrum. As the second and third mode, the fifth ($m = 3, n = 1$), seventh ($m = 4, n = 1$) and even the tenth mode ($m = 5, n = 1$), whose amplitudes can still be considered as enough high, assuming the right stroke, are very poor sound radiators and they contribute to the sounding spectrum. On the other hand, the sixth ($m = 1, n = 2$) and eight mode ($m = 2, n = 2$) do not participate in the sound when the timpani is hit correctly, although they decay quite slowly.

Now, considering the initial conditions and the sound radiation we have eliminated the modes which do not contribute to the musical sound of the drum. Thus we have obtained the following spectrum

Musical extract 4: Spectrum of membrane considering proper initial conditions and sound radiation



2	3	5	7	10
G♯2	C♯3	F3	A♭3	B♭3
104.25	139.72	173.58	206.45	238.64

We can see that this spectrum is still not a part of the harmonic series for what we want are only the integer multiples of some frequency and now we have the ratios 1 : 1.35 : 1.67 : 1.99 : 2.3.

Further aspect which has not yet been considered is the actual three-dimensionality of the membrane. A real membrane has to be modeled more like a plate. This means we have to consider also its bending stiffness and stiffness to shear. These two raise the frequencies of the overtones. However their effect is not of high importance (see [3]).

More important role plays the so called air loading. On both sides of a real membrane is air, the inner side is enclosed in the kettle and air in it also vibrates when the drum is struck. Air on the outer side of the membrane also plays its role. This effect lowers the frequencies of the low modes and it is the main factor which establishes the harmonicity of the spectrum.

The calculation of the effect of air loading is presented in the paper [1]. They model the drum as a rigid kettle of a cylindric shape with the length L and radius a, rigid bottom with a small circular vent hole and the membrane on the top. Moreover they model the vibrations of the drum using the wave equation not for the amplitude as in the preceding cases but for the acoustic pressure p

$$\frac{1}{c^2}\Delta p(\rho, \phi, z, t) - \frac{\partial^2 p(\rho, \phi, z, t)}{\partial t^2} = 0$$

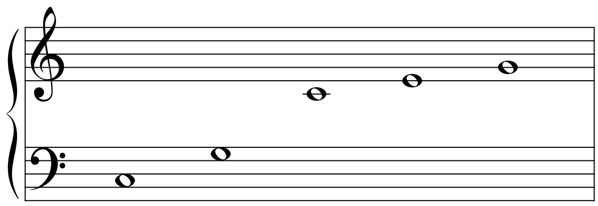
where Δ is the Laplace operator in the cylindrical coordinates

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

and c is the speed of the sound in air. To solve this problem they use the method of the Green functions.

The obtained results can be used to calculate the modes of vibration of an air loaded membrane. It shows that the important modes (those with $n = 1$) have frequencies in a ratio 1 : 1.51 : 1.99 : 2.46 : 2.93 for the fundamental frequency 150 Hz, in a ratio 1 : 1.5 : 1.97 : 2.44 : 2.89 for the frequency 107 Hz or in a ratio 1 : 1.51 : 1.98 : 2.44 : 2.9 for the frequency 145 Hz, where the fundamental frequency is the frequency of the first sounding mode ($m = 1, n = 1$) as we have shown before. In general it can be seen that for fundamental frequencies in a normal playing range for timpani (approximately 100 Hz $< f <$ 175 Hz), the modes that participate on the tone of the drum have frequencies in a ratio approximately 1 : 1.5 : 2 : 2.5 : 3, which is equal to 2 : 3 : 4 : 5 : 6 and thus can be considered as a beginning of the harmonic series without the first frequency. As we will show later the missing fundamental frequency makes no problem when perceiving the pitch of the sound. If we now take 130.8 Hz as the fundamental frequency we obtain the following spectrum

Musical extract 5: Spectrum of timpani



2	3	5	7	9
C3	G3	C4	E4	G4
130.8	196.2	261.6	327.0	392.4

There is a very interesting problem in psychoacoustics called the missing fundamental. It says that the human ear perceives the pitch of the tones not only by the fundamental frequency but, when the spectrum consists of the integer multiples of some frequency, then the important role play the differences between the individual frequencies and it does not matter whether a few of them are missing. For example a spectrum with the frequencies in a ratio 1 : 2 : 3 : 4 : 5 : 6 generates a harmonic series of the first frequency. The differences between the frequencies are equal to the first frequency. Thus the first frequency is the pitch we perceive. If we now remove for example the first frequency we obtain a ratio 2 : 3 : 4 : 5 : 6. As we can see the differences are still equal to the first frequency. Therefore we again perceive the pitch of the fundamental even when it is not included the spectrum. We could have also removed for example the fourth frequency obtaining 1 : 2 : 3 : 5 : 6. The dominant difference will still be equal to the fundamental and thus we will perceive the same pitch as before. But the problem is more complicated. Having 1 : 2 : 3 : 4 : 5 : 6 and removing all the even frequencies leading to 1 : 3 : 5 also generates a harmonic series of the fundamental even when the differences are equal to a tone an octave higher (a tone with a two times higher frequency). Thus the perceived pitch is the same as for 1 : 2 : 3 : 4 : 5 : 6, only the timbre is different. It is caused by the relative sizes of amplitudes being another important factor beside the differences. The amplitude of the fundamental is dominant and thus the fundamental is perceived as the pitch. This effect can be seen for example in the wind instruments having only one open end (the Dirichlet boundary conditions on the closed end and the Neumann boundary conditions on the open end, for example a clarinet) which causes its spectrum to be equivalent to the spectrum of an instrument twice as long having both ends open with all the even frequencies missing. This causes that having these two instruments with the same length, the one with the Neumann boundary conditions on both ends (for example a flute) would sound an octave higher (frequencies in a ratio 1 : 2 : 3 : 4 : 5 : 6...) than the one with the end with the Dirichlet boundary conditions (frequencies in a ratio 0.5 : 1.5 : 2.5 : 3.5...) because the sounding spectrum of the wind instrument with both ends open is equal to the spectrum of the string (the difference in the spectra is the missing zero eigenvalue in the string case which however does not contribute to the sound, having zero energy) as we said before and because the corresponding fundamentals (1 and 0.5) are in ratio 2 : 1 which is equal to the interval of octave.

If we now apply the effect of the missing fundamental to the spectrum of the timpani where the frequencies are approximately in a ratio $2 : 3 : 4 : 5 : 6$, we can see that the differences are equal to the missing first frequency of this harmonic series. Thus if the amplitudes of the individual modes would be in the right ratio, then the pitch of the drum could be perceived as of the fundamental (octave lower than the first sounding mode $m = 1, n = 1$) even when it is not contained in the spectrum at all.

Chapter 2

Dirichlet Laplacian

Our aim in this chapter is to correctly define the Laplace operator with the Dirichlet boundary conditions on any bounded domain Ω in $\mathbb{R}^d$ using quadratic forms. We will take an advantage of this approach in the following chapter.

Let Ω be a domain (an open connected set) in $\mathbb{R}^d$. We would like to define an operator Δ on $L^2(\Omega)$ such that

$$\begin{aligned} -\Delta\psi &= \lambda\psi \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

for all $\psi \in \mathrm{Dom}(\Delta)$.

2.1 Preliminaries

In this section we recall some definitions for unbounded operators and unbounded sesquilinear forms which can be found in the books [10], [11]. Let $\mathcal{H}$ be a separable complex Hilbert space with the inner product denoted by $(\cdot,\cdot)$ and by convention conjugate linear in the first argument and linear in the second.

Definition 2.1.1 (Densely defined operator). *Let $A : \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$. Then A is densely defined if* $\overline{\mathrm{Dom}(A)} = \mathcal{H}$.

Definition 2.1.2 (Symmetric operator). *An operator $A : \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$ is symmetric if it is densely defined and* $(\phi, A\psi) = (A\phi, \psi)$, $\forall \phi, \psi \in \mathrm{Dom}(A)$.

Definition 2.1.3 (Adjoint operator). *Let $A : \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$. We say that A^* is adjoint to A if the following two conditions are satisfied*

$$\begin{aligned} \mathrm{Dom}(A^*) &:= \{\phi \in \mathcal{H} : \exists \eta \in \mathcal{H}, \forall \psi \in \mathrm{Dom}(A), (\phi, A\psi) = (\eta, \psi)\} \\ A^*\psi &:= \eta. \end{aligned}$$

Definition 2.1.4 (Self-adjoint operator). *Let $A : \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$. Then A is self-adjoint if A is symmetric and* $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$

Definition 2.1.5 (Bounded below operator). *Operator $A : \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$ is bounded below if* $\exists c \in \mathbb{R}$, $\forall \psi \in \mathrm{Dom}(A)$, $(\psi, A\psi) \geq c\|\psi\|^2$, *where $\|\cdot\|$ is the norm on $\mathcal{H}$ induced by the inner product.*

Definition 2.1.6 (Compact operator). *Let A be an operator defined on the whole Hilbert space $\mathcal{H}$. Then A is compact if it maps all bounded subsets in $\mathcal{H}$ to a precompact subset (a subset with compact closure).*

Now we can proceed to the quadratic forms.

Definition 2.1.7 (Sesquilinear form). *A map $a : \mathrm{Dom}(a) \times \mathrm{Dom}(a) \to \mathbb{C}$ such that*

$$\psi \mapsto a(\phi, \psi) \text{ is linear}$$
$$\phi \mapsto a(\phi, \psi) \text{ is conjugate linear}$$
$$a(\phi, \psi) = \overline{a(\psi, \phi)} \quad \forall \phi, \psi \in \mathrm{Dom}(a)$$
$$a(\phi, \phi) \geq 0 \quad \forall \phi \in \mathrm{Dom}(a)$$

is called a sesquilinear form. We say that a is densely defined if $\mathrm{Dom}(a)$ is dense in $\mathcal{H}$.

From now on we would assume that all sesquilinear forms in this text are densely defined.

Definition 2.1.8 (Quadratic form). *Let a' be a sesquilinear form. Then $a : \mathrm{Dom}(a') \to \mathbb{C}$ defined by $a[\psi] := a'(\psi, \psi)$ is called a quadratic form and $\mathrm{Dom}(a) = \mathrm{Dom}(a')$.*

Using the polarization identities (see [11], Section 1.2) we can see that also every quadratic form $a[\phi]$ determines the sesquilinear form $a(\phi, \psi)$ uniquely, hence we can interchange between them.

Definition 2.1.9 (Bounded below quadratic form). *A quadratic form $a : \mathrm{Dom}(a) \to \mathbb{C}$ is bounded below if $\exists c \in \mathbb{R}$, $\forall \psi \in \mathrm{Dom}(a)$, $a[\psi] \geq c\|\psi\|^2$.*

Definition 2.1.10 (Closable quadratic form). *A quadratic form $a : \mathrm{Dom}(a) \to \mathbb{C}$ is closable if $\forall \{\psi_n\} \subset \mathrm{Dom}(a)$:*

$$\left(\lim_{n\to\infty} \psi_n = 0 \wedge \lim_{\substack{n\to\infty \\ m\to\infty}} a[\psi_n - \psi_m] = 0 \right) \Rightarrow \lim_{n\to\infty} a[\psi_n] = 0.$$

Definition 2.1.11 (Closed quadratic form). *A quadratic form $a : \mathrm{Dom}(a) \to \mathbb{C}$ is closed if $\forall \{\psi_n\} \subset \mathrm{Dom}(a)$, $\psi \in \mathcal{H}$:*

$$\left(\lim_{n\to\infty} \psi_n = \psi \wedge \lim_{\substack{n\to\infty \\ m\to\infty}} a[\psi_n - \psi_m] = 0 \right) \Rightarrow \left(\psi \in \mathrm{Dom}(a) \wedge \lim_{n\to\infty} a[\psi_n - \psi] = 0 \right).$$

Definition 2.1.12 (Closure of form). *Let $a : \mathrm{Dom}(a) \to \mathbb{C}$ be a closable quadratic form. Then its closure $\overline{a}$ is defined as*

$$\mathrm{Dom}(\overline{a}) := \left\{ \psi \in \mathcal{H} : \exists \{\psi_n\} \subset \mathrm{Dom}(a), \lim_{n\to\infty} \psi_n = \psi \wedge \lim_{\substack{n\to\infty \\ m\to\infty}} a[\psi_n - \psi_m] = 0 \right\}$$
$$\overline{a}(\phi, \psi) := \lim_{n\to\infty} a(\phi, \psi).$$

2.2 Definition of the Dirichlet Laplacian

First we introduce a one-by-one correspondence between below bounded self-adjoint operators and below bounded closed quadratic forms:

2.2.1 From operator to form

Let A be a below bounded self-adjoint operator. Then the quadratic form $\dot{a}$ associated with this operator using

$$\begin{aligned} \dot{a}[\psi] &:= (\psi, A\psi) \\ \mathrm{Dom}(\dot{a}) &= \mathrm{Dom}(A) \end{aligned}$$

is below bounded and closable (see [16], Thm. VI.1.27). Thus its closure satisfies the desired properties.

2.2.2 From form to operator

To get the below bounded self-adjoint operator we use the following important theorem.

Theorem 2.2.1 (Representation theorem, see [16], Thm. VI.2.2)**.** *Let a be below bounded closed form. Then the operator defined as*

$$\begin{aligned} \mathrm{Dom}(A) &:= \{\psi \in \mathrm{Dom}(a) : \exists \eta \in \mathcal{H}, \forall \phi \in \mathrm{Dom}(a), a(\phi, \psi) = (\phi, \eta)\} \\ A\psi &:= \eta \end{aligned}$$

is self-adjoint and below bounded.

2.2.3 Friedrichs extension

Using this correspondence we can now define the self-adjoint Dirichlet Laplacian using the quadratic forms.

Step 1 Let Ω be an arbitrary domain in $\mathbb{R}^d$. We can start with the operator

$$\begin{aligned} \dot{H}\psi &:= -\Delta\psi \\ \mathrm{Dom}(\dot{H}) &:= C_0^\infty(\Omega) \end{aligned} \tag{2.1}$$

where C_0^∞ is space of smooth functions with compact support and thus

$$\dot{H}(\psi) = 0 \quad \text{ on } \partial\Omega.$$

This operator is certainly densely defined, because $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$ (see [14], Thm. 2.1.8). The space C_0^∞ was also chosen because we wanted to avoid assumptions on the regularity of $\partial\Omega$.

At the same time we see that

$$(\psi, \dot{H}\psi) = -\int_\Omega \overline{\psi}\Delta\psi = \int_\Omega |\nabla\psi|^2 - \int_{\partial\Omega} \overline{\psi}\nabla\psi = \|\nabla\psi\|^2 \tag{2.2}$$

where the first equality is the definition of the inner product on $L^2(\Omega)$ space, the second equality is obtained using the Green identity (see [15], 8.4.1) and the third equality results from the Dirichlet boundary conditions. From this we can see that $(\psi, \dot{H}\psi)$ is greater or equal 0 which implies $\dot{H}$ is symmetric and below bounded. Unfortunately this operator is not self-adjoint.

Remark 2.2.2 (Notation)**.** *In* (2.2) *in the third integral*

$$\int_\Omega |\nabla\psi|^2$$

the absolute-value sign $|\cdot|$ *actually stands for the norm on* $\mathbb{R}^d$ *which is always positive and thus we omitted writing the absolute value which is needed in the definition of* $L^2(\Omega)$ *norm. The correct notation would be then*

$$(\psi, \dot{H}\psi) = -\int_\Omega \overline{\psi}\Delta\psi = \int_\Omega \|\nabla\psi\|^2_{\mathbb{R}^d} - \int_{\partial\Omega} \overline{\psi}\,\nabla\psi\cdot\boldsymbol{n} = \int_\Omega |\,\|\nabla\psi\|_{\mathbb{R}^d}\,|^2 = \|\,\|\nabla\psi\|_{\mathbb{R}^d}\,\|^2_{L^2(\Omega)}$$

which is certainly much less well-arranged and this is the reason why we rather use the shorter notation

$$(\psi, \dot{H}\psi) = \|\,\|\nabla\psi\|_{\mathbb{R}^d}\,\|^2_{L^2(\Omega)} =: \|\nabla\psi\|^2$$

where the last norm stands obviously for the norm on the space $L^2(\Omega)$*.*

Step 2 As a next step we can assign operator (2.1) to a quadratic form using the inner product and equality (2.2)

$$\begin{aligned}\dot{h}[\psi] &:= (\psi, \dot{H}\psi) = \|\nabla\psi\|^2\\ \mathrm{Dom}(\dot{h}) &:= \mathrm{Dom}(\dot{H}) = C_0^\infty(\Omega).\end{aligned}$$

From the definition we can see that this assignment is uniquely defined, below bounded and using [16], Thm. VI.1.27 we know that the form $\dot{h}$ is closable.

Step 3 The closability of $\dot{h}$ allows us to define a new form h as its closure

$$h := \overline{\dot{h}}$$

implying

$$h[\psi] = \|\nabla\psi\|^2$$

where now ∇ denotes the weak gradient and the domain of h is

$$\mathrm{Dom}(h) = \overline{C_0^\infty(\Omega)}^{|||\cdot|||}$$

with the norm

$$|||\psi|||^2 := \|\nabla\psi\|^2 + \|\psi\|^2.$$

The space $\overline{C_0^\infty(\Omega)}^{|||\cdot|||}$ usually denoted as $W_0^{1,2}(\Omega)$ is the Sobolev space (see [10], Sec. 6.1). Hence

$$\begin{aligned}h[\psi] &= \|\nabla\psi\|^2\\ \mathrm{Dom}(h) &= W_0^{1,2}(\Omega).\end{aligned} \tag{2.3}$$

Step 4 The form (2.3) is by the definition below bounded and closed. Thus Theorem 2.2.1 states that there exists a below bounded self-adjoint operator associated with this form denoted by H

$$\begin{gathered}\mathrm{Dom}(H) = \{\psi \in W_0^{1,2}(\Omega) : \exists \eta \in L^2(\Omega), \forall \phi \in W_0^{1,2}(\Omega), (\nabla\phi, \nabla\psi) = (\phi, \eta)\} \\ H\psi = \eta\end{gathered}$$

where again ∇ stands for the weak derivative. This operator is called the Friedrichs extension of $\dot{H}$. Notice that $(\nabla\phi, \nabla\psi) = (\phi, \eta)$ with $\phi \in C_0^\infty$ is the definition of the weak Laplacian. Hence finally we are able to define the self-adjoint Dirichlet Laplacian

$$\begin{gathered}-\Delta_D^\Omega := H \\ \mathrm{Dom}(-\Delta_D^\Omega) = \{\psi \in W_0^{1,2}(\Omega) : \Delta\psi \in L^2(\Omega)\} \\ -\Delta_D^\Omega\psi = -\Delta\psi.\end{gathered} \tag{2.4}$$

2.3 Spectrum of the Dirichlet Laplacian

In this section we define an alternative classification of the spectrum σ of the self-adjoint operator to the usual one (point, continuous and residual spectrum) and state the theorems which will be needed later.

Definition 2.3.1 (Discrete spectrum). *Let H be a self-adjoint operator. We define the discrete spectrum σ_{disc} of H as*

$$\sigma_{disc}(H) := \{\lambda \in \sigma_p(H) : \lambda \textit{ is isolated } \wedge m(\lambda) < \infty\}$$

where σ_p is the point spectrum and $m(\lambda)$ is the multiplicity of λ as an eigenvalue.

Definition 2.3.2 (Essential spectrum). *Let H be a self-adjoint operator. Then the essential spectrum σ_{ess} of H can be defined as*

$$\sigma_{ess}(H) := \sigma(H) \setminus \sigma_{disc}(H).$$

Remark 2.3.3. *A spectrum $\sigma(H)$ of a self-adjoint operator H can be expressed as a disjoint union of the discrete and essential spectrum*

$$\sigma(H) = \sigma_{disc}(H) \uplus \sigma_{ess}(H).$$

Theorem 2.3.4 (Min-max principle, see [10], Sec. 4.5). *Let H be a self-adjoint, below bounded operator on Hilbert space $\mathcal{H}$ and h the quadratic form associated with this operator using Theorem 2.2.1. We define $\{\lambda_n\}_{n=1}^\infty$ as*

$$\lambda_n = \inf_{\substack{\mathcal{L}^n \subset \mathrm{Dom}(H) \\ \dim \mathcal{L}^n = n}} \sup_{\psi \in \mathcal{L}^n} \frac{(\psi, H\psi)}{\|\psi\|^2} = \inf_{\substack{\mathcal{L}^n \subset \mathrm{Dom}(H) \\ \dim \mathcal{L}^n = n}} \sup_{\psi \in \mathcal{L}^n} \frac{h[\psi]}{\|\psi\|^2}$$

where $\|\cdot\|$ denotes the norm on $L^2(\mathcal{H})$. Then

$$\lambda_\infty := \lim_{n\to\infty} \lambda_n = \inf \sigma_{ess}(H)$$

and

$$\{\lambda_n\}_{n=1}^{\infty} \cap (-\infty, \lambda_\infty) = \sigma_{disc}(H) \cap (-\infty, \lambda_\infty)$$

with each $\lambda_n \in (-\infty, \lambda_\infty)$ *being an eigenvalue of* H *repeated a number of times equal to its multiplicity.*

Remark 2.3.5. λ_i *defined in the min-max principle are ordered, i.e.:*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\infty.$$

Recall the Dirichlet Laplacian (2.4) $-\Delta_D^\Omega$ on some domain Ω defined in the preceding chapter. We would like to prove that if the domain Ω is bounded, the spectrum of the Dirichlet Laplacian is only composed of the discrete part. First some definitions and theorems.

Theorem 2.3.6 (Monotonicity of Dirichlet eigenvalues, see [10], Thm. 6.2.3)**.** *Let* $-\Delta_D^{\Omega_1}, -\Delta_D^{\Omega_2}$ *be the Dirichlet Laplacians on* Ω_1, Ω_2*. Then* $\forall n \in \mathbb{N}$:

$$\Omega_1 \subset \Omega_2 \Rightarrow \lambda_n(-\Delta_D^{\Omega_1}) \geq \lambda_n(-\Delta_D^{\Omega_2}).$$

Definition 2.3.7 (Compact resolvent)**.** *Let* H *be a closed operator on Hilbert space* $\mathcal{H}$*. We say that* H *has a compact resolvent if*

$$\exists z \in \rho(H), \quad (H - z)^{-1} : \mathcal{H} \to \mathcal{H} \textit{ is compact}$$

where $\rho(H)$ *is the resolvent set, i.e., the complement of* $\sigma(H)$*. From [12], Thm. XIII.64 we know that if there exists such* z *then the resolvent operator is compact for all points in the resolvent set.*

Theorem 2.3.8 (see [13], Thm. IX.2.3)**.** *Let* H *be a self-adjoint operator with a compact resolvent then*

$$\sigma(H) = \sigma_{disc}(H).$$

Theorem 2.3.9 (General criteria for compact resolvent, see [12], Thm. XIII.64)**.** *Let* H *be a self-adjoint, below-bounded operator on Hilbert space* $\mathcal{H}$ *and* h *the quadratic form associated with this operator using Theorem 2.2.1. Then the following four statements are equivalent*

$$H \textit{ has a compact resolvent}$$
$$\mathrm{Dom}(H) \hookrightarrow \mathcal{H} \textit{ is compact}$$
$$\mathrm{Dom}(h) \hookrightarrow \mathcal{H} \textit{ is compact}$$
$$\lim_{n\to\infty} \lambda_n(H) = \infty.$$

Remark 2.3.10. *Considering the Dirichlet Laplacian* (2.4) *Theorems 2.3.8, 2.3.9 imply that*

$$\sigma(-\Delta_D^\Omega) = \sigma_{disc}(-\Delta_D^\Omega) \Leftrightarrow W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \textit{ is compact.}$$

Remark 2.3.11 (Dirichlet spectrum of parallelepiped, see [10], Lemma 6.2.1)**.** *Let* $a > 0$ *and* $Q(a)$ *be the d-dimensional parallelepiped of side* $2a$*, i.e.* $Q(a) = (-a, a) \times (-a, a) \times \cdots \times (-a, a)$*. Then using the Friedrichs extension we see that the Dirichlet Laplacian on* $Q(a)$ *is a self-adjoint operator* $-\Delta_D^{Q(a)} : \mathrm{Dom}\left(-\Delta_D^{Q(a)}\right) \to L^2(Q(a))$ *and its discrete spectrum can be expressed as*

$$\sigma_{disc}\left(-\Delta_D^{Q(a)}\right) = \left\{\left(\frac{n_1\pi}{2a}\right)^2 + \left(\frac{n_2\pi}{2a}\right)^2 + \cdots + \left(\frac{n_d\pi}{2a}\right)^2\right\}, \quad n_1, \ldots n_d \in \mathbb{N}$$

with the corresponding eigenfunctions

$$\psi^D_{n_1,\dots,n_d} = \psi^D_{n_1}\psi^D_{n_2}\cdots\psi^D_{n_d}$$

where $\psi^D_{n_i}$, $i \in \{1,\dots,d\}$, *is the eigenfunction of one-dimensional parallelepiped, i.e. the interval* $(-a,a)$:

$$\psi^D_{n_i}(x) = \begin{cases} \sqrt{\frac{1}{a}}\cos\left(\frac{n_i\pi}{2a}x\right) & n_i \text{ is odd} \\ \sqrt{\frac{1}{a}}\sin\left(\frac{n_i\pi}{2a}x\right) & n_i \text{ is even} \end{cases}.$$

The functions $\psi^D_{n_1,\dots,n_d}$ *form a complete system. It can be seen that*

$$\lim_{n_1,\dots,n_d\to\infty}\lambda_{n_1,\dots,n_d}\left(-\Delta_D^{Q(a)}\right) = \lim_{n_1,\dots,n_d\to\infty}\left(\left(\frac{n_1\pi}{2a}\right)^2 + \left(\frac{n_2\pi}{2a}\right)^2 + \cdots + \left(\frac{n_d\pi}{2a}\right)^2\right) = \infty$$

which by Theorem 2.3.9 implies that $\operatorname{Dom}\left(-\Delta_D^{Q(a)}\right) \hookrightarrow \mathcal{L}^2(Q(a))$ *is compact and* $-\Delta_D^{Q(a)}$ *has a compact resolvent. Using the second fact together with Theorem 2.3.8 we can see that the spectrum of the operator* $-\Delta_D^{Q(a)}$ *is composed only of its discrete part*

$$\sigma\left(-\Delta_D^{Q(a)}\right) = \sigma_{disc}\left(-\Delta_D^{Q(a)}\right).$$

Next using the preceding theorems we obtain the following statement.

Theorem 2.3.12. *Let* Ω *be a bounded domain in* $\mathbb{R}^d$ *and* $-\Delta_D^\Omega$ *the Dirichlet Laplacian on* $L^2(\Omega)$. *Then*

$$\sigma\left(-\Delta_D^\Omega\right) = \sigma_{disc}\left(-\Delta_D^\Omega\right).$$

Proof. Let Ω be a bounded domain in $\mathbb{R}^d$. Then there exists a constant $a > 0$ that $\Omega \subset Q(a)$. Let $-\Delta_D^\Omega$, respectively $-\Delta_D^{Q(a)}$ be the Dirichlet Laplacian on Ω, respectively $Q(a)$. Using the monotonicity theorem 2.3.6 we can see that

$$\lambda_n\left(-\Delta_D^\Omega\right) \geq \lambda_n\left(-\Delta_D^{Q(a)}\right).$$

From Remark 2.3.11 it can be seen that

$$\lim_{n\to\infty}\lambda_n\left(-\Delta_D^{Q(a)}\right) = \infty$$

which implies that also

$$\lim_{n\to\infty}\lambda_n\left(-\Delta_D^\Omega\right) = \infty.$$

Involving Theorem 2.3.9 we have that $-\Delta_D^{Q(a)}$ has a compact resolvent and finally Theorem 2.3.8 states that

$$\sigma\left(-\Delta_D^\Omega\right) = \sigma_{disc}\left(-\Delta_D^\Omega\right)$$

which proves our statement. □

And finally we can conclude with the variational formulation for the first eigenvalue of the Dirichlet Laplacian which will be further very useful in the formation of the first eigenvalue's upper bounds.

Remark 2.3.13 (Variational formulation for the first eigenvalue)**.** *Let $-\Delta_D^\Omega$ be the Dirichlet Laplacian on some bounded domain Ω in $\mathbb{R}^d$ and let h be the quadratic form associated with this operator using Theorem 2.2.1, i.e., $h[\psi] = \|\nabla\psi\|^2$. Now using the min-max principle 2.3.4, the following inequality holds*

$$\lambda_1(-\Delta_D^\Omega) = \inf \sigma(H) = \inf_{\psi \in W_0^{1,2}(\Omega)} \frac{\|\nabla\psi\|^2}{\|\psi\|^2}$$

where $\|\cdot\|$ stands for the norm on $L^2(\Omega)$. From Theorem 2.3.12 we have that the spectrum is purely discrete. Hence λ_1 is the first eigenvalue and the equality must be obtained for some function

$$\lambda_1(-\Delta_D^\Omega) = \min_{\psi \in W_0^{1,2}(\Omega)} \frac{\|\nabla\psi\|^2}{\|\psi\|^2}. \tag{2.5}$$

Substituting the first eigenfunction denoted by ψ_1 and using (2.2) *we obtain*

$$\lambda_1(-\Delta_D^\Omega) \leq \frac{\|\nabla\psi_1\|^2}{\|\psi_1\|^2} = \frac{(\psi_1, -\Delta_D^\Omega\psi_1)}{(\psi_1, \psi_1)} = \frac{(\psi_1, \lambda_1(-\Delta_D^\Omega)\psi_1)}{(\psi_1, \psi_1)} = \lambda_1(-\Delta_D^\Omega). \tag{2.6}$$

Thus the equality sign in (2.5) *is obtained if ψ is chosen as the first eigenfunction. On the other hand, if we substitute into* (2.5) *some function $\psi^* \neq \psi_1$ not being the first eigenfunction, it can be seen from* (2.6) *that we never obtain equality.*

Hence we can conclude with

$$\lambda_1(-\Delta_D^\Omega) = \min_{\psi \in W_0^{1,2}(\Omega)} \frac{\|\nabla\psi\|^2}{\|\psi\|^2}$$

where the equality sign is obtained if, and only if, ψ is chosen as the first eigenfunction.

Remark 2.3.14 (Dirichlet-Neumann Laplacian)**.** *For some proofs of the bounds for hollow domains (specifically the bound of Theorem 3.5.1) we will also need to correctly define the self-adjoint Dirichlet-Neumann Laplacian Δ_{DN} such as for a bounded domain Ω in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^2 and with outer boundary denoted by $\partial\Omega_0$ we have*

$$\begin{aligned} -\Delta_{DN}^\Omega u &= \lambda u \ \text{in } \Omega \\ u &= 0 \ \text{on } \partial\Omega_0 \\ \frac{\partial u}{\partial \boldsymbol{n}} &= 0 \ \text{on } \partial\Omega \setminus \partial\Omega_0 \end{aligned}$$

for $u \in \mathrm{Dom}(-\Delta_{DN}^\Omega)$.

Since this operator does not form a fundamental part of the thesis, we only mention how it is defined (and how its associated quadratic form looks) and not the procedure of its definition which is quite similar to the definition of the Dirichlet Laplacian shown above.

$$\mathrm{Dom}(-\Delta_{DN}^\Omega) = \left\{ u \in W^{1,2}(\Omega) : \Delta u \in L^2(\Omega) \wedge \frac{\partial u}{\partial \boldsymbol{n}}\bigg|_{\partial\Omega\setminus\partial\Omega_0} = 0 \wedge u\big|_{\partial\Omega_0} = 0 \right\}$$

$$-\Delta_{DN}^\Omega u = -\Delta u$$

where $W^{1,2}(\Omega)$ is the Sobolev space (see [10], Section 6.1) defined as

$$W^{1,2}(\Omega) = \{\psi \in L^2(\Omega) : \nabla\psi \in L^2(\Omega)\}.$$

The associated quadratic form can be written as follows

$$Q^{\Omega}_{DN}[\psi] = \|\nabla\psi\|^2$$
$$\mathrm{Dom}(Q^{\Omega}_{DN}) = \left\{ u \in W^{1,2}(\Omega) : u\big|_{\partial\Omega_0} = 0 \right\}$$

where ∇ *stands for the weak gradient.*

Chapter 3

Bounds

This is the main chapter of the thesis where we introduce various upper bounds for the first eigenvalue of Dirichlet Laplacian on bounded domains in arbitrary dimension. Henceforth let Ω be the bounded domain in $\mathbb{R}^d$, $-\Delta_D^\Omega$ be a Dirichlet Laplacian on $L^2(\Omega)$ defined in the previous chapter and $\lambda_1(\Omega) := \lambda_1(-\Delta_D^\Omega)$ its first eigenvalue.

Nevertheless we will start with one lower bound.

3.1 Faber-Krahn inequality

The Faber-Krahn or Rayleigh-Faber-Krahn inequality first conjectured by Lord Rayleigh in his 1877 book [2] and proved independently by Faber and Krahn is a lower bound for the first eigenvalue of Ω. It states that the first eigenvalue of Ω is equal to greater than the first eigenvalue of the ball with the same volume and the equality is obtained if, and only if, Ω is a ball.

Remark 3.1.1. *Let $R > 0$. Since from Remark 3.3.21 below we know that*

$$\lambda_1(B_R) = \frac{1}{R^2}\lambda_1(B_1)$$

where B_a is a ball of the radius a and using the well-known relation between the volume of B_R and B_1

$$|B_R| = R^d|B_1| \tag{3.1}$$

where $|\cdot|$ denotes the d-dimensional Lebesgue measure, we can express $\lambda_1(B_R)$ using $\lambda_1(B_1)$ and using volumes of B_1 and Ω by

$$\lambda_1(B_R) = \frac{1}{R^2}\lambda_1(B_1) = \left(\frac{|B_1|}{|B_R|}\right)^{2/d}$$

and finally choosing the radius R by the property $|B_R| = |\Omega|$ we obtain

$$\lambda_1(B_R) = \left(\frac{|B_1|}{|\Omega|}\right)^{2/d}.$$

Using this remark we can finally state the theorem.

Theorem 3.1.2. *Let Ω be a bounded domain in $\mathbb{R}^d$. Then the following lower bound holds*

$$\lambda_1(\Omega) \geq \lambda_1(B_1)\left(\frac{|B_1|}{|\Omega|}\right)^{2/d}.$$

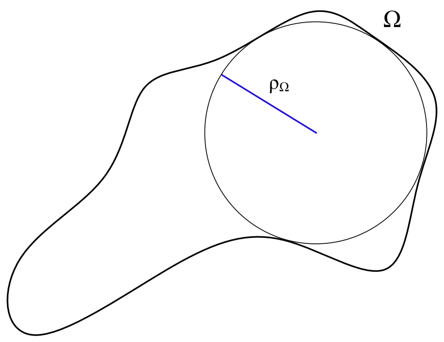


Figure 3.1: Inradius ρ_Ω of domain Ω

3.2 Trivial upper bound

Before we proceed to the bounds using shrinking or parallel coordinates we state the trivial upper bound which follows immediately from the monotonicity of the Dirichlet eigenvalues.

Let Ω be the bounded domain in $\mathbb{R}^d$ with inradius ρ_Ω (the inradius has the meaning of the radius of the biggest inscribed ball in the domain, see Figure 3.1). Then there exists a ball B_{ρ_Ω} with radius ρ_Ω such that

$$B_{\rho_\Omega} \subset \Omega.$$

Now recall the theorem 2.3.6. It states that the following implication holds for the first Dirichlet eigenvalue

$$B_{\rho_\Omega} \subset \Omega \Rightarrow \lambda_1(\Omega) \leq \lambda_1(B_{\rho_\Omega}).$$

Since the formulas for $\lambda_1(B_{\rho_\Omega})$ are explicitly known we can state the theorem.

Theorem 3.2.1 (Trivial upper bound)**.** *Let Ω be a bounded domain in $\mathbb{R}^d$ with inradius ρ_Ω. Then the following upper bound holds*

$$\lambda_1(\Omega) \leq \lambda_1(B_{\rho_\Omega}).$$

3.3 Pólya and Szegö's bound in arbitrary dimension

In this section we state the generalization to an arbitrary dimension of the sharp planar upper bound by Pólya and Szegö which appeared in their 1951 book [4]. This result was published in the paper [5] by Pedro Freitas and David Krejčiřík. The proof of this bound is based upon the use of the shrinking coordinates (see Figure 3.3). Before presenting the statement we have to introduce some notation and definitions involving the geometry of the domain.

Definition 3.3.1 (Lipschitz continuous, see [6], Def. 2.2.7)**.** *A map $f : X \to Y$, where (X, ρ_X) and (Y, ρ_Y) are metric spaces, is called Lipschitz continuous if, and only if, there exists a finite positive number M such that*

$$\rho_Y(f(a), f(b)) \leq M\rho_X(a, b), \quad \forall a, b \in X.$$

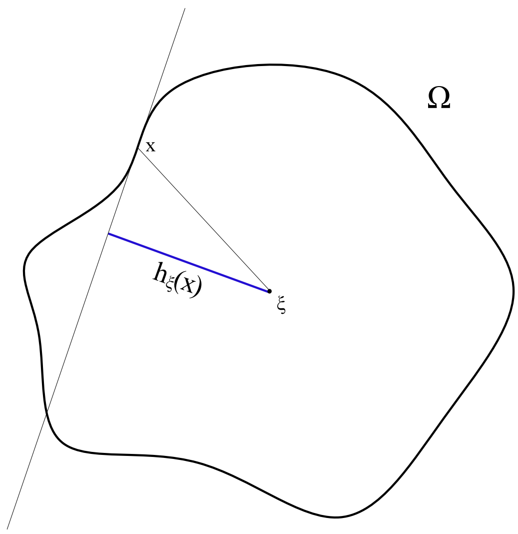


Figure 3.2: Geometrical interpretation of support function $h_\xi(x)$

Definition 3.3.2 (Locally Lipschitz continuous, see [6], Def. 2.2.7). *A map $f : X \to Y$, where (X, ρ_X) and (Y, ρ_Y) are metric spaces, is locally Lipschitz continuous if, and only if, for every $x \in X$ there exists a neighborhood $U, x \in U^o$ such that $f|_U$ is Lipschitz continuous.*

Definition 3.3.3 (Star-shaped domain). *A domain Ω is said to be star-shaped with respect to a point $\xi \in \Omega$ if for each point $x \in \partial\Omega$ the segment joining ξ with x lies in $\Omega \cup \{x\}$ and is transversal to $\partial\Omega$ at the point x.*

Theorem 3.3.4 (Rademacher, see [6], Thm. 3.1.6). *Let $\phi : U = U^o \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ be Lipschitz continuous, then ϕ is differentiable almost everywhere in U.*

From Rademacher theorem we can see that the outward unit normal vector field $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^d$ is uniquely defined almost everywhere on $\partial\Omega$.

Definition 3.3.5 (Support function). *Let Ω be a star-shaped domain with respect to ξ and $\partial\Omega$ its locally Lipschitz boundary. At the points $x \in \partial\Omega$ for which the outward unit normal vector field $\boldsymbol{n}(x)$ is uniquely defined a support function can be introduced*

$$h_\xi(x) := (x - \xi) \cdot \boldsymbol{n}(x)$$

with $\cdot$ denoting the standard inner product in $\mathbb{R}^d$.

Remark 3.3.6. *The support function $h_\xi(x) := (x - \xi) \cdot \boldsymbol{n}(x)$ can be interpreted as a scalar projection of $x - \xi$ in the direction of unit normal vector field $\boldsymbol{n}(x)$ or as the distance from ξ to the tangent space $T_x(\partial\Omega)$, see Figure 3.2.*

Definition 3.3.7 (Strictly star-shaped domain). *The domain Ω is strictly star-shaped with respect to the point $\xi \in \Omega$ if Ω is star-shaped with respect to ξ and the support function is uniformly positive, i.e.,*

$$\operatorname*{ess\,inf}_{x\in\partial\Omega} h_\xi(x) > 0.$$

Let us denote the set of points with respect to which Ω is strictly star-shaped as ω.

Definition 3.3.8 (Intrinsic quantity F). *Let Ω be the domain with locally Lipschitz boundary $\partial\Omega$, ω be the set with respect to which Ω is strictly star-shaped and h_ξ the corresponding support function of the domain. Then the intrinsic quantity F of the domain can be defined as*

$$F(\Omega) := \inf_{\xi\in\omega} \int_{\partial\Omega} h_\xi^{-1}.$$

Now we are ready to state the theorem.

Theorem 3.3.9 (PS bound in arbitrary dimension). *Let Ω be a bounded strictly star-shaped domain in $\mathbb{R}^d$ with locally Lipschitz boundary $\partial\Omega$. Then*

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{F(\Omega)}{d|\Omega|} \tag{3.2}$$

where $\lambda_1(B_1)$ denotes the first eigenvalue of the d-dimensional ball of unit radius and $|\Omega|$ denotes the d-dimensional Lebesgue measure of Ω.

Remark 3.3.10 (Dimension 1). *Let us assume $d = 1$. Then Ω reduces to some bounded interval (a, b), $a < b$, $a > -\infty$, $b < \infty$ and $|\Omega| = b - a$. From (4.1) below we know the explicit formula for $F(\Omega)$ for the parallelepiped of the side $2l$, implying*

$$F(\Omega) = \frac{|\Omega|}{l^2} = (b-a)\frac{4}{(b-a)^2} = \frac{4}{b-a}.$$

Since from 2.3.11 we have the spectrum of parallelepipeds explicitly we can write

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{4}{(b-a)^2}$$
$$\frac{\pi^2}{(b-a)^2} \leq \frac{\pi^2}{(b-a)^2}$$

which proves the bound in one dimension. Moreover the equality is obviously obtained for all intervals, making this bound sharp for all suitable domains in one dimension.

The previous remark proves Theorem 3.3.9 in one dimension, henceforth let us assume $d \geq 2$. The proof is based on the use of the shrinking coordinates (see Figure 3.3).

3.3.1 Shrinking coordinates

From this time forth, let Ω be a bounded strictly star-shaped domain in $\mathbb{R}^d$ with locally Lipschitz boundary $\partial\Omega$. The hypersurface $\partial\Omega$ is locally $C^{0,1}$-diffeomorphic to $\mathbb{R}^{d-1}$, i.e., for each point $x \in \partial\Omega$ there exists an open subset of $\mathbb{R}^d$ whose intersection with the boundary $\partial\Omega$ denoted by $V \subset \partial\Omega$ is $C^{0,1}$-diffeomorphic to an open subset $U \subset \mathbb{R}^{d-1}$ by a chart $\Gamma : U \to V$.

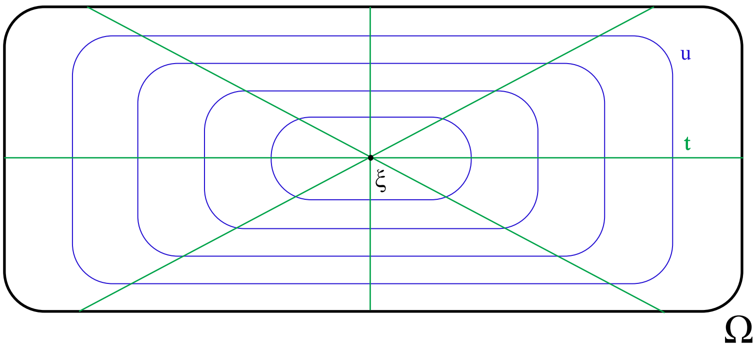


Figure 3.3: Shrinking coordinates

From Rademacher theorem 3.3.4 it can be seen that Γ is differentiable almost everywhere in U. Hence Γ induces the metric tensor g of $\partial\Omega$ by (see [7], Sec. 1.1.3)

$$g_{\mu\nu} := (\partial_\mu \Gamma) \cdot (\partial_\nu \Gamma) \qquad \mu, \nu \in \{1, \dots, d-1\}.$$

Now, let Ω be strictly star-shaped with respect to $\xi \in \omega$. We can parameterize $\Omega \setminus \{\xi\}$ by the natural mapping

$$\mathcal{L} : \partial\Omega \times (0,1) \to \Omega \setminus \{\xi\} : \{(x,t) \mapsto \xi + (x - \xi)t\} \tag{3.3}$$

or locally by $\mathfrak{L} = \mathcal{L} \circ (\Gamma \otimes 1)$ with 1 being the identity function on $(0,1)$

$$\mathfrak{L} : \mathbb{R}^{d-1} \times (0,1) \to \Omega \setminus \{\xi\} : \{(u,t) \mapsto \xi + (\Gamma(u) - \xi)t\}$$

where $u = (u^1, \dots, u^{d-1})$ are the local coordinates on U, $u^\mu = (\Gamma^{-1})^\mu(x)$. The coordinates u and t are also called "shrinking" which is motivated by their behavior. All the "shrunk" boundary $\mathcal{L}(\partial\Omega \times \{t\})$ is contained in Ω for $t \in (0,1)$. See Figure 3.3.

As a next step we need to compute the determinant of the Jacobi matrix J of this transformation

$$J(\cdot, t) = \begin{pmatrix} (\partial_1\Gamma^1)\, t & \dots & (\partial_{d-1}\Gamma^1)\, t & \Gamma^1 - \xi^1 \\ \vdots & & \vdots & \vdots \\ (\partial_1\Gamma^d)\, t & \dots & (\partial_{d-1}\Gamma^d)\, t & \Gamma^d - \xi^d \end{pmatrix}. \tag{3.4}$$

For this we will need to use the exterior product, homogeneity of the determinant in each row, a vector algebra identity and the fact that the transposition of a matrix does not change its determinant.

Remark 3.3.11 (Exterior product)**.** *Let $x_1, x_2, \dots, x_{d-1} \in \mathbb{R}^d$, then their exterior product denoted by $x_1 \wedge x_2 \wedge \dots \wedge x_{d-1}$ can expressed in coordinates as*

$$x_1 \wedge x_2 \wedge \dots \wedge x_{d-1} = \begin{vmatrix} x_1^1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & & \vdots \\ x_{d-1}^1 & x_{d-1}^2 & \dots & x_{d-1}^d \\ \boldsymbol{e_1} & \boldsymbol{e_2} & \dots & \boldsymbol{e_d} \end{vmatrix}$$

where $(\boldsymbol{e_1}, \boldsymbol{e_2}, \dots, \boldsymbol{e_d})$ is the standard basis in $\mathbb{R}^d$.

Remark 3.3.12 (Identity). *Let $x_1, x_2, \dots, x_{d-1} \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then*

$$\begin{vmatrix} x_1^1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & & \vdots \\ x_{d-1}^1 & x_{d-1}^2 & \dots & x_{d-1}^d \\ x^1 & x^2 & \dots & x^d \end{vmatrix} = (x_1 \wedge x_2 \wedge \dots \wedge x_{d-1}) \cdot x.$$

Proof of the remark. Let $x_1, x_2, \dots, x_{d-1} \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Starting from the right side and expressing the obtained determinant as a linear combination of the elements of the last row using the expansion formula by cofactors (see [9], Thm. 3.8) we get

$$\begin{aligned}(x_1 \wedge x_2 \wedge \dots \wedge x_{d-1}) \cdot x &= \begin{vmatrix} x_1^1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & & \vdots \\ x_{d-1}^1 & x_{d-1}^2 & \dots & x_{d-1}^d \\ \boldsymbol{e_1} & \boldsymbol{e_2} & \dots & \boldsymbol{e_d} \end{vmatrix} \cdot x \\
= \left[(-1)^{d+1} \boldsymbol{e_1} \begin{vmatrix} x_1^2 & \dots & x_1^d \\ \vdots & & \vdots \\ x_{d-1}^2 & \dots & x_{d-1}^d \end{vmatrix} + \dots + \boldsymbol{e_d} \begin{vmatrix} x_1^1 & x_1^2 & \dots & x_1^{d-1} \\ \vdots & \vdots & & \vdots \\ x_{d-1}^1 & x_{d-1}^2 & \dots & x_{d-1}^{d-1} \end{vmatrix} \right] \cdot x \\
= (-1)^{d+1} x_1 \begin{vmatrix} x_1^2 & \dots & x_1^d \\ \vdots & & \vdots \\ x_{d-1}^2 & \dots & x_{d-1}^d \end{vmatrix} + \dots + x_d \begin{vmatrix} x_1^1 & x_1^2 & \dots & x_1^{d-1} \\ \vdots & \vdots & & \vdots \\ x_{d-1}^1 & x_{d-1}^2 & \dots & x_{d-1}^{d-1} \end{vmatrix} \\
= \begin{vmatrix} x_1^1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & & \vdots \\ x_{d-1}^1 & x_{d-1}^2 & \dots & x_{d-1}^d \\ x^1 & x^2 & \dots & x^d \end{vmatrix}.\end{aligned}$$

□

Thus the determinant can be expressed as

$$\det J(\cdot, t) = (\partial_1 \Gamma \wedge \dots \wedge \partial_{d-1} \Gamma) \cdot (\Gamma - \xi)\, t^{d-1}.$$

Finally for the last adjustment of this formula we will have to use the following.

Remark 3.3.13. *Let g be the metric tensor on $\partial\Omega$ induced by the local diffeomorphisms Γ. Then $(\partial_1 \Gamma \wedge \dots \wedge \partial_{d-1} \Gamma)$ is perpendicular to $\partial\Omega$ and its magnitude is equal to $\sqrt{\det g}$.*

Proof of the remark. Since the perpendicularity to $\partial\Omega$ means the perpendicularity to the tangent space in every point which is formed by the tangent vectors $\partial_1 \Gamma, \dots, \partial_{d-1} \Gamma$ in the corresponding points, we can prove it by computing the inner product of $(\partial_1 \Gamma \wedge \dots \wedge \partial_{d-1} \Gamma)$ and an arbitrary tangent vector

$$(\partial_1 \Gamma \wedge \dots \wedge \partial_{d-1} \Gamma) \cdot \partial_\mu \Gamma = \begin{vmatrix} \partial_1 \Gamma^1 & \partial_1 \Gamma^2 & \dots & \partial_1 \Gamma^d \\ \vdots & \vdots & & \vdots \\ \partial_{d-1} \Gamma^1 & \partial_{d-1} \Gamma^2 & \dots & \partial_{d-1} \Gamma^d \\ \partial_\mu \Gamma^1 & \partial_\mu \Gamma^2 & \dots & \partial_\mu \Gamma^d \end{vmatrix} = 0$$

where $\mu \in \{1, \dots, d-1\}$, the first equality follows from the identity 3.3.12 and the second from the fact that the determinant of linearly dependent vectors is zero (see [9], Thm 3.1).

For the second part of the proof let us compute the Jacobi matrix of the map Γ

$$J^\Gamma = \begin{pmatrix} \partial_1\Gamma^1 & \partial_2\Gamma^1 & \dots & \partial_{d-1}\Gamma^1 \\ \vdots & \vdots & & \vdots \\ \partial_1\Gamma^d & \partial_2\Gamma^d & \dots & \partial_{d-1}\Gamma^d \end{pmatrix}.$$

Since

$$g_{\mu\nu} = (\partial_\mu\Gamma)\cdot(\partial_\nu\Gamma) = \sum_{k=1}^{d} \partial_\mu\Gamma^k \partial_\nu\Gamma^k = \sum_{k=1}^{d} (J^\Gamma)^T_{\mu k} J^\Gamma_{k\nu},$$

the matrix g can be written as $g = {J^\Gamma}^T \cdot J^\Gamma$ and thus

$$\det g = \det({J^\Gamma}^T \cdot J^\Gamma).$$

At the same time we denote

$$\tilde{n} := (\partial_1\Gamma \wedge \ldots \wedge \partial_{d-1}\Gamma) = \begin{vmatrix} \partial_1\Gamma^1 & \partial_1\Gamma^2 & \dots & \partial_1\Gamma^d \\ \vdots & \vdots & & \vdots \\ \partial_{d-1}\Gamma^1 & \partial_{d-1}\Gamma^2 & \dots & \partial_{d-1}\Gamma^d \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \dots & \boldsymbol{e}_d \end{vmatrix}$$

and from [20] we know that

$$|\det(\partial_1\Gamma, \ldots, \partial_{d-1}\Gamma, \tilde{n})| = |\tilde{n}|^2$$

where $\partial_1\Gamma, \ldots, \partial_{d-1}\Gamma, \tilde{n}$ form the columns of the matrix from which we compute the determinant. We can further adjust this formula using the fact that the determinant is invariant with respect to the transposition of the matrix

$$|\tilde{n}|^4 = \begin{vmatrix} \partial_1\Gamma^1 & \partial_1\Gamma^2 & \dots & \partial_1\Gamma^d \\ \vdots & \vdots & & \vdots \\ \partial_{d-1}\Gamma^1 & \partial_{d-1}\Gamma^2 & \dots & \partial_{d-1}\Gamma^d \\ \tilde{n}^1 & \tilde{n}^2 & \dots & \tilde{n}^d \end{vmatrix} \cdot \begin{vmatrix} \partial_1\Gamma^1 & \partial_2\Gamma^1 & \dots & \partial_{d-1}\Gamma^1 & \tilde{n}^1 \\ \vdots & \vdots & & \vdots & \\ \partial_1\Gamma^{d-1} & \partial_2\Gamma^{d-1} & \dots & \partial_{d-1}\Gamma^{d-1} & \tilde{n}^{d-1} \\ \partial_1\Gamma^d & \partial_2\Gamma^d & \dots & \partial_{d-1}\Gamma^d & \tilde{n}^d \end{vmatrix}$$

which can be rewritten using the block formalism as

$$|\tilde{n}|^4 = \begin{vmatrix} {J^\Gamma}^T \\ \tilde{n} \end{vmatrix} \cdot \begin{vmatrix} J^\Gamma & \tilde{n} \end{vmatrix}$$

and using the fact proved before that all tangent vectors are perpendicular to $\tilde{n} = (\partial_1\Gamma \wedge \ldots \wedge \partial_{d-1}\Gamma)$ we see

$$\begin{vmatrix} {J^\Gamma}^T \\ \tilde{n} \end{vmatrix} \cdot \begin{vmatrix} J^\Gamma & \tilde{n} \end{vmatrix} = \left|\begin{array}{c|c} {J^\Gamma}^T \cdot J^\Gamma & 0 \\ \hline 0 & |\tilde{n}|^2 \end{array}\right| = |\tilde{n}|^2 \det({J^\Gamma}^T \cdot J^\Gamma)$$

which implies

$$|\tilde{n}|^2 = \det({J^\Gamma}^T \cdot J^\Gamma) = \det g$$

and

$$|(\partial_1\Gamma \wedge \ldots \wedge \partial_{d-1}\Gamma)| = \sqrt{\det g}.$$

□

Remark 3.3.14 ([9], Sec 1.11). *Let $a, b \in \mathbb{R}^d$ then their inner product can be computed as*

$$a \cdot b = \|a\| \, \|b\| \cos\theta$$

where θ is the angle between a and b and $\|a\| \cos\theta$ is the scalar projection of a in the direction of b.

Applying these two facts we can conclude with the local formula for the absolute value of the Jacobian

$$|\det J(u,t)| = \sqrt{\det g(u)}\, h_\xi(\Gamma(u))\, t^{d-1}$$

for every $t \in (0,1)$ and almost every $u \in U$.

Now by the assumption that Ω is strictly star-shaped with respect to ξ, i.e., $\operatorname{ess\,inf}_{x\in\partial\Omega} h_\xi(x) > 0$, we can see that

$$|\det J(u,t)| \neq 0 \tag{3.5}$$

for every $t \in (0,1)$ and almost every $u \in U$ and thus involving the inverse function theorem $\mathcal{L} : \partial\Omega \times (0,1) \to \Omega \setminus \{\xi\}$ is a diffeomorphism.

Theorem 3.3.15 (Inverse function theorem, see [8], Thm. 12.4). *Let $p \in U = U^\circ \subset \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^d, f \in C^1$. If $\det Jf(p) \neq 0$, where Jf is the Jacobi matrix of f, then there exists $V = V^\circ \subset \mathbb{R}^d, f(p) \in V$ such that $f^{-1} : V \to U$ exists and $f^{-1} \in C^1$.*

Now we can proceed to the proof of the bound.

3.3.2 Proof of the bound

First we identify $\Omega \setminus \{\xi\}$ with a Riemannian manifold

$$M := (\partial\Omega \times (0,1), G)$$

where G is the metric tensor induced by $\mathcal{L}$ or locally by $\mathfrak{L}$. The coefficients of G are locally

$$G_{\mu\nu} := (\partial_\mu \mathfrak{L}) \cdot (\partial_\nu \mathfrak{L})$$

which can be further adjusted using the definition of the inner product in $\mathbb{R}^d$ and the Jacobi matrix J, noticing that, by the definition of the Jacobi matrix, $(\partial_\mu \mathfrak{L})$ is the μth column of J

$$G_{\mu\nu} = (\partial_\mu \mathfrak{L}) \cdot (\partial_\nu \mathfrak{L}) = \sum_{k=1}^{d} (\partial_\mu \mathfrak{L}^k)(\partial_\nu \mathfrak{L}^k) = \sum_{k=1}^{d} J_{k\mu} J_{k\nu} = \sum_{k=1}^{d} J^T_{\mu k} J_{k\nu}$$

therefore $G = J^T \cdot J$ and

$$G(\cdot,t) = \begin{pmatrix} g_{11}\, t^2 & \dots & g_{1d-1}\, t^2 & (\Gamma-\xi)\cdot(\partial_1\Gamma)\, t \\ \vdots & & \vdots & \vdots \\ g_{d-11}\, t^2 & \dots & g_{d-1d-1}\, t^2 & (\Gamma-\xi)\cdot(\partial_{d-1}\Gamma)\, t \\ (\Gamma-\xi)\cdot(\partial_1\Gamma)\, t & \dots & (\Gamma-\xi)\cdot(\partial_{d-1}\Gamma)\, t & |\Gamma-\xi|^2 \end{pmatrix}. \tag{3.6}$$

Remark 3.3.16 (Determinant of G). *From the formula $G = J^T \cdot J$ we can see that the determinant of G is*

$$\det G = (\det J)^2.$$

Indeed the manifold M is Riemannian, i.e., the quadratic form $x^\mu G_{\mu\nu} x^\nu$ is positive definite and thus it forms an inner product. The fact that $x^\mu G_{\mu\nu} x^\nu$ is positive semidefinite can be seen from

$$\begin{aligned} x^\mu G_{\mu\nu} x^\nu &= t^2 x^i g_{ij} x^j + |\Gamma - \xi|^2 (x^d)^2 + 2(\Gamma - \xi) \cdot (\partial_j \Gamma)\, t\, x^d x^j \\ &= \Big((\Gamma - \xi)\, x^d + t\, (\partial_j \Gamma)\, x^j \Big)^2 \geq 0 \end{aligned}$$

where $i, j \in \{1, \ldots, d-1\}$. Notice that from Remark 3.3.16 and (3.5) we have that $\det G \neq 0$. Finally recall that for positive semidefinite quadratic forms we have that the quadratic form is positive definite if, and only if, the matrix of the form is regular.

Remark 3.3.17 (Volume element of Riemannian manifold M, see [7], Sec. 1.1.1)**.** *Let M be an n-dimensional Riemannian manifold with the metric tensor G and $(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$ be an oriented basis of its cotangent space in the point x, then the volume element of M is*

$$\mathrm{dvol} = \sqrt{|\det G|}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n .$$

Using Remarks 3.3.16 and 3.3.17 we can express the volume element of our manifold as

$$\mathrm{dvol}(u, t) = \sqrt{\det g(u)}\, h_\xi(\Gamma(u))\, \mathrm{d}u\, t^{d-1}\, \mathrm{d}t$$

or

$$\mathrm{dvol}(x, t) = h_\xi(x)\, \mathrm{d}\sigma(x)\, t^{d-1}\, \mathrm{d}t \tag{3.7}$$

where $\mathrm{d}\sigma$ is the measure on $\partial\Omega$, $\mathrm{d}u$ is the measure on U and $\mathrm{d}t$ is the measure on $(0, 1)$. We will use the second formula later.

The last step before we proceed to the upper bound for the first eigenvalue is to compute the norm of the gradient in M of some radially symmetric test function. Hence let us take an arbitrary function $\tilde{\eta} = \tilde{\eta}(t)$ of the form

$$\tilde{\eta} = \psi \otimes 1 \tag{3.8}$$

where ψ is any differentiable function on $(0, 1)$ and 1 denotes a function constantly equal to 1 on $\partial\Omega$. For the computation of the norm of the gradient the matrix inverse to G would be needed. We denote the elements of the inverse matrix by upper indices. As we will see later only the element G^{dd} would be necessary. This element can be easily obtained using the adjugate matrix.

Remark 3.3.18 (Element of inverse matrix using adjugate matrix, see [9])**.** *Element G^{kl} of matrix inverse to G can be computed as*

$$G^{kl} = \frac{1}{\det G} A_{kl}$$

where the matrix A with coefficients

$$A_{kl} = (-1)^{k+l} B_{lk}$$

is the adjugate matrix of G and the element B_{lk} is the determinant of G without kth row and lth column.

Now we can try to compute element G^{dd}. From the preceding remark we can see that locally

$$\begin{aligned}
G^{dd}(u,t) &= \frac{1}{\det G(u,t)} B_{dd}(u,t) \\
&= \frac{1}{\det g(u)\, h_\xi^2(\Gamma(u))\, t^{2d-2}} \begin{vmatrix} g_{11}\, t^2 & \dots & g_{1d-1}\, t^2 \\ \vdots & & \vdots \\ g_{d-11}\, t^2 & \dots & g_{d-1d-1}\, t^2 \end{vmatrix} (u) \\
&= \frac{1}{\det g(u)\, h_\xi^2(\Gamma(u))} \begin{vmatrix} g_{11} & \dots & g_{1d-1} \\ \vdots & & \vdots \\ g_{d-11} & \dots & g_{d-1d-1} \end{vmatrix} (u) \\
&= \frac{\det g_{ij}(u)}{\det g(u)\, h_\xi^2(\Gamma(u))} \\
&= h_\xi^{-2}(\Gamma(u)).
\end{aligned}$$

At last we can proceed to the norm of the gradient.

Remark 3.3.19 (Norm of gradient in Riemannian manifold M)**.** *In the Riemannian manifold M with the metric tensor G the norm of the gradient of the function $F : M \to \mathbb{R}$ is equal to*

$$\|\nabla_G F\|_G^2 = \partial_\mu F\, G^{\mu\nu} \partial_\nu F$$

where $\|\cdot\|_G$ is the norm in the manifold M.

Proof of the remark. Let x^i be the Cartesian coordinates in $\mathbb{R}^d$, more specifically in Ω, $x^i := \mathfrak{L}^i(q^1, \dots, q^d)$, where q^μ are the coordinates in M, $q^\mu := (\mathfrak{L}^{-1})^\mu(x^1, \dots, x^d)$. First we express the norm of the gradient of the function $f = F \circ \mathfrak{L}^{-1}$ in the coordinates x^i as

$$|\nabla f|^2 = \frac{\partial f}{\partial x^i} \delta^{ij} \frac{\partial f}{\partial x^j}$$

where δ^{ij} is the Kronecker delta. By the change of the coordinates we obtain

$$\frac{\partial f}{\partial x^i} \delta^{ij} \frac{\partial f}{\partial x^j} = \frac{\partial (f \circ \mathfrak{L})}{\partial q^\mu} \frac{\partial q^\mu}{\partial x^i} \delta^{ij} \frac{\partial q^\nu}{\partial x^j} \frac{\partial (f \circ \mathfrak{L})}{\partial q^\nu} = \frac{\partial F}{\partial q^\mu} \frac{\partial q^\mu}{\partial x^i} \frac{\partial q^\nu}{\partial x^i} \frac{\partial F}{\partial q^\nu}$$

and using the definition of G which implies

$$\frac{\partial q^\mu}{\partial x^i} \frac{\partial q^\nu}{\partial x^i} = G^{\mu\nu}$$

we conclude with

$$\|\nabla_G F\|_G^2 = \frac{\partial F}{\partial q^\mu} G^{\mu\nu} \frac{\partial F}{\partial q^\nu}.$$

□

Thus substituting the function $\tilde{\eta}$ into this formula we obtain

$$\begin{aligned}
\|\nabla_G (1 \otimes \psi)\|_G^2 &= \partial_\mu (1 \otimes \psi) G^{\mu\nu}\, \partial_\nu (1 \otimes \psi) \\
&= \partial_d (1 \otimes \psi) G^{dd}\, \partial_d (1 \otimes \psi) \\
&= h_\xi^{-2} |\psi'|^2
\end{aligned}$$

because due to the constant function 1 all other terms are equal to zero. Hence

$$\|\nabla_G(1 \otimes \psi)\|_G = h_\xi^{-1}|\psi'|. \tag{3.9}$$

With the last result we already have all the geometric preliminaries to embark on the spectral problem.

Recall that the Dirichlet Laplacian (2.4) correctly defined in the previous chapter is uniquely associated with the quadratic form (2.3)

$$h_D^\Omega[\psi] = \|\nabla\psi\|^2$$
$$\mathrm{Dom}(h_D^\Omega) = W_0^{1,2}(\Omega)$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm and the Sobolev space $W_0^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$(\|\nabla\cdot\|^2 + \|\cdot\|^2)^{1/2}.$$

At the same time using the geometric preliminaries the Hilbert space $L^2(\Omega)$ can be identified with $L^2(M) := L^2(\partial\Omega \times (0,1), \mathrm{dvol})$ using a transformation

$$U : L^2(\Omega) \to L^2(M) : \{f \mapsto f \circ \mathfrak{L}\}$$

and thus the Dirichlet Laplacian is unitarily equivalent to the operator

$$H := U(-\Delta_D^\Omega)U^{-1}.$$

The quadratic form (2.3) associated with this operator can be expressed using 3.3.19 as

$$h[\psi] = h_D^\Omega[U^{-1}\psi] = \int\limits_{\partial\Omega\times(0,1)} \|\nabla_G\psi\|_G^2 \,\mathrm{dvol} = \| \, \|\nabla_G\psi\|_G \, \|_{L^2(M)}^2$$
$$\mathrm{Dom}(h) := UD(h_D^\Omega) = W_0^{1,2}(M)$$

where the $\|\cdot\|_{L^2(M)}$ denotes the norm on the space $L^2(M)$

$$\|\psi\|_{L^2(M)} = \left(\int\limits_{\partial\Omega\times(0,1)} |\psi(t,x)|^2 \, h_\xi(x) \, t^{d-1} \, \mathrm{d}\sigma(x) \, \mathrm{d}t \right)^{1/2} \tag{3.10}$$

and

$$W_0^{1,2}(M) = \overline{C_0^\infty(\partial\Omega \times (0,1))}^{\sqrt{\|\nabla_G\psi\|_{L^2(M)}^2 + \|\psi\|_{L^2(M)}^2}}.$$

Employing the identification of the two L^2 spaces into the variational formulation for the first eigenvalue 2.3.13, we obtain

$$\lambda_1(\Omega) \le \frac{\| \, \|\nabla_G \,\psi\|_G \, \|_{L^2(M)}^2}{\|\psi\|_{L^2(M)}^2}, \quad \psi \in W_0^{1,2}(M).$$

Now let us take some radially symmetric function $\eta = \eta(t)$ of the form

$$\eta(t) = \psi(t) \otimes 1$$

where $\psi \in W_0^{1,2}((0,1), t^{d-1}\,\mathrm{d}t)$ and 1 denotes a function constantly equal to 1 on $\partial\Omega$. Recall the requirements for the radially symmetric function $\tilde{\eta}$ introduced before (see (3.8)). The function η definitely satisfies these requirements and thus we can compute the norm of its gradient using formula (3.9). Substituting η into (3.10) we can use the Fubini's theorem obtaining

$$\|\eta\|^2_{L^2(M)} = \int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi(t)|^2\, t^{d-1}\,\mathrm{d}t$$

and analogically the term $\|\,\|\nabla_G\,\eta\|_G\,\|^2_{L^2(M)}$ can be expressed as

$$\|\,\|\nabla_G\,\eta\|_G\,\|^2_{L^2(M)} = \int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi'(t)|^2\, t^{d-1}\,\mathrm{d}t.$$

Hence we obtain

$$\lambda_1(\Omega) \le \frac{\displaystyle\int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi'(t)|^2\, t^{d-1}\,\mathrm{d}t}{\displaystyle\int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi(t)|^2\, t^{d-1}\,\mathrm{d}t}. \tag{3.11}$$

Let us define a functional $\varphi = \varphi(\Omega;\psi,\xi)$ as the right hand side of this inequality

$$\varphi(\Omega;\psi,\xi) := \frac{\displaystyle\int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi'(t)|^2\, t^{d-1}\,\mathrm{d}t}{\displaystyle\int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi(t)|^2\, t^{d-1}\,\mathrm{d}t}.$$

From Remark 2.3.13 we know that the equality is obtained if, and only if, $\eta = \psi \otimes 1$ is chosen as the first eigenfunction. Our aim is to obtain a sharp bound for the first eigenvalue, thus we would like to find some geometric object for which the equality is attained, i.e. its first eigenfunction minimizes the functional φ with respect to ψ. Let us try as the domain a ball of unit radius with center at ξ denoted by B_1. From the geometric interpretation of the support function as the distance from ξ to the tangent space $T_x(\partial B_1)$ we can see that (figure 3.4) the support function of the ball is equal to its radius, i.e.

$$h_\xi(x) = 1, \quad x \in \partial B_1$$

which implies

$$\lambda_1(B_1) \le \frac{\displaystyle\int_0^1 |\psi'(t)|^2\, t^{d-1}\,\mathrm{d}t}{\displaystyle\int_0^1 |\psi(t)|^2\, t^{d-1}\,\mathrm{d}t}$$

where $\lambda_1(B_1)$ is a known constant. Notice that the two integrations over the border disappeared. The consequence of this fact is that the equality is obtained if, and only if, ψ is the radial component of the first eigenfunction of $-\Delta_D^{B_1}$ denoted by ψ^*. Indeed, the first eigenfunction of the ball is radially symmetric and an element of $W_0^{1,2}((0,1))$ and thus can be written in the form η has and used in the variational formulation

$$\lambda_1(B_1) = \frac{\displaystyle\int_0^1 |\psi^{*\prime}(t)|^2\, t^{d-1}\,\mathrm{d}t}{\displaystyle\int_0^1 |\psi^*(t)|^2\, t^{d-1}\,\mathrm{d}t}.$$

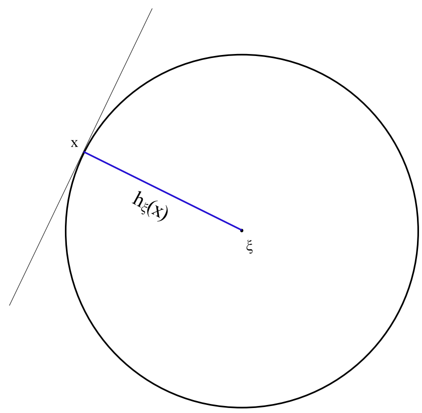


Figure 3.4: Support function of ball centered at ξ

After substituting this function into the functional φ we see that the functional no longer depends on it

$$\varphi(\Omega;\psi^*,\xi) := \frac{\displaystyle\int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi^{*\prime}(t)|^2\, t^{d-1}\,\mathrm{d}t}{\displaystyle\int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x) \int_0^1 |\psi^*(t)|^2\, t^{d-1}\,\mathrm{d}t} = \lambda_1(B_1)\frac{\displaystyle\int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x)}{\displaystyle\int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x)}.$$

Hence we have found the object whose first eigenfunction minimizes the functional φ with respect to ψ

$$\min_{\psi\in W_0^{1,2}((0,1),t^{d-1}\,\mathrm{d}t)} \varphi(\Omega;\psi,\xi) = \lambda_1(B_1)\frac{\displaystyle\int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x)}{\displaystyle\int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x)}. \tag{3.12}$$

Further adjustments of this formula are possible.

First, recall the volume element of the manifold M (3.7)

$$\mathrm{dvol}(x,t) = h_\xi(x)\,\mathrm{d}\sigma(x)\, t^{d-1}\,\mathrm{d}t.$$

Let us integrate the volume element over the whole domain obtaining its volume

$$|\Omega| = \int_{\partial\Omega\times(0,1)} \mathrm{dvol} = \int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x) \int_0^1 t^{d-1}\,\mathrm{d}t = \frac{1}{d}\int_{\partial\Omega} h_\xi(x)\,\mathrm{d}\sigma(x).$$

Substituting this result into (3.12) we obtain

$$\min_{\psi\in W_0^{1,2}((0,1))} \varphi(\Omega;\psi,\xi) = \lambda_1(B_1)\frac{\displaystyle\int_{\partial\Omega} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x)}{d\,|\Omega|}.$$

Finally we can minimize the last remaining integral with respect to ξ over the set ω obtaining the intrinsic quantity $F(\Omega)$ (see 3.3.8) and concluding the proof of the bound of Theorem 3.3.9

$$\lambda_1(\Omega) \leq \min_{\psi \in W_0^{1,2}((0,1),t^{d-1}\,\mathrm{d}t)} \varphi(\Omega;\psi,\xi) = \lambda_1(B_1)\frac{F(\Omega)}{d\,|\Omega|}.$$

□

3.3.3 Weaker version for convex domains

Let Ω be a bounded convex domain in $\mathbb{R}^d$. From the book [13], Sec V.4.1 we know that the boundary $\partial\Omega$ is locally Lipschitz. From the geometrical interpretation of $h_\xi(x)$ we have

$$\operatorname*{ess\,inf}_{x\in\partial\Omega} h_\xi(x) \geq \mathrm{dist}(\xi,\partial\Omega)$$

which implies that the set to which Ω is strictly star-shaped denoted by ω is equal to Ω, i.e., $\omega = \Omega$ and also

$$F(\Omega) = \inf_{\xi\in\omega=\Omega} \int_{\partial\Omega} h_\xi^{-1} \leq \int_{\partial\Omega} h_\xi^{-1}$$

for any $\xi \in \omega = \Omega$ and taking ξ as the center of the inscribed ball we obtain

$$\int_{\partial\Omega} h_\xi^{-1} \leq \int_{\partial\Omega} \frac{1}{\rho_\Omega} = \frac{|\partial\Omega|}{\rho_\Omega}$$

where ρ_Ω is the inradius of Ω (the radius of the inscribed ball, see 3.1) and $|\partial\Omega|$ is the $(d-1)$-dimensional Hausdorff measure of the boundary $\partial\Omega$. Hence we have obtained a simple upper bound for $F(\Omega)$

$$F(\Omega) \leq \frac{|\partial\Omega|}{\rho_\Omega}.$$

Since all the requirements of the theorem (3.3.9) are satisfied we have the bound

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{F(\Omega)}{d\,|\Omega|}$$

and employing the obtained bound for $F(\Omega)$ we conclude with

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}.$$

Hence we can state the weaker version of the PS bound in an arbitrary dimension which holds for convex domains.

Theorem 3.3.20. *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Then*

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}.$$

3.3.4 Remarks

Remark 3.3.21 (Sharp for balls). *From the proof it can be seen that the bound of Theorem 3.3.9 is sharp for the ball of unit radius. Now let us verify that it is sharp for all balls. For this we need to express the first eigenvalue for a ball with an arbitrary radius R denoted by B_R using the first eigenvalue of unit ball. Without loss of generality we can assume that B_R has the center at the origin,i.e., $B_R = B_R(0)$. We will use the variational formulation for the first eigenvalue 2.3.13*

$$\lambda_1(B_R) = \frac{\|\nabla\psi_1\|^2}{\|\psi_1\|^2} = \frac{\displaystyle\int_{B_R} |\nabla_x\psi_1(x)|^2 \,\mathrm{d}x}{\displaystyle\int_{B_R} |\psi_1(x)|^2 \,\mathrm{d}x}$$

where ψ_1 is the first eigenfunction of B_R. As a next step we use the substitution

$$x = Ry$$
$$\mathrm{d}x = R^d \mathrm{d}y$$

implying

$$\frac{\displaystyle\int_{B_R} |\nabla_x\psi_1(x)|^2 \,\mathrm{d}x}{\displaystyle\int_{B_R} |\psi_1(x)|^2 \,\mathrm{d}x} = \frac{1}{R^2}\frac{\displaystyle\int_{B_1} |\nabla_y\psi_1(Ry)|^2 \,\mathrm{d}y}{\displaystyle\int_{B_1} |\psi_1(Ry)|^2 \,\mathrm{d}y}.$$

Since it is apparent that the equality $\phi_1(y) = \psi_1(Ry)$ holds, where ϕ_1 is the first eigenfunction of the unit ball, we can see that

$$\frac{\displaystyle\int_{B_1} |\nabla_y\psi_1(Ry)|^2 \,\mathrm{d}y}{\displaystyle\int_{B_1} |\psi_1(Ry)|^2 \,\mathrm{d}y} = \frac{\displaystyle\int_{B_1} |\nabla_y\phi_1(y)|^2 \,\mathrm{d}y}{\displaystyle\int_{B_1} |\phi_1(y)|^2 \,\mathrm{d}y} = \lambda_1(B_1)$$

and thus

$$\lambda_1(B_R) = \frac{1}{R^2}\lambda_1(B_1). \tag{3.13}$$

At the same time from (4.3) *below we know the explicit formula for $F(\Omega)$ when Ω is a ball of radius R which is a special case of an ellipsoid*

$$F(B_R) = |B_R|\frac{d}{R^2}. \tag{3.14}$$

Now we have everything ready to take $\Omega = B_R$ and substitute (3.13) *and* (3.14) *into* (3.2)

$$\lambda_1(B_R) \le \lambda_1(B_1)\frac{F(B_R)}{d|B_R|} = R^2\lambda_1(B_R)\frac{d|B_R|}{R^2\, d|B_R|} = \lambda_1(B_R) \tag{3.15}$$

and thus the bound of Theorem 3.3.9 is sharp for all balls.

Remark 3.3.22 (Conjecture 1). *In the paper [5] it was conjectured that the upper bound for the bounded convex domains 3.3.20 holds for general bounded domains in $\mathbb{R}^d$. We will use this conjecture for comparison with other bounds for some domains which are strictly star-shaped but not convex in the subsequent chapter.*

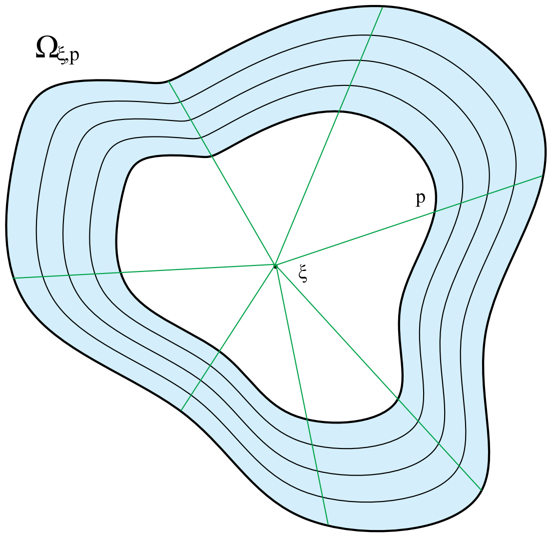


Figure 3.5: Domain $\Omega_{\xi,p}$ generated from Ω

3.4 Generalization of PS bound for particular hollow domains

In this section we would like to introduce our own result, the generalization of the Pólya and Szegö's bound in arbitrary dimension for some hollow domains of a particular form.

To create such a domain let us take the bounded, strictly star-shaped domain Ω in $\mathbb{R}^d$ with locally Lipschitz boundary $\partial\Omega$, the set ω containing the points with respect to which Ω is strictly star-shaped and choose an arbitrary point $\xi \in \omega$. Recall the transformation $\mathcal{L}$ parameterizing $\Omega \setminus \{\xi\}$ and identifying it with the Riemannian manifold M

$$\mathcal{L} : \partial\Omega \times (0, 1) \to \Omega \setminus \{\xi\} : \{(x, t) \mapsto \xi + (x - \xi)t\}.$$

Let p be some fixed value of the shrinking coordinate $t \in (0, 1)$. Now we are ready to define a new domain $\Omega_{\xi,p}$ (see Figure 3.5) with parameters ξ and p using a modification of the map $\mathcal{L}$:

$$\mathcal{L}_p : \partial\Omega \times (p, 1) \to \Omega_{\xi,p} : \{(x, t) \mapsto \xi + (x - \xi)t\}.$$

The bounded hollow domain with locally Lipschitz boundary $\Omega_{\xi,p}$ is then the domain Ω with a hole of the "size" p and "centered" at the point ξ. The boundary of the hole is equal to the shrunk boundary $\mathcal{L}(\partial\Omega \times \{p\})$ and the outer boundary is equal to the boundary $\partial\Omega$ of the domain Ω. Let us denote the outer boundary by $\partial\Omega^1_{\xi,p} := \partial\Omega$. Since $\partial\Omega^1_{\xi,p}$ is by assumption locally Lipschitz, from Rademacher theorem 3.3.4 we again have that the outward unit normal vector field $\boldsymbol{n} : \partial\Omega^1_{\xi,p} \to \mathbb{R}^d$ is uniquely defined almost everywhere on $\partial\Omega^1_{\xi,p}$. Some definitions are needed before we can proceed to the statement of the bound.

Definition 3.4.1 (Support function of $\Omega_{\xi,p}$)**.** *Let $\Omega_{\xi,p}$ be the domain generated from Ω in the preceding paragraph with the outer boundary denoted by $\partial\Omega^1_{\xi,p}$, ($\partial\Omega^1_{\xi,p} = \partial\Omega$). At such points* *where the outward unit normal vector field* $\boldsymbol{n}$ *is defined, a support function of the domain can be introduced as*

$$h_\xi(x) := (x - \xi) \cdot \boldsymbol{n}(x).$$

Remark 3.4.2 (Difference between support function of $\Omega_{\xi,p}$ and Ω). *Recall the definition of the support function for the domain* Ω*, Definition 3.3.5, and only for purpose of this remark let us denote the function by* h_ξ^Ω*. It can be seen that the only difference between the definition of support function for* $\Omega_{\xi,p}$ *and for* Ω *is in the point* ξ*. The center of the shrinking coordinates* ξ *lies in* Ω *but by the definition of* $\Omega_{\xi,p}$ *it does not lie in* $\Omega_{\xi,p}$*. Hence for all points* $x \in \partial\Omega^1_{\xi,p} = \partial\Omega$ *we have an equality*

$$h_\xi(x) = h_\xi^\Omega(x).$$

Definition 3.4.3 (Centered intrinsic quantity $F_\xi(\Omega)$). *Let* Ω *be strictly star-shaped domain with respect to the point* $\xi \in \Omega$*. Then we define the centered intrinsic quantity of the domain* Ω *with the center at* ξ *as*

$$F_\xi(\Omega) := \int_{\partial\Omega} h_\xi^{-1}$$

where h_ξ *is the support function of the domain* Ω*.*

Definition 3.4.4 (Centered intrinsic quantity $F_\xi(\Omega_{\xi,p})$). *Using the previous definition and denotations we define the centered intrinsic quantity for* $\Omega_{\xi,p}$ *by*

$$F_\xi(\Omega_{\xi,p}) := \int_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}$$

where h_ξ *is the support function of the domain* $\Omega_{\xi,p}$*.*

Now we can state the theorem.

Theorem 3.4.5. *Let* Ω *be a bounded domain in* $\mathbb{R}^d$ *strictly star-shaped with respect to a point* $\xi \in \Omega$ *and with locally Lipschitz boundary* $\partial\Omega$*. Let* $\Omega_{\xi,p}$ *be the domain generated from* Ω *in the preceding paragraph and* $A_{a,b}$ *be an annulus with radii such that* $\frac{a}{b} = p$ *and* $|A_{a,b}| = |\Omega_{\xi,p}|$*. Then the following upper bound for the first eigenvalue of* $\Omega_{\xi,p}$ *holds*

$$\lambda_1(\Omega_{\xi,p}) \leq \lambda_1(A_{a,b}) \frac{b^2 F_\xi(\Omega_{\xi,p})}{d|B_b|}.$$

3.4.1 Proof of the bound

Let $\Omega_{\xi,p}$ be the hollow domain with outer boundary $\partial\Omega^1_{\xi,p}$ generated using the bounded domain Ω as was shown in the beginning of this section. Recall the proof of Theorem 3.3.9. Using the same argumentation as in the proof we can introduce the locally Lipschitz continuous chart Γ mapping an open subset U of $\mathbb{R}^{d-1}$ to the intersection of $\mathbb{R}^d$ and $\partial\Omega^1_{\xi,p}$. This chart is from Rademacher theorem 3.3.4 differentiable almost everywhere and thus induces the metric tensor g of $\partial\Omega^1_{\xi,p}$

$$g_{\mu\nu} := (\partial_\mu\Gamma) \cdot (\partial_\nu\Gamma) \quad \mu, \nu \in \{1, \ldots, d-1\}.$$

From the definition of $\Omega_{\xi,p}$ we see that it can be parameterized by the mapping

$$\mathcal{L}_p : \partial\Omega^1_{\xi,p} \times (p, 1) \to \Omega_{\xi,p} : \{(x, t) \mapsto \xi + (x - \xi)t\}$$

or locally by

$$\mathfrak{L}_p : \mathbb{R}^{d-1} \times (p,1) \to \Omega_{\xi,p} : \{(u,t) \mapsto \xi + (\Gamma(u) - \xi)t\}.$$

Indeed, the Jacobi matrix $J(\cdot,t)$ of the transformation $\mathfrak{L}_p$ is equal to the Jacobi matrix of $\mathfrak{L}$ (see (3.4))

$$J(\cdot,t) = \begin{pmatrix} (\partial_1\Gamma^1)\,t & \dots & (\partial_{d-1}\Gamma^1)\,t & \Gamma^1 - \xi^1 \\ \vdots & & \vdots & \vdots \\ (\partial_1\Gamma^d)\,t & \dots & (\partial_{d-1}\Gamma^d)\,t & \Gamma^d - \xi^d \end{pmatrix}$$

and the absolute value of its determinant is locally equal to

$$|\det J(u,t)| = \sqrt{\det g(u)}\, h_\xi(\Gamma(u))\, t^{d-1}$$

where h_ξ is the support function of the domain $\Omega_{\xi,p}$.

Again using the same argumentation as in the preceding proof we see that $\Omega_{\xi,p}$ can be identified with the Riemannian manifold

$$M_p := (\partial\Omega^1_{\xi,p} \times (p,1), G)$$

where G is the metric tensor induced by $\mathcal{L}_p$ and is also equal to the metric tensor induced by $\mathcal{L}$ (see (3.6))

$$G(\cdot,t) = \begin{pmatrix} g_{11}\,t^2 & \dots & g_{1d-1}\,t^2 & (\Gamma-\xi)\cdot(\partial_1\Gamma)\,t \\ \vdots & & \vdots & \vdots \\ g_{d-11}\,t^2 & \dots & g_{d-1d-1}\,t^2 & (\Gamma-\xi)\cdot(\partial_{d-1}\Gamma)\,t \\ (\Gamma-\xi)\cdot(\partial_1\Gamma)\,t & \dots & (\Gamma-\xi)\cdot(\partial_{d-1}\Gamma)\,t & |\Gamma-\xi|^2 \end{pmatrix}.$$

Recall the definition of the volume element of the Riemannian manifold (3.3.17), whereas the corresponding terms are equal for Ω and $\Omega_{\xi,p}$ we see that the volume element of $\Omega_{\xi,p}$ is

$$\mathrm{dvol}(x,t) = h_\xi(x)\,\mathrm{d}\sigma(x)\,t^{d-1}\,\mathrm{d}t.$$

As a next step let us take some test function η of the form

$$\eta = \psi \otimes 1 \tag{3.16}$$

where 1 denotes a function constantly equal to 1 on $\partial\Omega^1_{\xi,p}$ and as we saw at the end of the preceding proof we can now assume $\psi \in W^{1,2}_0((p,1), t^{d-1}\mathrm{d}t)$. We would like to compute its norm of the gradient. Indeed as in the preceding formulas it can be seen that the result is the same as for domain Ω

$$\|\nabla_G(1 \otimes \psi)\|_G = h_\xi^{-1}|\psi'|.$$

Now we can proceed to the spectral problem. Recall the Dirichlet Laplacian $-\Delta^\Omega_D$ defined in the previous chapter and the quadratic form associated with it

$$h[\psi] = \|\nabla\psi\|^2$$
$$\mathrm{Dom}(h) = W^{1,2}_0(\Omega_{\xi,p})$$

with $\|\cdot\|$ being the $L^2(\Omega_{\xi,p})$ norm. We again introduce the identification of the Hilbert space $L^2(\Omega_{\xi,p})$ with $L^2(M_p) := L^2(\partial\Omega^1_{\xi,p} \times (p,1), \mathrm{dvol})$. The space $L^2(M_p)$ is equipped with the norm

$$\|\psi\|_{L^2(M_p)} = \left(\int\limits_{\partial\Omega^1_{\xi,p}\times(p,1)} |\psi(t,x)|^2 \, h_\xi(x) \, t^{d-1} \, \mathrm{d}\sigma(x) \, \mathrm{d}t \right)^{1/2}.$$

Thus we can express the quadratic form as

$$h_D^{\Omega_{\xi,p}}[\psi] = \| \, \|\nabla_G \psi\|_G \, \|^2_{L^2(M_p)}$$
$$\mathrm{Dom}(h) = W_0^{1,2}(M_p)$$

where

$$W_0^{1,2}(M_p) = \overline{C_0^\infty(\partial\Omega^1_{\xi,p} \times (p,1))}^{\sqrt{\|\nabla_G\psi\|^2_{L^2(M_p)} + \|\psi\|^2_{L^2(M_p)}}}.$$

Recall the variational formulation of the first eigenvalue 2.3.13. Employing the identification of the two Hilbert spaces we obtain

$$\lambda_1(\Omega_{\xi,p}) \le \frac{\| \, \|\nabla_G \psi\|_G \, \|^2_{L^2(M_p)}}{\|\psi\|^2_{L^2(M_p)}}, \quad \psi \in W_0^{1,2}(M_p).$$

As the test function in this formulation we would like to use the function η (3.16). Indeed this function by the definition belongs to the space $W_0^{1,2}(M_p)$. The integrals of η appearing in the formulation can be computed as

$$\|\eta\|^2_{L^2(M_p)} = \int\limits_{\partial\Omega^1_{\xi,p}} h_\xi(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi(t)|^2 \, t^{d-1} \, \mathrm{d}t$$

and

$$\| \, \|\nabla_G \, \eta\|_G \, \|^2_{L^2(M_p)} = \int\limits_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi'(t)|^2 \, t^{d-1} \, \mathrm{d}t.$$

Hence

$$\lambda_1(\Omega_{\xi,p}) \le \frac{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi'(t)|^2 \, t^{d-1} \, \mathrm{d}t}{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi(t)|^2 \, t^{d-1} \, \mathrm{d}t} \tag{3.17}$$

and we can define the functional φ as

$$\varphi(\Omega_{\xi,p}; \psi) := \frac{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi'(t)|^2 \, t^{d-1} \, \mathrm{d}t}{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi(t)|^2 \, t^{d-1} \, \mathrm{d}t}. \tag{3.18}$$

We would again like to minimize this functional by finding some object for whose first eigenfunction we obtain equality in (3.17). Let us try the annulus of radii p and 1 denoted by $A_{p,1}$, where

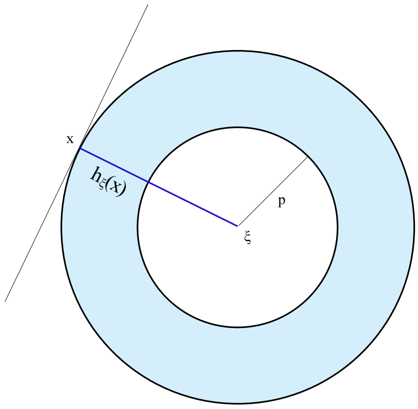


Figure 3.6: Support function of annulus $A_{p,1}$ centered at ξ

$p < 1$, generated from the unit ball with the same center being also the point ξ and using the parameter p, i.e., in our notation $A_{p,1} = (B_1)_{\xi,p}$. Recall that the support function of $A_{p,1}$ which is equal to the support function of B_1 is (see Figure 3.6)

$$h_\xi(x) = 1, \quad x \in \partial A^1_{p,1}$$

where the upper index has the meaning of the outer boundary of the annulus. Therefore

$$\lambda_1(A_{p,1}) \leq \frac{\displaystyle\int_p^1 |\psi'(t)|^2 \, t^{d-1} \, \mathrm{d}t}{\displaystyle\int_p^1 |\psi(t)|^2 \, t^{d-1} \, \mathrm{d}t}$$

where the equality is obtained if, and only if, $\eta = \psi \otimes 1$ is chosen as the first eigenfunction of $A_{p,1}$ which is certainly radially symmetric and thus can be written in the form the function η requires. Denoting its radial component as ψ^* we obtain

$$\lambda_1(A_{p,1}) = \frac{\displaystyle\int_p^1 |\psi^{*\prime}(t)|^2 \, t^{d-1} \, \mathrm{d}t}{\displaystyle\int_p^1 |\psi^*(t)|^2 \, t^{d-1} \, \mathrm{d}t}.$$

Substituting ψ^* into functional φ we, as in the preceding proof, loose the dependence on the test function ψ

$$\varphi(\Omega_{\xi,p}; \psi^*) := \frac{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi^{*\prime}(t)|^2 \, t^{d-1} \, \mathrm{d}t}{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi(x) \, \mathrm{d}\sigma(x) \int_p^1 |\psi^*(t)|^2 \, t^{d-1} \, \mathrm{d}t} = \lambda_1(A_{p,1}) \frac{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}(x) \, \mathrm{d}\sigma(x)}{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi(x) \, \mathrm{d}\sigma(x)}$$

and thus we have successfully minimized the functional with respect to ψ

$$\min_{\psi \in W_0^{1,2}((p,1),t^{d-1}\mathrm{d}t)} \varphi(\Omega_{\xi,p};\psi) = \lambda_1(A_{p,1}) \frac{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi^{-1}(x)\,\mathrm{d}\sigma(x)}{\displaystyle\int_{\partial\Omega^1_{\xi,p}} h_\xi(x)\,\mathrm{d}\sigma(x)}.$$

As a next step we would like to cancel the integral over the support function using the volume of the domain $\Omega_{\xi,p}$

$$|\Omega_{\xi,p}| = \int_{\partial\Omega^1_{\xi,p}\times(p,1)} \mathrm{dvol} = \int_{\partial\Omega^1_{\xi,p}} h_\xi(x)\,\mathrm{d}\sigma(x) \int_p^1 t^{d-1}\,\mathrm{d}t = \frac{1-p^d}{d}\int_{\partial\Omega^1_{\xi,p}} h_\xi(x)\,\mathrm{d}\sigma(x).$$

Recalling the centered intrinsic quantity of the domain $F_\xi(\Omega_{\xi,p})$ we can conclude with

$$\lambda_1(\Omega_{\xi,p}) \le \lambda_1(A_{p,1})\frac{F_\xi(\Omega_{\xi,p})}{d|\Omega_{\xi,p}|}(1-p^d).$$

Now we take an annulus $A_{a,b}$ with radii a and b such that $\frac{a}{b} = p$ and $|A_{a,b}| = |\Omega_{\xi,p}|$. Using (3.1) we have

$$|A_{a,b}| = |B_b| - |B_a| = (b^d - a^d)|B_1|. \tag{3.19}$$

Finally using the equality $\lambda_1(A_{p,1}) = \frac{1}{b^2}\lambda_1(A_{a,b})$ which can be proven analogically as (3.13) we can conclude with

$$\lambda_1(\Omega_{\xi,p}) \le b^2\lambda_1(A_{a,b})\frac{F_\xi(\Omega_{\xi,p})}{d|B_1|(b^d-a^d)}(1-p^d) = \lambda_1(A_{a,b})\frac{b^2F_\xi(\Omega_{\xi,p})}{d|B_b|}$$

which proves Theorem 3.4.5. □

3.4.2 Remarks

Remark 3.4.6 (Existence of $A_{a,b}$). *Our aim is to find the two radii a and b such that $\frac{a}{b} = p$ and $|A_{a,b}| = |\Omega_{\xi,p}|$. From* (3.19) *and* (3.13) *we get*

$$|\Omega_{\xi,p}| = (b^d - a^d)\,|B_1| = (1-p^d)\,|B_1|\,b^d$$

which implies

$$b = \left(\frac{|\Omega_{\xi,p}|}{(1-p^d)|B_1|}\right)^{\frac{1}{d}}.$$

This together with $a = bp$ defines the annulus $A_{a,b}$.

Remark 3.4.7 (Sharp for arbitrary annulus). *From the proof we can see that this bound is sharp for $A_{p,1}$. Let $A_{m,n}$ be an annulus centered at the point ξ. It can be interpreted as a domain $B_{\xi,\frac{m}{n}}$ for Theorem 3.4.5. Since from* (4.3) *the intrinsic quantity $F_\xi(A_{m,n})$ can be expressed as*

$$F_\xi(A_{m,n}) = F(B_n) = |B_n|\frac{d}{n^2}, \tag{3.20}$$

we can write

$$\lambda_1(A_{m,n}) \le \lambda_1(A_{m,n})\frac{n^2F_\xi(A_{m,n})}{d|B_n|} = \lambda_1(A_{m,n}).$$

Remark 3.4.8 (Bound as a fraction of two intrinsic quantities). *Using* (3.20) *we can write*

$$\lambda_1(\Omega) \le \lambda_1(A_{a,b})\frac{F_\xi(\Omega_{\xi,p})}{F(B_b)} = \lambda_1(A_{a,b})\frac{F_\xi(\Omega_{\xi,p})}{F_\xi(A_{a,b})}.$$

3.5 Payne and Weinberger's planar bound

In this chapter we introduce Payne and Weinberger's planar bound which originally appeared in their paper [17]. The proof of this bound is based upon the use of the parallel coordinates (see Figure 3.7). In this bound we use a modern approach to this coordinates developed by Savo in [18] which appeared also in the paper [19] by Pedro Freitas and David Krejčiřík.

This bound as presented in the thesis differs from the others when acting on not simply connected domains. It still works for them but on the inner parts of the boundary we can demand only the Neumann boundary conditions. Thus for simply connected domains we rather get an upper bound for the first eigenvalue of the Dirichlet-Neumann operator defined in the Remark 2.3.14.

Theorem 3.5.1 (Payne and Weinberger's planar bound)**.** *Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with C^2 boundary $\partial\Omega$. Let $|\Omega|$ be the 2-dimensional Lebesgue measure of Ω and $|\partial\Omega|$ be the 1-dimensional Hausdorff measure of the boundary $\partial\Omega$. Denote by p the value*

$$p := 1 - \frac{4\pi|\Omega|}{|\partial\Omega|^2}$$

and by $k = k(p)$ the first zero of the transcendental equation

$$J_0(k)Y_1(\sqrt{p}k) = Y_0(k)J_1(\sqrt{p}k) \tag{3.21}$$

where J_0, respectively J_1 stands for the Bessel function of the first kind of the first, respectively second order and Y_0, respectively Y_1 stands for the Bessel function of the second kind of the first, respectively second order. Then the following bound holds

$$\lambda_1(\Omega) \leq \frac{4\pi^2}{|\partial\Omega|^2}k(p)^2.$$

First we introduce the parallel coordinates.

3.5.1 Parallel coordinates

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$ of class C^2. The boundary $\partial\Omega$ can be interpreted as a Jordan curve (i.e., simple and closed curve) of class C^2 denoted by Γ_0.

First we define the map

$$\Phi : \Gamma_0 \times [0, \infty) \to \mathbb{R}^2 : \{(s,t) \mapsto s - \boldsymbol{n}(s)\,t\}$$

and locally, denoting by γ the natural parametrization by arc length of the curve Γ_0, $\gamma : [a, b] \to \Gamma_0$ with p being the coordinate on $[a, b]$,

$$\Phi \circ (\gamma \times \mathbf{1}) : \mathbb{R} \times (0, \infty) \to \mathbb{R}^2 : \{(p,t) \mapsto \gamma(p) - \boldsymbol{n}(\gamma(p))\,t\}$$

where $\mathbf{1}$ is an identity function on $(0, \infty)$ and $\boldsymbol{n}$ is again the outward unit normal to $\partial\Omega$. Next we define the so called cut-radius map $c : \Gamma_0 \to (0, \infty)$ by the property that the segment mapping $t \mapsto \Phi(s, t)$ minimises the distance from Γ_0 if, and only if, $t \in [0, c(s)]$. This map is known to be continuous and denoting by ρ_Ω the inner radius of Ω we clearly have

$$\max_{s\in\Gamma_0} c(s) = \rho_\Omega.$$

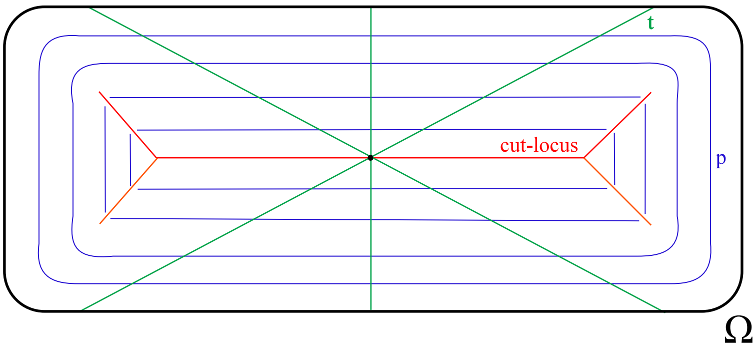


Figure 3.7: Parallel coordinates

Finally we define the cut-locus

$$\mathcal{C}(\Gamma_0) := \{\Phi(s, c(s)) : s \in \Gamma_0\}$$

being a closed subset of Ω of measure zero. If we now restrict the map Φ to the open set

$$U := \{(s,t) \in \Gamma_0 \times (0,\infty) : 0 < t < c(s)\}$$

we obtain a diffeomorphism between U and $\Omega \setminus \mathcal{C}(\Gamma_0)$. The coordinates s and t based at Γ_0 are also called "parallel" which is again motivated by their behavior (see Figure 3.7).

For the purposes of the subsequent proof we also need to compute the determinant of the Jacobi matrix of the transformation Φ. First recall that the unit tangent and normal vector to Γ_0 can be expressed in the point s_0 using the natural parametrization $\gamma(p), \gamma(p_0) = s_0$ as

$$\tau(s_0) = \left.\frac{\partial \gamma(p)}{\partial p}\right|_{p_0}$$
$$n(s_0) = \frac{\left.\frac{\partial^2 \gamma(p)}{\partial p^2}\right|_{p_0}}{\left\|\left.\frac{\partial^2 \gamma(p)}{\partial p^2}\right|_{p_0}\right\|}.$$

Using these formulas the Jacobi matrix can be computed as

$$J(p_0, t_0) = \begin{pmatrix} \tau^1(s_0) - \left.\frac{\partial n^1}{\partial p}\right|_{s_0} t & n^1(s_0) \\ \tau^2(s_0) - \left.\frac{\partial n^2}{\partial p}\right|_{s_0} t & n^2(s_0) \end{pmatrix}$$

and its determinant as

$$\det J(p_0, t_0) = \tau^1(s_0)n^2(s_0) - \tau^2(s_0)n^1(s_0) - \left(\left.\frac{\partial n^1}{\partial p}\right|_{s_0} n^2(s_0) - \left.\frac{\partial n^2}{\partial p}\right|_{s_0} n^1(s_0) \right) t.$$

First we will analyse the term

$$\tau^1(s_0)n^2(s_0) - \tau^2(s_0)n^1(s_0).$$

Denoting by $\tau_\perp(s_0)$ the vector perpendicular to $\tau(s_0)$, i.e., $\tau_\perp(s_0) := (-\tau^2(s_0), \tau^1(s_0))$, the first term is equal to

$$n(s_0) \cdot \tau_\perp(s_0)$$

and since $n(s_0)$ and $\tau_\perp(s_0)$ are co-directional unit vectors, we can see from the remark 3.3.14 that

$$\tau^1(s_0)n^2(s_0) - \tau^2(s_0)n^1(s_0) = n(s_0) \cdot \tau_\perp(s_0) = 1. \tag{3.22}$$

Subsequently recall the Frenet equation

$$\begin{pmatrix} \frac{\partial \tau}{\partial p} \\ \frac{\partial n}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \tau \\ n \end{pmatrix}$$

defining the curvature κ of the curve Γ_0. Using this equation we can adjust the second term

$$\left(\frac{\partial n^1}{\partial p}\Big|_{p_0} n^2(s_0) - \frac{\partial n^2}{\partial p}\Big|_{p_0} n^1(s_0) \right) t = -\kappa(s_0) \left(\tau^1(s_0)n^2(s_0) - \tau^2(s_0)n^1(s_0)\right) t$$

which is equal to $-\kappa t$ by applying (3.22). Hence we can conclude with

$$\det J(s,t) = 1 - \kappa(s)\, t. \tag{3.23}$$

Now we can proceed to the proof of the bound.

3.5.2 Proof of the bound

Using the expression for the determinant of the Jacobi matrix (3.23) we can obtain the uniform bound

$$\| \det J(s,t) \|_{L^\infty(U)} \leq 1 + \|\kappa\|_{L^\infty(\Gamma_0)}\, \rho_\Omega. \tag{3.24}$$

Moreover we introduce the distance function from the boundary Γ_0

$$\rho : \Omega \to (0,\infty) : \{x \mapsto \operatorname{dist}(x,\Gamma_0) = \inf_{s\in\Gamma_0} \|s - x\|\}$$

and the function $A(t)$ of the area of the shell $\{x \in \Omega : 0 < \rho(x) < t\}$, i.e.,

$$A(t) = |\{x \in \Omega : 0 < \rho(x) < t\}|.$$

Clearly $A_0 := A(\rho_\Omega) = |\Omega|$. Finally we define the length of the boundary curve $\{\rho(x) = t\}$ lying in Ω by

$$L(t) := \int\limits_{\{s\in\Gamma_0, t<c(s), \Phi(s,t)\in\Omega\}} \det J(s,t)\, \mathrm{d}s = \int\limits_{\{s\in\Gamma_0, t<c(s), \Phi(s,t)\in\Omega\}} 1 - \kappa(s)\, t\, \mathrm{d}s. \tag{3.25}$$

It can be seen that $L_0 := L(0) = |\Gamma_0|$, where now $|\Gamma_0|$ denotes the one-dimensional Hausdorff measure of the outer boundary. This together with the uniform bound for the Jacobian (3.24) leads to the crude bound for $L(t)$

$$L(t) \leq L_0(1 + \|\kappa\|_{L^\infty(\Gamma_0)}\, \rho_\Omega).$$

Using the co-area formula (see [6]) we can write

$$|A(t_2) - A(t_1)| = \left| \int_{t_1}^{t_2} L(t)\, \mathrm{d}t \right|$$

from which we see that $A(t)$ is Lipschitz on $[0, \rho_\Omega]$ and for almost every t (see 3.3.4)

$$A'(t) = L(t). \tag{3.26}$$

Now we would like to use the min-max principle 2.3.4 to estimate the first eigenvalue. For this let us take some smooth function $\phi : [0, A_0] \to \mathbb{R}$ and consider the test function $u = \phi \circ A \circ \rho$, Lipschitz in Ω and depending only on the distance from the boundary Γ_0.

By the change of the coorinates from Cartesian to parallel and using the fact that the test function u depends only on the distance from the outer boundary, we can compute $\|u\|^2_{L^2(\Omega)}$

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega |u(x)|^2 \,\mathrm{d}x = \int_0^{L_0} \mathrm{d}s \int_0^{c(s)} \mathrm{d}t \, |\phi(A(t))|^2 \,(1 - \kappa(s)\,t),$$

then denoting $M := \{s \in \Gamma_0, t < c(s), \Phi(s,t) \in \Omega\}$ and using the Fubini theorem

$$\int_0^{L_0} \mathrm{d}s \int_0^{c(s)} \mathrm{d}t \, |\phi(A(t))|^2 \,(1 - \kappa(s)\,t) = \int_0^{\rho_\Omega} \mathrm{d}t \int_M \mathrm{d}s \, |\phi(A(t))|^2 \,(1 - \kappa(s)\,t).$$

Since the last integral is from (3.25) equal to $L(t)$, we can write using (3.26)

$$\int_0^{R} \mathrm{d}t \, |\phi(A(t))|^2 \int_M \mathrm{d}s \,(1 - \kappa(s)\,t) = \int_0^{\rho_\Omega} \mathrm{d}t \, |\phi(A(t))|^2 A'(t)$$

and thus

$$\|u\|^2_{L^2(\Omega)} = \int_0^{\rho_\Omega} \mathrm{d}t \, \phi(A(t))^2 A'(t). \tag{3.27}$$

Analogically we can also compute

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_0^{\rho_\Omega} \mathrm{d}t \, \phi'(A(t))^2 A'(t)^3. \tag{3.28}$$

To continue we will use a remarkable idea introduced by Payne and Weinberger in [17] to use the change of the coordinates

$$r(t) := \frac{\sqrt{L_0^2 - 4\pi A(t)}}{2\pi}, \quad t \in [0, R] \tag{3.29}$$

with

$$r1 := r(\rho_\Omega) = \frac{\sqrt{L_0^2 - 4\pi A_0}}{2\pi}, \tag{3.30}$$

$$r_2 := r(0) = \frac{L_0}{2\pi}. \tag{3.31}$$

Recall the isoperimetric inequality for planar bounded domains.

Remark 3.5.2 (Isoperimetric inequality, see [4])**.** *Let Ω be a bounded domain in $\mathbb{R}^2$ and L_0 be the Hausdorff measure of its boundary. Then the following inequality holds*

$$L_0^2 \geq 4\pi|\Omega|$$

where $|\Omega|$ stands for the 2-dimensional Lebesgue measure of Ω. The equality is obtained if, and only if, the domain is a ball.

Thanks to this inequality, the transformation (3.29) is well-defined on $[0, \rho_\Omega]$ and we can use it as a substitution in the integrals (3.27) and (3.28). Also note that the transformation was chosen so that the area of annulus with radii r_1 and r_2, A_{r_1,r_2}, is equal to the area of Ω, i.e., $|A_{r_1,r_2}| = A_0$. Defining

$$\psi(r) := \phi\left(\frac{L_0^2}{4\pi} - \pi r^2\right)$$

we obtain

$$\|u\|^2_{L^2(\Omega)} = 2\pi \int_{r_1}^{r_2} \mathrm{d}r\, \psi(r)^2\, r \tag{3.32}$$

and

$$\|\nabla u\|^2_{L^2(\Omega)} = 2\pi \int_{r_1}^{r_2} \mathrm{d}r\, \psi'(r)^2\, r'(t)^2\, r. \tag{3.33}$$

Our aim is to compare Ω with annulus A_{r_1,r_2}. For this we would like to estimate the term $r'(t)^2$ by 1. The following theorem claims that this estimate is possible.

Theorem 3.5.3. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then for the function $r(t)$ defined above* (3.29), *we have the following bound for almost every $t \in [0, \rho_\Omega]$*

$$|r'(t)| \leq 1.$$

Proof. First we compute the derivative of $r(t)$

$$r'(t) = -\frac{L(t)}{\sqrt{L_0^2 - 4\pi A(t)}}$$

for almost every $t \in [0, \rho_\Omega]$. Recall that for any Jordan curve we have

$$\int_{\Gamma_0} \kappa(s)\, \mathrm{d}s = 2\pi.$$

Hence from (3.25) we can obtain

$$L(t) \leq L_0 - 2\pi t$$

and using (3.26) also

$$A(t) \leq L_0 t - \pi t^2.$$

From the last bound we can express t as a function of $A(t)$ and L_0 since it can be reduced to a problem of solving a quadratic equation

$$\pi t^2 - L_0 t + A(t) \leq 0 \tag{3.34}$$

and thus the roots of the associated equation are

$$t_{1,2} = \frac{L_0 \mp \sqrt{L_0^2 - 4\pi A(t)}}{2\pi}$$

and the inequality (3.34) is fulfilled for $t \in [t_1, t_2]$ (see Figure 3.8). Now we can proceed to the final estimate

$$L(t) \leq L_0 - 2\pi t \leq L_0 - 2\pi t_1 = \sqrt{L_0^2 - 4\pi A(t)}$$

which concludes the proof. □

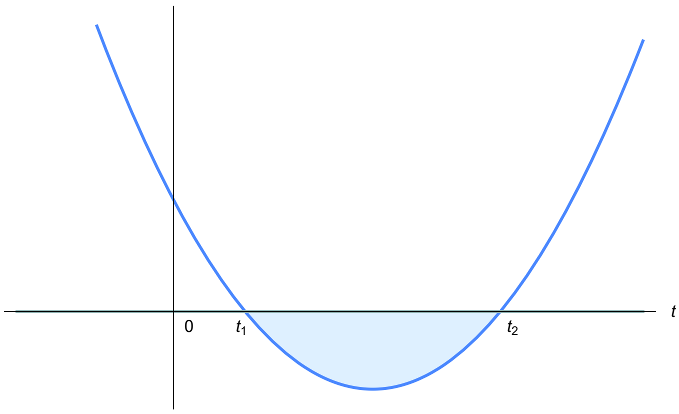


Figure 3.8: Plot of inequality (3.34)

Recall the Dirichlet Laplacian defined in the previous chapter. Using the variational formulation for the first eigenvalue, i.e., Remark 2.3.13, and substituting (3.32) and (3.33) and using Theorem 3.5.3 we get

$$\lambda_1(\Omega) = \inf \frac{\int_{r_1}^{r_2} \mathrm{d}r\, \psi'(r)^2\, r}{\int_{r_1}^{r_2} \mathrm{d}r\, \psi(r)^2\, r}$$

where the infimum is taken over all smooth non-zero functions ψ. At the same time recall the Dirichlet-Neumann Laplacian defined in Remark 2.3.14. Employing this self-adjoint operator into the min-max principle 2.3.4 and taking as the domain the annulus A_{r_1,r_2} we obtain

$$\lambda_1^{DN}(A_{r_1,r_2}) = \frac{\|\nabla\psi_1\|^2_{L^2(A_{r_1,r_2})}}{\|\psi_1\|^2_{L^2(A_{r_1,r_2})}} = \frac{\int_{r_1}^{r_2} \mathrm{d}r\, \psi_1'(r)^2\, r}{\int_{r_1}^{r_2} \mathrm{d}r\, \psi_1(r)^2\, r}$$

where λ_1^{DN} is the first Dirichlet-Neumann eigenvalue of A_{r_1,r_2} and $\psi_1(r)$ is the first Dirichlet-Neumann eigenfunction of A_{r_1,r_2}. Since the radially symmetric function $\psi_1(r)$ is definitely smooth we can write

$$\lambda_1(\Omega) \leq \frac{\int_{r_1}^{r_2} \mathrm{d}r\, \psi_1'(r)^2\, r}{\int_{r_1}^{r_2} \mathrm{d}r\, \psi_1(r)^2\, r} = \lambda_1^{DN}(A_{r_1,r_2})$$

and $\lambda_1^{DN}(A_{r_1,r_2})$ is then the upper bound for $\lambda_1(\Omega)$.

Remark 3.5.4. *Using the definitions from the preceding proof let us denote by p the value*

$$p := 1 - \frac{4\pi|\Omega|}{|\partial\Omega|^2}$$

and let $k = k(p)$ be the first zero of the transcendental equation (3.21) *then*

$$\frac{4\pi^2}{|\partial\Omega|^2} k(p)^2 = \lambda_1^{DN}(A_{r_1,r_2}) \tag{3.35}$$

Proof of the remark. First let us compute $\lambda_1(A_{r_1,r_2})$. We solve this problem in polar coordinates ρ and φ. Hence

$$\begin{aligned} -\Delta u &= \lambda u \\ u(r_2,\varphi) &= 0 \\ \left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{r_1,\varphi} &= 0 \\ u(r,0) &= u(r,2\pi) \\ \left.\frac{\partial u}{\partial \varphi}\right|_{r,0} &= \left.\frac{\partial u}{\partial \varphi}\right|_{r,2\pi} \end{aligned}$$

for all $\varphi \in [0,2\pi]$ and $r \in (r_1,r_2)$. Writing $-\Delta$ in polar coordinates we obtain

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = -\lambda u \tag{3.36}$$

and using the separation of variables $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ we have

$$\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} + \lambda\rho^2 = -\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2}. \tag{3.37}$$

Since the left-hand side of this equation does not depend on φ and also the right-hand side does not depend on ρ, we see that both sides are equal to a constant m^2 obtaining

$$\begin{aligned} \rho^2\frac{\partial^2 R}{\partial^2 \rho} + \rho\frac{\partial R}{\partial \rho} + \lambda\rho^2 R - m^2 R &= 0 \\ \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} &= -m^2. \end{aligned}$$

The second equation together with the cyclic boundary conditions

$$\begin{aligned} \Phi(0) &= \Phi(2\pi) \\ \Phi'(0) &= \Phi'(2\pi) \end{aligned}$$

has the harmonic solution

$$\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi)$$

for $m \in \mathbb{Z}$.

The first equation is the Bessel equation with the solution

$$R(\rho) = C_1 J_m(\sqrt{\lambda}\rho) + C_2 Y_m(\sqrt{\lambda}\rho).$$

Applying the two boundary conditions

$$R(r_2) = 0$$

respectively

$$\left.\frac{\partial R}{\partial \rho}\right|_{r_1} = 0$$

we get

$$C_1 J_m(\sqrt{\lambda} r_2) + C_2 Y_m(\sqrt{\lambda} r_2) = 0$$

respectively

$$C_1\sqrt{\lambda}(J_{-1+m}(\sqrt{\lambda}r_1) - J_{1+m}(\sqrt{\lambda}r_1)) + C_2\sqrt{\lambda}(Y_{-1+m}(\sqrt{\lambda}r_1) - Y_{1+m}(\sqrt{\lambda}r_1)) = 0$$

and by eliminating the constants C_1 and C_2 we have

$$Y_m(\sqrt{\lambda}r_2)(J_{-1+m}(\sqrt{\lambda}r_1) - J_{1+m}(\sqrt{\lambda}r_1)) - J_m(\sqrt{\lambda}r_2)(Y_{-1+m}(\sqrt{\lambda}r_1) - Y_{1+m}(\sqrt{\lambda}r_1)) = 0.$$

Moreover putting $m = 0$ (we are interested in the first eigenvalue) and thanks to the properties of the Bessel functions written as

$$J_1(\sqrt{\lambda_1}r_1)Y_0(\sqrt{\lambda_1}r_2) = J_0(\sqrt{\lambda_1}r_2)Y_1(\sqrt{\lambda_1}r_1). \tag{3.38}$$

Finally we check the equality of the two equations (3.21) and (3.38). First we take a look at the term $\sqrt{\lambda_1}r_1$. Using (3.30) we can see that

$$\sqrt{\lambda_1}r_1 = \sqrt{\lambda_1}\frac{\sqrt{|\partial\Omega|^2 - 4\pi|\Omega|}}{2\pi} = \sqrt{\lambda_1}\frac{\sqrt{|\partial\Omega|^2 - 4\pi|\Omega|}}{|\partial\Omega|}\frac{|\partial\Omega|}{2\pi} = \sqrt{p}k.$$

Analogically using (3.31) we get

$$\sqrt{\lambda_1}r_2 = \sqrt{\lambda_1}\frac{|\partial\Omega|}{2\pi} = k$$

which proves the remark. □

The last remark concludes the proof of Theorem 3.5.1.

3.5.3 Remarks

Remark 3.5.5. *The parallel coordinates introduced above can also be built for bounded but not simply-connected domain Ω. In this case we however obtain an upper bound for $\lambda_1^{DN}(\Omega)$, i.e., we have the Dirichlet boundary conditions on the outer boundary and the Neumann boundary conditions on the inner boundary of Ω.*

Remark 3.5.6. *The parallel coordinates can also be built for not simply-connected domain Ω based on its whole boundary (not only on the outer boundary as in the preceding remark). This procedure leads to the upper bound for the first Dirichlet eigenvalue of Ω however in the final part of the proof we are not able to prove that $|r'(t)| \leq 1$. Indeed it can be shown that $|r'(t)|$ may be larger than 1 for some not simply-connected domains and thus we cannot compare $\lambda_1(\Omega)$ with $\lambda_1^{DN}(A_{r_1,r_2})$ and obtain the upper bound.*

Remark 3.5.7 (Sharp for balls). *Let $\Omega = B_R$. Then $|B_R| = \pi R^2$ and $|\partial B_R| = 2\pi R$. Substituting these into the definitions of the radii r_1 and r_2, (3.30) and (3.31), we obtain the annulus $A_{0,R}$, i.e. the ball B_R. Since B_R has no inner boundary, we have $\lambda_1^{DN}(B_R) = \lambda_1(B_R)$ which implies the sharpness of the PW bound for the balls.*

3.6 Antunes and Freitas conjecture

Finally we introduce the planar conjecture based on numerical studies of Antunes and Freitas appearing in their paper [21].

Theorem 3.6.1 (Conjecture 2). *Let Ω be a planar simply-connected domain. Then the following inequality holds*

$$\lambda_1(\Omega) \leq \frac{\pi j_{01}^2}{|\Omega|} + \frac{\pi^2}{4}\frac{|\partial\Omega|^2 - 4\pi|\Omega|}{|\Omega|^2}$$

with j_{01} being again the first positive zero of the Bessel function of the first kind of order one. The equality is obtained for balls and asymptotically for infinite rectangular strips.

3.7 Summary

Let Ω be a bounded domain in $\mathbb{R}^d$.

Faber-Krahn inequality

$$\lambda_1(\Omega) \geq \lambda_1(B_1)\left(\frac{|B_1|}{|\Omega|}\right)^{2/d}$$

Trivial upper bound

$$\lambda_1(\Omega) \leq \lambda_1(B_{\rho_\Omega})$$

Pólya and Szegö's bound (Ω strictly star-shaped with locally Lipschitz boundary)

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{F(\Omega)}{d|\Omega|}$$

Pólya and Szegö's bound for convex domains (Ω convex)

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}$$

Generalization of Pólya and Szegö's bound for particular hollow domains (Ω strictly star-shaped with locally Lipschitz boundary, $\Omega_{\xi,p}$ generated from Ω, annulus $A_{a,b}$, $\frac{a}{b} = p$, $|A_{a,b}| = |\Omega_{\xi,p}|$, see Theorem 3.4.5)

$$\lambda_1(\Omega_{\xi,p}) \leq \lambda_1(A_{a,b})\frac{b^2 F_\xi(\Omega_{\xi,p})}{d|B_b|}$$

Payne and Weinberger's planar bound ($\Omega \subset \mathbb{R}^2$, simply-connected with C^2 boundary)

$$\lambda_1(\Omega) \leq \frac{4\pi^2}{|\partial\Omega|^2}k(p)^2$$

where

$$p := 1 - \frac{4\pi|\Omega|}{|\partial\Omega|^2}$$

and $k = k(p)$ is the first zero of the transcendental equation (3.21).

Conjecture 1

$$\lambda_1(\Omega) \leq \lambda_1(B_1)\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}$$

Antunes and Freitas conjecture ($\Omega \subset \mathbb{R}^2$ simply connected)

$$\lambda_1(\Omega) \leq \frac{\pi j_{01}^2}{|\Omega|} + \frac{\pi^2}{4}\frac{|\partial\Omega|^2 - 4\pi|\Omega|}{|\Omega|^2}$$

Chapter 4

Examples

This is the last chapter of the thesis where we compare the bounds and conjectures introduced in the preceding part for some particular domains, more precisely for rectangular parallelepipeds, ellipsoids, stadiums and swiss crosses.

4.1 Simply-connected domains

For every particular domain we first compute its intrinsic quantity F appearing in Theorem 3.3.9 and then we compare the bounds of Theorems 3.2.1 (Trivial bound), 3.3.9 (Pólya and Szegö's bound, denoted by PS), 3.3.20 (Pólya and Szegö's bound for convex domains, denoted by PS convex), 3.5.1 (Payne and Weinberger's bound, denoted by PW), the Conjectures 3.3.22 (denoted by C1) and 3.6.1 (denoted by AF) and specifically for the parallelepipeds we can also use for the comparison the actual eigenvalues (Remark 2.3.11, denoted by AE). Since the PW bound which uses the parallel coordinates works only for planar domains, we have to restrict ourselves to domains in $\mathbb{R}^2$.

4.1.1 Rectangular parallelepipeds

Let $a_1, a_2, \ldots, a_d \in \mathbb{R}^+$ and $\mathcal{R} := (-a_1, a_1) \times \cdots \times (-a_d, a_d)$ be the rectangular parallelepiped in $\mathbb{R}^d$. First let us compute the intrinsic quantity $F(\mathcal{R})$. Let $\xi \in \mathcal{R}$ be a point to which $\mathcal{R}$ is strictly star-shaped. We have for every $k \in \{1, \ldots, d\}$,

$\forall x \in \partial\mathcal{R}$ such that $x_k = a_k$

$$\begin{aligned} h_\xi(x) &= (x - \xi) \cdot \boldsymbol{n}(x) \\ &= (x_1 - \xi_1, \ldots, x_{k-1} - \xi_{k-1}, a_k - \xi_k, x_{k+1} - \xi_{k+1}, \ldots, x_d - \xi_d) \cdot (0, \ldots, 0, 1, 0, \ldots, 0) \\ &= a_k - \xi_k \end{aligned}$$

and $\forall x \in \partial\mathcal{R}$ such that $x_k = -a_k$

$$\begin{aligned} h_\xi(x) &= (x - \xi) \cdot \boldsymbol{n}(x) \\ &= (x_1 - \xi_1, \ldots, x_{k-1} - \xi_{k-1}, -a_k - \xi_k, x_{k+1} - \xi_{k+1}, \ldots, x_d - \xi_d) \cdot (0, \ldots, 0, -1, 0, \ldots, 0) \\ &= a_k + \xi_k. \end{aligned}$$

Hence

$$\begin{aligned}F(\mathcal{R}) &= \inf_{\xi\in\omega}\int_{\partial\mathcal{R}} h_\xi^{-1}\\ &= \inf_{\xi\in\omega}\sum_{k=1}^{d}\int_{-a_1}^{a_1}\mathrm{d}x_1\ldots\int_{-a_{k-1}}^{a_{k-1}}\mathrm{d}x_{k-1}\int_{-a_{k+1}}^{a_{k+1}}\mathrm{d}x_{k+1}\ldots\int_{-a_d}^{a_d}\mathrm{d}x_d\left(\frac{1}{a_k-\xi_k}+\frac{1}{a_k+\xi_k}\right)\\ &= \inf_{\xi\in\omega}\sum_{k=1}^{d}\frac{2^d a_1a_2\ldots a_d}{a_k^2-\xi_k^2}\\ &= |\mathcal{R}|(a_1^{-2}+\cdots+a_d^{-2}).\end{aligned}\tag{4.1}$$

For the comparison we now take the two dimensional rectangular parallelepiped of sides a and b, $a<b$, i.e., $\mathcal{R}_2=(-\frac{a}{2},\frac{a}{2})\times(-\frac{b}{2}\times\frac{b}{2})$. Hence $A:=|\mathcal{R}_2|=a\cdot b$, $L:=|\partial\mathcal{R}_2|=2(a+b)$ and $\rho_{\mathcal{R}_2}=\frac{a}{2}$. Recall that in two dimensions $\lambda_1(B_1)=j_{01}^2$ (see (1.18)). The rectangle $\mathcal{R}_2$ is certainly bounded and convex (therefore strictly star-shaped) and thus we can use for the comparison the trivial bound, PS bound, PS bound for convex domains, PW bound, AF conjecture and since they can be explicitly computed, also the actual eigenvalues.

Trivial bound

$$\lambda_1(\mathcal{R}_2)\le\frac{4j_{01}^2}{a^2}$$

PS bound

$$\lambda_1(\mathcal{R}_2)\le 2j_{01}^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

PS bound for convex domains

$$\lambda_1(\mathcal{R}_2)\le 2j_{01}^2\frac{a+b}{a^2b}$$

PW bound

$$\lambda_1(\mathcal{R}_2)\le\frac{2\pi^2}{(a+b)^2}k(p)^2$$

with

$$p=1-\frac{\pi ab}{(a+b)^2}$$

and $k=k(p)$ be the first zero of the transcendental equation (3.21).

AF conjecture

$$\lambda_1(\mathcal{R}_2)\le\frac{\pi j_{01}^2}{ab}+\frac{\pi^2}{4}\frac{(a+b)^2-4\pi ab}{a^2b^2}$$

Actual eigenvalue

$$\lambda_1(\mathcal{R}_2)=\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

Setting $c:=\frac{a}{b}$ we can plot the obtained bounds with respect to the constant c (see Figure 4.1). We can see for example that the AF conjecture is for all values of the parameter c better than the PW bound. Also the PW bound behaves worse for square-like rectangles ($c\approx 1$) than the PS bound and even than the trivial bound.

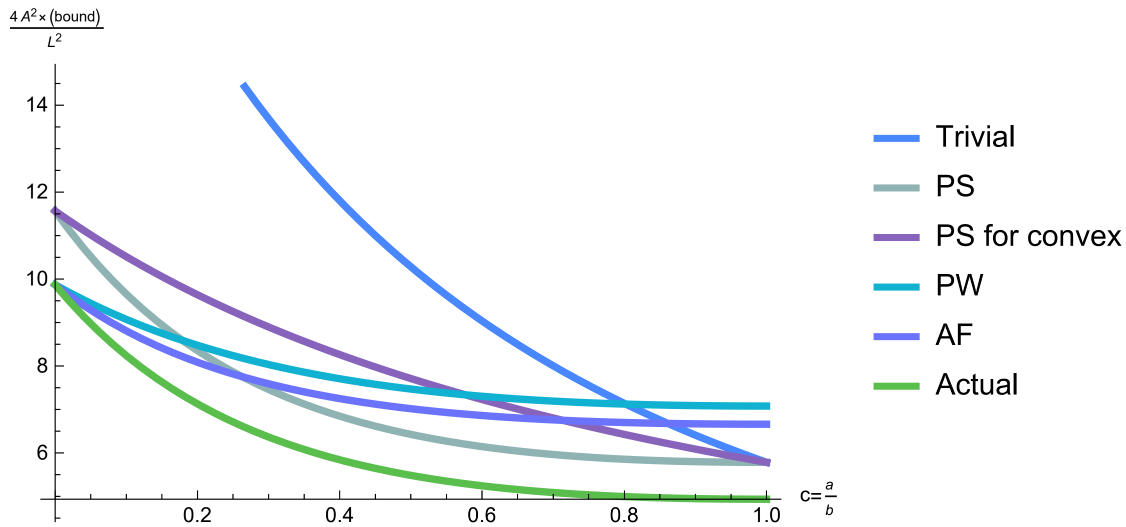


Figure 4.1: Plot of bounds for rectangle with sides a and b, with area A and length of boundary curve L

4.1.2 Ellipsoids

Let $a_1, a_2, \dots, a_d \in \mathbb{R}^+$ and $\mathcal{E} := \left\{ x \in \mathbb{R}^d : \frac{x_1^2}{a_1^2} + \dots + \frac{x_d^2}{a_d^2} < 1 \right\}$ be the domain enclosed by an ellipsoid in $\mathbb{R}^d$. Next we compute the intrinsic quantity $F(\mathcal{E})$. The ellipsoid is described by the implicit equation

$$f(x) := \frac{x_1^2}{a_1^2} + \dots + \frac{x_d^2}{a_d^2} - 1 = 0.$$

From the symmetry we can conclude, as in the preceding case, that the minimum value in the definition of the intrinsic quantity is attained for $\xi = 0$. Recall that the normalized gradient $\frac{\nabla f}{|\nabla f|}$ is uniformly equivalent to $\boldsymbol{n}$ or $-\boldsymbol{n}$ on the ellipsoid, thus

$$\nabla f = \boldsymbol{n}|\nabla f| \Rightarrow \nabla f \cdot \boldsymbol{n} = |\nabla f|$$

or

$$\nabla f = -\boldsymbol{n}|\nabla f| \Rightarrow -\nabla f \cdot \boldsymbol{n} = |\nabla f|.$$

Using this we obtain

$$h_0^{-1}(x) = \frac{1}{x \cdot \boldsymbol{n}(x)} = \frac{|\nabla f(x)|}{x \cdot \nabla f(x)} = \boldsymbol{n}(x) \cdot \frac{\nabla f(x)}{x \cdot \nabla f(x)}. \tag{4.2}$$

Substituting

$$\nabla f = \left(\frac{2x_1}{a_1^2}, \dots, \frac{2x_d}{a_d^2} \right)$$

and

$$x \cdot \nabla f(x) = 2(f(x) + 1) = 2$$

into (4.2) we have

$$h_0^{-1} = \boldsymbol{n}(x) \cdot \left(\frac{x_1}{a_1^2}, \dots, \frac{x_d}{a_d^2} \right).$$

The desired result can be obtained using the Divergence theorem (see [22], Thm. 5.8)

$$F(\mathcal{E}) = \int_{\partial\mathcal{E}} h_0^{-1} = \int_{\partial\mathcal{E}} \boldsymbol{n}(x)\cdot\left(\frac{x_1}{a_1^2},\dots,\frac{x_d}{a_d^2}\right)\mathrm{d}S = \int_{\mathcal{E}} \nabla\cdot\left(\frac{x_1}{a_1^2},\dots,\frac{x_d}{a_d^2}\right)\mathrm{d}V = |\mathcal{E}|(a_1^{-2}+\dots+a_d^{-2}). \tag{4.3}$$

Again for the comparison we need a two dimensional ellipse with axis a and b,$a > b$, i.e., $\mathcal{E}_2 = \left\{x \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1\right\}$. We have

$$A := |\mathcal{E}_2| = \pi ab$$

$$L := |\partial\mathcal{E}_2| = 4a\int_0^{\frac{\pi}{2}} \sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2\theta}\,\mathrm{d}\theta$$

and also $\rho_{\mathcal{E}_2} = b$. The ellipse is again bounded and convex and thus we can use for the comparison the same bounds as for the rectangle with the exception that the actual eigenvalues for ellipsoids are not known explicitly.

Trivial bound

$$\lambda_1(\mathcal{E}_2) \le \frac{j_{01}^2}{b^2}$$

PS bound

$$\lambda_1(\mathcal{E}_2) \le \frac{j_{01}^2}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

PS bound for convex domains

$$\lambda_1(\mathcal{E}_2) \le j_{01}^2\frac{|\partial\mathcal{E}_2|}{2\pi ab^2}$$

PW bound

$$\lambda_1(\mathcal{E}_2) \le \frac{4\pi^2}{|\partial\mathcal{E}_2|^2}k(p)^2$$

with

$$p = 1 - \frac{4\pi^2 ab}{|\partial\mathcal{E}_2|^2}$$

and $k = k(p)$ be the first zero of the transcendental equation (3.21).

AF conjecture

$$\lambda_1(\mathcal{E}_2) \le \frac{j_{01}^2}{ab} + \frac{1}{4}\frac{|\partial\mathcal{E}_2|^2 - 4\pi^2 ab}{(ab)^2}$$

Notice that the PS bound for ellipse with axes a and b is the same as for the rectangle with sides a and b. Setting $c := \frac{b}{a}$ we obtain a plot of bounds with respect to the constant c (see Figure 4.2). We can see that the PS bound is better than all the other bounds for all the values of the parameter c. Also the PS bound for convex domains is better than conjecture AF for $c \in (0, 0.1]$.

4.1.3 Stadium

We proceed to another type of domain called the stadium (see Figure 4.3) defined from the beginning only in the planar case. Let $a, b \in \mathbb{R}^+$ and let the stadium $\mathcal{S} \subset \mathbb{R}^2$ be the union of the rectangle $(-b, b) \times (-a, a)$ and two discs of radius a centered at the points $(-b, 0)$ and $(b, 0)$. Let $c := \frac{b}{a} \in [0, +\infty)$. We now compute the intrinsic quantity $F(\mathcal{S})$. By the symmetry we can

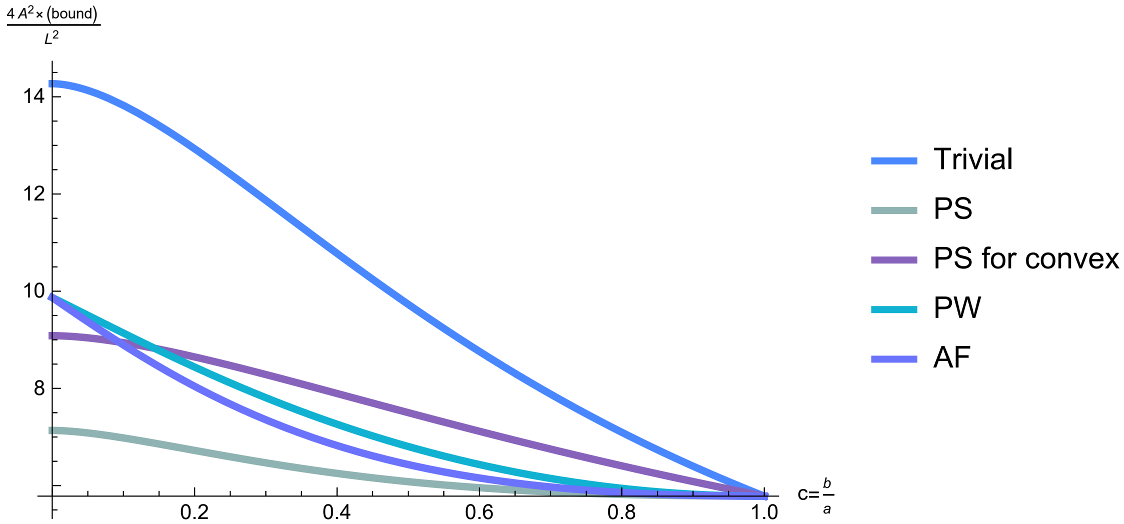


Figure 4.2: Plot of bounds for ellipse with axes a and b, with area A and length of boundary curve L

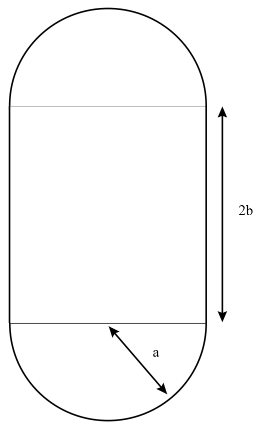


Figure 4.3: Stadium with parameters a and b

again conclude that the infimum is attained for $\xi = 0$. First we compute the integral over the straight line segments of $\partial\mathcal{S}$. For all $x \in \partial\mathcal{S}$ such that $x_1 = a$ we have

$$\int_{-b}^{b} h_0^{-1} = \int_{-b}^{b} \frac{\mathrm{d}x_2}{(a, x_2)\cdot(1,0)} = \int_{-b}^{b} \frac{\mathrm{d}x_2}{a} = \frac{2b}{a}$$

and analogically for all $x \in \partial\mathcal{S}$ such that $x_1 = -a$

$$\int_{-b}^{b} h_0^{-1} = \int_{-b}^{b} \frac{\mathrm{d}x_2}{(-a, x_2)\cdot(-1,0)} = \int_{-b}^{b} \frac{\mathrm{d}x_2}{a} = \frac{2b}{a}.$$

Next we compute the integral over the two arc segments. We start with the upper one which can be parameterized using the polar coordinates as

$$\begin{aligned} x_1 &= a\cos\varphi \\ x_2 &= a\sin\varphi + b \end{aligned}$$

where $\varphi \in (0, \pi)$. Also the normal $\boldsymbol{n}$ can be expressed as $(\cos\varphi, \sin\varphi)$. Thus we have

$$\int_0^{\pi} \frac{a\,\mathrm{d}\varphi}{(a\cos\varphi, a\sin\varphi + b)\cdot(\cos\varphi, \sin\varphi)} = \int_0^{\pi} \frac{\mathrm{d}\varphi}{1 + c\sin\varphi}.$$

Now we use the substitution

$$\begin{aligned} t &= \tan\frac{\varphi}{2} \\ \frac{2}{1+t^2}\,\mathrm{d}t &= \mathrm{d}\varphi \\ 2\arctan t &= \varphi \\ \sin\varphi &= \frac{2t}{1+t^2} \end{aligned}$$

obtaining

$$\int_0^{\pi} \frac{\mathrm{d}\varphi}{1 + c\sin\varphi} = \int_0^{+\infty} \frac{2\,\mathrm{d}t}{t^2 + 2ct + 1} = \int_0^{+\infty} \frac{2\,\mathrm{d}t}{(t+c)^2 + 1 - c^2}.$$

At this time we have to distinguish between $c < 1$, $c = 1$ and $c > 1$.

- $c = 1$

$$\int_0^{+\infty} \frac{2\,\mathrm{d}t}{(t+1)^2} = 2$$

- $c < 1$

$$\begin{aligned} \frac{2}{1-c^2}\int_0^{+\infty} \frac{\mathrm{d}t}{\left(\frac{t+c}{\sqrt{1-c^2}}\right)^2 + 1} &= \frac{2}{\sqrt{1-c^2}}\left[\arctan\frac{t}{\sqrt{1-c^2}}\right]_c^{+\infty} \\ &= \frac{2}{\sqrt{1-c^2}}\left(\frac{\pi}{2} - \arctan\frac{c}{\sqrt{1-c^2}}\right) \end{aligned}$$

Finally using trigonometric identities

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x, \quad x > 0$$
$$\arctan x = 2\arctan\frac{x}{1+\sqrt{1+x^2}}$$

we obtain the desired result

$$\frac{2}{\sqrt{1-c^2}}\left(\frac{\pi}{2} - \arctan\frac{c}{\sqrt{1-c^2}}\right) = \frac{2}{\sqrt{1-c^2}}\arctan\frac{\sqrt{1-c^2}}{c}$$
$$= \frac{4}{\sqrt{1-c^2}}\arctan\frac{\frac{\sqrt{1-c^2}}{c}}{1+\sqrt{1+\frac{1-c^2}{c^2}}} = \frac{4}{\sqrt{1-c^2}}\arctan\sqrt{\frac{1-c}{1+c}}.$$

- $c > 1$

$$\frac{2}{c^2-1}\int_0^{+\infty}\frac{\mathrm{d}t}{\left(\frac{t+c}{\sqrt{c^2-1}}\right)^2-1} = \frac{2}{\sqrt{c^2-1}}\int_c^{+\infty}\frac{\mathrm{d}\left(\frac{t}{\sqrt{c^2-1}}\right)}{\left(\frac{t}{\sqrt{c^2-1}}\right)^2-1}.$$

The integral of type $\int\frac{\mathrm{d}x}{x^2-1}$ can be computed using the partial fractions

$$\int\frac{\mathrm{d}x}{x^2-1} = \log\sqrt{\left|\frac{x-1}{x+1}\right|} + C.$$

Employing this we can conclude with

$$\frac{2}{\sqrt{c^2-1}}\int_c^{+\infty}\frac{\mathrm{d}\left(\frac{t}{\sqrt{c^2-1}}\right)}{\left(\frac{t}{\sqrt{c^2-1}}\right)^2-1} = \frac{2}{\sqrt{c^2-1}}\left[\log\sqrt{\frac{\frac{t}{\sqrt{c^2-1}}-1}{\frac{t}{\sqrt{c^2-1}}+1}}\right]_c^{+\infty}$$
$$= \frac{2}{\sqrt{c^2-1}}\log(c+\sqrt{c^2-1}).$$

If we now take a look at the second arc which can be parametrized as

$$x_1 = a\cos\varphi$$
$$x_2 = a\sin\varphi - b$$

for $\varphi \in (\pi, 2\pi)$ with the normal vector $\boldsymbol{n} = (\cos\varphi, \sin\varphi)$, we obtain the integral

$$\int_\pi^{2\pi}\frac{\mathrm{d}\varphi}{1-c\sin\varphi}.$$

Since $\sin(\varphi) = -\sin(\varphi-\pi)$, we get

$$\int_\pi^{2\pi}\frac{\mathrm{d}\varphi}{1-c\sin\varphi} = \int_0^\pi\frac{\mathrm{d}\varphi}{1+c\sin\varphi},$$

i.e., the integrals over the two arcs are equal. Summarizing

$$F(\mathcal{S}) = \begin{cases} 4c + \frac{8}{\sqrt{1-c^2}} \arctan \sqrt{\frac{1-c}{1+c}} & c < 1, \\ 8 & c = 1, \\ 4c + \frac{4}{\sqrt{c^2-1}} \log(c + \sqrt{c^2-1}) & c > 1. \end{cases} \tag{4.4}$$

For the stadium we also have

$$A := |\mathcal{S}| = 4ab + \pi a^2$$
$$L := |\partial \mathcal{S}| = 4b + 2\pi a$$
$$\rho_{\mathcal{S}} = a.$$

The stadium is obviously bounded and convex and thus we can use for the comparison the same bounds as for the ellipse since the actual eigenvalues for stadiums are also not known explicitly.

Trivial bound

$$\lambda_1(\mathcal{S}) \le \frac{j_{01}^2}{a^2}$$

PS bound

$$\lambda_1(\mathcal{S}) \le j_{01}^2 \frac{F(\mathcal{S})}{2(4ab + \pi a^2)}$$

PS bound for convex domains

$$\lambda_1(\mathcal{S}) \le j_{01}^2 \frac{b + \pi a}{a(4ab + \pi a^2)}$$

PW bound

$$\lambda_1(\mathcal{S}) \le \frac{\pi^2}{(2b + \pi a)^2} k(p)^2$$

with

$$p = 1 - \frac{\pi(4ab + \pi a^2)}{(2b + \pi a)^2}$$

and $k = k(p)$ be the first zero of the transcendental equation (3.21).

AF conjecture

$$\lambda_1(\mathcal{S}) \le \frac{\pi j_{01}^2}{4ab + \pi a^2} + \pi^2 \frac{(2b + \pi a)^2 - \pi(4ab + \pi a^2)}{(4ab + \pi a^2)^2}$$

We again plot the preceding results (see Figure 4.4). It can be seen that for example the PS bound for convex domains is worse than all the other bounds and conjecture AF (except for the trivial bound) for all the values of the parameter c.

4.1.4 Swiss cross

Finally we use another planar domain called the swiss cross (see Figure 4.5) which is strictly star-shaped with respect to the origin but non-convex. Let $a, b \in \mathbb{R}^+$ and let the swiss cross $\mathcal{C} \subset \mathbb{R}^2$ be the union of the two rectangles $(-b-a, b+a) \times (-a, a)$ and $(-a, a) \times (-b-a, b+a)$. We again set $c := \frac{b}{a} \in [0, +\infty)$. Next we compute the intrinsic quantity $F(\mathcal{C})$.

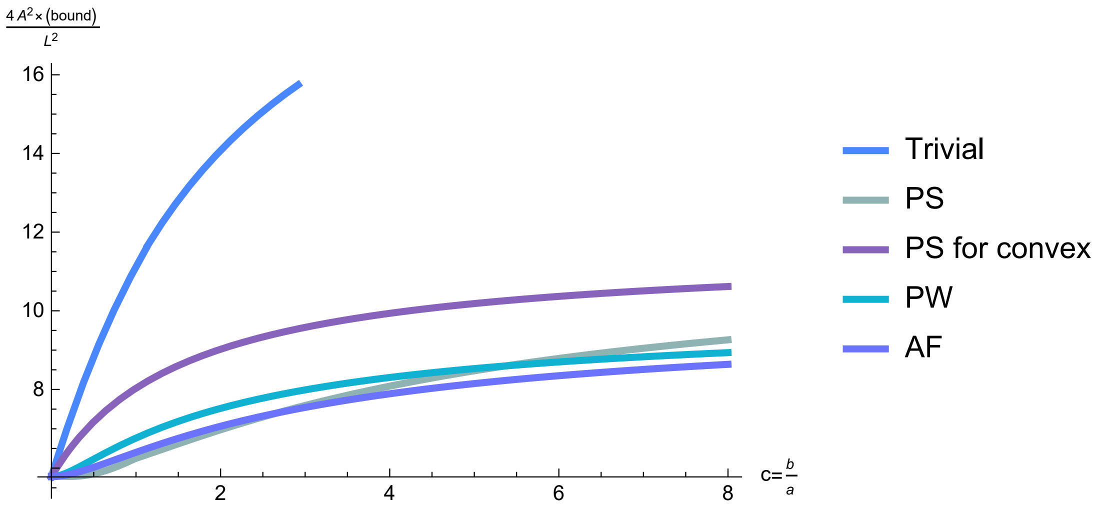


Figure 4.4: Plot of bounds for stadium $\mathcal{S}$ with area A and length of boundary curve L

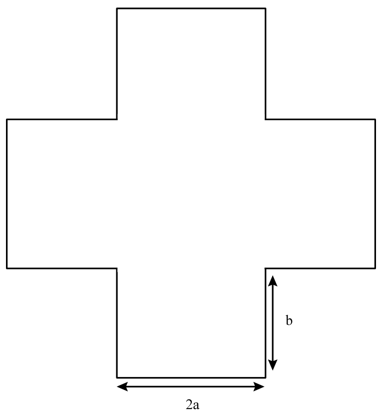


Figure 4.5: Swiss cross with parameters a and b

Recall the computation of the intrinsic quantity for the rectangular parallelepipeds. For each side of the swiss cross we will use the same procedure as for the side of the parallelepiped. Thus we obtain

$$\begin{aligned}\inf_{\xi\in\omega}\int_{\partial\mathcal{C}} h_\xi^{-1} &= \inf_{\xi\in\omega}\left(\int_{-a}^{a}\frac{\mathrm{d}x_2}{b+a-\xi_1}+\int_{a}^{a+b}\frac{\mathrm{d}x_1}{a-\xi_2}+\int_{a}^{a+b}\frac{\mathrm{d}x_2}{a-\xi_1}+\int_{-a}^{a}\frac{\mathrm{d}x_1}{a+b-\xi_2}\right.\\ &+\int_{a}^{a+b}\frac{\mathrm{d}x_2}{a+\xi_1}+\int_{-a-b}^{-a}\frac{\mathrm{d}x_1}{a-\xi_2}+\int_{-a}^{a}\frac{\mathrm{d}x_2}{a+b+\xi_1}+\int_{-a-b}^{-a}\frac{\mathrm{d}x_1}{a+\xi_2}\\ &\left.+\int_{-a-b}^{-a}\frac{\mathrm{d}x_2}{a+\xi_1}+\int_{-a}^{a}\frac{\mathrm{d}x_1}{a+b+\xi_2}+\int_{-a-b}^{-a}\frac{\mathrm{d}x_2}{a-\xi_1}+\int_{a}^{a+b}\frac{\mathrm{d}x_1}{a+\xi_2}\right) \qquad (4.5)\\ &= \inf_{\xi\in\omega}\left(\frac{4a(a+b)}{(a+b)^2-\xi_1^2}+\frac{4ab}{a^2-\xi_2^2}+\frac{4ab}{a^2-\xi_1^2}\right)\\ &= 8\left(\frac{a(a+b)}{(a+b)^2}+\frac{ba}{a^2}\right)=8\frac{1+c+c^2}{1+c}.\end{aligned}$$

For the calculations we will also need

$$\begin{aligned}A &:= |\mathcal{C}| = 8ab+4a^2\\ L &:= |\partial\mathcal{C}| = 8(a+b)\\ \rho_{\mathcal{C}} &= \begin{cases} a+b & b<(\sqrt{2}-1)a\\ \sqrt{2}a & b\geq(\sqrt{2}-1)a.\end{cases}\end{aligned}$$

The swiss cross is obviously bounded but not convex, still instead of the PS bound for convex domains we can use the conjecture C1 formally identical to the PS bound for convex domains. The actual eigenvalues are not known explicitly.

Trivial bound

$$\lambda_1(\mathcal{C}) \leq \frac{j_{01}^2}{\rho_{\mathcal{C}}^2}$$

PS bound

$$\lambda_1(\mathcal{C}) \leq j_{01}^2\frac{1+\frac{b}{a}+\left(\frac{b}{a}\right)^2}{\left(1+\frac{b}{a}\right)(2ab+a^2)}$$

PW bound

$$\lambda_1(\mathcal{C}) \leq \frac{\pi^2}{16(a+b)^2}k(p)^2$$

with

$$p = 1-\frac{\pi(2ab+a^2)}{4(a+b)^2}$$

and $k=k(p)$ be the first zero of the transcendental equation (3.21).

AF conjecture

$$\lambda_1(\mathcal{C}) \leq \frac{\pi j_{01}^2}{8ab+4a^2}+\pi^2\frac{4(a+b)^2-\pi(2ab+a^2)}{(4ab+2a^2)^2}$$

C1 conjecture

$$\lambda_1(\mathcal{C}) \leq j_{01}^2\frac{a+b}{\rho_{\mathcal{C}}(2ab+a^2)}$$

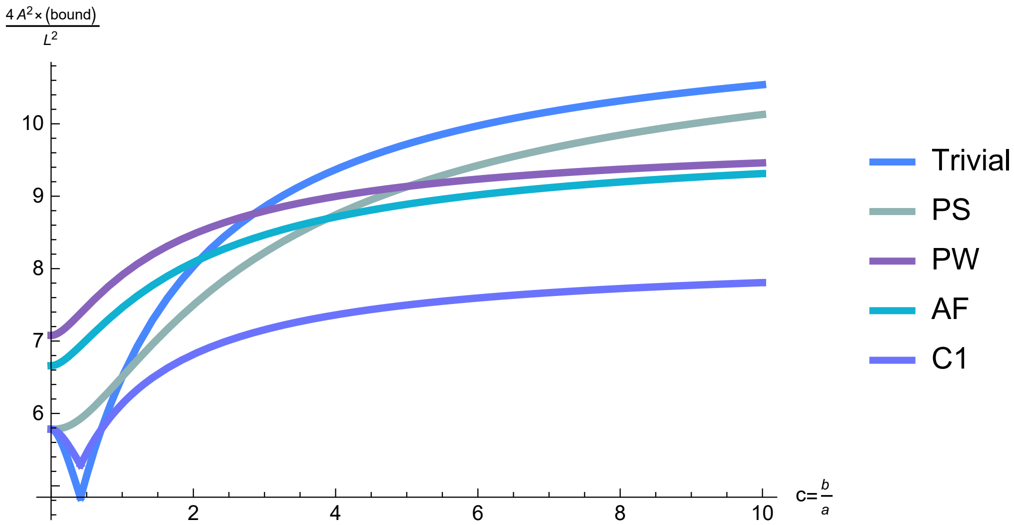


Figure 4.6: Plot of bounds for swiss cross $\mathcal{C}$ with area A and length of boundary curve L

Plotting these results leads to Figure 4.6. We can see that for very small values of parameter c the trivial bound is better than all the others. Also the conjecture C1 is better than all non-trivial bounds for all the values of c. Finally the PS bound acts better than the conjecture AF for $c \in [0, 3.8]$.

4.2 Domains with particular holes

At the end we would like to show some examples of domains on which our own result, Theorem 3.4.5, can be applied. We take the already introduced planar shapes (planar only due to simplicity) and create a hole of size $p > 0$ in them (see Section 3.4). Since all of the others preceding bounds (except for the Trivial bound) demand simply-connected domains, we only plot the dependence of our bound on the size of the domain (as in the previous plots) and on the parameter p.

Before we proceed to the particular shapes of the domains we need to compute the first eigenvalue of the Dirichlet Laplacian for some arbitrary annulus $A_{s,t}$, i.e., we have the following spectral problem (in the polar coordinates)

$$\begin{aligned} -\Delta u &= \lambda u \\ u(s,\varphi) &= 0 \\ u(t,\varphi) &= 0 \\ u(r,0) &= u(r,2\pi) \\ \frac{\partial u}{\partial \varphi}\bigg|_{r,0} &= \frac{\partial u}{\partial \varphi}\bigg|_{r,2\pi} \end{aligned}$$

for all $\varphi \in [0, 2\pi]$ and $r \in (s,t)$. Using the same procedure as in the proof of Remark 3.5.4 we obtain the angular equation

$$\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = -m^2$$

with boundary conditions

$$\Phi(0) = \Phi(2\pi)$$
$$\Phi'(0) = \Phi'(2\pi)$$

and having the harmonic solution

$$\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi)$$

for $m \in \mathbb{Z}$ and we also get the Bessel equation

$$R(\rho) = C_1 J_m(\sqrt{\lambda}\rho) + C_2 Y_m(\sqrt{\lambda}\rho)$$

with the boundary conditions

$$R(s) = R(t) = 0,$$

i.e.,

$$C_1 J_m(\sqrt{\lambda}s) + C_2 Y_m(\sqrt{\lambda}s) = C_1 J_m(\sqrt{\lambda}t) + C_2 Y_m(\sqrt{\lambda}t) = 0.$$

Reducing the constants and choosing $m = 0$ (because we are interested in the first eigenvalue) we get

$$J_0(\sqrt{\lambda}s)Y_0(\sqrt{\lambda}t) = Y_0(\sqrt{\lambda}s)J_0(\sqrt{\lambda}t).$$

The first eigenvalue of $A_{s,t}$ is the first zero of this equation.

4.2.1 Rectangles with hole

Let $\mathcal{R}_2 = (-\frac{a_1}{2}, \frac{a_1}{2}) \times (-\frac{a_2}{2}, \frac{a_2}{2})$ be the rectangle with sides a_1 and a_2, $a_1 < a_2$, as in Subsection 4.1.1. Let $\xi = 0$ (we choose the center of our hole to be the origin) and $p \in (0,1)$ be the parameters of the domain $\mathcal{R}_{20,p}$ created from $\mathcal{R}_2$ (see Section 3.4). The annulus $A_{a,b}$ from Theorem 3.4.5 can be found using Remark 3.4.6

$$b = \sqrt{\frac{|\mathcal{R}_{20,p}|}{(1-p^2)|B_1|}} = \sqrt{\frac{a_1 a_2 (1-p^2)}{(1-p^2)\pi}} = \sqrt{\frac{a_1 a_2}{\pi}}$$
$$a = bp.$$

At the same time we have from (4.1)

$$F_0(\mathcal{R}_{20,p}) = F(\mathcal{R}_2) = 4\frac{a_1^2 + a_2^2}{a_1 a_2}.$$

Hence we obtain the following bound

$$\lambda_1(\mathcal{R}_{20,p}) \leq \lambda_1(A_{a,b})\frac{b^2 F_0(\mathcal{R}_{20,p})}{d|B_b|} = \lambda_1(A_{a,b})\frac{2(a_1^2 + a_2^2)}{\pi a_1 a_2}$$

where $\lambda_1(A_{a,b})$ is the first zero of the equation

$$J_0(\sqrt{\lambda}a)Y_0(\sqrt{\lambda}b) = Y_0(\sqrt{\lambda}a)J_0(\sqrt{\lambda}b). \tag{4.6}$$

Plot of this bound for $p \in (0,1)$ and $c := \frac{1}{a_2} \in (0,1)$ can be seen on Figure 4.7. We can see that for $p \approx 0$ (the hole is very small) we get very similar behavior as for the PS bound for rectangles without hole.

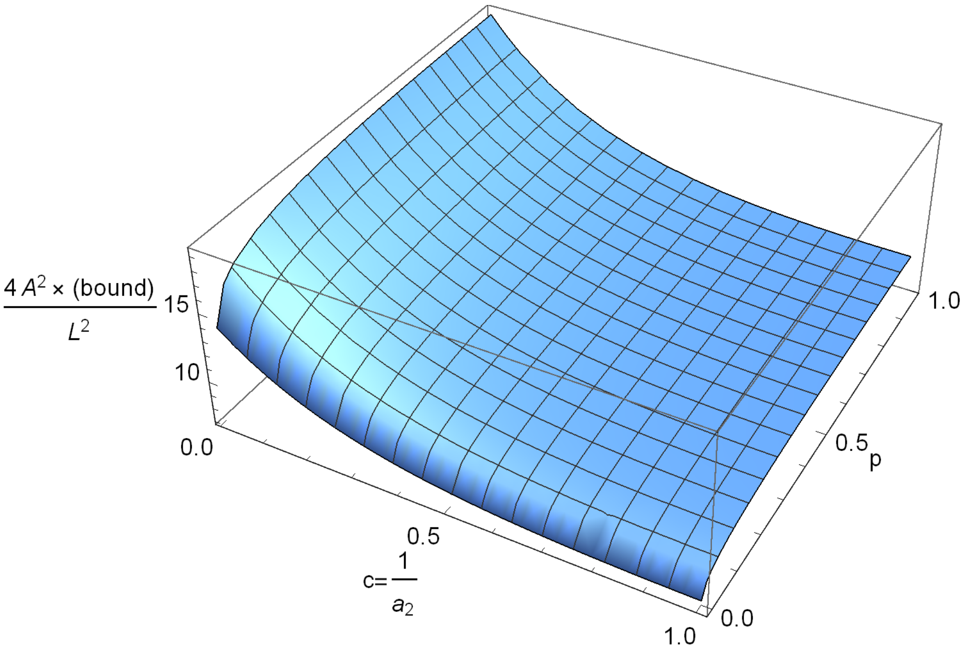


Figure 4.7: Generalized PS bound for rectangle with sides $a_1 = 1$ and a_2 and with hole of size p. A is area and L is length of boundary of rectangle

4.2.2 Ellipses with hole

Let $\mathcal{E}_2 = \left\{ x \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} < 1 \right\}$ be an ellipse with axis a_1 and a_2,$a_1 > a_2$. Again let $\xi = 0$ and $p \in (0,1)$ be the parameters of the domain with the hole $\mathcal{E}_{20,p}$. First we find the annulus $A_{a,b}$

$$b = \sqrt{\frac{|\mathcal{E}_{20,p}|}{(1-p^2)|B_1|}} = \sqrt{\frac{\pi a_1 a_2 (1-p^2)}{(1-p^2)\pi}} = \sqrt{a_1 a_2}$$

$$a = bp.$$

Also using (4.3) we have

$$F_0(\mathcal{E}_{20,p}) = F(\mathcal{E}_2) = \pi \frac{a_1^2 + a_2^2}{a_1 a_2}$$

and thus the bound is

$$\lambda_1(\mathcal{E}_{20,p}) \leq \lambda_1(A_{a,b}) \frac{b^2 F_0(\mathcal{E}_{20,p})}{d|B_b|} = \lambda_1(A_{a,b}) \frac{a_1^2 + a_2^2}{2a_1 a_2}$$

where $\lambda_1(A_{a,b})$ is again the first zero of equation (4.6). Plot for $p \in (0,1)$ and $c := \frac{1}{a_1} \in (0,1)$ can be seen on Figure 4.8.

4.2.3 Stadium with hole

Let the stadium $\mathcal{S}$ be the union of the rectangle $(-a_2, a_2) \times (-a_1, a_1)$ and two discs of radius a_1 centered at the points $(-a_2, 0)$ and $(a_2, 0)$. Let $c := \frac{a_2}{a_1} \in [0, +\infty)$, $\xi = 0$ and $p \in (0,1)$. Assume

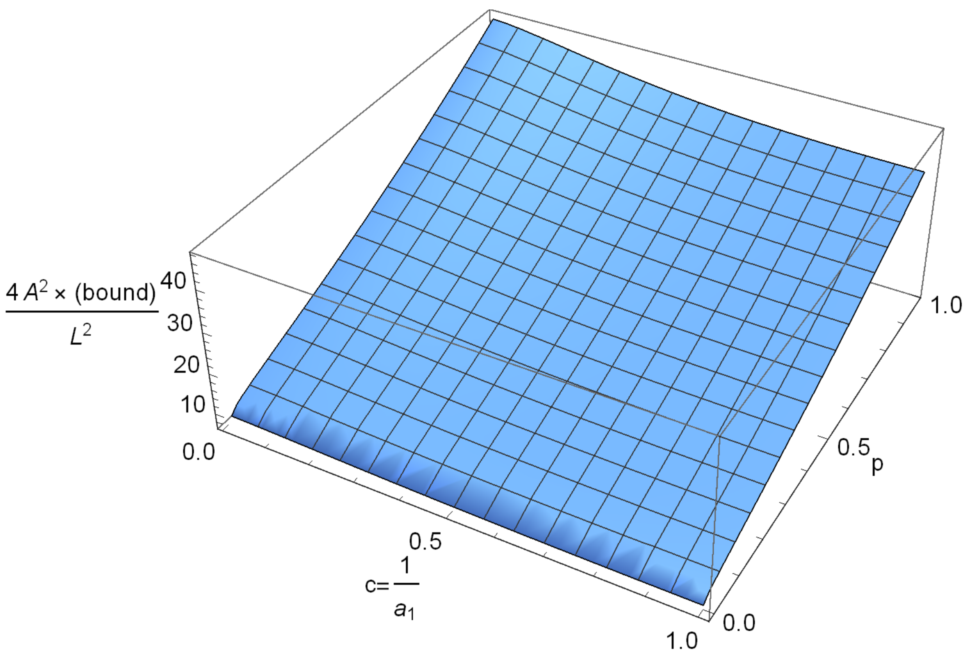


Figure 4.8: Generalized PS bound for ellipse with axes a_1 and $a_2 = 1$ and with hole of size p. A is area and L is length of boundary curve of ellipse

the domain $\mathcal{S}_{0,p}$, i.e., the stadium $\mathcal{S}$ with the hole of size p. As in the preceding subsections we find the annulus $A_{a,b}$

$$b = \sqrt{\frac{|\mathcal{S}_{0,p}|}{(1-p^2)|B_1|}} = \sqrt{\frac{(4ab+\pi a^2)(1-p^2)}{(1-p^2)\pi}} = \sqrt{\frac{4ab+\pi a^2}{\pi}}$$
$$a = bp.$$

The intrinsic quantity $F_0(\mathcal{S}_{0,p})$ is again equal to the intrinsic quantity of the stadium $F(\mathcal{S})$, see (4.4). The obtained bound is then

$$\lambda_1(\mathcal{S}_{0,p}) \leq \lambda_1(A_{a,b})\frac{b^2 F_0(\mathcal{S}_{0,p})}{d|B_b|} = \lambda_1(A_{a,b})\frac{F_0(\mathcal{S}_{0,p})}{2\pi}.$$

This bound is plotted for $a_1 = 1$, i.e., $c = a_2 \in (0, 8)$ and $p \in (0, 1)$ on Figure 4.9.

4.2.4 Swiss cross with hole

In the end we introduce the swiss cross with hole. Let the swiss cross $\mathcal{C}$ be the union of the two rectangles $(-a_2 - a_1, a_2 + a_1) \times (-a_1, a_1)$ and $(-a_1, a_1) \times (-a_2 - a_1, a_2 + a_1)$. Let $c := \frac{a_2}{a_1} \in [0, +\infty)$, the center of the hole $\xi = 0$ and $p \in (0, 1)$. So we have the hollow domain $\mathcal{C}_{0,p}$. The required annulus $A_{a,b}$ is

$$b = \sqrt{\frac{|\mathcal{C}_{0,p}|}{(1-p^2)|B_1|}} = \sqrt{\frac{(8ab+4a^2)(1-p^2)}{(1-p^2)\pi}} = \sqrt{\frac{8ab+4a^2}{\pi}}$$
$$a = bp.$$

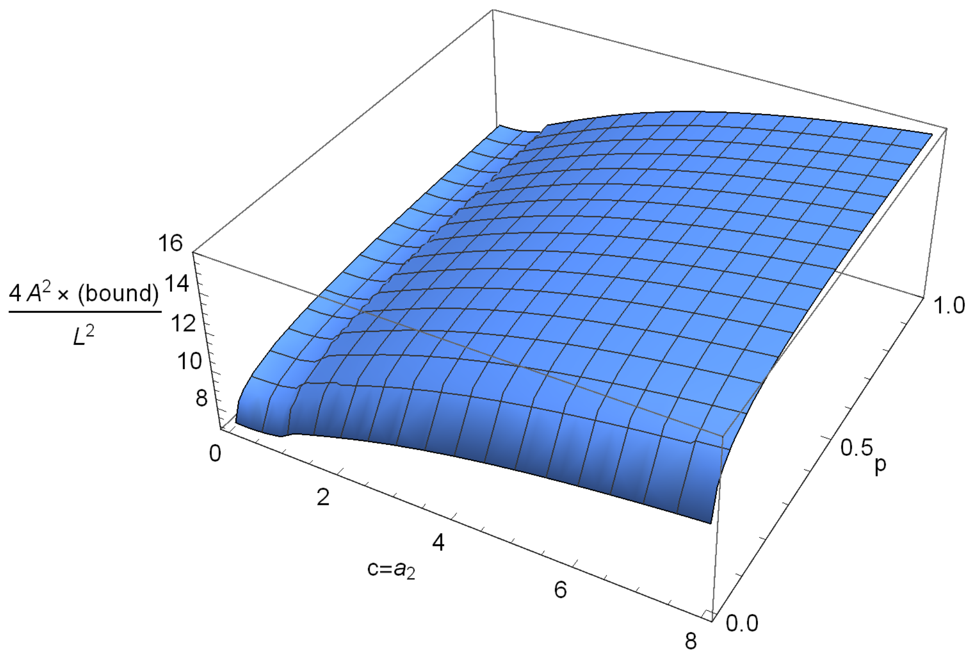


Figure 4.9: Generalized PS bound for stadium with parameters $a_1 = 1$ and a_2 and with hole of size p. A is area and L is length of boundary curve of stadium

Also its intrinsic quantity $F_0(\mathcal{C}_{0,p})$ can be written as

$$F_0(\mathcal{C}_{0,p}) = F(\mathcal{C}) = 8\frac{1+c+c^2}{1+c},$$

see (4.5). Therefore we have the bound

$$\lambda_1(\mathcal{C}_{0,p}) \le \lambda_1(A_{a,b})\frac{b^2 F_0(\mathcal{C}_{0,p})}{d|B_b|} = \lambda_1(A_{a,b})\frac{4(1+c+c^2)}{\pi(1+c)}.$$

The plot for $a_1 = 1$, i.e., $c = a_2 \in (0, 8)$ and $p \in (0, 1)$ is on Figure 4.10.

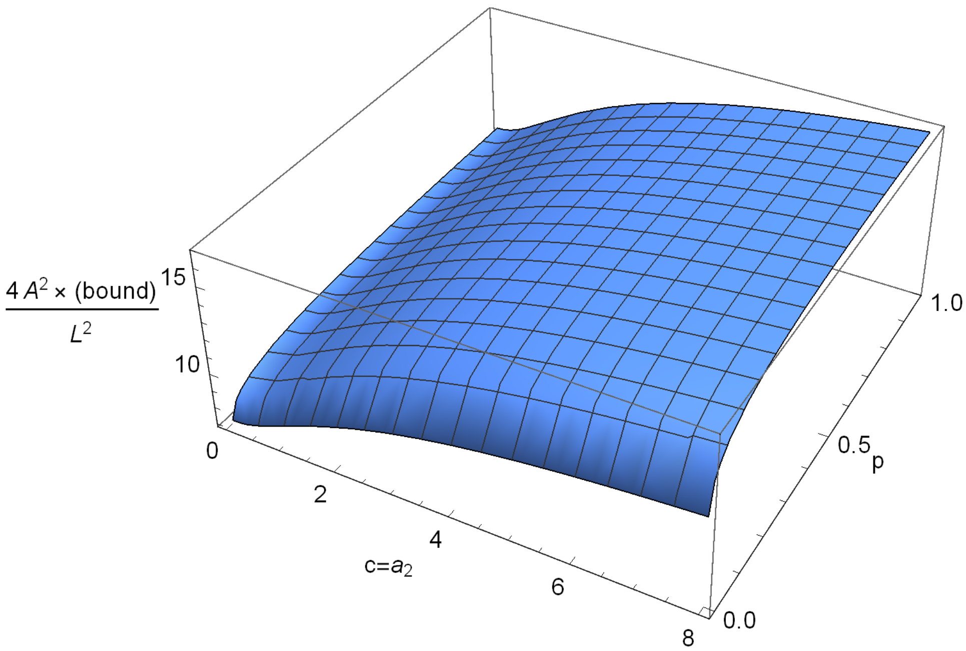


Figure 4.10: Generalized PS bound for swiss cross with parameters $a_1 = 1$ and a_2 and with hole of size p. A is area and L is length of boundary curve of swiss cross

Conclusion

In the thesis we introduced the Laplace operator with the Dirichlet boundary conditions and showed its huge importance in the musical theory. Then we correctly defined the self-adjoint Dirichlet Laplacian on bounded domains and stated some of its spectral properties.

We used the min-max principle and the shrinking and parallel coordinates to obtain the two non-trivial and sharp (for balls) upper bounds for the first eigenvalue of the Dirichlet Laplacian, the Pólya and Szëgo's (in arbitrary dimension) and Payne and Weinberger's (planar) bound. Moreover we introduced our own result, the generalization of the Pólya and Szëgo's bound for particular not simply-connected domains. We also stated the trivial bound, two conjectures and one lower bound, the Faber-Krahn inequality.

In the end we applied the obtained bounds to some types of domains (particularly rectangular parallelepipeds, ellipsoids, stadiums and swiss crosses) and compared them. And since none of the others bounds is applicable on not simply-connected domains which we created from the preceding types, we plotted the dependence of our bound on the size of the domain and the size of the hole.

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