České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Katedra fyziky
Obor: Matematická fyzika

Vlastnosti a klasifikace třídy řešitelných Lieových algeber s vícerozměrnými centry

Properties and classification of a class of solvable Lie algebras with higher dimensional centres

DIPLOMOVÁ PRÁCE

Vypracoval: Bc. Jindřich Prokop
Vedoucí práce: doc. Ing. Libor Šnobl, Ph.D.
Rok: 2016

Prohlášení

Prohlašuji, že jsem svoji diplomovou práci vypracoval samostatně a použil jsem pouze podklady uvedené v přiloženém seznamu.

Nemám závažný důvod proti užití tohoto školního díla ve smyslu §60 Zákona č.121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Praze dne ....

Jindřich Prokop

Poděkování

Děkuji vedoucímu práce doc. Liboru Šnoblovi za trpělivost a vydatnou pomoc při tvorbě diplomové práce.

Bc. Jindřich Prokop

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS13/217/OHK4/3T/14 and grant No. SGS16/239/OHK4/3T/14.

Název práce: **Vlastnosti a klasifikace třídy řešitelných Lieových algeber s vícerozměrnými centry**
Autor: Bc. Jindřich Prokop
Obor: Matematická fyzika
Druh práce: Diplomová práce
Vedoucí práce: doc. Ing. Libor Šnobl, Ph.D.
Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Abstrakt: Práce se zabývá hledáním řešitelných rozšíření zvolené posloupnosti nilpotentních Lieových algeber liché dimenze. V první části je prozkoumána nejjednodušší nilpotentní, tj. sedmirozměrná, algebra a její řešitelná rozšíření, ve druhé je pak rozebrán případ obecného členu zvolené posloupnosti. Poslední část obsahuje zobecněné Casimirovy invarianty nalezených algeber. Cílem práce bylo přispět k programu klasifikace řešitelných Lieových algeber vyšších dimenzí.
Klíčová slova: nilpotentní Lieovy algebry, řešitelné Lieovy algebry, řešitelná rozšíření, klasifikace řešitelných Lieových algeber

Title: **Properties and classification of a class of solvable Lie algebras with higher dimensional centres**
Author: Bc. Jindřich Prokop

Abstract: The aim of this work is to find all solvable extensions of the given series of nilpotent Lie algebras of odd dimension. The first part focuses on the simplest case of the seven-dimensional nilpotent algebra and its solvable extensions. The second part generalizes the results of the former to an arbitrary element of the chosen series. The generalized Casimir invariants of the constructed algebras are given in the last part. Our goal was to contribute to the classification of solvable Lie algebras of higher dimensions.
Key words: nilpotent Lie algebras, solvable Lie algebras, solvable extensions, classification of solvable Lie algebras

Contents

Introduction **15**

1 Theoretical Background and Series of Nilradicals **17**

1.1 Elementary Notions and Notation 17
1.2 Solvable Extensions of Nilpotent Lie Algebra 21
1.3 Previous Results 22
1.4 Series of Nilradicals 22

2 Solvable Extensions of Seven-dimensional Nilradical **23**

2.1 Properties of $\mathfrak{r}_7$ 23
2.1.1 Inner Derivations 24
2.1.2 Derived Series, Flag of Ideals 24
2.1.3 Automorphisms 25
2.1.4 Generic Outer Derivation 27
2.2 Extensions 28
2.2.1 Triangular Form of Derivation 28
2.2.2 Outer Derivation Classes 30
2.2.3 Extensions by Two Vectors 49
2.2.4 Extensions by Three Vectors 65

3 Solvable Extensions of Nilradical of Arbitrary Dimension **69**

3.1 Properties of $\mathfrak{r}_{2k-1}$ 69
3.1.1 Inner Derivations 69
3.1.2 Derived Series, Flag of Ideals 70
3.1.3 Automorphisms 70
3.1.4 Generic Outer Derivation 71
3.2 Extensions 72
3.2.1 Extensions by One Element 72
3.2.2 Extensions by Two Elements 90
3.2.3 Extensions by Three Elements 101

4 Generalized Casimir Invariants **105**
4.1 Method of Characteristics 106
4.2 Method of Moving Frames 107
4.3 The Invariants 108
4.3.1 Invariants of $\mathfrak{r}_{2k-1}$ 108
4.3.2 Invariants of Extensions by One Element 109
4.3.3 Invariants of Extension by Two Elements 123
4.3.4 Invariants of Extension by Three Elements 132

Conclusion **135**

Bibliography **137**

Introduction

In the last third of the nineteenth century a new mathematical theory gradually emerged. In the modern terminology several central notions and the theory itself carry the name of its father, the Norwegian mathematician Sophus Lie. In the beginning, the main motivation was to construct continuous groups describing the symmetries of differential equations. The (smooth) manifolds that are simultaneously groups, today called Lie groups, were the answer. Many results were obtained by Lie himself, arguably, the most important one being a discovery of an object inherently tied to any Lie group – a tangent space to the group in group identity with additional structure induced by the group law – Lie bracket. Such vector space is now called the Lie algebra and it allows one to learn about the properties of the group, or more generally about the problem which was initially described by the group, while working with the linear space instead of the manifold. Furthermore, since the algebra is a linear space, it is possible to construct representations of both the group and the algebra on the algebra and on its dual space. Consequently, the Lie theory proves to be very rich and applicable to many different problems with symmetries of differential equations being only one of them.

When a Lie algebra arises in an application on a given problem, it is useful to identify it as being isomorphic to some known algebra. Thus the classification of Lie algebras is of importance for numerous fields of mathematics and physics. According to Levi's theorem every Lie algebra $\mathfrak{g}$ is decomposable into a semidirect sum of its radical $\mathfrak{s}_{\mathfrak{g}}$ and a semisimple Lie algebra $\mathfrak{p}$ usually called Levi factor, that is

$$\mathfrak{g} = \mathfrak{p} \ni\!\!\!+ \mathfrak{s}_{\mathfrak{g}}, \qquad ([\mathfrak{p}, \mathfrak{s}_{\mathfrak{g}}] \subset \mathfrak{s}_{\mathfrak{g}}, \quad \mathfrak{p} \cong \mathfrak{g}/\mathfrak{s}_{\mathfrak{g}}).$$

Via Levi's decomposition the problem of classifying all Lie algebras is simplified to classification of all semisimple ones, all solvable ones and all possible combinations of these in the sense of Levi's decomposition, that is all possible actions of semisimple algebras on solvable algebras. The semisimple algebras were fully classified by the end of the nineteenth century by É. Cartan and

W. Killing. On the other hand, more than a century later there is no complete classification of solvable algebras, nor is it in principle possible to obtain complete classification to isomorphism classes. Partial classification begun with attempts on complete classification of algebras with low fixed dimension [7, 8, 9, 14, 18, 20, 21], a compact list of all solvable algebras of maximal dimension six is provided in [29]. However, this approach stops being feasible for rather low dimensions; in [19] more than twenty thousand different classes of nine-dimensional nilpotent Lie algebras with maximal abelian ideal of dimension seven were found. An alternative approach was concocted consisting of constructing a series of nilpotent Lie algebras and consequent classification of all solvable algebras with nilradical equal to an element of the series. This approach was devised and first used in [13] and has been since applied in [1, 10, 11, 16, 17, 22, 23, 24, 25, 26, 27, 28] (the list is not exhaustive). This work is a continuation of [12] exploring odd-dimensional counterpart of the series of nilradicals of even dimensions investigated in [12].

The first chapter gives a basic summary of underlying theory and notation used, it explains how the solvable extensions to a nilpotent algebra can be found and it contains the definition of the series of nilpotent algebras. The second chapter is the analysis of the task at hand for the first element of the series, a seven-dimensional nilpotent algebra. Its solvable extensions are given there. The results of the second chapter are then generalized in the following chapter to an arbitrary dimension of the element of the series. The generalized Casimir invariants for most of the constructed algebras are given in the last chapter.

Chapter 1

Theoretical Background and Series of Nilradicals

Summary of basic concepts of Lie algebra theory along with the notation used within the text is given, detailed introduction into the topic can be found e.g. in [3, 6, 15]. The process for finding the solvable Lie algebras with a given nilradical used in this work is outlined. The central idea of the process is construction of a series of nilpotent Lie algebras with similar structure and consequent classification of all solvable Lie algebras with its nilradical equal to some element of the series. The series of nilradicals is defined at the end of this chapter. Further, we recall the results already obtained in [12] for a nilradicals series, which is in a sense explained below an even-dimensional nilradicals counterpart to the task of finding the extensions to the odd-dimensional nilradicals here.

1.1 Elementary Notions and Notation

Throughout the text Einstein summation convention is used; index ι always runs over $\{1, 2\}$, range for any other index is either explicitly given or evident from the context. $\mathbb{F}$ is a field; we assume it to be either real ($\mathbb{R}$), or complex ($\mathbb{C}$) numbers. Furthermore, we define the following one-element sets

$$\theta_i \equiv \Big\{(0, \dots, 0)\Big\} \subset \mathbb{F}^i.$$

An abstract Lie algebra $\mathfrak{g}$ is a vector space with additional structure given by a bilinear map

$$[\bullet, \bullet] \equiv \mathrm{ad} : \ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} : \ (x, y) \longmapsto [x, y] \equiv \mathrm{ad}_y(x),$$

such that

$$[x,y] = -[y,x] \qquad \forall x,y \in \mathfrak{g}, \tag{1.1}$$
$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \qquad \forall x,y,z \in \mathfrak{g}. \tag{1.2}$$

This map is called Lie bracket, it is antisymmetric (1.1), and satisfies Jacobi identity (1.2). The image $[x,y]$ is called a commutator of x and y. The image of $\mathfrak{h} \times \mathfrak{h}'$ under the Lie bracket for any $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$, or else their commutator, is denoted by

$$[\mathfrak{h}, \mathfrak{h}'] \equiv \operatorname{span}\{[x,y] \mid x \in \mathfrak{h},\ y \in \mathfrak{h}'\}.$$

A vector subspace of $\mathfrak{g}$ closed under the bracket is called a subalgebra of $\mathfrak{g}$. The set of all vectors that commute with any vector from the algebra, that is

$$\mathfrak{z}_{\mathfrak{g}} \equiv \mathfrak{z}_{\mathfrak{g}}^1 \equiv \{x \in \mathfrak{g} \mid [x,y] = 0 \quad \forall y \in \mathfrak{g}\}, \tag{1.3}$$

is the centre of $\mathfrak{g}$, while the algebra $\mathfrak{g}$ may be dropped from the notation if evident from the context. Any subalgebra $\mathfrak{i}$ of the Lie algebra $\mathfrak{g}$ such that

$$[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$$

is an ideal of $\mathfrak{g}$. It is evident that $\mathfrak{g}$, $\{0\}$ and $\mathfrak{z}(\mathfrak{g})$ are always ideals of $\mathfrak{g}$. Furthermore, the commutator of two ideals is an ideal as well. Thus we can recursively define characteristic series of Lie algebra $\mathfrak{g}$ in the following way. The derived series is defined as

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g}, \\ \mathfrak{g}^{(k+1)} &= [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \qquad \forall k \in \mathbb{N}_0, \end{aligned} \tag{1.4}$$

the lower central series is given by

$$\begin{aligned} \mathfrak{g}^1 &= \mathfrak{g}, \\ \mathfrak{g}^{k+1} &= [\mathfrak{g}^k, \mathfrak{g}] \qquad \forall k \in \mathbb{N}, \end{aligned} \tag{1.5}$$

and the upper central series is defined as

$$\begin{aligned} \mathfrak{z}^1 &= \text{the centre defined above}, \\ \mathfrak{z}^{k+1} &= \{x \in \mathfrak{g} \mid [x,y] \in \mathfrak{z}^k \quad \forall y \in \mathfrak{g}\}. \end{aligned} \tag{1.6}$$

Lie algebra with $\{0\}$ and itself as its sole ideals is called simple. Lie algebra whose derived (lower central) series terminates at some point is called solvable (nilpotent). Every nilpotent algebra is automatically solvable. If $\mathfrak{s}, \mathfrak{s}'$ $(\mathfrak{r}, \mathfrak{r}')$ are solvable (nilpotent) ideals, then $\mathfrak{s} + \mathfrak{s}'$ $(\mathfrak{r} + \mathfrak{r}')$ is a solvable (nilpotent) ideal as well. The maximal solvable (nilpotent) ideal $\mathfrak{s}_\mathfrak{g}$ ($\mathfrak{r}_\mathfrak{g}$) of the given Lie algebra $\mathfrak{g}$ is called the radical (nilradical) of $\mathfrak{g}$. Lie algebra with vanishing radical (and thus with vanishing nilradical) is called semisimple.

Let us consider a subspace $\mathfrak{h}$ and any subset $\mathfrak{h}'$ of $\mathfrak{g}$. Then we define the normalizer of $\mathfrak{h}$ in $\mathfrak{h}'$ as

$$N_{\mathfrak{h}'}(\mathfrak{h}) \equiv \{x \in \mathfrak{h}' \mid [x, y] \in \mathfrak{h} \quad \forall y \in \mathfrak{h}\}. \tag{1.7}$$

Additionally, if $\mathfrak{h}$ is a subalgebra, we may define the centralizer of $\mathfrak{h}'$ in $\mathfrak{h}$ as

$$Z_{\mathfrak{h}'}(\mathfrak{h}) \equiv \{x \in \mathfrak{h}' \mid [x, y] = 0 \quad \forall y \in \mathfrak{h}\}. \tag{1.8}$$

A linear mapping Φ from the Lie algebra $\mathfrak{g}$ onto itself respecting its Lie bracket, that is

$$\Phi([x, y]) = [\Phi(x), \Phi(y)],$$

is an automorphism of $\mathfrak{g}$. The set of all automorphisms of $\mathfrak{g}$ with composition as a group law forms a Lie group $\mathrm{Aut} \equiv \mathrm{Aut}(\mathfrak{g})$. Let V be a vector space. Then any map

$$\rho : \ \mathfrak{g} \longrightarrow \mathcal{L}(V) : \ x \longmapsto \rho(x) : \qquad \rho([x, y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x),$$

where $\mathcal{L}(V)$ is a space of linear operators on V, is called a representation of $\mathfrak{g}$ on V. The Lie bracket on any given Lie algebra $\mathfrak{g}$ allows us to define an important representation for which the role of the vector space V is assumed by $\mathfrak{g}$ itself and the mapping is given by

$$\mathrm{ad} : \ \mathfrak{g} \longrightarrow \mathcal{L}(\mathfrak{g}) : \ x \longmapsto \mathrm{ad}_x \equiv [\bullet, x]. \tag{1.9}$$

A linear mapping D from the Lie algebra $\mathfrak{g}$ into itself acting in the following way

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

with respect to the bracket is a derivation of $\mathfrak{g}$. All derivations of $\mathfrak{g}$ with Lie bracket given by

$$[D, D'] = D \circ D' - D' \circ D \qquad \forall D, D' \text{ derivation of } \mathfrak{g}$$

form a Lie algebra $\mathfrak{Der} \equiv \mathfrak{Der}(\mathfrak{g})$. This algebra can be obtained as Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ and contains a subalgebra of inner derivations

$$\mathfrak{Inn} \equiv \mathfrak{Inn}(\mathfrak{g}) \equiv \{D \mid \exists x \in \mathfrak{g} : \ D = \mathrm{ad}_x\}; \tag{1.10}$$

any derivation from $\mathfrak{Der} \setminus \mathfrak{Inn}$ is an outer derivation.

An important result for classification of Lie algebras is the Levi theorem. Having a finite-dimensional Lie algebra $\mathfrak{g}$, there is unique (up to automorphisms of $\mathfrak{g}$) semisimple algebra $\mathfrak{p}$ such that

$$\begin{aligned} &\mathfrak{g} = \mathfrak{p} \niplus \mathfrak{s}_{\mathfrak{g}}, \text{ where} \\ &\mathfrak{p} \cong \mathfrak{g}/\mathfrak{s}_{\mathfrak{g}}. \end{aligned}$$

This algebra is called Levi factor of $\mathfrak{g}$ and the corollary to this fact is that to classify the Lie algebras of finite dimension it suffices to classify all semisimple and solvable Lie algebras and all possible linear actions of a semisimple algebra $\mathfrak{p}$ on a solvable algebra $\mathfrak{s}$, such that $\mathfrak{p} + \mathfrak{s}$ is a Lie algebra with $\mathfrak{s}$ as its radical. The semisimple algebras were fully classified, while the complete classification of solvable Lie algebras into isomorphism classes is not possible in principle. The next section explains an approach to partial classification of solvable Lie algebras with a given nilradical.

There are another three significant representations connected with a given Lie algebra, apart from the adjoint representation of the Lie Algebra $\mathfrak{g}$ on itself defined in (1.9). Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Using the left and the right action of G on itself

$$\begin{aligned} L:\ G \times G \longrightarrow G:\ (g,h) \longmapsto L_g(h) \equiv gh, \\ R:\ G \times G \longrightarrow G:\ (g,h) \longmapsto R_g(h) \equiv hg, \end{aligned} \tag{1.11}$$

we can define the adjoint representation Ad of the group G on $\mathfrak{g}$ as

$$\mathrm{Ad}:\ G \longrightarrow \mathcal{L}(\mathfrak{g}):\ g \longmapsto \mathrm{Ad}_g \equiv L_{g*} \circ R_{g^{-1}*}. \tag{1.12}$$

If follows that

$$\left.\begin{aligned} \mathrm{e}^{\mathrm{Ad}_g x} &= g\ \mathrm{e}^x g^{-1} \\ \mathrm{Ad}_{\mathrm{e}^{tx}} &\equiv \mathrm{e}^{t\mathrm{ad}_x} \end{aligned}\right. \qquad \forall x \in \mathfrak{g}.$$

This allows to view the adjoint representation of $\mathfrak{g}$ from a different perspective. Letting Ad act on one-parametric subgroups of G, we can obtain ad as

$$\mathrm{ad}_x y = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\mathrm{e}^{tx}} y \qquad \forall x, y \in \mathfrak{g}. \tag{1.13}$$

Two more representation can be obtained if we take the dual space $\mathfrak{g}^*$ into account. The coadjoint representation of G on $\mathfrak{g}^*$, $\mathrm{Ad}^*:\ G \longrightarrow \mathcal{L}(\mathfrak{g}^*)$, is given by

$$\mathrm{Ad}^*_g \alpha = \alpha \circ \mathrm{Ad}_{g^{-1}} \qquad \forall g \in G,\ \forall \alpha \in \mathfrak{g}^*. \tag{1.14}$$

Finally, through the coadjoint representation of the group G we define the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ as

$$\mathrm{ad}^*_x \alpha = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\mathrm{e}^{tx}} \alpha \qquad \forall x \in \mathfrak{g},\ \forall \alpha \in \mathfrak{g}^*, \tag{1.15}$$

which is equivalent to

$$(\mathrm{ad}_x^*\alpha)y = -\alpha([x, y]). \tag{1.16}$$

1.2 Solvable Extensions of Nilpotent Lie Algebra

Let $\mathfrak{r}$ be a nilpotent Lie algebra with the basis $(e_i)_1^n$ and the commutation relations

$$[e_i, e_j] = c_{ij}{}^k e_k.$$

We are interested in finding solvable Lie algebras such that $\mathfrak{r}$ is their nilradical. Let us assume that $\mathfrak{s}$ is a solvable Lie algebra with the basis $\mathrm{span}(e_i, f_a)_{i=1,a=1}^{n,m}$, nilradical $\mathfrak{r}$, and ad-action of elements f_a of the form

$$\begin{aligned} [e_i, f_a] &= (\mathbb{D}_a)_i{}^k e_k \qquad && \forall i \in \{1, \dots, n\},\ \forall a \in \{1, \dots, m\}, \\ [f_a, f_b] &= \gamma_{ab}{}^k e_k && \forall a, b \in \{1, \dots, m\}. \end{aligned} \tag{1.17}$$

The requirement of Jacobi identities (1.2) for (e_i, e_j, f_a) for all $i, j \in \{1, \dots, n\}$ is equivalent to $\mathbb{D}_a$ being a derivation of the nilpotent algebra $\mathfrak{r}$. Let $\mathfrak{s}_1$ be an extension of $\mathfrak{r}$ by one element with the basis $(e_i, f_1)_1^n$. Then assuming that $\mathfrak{s}_1$ is solvable Lie algebra, there are two possible nilradicals of $\mathfrak{s}_1$. Either $\mathfrak{s}_1$ is nilpotent and thus it is its own nilradical or it is not nilpotent and $\mathfrak{r}$ is its nilradical. We are interested in the latter, which is equivalent to $\mathbb{D}_1$ being an outer non-nilpotent derivation of $\mathfrak{r}$. Using this fact, we can look for extensions of a nilpotent Lie algebra $\mathfrak{r}$ by one element by finding its outer derivations. Given two outer derivations, we have two corresponding extensions, which are isomorphic if and only if the derivations are from the same class w. r. t. addition from the algebra of inner derivations $\mathfrak{Inn} \equiv \mathfrak{Inn}(\mathfrak{r})$, multiplication by number from the field $\mathbb{F}$, and conjugation by any automorphism of $\mathfrak{r}$. Thus, classifying the classes of solvable extensions by one element w. r. t. isomorphisms is equivalent to classifying these outer derivation classes. If we wish to find extensions by more elements, we must ensure that the structure constants γ_{ab}^k are such that the Jacobi identities still hold. This requires that the commutators of $\mathbb{D}_i, \mathbb{D}_j$ fall into the algebra $\mathfrak{Inn}$. We also demand that the nilradical of the extension is not larger than $\mathfrak{r}$, which is equivalent to any two derivations being linearly nil-independent. The process of finding extensions by two elements is described in more detail in the beginning of section 2.2.3 and the generalization to arbitrary number of extending elements is straightforward.

1.3 Previous Results

In the past work [12], we have found all solvable extensions of a series of nilpotent algebras $(\mathfrak{r}_{2k})_3^\infty$ as described in the section (1.2), where $\mathfrak{r}_{2k} \equiv \text{span}(e_i)_1^{2k}$ with the sole nonvanishing commutators

$$[e_i, e_{2k}] = e_{i-2} \qquad \forall i \in \{3, \dots, 2k-1\}. \tag{1.18}$$

There were no solvable extensions by more than three elements, exactly one extension (class) by three elements, $2k + 12$ families of extensions by two elements with varying number of continuous parameters and many more extension classes/families of classes of extensions by one element.

1.4 Series of Nilradicals

In the two following chapters, we seek extensions to nilpotent algebras $\mathfrak{r}_{2k-1} \equiv \text{span}\{e_i\}_1^{2k-1}$ with commutation relations given by

$$[e_i, e_{2k-1}] = e_{i-2} \qquad \forall i \in \{3, \dots, 2k-2\} \tag{1.19}$$

for any $k \geq 4$. Comparing the relations (1.19) with the relations (1.18), it is evident that the structure of the elements of the series $(\mathfrak{r}_{2k})$ from the preceding section and of the series $(\mathfrak{r}_{2k-1})$ is rather similar. Together they can be combined in one series $(\mathfrak{r}_6, \mathfrak{r}_7, \mathfrak{r}_8, \dots)$. There is one important difference affecting the process of finding the extensions. In the case of the nilradicals $\mathfrak{r}_{2k}$ of even dimension, the sets

$$\text{span}(e_1), \text{span}(e_1, e_2), \dots, \text{span}(e_1, \dots, e_{2k-1}), \text{span}(e_1, \dots, e_{2k}) \tag{1.20}$$

were invariant ideals of $\mathfrak{r}_{2k}$, which in turn yielded the automorphisms and derivations matrices of $\mathfrak{r}_{2k}$ triangular w. r. t. the defining basis $(e_i)_1^{2k}$. In the following, we will see that this is not the case for odd-dimensional nilradicals $\mathfrak{r}_{2k-1}$.

Chapter 2

Solvable Extensions of Seven-dimensional Nilradical

To get some intuition for more general case we first find all outer derivation classes and solvable extension of $\mathfrak{r} \equiv \mathfrak{r}_7 = \operatorname{span}(e_1, \dots, e_7)$ given by relations

$$[e_i, e_7] = e_{i-2}, \qquad \forall i \in \{3, 4, 5, 6\}. \tag{2.1}$$

Towards this aim we will use methods described in the previous chapter; firstly we must examine the structure of $\mathfrak{r}_7$ itself. This is done in the first section. In the second section, all outer derivation classes of $\mathfrak{r}_7$ are found, while the corresponding solvable extensions by one element are omitted as they can be easily deduced from the derivation classes and are covered in the following chapter on general dimension. The extensions by two and three elements are found and explicitly listed with the dimensions of the members of their characteristic series.

2.1 Properties of $\mathfrak{r}_7$

Basic properties of $\mathfrak{r}_7$ are found in this section. The form of a generic outer derivation and a generic automorphism are explicitly listed as they will be needed in the next section.

2.1.1 Inner Derivations

The inner derivations are easily found to be

$$
\begin{aligned}
&\mathrm{ad}_{e_1} = 0, && \mathrm{ad}_{e_2} = 0, \\
&\mathrm{ad}_{e_3} = \begin{pmatrix} 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, && \mathrm{ad}_{e_4} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \\
&\mathrm{ad}_{e_5} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, && \mathrm{ad}_{e_6} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \\
&\mathrm{ad}_{e_7} = \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}.
\end{aligned}
\tag{2.2}
$$

2.1.2 Derived Series, Flag of Ideals

It is readily seen that

$$\mathfrak{r}_7^2 = \mathfrak{r}_7^{(1)} = \mathrm{span}(e_1, e_2, e_3, e_4), \quad \mathfrak{r}_7^{(2)} = 0, \quad \mathfrak{r}_7^3 = \mathrm{span}(e_1, e_2), \quad \mathfrak{r}_7^4 = 0,$$

and

$$\mathfrak{z} = \mathrm{span}(e_1, e_2), \quad \mathfrak{z}^2 = \mathrm{span}(e_1, e_2, e_3, e_4), \quad \mathfrak{z}^3 = \mathfrak{r}_7.$$

As opposed to the even-dimensional case, the odd-dimensional algebra does not contain a flag of invariant ideals with codimensions equal to one. Nevertheless, there is one more invariant ideal to be found:

$$Z_{\mathfrak{r}_7}(\mathfrak{r}_7^2) = \mathrm{span}(e_1, \dots, e_6).$$

To see that the remaining subspaces $\mathrm{span}(e_1)$, $\mathrm{span}(e_1, e_2, e_3)$ and $\mathrm{span}(e_i)_1^5$ are not invariant, we consider the automorphism

$$\Phi : e_1 \leftrightarrow e_2, \quad e_3 \leftrightarrow e_4, \quad e_5 \leftrightarrow e_6, \quad e_7 \mapsto e_7$$

of $\mathfrak{r}_7$. It is evident that

$$\Phi(\mathrm{span}(e_1, \dots, e_{(2\ell-1)})) = \mathrm{span}(e_1, \dots, e_{(2\ell-2)}, e_{2\ell}) \qquad \forall \ell \in 1, 2, 3.$$

2.1.3 Automorphisms

Using the knowledge of invariant ideals from the preceding section, we find a general form of automorphism of $\mathfrak{r}_7$. Consider an automorphism Φ of $\mathfrak{r}_7$ and assume that

$$\Phi e_7 = \sum_{i=1}^{7} \alpha_i e_i, \quad \Phi e_6 = \sum_{i=1}^{6} \beta_i e_i, \quad \Phi e_5 = \sum_{i=1}^{6} \gamma_i e_i.$$

Then by the commutation relations (2.1) we get

$$\begin{aligned}
\Phi e_4 &= \Phi[e_6, e_7] = [\Phi e_6, \Phi e_7] = \left[\sum_{i=1}^{6} \beta_i e_i, \sum_{j=1}^{7} \alpha_j e_j\right] = \sum_{i=3}^{6} \alpha_7 \beta_i e_{i-2}, \\
\Phi e_3 &= \Phi[e_5, e_7] = [\Phi e_5, \Phi e_7] = \left[\sum_{i=1}^{6} \gamma_i e_i, \sum_{j=1}^{7} \alpha_j e_j\right] = \sum_{i=3}^{6} \alpha_7 \gamma_i e_{i-2}, \\
\Phi e_2 &= \Phi[e_4, e_6] = [\Phi e_4, \Phi e_6] = \sum_{i=5}^{6} \alpha_7^2 \beta_i e_{i-4}, \\
\Phi e_1 &= \Phi[e_3, e_6] = [\Phi e_3, \Phi e_6] = \sum_{i=3}^{6} \alpha_7^2 \beta_i e_{i-4},
\end{aligned}$$

and the generic automorphism in the form

$$\begin{pmatrix}
\alpha_7^2\gamma_5 & \alpha_7^2\beta_5 & \alpha_7\gamma_3 & \alpha_7\beta_3 & \gamma_1 & \beta_1 & \alpha_1 \\
\alpha_7^2\gamma_6 & \alpha_7^2\beta_6 & \alpha_7\gamma_4 & \alpha_7\beta_4 & \gamma_2 & \beta_2 & \alpha_2 \\
0 & 0 & \alpha_7\gamma_5 & \alpha_7\beta_5 & \gamma_3 & \beta_3 & \alpha_3 \\
0 & 0 & \alpha_7\gamma_6 & \alpha_7\beta_6 & \gamma_4 & \beta_4 & \alpha_4 \\
0 & 0 & 0 & 0 & \gamma_5 & \beta_5 & \alpha_5 \\
0 & 0 & 0 & 0 & \gamma_6 & \beta_6 & \alpha_6 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_7
\end{pmatrix}. \tag{2.3}$$

Assuming that $\gamma_5 \neq 0$ we can decompose Φ into lower unitriangular, diagonal and upper unitriangular matrix in the following way:

$$\Phi = \Phi_L \cdot \Phi_D \cdot \Phi_U,$$

where

$$\Phi_L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma_6}{\gamma_5} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma_6}{\gamma_5} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\gamma_6}{\gamma_5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Phi_D = \begin{pmatrix} \alpha_7^2\gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_7^2}{\gamma_5}(\gamma_5\beta_6 - \gamma_6\beta_5) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_7\gamma_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_7}{\gamma_5}(\gamma_5\beta_6 - \gamma_6\beta_5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_6 - \frac{\gamma_6\beta_5}{\gamma_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 \end{pmatrix},$$

$$\Phi_U = \begin{pmatrix} 1 & \frac{\beta_5}{\gamma_5} & \frac{\gamma_3}{\alpha_7\gamma_5} & \frac{\beta_3}{\alpha_7\gamma_5} & \frac{\gamma_1}{\alpha_7^2\gamma_5} & \frac{\beta_1}{\alpha_7^2\gamma_5} & \frac{\alpha_1}{\alpha_7^2\gamma_5} \\ 0 & 1 & \frac{\gamma_5\gamma_4-\gamma_6\gamma_3}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} & \frac{\gamma_5\beta_4-\gamma_6\beta_3}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} & \frac{\gamma_5\gamma_2-\gamma_6\gamma_1}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} & \frac{\gamma_5\beta_2-\gamma_6\beta_1}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} & \frac{\gamma_5\alpha_2-\gamma_6\alpha_1}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} \\ 0 & 0 & 1 & \frac{\beta_5}{\gamma_5} & \frac{\gamma_3}{\alpha_7\gamma_5} & \frac{\beta_3}{\alpha_7\gamma_5} & \frac{\alpha_3}{\alpha_7\gamma_5} \\ 0 & 0 & 0 & 1 & \frac{\gamma_5\gamma_4-\gamma_6\gamma_3}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} & \frac{\gamma_5\beta_4-\gamma_6\beta_3}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} & \frac{\gamma_5\alpha_4-\gamma_6\alpha_3}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} \\ 0 & 0 & 0 & 0 & 1 & \frac{\beta_5}{\gamma_5} & \frac{\alpha_5}{\gamma_5} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{\gamma_5\alpha_6-\gamma_6\alpha_5}{\alpha_7(\gamma_5\beta_6-\gamma_6\beta_5)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

These are again automorphisms, thus changing the names of the variables, we can rewrite them in the form:

$$\Phi_L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (2.4)$$

$$\Phi_D = \begin{pmatrix} \alpha_7^2\gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_7^2\beta_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_7\gamma_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_7\beta_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 \end{pmatrix}, \qquad (2.5)$$

$$\Phi_U = \begin{pmatrix} 1 & \beta_5 & \gamma_3 & \beta_3 & \gamma_1 & \beta_1 & \alpha_1 \\ 0 & 1 & \gamma_4 & \beta_4 & \gamma_2 & \beta_2 & \alpha_2 \\ 0 & 0 & 1 & \beta_5 & \gamma_3 & \beta_3 & \alpha_3 \\ 0 & 0 & 0 & 1 & \gamma_4 & \beta_4 & \alpha_4 \\ 0 & 0 & 0 & 0 & 1 & \beta_5 & \alpha_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

Finally, let us denote Φ_a the special case of (2.3) with $\gamma_5 = 0$, that is

$$\Phi_a = \begin{pmatrix} 0 & \alpha_7^2\beta_5 & \alpha_7\gamma_3 & \alpha_7\beta_3 & \gamma_1 & \beta_1 & \alpha_1 \\ \alpha_7^2\gamma_6 & \alpha_7^2\beta_6 & \alpha_7\gamma_4 & \alpha_7\beta_4 & \gamma_2 & \beta_2 & \alpha_2 \\ 0 & 0 & 0 & \alpha_7\beta_5 & \gamma_3 & \beta_3 & \alpha_3 \\ 0 & 0 & \alpha_7\gamma_6 & \alpha_7\beta_6 & \gamma_4 & \beta_4 & \alpha_4 \\ 0 & 0 & 0 & 0 & 0 & \beta_5 & \alpha_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \beta_6 & \alpha_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 \end{pmatrix}. \tag{2.7}$$

2.1.4 Generic Outer Derivation

We obtain generic outer derivation by differentiating generic automorphism (2.3) at the point $\mathbb{1}$. Evidently $\Phi = \Phi_L \circ \Phi_D \circ \Phi_U = \mathbb{1} \iff \alpha_7 = \beta_6 = \gamma_5 = 1$ and $\alpha_i = \beta_j = \gamma_k = 0$ for any $i \neq 7,\ j \neq 6,\ k \neq 5$. Thus we vary the parameters around these values, where we denote the variation of the original parameter $(\alpha_i, \beta_j, \gamma_k)$ with the corresponding Latin letter, e. g. $\alpha_7 = 1 + a_7$. From the resulting matrix we subtract $\mathbb{1}$ to obtain a generic derivation in the form

$$\begin{pmatrix} 2a_7 + c_5 & b_5 & c_3 & b_3 & c_1 & b_1 & a_1 \\ c_6 & 2a_7 + b_6 & c_4 & b_4 & c_2 & b_2 & a_2 \\ 0 & 0 & a_7 + c_5 & b_5 & c_3 & b_3 & a_3 \\ 0 & 0 & c_6 & a_7 + b_5 & c_4 & b_4 & a_4 \\ 0 & 0 & 0 & 0 & c_5 & b_5 & a_5 \\ 0 & 0 & 0 & 0 & c_6 & b_6 & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix}.$$

We may eliminate the parameters a_1, a_2, a_3, and a_4 by subtracting multiples of inner derivations $\mathrm{ad}_{e_3}, \mathrm{ad}_{e_4}, \mathrm{ad}_{e_5}$, and ad_{e_6} given in subsection 2.2 and without loss of generality we may eliminate b_4 by subtracting a multiple of ad_{e_7}; this changes $c_3 \to c_3 - b_4$ so we rename this parameter accordingly to obtain the generic form of the outer derivation of $\mathfrak{r}_7$:

$$D := \begin{pmatrix} 2a_7+c_5 & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ c_6 & 2a_7+b_6 & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a_7+c_5 & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & c_6 & a_7+b_6 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & b_5 & a_5 \\ 0 & 0 & 0 & 0 & c_6 & b_6 & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix}. \tag{2.8}$$

2.2 Extensions

We are interested in equivalence classes of outer derivations w. r. t. conjugation by arbitrary automorphism (2.3), multiplication by an arbitrary constant, and addition of linear combinations of inner derivations. Note that either the component γ_5 of the automorphism is zero or the automorphism can be decomposed to $\Phi_L \Phi_D \Phi_U$. These classes correspond to the extensions by one vector. We first resolve under which conditions it is possible to simplify the derivations in subsection 2.2.1, then we find all nonequivalent classes in subsection 2.2.2. The extensions by two vectors, i.e. the nine-dimensional extensions are found in the subsection 2.2.3, the extensions by three vectors are found in the subsection 2.2.4. Henceforth, equalities are understood mod inner derivations, where appropriate.

2.2.1 Triangular Form of Derivation

We start by observing that there is a special automorphism

$$\Phi_P := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.9}$$

which transforms a generic derivation in the following way:

$$\Phi_P^{-1} D \Phi_P = \begin{pmatrix} 2a_7+b_6 & c_6 & 0 & c_4 & b_2 & c_2 & 0 \\ b_5 & 2a_7+c_5 & b_3 & c_3 & b_1 & c_1 & 0 \\ 0 & 0 & a_7+b_6 & c_6 & 0 & c_4 & 0 \\ 0 & 0 & b_5 & a_7+c_5 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_6 & c_6 & a_6 \\ 0 & 0 & 0 & 0 & b_5 & c_5 & a_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix},$$

that is, it pairwise swaps first and second, third and fourth, and fifth and sixth columns and rows, or else it swaps the parameters

$$b_1 \leftrightarrow c_2, \quad b_2 \leftrightarrow c_1, \quad b_3 \leftrightarrow c_4, \quad b_4 \leftrightarrow c_3, \quad b_5 \leftrightarrow c_6, \quad b_6 \leftrightarrow c_5, \quad a_6 \leftrightarrow a_5.$$

Thus, whatever can be achieved for one parameter in a given pair, can be achieved for the other if it is more convenient.

To simplify the classification, it is desirable to transform the derivation matrix to a triangular form. From the preceding, we readily see that in case $b_5 = 0$ and $c_6 \neq 0$ we simply swap these parameters to achieve upper-triangular form of the derivation. If that is not the case, we conjugate the derivation (2.8) with the automorphism (2.3) which yields the expression:

$$\left(\Phi^{-1} D \Phi\right)_{21} = \frac{b_5\gamma_6^2 - (b_6+c_5)\gamma_5\gamma_6 - c_6\gamma_5^2}{\beta_5\gamma_6 - \beta_6\gamma_5}$$

in place of the parameter c_6. Since we assume $b_5 \neq 0$, we can make the above equal to 0 if and only if $\sqrt{4b_5c_6 + (b_6-c_5)^2}$ is an element of the field $\mathbb{F}$ over which we consider the algebra $\mathfrak{r}_7$. In case of $\mathbb{F} = \mathbb{C}$ the root always exists. In case of $\mathbb{F} = \mathbb{R}$ the inequality

$$4b_5c_6 + (b_6-c_5)^2 \geq 0 \tag{2.10}$$

must hold to allow for the triangulation of the derivation matrix. In the case of complex field or the inequality (2.10) being true, using the special case of automorphism (2.4) with $\gamma_6 = \frac{1}{2b_5}\left(b_6 - c_5 + \sqrt{4b_5c_6 + (b_6-c_5)^2}\right)$ we transform the derivation (2.8) to the form

$$\begin{pmatrix} 2a_7+c_5 & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a_7+b_6 & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a_7+c_5 & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a_7+b_6 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & b_5 & a_5 \\ 0 & 0 & 0 & 0 & 0 & b_6 & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix}. \tag{2.11}$$

Applying Φ_L with $\gamma_6 \neq 0$ disturbs the triangularity of the derivation matrix whenever $b_5 \neq 0$ and $\Phi_L|_{\gamma_6=0} = \mathbb{1}$. Similarly, setting $\gamma_6 := 0$ within Φ_a is forbidden since we require that Φ_a is an automorphism and hence regular and applying Φ_a with $\gamma_6 \neq 0$ would transform the derivation matrix into a non-triangular form whenever $b_5 \neq 0$. Thus we are left with upper-triangular derivation matrix (2.11) and automorphisms of the form Φ_U to simplify the above triangular derivation matrix, unless we have $b_5 = 0$.

2.2.2 Outer Derivation Classes

Throughout this subsection special cases of automorphisms are used. Any parameter which is not specifically listed is assumed to be 1 in the case of automorphism (2.5) and 0 in the case of any other automorphism.

Non-triangular Case

Let us have $\mathbb{F} = \mathbb{R}$, and

$$b_5 \neq 0 \quad \wedge \quad 4b_5c_6 < -(b_6 - c_5)^2. \tag{2.12}$$

Then there is no possibility of eliminating c_6 and we do not restrict ourselves in using the available automorphisms as there is nothing to be preserved. Using the automorphism (2.6) with

$$\alpha_5 = \frac{a_5a_7 - a_5b_6 + a_6b_5}{(a_7 - b_6)(a_7 - c_5) - b_5c_6}, \qquad \alpha_6 = \frac{a_5c_6 + a_6a_7 - a_6c_5}{(a_7 - b_6)(a_7 - c_5) - b_5c_6},$$

we eliminate parameters a_5 and a_6 from the derivation and after subtracting the inner derivations we obtain the derivation in the form

$$\begin{pmatrix} 2a_7 + c_5 & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ c_6 & 2a_7 + b_6 & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a_7 + c_5 & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & c_6 & a_7 + b_6 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & b_5 & 0 \\ 0 & 0 & 0 & 0 & c_6 & b_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{pmatrix},$$

where the remaining parameters have been renamed accordingly. The denominator of α_5 and α_6 above is the same and is equal to zero if and only if

$$2a_7 = b_6 + c_5 - \sqrt{4\,b_5c_6 + (b_6 - c_5)^2}.$$

The root cannot be found due to the inequalities (2.12) and thus the denominators are not equal to zero. In the next step we apply the automorphism (2.7) with

$$\alpha_7 = 1, \quad \beta_5 = 1, \quad \beta_6 = \frac{b_6 - c_5}{2b_5}, \quad \gamma_6 = -\frac{\sqrt{-4b_5c_6 - (b_6 - c_5)^2}}{2b_5}.$$

After renaming parameters, the derivation takes up the form

$$\begin{pmatrix} 2a+c & b & c_3 & b_3 & c_1 & b_1 & 0 \\ -b & 2a+c & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & b & c_3 & b_3 & 0 \\ 0 & 0 & -b & a+c & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b & 0 \\ 0 & 0 & 0 & 0 & -b & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Again, the denominators in the parameters above are not null due to the inequalities (2.12). Moreover, b in the resulting form is necessarily different from zero. Hence, we can readily use the automorphism (2.6) with $\beta_4 = -\frac{c_4}{b}$ to eliminate the parameter c_4 in the derivation. After multiplying by b^{-1} the derivation matrix takes up the following form:

$$\begin{pmatrix} 2a+c & 1 & c_3 & b_3 & c_1 & b_1 & 0 \\ -1 & 2a+c & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & 1 & c_3 & b_3 & 0 \\ 0 & 0 & -1 & a+c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{2.13}$$

i. $a \neq 0$

In the case of $a \neq 0$ we can use automorphism (2.6) with

$$\begin{aligned} \beta_3 &= -\frac{a^2b_3 + ac_3 + 2\,b_3}{a\,(a^2+4)}, &\quad \beta_4 &= -\frac{ab_3 + 2\,c_3}{a\,(a^2+4)}, \\ \gamma_3 &= -\frac{a^2c_3 - ab_3 + 2\,c_3}{a\,(a^2+4)}, &\quad \gamma_4 &= -\frac{ac_3 - 2\,b_3}{a\,(a^2+4)} \end{aligned} \tag{2.14}$$

to eliminate parameters b_3, c_3 from the derivation matrix. To eliminate the remaining parameters b_1, b_2, c_1, c_2 we conjugate the derivation matrix by the automorphism (2.6) with

$$\beta_1 = \frac{2\,a^2b_1 - ab_2 + abc_1 + b^2b_1 - b^2c_2}{4a^3},$$
$$\gamma_1 = \frac{2\,a^2c_1 - ab_1 - abc_2 + b^2b_2 + b^2c_1}{4a^3},$$
$$\beta_2 = \frac{2\,a^2b_2 + ab_1 + abc_2 + b^2b_2 + b^2c_1}{4a^3},$$
$$\gamma_2 = \frac{2\,a^2c_2 - ab_2 + abc_1 - b^2b_1 + b^2c_2}{4a^3}$$

thus simplifying the derivation to the form

$$\begin{pmatrix} 2a+c & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2a+c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+c & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & a+c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Multiplying by $\mathrm{sgn}(c)$ and using the automorphism (2.9) if necessary we get the resulting class of derivations given by:

$$[D_1] = \left\{ \begin{pmatrix} 2a+c & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2a+c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+c & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & a+c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \middle| \begin{array}{c} c \in \mathbb{R}_0^+ \\ a \in \mathbb{R} \end{array} \right\}. \quad (2.15)$$

ii. $a = 0$

We first apply the automorphism (2.6) with $\beta_3 = -\frac{c_3}{2}$ to get the derivation in the form:

$$\begin{pmatrix} c & 1 & -\frac{c_3}{2} & b_3 & c_1 & b_1 & 0 \\ -1 & c & 0 & -\frac{c_3}{2} & c_2 & b_2 & 0 \\ 0 & 0 & c & 1 & -\frac{c_3}{2} & b_3 & 0 \\ 0 & 0 & -1 & c & 0 & -\frac{c_3}{2} & 0 \\ 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

thus producing same value in the place of b_4 and c_3. Then, by adding $\frac{c3}{2}\mathrm{ad}_{e_7}$, we eliminate the parameter c_3 from the derivation entirely. From the remaining parameters b_1, c_2 and c_1, b_2, we can eliminate one of the parameters from each pair by using (2.6). We choose to eliminate b_1 and c_1 by setting $\gamma_1 = b_1$ and $\gamma_2 = -c_1$ in (2.6) and we arrive at

$$\begin{pmatrix} c & 1 & 0 & b_3 & 0 & 0 & 0 \\ -1 & c & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & c & 1 & 0 & b_3 & 0 \\ 0 & 0 & -1 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $c \geq 0$. Depending on the values of b_3, b_2 and c_2 we can use the diagonal the automorphism (2.5) to scale some of them. To preserve the already fixed lower 3×3 block, we demand that $\gamma_5 = \beta_6$, while α_7 is arbitrary and can be used to scale one of the parameters. Scaling b_3 if it is non-zero, c_2 if b_3 is 0 and c_2 is non-zero, and b_2 if both c_2 and b_3 are 0 and b_2 is not, we arrive at the following family:

$$[D_2] = \left\{ \begin{pmatrix} c & 1 & 0 & b_3 & 0 & 0 & 0 \\ -1 & c & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & c & 1 & 0 & b_3 & 0 \\ 0 & 0 & -1 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} c \in \mathbb{R}_0^+ \\ (b_3, c_2, b_2) \in \{1\} \times \mathbb{R}^2 \cup \\ \{0, \pm 1\} \times \mathbb{R} \cup \{0, 0, \pm 1\} \end{array} \right\}, \tag{2.16}$$

If all three parameters b_3, c_2, and b_2 were 0, we would obtain a special case of $[D_1]$.

Triangular Case

If we have either $\mathbb{F} = \mathbb{C}$ or

$$4b_5c_6 \geq -(b_6 - c_5)^2 \tag{2.17}$$

or $b_5 = 0$ or $c_6 = 0$, we can assume the derivation matrix in the upper triangular form and restrict ourselves to automorphisms preserving this property. To simplify the notation, indices of diagonal elements are dropped. Thus we begin with the derivation in the form:

$$\begin{pmatrix} 2a+c & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a+b & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b_5 & a_5 \\ 0 & 0 & 0 & 0 & 0 & b & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Depending on the relations between the parameters we can simplify the derivation using the allowed transforms to find the classes. The transformations for different cases are described step by step below.

1. $a \neq b \neq c \neq a$
Automorphism (2.6) with

$$\alpha_5 = \frac{1}{a-c}\left(\frac{a_6 b_5}{a-b} + a_5\right), \ \alpha_6 = \frac{a_6}{a-b}, \ \beta_5 = \frac{b_5}{b-c} \tag{2.18}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & 0 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a+b & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \tag{2.19}$$

1.1. $a + b \neq c$
Automorphism (2.6) with

$$\gamma_4 = \frac{c_4}{c-a-b}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & 0 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}$$

1.1.1. $a \neq 0$, $b \neq a+c$, $b \neq 2a+c$, $c \neq 2a+b$
Automorphism (2.6) with

$$\gamma_2 = -\frac{c_2}{2a+b-c}, \quad \beta_2 = -\frac{b_2}{2a}, \quad \gamma_3 = -\frac{c_3}{a}, \quad \beta_3 = -\frac{b_3}{a+c-b}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & 0 & 0 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\gamma_1 = -\frac{c_1}{2a}, \quad \beta_1 = -\frac{b_1}{2a+c-b}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Multiplying by a^{-1} yields the derivation class

$$[D_3] = \left\{ \begin{pmatrix} 2+c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} b \in \mathbb{F} \\ c \in \mathbb{F} \end{array} \right\}. \tag{2.20}$$

1.1.2. $b = a+c \quad (\implies a \neq 0,\, c \neq 2a+b, b = 2a+c)$
Automorphism (2.6) with

$$\gamma_2 = -\frac{c_2}{3a}, \quad \beta_2 = -\frac{b_2}{2a}, \quad \gamma_3 = -\frac{c_3}{a}, \quad \beta_3 = 0$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & 0 & b_3 & c_1 & b_1 & 0 \\ 0 & 3a+c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+c & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 2a+c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a+c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\gamma_1 = -\frac{c_1}{2a}, \quad \beta_1 = -\frac{b_1}{a}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & 0 & b_3 & 0 & 0 & 0 \\ 0 & 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+c & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

We multiply the matrix by a^{-1} and scale b_3 to one by applying the automorphism (2.5) with $\gamma_5 = b_3$; the resulting family is given by:

$$[D_4] = \left\{ \begin{pmatrix} 2+c & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3+c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2+c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; c \in \mathbb{F} \right\}. \tag{2.21}$$

The case of $b_3 = 0$ falls back to $[D_3]$ with $b = 1 + c$.

1.1.3. $c = 2a + b \quad (\implies a \neq 0, b \neq 2a + c, b \neq 2a + c)$

Automorphism (2.6) with

$$\gamma_2 = 0, \quad \beta_2 = -\frac{b_2}{2a}, \quad \gamma_3 = -\frac{c_3}{a}, \quad \beta_3 = -\frac{b_3}{3a}$$

transforms the derivation to the form

$$\begin{pmatrix} 4a+b & 0 & 0 & 0 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 3a+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\gamma_1 = -\frac{c_1}{2a}, \quad \beta_1 = -\frac{b_1}{a}$$

transforms the derivation to the form

$$\begin{pmatrix} 4a+b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a+b & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 3a+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

We multiply the matrix by a^{-1} and scale c_2 to one by applying the automorphism (2.5) with $\beta_6 = c_2$; the resulting family is given by:

$$[D_5] = \left\{ \begin{pmatrix} 4+b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; b \in \mathbb{F} \right\}. \tag{2.22}$$

The case of $c_2 = 0$ falls back to $[D_3]$ with $c = 2 + b$.

1.1.4. $b = 2a + c \quad (\implies a \neq 0, b \neq 2a + c, c \neq 2a + b)$

Automorphism Φ_P (2.9) transforms the derivation to the form:

$$\begin{pmatrix} 4a+b & 0 & 0 & 0 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 3a+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & c_4 & b_4 & 0 \\ 0 & 0 & 0 & 0 & 2a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix},$$

where the parameters have been renamed to reflect their new position. We subtract $b_4\mathrm{ad}_{e_7}$ and rename b_4 to c_3 and eliminate c_4 again as described in 1.1 to obtain

$$\begin{pmatrix} 4a+b & 0 & 0 & 0 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 3a+b & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix},$$

which is a special case of 1.1.3.

1.1.5. $a = 0 \quad (\implies b \neq a + c,\, b \neq 2a + c,\, c \neq 2a + b)$

Automorphism (2.6) with

$$\gamma_2 = -\frac{c_2}{b-c}, \quad \beta_3 = -\frac{b_3}{c-b}$$

transforms the derivation to the form

$$\begin{pmatrix} c & 0 & c_3 & 0 & c_1 & b_1 & 0 \\ 0 & b & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & c & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Automorphism (2.6) with

$$\beta_1 = -\frac{b_1}{c-b}$$

transforms the derivation to the form

$$\begin{pmatrix} c & 0 & c_3 & 0 & c_1 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & c & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c_3 \neq 0$, we can scale it to 1 using the automorphism (2.5) with $\alpha_7 = c_3$. Multiplying by b^{-1} in case $b \neq 0$ and by c^{-1} in case $b = 0$, we obtain the family of derivation classes:

$$[D_6] = \left\{ \begin{pmatrix} c & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & c & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} c_1 \in \mathbb{F} \\ b_2 \in \mathbb{F} \\ (b,c) \in \{1\} \times \mathbb{F} \ \cup \ \{0,1\} \end{array} \right\}. \tag{2.23}$$

The case of $c = b = 0$ is not allowed by the conditions assumed and irrelevant as the derivation would be nilpotent. Without loss of generality we may assume that $b \neq 0$ and multiply the derivation matrix by b^{-1}. Assuming that $c_3 = 0$ and $b_2 \neq 0$ we scale b_2 to ± 1 if $\mathbb{F} = \mathbb{R}$ or to 1 if $\mathbb{F} = \mathbb{C}$. In case both c_3 and b_2 vanish and $c_1 \neq 0$ we scale c_1 to $(\pm)1$. To describe all cases at once we leave the parameters in the derivation matrix and construct the corresponding ranges for them.

$$[D_6'] = \left\{ \begin{pmatrix} c & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} \qquad\qquad c \in \mathbb{F} \\ \mathbb{F} = \mathbb{R} \implies : \\ \quad (b_2, c_1) \in \{\pm 1\} \times \mathbb{R} \cup \{0\} \times \{\pm 1\} \\ \mathbb{F} = \mathbb{C} \implies : \\ \quad (b_2, c_1) \in \{1\} \times \mathbb{C} \cup \{0\} \times \{1\} \end{array} \right\}. \tag{2.24}$$

1.2. $a + b = c \quad (\implies a \neq 0, c \neq 2a + b, b \neq a + c, b \neq 2a + c)$
We take analogous steps as in case 1.1.4 to come from this case to 1.1.2.

2. $a = b \neq c$
Automorphism (2.6) with

$$\beta_5 = \frac{b_5}{a - c}, \quad \alpha_5 = \frac{a_5 - \frac{b_5 a_6}{a-c}}{a - c}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a + c & 0 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 3a & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a + c & 0 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & 2a & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{2.25}$$

2.1. $c \neq 2a$
Automorphism (2.6) with

$$\gamma_4 = \frac{c_4}{c - 2a}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+c & 0 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 3a & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & 0 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

2.1.1. $a \neq 0$, $c \neq 0$, $c \neq 3a$, $c \neq -a$
Taking same steps as in 1.1.1 with $b = a$ we eliminate c_1, c_2, c_3, b_1, b_2, b_3. Setting $\beta_6 = a_6$ in the automorphism (2.5) we scale a_6 to 1 and obtain the resulting family of derivation classes:

$$[D_7] = \left\{ \begin{pmatrix} 2+c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; c \in \mathbb{F} \right\}. \tag{2.26}$$

The case of $a_6 = 0$ falls back to 1.1.1.
2.1.2. $c = 0 \quad (\implies a \neq 0, c \neq 3a, c \neq -a)$
Taking same steps as in 1.1.2 we eliminate c_1, c_2, c_3, b_1, b_2. Setting $\beta_6 = a_6$ and $\gamma_5 = b_3$ in the automorphism (2.5) we scale a_6 and b_3 to 1 and obtain the resulting derivation class:

$$[D_8] = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.27}$$

If either of the parameters a_6, b_3 was equal to 0, we would obtain a class from one of the families already found, namely, for $b_3 = 0$, $a_6 \neq 0$ we would obtain $[D_7]$ with $c = 0$, for $b_3 \neq 0$, $a_6 = 0$ we would obtain $[D_4]$ with $c = 0$, and for both parameters vanishing we would obtain $[D_3]$ with $b = 1$ and $c = 0$.

2.1.3. $c = 3a \quad (\implies a \neq 0, c \neq 0, c \neq -a)$
Taking same steps as in 1.1.3 we eliminate c_1, c_3, b_1, b_2, b_3. Setting $\alpha_7 = a_6^{-1}$ and $\gamma_5 = a_6^2 c_2^{-1}$ in the automorphism (2.5) we scale a_6 and c_2 to 1 and obtain the resulting derivation class:

$$[D_9] = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.28}$$

Again, if either of the parameters a_6, c_2 was equal to 0, we would obtain a class from one of the families already found.

2.1.4. $c = -a \quad (\implies a \neq 0, c \neq 0, c \neq 3a)$
Automorphism (2.6) with

$$\gamma_2 = -\frac{c_2}{4a}, \quad \beta_2 = -\frac{b_2}{2a}, \quad \gamma_3 = -\frac{c_3}{a}, \quad \beta_3 = \frac{b_3}{a}$$

transforms the derivation to the form

$$\begin{pmatrix} a & 0 & 0 & 0 & c_1 & b_1 & 0 \\ 0 & 3a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\gamma_1 = -\frac{c_1}{2a}$$

transforms the derivation to the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & b_1 & 0 \\ 0 & 3a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Taking same steps as in 1.1.3 we eliminate c_1, c_3, b_1, b_2, b_3. Setting $\alpha_7 = a_6^{-1}$ and $\gamma_5 = a_6^{-2}c_2$ in the automorphism (2.5) we scale a_6 and b_1 to 1 and obtain the resulting derivation class:

$$[D_{10}] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.29}$$

Again, if either of the parameters a_6, b_1 was equal to 0, we would obtain a class from one of the families already found.

2.1.5. $a = 0 \quad (\implies c \neq -a, c \neq 0, c \neq 3a)$

Taking same steps as in 1.1.5 with $b := a$ we eliminate c_2, b_1, b_3. In case $c_3 \neq 0$, we scale it to 1 along with a_6 by applying the automorphism (2.5) with $\alpha_7 = c_3$ and $\beta_6 = a_6 c_3$, and multiply the derivation matrix by c^{-1} thus obtaining the resulting family:

$$[D_{11}] = \left\{ \begin{pmatrix} 1 & 0 & 1 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{matrix} c_1 \in \mathbb{F} \\ b_2 \in \mathbb{F} \end{matrix} \right\}. \tag{2.30}$$

In case $c_3 = 0$ the argument is analogous to that in 1.1.5 with the difference in the scaling the automorphism, α_7 is set to the same value, while $\beta_6 :=$ scaled parameter $\cdot\, a_6$ to scale a_6 to one as well. The resulting family is given by:

$$[D'_{11}] = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{matrix} \mathbb{F} = \mathbb{R} \implies : \\ \quad (c_1, b_2) \in \{\pm 1\} \times \mathbb{R} \cup \{0\} \times \{\pm 1\} \\ \mathbb{F} = \mathbb{C} \implies : \\ \quad (c_1, b_2) \in \{1\} \times \mathbb{C} \cup \{0\} \times \{1\} \end{matrix} \right\}. \tag{2.31}$$

Similarly to the preceding, $a_6 = 0$ would yield one of the families already found.

2.2. $c = 2a \quad (\implies c \neq -a \neq 0, c \neq 3a)$
Automorphism (2.6) with

$$\gamma_3 = -\frac{c_3}{a}, \quad \beta_3 = -\frac{b_3}{a+c-b}$$

transforms the derivation to the form

$$\begin{pmatrix} 4a & 0 & 0 & 0 & c_1 & b_1 & 0 \\ 0 & 3a & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 3a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{2.32}$$

Automorphism (2.6) with

$$\gamma_1 = -\frac{c_1}{2a}, \quad \beta_1 = -\frac{b_1}{3a}, \quad \gamma_2 = -\frac{c_2}{a}, \quad \beta_2 = -\frac{b_2}{2a}$$

transforms the derivation to the form

$$\begin{pmatrix} 4a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3a & c_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{2.33}$$

As $a \neq 0$, we first multiply the matrix by a^{-1}. Both remaining nondiagonal parameters can be scaled to one using the automorphism (2.5) with $\gamma_5 = c_4^{-1}$ to scale c_4 to 1 and $\beta_6 = a_6$ to scale a_6 to 1 in case they are not equal to zero. In case $a_6 = 0$ we can apply the automorphism (2.9) and thus come back to a special case 1.1.2. In case $c_4 = 0$ we have obtained a special case of 2.1.1. Thus, we obtain the only 'missing' class:

$$[D_{12}] = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.34}$$

3. $a = c \neq b$
Automorphism (2.6) with

$$\alpha_6 = \frac{a_6}{a-b}, \ \beta_5 = \frac{b_5}{b-a} \tag{2.35}$$

eliminates b_5 and a_6 from the derivation matrix. Applying the automorphism (2.9) and renaming parameters accordingly we reduce this case to case 2.

4. $c = b \neq a$
Automorphism (2.6) with

$$\alpha_5 = \frac{1}{a-b}\left(\frac{a_6 b_5}{a-b} + a_5\right), \ \alpha_6 = \frac{a_6}{a-b}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+b & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+b & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a+b & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

4.1. $a \neq 0$
Automorphism (2.6) with

$$\gamma_4 = -\frac{c_4}{a}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+b & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+b & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\gamma_3 = -\frac{c_3}{a}, \quad \gamma_2 = -\frac{c_2}{2a}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+b & b_5 & 0 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & a+b & b_5 & 0 & b_3 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\gamma_1 = -\frac{c_1}{2a}, \quad \beta_2 = -\frac{b_2}{2a}, \quad \beta_3 = -\frac{b_3}{a}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+b & b_5 & 0 & 0 & 0 & b_1 & 0 \\ 0 & 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+b & b_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Automorphism (2.6) with

$$\beta_1 = -\frac{b_1}{2a}$$

transforms the derivation to the form

$$\begin{pmatrix} 2a+b & b_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a+b & b_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Multiplying by a^{-1} and scaling b_5 to 1 using the automorphism (2.5) with $\gamma_5 = b_5$ we obtain the resulting family of clasess:

$$[D_{13}] = \left\{ \begin{pmatrix} 2+b & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{F} \right\}. \tag{2.36}$$

If $b_5 = 0$ we obtain a special case of $[D_2]$.

4.2. $a = 0$

4.2.1. $b_5 \neq 0$
Automorphism (2.6) with

$$\gamma_4 = -\frac{c_3}{2b_5}$$

transforms the derivation to the form

$$\begin{pmatrix} b & b_5 & -\frac{c_3}{2b_5} & b_3 & c_1 & b_1 & 0 \\ 0 & b & c_4 & -\frac{c_3}{2b_5} & c_2 & b_2 & 0 \\ 0 & 0 & b & b_5 & -\frac{c_3}{2b_5} & b_3 & 0 \\ 0 & 0 & 0 & b & c_4 & -\frac{c_3}{2b_5} & 0 \\ 0 & 0 & 0 & 0 & b & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

adding $\frac{c_3}{2}\mathrm{ad}_{e_7}$ afterwards eliminates the parameter c_3. Using the automorphisms (2.6) with

$$\gamma_i = \frac{b_i}{b_5}$$

we eliminate b_i for i running from 3 to 1. It is possible to scale two non-diagonal parameters to $(\pm)1$ by applying the diagonal automorphism with $\gamma_5 = b_5$, $\beta_6 = 1$, and $\alpha_7 = c_4 b_5$ for $c_4 \neq 0$ or $\alpha_7^2 = c_i b_5$ for $c_4 = 0$ and $c_i \neq 0$, where we pick the greatest i for which $c_i \neq 0$. Thus we obtain the family:

$$[D'_{14}] = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 1 & c_4 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} \mathbb{F} = \mathbb{R} \implies: \\ \quad (c_4, c_2, c_1) \in \{1\} \times \mathbb{R}^2 \cup \\ \quad \cup \{0\} \times \{\pm 1\} \times \mathbb{R} \cup \{0, 0, \pm 1\} \\ \mathbb{F} = \mathbb{C} \implies: \\ \quad (c_4, c_2, c_1) \in \{1\} \times \mathbb{C}^2 \cup \\ \quad \cup \{0\} \times \{1\} \times \mathbb{C} \cup \{0, 0, 1\} \end{array} \right\}. \tag{2.37}$$

4.2.2. $b_5 = 0$, $c_4 \neq 0$
Automorphism (2.6) with

$$\beta_5 = \frac{c_3}{2c_4}$$

transforms the derivation to the form

$$\begin{pmatrix} b & 0 & -\frac{c_3}{2b_5} & b_3 & c_1 & b_1 & 0 \\ 0 & b & c_4 & -\frac{c_3}{2b_5} & c_2 & b_2 & 0 \\ 0 & 0 & b & 0 & -\frac{c_3}{2b_5} & b_3 & 0 \\ 0 & 0 & 0 & b & c_4 & -\frac{c_3}{2b_5} & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

adding $\frac{c_3}{2}\mathrm{ad}_{e_7}$ afterwards eliminates the parameter c_3. Automorphism (2.6) with

$$\beta_4 = \frac{c_2}{c_4}, \quad \beta_3 = \frac{c_1}{c_4}$$

transforms the derivation to the form

$$\begin{pmatrix} b & 0 & 0 & b_3 & 0 & b_1 & 0 \\ 0 & b & c_4 & 0 & 0 & b_2 & 0 \\ 0 & 0 & b & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

We multiply the matrix by b^{-1} and apply the automorphism (2.5) with $\beta_6 = c_4$ thus scaling the parameter c_4 to 1, it is again possible to scale one of the remaining parameters b_1, b_2, or b_3. To preserve scaling of c_4 we set $\gamma_5 = \alpha_7$ and $\beta_6 = 1$. Then the diagonal automorphism transforms the parameters in the following way:

$$b_1 \mapsto \frac{b_1}{\alpha_7^3}, \quad b_2 \mapsto \frac{b_2}{\alpha_7^2}, \quad b_3 \mapsto \frac{b_3}{\alpha_7^2},$$

therefore, we set α_7 to $\sqrt{(\mp)b_3}$, $\sqrt{(\mp)b_2}$, or $\sqrt[3]{b_1}$ thus scaling b_3, b_2 or b_1 to $(\pm)1$. The $(\mp)$ and $(\pm)$ only applies if $\mathbb{F} = \mathbb{R}$ and the degree of the root is even. The resulting family is given by the matrix and the ranges for the parameters:

$$[D'_{15}] = \left\{ \begin{pmatrix} 1 & 0 & 0 & b_3 & 0 & b_1 & 0 \\ 0 & 1 & 1 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} \mathbb{F} = \mathbb{R} \implies: \\ \quad (b_1, b_3, b_2) \in \{1\} \times \mathbb{R}^2 \cup \\ \quad \cup \{0\} \times \{\pm 1\} \times \mathbb{R} \cup \{0, 0, \pm 1\} \\ \mathbb{F} = \mathbb{C} \implies: \\ \quad (b_3, b_2, b_1) \in \{1\} \times \mathbb{C}^2 \cup \\ \quad \cup \{0\} \times \{1\} \times \mathbb{C} \cup \{0, 0, 1\} \end{array} \right\}. \tag{2.38}$$

4.2.3. $b_5 = c_4 = 0$

In case $b_3 \neq 0$, we can apply the automorphism (2.9) and subtract $c_3\mathrm{ad}_{e_7}$ to come back to the preceding case. Thus, we assume $b_3 = 0$ in the following. Assuming that $c_3 \neq 0$ we can use automorphism (2.6) with

$$\gamma_4 = \frac{c_2}{c_3}, \quad \beta_3 = -\frac{b_1}{c_3}$$

to eliminate c_2 and b_1. Applying the automorphism (2.5) with $\alpha_7 = c_3$ we scale c_3 to 1 thus obtaining the derivation in the form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $c_3 = 0$ and $c_2 \neq 0$ we eliminate the parameter c_1 using the automorphism (2.6) with $\beta_5 = \frac{c_1}{c_2}$ and then scale c_2 to 1. One of the remaining parameters b_1, b_2 can be scaled up to the sign as well. In case $c_2 = 0$ (and $b_1 \neq 0$, otherwise we would use the same argument as with $c_4 = 0$ and $b_3 \neq 0$) none of the remaining parameters may be eliminated, however, one of them can be scaled up to its sign. Describing all these cases together, we get the family

$$[D'_{16}] = \left\{ \begin{pmatrix} 1 & 0 & c_3 & 0 & c_1 & b_1 & 0 \\ 0 & 1 & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 1 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| (c_3, c_2, b_2, c_1, b_1) \in \mathcal{A}_{16} \right\}, \tag{2.39}$$

where

$$\begin{aligned} \mathcal{A}_{16} =& \{(1,0)\} \times \mathbb{F}^2 \times \{0\} \\ & \cup \{(0,1)\} \times \{((\pm)1, 0) \times \mathbb{F}, (0, 0, (\pm)1), \theta_3\} \\ & \cup \theta_2 \times \Big\{\{(\pm)1\} \times \mathbb{F}, \{(0, (\pm)1)\}\Big\} \times \theta_1 \end{aligned} \tag{2.40}$$

5. $a = c = b$

The case of $a = 0$ is of no interest as we seek non-nilpotent outer derivations. As a first step we multiply the derivation matrix by a^{-1}. Regardless of the lower 3×3 block we can eliminate the other nondiagonal parameters using a series of automorphisms of the form (2.6) with

$$\gamma_4 = -c_4$$

to eliminate c_4,

$$\gamma_3 = -c_3, \quad \gamma_2 = -\frac{c_2}{2}$$

to eliminate c_3 and c_2,

$$\beta_2 = -\frac{b_2}{2}, \quad \beta_3 = -b_3, \quad \gamma_1 = -\frac{c_2}{2}$$

to eliminate b_3, b_2 and c_1, and

$$\beta_1 = -\frac{b_1}{2}$$

to eliminate b_1. In the lower block, we can assume that $b_5 \neq 0$ and $a_6 \neq 0$ as we would obtain one of the previous cases otherwise. The parameter a_5 can be eliminated by applying the automorphism (2.6) with $\alpha_6 = -\frac{a_5}{a_6}$. We can scale the nondiagonal parameters to one using the automorphism (2.5) with $\alpha_7 = a_6^{-1}$ and $\gamma_5 = b_5$. The resulting class is given by

$$[D_{19}] = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.41}$$

2.2.3 Extensions by Two Vectors

All solvable algebras with the nilradical $\mathfrak{r}$ of dimension nine are found in this section. To accomplish this we must seek all non-equivalent compatible outer derivations of $\mathfrak{r} \equiv \mathfrak{r}_7$, that is:

i) The derivations must be linearly nil-independent. Otherwise we would either get eight-dimensional extension or the nilradical would be larger. In the case of upper-triangular matrices this translates as the matrices having linearly independent diagonals.

ii) Commutator of the derivations must lie in $\mathfrak{Inn} \equiv \mathfrak{Inn}(\mathfrak{r})$.

iii) We may apply automorphism from $\mathrm{Aut}(\mathfrak{r})$ simultaneously on both derivations to obtain an equivalent extension.

iv) Assuming that we have two compatible derivations D and d we can choose arbitrary $\tilde{D}, \tilde{d}$ from the plane $\mathrm{span}\{D, d\}$ as long as they keep generating this plane. Of course, extension obtained using D, d is the same as that obtained using $\tilde{D}, \tilde{d}$. It is readily seen that if D and d conform to i) and ii) then $\tilde{D}$ and $\tilde{d}$ conform to these rules as well.

v) We can add an arbitrary inner derivation to D or d.

vi) $[D, d]$ is an ad-preimage of the commutator $[f_1, f_2]$ of the extending vectors f_1 and f_2. In all the cases below requirement ii) leads to $[D, d] = 0$, which implies that in all cases $[f_1, f_2]$ must be in $Z(\mathfrak{r}) = \mathrm{span}\{e_1, e_2\}$. Assume that

$$[f_1, f_2] = \alpha_1 e_1 + \alpha_2 e_2, \quad [f_1, e_1] = \beta_1 e_1 + \beta_2 e_2, \quad [f_1, e_2] = \gamma_1 e_1 + \gamma_2 e_2,$$
$$[f_2, e_1] = \delta_1 e_1 + \delta_2 e_2, \quad [f_2, e_2] = \varepsilon_1 e_1 + \varepsilon_2 e_2.$$

We can add an arbitrary vector from $Z(\mathfrak{r})$ to f_1 and f_2 without changing the ad-action of f_1 or f_2 on $\mathfrak{r}$. Thus we try and make $[f_1, f_2]$ as simple as possible by changing $f_1 \to f_1 + A_1 e_1 + A_2 e_2$ and $f_2 \to f_2 + B_1 e_1 + B_2 e_2$. We find that

$$[f_1 + A_i e_i, f_2 + B_j e_j] = (\alpha_\iota - A_1 \delta_\iota - A_2 \varepsilon_\iota + B_1 \beta_\iota + B_2 \gamma_\iota) e_\iota,$$

where sum over $i, j, \iota = 1, 2$ applies. Thus for given D, d all extensions with nonvanishing $[f_1, f_2]$ are equivalent to the otherwise same extension with $[f_1, f_2] = 0$ if for arbitrary α_1, α_2 we are able to find A_1, B_1, C_1, D_1 such that the equations

$$\alpha_\iota = A_1 \delta_\iota + A_2 \varepsilon_\iota - B_1 \beta_\iota - B_2 \gamma_\iota \qquad \iota = 1, 2$$

are satisfied. If $\beta_\iota, \gamma_\iota, \delta_\iota, \varepsilon_\iota = 0$ then ι. equation may not be satisfied and we must discriminate non-equivalent extensions with parameter α_ι. On the other hand if

$$\mathrm{rank} \begin{pmatrix} \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 \\ \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 \end{pmatrix} = 2$$

it is possible to satisfy both equations and thus we may set $[f_1, f_2] := 0$ without any loss of generality.

Throughout this section we denote one derivation D, its parameters with capital letters, the other is denoted d, its parameters with lower case letters:

$$D := \begin{pmatrix} 2A+C & B_5 & C_3 & B_3 & C_1 & B_1 & 0 \\ C_6 & 2A+B & C_4 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & A+C & B_5 & C_3 & B_3 & 0 \\ 0 & 0 & C_6 & A+B & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & B_5 & A_5 \\ 0 & 0 & 0 & 0 & C_6 & B & A_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & A \end{pmatrix}, \tag{2.42}$$

$$d := \begin{pmatrix} 2a+c & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ c_6 & 2a+b & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & c_6 & a+b & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b_5 & a_5 \\ 0 & 0 & 0 & 0 & c_6 & b & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}. \tag{2.43}$$

The lower 3×3 submatrices of D, d are denoted S, s respectively.

Nontriangular Derivations

From the previous section we know that it is sufficient to consider nontriangular derivations only for $\mathbb{F} = \mathbb{R}$. Assume that we have both D and d nontriangular, d of the most general form (2.8) and D of the form (2.13) which was obtained using the automorphisms. Then we can use

$$\tilde{d} := d + c_6 D$$

instead of d. Thus we can restrict ourselves to one nontriangular matrix only and, additionally, assume it already in the form (2.13); that is, we want to find all nonequivalent pairs of derivation matrices (D, d) of the form

$$D := \begin{pmatrix} 2A+C & 1 & C_3 & B_3 & C_1 & B_1 & 0 \\ -1 & 2A+C & 0 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & A+C & 1 & C_3 & B_3 & 0 \\ 0 & 0 & -1 & A+C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A \end{pmatrix},$$

$$d := \begin{pmatrix} 2a+c & b_5 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 2a+b & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & a+c & b_5 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & a+b & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b_5 & a_5 \\ 0 & 0 & 0 & 0 & 0 & b & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Given the algebra of inner derivations, the requirement $[D, d] \overset{!}{\in} \mathfrak{Inn}$ translates to the lower blocks as $[S, s] \overset{!}{=} 0$, that is:

$$\begin{pmatrix} b_5 & b-c & (C-A)a_5 + a_6 \\ b-c & b_5 & (C-A)a_6 - a_5 \\ 0 & 0 & 0 \end{pmatrix} \overset{!}{=} 0.$$

It follows that $c \overset{!}{=} b$, $b_5 \overset{!}{=} 0$. Let us assume that $a_6 \neq 0$. Then substituting $a_6 = (A-C)a_5$ into the equation $(C-A)a_6 - a_5 \overset{!}{=} 0$ yields

$$(A-C)^2 \overset{!}{=} -1,$$

which cannot be satisfied with $A, C \in \mathbb{R}$. Thus $a_6 \overset{!}{=} 0$, and it follows that $a_5 \overset{!}{=} 0$ as well.

i) $a \neq 0$
By point iv) from the beginning of this subsection we can use

$$\tilde{D} := D - \frac{A-1}{a} d$$

instead of D. Using automorphisms described in the subsection 2.2.2, we can simplify the nontriangular derivation D to the form:

$$D = \begin{pmatrix} 2+C & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2+C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+C & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1+C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The commutator $[D, d]$ is still required to be in $\mathfrak{Inn}$, that is

$$\begin{pmatrix} 0 & 0 & c_3+c_4+b_3 & b_3-c_3 & 2c_1+c_2+b_1 & 2b_1+b_2-c_1 & 0 \\ 0 & 0 & -c_3+c_4 & -b_3-c_4 & -c_1+2c_2+b_2 & -b_1+2b_2-c_2 & 0 \\ 0 & 0 & 0 & 0 & c_3+c_4+b_3 & b_3-c_3 & 0 \\ 0 & 0 & 0 & 0 & -c_3+c_4 & -b_3-c_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & u & 0 & 0 & 0 & v \\ 0 & 0 & 0 & u & 0 & 0 & w \\ 0 & 0 & 0 & 0 & u & 0 & x \\ 0 & 0 & 0 & 0 & 0 & u & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $u, v, w, x, y \in \mathrm{R}$. The only solution to this system is $b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = c_4 = 0$. Multiplying d by a^{-1} we scale a to 1. After subtracting d from D and renaming the parameters, we obtain the resulting matrices

$$D = \begin{pmatrix} C & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$d = \begin{pmatrix} 2+b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.44}$$

The corresponding extensions are given by

$\mathfrak{s}_{7+2,1}(C, b)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$Ce_1 - e_2$	$e_1 + Ce_2$	$Ce_3 - e_4$	$e_3 + Ce_4$	$Ce_5 - e_6$	$e_5 + Ce_6$	0	0
f_2	$(2+b)e_1$	$(2+b)e_2$	$(1+b)e_3$	$(1+b)e_4$	be_5	be_6	e_7	0

the dimensions of the elements of their characteristic series are

$$CS = [9,7], \quad DS = [9,7,4,0], \quad US = [0].$$

ii) $a = 0 \ \wedge \ A = 0$
From the condition of linear nil-independence we immediately see that $c \neq 0$ and thus use $c^{-1}d$ instead. Hence, the derivations are of the form

$$D = \begin{pmatrix} C & 1 & C_3 & B_3 & C_1 & B_1 & 0 \\ -1 & C & 0 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & C & 1 & C_3 & B_3 & 0 \\ 0 & 0 & -1 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ d = \begin{pmatrix} 1 & 0 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 1 & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 1 & 0 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & 1 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$[D,d] \overset{!}{\in} \mathfrak{Inn}$ yields $c_3 \overset{!}{=} 0$ and $b_3 \overset{!}{=} -c_4$. Assuming that $c_4 \neq 0$ we change from D to $D + \frac{B_3}{2c_4}d$ which after renaming the parameters yields

$$D = \begin{pmatrix} C & 1 & C_3 & B_3 & C_1 & B_1 & 0 \\ -1 & C & B_3 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & C & 1 & C_3 & B_3 & 0 \\ 0 & 0 & -1 & C & B_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Applying the automorphism (2.6) with $\gamma_3 = B_3$ we eliminate the parameter B_3. Using automorphisms described in the beginning of the subsection 2.2.2, we further simplify D to the form:

$$D = \begin{pmatrix} C & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & C & 0 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & C & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Taking the commutator once more we find that $c_1 \overset{!}{=} b_2$ and $b_1 \overset{!}{=} -c_2$. Applying the diagonal automorphism (2.5) with $\alpha_7 = c_4$ (and $\beta_6 = \gamma_5$ to preserve the form of D) we scale c_4 to 1.The derivation d then becomes

$$d = \begin{pmatrix} 1 & 0 & 0 & -1 & b_2 & -c_2 & 0 \\ 0 & 1 & 1 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding extensions are given by

$\mathfrak{s}_{7+2,2}(C, C_2, B_2, b_2, c_2)$	f_1	f_2
e_1	$Ce_1 - e_2$	e_1
e_2	$e_1 + Ce_2$	e_2
e_3	$Ce_3 - e_4$	$e_2 + e_3$
e_4	$e_3 + Ce_4$	$-e_1 + e_4$
e_5	$C_2e_2 + Ce_5 - e_6$	$b_2e_1 + c_2e_2 + e_4 + e_5$
e_6	$B_2e_2 + e_5 - Ce_6$	$-c_2e_1 + b_2e_2 + e_6$
e_7	0	0
f_1	0	0

the dimensions of the elements of their derived series are

$$CS = [9, 6], \quad DS = [9, 6, 0], \quad US = [0].$$

In case of $c_4 = 0$ we subtract Cd from D to eliminate parameter C. Using automorphisms we can further eliminate parameters C_1, B_1, and C_3. Similarly to the case of $c_4 \neq 0$ the condition $[D, d] \overset{!}{\in} \mathfrak{Inn}$ gives $c_4 = 0$, $c_1 \overset{!}{=} b_2$, $b_1 \overset{!}{=} -c_2$, and $b_3 = 0$. The resulting form of the derivations is:

$$D = \begin{pmatrix} 0 & 1 & 0 & B_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & B_3 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} 1 & 0 & 0 & 0 & b_2 & -c_2 & 0 \\ 0 & 1 & 0 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can again apply the diagonal automorphism (2.5) with $\beta_6 = \gamma_5$ and α_5 such that it scales one of the remaining parameters. We thus obtain a family of derivation pairs with representatives given above and parameters such that

$$(B_3, C_2, B_2, c_2, b_2) \in \{1\} \times \mathbb{R}^4 \ \cup\ \{0, \pm 1\} \times \mathbb{R}^3 \ \cup\ \{0, 0, \pm 1\} \times \mathbb{R}^2 \ \cup \\ \cup\ \{0, 0, 0, \pm 1\} \times \mathbb{R} \ \cup\ \{0, 0, 0, 0, \pm 1\}$$

The corresponding extensions are given by

$\mathfrak{s}_{7+2,3}(C, C_2, B_2, b_2, c_2)$	f_1	f_2
e_1	$-e_2$	e_1
e_2	e_1	e_2
e_3	$-e_4$	e_3
e_4	$B_3e_1 + e_3$	e_4
e_5	$C_2e_2 - e_6$	$b_2e_1 + c_2e_2 + e_5$
e_6	$B_2e_2 + B_3e_3 + e_5$	$-c_2e_1 + b_2e_2 + e_6$
e_7	0	0
f_1	0	0

the dimensions of the elements of their derived series are

$$CS = [9, 6], \quad DS = [9, 6, 0], \quad US = [0].$$

iii) $a = 0 \ \wedge A \ \neq 0$
Using the automorphisms described in the subsection 2.2.2 we simplify D to a quasidiagonal form:

$$D = \begin{pmatrix} 2A+C & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2A+C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A+C & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & A+C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A \end{pmatrix}.$$

Taking the commutator of D and d we see that d must be diagonal, that is

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding extensions are given by

$\mathfrak{s}_{7+2,4}(A,C)$	f_1	f_2
e_1	$(2A+C)e_1-e_2$	e_1
e_2	$e_1+(2A+C)e_2$	e_2
e_3	$(A+C)e_3-e_4$	e_3
e_4	$e_3+(A+C)e_4$	e_4
e_5	Ce_5-e_6	e_5
e_6	e_5-Ce_6	e_6
e_7	Ae_7	0
f_1	0	0

the dimensions of the elements of their derived series are

$$CS=[9,7],\quad DS=[9,7,4,0],\quad US=[0].$$

Triangular Derivations

It remains to find the extensions corresponding to pairs of triangular derivations. The possibilities of simplifying the derivations depend on the parameters. The division to cases at the lowest level is given by division in the subsection 2.2.2 on the derivation classes as it determines the availability of automorphisms applied to derivations.

1. $A\neq 0\ \vee\ a\neq 0$
The condition allows to change to

$$S=\begin{pmatrix} C & B_5 & A_5\\ 0 & B & A_6\\ 0 & 0 & 2\end{pmatrix},\, s=\begin{pmatrix} c & b_5 & a_5\\ 0 & b & a_6\\ 0 & 0 & 0\end{pmatrix}.$$

1.1. $b\neq 0$
Given that $b\neq 0$ we can transform the derivations so that

$$S=\begin{pmatrix} C & B_5 & A_5\\ 0 & 1 & A_6\\ 0 & 0 & 2\end{pmatrix},\, s=\begin{pmatrix} c & b_5 & a_5\\ 0 & 1 & a_6\\ 0 & 0 & 0\end{pmatrix}.$$

1.1.1. $C\neq \pm 1, 2, \pm 3, 5$
The conditions allow for complete diagonalization of D as described in point 1.1.1 in the subsection 2.2.2. Taking the commutator we see that the other derivation must be diagonal as well. Finally, we change from D to $D-d$.

The resulting family is given by:

$$D(C) = \begin{pmatrix} 4+C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2+C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \ d(c) = \begin{pmatrix} c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

According to vi) it is possible to set $[f_1, f_2] := 0$ if $C \neq -4$ or $c \neq 0$. In that case we obtain the extension:

$\mathfrak{s}_{7+2,5}(C,c)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$(4+C)e_1$	$4e_2$	$(2+C)e_3$	$2e_4$	Ce_5	0	$2e_7$	0
f_2	ce_1	e_2	ce_3	e_4	ce_5	e_6	0	0

$$C \neq 0 \ \vee \ c \neq 0 \quad \Longrightarrow \quad CS = [9,7], \quad DS = [9,7,4,0],$$
$$C = 0 \ \wedge \ c = 0 \quad \Longrightarrow \quad CS = [9,6], \quad DS = [9,6,3,0],$$
$$C \neq -4 \ \vee \ c \neq 0 \quad \Longrightarrow \quad US = [0],$$
$$C = -4 \ \wedge \ c = 0 \quad \Longrightarrow \quad US = [1].$$

In the case $C = -4, \ c = 0$ we get $[f_1, f_2] = \alpha_1 e_1$. The action of f_1 and f_2 on $\mathfrak{r}$ is diagonal; hence, assuming that $\alpha_1 \neq 0$, it will be the same in the basis $(\tilde{e}_1, e_2, \tilde{e}_3, e_4, \tilde{e}_5, e_6, e_7)$ with $\tilde{e}_i = \alpha_1 e_i$ for $i \in \{1,3,5\}$ and the action of e_7 will remain the same as well. But then $[f_1, f_2] = \tilde{e}_1$ and omitting the tildes we get the extension:

$\mathfrak{s}'_{7+2,5}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	0	$4e_2$	$-2e_3$	$2e_4$	$-4e_5$	0	$2e_7$	0
f_2	0	e_2	0	e_4	0	e_6	0	e_1

$$CS = [9,7], \quad DS = [9,7,4,0], \quad US = [1].$$

The case of $C = -4$, $c = 0$ and $[f_1, f_2] = 0$ was already covered under $\mathfrak{s}_{2k+1,5}$ above.

1.1.2. $C = -1$

Using automorphisms described in the preceding we can eliminate all parameters in D but B_3. The condition $[D, d] \overset{!}{\in} \mathfrak{Inn}$ demands that all non-diagonal parameters in d but b_3 vanish as well. Furthermore, if $c \neq 1$ we can apply additional automorphism to eliminate b_3 while keeping the form of D unchanged. Taking the commutator again we find that $B_3 \overset{!}{=} 0$ as well thus obtaining a special case of previous, $\mathfrak{s}_{7+2,5}(-1,c)$. Assuming that $c=1$ and $b_3\neq 0$ we apply the diagonal automorphism to scale b_3 to 1. Additionally we change from D to $D-B_3d$, introduce $B:=-\frac{B_3-1}{2}$ and take half of D to obtain

$$D(B)=\begin{pmatrix} B+1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & B+2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & B & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & B+1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & B-1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & B & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$d=\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

with the corresponding family of extensions

$\mathfrak{s}_{7+2,6}(B)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$(B+1)e_1$	$(B+2)e_2$	Be_3	$(B+1)e_4$	$(B-1)e_5$	Be_6	e_7	0
f_2	e_1	e_2	e_3	e_4	e_5	e_3+e_6	0	0

$$CS=[9,7],\quad DS=[9,7,4,0],\quad US=[0].$$

If on the other hand $b_3=0$ and $B_3\neq 0$, we scale B_3 to 2 and change from D to $\frac{1}{2}(D-d)$ to obtain

$$D=\begin{pmatrix} 1&0&0&1&0&0&0\\ 0&2&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, d=\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

with the corresponding extension

$\mathfrak{s}_{7+2,7}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	e_1	$2e_2$	0	e_1+e_4	$-e_5$	e_3	e_7	0
f_2	e_1	e_2	e_3	e_4	e_5	e_6	0	0

$$CS=[9,7],\quad DS=[9,7,4,0],\quad US=[0].$$

1.1.3. $C = -3$

This case is analogous to the preceding one. All parameters in D but B_1 can be eliminated. Unless $c \neq 1$ we can apply additional automorphism to eliminate b_1 and arrive to $\mathfrak{s}_{7+2,5}(-3, c)$. Assuming that $c = 1$ and $b_1 \neq 0$ we eliminate all parameters but b_1 and B_1, scale b_1 to one, subtract $B_1 d$ from D, multiply D by $\frac{1}{2}$ and rename the remaining parameter accordingly to arrive at

$$D(B) = \begin{pmatrix} B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B+2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the corresponding family of extension

$\mathfrak{s}_{7+2,8}(B)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	Be_1	$(B+2)e_2$	$(B-1)e_3$	$(B+1)e_4$	$(B-2)e_5$	Be_6	e_7	0
f_2	e_1	e_2	e_3	e_4	e_5	$e_1 + e_6$	0	0

$$CS = [9, 7], \quad DS = [9, 7, 4, 0], \quad US = [0].$$

In case of $b_1 = 0$ we scale B_1 to 1 instead and then change from D to $\frac{1}{2}(D - d)$ to obtain the derivations

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the corresponding extension

$\mathfrak{s}_{7+2,9}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	0	$2e_2$	$-e_3$	e_4	$-2e_5$	e_1	e_7	0
f_2	e_1	e_2	e_3	e_4	e_5	e_6	0	0

$$CS = [9, 7], \quad DS = [9, 7, 4, 0], \quad US = [0].$$

1.1.4. $C = 5$
Applying the automorphism (2.9) and subtracting $4d$ from D we come back to the preceding case.
1.1.5. $C = 3$
Applying the automorphism (2.9) and subtracting $2d$ from D we come back to the case 1.1.2.
1.1.6. $C = 2$
In case of $c \neq 0$ this leads to $\mathfrak{s}_{7+2,5}(2, c)$. Hence, we assume $c = 0$. The automorphisms allow us to eliminate every parameter in D but A_5, the condition $[D, d] \overset{!}{\in} \mathfrak{Inn}$ forces all parameters in d but a_5 to vanish. In case $a_5 \neq 0$ we scale it 1 and subtract $A_5 d$ from D and introduce $B := \frac{1-A_5}{2}$. Taking half of D then yields

$$D(B) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B+2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the corresponding family of extensions

$\mathfrak{s}_{7+2,12}(B)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$3e_1$	$(B+2)e_2$	$2e_3$	$(B+1)e_4$	e_5	Be_6	e_7	0
f_2	0	e_2	0	e_4	0	e_6	e_5	0

$$CS = [9, 7], \quad DS = [9, 7, 4, 0], \quad US = [0].$$

If on the other hand $a_5 = 0$ and $A_5 \neq 0$, we scale A_5 to 2 and change from D to $\frac{1}{2}(D-d)$ to obtain

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the corresponding extension

$\mathfrak{s}_{7+2,13}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$3e_1$	$2e_2$	$2e_3$	e_4	e_5	0	e_5+e_7	0
f_2	0	e_2	0	e_4	0	e_6	0	0

$$CS = [9,7], \quad DS = [9,7,4,0], \quad US = [0].$$

1.1.7. $C = 1$

Similarly to preceding cases, if $c \neq 1$ then this falls back to $\mathfrak{s}_{7+2,5}(1,c)$. Thus we assume that $c = 1$. In case of $b_5 \neq 0$ we scale the parameter to 1 and subtract $B_5 d$ from D. Introducing $B := \frac{1-B_5}{2}$ and taking half of D instead of D we obtain

$$D(B) = \begin{pmatrix} B+2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B+2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$d = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the corresponding family of extensions

$\mathfrak{s}_{7+2,14}(B)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$(B+2)e_1$	$(B+2)e_2$	$(B+1)e_3$	$(B+1)e_4$	Be_5	Be_6	e_7	0
f_2	e_1	e_1+e_2	e_3	e_3+e_4	e_5	e_5+e_6	0	0

$$CS = [9,7], \quad DS = [9,7,4,0], \quad US = [0].$$

If on the other hand $b_5 = 0$ and $B_5 \neq 0$, we scale B_5 to 2 and change from D to $\frac{1}{2}(D-d)$ to obtain

$$D = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the corresponding extension

$\mathfrak{s}_{7+2,15}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	$2e_1$	e_1+2e_2	e_3	e_3+e_4	0	e_5	e_7	0
f_2	e_1	e_2	e_3	e_4	e_5	e_6	0	0

$$CS = [9,7], \quad DS = [9,7,4,0], \quad US = [0].$$

1.2. $b = 0$

It is evident that $c \neq 0$ since d must not be nilpotent, hence we can change basis to the linear combinations such that the lower submatrices take up the form

$$S = \begin{pmatrix} C & B_5 & A_5 \\ & B & A_6 \\ & & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & b_5 & a_5 \\ & 0 & a_6 \\ & & 0 \end{pmatrix}.$$

Applying the automorphism (2.6) with $\beta_5 = -b_5$ we eliminate the parameter b_5 in d. Since $[S,s] \overset{!}{=} 0$ we see that the parameter in position of B_5 in D must be equal to 0 as well. Applying the automorphism (2.9) we obtain the derivations with the following lower submatrices

$$S = \begin{pmatrix} B & 0 & A_6 \\ & C & A_5 \\ & & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & a_6 \\ & 1 & a_5 \\ & & 0 \end{pmatrix},$$

which after subtracting $(C-1)d$ from D and renaming the parameters becomes

$$S = \begin{pmatrix} C & 0 & A_5 \\ & 1 & A_6 \\ & & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & a_5 \\ & 1 & a_6 \\ & & 0 \end{pmatrix}.$$

We thus came back to the case 1.1 with $c = B_5 = b_5 = 0$.

2. $A = 0 \ \wedge \ a = 0$
There exist linear combinations of the two derivations such that their lower 3×3 submatrices are

$$S := \begin{pmatrix} 1 & B_4 & A_4 \\ & 2 & A_5 \\ & & 0 \end{pmatrix}, \ s := \begin{pmatrix} 0 & b_4 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix},$$

since we demand that they are linearly nil-independent. We can use automorphism to simplify them to the form

$$S := \begin{pmatrix} 1 & 0 & 0 \\ & 2 & 0 \\ & & 0 \end{pmatrix}, \ s := \begin{pmatrix} 0 & b_4 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix}.$$

Commuting S and s we find $b_4, a_4, a_5 \overset{!}{=} 0$. The derivation are thus of the form

$$D := \begin{pmatrix} 1 & 0 & C_3 & B_3 & C_1 & B_1 & 0 \\ 0 & 2 & C_4 & 0 & C_2 & B_2 & 0 \\ 0 & 0 & 1 & 0 & C_3 & B_3 & 0 \\ 0 & 0 & 0 & 2 & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ d := \begin{pmatrix} 0 & 0 & c_3 & b_3 & c_1 & b_1 & 0 \\ 0 & 1 & c_4 & 0 & c_2 & b_2 & 0 \\ 0 & 0 & 0 & 0 & c_3 & b_3 & 0 \\ 0 & 0 & 0 & 1 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using automorphisms we eliminate B_1, C_2, B_3, C_4. Taking the commutator we find that b_1, c_2, b_3, and c_4 vanish as well. In case $c_3 \neq 0$, we scale it to one using the diagonal automorphism (2.5) and we change from D to $D - C_3 d$ obtaining D in the form

$$D := \begin{pmatrix} 1 & 0 & 0 & 0 & C_1 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where we introduced $B := 2 - C_3$. We proceed analogously in case $c_3 = 0$ and $b_2 \neq 0$ and in case $c_3 = 0$, $b_2 = 0$, and $c_1 \neq 0$ with the difference in scaling the parameter to ±1 if we assume the algebra to be above field of real numbers. In case all three parameters c_3, b_2, and c_1 are null, we scale C_3, B_2, or C_1 depending on which parameter is not null. Whenever we scale a parameter p to ±1 and there is still another nonvanishing parameter q that could have been scaled to 1 instead, we can still choose the diagonal automorphism so that $q \geq 0$. The resulting family of extensions is given by matrices

$$D := \begin{pmatrix} 1 & 0 & C_3 & 0 & C_1 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & 1 & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ d := \begin{pmatrix} 0 & 0 & c_3 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} &(c_3, b_2, c_1, B, C_3, B_2, C_1) \in \\ &\quad \{1\} \times \mathbb{F}^3 \times \{0\} \times \mathbb{F}^2 \ \cup \ \{(0, (\pm)1)\} \times \mathbb{F}^2 \times \mathbb{F}_0^+ \times \{0\} \times \mathbb{F} \cup \\ &\quad \{(0, 0, (\pm)1)\} \times \mathbb{F} \times \mathbb{F}_0^+ \times \mathbb{F} \times \{0\} \ \cup \ \{(0, 0, 0, 2, 1)\} \times \mathbb{F}^2 \cup \\ &\quad \{(0, 0, 0, 2, 0, (\pm)1)\} \times \mathbb{F} \ \cup \ \{(0, 0, 0, 2, 0, 0, (\pm)1)\} \end{aligned} \tag{2.45}$$

The commutation relations and dimensions of members of characteristic series are as follows

$\mathfrak{s}_{7+2,24}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1
f_1	e_1	Be_2	$C_3e_1 + e_3$	Be_4	$C_1e_1 + C_3e_3 + e_5$	$B_2e_2 + Be_6$	0	0
f_2	0	e_2	c_3e_1	e_4	$c_1e_1 + c_3e_3$	$b_2e_2 + e_6$	0	0

$$CS = [9, 6], \quad DS = [9, 6, 0], \quad US = [0].$$

2.2.4 Extensions by Three Vectors

Nontriangular Derivations

By the same argument as in the preceding subsection 2.2.3, it suffices to assume only one of the derivations in the non-triangular form and only in case of $\mathbb{F} = \mathbb{R}$. Let us denote the derivation matrices by $\mathcal{D}, D, d$ and the lower 3×3 submatrices

$$\mathcal{S} = \begin{pmatrix} \mathcal{C} & 1 & 0 \\ -1 & \mathcal{C} & 0 \\ 0 & 0 & \mathcal{A} \end{pmatrix}, \ S = \begin{pmatrix} C & B_5 & A_5 \\ 0 & B & A_6 \\ 0 & 0 & A \end{pmatrix}, s = \begin{pmatrix} c & b_5 & a_5 \\ 0 & b & a_6 \\ 0 & 0 & a \end{pmatrix}.$$

By the conditions $[\mathcal{S}, S] \stackrel{!}{=} 0$ and $[\mathcal{S}, s]$ the nondiagonal parameters in S and s vanish and $C \stackrel{!}{=} B$ and $c \stackrel{!}{=} b$. Since the matrices must be linearly nil-independent, we require that (A, B) and (a, b) are linearly independent. Hence, we can change to triangular matrices of the form

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and subtracting $(\mathcal{A}-1)S+\mathcal{C}s$ from $\mathcal{S}$ we obtain the nontriangular derivation submatrix in the form

$$\mathcal{S} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding derivation matrix $\mathcal{D}$ can be quasidiagonalized using the automorphisms, so that it takes up the form

$$\mathcal{D} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Looking at the commutators, we see that all nondiagonal parameters in both D and d vanish and we obtain

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the matrices, we can see the actions of $f_1 \leftrightarrow \mathcal{D}$, $f_2 \leftrightarrow D$, and $f_3 \leftrightarrow d$ on $\mathfrak{r}$. Assuming that

$$[f_1, f_2] = \alpha_\iota e_\iota, \quad [f_1, f_3] = \beta_\iota e_\iota, \quad [f_2, f_3] = \gamma_\iota e_\iota.$$

we can change from f_1, f_2, f_3 to

$$\tilde{f}_1 := f_1 + A_\iota e_\iota, \quad \tilde{f}_2 := f_1 + B_\iota e_\iota, \quad \tilde{f}_3 := f_1 + C_\iota e_\iota$$

and we can find values of A_ι, B_ι, and C_ι such that some commutators of $(\tilde{f}_i)_1^3$ vanish. Namely, we set

$$C_\iota := 0, \quad B_\iota := -\gamma_\iota, \quad A_\iota := -\beta_\iota \qquad \forall \iota \in \{1, 2\}$$

which after dropping the tildes yields $[f_1, f_3] = 0$ and $[f_2, f_3] = 0$. From the Jacobi identity we have that $[f_1, f_2] = 0$ as well. Thus we have obtained the extension

$\mathfrak{s}_{7+3,1}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1	f_2
f_1	$2e_1 - e_2$	$e_1 + 2e_2$	$e_3 - e_4$	$e_3 + e_4$	$-e_6$	e_5	e_7	0	0
f_2	$2e_1$	$2e_2$	e_3	e_4	0	0	e_7	0	0
f_3	e_1	e_2	e_3	e_4	e_5	e_6	0	0	0

$$CS = [10, 7], \quad DS = [10, 7, 4, 0], \quad US = [0].$$

Triangular Derivations

Assuming all three derivations $\mathcal{D}$, D, and d are in the triangular form, the condition of linear nil-independence translates to linear independence of the diagonals of the derivation matrices which allows arbitrary choice of diagonal parameters due to the generic form (2.11). We choose the parameters so that $diag(\mathcal{S}) = (1, 0, 2)$, $diag(S) = (0, 1, 0)$, and $diag(s) = (1, 0, 0)$. Then the first derivation $\mathcal{D}$ can be fully diagonalized using the automorphisms. From the condition $[\mathcal{D}, D], [\mathcal{D}, d] \in \mathfrak{Inn}$ we see that the nondiagonal parameters in D and d vanish as well. After changing from $\mathcal{D}$ to $\frac{1}{2}(\mathcal{D} + d)$, we have the resulting triple in the form

$$\mathcal{D} = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The argument for vanishing commutators of f_1, f_2, f_3 follows the same route as in the previous case. Thus, we have obtained the second extension of $\mathfrak{s}$ by three vectors:

$\mathfrak{s}_{7+3,2}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_1	f_2
f_1	$5e_1$	$4e_2$	$3e_3$	$2e_4$	e_5	0	e_7	0	0
f_2	0	e_2	0	e_4	0	e_6	0	0	0
f_3	e_1	0	e_3	0	e_5	0	0	0	0

$$CS = [10, 7], \quad DS = [10, 7, 4, 0], \quad US = [0].$$

Chapter 3

Solvable Extensions of Nilradical of Arbitrary Dimension

All solvable extensions of $\mathfrak{r} \equiv \mathfrak{r}_{2k-1} \equiv \text{span}(e_1, \dots, e_{2k-1})$ with the sole non-trivial ad-action $[e_i, e_{2k-1}] = e_{i-2}$ for every $i \in \{3, \dots, 2k-2\}$ and for all $k \in \{4, 5, \dots\}$ are found in this chapter. Results from the preceding one are used extensively.

3.1 Properties of $\mathfrak{r}_{2k-1}$

We first examine the nilpotent algebra $\mathfrak{r}_{2k-1}$ analogously to the special case of $k = 4$.

3.1.1 Inner Derivations

The inner derivations corresponding to the basal elements as defined above are

$$\text{ad}_{e_1} = 0, \qquad \text{ad}_{e_2} = 0, \tag{3.1}$$

$$(\text{ad}_{e_i})_{j\ell} = \begin{cases} -1 & \text{for } j = i-2,\ \ell = 2k-1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in \{3, \dots, 2k-2\}, \tag{3.2}$$

$$(\text{ad}_{e_{2k-1}})_{ij} = \begin{cases} 1 & \text{for } i = j-2 \neq 2k-3 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

3.1.2 Derived Series, Flag of Ideals

The characteristic series are as follows

$$\mathfrak{r}_{2k-1}^{\ell} = \mathrm{span}(e_1, \dots, e_{2(k-\ell)}) \quad \forall \ell \in (2, \dots, k-1),$$
$$\mathfrak{r}_{2k-1}^{(1)} = \mathrm{span}(e_1, \dots, e_{2k-4}), \qquad \mathfrak{r}_{2k-1}^{(2)} = 0,$$
$$\mathfrak{z}^{\ell} = \mathrm{span}(e_1, \dots, e_{2\ell}) \quad \forall \ell \in (1, \dots, k-2), \qquad \mathfrak{z}^{k-1} = \mathfrak{r}_{2k-1}.$$

Analogously to the seven-dimensional case, there is one more invariant ideal to be found

$$Z_{\mathfrak{r}}(\mathfrak{r}^2) = \mathrm{span}(e_1, \dots, e_{2k-2}).$$

The argument showing that the subspaces $\mathrm{span}(e_i)_1^{\ell}$, $\ell \in \{1, 3, \dots, 2k-3\}$ are not invariant ideals follows the same route as in the preceding chapter, that is we apply the automorphism swapping $e_i \leftrightarrow e_{i+1}$ for every $i \in \{1, 3, \dots, 2k-3\}$ on each of these subspaces and see that the image is outside of the given subspace.

3.1.3 Automorphisms

Based on commutation relations defining $\mathfrak{r}$, we find the generic form of automorphism of $\mathfrak{r}$ to be

$$\Phi = \begin{pmatrix} \alpha_{2k-1}^{k-2}\gamma_{2k-3} & \alpha_{2k-1}^{k-2}\beta_{2k-3} & \cdots & \gamma_3 & \beta_3 & \gamma_1 & \beta_1 & \alpha_1 \\ \alpha_{2k-1}^{k-2}\gamma_{2k-2} & \alpha_{2k-1}^{k-2}\beta_{2k-2} & \cdots & \gamma_4 & \beta_4 & \gamma_2 & \beta_2 & \alpha_2 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & \alpha_{2k-1}\gamma_{2k-3} & \alpha_{2k-1}\beta_{2k-3} & \gamma_{2k-5} & \beta_{2k-5} & \alpha_{2k-5} \\ & & & \alpha_{2k-1}\gamma_{2k-2} & \alpha_{2k-1}\beta_{2k-2} & \gamma_{2k-4} & \beta_{2k-4} & \alpha_{2k-4} \\ & & & & & \gamma_{2k-3} & \beta_{2k-3} & \alpha_{2k-3} \\ & & & & & \gamma_{2k-2} & \beta_{2k-2} & \alpha_{2k-2} \\ & & & & & & & \alpha_{2k-1} \end{pmatrix} \tag{3.4}$$

Given that $\gamma_{2k-3} \neq 0$, we can decompose the automorphism into

$$\Phi = \Phi_L \circ \Phi_D \circ \Phi_U, \tag{3.5}$$

where Φ_L, Φ_D, and Φ_U are analogous to the seven-dimensional case. After renaming the parameters, they take up the following form

$$\Phi_L = \begin{pmatrix} 1 & 0 & & & 0 & \\ \gamma_{2k-2} & 1 & & & 0 & \\ & & \ddots & & 0 & \\ & & & 1 & 0 & \\ & & & \gamma_{2k-2} & 1 & \\ & & & & 0 & 1 \end{pmatrix}, \tag{3.6}$$

$$\Phi_D = \begin{pmatrix} \alpha_{2k-1}^{k-2}\gamma_{2k-3} & & & & & & \\ & \alpha_{2k-1}^{k-2}\beta_{2k-2} & & & & & \\ & & \ddots & & & & \\ & & & \alpha_{2k-1}\gamma_{2k-3} & & & \\ & & & & \alpha_{2k-1}\beta_{2k-2} & & \\ & & & & & \gamma_{2k-3} & \\ & & & & & & \beta_{2k-2} & \\ & & & & & & & \alpha_{2k-1} \end{pmatrix}, \tag{3.7}$$

$$\Phi_U = \begin{pmatrix} 1 & \beta_{2k-3} & \gamma_{2k-5} & \cdots & \gamma_1 & \beta_1 & \alpha_1 \\ & 1 & \gamma_{2k-4} & \cdots & \gamma_2 & \beta_2 & \alpha_2 \\ & & 1 & \ddots & \vdots & \vdots & \vdots \\ & & & \ddots & \gamma_{2k-4} & \beta_{2k-4} & \alpha_{2k-4} \\ & & & & 1 & \beta_{2k-3} & \alpha_{2k-3} \\ & & & & & 1 & \alpha_{2k-2} \\ & & & & & & 1 \end{pmatrix}. \tag{3.8}$$

There is one more significant automorphism used widely in the following section

$$\Phi_P = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \end{pmatrix}. \tag{3.9}$$

This automorphism pairwise swaps $(2\ell-1)$-th and 2ℓ-th row and column for every $\ell \in \{1, \dots, k-1\}$.

3.1.4 Generic Outer Derivation

Varying the parameters of the generic automorphism Φ (3.4) around $\Phi = \mathbb{1}$ yields the outer derivation in the following form

$$\begin{pmatrix} (k-2)a_{2k-1} + c_{2k-3} & b_{2k-3} & \cdots & c_3 & b_3 & c_1 & b_1 & a_1 \\ c_{2k-2} & (k-2)a_{2k-1} + b_{2k-2} & & c_4 & b_4 & c_2 & b_2 & a_2 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & a_{2k-1} + c_{2k-3} & b_{2k-3} & c_{2k-5} & b_{2k-5} & a_{2k-5} \\ & & & c_{2k-2} & a_{2k-1} + b_{2k-2} & c_{2k-4} & b_{2k-4} & a_{2k-4} \\ & & & & & c_{2k-3} & b_{2k-3} & a_{2k-3} \\ & & & & & c_{2k-2} & b_{2k-2} & a_{2k-2} \\ & & & & & & & a_{2k-1} \end{pmatrix}$$

Subtracting $a_i \mathrm{ad}_{e_{i+2}}$ for every $i \in 1, \dots, 2k-4$ and $b_{2k-4}\mathrm{ad}_{e_{2k-1}}$ we can eliminate the parameters $a_1, \dots, a_{2k-4}$, and b_{2k-4}. The form of the matrix is such that the whole derivation is determined by its last three columns as all the parameters are contained in them. To make the notation more feasible, we assume the derivation matrices in the following to observe the general form and only write the last three columns explicitly. The generic outer derivation is thus given by

$$D = \begin{pmatrix} \cdots & c_1 & b_1 & 0 \\ \cdots & c_2 & b_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & c_{2k-5} & b_{2k-5} & 0 \\ \cdots & c_{2k-4} & 0 & 0 \\ \cdots & c_{2k-3} & b_{2k-3} & a_{2k-3} \\ \cdots & c_{2k-2} & b_{2k-2} & a_{2k-2} \\ \cdots & & & a_{2k-1} \end{pmatrix} \equiv$$

$$\begin{pmatrix} (k-2)a_{2k-1}+c_{2k-3} & b_{2k-3} & \cdots & c_3 & b_3 & c_1 & b_1 & 0 \\ c_{2k-2} & (k-2)a_{2k-1}+b_{2k-2} & & c_4 & b_4 & c_2 & b_2 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & a_{2k-1}+c_{2k-3} & b_{2k-3} & c_{2k-5} & b_{2k-5} & 0 \\ & & & c_{2k-2} & a_{2k-1}+b_{2k-2} & c_{2k-4} & 0 & 0 \\ & & & & & c_{2k-3} & b_{2k-3} & a_{2k-3} \\ & & & & & c_{2k-2} & b_{2k-2} & a_{2k-2} \\ & & & & & & & a_{2k-1} \end{pmatrix} \tag{3.10}$$

3.2 Extensions

We seek to generalize the results of the section 2.2 on extensions from the preceding chapter to an arbitrary dimension of $\mathfrak{r}$. The possibilities of simplifying the lower $j \times j$ submatrix are the same as in seven-dimensional case for any $j \leq 7$. This is specifically true for $j = 3$ and $j = 4$ which allows us to follow the same division by relations among the parameters of these submatrices leading to different simplification options for them.

3.2.1 Extensions by One Element

Non-triangular Case

The inequality (2.12) determining the possibility of transforming the derivation into upper triangular matrix generalizes to the following form for k arbitrary

$$4b_{2k-3}c_{2k-2} < -(b_{2k-2} - c_{2k-3})^2. \tag{3.11}$$

After steps analogous to the seven-dimensional case, we have the derivation matrix in the form

$$\begin{pmatrix} (k-2)a+c & 1 & c_{2k-5} & b_{2k-5} & \cdots & c_3 & b_3 & c_1 & b_1 & 0 \\ -1 & (k-2)a+c & 0 & 0 & \cdots & c_4 & b_4 & c_2 & b_2 & 0 \\ & & (k-3)a+c & 1 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & -1 & (k-3)a+c & \ddots & c_{2k-5} & b_{2k-5} & c_{2k-7} & b_{2k-7} & 0 \\ & & & & \ddots & 0 & 0 & c_{2k-6} & b_{2k-6} & 0 \\ & & & & & a+c & 1 & c_{2k-5} & b_{2k-5} & 0 \\ & & & & & -1 & a+c & 0 & 0 & 0 \\ & & & & & & & c & 1 & 0 \\ & & & & & & & -1 & c & 0 \\ & & & & & & & & & a \end{pmatrix} \tag{3.12}$$

i. $a \neq 0$

We eliminate the remaining nondiagonal parameters in $k-3$ steps. In every step we apply the automorphism (3.8) with

$$\begin{aligned} \beta_{2(k-\ell)-1} &= \frac{\ell^2 a^2 b_{2(k-\ell)-1} - \ell a(b_{2(k-\ell)} - c_{2(k-\ell)-1}) + 2b_{2(k-\ell)-1} - 2c_{2(k-\ell)}}{-\ell a(\ell^2 a^2 + 4)}, \\ \gamma_{2(k-\ell)-1} &= \frac{\ell^2 a^2 c_{2(k-\ell)-1} - \ell a(b_{2(k-\ell)-1} + c_{2(k-\ell)}) + 2c_{2(k-\ell)-1} + 2b_{2(k-\ell)}}{-\ell a(\ell^2 a^2 + 4)}, \\ \beta_{2(k-\ell)} &= \frac{\ell^2 a^2 b_{2(k-\ell)} + \ell a(b_{2(k-\ell)-1} + c_{2(k-\ell)}) + 2c_{2(k-\ell)-1} + 2b_{2(k-\ell)}}{-\ell a(\ell^2 a^2 + 4)}, \\ \beta_{2(k-\ell)} &= \frac{\ell^2 a^2 c_{2(k-\ell)} - \ell a(b_{2(k-\ell)} - c_{2(k-\ell)-1}) - 2c_{2(k-\ell)-1} + 2b_{2(k-\ell)}}{-\ell a(\ell^2 a^2 + 4)} \end{aligned}$$

to remove parameters $b_{2(k-\ell)-1}$, $b_{2(k-\ell)}$, $c_{2(k-\ell)-1}$, and $c_{2(k-\ell)}$ with ℓ running from 3 to $k-1$. After the transformations, we obtain the resulting family of classes

$$[D_1] = \left\{ \left. \begin{pmatrix} (k-2)a+c & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -1 & (k-2)a+c & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & a+c & 1 & 0 & 0 & 0 \\ & & & -1 & a+c & 0 & 0 & 0 \\ & & & & & c & 1 & 0 \\ & & & & & -1 & c & 0 \\ & & & & & & & a \end{pmatrix} \right| \begin{matrix} c \in \mathbb{R}_0^+ \\ a \in \mathbb{R} \end{matrix} \right\}, \tag{3.13}$$

and the corresponding extensions by one element

$$\mathfrak{s}_{2k,1}(a,c) = \operatorname{span}(e_1, \dots, e_{2k-1}, f_1),$$

where adjoint action of f_1 on the remaining elements is given by

$$\begin{aligned}
[e_{2(k-\ell)-1}, f_1] &= ((\ell-1)a+c)e_{2(k-\ell)-1} - e_{2(k-\ell)} && \forall \ell \in \{k-1, \dots, 1\}, \\
[e_{2(k-\ell)}, f_1] &= e_{2(k-\ell)-1} + ((\ell-1)a+c)e_{2(k-\ell)} && \forall \ell \in \{k-1, \dots, 1\}, \\
[e_{2k-1}, f_1] &= ae_{2k-1}.
\end{aligned} \tag{3.14}$$

It follows that the dimensions of the derived series of these solvable extensions are

$$\begin{aligned}
a \neq 0 &\implies \text{CS} = [2k, 2k-1], \ \text{DS} = [2k, 2k-1, 2k-3, 0], \\
a = 0 &\implies \text{CS} = [2k, 2k-2], \ \text{DS} = [2k, 2k-2, 0], \\
&\qquad\qquad \text{US} = [0].
\end{aligned} \tag{3.15}$$

ii. $a = 0$

The first step is to apply the automorphism (3.8) with $\beta_{2k-5} = -\frac{c_{2k-5}}{2}$ and add $\frac{c_{2k-5}}{2}\mathrm{ad}_{e_{2k-1}}$ to eliminate c_{2k-5}. Analogously to the seven-dimensional case we can eliminate only two out of four parameters in the blocks of $c_{2\ell-1}$, $c_{2\ell}$, $b_{2\ell-1}$, $b_{2\ell}$. We do this in $k-3$ steps eliminating c_{2k-7} and b_{2k-7} in the first step and c_1 and b_1 in the last one. The automorphisms applied are of the form (3.8) with

$$\gamma_{2\ell-1} = b_{2\ell-1}, \qquad \gamma_{2\ell} = -c_{2\ell-1}, \tag{3.16}$$

in $(k-\ell-2)$th step, that is ℓ runs from $k-3$ in the first step to 1 in the last step. The resulting matrix is given by

$$\begin{pmatrix}
\cdots & 0 & 0 & 0 \\
\cdots & c_2 & b_2 & 0 \\
\cdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 \\
\cdots & c_{2k-6} & b_{2k-6} & 0 \\
\cdots & 0 & b_{2k-5} & 0 \\
\cdots & 0 & 0 & 0 \\
\cdots & c & 1 & 0 \\
\cdots & -1 & c & 0 \\
\cdots & 0 & 0 & 0
\end{pmatrix} \tag{3.17}$$

It is possible to scale one of the remaining parameters using the diagonal automorphism (3.7); fixing $\gamma_{2k-3} = \beta_{2k-2}$ so that the lower 3×3 block is not disrupted, the remaining parameters are transformed as

$$b_{2k-5} \longmapsto \frac{b_{2k-5}}{\alpha_{2k-1}}, \qquad c_{2(k-i)} \longmapsto \frac{c_{2(k-i)}}{\alpha_{2k-1}^{i-1}}, \qquad b_{2(k-i)} \longmapsto \frac{b_{2(k-i)}}{\alpha_{2k-1}^{i-1}},$$

for every i in $\{3, \dots, k-1\}$. We choose which of the parameters we scale as follows: If there is at least one nonvanishing parameter that can be scaled to 1, we pick one with the greatest index, taking c_i before b_i. If there is no parameter with odd power of α_{2k-1} left, we scale the nonvanishing parameter with greatest index (again, taking c_i before b_i) and scale it to ± 1. The resulting family of the classes of the derivation matrices is given by

$$[D_2] = \left\{ \begin{pmatrix} \cdots & 0 & 0 & 0 \\ \cdots & c_2 & b_2 & 0 \\ \cdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & c_{2k-6} & b_{2k-6} & 0 \\ \cdots & 0 & b_{2k-5} & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & c & 1 & 0 \\ \cdots & -1 & c & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \; c \in \mathbb{R}_0^+, \begin{pmatrix} b_{2k-5} \\ c_{2k-8} \\ b_{2k-8} \\ \vdots \\ c_{2(2-k+2\lfloor \frac{k}{2} \rfloor)} \\ b_{2(2-k+2\lfloor \frac{k}{2} \rfloor)} \\ c_{2k-6} \\ b_{2k-6} \\ \vdots \\ c_{2(1+k+2\lfloor \frac{k}{2} \rfloor)} \\ b_{2(1+k+2\lfloor \frac{k}{2} \rfloor)} \end{pmatrix} \in \mathcal{A}_2 \right\}, \tag{3.18}$$

where

$$\begin{aligned} \mathcal{A}_2 \equiv \{1\} \times \mathbb{R}^{2k-6} \; \cup \; \dots \; \cup \; \theta_{2\lfloor \frac{k-3}{2} \rfloor} \times \{1\} \times \mathbb{R}^{2\lfloor \frac{k-2}{2} \rfloor} \; \cup \\ \theta_{2\lfloor \frac{k-3}{2} \rfloor + 1} \times \{\pm 1\} \times \mathbb{R}^{2\lfloor \frac{k-2}{2} \rfloor - 1} \; \cup \; \dots \; \cup \; \theta_{2k-6} \times \{\pm 1\}. \end{aligned} \tag{3.19}$$

The corresponding extensions $\mathfrak{s}_{2k,2}(c, \vec{c}, \vec{b})$ are given by the set $\mathcal{A}_2$ and the extending commutation relations

$$\begin{aligned} [e_{2k-1}, f_1] &= 0, \\ \forall \ell \in \{1, \dots, k-3\} : & \\ [e_{2(k-\ell)}, f_1] &= \sum_{i=1}^{k-2-\ell} \left(b_{2(i-1+\ell)} e_{2i} \right) + b_{2k-5} e_{2(k-\ell)-3} + e_{2(k-\ell)-1} + c e_{2(k-\ell)}, \\ \forall \ell \in \{1, \dots, k-3\} : & \\ [e_{2(k-\ell)-1}, f_1] &= \sum_{i=1}^{k-2-\ell} \left(c_{2(i-1+\ell)} e_{2i} \right) + c e_{2(k-\ell)-1} - e_{2(k-\ell)}, \\ [e_4, f_1] &= b_{2k-5} e_1 + e_3 + c e_4, \\ [e_3, f_1] &= c e_3 - e_4, \\ [e_2, f_1] &= e_1 + c e_2, \\ [e_1, f_1] &= c e_1 - e_2. \end{aligned} \tag{3.20}$$

The dimensions of the elements of the derived series are

$$\mathrm{CS} = [2k, 2k-2], \quad \mathrm{DS} = [2k, 2k-2, 0], \quad \mathrm{US} = [0]. \tag{3.21}$$

Triangular Derivations

We proceed similarly as in the seven-dimensional case, that is at the first level we divide the cases leading to different 3×3 lower submatrices, at the second level we discriminate whether it is possible to eliminate c_{2k-3} and the third one corresponds to the rest of the parameters of the derivation. The process for obtaining the derivation in the triangular form is described in the preceding chapter. The resulting triangular matrix is of the form

$$\begin{pmatrix} (k-2)a+c & b_{2k-3} & \cdots & c_3 & b_3 & c_1 & b_1 & 0 \\ & (k-2)a+b & & c_4 & b_4 & c_2 & b_2 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & a+c & b_{2k-3} & c_{2k-5} & b_{2k-5} & 0 \\ & & & & a+b & c_{2k-4} & 0 & 0 \\ & & & & & c & b_{2k-3} & a_{2k-3} \\ & & & & & & b & a_{2k-2} \\ & & & & & & & a \end{pmatrix} \tag{3.22}$$

1. $a \neq b \neq c \neq a$
The lower 3×3 submatrix is diagonalized.

1.1. $c \neq a + b$
The parameter c_{2k-4} is eliminated.
1.1.1. $a \neq 0$, $ja + b \neq c$ for $j \in \{2, \dots, k-2\}$ and $\ell a + c \neq b$ for every $\ell \in \{1, \dots, k-2\}$
The derivation is diagonalized in $k-2$ steps using the automorphism (3.8) with

$$\gamma_{2(k-\ell)-4} = -\frac{c_{2(k-\ell)-4}}{(\ell+1)a + b - c}, \qquad \beta_{2(k-\ell)-4} = -\frac{b_{2(k-\ell)-4}}{(\ell+1)a},$$
$$\gamma_{2(k-\ell)-3} = -\frac{b_{2(k-\ell)-3}}{\ell a}, \qquad \beta_{2(k-\ell)-3} = -\frac{b_{2(k-\ell)-3}}{\ell a + c - b}$$

in the ℓ-th step for $\ell = 1, \dots, k-3$ and with $\gamma_1 = -\frac{c_1}{(k-2)a}$, $\beta_1 = -\frac{b_1}{(k-2)a+c-b}$ in the last step. Since $a \neq 0$ we can further multiply by a^{-1} to get a two-parametric family of classes

$$[D_3] = \left\{ \begin{pmatrix} \ddots & & & \\ & c & & \\ & & b & \\ & & & 1 \end{pmatrix} \middle| \begin{matrix} b \in \mathbb{F} \\ c \in \mathbb{F} \end{matrix} \right\}$$

and the corresponding extensions $\mathfrak{s}_{2k,3}(b,c) = \operatorname{span}(e_1, \dots, e_{2k}, f_1)$ with

$$\begin{aligned} [e_{2k-1}, f_1] &= e_{2k-1}, \\ [e_{2\ell}, f_1] &= (k-\ell-1+b)e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2\ell-1}, f_1] &= (k-\ell-1+c)e_{2\ell-1}, && \forall \ell \in \{1, \dots, k-1\} \end{aligned}$$

and dimensions of elements of lower and derived series

$$\begin{aligned} b, c \neq 0 \implies CS = [2k, 2k-1], \quad DS = [2k, 2k-1, 2k-4, 0], \\ b \cdot c = 0,\ b+c \neq 0 \implies CS = [2k, 2k-2], \quad DS = [2k, 2k-2, 2k-5, 0], \\ b, c = 0 \implies CS = [2k, 2k-3], \quad DS = [2k, 2k-3, 2k-6, 0] \end{aligned}$$

and upper central series

$$\begin{aligned} b \neq 2-k,\ c \neq 2-k \implies US = [0], \\ b = 2-k,\ c \neq 2-k \ \vee\ b \neq 2-k,\ c = 2-k \implies US = [1], \\ b = 2-k,\ c = 2-k \implies US = [2]. \end{aligned}$$

1.1.2. $b = ia + c$ for some $i \in \{1, \dots, k-2\} \implies a \neq 0$ and $c \neq ja + b$ for $j = 2, \dots, k-2$

We first justify the implication above. Since $ia + c = b$ and $b \neq c$ it is evident that $a \neq 0$. Further, assume that $ja + b = c$ for some j. Then $ia + c = ia + ja + b = b$, that is $i + j \stackrel{!}{=} 0$ since $a \neq 0$, thus $j = -i$. Therefore, the inequality $ja + b \neq c$ holds for every positive j.

In this case we are unable to eliminate parameter $b_{2(k-i)-3}$, however we can eliminate all the other parameters using the same automorphisms as in the preceding case with the exception of setting $\beta_{2(k-i)-3} = 0$ instead of the value above. We multiply the derivation by a^{-1} and apply the automorphism (3.7) with $\gamma_{2k-3} = b_{2(k-i)-3}$ thus scaling the nondiagonal parameter to 1. We have obtained $k-2$ different one-parametric sets of classes

$$[D_4^{(i)}] = \left\{ \left(\begin{array}{ccccc} & & \vdots & & \\ & & 1 & & \\ & \ddots & \vdots & & \\ & & c & & \\ & & & i+c & \\ & & & & 1 \end{array} \right) \middle| \begin{array}{c} i \in \{1, \dots, k-2\} \\ c \in \mathbb{F} \end{array} \right\}$$

and the corresponding extensions $\mathfrak{s}_{2k,4}^{(i)}(c)$ given by the relations

$$\begin{aligned} [e_{2k-1}, f_1] &= e_{2k-1}, \\ [e_{2(k-\ell)}, f_1] &= e_{2(k-\ell-i)-1} + (i+\ell-1+c)e_{2(k-\ell)} && \forall \ell \in \{1, \dots, k-i-1\}, \\ [e_{2(k-\ell)}, f_1] &= (i+\ell-1+c)e_{2(k-\ell)} && \forall \ell \in \{k-i, \dots, k-1\}, \\ [e_{2\ell-1}, f_1] &= (k-\ell-1+c)e_{2\ell-1}, && \forall \ell \in \{1, \dots, k-1\} \end{aligned}$$

with dimensions of the elements of their lower and derived series

$$c \notin \{0,-i\} \implies CS = [2k, 2k-1], \quad DS = [2k, 2k-1, 2k-4, 0],$$
$$c \in \{0,-i\} \implies CS = [2k, 2k-2], \quad DS = [2k, 2k-2, 2k-5, 0]$$

and of their upper central series

$$c \notin \{k-2, k-2-i\} \implies US = [0],$$
$$c \in \{k-2, k-2-i\} \implies US = [1].$$

1.1.3. $c = ia + b$ for some $i \in \{2, \dots, k-2\}$
Applying the automorphism (3.9) we come back to the preceding case.
1.1.4. $a = 0$
It is possible to eliminate b_1, c_2, b_3, ..., c_{2k-6}, b_{2k-5} in $k-2$ steps using the automorphism (3.8) with

$$\beta_{2(k-\ell)-3} = \frac{b_{2(k-\ell)-3}}{c-b}, \qquad \gamma_{2(k-\ell-2)} = \frac{c_{2(k-\ell-2)}}{c-b}$$

in ℓ-th step for $\ell = 1, \dots, k-3$ and $\beta_1 = \frac{b_1}{c-b}$ in the last step. Depending on the remaining parameters c_{2k-5}, b_{2k-6}, c_{2k-7}, ..., b_2, c_1 we can scale one of them to $(\pm)1$; we pick the parameter to scale in the same manner as in the case of $\mathfrak{s}_{2k,2}$ above, which yields the range $\mathcal{A}_5$ (3.23). Multiplying by b^{-1} and scaling the first non-zero parameter we obtain families given by

$$[D_5] = \left\{ \begin{pmatrix} \cdots & c_1 & 0 & 0 \\ \cdots & 0 & b_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 0 & b_{2k-6} & 0 \\ \cdots & c_{2k-5} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & c & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \; c \in \mathbb{F}, \begin{pmatrix} c_{2k-5} \\ b_{2k-8} \\ c_{2k-9} \\ \vdots \\ b_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \\ b_{2k-6} \\ c_{2k-7} \\ \vdots \\ b_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \end{pmatrix} \in \mathcal{A}_5 \right\},$$

where

$$\begin{aligned} \mathcal{A}_5 \equiv \{1\} \times \mathbb{F}^{2k-6} \;\cup\; \dots \;\cup\; \theta_{2\left\lfloor \frac{k-3}{2} \right\rfloor} \times \{1\} \times \mathbb{F}^{2\left\lfloor \frac{k-2}{2} \right\rfloor} \;\cup \\ \theta_{2\left\lfloor \frac{k-3}{2} \right\rfloor+1} \times \{(\pm)1\} \times \mathbb{F}^{2\left\lfloor \frac{k-2}{2} \right\rfloor-1} \;\cup\; \dots \;\cup\; \theta_{2k-6} \times \{(\pm)1\}. \end{aligned} \tag{3.23}$$

with $\pm$ valid for $\mathbb{F}=\mathbb{R}$. The commutation relations defining the corresponding extensions $\mathfrak{s}_{2k,5}(c, c_1, b_2, \dots, c_{2k-5})$ are

$$\begin{aligned}
[e_{2k-1}, f_1] &= 0, \\
[e_{2(k-\ell)}, f_1] &= \sum_{i=1}^{k-2-\ell} b_{2(i-1+\ell)} e_{2i} + e_{2(k-\ell)} && \forall \ell \in \{1, \dots, k-3\}, \\
[e_{2(k-\ell)-1}, f_1] &= \sum_{i=1}^{k-1-\ell} c_{2(i-1+\ell)-1} e_{2i-1} + c e_{2(k-\ell)-1} && \forall \ell \in \{1, \dots, k-2\}, \\
[e_4, f_1] &= e_4, \quad [e_2, f_1] = e_2, \quad [e_1, f_1] = c e_1,
\end{aligned}$$

the dimensions of the elements of their characteristic series are

$$\begin{aligned}
c \neq 0 &\implies CS = [2k, 2k-2],\ DS = [2k, 2k-2, 0],\ US = [0], \\
c = 0 &\implies CS = [2k, 2k-3],\ DS = [2k, 2k-3, 0],\ US = [1, \dots, k-2],
\end{aligned}$$

1.2. $c = a + b$

Applying the automorphism (3.9) we come back to the case 1.1.2.

2. $a = b \neq c$

The parameters b_{2k-3}, a_{2k-3} are eliminated, while a_{2k-2} remains.

2.1. $c \neq 2a$

The parameter c_{2k-4} is eliminated.

2.1.1. $a \neq 0$, $c \neq ia$ for all $i \in \{-k+3, \dots, 0\} \cup \{3, \dots, k-2\}$

Following the same steps as in 1.1.1, we eliminate all remaining nondiagonal parameters but a_{2k-2}. Since $a \neq 0$, we can multiply the derivation by a^{-1}. Furthermore, we can assume that $a_{2k-2} \neq 0$, as we would obtain the extension $\mathfrak{s}_{2k,3}(1, c)$ otherwise. Hence, it is possible to scale it to one. We thus obtain one-parametric family of derivation classes

$$[D_6] = \left\{ \left. \begin{pmatrix} \ddots & & & \\ & c & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \right| c \in \mathbb{F} \right\}$$

and the corresponding extensions $\mathfrak{s}_{2k,6}(c) = \operatorname{span}(e_1, \dots, e_{2k}, f_1)$ with

$$\begin{aligned}
[e_{2k-1}, f_1] &= e_{2k-2} + e_{2k-1}, \\
[e_{2\ell}, f_1] &= (k - \ell - 1 + b) e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\
[e_{2\ell-1}, f_1] &= (k - \ell - 1 + c) e_{2\ell-1}, && \forall \ell \in \{1, \dots, k-1\}
\end{aligned}$$

and the dimensions of the elements of their lower and derived series

$$\begin{aligned} c \neq 0 &\implies CS = [2k, 2k-1], \quad DS = [2k, 2k-1, 2k-4, 0], \\ c = 0 &\implies CS = [2k, 2k-2], \quad DS = [2k, 2k-2, 2k-5, 0], \end{aligned}$$

and of their upper central series

$$\begin{aligned} c \neq 2-k &\implies US = [0], \\ c = 2-k &\implies US = [1]. \end{aligned}$$

2.1.2. $c = ia$ for some $i \in \{-k+3, \dots, 0\}$
This is again analogous to the case 1.1.2; it is not possible to eliminate the parameter $b_{2(k+i)-5}$, while all the other remaining parameters are eliminated using the same steps as in 1.1.2. Once more, we multiply by a^{-1} and scale the remaining parameters to 1 using the automorphism (3.7) with $\beta_{2k-2} = a_{2k-2}$ and $\gamma_{2k-3} = b_{2(k+i)-5}a_{2k-2}$. The resulting classes are of the form

$$[D_7^{(i)}] = \left\{ \left(\begin{array}{cccc} & & \vdots & \\ & & 1 & \\ \ddots & & \vdots & \\ & i & & \\ & & 1 & 1 \\ & & & 1 \end{array} \right) \middle| \, i \in \{-k+3, \dots, 0\} \right\},$$

the corresponding extensions $\mathfrak{s}_{2k,7}^{(i)}$ are given by the relations

$$\begin{aligned} [e_{2k-1}, f_1] &= e_{2k-2} + e_{2k-1}, & & \\ [e_{2(k-\ell)}, f_1] &= e_{2(k-\ell+i)-3} + \ell e_{2(k-\ell)} & & \forall \ell \in \{1, \dots, k+i-2\}, \\ [e_{2(k-\ell)}, f_1] &= \ell e_{2(k-\ell)} & & \forall \ell \in \{k+i-1, \dots, k-1\}, \\ [e_{2\ell-1}, f_1] &= (k-\ell-1+i)e_{2\ell-1}, & & \forall \ell \in \{1, \dots, k-1\}; \end{aligned}$$

the dimensions of the elements of their characteristic series are

$$\begin{aligned} i \neq 0 &\implies CS = [2k, 2k-1],\ DS = [2k, 2k-1, 2k-4, 0],\ US = [0], \\ i = 0 &\implies CS = [2k, 2k-2],\ DS = [2k, 2k-2, 2k-5, 0],\ US = [0]. \end{aligned}$$

2.1.3. $c = ia$ for some $i \in \{3, \dots, k-2\}$
Following the same steps as in 1.1.1, except for setting $\gamma_{2(k-i)} = 0$, we eliminate all parameters except $c_{2(k-i)}$. It is again possible to scale both parameters to 1 simultaneously and the cases of either of them being zero were

already covered. Hence, we come to the classes of derivations given by

$$[D_8^{(i)}] = \left\{ \left(\begin{array}{cccc} & \vdots & & \\ & 1 & & \\ \ddots & \vdots & & \\ & i & & \\ & & 1 & 1 \\ & & & 1 \end{array} \right) \middle| \, i \in \{3, \dots, k-2\} \right\}$$

with the corresponding extensions $\mathfrak{s}_{2k,8}^{(i)}$ given by the relations

$$\begin{aligned} [e_{2k-1}, f_1] &= e_{2k-2} + e_{2k-1}, && \\ [e_{2\ell}, f_1] &= (k-\ell-1)e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2(k-\ell)-1}, f_1] &= e_{2(k-i-\ell+1)} + (i+\ell-1)e_{2(k-\ell)-1} && \forall \ell \in \{1, \dots, k-i\}, \\ [e_{2(k-\ell)-1}, f_1] &= (i+\ell-1)e_{2(k-\ell)-1} && \forall \ell \in \{k-i, \dots, k-2\}; \end{aligned}$$

the dimensions of the elements of their characteristic series are

$$CS = [2k, 2k-1], \; DS = [2k, 2k-1, 2k-4, 0], \; US = [0].$$

2.1.4. $a = 0$
Taking same steps as in 1.1.4 we arrive to

$$[D_9] = \left\{ \begin{pmatrix} \cdots & c_1 & 0 & 0 \\ \cdots & 0 & b_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 0 & b_{2k-6} & 0 \\ \cdots & c_{2k-5} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} c_{2k-5} \\ b_{2k-8} \\ c_{2k-9} \\ \vdots \\ b_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \\ b_{2k-6} \\ c_{2k-7} \\ \vdots \\ b_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \end{pmatrix} \in \mathcal{A}_5 \right\},$$

where $\mathcal{A}_5$ is given in (3.23). The commutation relations for the corresponding extensions $\mathfrak{s}_{2k,9}(c_1, b_2, \dots, c_{2k-5})$ are

$$
\begin{aligned}
[e_{2k-1}, f_1] &= e_{2k-2}, \\
[e_{2(k-\ell)}, f_1] &= \sum_{i=1}^{k-2-\ell} b_{2(i+1-\ell)} e_{2i} && \forall \ell \in \{1, \dots, k-3\}, \\
[e_{2(k-\ell)-1}, f_1] &= \sum_{i=1}^{k-1-\ell} c_{2(i+1-\ell)-1} e_{2i-1} + e_{2(k-\ell)-1} && \forall \ell \in \{1, \dots, k-2\}, \\
[e_4, f_1] &= 0, \quad [e_2, f_1] = 0, \quad [e_1, f_1] = c e_1,
\end{aligned}
$$

the dimensions of the elements of their characteristic series are

$$CS = [2k, 2k-2],\ DS = [2k, 2k-2, 0],\ US = [1, \dots, k-2].$$

2.2. $c = 2a$

Skipping the automorphism applied in 1.1, we follow the same steps as in 1.1.1 with $c = 2a = 2b$. After multiplying by a^{-1} and scaling the remaining parameters to 1 using the diagonal automorphism (3.7) with $\beta_{2k-2} = a_{2k-2}$ and $\gamma_{2k-3} = a_{2k-2} c_{2k-4}^{-1}$ we obtain the derivation class given by

$$
[D_{10}] = \begin{pmatrix}
\ddots & \ddots & & & \\
& 2 & 1 & & \\
& & 2 & 0 & \\
& & & 1 & 1 \\
& & & & 1
\end{pmatrix},
$$

with the corresponding extension $\mathfrak{s}_{2k,10}$ given by the commutation relations

$$
\begin{aligned}
[e_{2\ell-1}, f_1] &= e_{2\ell-2} + (k-\ell+1) e_{2\ell-1} && \forall \ell \in \{2, \dots, k\}, \\
[e_{2\ell}, f_1] &= (k-\ell) e_{2\ell} && \forall \ell \in \{1, \dots, k-1\}, \\
[e_1, f_1] &= k e_1.
\end{aligned}
$$

The dimensions of the elements of the characteristic series are

$$CS = [2k, 2k-2],\ DS = [2k, 2k-2, 0],\ US = [0].$$

3. $a = c \neq b$

Analogously to the seven-dimensional case, we eliminate the parameters a_{2k-2} and b_{2k-3} and apply the swapping automorphism (3.9) thus reducing this case to the case 2.

4. $a \neq b = c$

The parameters a_{2k-3}, a_{2k-2} are eliminated, while the parameter b_{2k-3} remains.

4.1. $a \neq 0$

By applying the automorphism (3.8) with $\gamma_{2k-4} = -\frac{c_{2k-4}}{a}$, the parameter c_{2k-4} is eliminated. The other parameters are eliminated in $2k-5$ steps, using the automorphism (3.8) in every one of them. In the first step we set the automorphism parameters to:

$$\gamma_{2k-5} = -\frac{c_{2k-5}}{a}, \quad \gamma_{2k-6} = -\frac{c_{2k-6}}{2a}.$$

As a result, the parameters c_{2k-5} and c_{2k-6} vanish from the derivation. In the second step we eliminate the parameters c_{2k-7}, b_{2k-5}, and b_{2k-6} by setting the automorphism parameters to

$$\gamma_{2k-7} = -\frac{c_{2k-7}}{2a}, \quad \beta_{2k-5} = -\frac{b_{2k-5}}{a}, \quad \beta_{2k-6} = -\frac{b_{2k-6}}{2a}.$$

In the ℓ-th step, the automorphism used is given by:

$$\begin{aligned} \ell \text{ odd}: \qquad & \gamma_{2k-\ell} = -\frac{c_{2k-\ell}}{\ell a}, \quad \beta_{2k-\ell} = -\frac{b_{2k-\ell}}{(\ell-1)a}, \\ \ell \text{ even}: \qquad & \gamma_{2k-\ell} = -\frac{c_{2k-\ell}}{\ell a}, \quad \beta_{2k-\ell} = -\frac{b_{2k-\ell}}{\ell a}, \end{aligned}$$

where ℓ runs from 3 to $2k-6$. After these steps we are left with b_1 to eliminate; this is achieved by setting

$$\beta_1 = -\frac{b_1}{(k-2)a}$$

in the automorphism. After multiplication by a^{-1}, we scale b_{2k-3} to 1 and obtain family of derivations given by

$$[D_{11}(b)] = \left\{ \left(\begin{array}{ccccc} \ddots & \ddots & & & \\ & b+1 & 0 & & \\ & & b & 1 & \\ & & & b & 0 \\ & & & & 1 \end{array} \right) \middle| \, b \in \mathbb{F} \right\}.$$

The corresponding extensions $\mathfrak{s}_{2k,11}(b)$ are given by commutation relations

$$\begin{aligned} [e_{2k-1}, f_1] &= e_{2k-1}, \\ [e_{2\ell}, f_1] &= e_{2\ell-1} + (k-1-\ell+b)e_{2\ell} \qquad && \forall \ell \in \{1, \ldots, k-1\}, \\ [e_{2\ell-1}, f_1] &= (k-1-\ell+b)e_{2\ell-1} && \forall \ell \in \{1, \ldots, k-1\}. \end{aligned}$$

The dimensions of the elements of their derived and lower central series are

$$\begin{aligned} b \neq 0 &\implies CS = [2k, 2k-1], \quad DS = [2k, 2k-1, 2k-4, 0], \\ b = 0 &\implies CS = [2k, 2k-2], \quad DS = [2k, 2k-2, 2k-5, 0], \end{aligned}$$

while the dimensions of the elements of the upper central series are

$$\begin{aligned} b \neq 2-k &\implies US = [0], \\ b = 2-k &\implies US = [1, 2]. \end{aligned}$$

4.2. $a = 0$
Due to the requirement of non-nilpotency of the derivation, the parameter b must not be null. Hence, we multiply by b^{-1} and the lower 3×3 block changes to

$$\begin{pmatrix} 1 & b_{2k-3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The possibilities of simplifying the derivation matrix depend on whether certain parameters in the matrix are null. Any block consisting of c_{2i-1}, b_{2i-1}, c_{2i}, b_{2i} can be transformed as

$$\begin{pmatrix} c_{2i-1} & b_{2i-1} \\ c_{2i} & b_{2i} \end{pmatrix} \overset{\Phi_P}{\longmapsto} \begin{pmatrix} b_{2i} & c_{2i} \\ b_{2i-1} & c_{2i-1} \end{pmatrix} \tag{3.24}$$

along with the rest of the matrix transforming accordingly. Hence, if $c_{2i} = 0$ and $b_{2i-1} \neq 0$ we can apply Φ_P and rename the parameters accordingly to obtain $c_{2i} \neq 0$ (and $b_{2i-1} = 0$). This allows us to only consider the case of $c_{2i} \neq 0$, b_{2i-1} arbitrary and both $c_{2i} = 0$, $b_{2i-1} = 0$ provided that the change in the remainder of the matrix is irrelevant – that is if any other two swapped parameters are either both the same or both arbitrary. The same argument holds for b_{2i} and c_{2i-1}.

4.2.1. $b_{2k-3} \neq 0$
We first apply the automorphism (3.8) with $\gamma_{2k-4} = -\frac{c_{2k-5}}{2b_{2k-3}}$ and then subtract $\frac{c_{2k-5}}{2}\mathrm{ad}_{e_{2k-1}}$ to eliminate parameter c_{2k-5}. Then we apply $2k-5$ automorphisms of the form (3.8) with

$$\gamma_i = \frac{b_i}{2b_{2k-3}},$$

with i running from $2k-5$ to 1 thus eliminating the parameters $b_{2k-5}, \dots, b_1$. After scaling the parameter b_{2k-3} to 1 the derivation matrix takes up the form

$$\begin{pmatrix} \cdots & c_1 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & c_{2k-7} & 0 & 0 \\ \cdots & c_{2k-6} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & c_{2k-4} & 0 & 0 \\ \cdots & 1 & 1 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix}.$$

One of the remaining parameters can be scaled to $(\pm)1$, where $\pm$ applies to $c_{2(k-i)}$, $c_{2(k-i)-1}$ for i even and $\mathbb{F} = \mathbb{R}$. The resulting family of derivation matrices is given by

$$[D_{12}] = \left\{ \begin{pmatrix} \cdots & c_1 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & c_{2k-7} & 0 & 0 \\ \cdots & c_{2k-6} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & c_{2k-4} & 0 & 0 \\ \cdots & 1 & 1 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} c_{2k-4} \\ c_{2k-8} \\ c_{2k-9} \\ \vdots \\ c_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \\ c_{2k-6} \\ c_{2k-7} \\ \vdots \\ c_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \end{pmatrix} \in \mathcal{A}_5 \right\},$$

where the range for the parameters, $\mathcal{A}_5$, is given in (3.23). The corresponding extensions $\mathfrak{s}_{2k,12}(c_1, \dots, c_{2k-6}, c_{2k-7})$ are given by commutation relations

$$\begin{aligned} &[e_{2k-1}, f_1] = 0, \\ \forall \ell \in \{1, \dots, k-1\} : &\\ &[e_{2\ell}, f_1] = e_{2\ell-1} + e_{2\ell} \\ \forall \ell \in \{1, \dots, k-3\} : &\\ &[e_{2(k-\ell)-1}, f_1] = \sum_{i=1}^{2(k-\ell-1)} c_{i+2\ell-2} e_i + c_{2k-4} e_{2(k-\ell-1)} + e_{2(k-\ell)-1}, \\ &[e_3, f_1] = c_{2k-4} e_2 + e_3, \quad [e_1, f_1] = e_1 \end{aligned}$$

and the range for the parameters given above. The dimensions of the elements of the characteristic series do not depend on the parameters and are following

$$CS = [2k, 2k-2], \; DS = [2k, 2k-2, 0], \; US = [0].$$

4.2.2. $b_{2k-3} = 0$, $c_{2k-4} \neq 0$

As in the preceding case, we first eliminate the parameter c_{2k-5} by applying the automorphism (3.8) with $\beta_{2k-3} = \frac{c_{2k-5}}{2c_{2k-4}}$ and adding $\frac{c_{2k-5}}{2}\mathrm{ad}_{2k-4}$. Then we apply a series of $k-3$ automorphisms of the form (3.8) with

$$\beta_{2i} = \frac{c_{2i-2}}{c_{2k-4}}, \quad \beta_{2i-1} = \frac{c_{2i-3}}{c_{2k-4}}$$

and i running from $k-2$ to 2. This eliminates the parameters $c_{2k-6}, \dots, c_1$. Using the automorphism (3.7) with $\beta = c_{2k-4}$, we scale the parameter c_{2k-4} to 1. It is further possible to scale one of the remaining parameters to $(\pm)1$, the resulting family is given by

$$[D_{13}] = \left\{ \begin{pmatrix} \cdots & 0 & b_1 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 0 & b_{2k-6} & 0 \\ \cdots & 0 & b_{2k-5} & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} b_{2k-5} \\ b_{2k-8} \\ b_{2k-9} \\ \vdots \\ b_{2\left(2-k+2\lfloor\frac{k}{2}\rfloor\right)} \\ b_{2\left(2-k+2\lfloor\frac{k}{2}\rfloor\right)-1} \\ b_{2k-6} \\ b_{2k-7} \\ \vdots \\ b_{2\left(1+k+2\lfloor\frac{k}{2}\rfloor\right)} \\ b_{2\left(1+k+2\lfloor\frac{k}{2}\rfloor\right)-1} \end{pmatrix} \in \mathcal{A}_5 \right\},$$

where the range for the parameters, $\mathcal{A}_5$, is given in (3.23). The corresponding extensions $\mathfrak{s}_{2k,13}(b_1, \dots, b_{2k-5})$ are given by commutation relations

$$\begin{aligned} [e_{2k-1}, f_1] &= 0, \\ [e_{2(k-\ell)}, f_1] &= \sum_{i=1}^{2(k-\ell)-3} b_{i+2\ell-2} e_i + e_{2(k-\ell)} \qquad && \forall \ell \in \{1, \dots, k-3\}, \\ [e_{2(k-\ell)-1}, f_1] &= e_{2(k-\ell-1)} + e_{2(k-\ell)-1} && \forall \ell \in \{1, \dots, k-2\}, \end{aligned}$$

$$[e_4, f_1] = e_4, \quad [e_2, f_1] = e_2, \quad [e_1, f_1] = e_1$$

and the range for the parameters given above. The dimensions of the elements of the characteristic series do not depend on the parameters and are following

$$CS = [2k, 2k-2],\ DS = [2k, 2k-2, 0],\ US = [0].$$

4.2.3. $b_{2k-3} = 0$, $c_{2k-4} = 0$, $c_{2k-5} \neq 0$
We apply $k-3$ automorphisms of the form (3.8) with

$$\gamma_{2(k-i-1)} = \frac{c_{2(k-i-2}}{c_{2k-5}}, \quad \beta_{2(k-i)-3} = -\frac{b_{2(k-i)-5}}{c_{2k-5}}$$

in i-th step. This eliminates the parameters b_1, c_2, ..., b_{2k-7}, c_{2k-6}. One of the remaining parameters may be scaled to $(\pm)1$. The resulting family is given by

$$[D_{14}] = \left\{ \begin{pmatrix} \cdots & c_1 & 0 & 0 \\ \cdots & 0 & b_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & c_{2k-7} & 0 & 0 \\ \cdots & 0 & b_{2k-6} & 0 \\ \cdots & c_{2k-5} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} c_{2k-5} \\ b_{2k-8} \\ c_{2k-9} \\ \vdots \\ b_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(2-k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \\ b_{2k-6} \\ c_{2k-7} \\ \vdots \\ b_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)} \\ c_{2\left(1+k+2\left\lfloor \frac{k}{2} \right\rfloor\right)-1} \end{pmatrix} \in \mathcal{A}_5 \right\},$$

where the range for the parameters, $\mathcal{A}_5$, is given in (3.23). The corresponding extensions $\mathfrak{s}_{2k,14}(c_1, b_2, \ldots, b_{2k-6}, c_{2k-5})$ are given by commutation relations

$$\begin{aligned} [e_{2k-1}, f_1] &= 0, \\ [e_{2(k-\ell)}, f_1] &= \sum_{i=1}^{k-\ell-2} b_{2(\ell-1+i)} e_{2i} + e_{2(k-\ell)} && \forall \ell \in \{1, \ldots, k-3\}, \\ [e_{2(k-\ell)-1}, f_1] &= \sum_{i=1}^{k-\ell-1)} c_{2(\ell+i)-3} e_{2i-1} + e_{2(k-\ell)-1} && \forall \ell \in \{1, \ldots, k-2\}, \end{aligned}$$

$$[e_4, f_1] = e_4, \quad [e_2, f_1] = e_2, \quad [e_1, f_1] = e_1$$

and the range for the parameters given above. The dimensions of the elements of the characteristic series do not depend on the parameters and are following

$$CS = [2k, 2k-2],\ DS = [2k, 2k-2, 0],\ US = [0].$$

4.2.4. $\exists i \in \{3, \ldots, k-1\} : c_{2(k-i)} \neq 0 \wedge (\forall j > i : c_{2j} = 0 \wedge b_{2j-1} = 0 \wedge b_{2j} = c_{2j-1})$

In this case we apply $k-i$ automorphism of the form (3.8) with $\beta_{2k-3} = \frac{c_{2(k-i)-1}}{c_{2(k-i)}}$ in the first step and with

$$\beta_{2(k-\ell+1)+1} = \frac{c_{2(k-i-\ell)+1}}{c_{2(k-i)}}, \quad \beta_{2(k-\ell+1)} = \frac{c_{2(k-i-\ell)}}{c_{2(k-i)}}$$

in the subsequent steps with ℓ running from 2 to $k-i$. This eliminates the parameters $c_1, \dots, c_{2(k-i)-1}$. The resulting family is given by

$$[D_{15}^{(i)}] = \left\{ \begin{pmatrix} \cdots & 0 & b_1 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 0 & b_{2(k-i)-1} & 0 \\ \cdots & 1 & b_{2(k-i)} & 0 \\ \cdots & b_{2(k-i+1)} & 0 & 0 \\ \cdots & 0 & b_{2(k-i+1)} & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & b_{2k-6} & 0 & 0 \\ \cdots & 0 & b_{2k-6} & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} b_{2k-8} \\ b_{2k-12} \\ \vdots \\ b_{2\left(2-k+2\lfloor \frac{k}{2} \rfloor\right)} \\ b_{2(k-i)-3} \\ b_{2(k-i)-7} \\ \vdots \\ b_{2\left(1+k+i-2\lfloor \frac{k+i}{2} \rfloor\right)} \\ b_{2k-6} \\ b_{2k-10} \\ \vdots \\ b_{2\left(1+k-2\lfloor \frac{k}{2} \rfloor\right)} \\ b_{2(k-i)-1} \\ b_{2(k-i)-5} \\ \vdots \\ b_{2\left(2-k-i+2\lfloor \frac{k+i}{2} \rfloor\right)} \end{pmatrix} \in \mathcal{A}_{15}^{(i)} \right\},$$

where

$$\begin{aligned} \mathcal{A}_{15}^{(i)} = \ & \{1\} \times \mathbb{F}^{2k-i-4} \ \cup \ \dots \ \cup \ \theta_{\lfloor \frac{k-3}{2} \rfloor + \lfloor \frac{k-i}{2} \rfloor - 1} \times \{1\} \times \mathbb{F}^{\lfloor \frac{k-2}{2} \rfloor + \lfloor \frac{k-i+1}{2} \rfloor} \ \cup \\ & \theta_{\lfloor \frac{k-3}{2} \rfloor + \lfloor \frac{k-i}{2} \rfloor} \times \{(\pm)1\} \times \mathbb{F}^{\lfloor \frac{k-2}{2} \rfloor + \lfloor \frac{k-i+1}{2} \rfloor - 1} \ \cup \ \dots \ \cup \ \theta_{2k-i-4} \times \{(\pm)1\} \end{aligned} \tag{3.25}$$

The corresponding extensions $\mathfrak{s}_{2k,15}^{(i)}(b_1, \dots, b_{2(k-i)})$ are given by the range for the parameters given above and the following commutation relations:

$$[e_{2k-1}, f_1] = 0,$$

$$\forall \ell \in \{1, \dots, k-i\}:$$

$$[e_{2(k-\ell)}, f_1] = \sum_{j=1}^{2(k-i+1-\ell)} b_{j+2\ell-2} e_j \ + \sum_{j=k-i-1}^{k-2-\ell} b_{2j-1+\ell} e_{2j} \ + \ e_{2(k-\ell)},$$

$$[e_{2(k-\ell)-1}, f_1] = e_{2(k-i+1-\ell)} + \sum_{j=k-i-1}^{k-2-\ell} b_{2j-1+\ell} e_{2j-1} \ + \ e_{2(k-\ell)-1},$$

where sums with upper limit lower than the lower limit are to be understood as 0. The dimensions of the elements of the characteristic series do not depend on the parameters and are following

$$CS = [2k, 2k-2],\ DS = [2k, 2k-2, 0],\ US = [0].$$

4.2.5. $\exists i \in \{3, \dots, k-1\} : b_{2(k-i)} \neq c_{2(k-i)-1} \wedge (\forall j \geq i : c_{2j} = 0 \wedge b_{2j-1} = 0)$ $\wedge(\forall j > i : b_{2j} = c_{2j-1})$
This case can reduced to the preceding one in the following way. We first apply the automorphism (3.8) with $\beta_{2k-3} = \frac{1}{c_{2(k-i)-1} - b_{2(k-i)}}$. This produces 1 in place of $b_{2(k-i)-1}$ and after applying the automorphism (3.9) it moves to the place of $c_{2(k-i)}$.

5. $a = b = c$
Since we require the derivation matrix to be non-nilpotent, we can multiply the derivation by a^{-1}. It is again possible to eliminate all nondiagonal parameters outwith the lower 3×3 block regardless of the parameters within. The parameter c_{2k-4} is eliminated using the automorphism (3.8) with $\gamma_{2k-4} = -c_{2k-4}$. The rest is eliminated using the automorphism of the same form with

$$\gamma_{2k-5} = -c_{2k-5}, \quad \gamma_{2k-6} = -\frac{c_{2k-6}}{2}$$

in the first step, with

$$\begin{aligned} \gamma_{2(k-\ell)-3} &= -\frac{c_{2(k-\ell)-3}}{\ell}, & \gamma_{2(k-\ell-2)} &= -\frac{c_{2(k-\ell-2)}}{\ell+1}, \\ \beta_{2(k-\ell)-1} &= -\frac{b_{2(k-\ell)-1}}{\ell-1}, & \beta_{2(k-\ell-1)} &= -\frac{b_{2(k-\ell-1)}}{\ell} \end{aligned}$$

in ℓ-th step for $\ell \in \{2, \dots, k-3\}$, with

$$\gamma_1 = -\frac{c_1}{k-2}, \quad \beta_3 = -\frac{b_3}{k-3}, \quad \beta_2 = -\frac{b_2}{k-2}$$

in $(k-2)$-th step, and with

$$\beta_1 = -\frac{b_1}{k-2}$$

in the last step. As in the seven-dimensional case the case of either of b_{2k-3}, a_{2k-2} being zero is already covered by the preceding. The parameter a_{2k-3} is eliminated by applying the automorphism (3.8) with $\alpha_{2k-2} = -\frac{a_{2k-3}}{a_{2k-2}}$. The remaining parameters are scaled to one using the automorphism (3.7) with

$\alpha_{2k-1} = a_{2k-2}^{-1}$ and $\gamma_{2k-3} = b_{2k-3}$. We have thus obtained a class of derivations given by

$$[D_{16}] = \begin{pmatrix} \ddots & \ddots & & & & \\ & & 2 & 0 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$$

The corresponding extensions $\mathfrak{s}_{2k,16}(b)$ are given by commutation relations

$$\begin{aligned} [e_{2k-1}, f_1] &= e_{2k-2} + e_{2k-1}, & \\ [e_{2\ell}, f_1] &= e_{2(k-\ell)-1} + (k-\ell)e_{2(k-\ell)} & \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2\ell-1}, f_1] &= (k-\ell)e_{2(k-\ell)-1} & \forall \ell \in \{1, \dots, k-1\}. \end{aligned}$$

The dimensions of the elements of their characteristic series are

$$CS = [2k, 2k-1],\ DS = [2k, 2k-1, 2k-4, 0],\ US = [0].$$

3.2.2 Extensions by Two Elements

All solvable extensions of $\mathfrak{r}_{2k-1}$ are found in this subsection. The approach is very similar to that in subsection 2.2.3, the points i) through vi) from the beginning of the aforementioned subsection are referenced throughout this section and the notation D, d and S, s has the analogous meaning. It is easily seen that the behaviour in the lower 4×4 submatrices is exactly the same regardless of k. Thus the previous results obtained for the seven-dimensional ($k = 4$) case are used here without special reference and the cases are divided analogously.

Nontriangular Derivations

i) $a \neq 0$

Having the nontriangular derivation in the form (3.12), we apply the automorphisms described in point i) in the beginning of the subsection 3.2.1 to simplify the derivation to the form (3.13) with $a = 1$. The commutator of the triangular and nontriangular derivation is of the form of generic derivation (3.10) and thus given by its last three columns

$$\begin{pmatrix}
\cdots & (k-2)c_1+c_2+b_1 & (k-2)b_1+b_2-c_1 & 0\\
\cdots & -c_1+(k-2)c_2+b_2 & -b_1+(k-2)b_2-c_2 & 0\\
\cdots & (k-3)c_3+c_4+b_3 & (k-3)b_3+b_4-c_3 & 0\\
\cdots & -c_1+(k-2)c_2+b_4 & -b_3+(k-3)b_4-c_4 & 0\\
\cdots & \vdots & \vdots & \vdots\\
\cdots & 2c_{2k-7}+c_{2k-6}+b_{2k-7} & 2b_{2k-7}+b_{2k-6}-c_{2k-7} & 0\\
\cdots & -c_{2k-7}+2c_{2k-6}+b_{2k-6} & -b_{2k-7}+2b_{2k-6}-c_{2k-6} & 0\\
\cdots & c_{2k-5}+c_{2k-4}+b_{2k-5} & b_{2k-5}-c_{2k-5} & 0\\
\cdots & -c_{2k-5}+c_{2k-4} & -b_{2k-5}-c_{2k-4} & 0\\
\cdots & 0 & 0 & 0\\
\cdots & 0 & 0 & 0\\
\cdots & 0 & 0 & 0
\end{pmatrix}.$$

Comparing the lowest two non-zero rows with the inner derivations, we see that

$$b_{2k-5}-c_{2k-5}=0, \quad -c_{2k-5}+c_{2k-4}=0,$$
$$c_{2k-5}+c_{2k-4}+b_{2k-5}=-b_{2k-5}-c_{2k-4},$$

which leads to $c_{2k-4}=c_{2k-5}=b_{2k-5}=0$. All the remaining cells must be equal to 0, while they form two by two blocks following the same pattern:

$$\begin{pmatrix} \ell c_{2i-1}+c_{2i}+b_{2i-1} & \ell b_{2i-1}+b_{2i}-c_{2i-1}\\ -c_{2i-1}+\ell c_{2i}+b_{2i} & -b_{2i-1}+\ell b_{2i}-c_{2i}\end{pmatrix},$$

where i runs from 1 to $k-3$ and ℓ is an integer. Comparing the antidiagonal expressions, we get that $b_{2i-1}=c_{2i}$, comparing the diagonal expressions we get that $b_{2i}=-c_{2i-1}$. Substituting these results back to the first column and comparing the expressions, we get $(\ell^2+4)c_{2i}=0$, but since ℓ is a positive integer, it follows that $c_{2i}=0$ and thus also $b_{2i-1}=0$. Substituting this back into the original expressions we immediately see that $c_{2i-1}=b_{2i}=0$ as well. Hence, the triangular matrix must be diagonal. Taking a^{-1} multiple of it and subtracting from D we obtain the following pair

$$D=\begin{pmatrix} \ddots & 0 & & \\ & C & 1 & \\ & -1 & C & 0\\ & 0 & 0 & 0\end{pmatrix}, \qquad d=\begin{pmatrix} \ddots & & & \\ & b & & \\ & & b & \\ & & & 1\end{pmatrix}$$

with the corresponding family of extensions $\mathfrak{s}_{2k+1,1}(C,b)$ given by

$\mathfrak{s}_{2k+1,1}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k-1\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k-1}
f_1	$Ce_{2\ell-1}-e_{2\ell}$	$e_{2\ell-1}+Ce_{2\ell}$	0
f_2	$(k-1-\ell+b)e_{2\ell-1}$	$(k-1-\ell+b)e_{2\ell}$	e_{2k-1}

the dimensions of the elements of their characteristic series are

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0], \quad US = [0].$$

ii) $a = 0 \ \wedge \ A = 0$
We proceed analogously to the seven-dimensional case. The lower 4×4 blocks are of the form:

$$\begin{pmatrix} C & 0 & 0 & 0 \\ 0 & C & 1 & 0 \\ 0 & -1 & C & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & c_{2k-4} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, the condition of the commutator being in the algebra of inner derivations yields $c_{2k-5} = 0$ and $b_{2k-5} = c_{2k-4}$ Let us first assume that $c_{2k-4} \neq 0$. Then we change from D to $D + \frac{B_{2k-5}}{2c_{2k-4}}$ and apply the automorphism (3.8) with $\gamma_{2k-5} = \frac{B_{2k-5}}{2}$ to D thus eliminating B_{2k-5}. After applying the series of automorphisms described in point ii. of the subsection 3.2.1, we get the nontriangular derivation in the form

$$\begin{pmatrix} \cdots & 0 & 0 & 0 \\ \cdots & C_2 & B_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & C_{2k-6} & B_{2k-6} & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & C & 1 & 0 \\ \cdots & -1 & C & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix}.$$

The condition $[D, d] \overset{!}{\in} \mathfrak{Inn}$ yields

$$c_{2k-5} \overset{!}{=} 0, \qquad c_{2k-7} = b_{2k-6}, \qquad b_{2k-7} = -c_{2k-6},$$

$$\begin{aligned} &\forall \ell \in \{4, \dots, k-1\} : \\ &\quad c_{2(k-\ell)-1} \overset{!}{=} \tilde{c}_{2(k-\ell)-1} \equiv \sum_{i=1}^{\ell-3} B_{2(k+i-\ell)} c_{2(k-i-1)} + b_{2(k-\ell)}, \\ &\quad b_{2(k-\ell)-1} \overset{!}{=} \tilde{b}_{2(k-\ell)-1} \equiv \sum_{i=1}^{\ell-3} C_{2(k+i-\ell)} c_{2(k-i-1)} - c_{2(k-\ell)}. \end{aligned} \tag{3.26}$$

After scaling c_{2k-4} to 1 using the automorphism (3.7) with $\alpha_{2k-1} = c_{2k-4}$, we obtain the triangular derivation in the final form

$$\begin{pmatrix}
\cdots & \tilde{c}_1 & \tilde{b}_1 & 0 \\
\cdots & c_2 & b_2 & 0 \\
 & \vdots & \vdots & \vdots \\
\cdots & \tilde{c}_{2k-9} & \tilde{b}_{2k-9} & 0 \\
\cdots & c_{2k-8} & b_{2k-8} & 0 \\
\cdots & b_{2k-6} & -c_{2k-6} & 0 \\
\cdots & c_{2k-6} & b_{2k-6} & 0 \\
\cdots & 0 & -1 & 0 \\
\cdots & 1 & 0 & 0 \\
\cdots & 1 & 0 & 0 \\
\cdots & 0 & 1 & 0 \\
\cdots & 0 & 0 & 0
\end{pmatrix},$$

where $\tilde{c}_\ell$, $\tilde{b}_\ell$ are given above in (3.26).
The corresponding algebras $\mathfrak{s}_{2k+1,2}(C, C_{2i}, B_{2i}, c_{2i}, b_{2i})_{i=1}^{k-3}$ are given by relations

$$\begin{aligned}
&[e_{2k-1}, f_1] = [e_{2k-1}, f_2] = 0, \\
&\forall \ell \in \{1, \dots, k-3\} : \\
&[e_{2(k-\ell)}, f_1] = \sum_{i=1}^{k-2-\ell} \Big(B_{2(i-1+\ell)} e_{2i}\Big) + e_{2(k-\ell)-1} + C e_{2(k-\ell)}, \\
&[e_{2(k-\ell)}, f_2] = \sum_{i=1}^{k-3-\ell} \Big(\tilde{b}_{2(i-1+\ell)-1} e_{2i-1}\Big) + \sum_{i=1}^{k-2-\ell} \Big(b_{2(i-1+\ell)} e_{2i}\Big) \\
&\qquad - c_{2k-6} e_{2(k-\ell)-5} - e_{2(k-\ell)-3} + e_{2(k-\ell)}, \\
&\forall \ell \in \{1, \dots, k-3\} : \\
&[e_{2(k-\ell)-1}, f_1] = \sum_{i=1}^{k-2-\ell} \Big(C_{2(i-1+\ell)} e_{2i}\Big) + C e_{2(k-\ell)-1} - e_{2(k-\ell)}, \\
&[e_{2(k-\ell)-1}, f_2] = \sum_{i=1}^{k-3-\ell} \Big(\tilde{c}_{2(i-1+\ell)-1} e_{2i-1}\Big) + \sum_{i=1}^{k-2-\ell} \Big(c_{2(i-1+\ell)} e_{2i}\Big) \\
&\qquad + b_{2k-6} e_{2(k-\ell)-5} + e_{2(k-\ell-1)} + e_{2(k-\ell)-1}, \\
&[e_4, f_1] = e_3 + C e_4, \quad [e_4, f_2] = -e_1 + e_4, \\
&[e_3, f_1] = C e_3 - e_4, \quad [e_3, f_2] = e_2 + e_3, \\
&[e_2, f_1] = e_1 + C e_2, \quad [e_2, f_2] = e_2, \\
&[e_1, f_1] = C e_1 - e_2, \quad [e_1, f_2] = e_1.
\end{aligned} \tag{3.27}$$

Regardless of the values of the parameters, the dimensions of the elements of their characteristic series are

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0], \quad US = [0].$$

If on the other hand $c_{2k-4} = 0$, then we subtract Cd from D thus eliminating the parameter C. Following the steps described in the beginning of the preceding subsection under point ii. we eliminate parameters $C_{2\ell-1}$, $B_{2\ell-1}$ for every $\ell \in \{1, \dots, k-3\}$, the nontriangular derivation is then of the form

$$\begin{pmatrix} \cdots & 0 & 0 & 0 \\ \cdots & C_2 & B_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & C_{2k-6} & B_{2k-6} & 0 \\ \cdots & 0 & B_{2k-5} & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & -1 & 0 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix}.$$

The condition $[D, d]$ yields similar results as in the case of $c_{2k-4} \neq 0$:

$$c_{2k-5} \stackrel{!}{=} 0, \qquad c_{2k-7} = b_{2k-6}, \qquad b_{2k-7} = -c_{2k-6},$$

$$\begin{aligned} &\forall \ell \in \{4, \dots, k-1\} : \\ &c_{2(k-\ell)-1} \stackrel{!}{=} \tilde{c}_{2(k-\ell)-1} \equiv \sum_{i=1}^{\ell-4} B_{2(k-\ell-1+i)} c_{2(k-2-i)} + b_{2(k-\ell)}, \\ &b_{2(k-\ell)-1} \stackrel{!}{=} \tilde{b}_{2(k-\ell)-1} \equiv \sum_{i=1}^{\ell-4} C_{2(k-\ell-1+i)} c_{2(k-2-i)} - B_{2k-5} c_{2(k-\ell-1)} - c_{2(k-\ell)}. \end{aligned} \tag{3.28}$$

The triangular derivation thus becomes

$$\begin{pmatrix} \cdots & \tilde{c}_1 & \tilde{b}_1 & 0 \\ \cdots & c_2 & b_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & \tilde{c}_{2k-9} & \tilde{b}_{2k-9} & 0 \\ \cdots & c_{2k-8} & b_{2k-8} & 0 \\ \cdots & b_{2k-6} & -c_{2k-6} & 0 \\ \cdots & c_{2k-6} & b_{2k-6} & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix}.$$

where $\tilde{c}_\ell$, $\tilde{b}_\ell$ are given above in (3.26). Leaving the parameters unscaled, we describe all families this case would lead to at once as $\mathfrak{s}'_{2k+1,3}$ with the following commutation relations

$$\begin{aligned}
&[e_{2k-1}, f_1] = [e_{2k-1}, f_2] = 0, \\
&\forall \ell \in \{1, \dots, k-3\}: \\
&\quad [e_{2(k-\ell)}, f_1] = \sum_{i=1}^{k-2-\ell} \left(B_{2(i-1+\ell)} e_{2i}\right) + B_{2k-5} e_{2(k-\ell)-3} + e_{2(k-\ell)-1}, \\
&\quad [e_{2(k-\ell)}, f_2] = \sum_{i=1}^{k-3-\ell} \left(\tilde{b}_{2(i-1+\ell)-1} e_{2i-1}\right) + \sum_{i=1}^{k-2-\ell} \left(b_{2(i-1+\ell)} e_{2i}\right) \\
&\qquad - c_{2k-6} e_{2(k-\ell)-5} + e_{2(k-\ell)}, \\
&\forall \ell \in \{1, \dots, k-3\}: \\
&\quad [e_{2(k-\ell)-1}, f_1] = \sum_{i=1}^{k-2-\ell} \left(C_{2(i-1+\ell)} e_{2i}\right) - e_{2(k-\ell)}, \\
&\quad [e_{2(k-\ell)-1}, f_2] = \sum_{i=1}^{k-3-\ell} \left(\tilde{c}_{2(i-1+\ell)-1} e_{2i-1}\right) + \sum_{i=1}^{k-2-\ell} \left(c_{2(i-1+\ell)} e_{2i}\right) \\
&\qquad + b_{2k-6} e_{2(k-\ell)-5} + e_{2(k-\ell)-1},
\end{aligned}$$

$$\begin{aligned}
&[e_4, f_1] = B_{2k-5} e_1 + e_3, \quad && [e_4, f_2] = e_4, \quad [e_3, f_1] = -e_4, \quad [e_3, f_2] = e_3, \\
&[e_2, f_1] = e_1, && [e_2, f_2] = e_2, \quad [e_1, f_1] = -e_2, \quad [e_1, f_2] = e_1.
\end{aligned}$$

Regardless of the values of the parameters, the dimensions of the elements of their characteristic series are

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0], \quad US = [0].$$

iii) $a = 0 \wedge A \neq 0$

In this case, the nontriangular derivation is quasidiagonalized using the automorphisms. Since d must not be nilpotent, we know that $b \neq 0$ and consequently, we can use $b^{-1}d$ instead of d. Comparing the commutator $[D, d]$ with inner derivations, we see that the triangular derivation is in fact diagonal. The resulting family of extensions $\mathfrak{s}_{2k+1,4}(A, C)$ is described by

$\mathfrak{s}_{2k+1,4}$	$e_{2\ell-1},\ \ell \in \{1, \dots, k-1\}$	$e_{2\ell},\ \ell \in \{1, \dots, k-1\}$	e_{2k-1}
f_1	$((k-1-\ell)A + C)e_{2\ell-1} - e_{2\ell}$	$e_{2\ell-1} + ((k-1-\ell)A + C)e_{2\ell}$	Ae_{2k-1}
f_2	$e_{2\ell-1}$	$e_{2\ell}$	0

the dimensions of the elements of the characteristic series are

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0], \quad US = [0].$$

Triangular Derivations

We proceed analogously to the case of $k = 4$.

1. $A \neq 0 \ \vee\ a \neq 0$

The condition allows to change to derivations with the lower submatrices of the form

$$S = \begin{pmatrix} C & B_{2k-3} & A_{2k-3} \\ 0 & B & A_{2k-2} \\ 0 & 0 & 2 \end{pmatrix}, s = \begin{pmatrix} c & b_{2k-3} & a_{2k-3} \\ 0 & b & a_{2k-2} \\ 0 & 0 & 0 \end{pmatrix}.$$

1.1. $b \neq 0$

Given that $b \neq 0$ we can transform the derivations so that the submatrices are

$$S = \begin{pmatrix} C & B_{2k-3} & A_{2k-3} \\ 0 & 1 & A_{2k-2} \\ 0 & 0 & 2 \end{pmatrix}, s = \begin{pmatrix} c & b_{2k-3} & a_{2k-3} \\ 0 & 1 & a_{2k-2} \\ 0 & 0 & 0 \end{pmatrix}.$$

1.1.1. $C \neq 2,\ C \notin \{1, 3, \dots, 2k-3\},\ C \notin \{-2k+5, -2k+7, \dots, -1\}$

The conditions allow for complete diagonalization of D as described in point 1.1.1 in the subsection 3.2.1. Taking the commutator we see that the other derivation must be diagonal as well. Finally, we change from D to $D - d$. The resulting family of pairs of derivations is given by:

$$D = \begin{pmatrix} \ddots & & & \\ & C & & \\ & & 0 & \\ & & & 2 \end{pmatrix}, \quad d = \begin{pmatrix} \ddots & & & \\ & c & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

By the same argument as with seven-dimensional nilradical, we split the corresponding extensions to two cases based on the possibility of simplification of $[f_1, f_2]$. In case $C \neq 4-2k$ or $c \neq 0$, we can set the commutator to 0, thus obtaining the extensions $\mathfrak{s}_{2k+1,5}(C, c)$

$\mathfrak{s}_{2k+1,5}$	$e_{2\ell-1},\ \ell \in \{1, \dots, k-1\}$	$e_{2\ell},\ \ell \in \{1, \dots, k-1\}$	e_{2k-1}	f_1
f_1	$(2(k-1-\ell)+C)e_{2\ell-1}$	$2(k-1-\ell)e_{2\ell}$	$2e_{2k-1}$	0
f_2	$ce_{2\ell-1}$	$e_{2\ell}$	0	0

$$C \neq 0 \ \vee\ c \neq 0 \implies CS = [2k+1, 2k-1],\ DS = [2k+1, 2k-1, 2k-4, 0],$$

$$C = 0 \ \wedge\ c = 0 \implies CS = [2k+1, 2k-2],\ DS = [2k+1, 2k-2, 2k-5, 0],$$

$$C \neq 4-2k \ \vee\ c \neq 0 \implies US = [0], \quad C = 4-2k \ \wedge\ c = 0 \implies US = [1].$$

In the case of $C = 4-2k$ and $c = 0$, we obtain another extension $\mathfrak{s}'_{2k+1,5}$

$\mathfrak{s}'_{2k+1,5}$	$e_{2\ell-1},\ \ell \in \{1, \dots, k-1\}$	$e_{2\ell},\ \ell \in \{1, \dots, k-1\}$	e_{2k-1}	f_1
f_1	$(2-2\ell)e_{2\ell-1}$	$2(k-1-\ell)e_{2\ell}$	$2e_{2k-1}$	0
f_2	0	$e_{2\ell}$	0	e_1

$$CS = [2k+1, 2k-1],\ DS = [2k+1, 2k-1, 2k-4, 0],\ US = [0].$$

1.1.2. $C = 1-2i$ for some $i \in \{1, \dots, k-2\}$

Applying the automorphisms described above, we eliminate all nondiagonal parameters in D with the exception of $B_{2(k-i)-3}$. Taking the commutator $[D, d]$, we see that all nondiagonal parameters in d but $b_{2(k-i)-3}$ vanish as well. Given that $c \neq 1$, we eliminate $b_{2(k-i)-3}$ from d using the automorphism which preserves the form of D. Taking the commutator again, we see that $B_{2(k-i)-3} = 0$. After subtracting d from D this leads to $\mathfrak{s}_{2k+1,5}(1-2i-c, c)$. Hence we assume that $c = 1$. In case the parameter $b_{2(k-i)-3}$ is not null, we scale it to 1 and then change from D to $D - B_{2(k-i)-3}d$ and introduce parameter $B = 1 - B_{2(k-i)-3}$, thus obtaining the derivations in the form

$$D(B) = \begin{pmatrix} \ddots & & & \\ & B-2i & & \\ & & B & \\ & & & 2 \end{pmatrix}, d = \begin{pmatrix} & & \vdots & \\ & & 1 & \\ \ddots & & \vdots & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

The corresponding extensions $\mathfrak{s}^{(i)}_{2k+1,6}(B)$ are given by

$\mathfrak{s}^{(i)}_{2k+1,6}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k-1\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$(2(k-1-i-\ell)+B)e_{2\ell-1}$	$(2(k-1-\ell)+B)e_{2\ell}$	$2e_{2k-1}$	0
f_2	$e_{2\ell-1}$	$e_{2(\ell-i)-1}+e_{2\ell}$	0	0

$$CS=[2k+1,2k-1],\ DS=[2k+1,2k-1,2k-4,0],\ US=[0].$$

In case the parameter $b_{2(k-i)-3}$ vanishes, we scale $B_{2(k-i)-3}$ to 1 and change from D to $\frac{1}{2}(D-d)$ thus obtaining additional extension classes $\mathfrak{s}'^{(i)}_{2k+1,6}$

$\mathfrak{s}'^{(i)}_{2k+1,6}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k-1\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$(k-1-i-\ell)e_{2\ell-1}$	$e_{2(\ell-i)-1}+(k-1-\ell)e_{2\ell}$	e_{2k-1}	0
f_2	$e_{2\ell-1}$	$e_{2\ell}$	0	0

$$CS=[2k+1,2k-1],\ DS=[2k+1,2k-1,2k-4,0],\ US=[0].$$

1.1.3. $C=2i+1$ for some $i\in\{1,\dots,k-2\}$
Applying the automorphism (3.9) and subtracting $2id$ from D this case is reduced to the preceding one with the same i.
1.1.4. $C=2$
All remaining parameters in D except for A_{2k-3} are eliminated using the automorphisms. Taking the commutator $[D,d]$ we see that all nondiagonal parameters in d but a_{2k-3} vanish too. Similarly to the preceding cases if $c\neq 0$, we can eliminate the parameter a_{2k-3} from d using automorphism and then comparing the commutator of the derivation matrices with the algebra of inner derivations again, we obtain that A_{2k-3} vanishes as well. This would lead to $\mathfrak{s}_{2k+1,5}(2,c)$; hence, we assume that $c=0$. In the case of $a_{2k-3}\neq 0$, we scale this parameter to 1 via the diagonal automorphism (3.7). After subtracting $A_{2k-3}d$ from D and introducing $B=\frac{1}{2}(1-A_{2k-3})$, we obtain the derivations in the form:

$$D(B)=\begin{pmatrix}\ddots &&&\\ &1&&\\ &&B&\\ &&&1\end{pmatrix},\quad d=\begin{pmatrix}\ddots &&&\\ &0&&1\\ &&1&\\ &&&0\end{pmatrix}.$$

The corresponding extensions $\mathfrak{s}_{2k+1,7}(B)$ are given by

$\mathfrak{s}_{2k+1,7}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k-1\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$(k-\ell)e_{2\ell-1}$	$(k-\ell-1+B)e_{2\ell}$	e_{2k-1}	0
f_2	$e_{2\ell-1}$	0	e_{2k-3}	0

$$CS=[2k+1,2k-1],\ DS=[2k+1,2k-1,2k-4,0],\ US=[0].$$

In the case of $a_{2k-3} = 0$, we scale A_{2k-3} to 2. The case of both parameters being zero was already covered above. After subtracting d from D and multiplying D by $\frac{1}{2}$, we obtain the derivation matrices in the form

$$D(B) = \begin{pmatrix} \ddots & & & \\ & 1 & & 1 \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \ddots & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

The corresponding extension $\mathfrak{s}_{2k+1,8}$ is given by

$\mathfrak{s}_{2k+1,8}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k-1\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$(k-\ell)e_{2\ell-1}$	$(k-\ell-1)e_{2\ell}$	$e_{2k-3}+e_{2k-1}$	0
f_2	$e_{2\ell-1}$	0	0	0

$$CS = [2k+1, 2k-1],\ DS = [2k+1, 2k-1, 2k-4, 0],\ US = [0].$$

1.1.5. $C = 1$

By the same argument as in the preceding, the case of $c \neq 1$ leads back to $\mathfrak{s}_{2k+1,5}(1,c)$. Hence we assume that $c = 1$. Eliminating all remaining parameters in D but B_{2k-3} and comparing the commutator $[D,d]$ with $\mathfrak{Inn}$ we have that all the remaining parameters in d but b_{2k-3} vanish as well. Again, this case splits into two subcases, the first one being $b_{2k-3} \neq 0$. Subtracting $B_{2k-3}d$ from D, introducing $B = \frac{1}{2}(1-B_{2k-3})$ and taking half of thus obtained D, we arrive at

$$D(B) = \begin{pmatrix} \ddots & & & \\ & B & & \\ & & B & \\ & & & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \ddots & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

The corresponding extensions $\mathfrak{s}_{2k+1,9}(B)$ are given by

$\mathfrak{s}_{2k+1,9}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k-1\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$(k-\ell-1-B)e_{2\ell-1}$	$(k-\ell-1+B)e_{2\ell}$	e_{2k-1}	0
f_2	$e_{2\ell-1}$	$e_{2\ell-1}+e_{2\ell}$	0	0

$$CS = [2k+1, 2k-1],\ DS = [2k+1, 2k-1, 2k-4, 0],\ US = [0].$$

In the second subcase, $b_{2k-3} = 0$, we scale B_{2k-3} to 2, subtract d from D and take half of D, thus obtaining the matrices in the form

$$D(B) = \begin{pmatrix} \ddots & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \ddots & & & \\ & 1 & 0 & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

The corresponding extension $\mathfrak{s}_{2k+1,10}$ is given by

$\mathfrak{s}_{2k+1,10}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k-1\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$(k-1-\ell)e_{2\ell-1}$	$e_{2\ell-1}+(k-1-\ell)e_{2\ell}$	e_{2k-1}	0
f_2	$e_{2\ell-1}$	$e_{2\ell}$	0	0

$$CS = [2k+1, 2k-1],\ DS = [2k+1, 2k-1, 2k-4, 0],\ US = [0].$$

1.2. $b = 0$
Following the same argument as in the seven-dimensional case we reduce this case to the case 1.1 with $c = B_{2k-3} = A_{2k-3} = A_{2k-2}$.

2. $A = 0 \wedge a = 0$
Following the same steps as in the seven-dimensional case we transform the derivation matrices so that their lower 3×3 submatrices take up the form

$$S = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

Using a series of automorphisms described in the subsection 3.2.1 on outer derivation classes we can eliminate the parameters C_1, B_2, ..., C_{2k-7}, B_{2k-6}, C_{2k-5} from D. Comparing the commutator $[D, d]$ with the algebra of inner derivations we see that the parameters $c_1, b_2, \dots, c_{2k-7}, b_{2k-6}, c_{2k-5}$ vanish as well. Then the derivations are in the form

$$D = \begin{pmatrix} \cdots & C_1 & 0 & 0 \\ \cdots & 0 & B_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & C_{2k-7} & 0 & 0 \\ \cdots & 0 & B_{2k-6} & 0 \\ \cdots & C_{2k-5} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & 2 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \cdots & c_1 & 0 & 0 \\ \cdots & 0 & b_2 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & c_{2k-7} & 0 & 0 \\ \cdots & 0 & b_{2k-6} & 0 \\ \cdots & c_{2k-5} & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{3.29}$$

We can apply the diagonal automorphism (3.7) to scale one of the parameters to $(\pm)1$. Analogously to the extensions by one element, our first choice is the bottom-most nonvanishing parameter that can be scaled including its sign if $\mathbb{F} = \mathbb{R}$. Here, we add the preference of scaling the parameter in the second derivation (d) if possible. This yields the following range

$$\Big(c_{2k-5}, C_{2k-5}, b_{2k-8}, B_{2k-8}, c_{2k-9}, C_{2k-9}, \dots, b_{2\left(2-k+2\left\lfloor\frac{k}{2}\right\rfloor\right)}, B_{2\left(2-k+2\left\lfloor\frac{k}{2}\right\rfloor\right)}, c_{2\left(2-k+2\left\lfloor\frac{k}{2}\right\rfloor\right)-1}, C_{2\left(2-k+2\left\lfloor\frac{k}{2}\right\rfloor\right)-1}, b_{2k-6}, B_{2k-6}, c_{2k-7}, C_{2k-7}, \dots, b_{2\left(1+k-2\left\lfloor\frac{k}{2}\right\rfloor\right)}, B_{2\left(1+k-2\left\lfloor\frac{k}{2}\right\rfloor\right)}, c_{2\left(1+k-2\left\lfloor\frac{k}{2}\right\rfloor\right)-1}, C_{2\left(1+k-2\left\lfloor\frac{k}{2}\right\rfloor\right)-1}\Big) \in \mathcal{A}_{2,11} \equiv \{1\} \times \mathbb{F}^{4k-11} \cup \dots \cup \theta_{4\left\lfloor\frac{k-3}{2}\right\rfloor+1} \times \{1\} \times \mathbb{F}^{4\left\lfloor\frac{k-2}{2}\right\rfloor} \cup \theta_{4\left\lfloor\frac{k-3}{2}\right\rfloor+2} \times \{(\pm)1\} \times \mathbb{F}^{4\left\lfloor\frac{k-2}{2}\right\rfloor-1} \cup \dots \cup \theta_{4k-11} \times \{(\pm)1\} \tag{3.30}$$

for the parameters. This allows us to describe all of the families by two matrices (3.29) above combined with range (3.30). Each of the corresponding $4k-9$ families of extensions $\mathfrak{s}'_{2k+1,11}$ is given by relations

$\mathfrak{s}_{2k+1,11}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k-1\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k-1}	f_1
f_1	$\sum_{i=1}^{\ell-1} C_{2(i+k-1+\ell)-1} e_{2i-1} + e_{2\ell-1}$	$\sum_{i=1}^{\ell-2} B_{2(i+k-1+\ell)} e_{2i} + 2e_{2\ell}$	0	0
f_2	$\sum_{i=1}^{\ell-1} c_{2(i+k-1+\ell)-1} e_{2i-1}$	$\sum_{i=1}^{\ell-2} b_{2(i+k-1+\ell)} e_{2i} + e_{2\ell}$	0	0

and one of the sets $\mathcal{A}_{\text{even}}$, $\mathcal{A}_{\text{odd}}$ for k even, k odd respectively. All of these extensions have the same elements of the characteristic series (with the exception of extensions themselves), their dimensions are

$$CS = [2k+1, 2k-2],\ DS = [2k+1, 2k-2, 0],\ US = [0].$$

3.2.3 Extensions by Three Elements

Similarly to the seven-dimensional counterpart of this subsection in the previous chapter, we seek three linearly nil-independent outer derivations which we denote $\mathcal{D}$, D, and d, their lower 3×3 submatrices $\mathcal{S}$, S, s and the corresponding extending elements f_1, f_2, and f_3 respectively. There are two extensions to be found, one with one of the matrices being nontriangular and one with all three matrices triangular.

Nontriangular Derivations

Following the same argument used in the last subsection of the preceding chapter, we come to the matrices $\mathcal{D}$, D, and d with lower 3×3 submatrices of the form

$$\mathcal{S} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The nontriangular matrix $\mathcal{D}$ is fully diagonalized using automorphisms described in the beginning of subsection 3.2.1. Comparing the commutators $[\mathcal{D}, D]$ and $[\mathcal{D}, d]$ with the algebra of inner derivations we see that both D and d are diagonal. After subtracting D from $\mathcal{D}$, we obtain the resulting triple in the form

$$\mathcal{D} = \begin{pmatrix} \ddots & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \ddots & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix},$$

with the corresponding extension $\mathfrak{s}_{2k+2,1}$ given by

$\mathfrak{s}_{2k+2,1}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k-2\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k-1}	f_1	f_2
f_1	$-e_{2\ell}$	$e_{2\ell-1}$	0	0	0
f_2	$(k-1-\ell)e_{2\ell-1}$	$(k-1-\ell)e_{2\ell}$	e_{2k-1}	0	0
f_3	$e_{2\ell-1}$	$e_{2\ell}$	0	0	0

The argument for vanishing commutators of the extending elements is the same as in the seven-dimensional case. The dimensions of the elements of the characteristic series are

$$CS = [2k+2, 2k-1],\ DS = [2k+2, 2k-1, 2k-4, 0],\ US = [0].$$

Triangular Derivations

Once more we follow the steps given in the case of $k = 4$ to obtain the derivation matrices with

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first derivation, $\mathcal{D}$, can be diagonalized using the automorphisms given in the preceding. The condition $[\mathcal{D}, D], [\mathcal{D}, d] \overset{!}{\in} \mathfrak{Inn}$ requires that the remaining matrices are diagonal as well. We thus obtain the derivations given by

$$\mathcal{D} = \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \ddots & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Similarly as with nontriangular derivation in the triplet, the commutators of the extending elements can be set to zero as shown in the seven-dimensional case. The resulting extension $\mathfrak{s}_{2k+2,2}$ is thus given by

$\mathfrak{s}_{2k+2,2}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k-2\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k-1}	f_1	f_2
f_1	$(k-\ell)e_{2\ell-1}$	$(k-1-\ell)e_{2\ell}$	e_{2k-1}	0	0
f_2	0	$e_{2\ell}$	e_{2k-1}	0	0
f_3	$e_{2\ell-1}$	0	0	0	0

$$CS = [2k+2, 2k-1],\ DS = [2k+2, 2k-1, 2k-4, 0],\ US = [0].$$

Chapter 4

Generalized Casimir Invariants

The generalized Casimir invariants of the nilpotent algebras $\mathfrak{r}_{2k-1}$ and of their solvable extensions, with the exception of the most degenerate cases, are found in this chapter.

Let us consider a Lie group G and its Lie algebra $\mathfrak{g}$. Then we have the coadjoint representation Ad^* as given in (1.14) acting on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. Let I be a smooth function on $\mathfrak{g}^*$, that is $I : \mathfrak{g}^* \longrightarrow \mathbb{F}$. Thus, it is possible to compose Ad^*_g and I for any fixed $g \in G$ as

$$I \circ \mathrm{Ad}^*_g : \ \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \longrightarrow \mathbb{F}; \tag{4.1}$$

if I is invariant under the coadjoint action of G, that is if

$$I \circ \mathrm{Ad}^*_g \equiv I \text{ on } \mathfrak{g}^* \quad \forall g \in G, \tag{4.2}$$

we call the function I the generalized Casimir invariant of $\mathfrak{g}$. Using one-parametric subgroups, we can rewrite the relation as

$$I \circ \mathrm{Ad}^*_{\mathrm{e}^{tx}} \equiv I \text{ on } \mathfrak{g}^* \quad \forall x \in \mathfrak{g}. \tag{4.3}$$

Taking derivative by t in $t = 0$, we obtain

$$I \circ \mathrm{ad}^*_x \equiv 0 \text{ on } \mathfrak{g}^* \quad \forall x \in \mathfrak{g}, \tag{4.4}$$

where ad^* is given in (1.9).

Let $(x_i)_1^n$ be the basis of the Lie algebra $\mathfrak{g}$ and let c_{ij}^k be the structure constants w. r. t. this basis, that is

$$[x_i, x_j] = c_{ij}^k x_k \qquad \forall i, j \in \{1, \dots, n\}.$$

Using the basis and one-to-one correspondence between the Lie algebra $\mathfrak{g}$ and the left-invariant vector fields on G, we can rewrite the relation (4.4) as

$$\hat{X}_k I \equiv 0 \qquad \forall k \in \{1, \dots, n\}, \tag{4.5}$$

where the vector fields $\hat{X}$ are given by

$$\hat{X}_k = x_i c^i_{jk} \frac{\partial}{\partial x_j} \qquad \forall k \in \{1, \dots, n\}. \tag{4.6}$$

4.1 Method of Characteristics

Method of characteristics allows to solve 1st order linear homogeneous PDE by transforming the problem of one PDE into a system of ODEs. Let $\vec{x} = (x_i)_1^n$ and let us have a linear PDE

$$\hat{X} I(\vec{x}) = f_j(\vec{x}) \frac{\partial}{\partial x_j} I(\vec{x}) = 0, \tag{4.7}$$

then solving the following system

$$\begin{aligned} &\dot{\alpha}_j(t) = f_j(\alpha_1(t), \dots, \alpha_n(t)) \\ &\alpha_j(0) = x_j \end{aligned} \qquad \forall j \in \{1, \dots, n\} \tag{4.8}$$

of ODEs yields n functions $\alpha_i(t)$. These are n components of the flow of the vector field $\hat{X}$. The integral curves of this field are called the characteristics of the PDE (4.7). By setting any suitable, that is one that does not depend on t, α_i to an arbitrary fixed value we obtain the corresponding value of t. Substituting t back into the remaining solutions α_j, $j \neq i$ then provides $n-1$ functionally independent solutions of equation (4.7).

To solve a system of linear PDEs (4.5) we first pick one of these equations and find its solutions $\vec{\alpha} \equiv (\alpha_1, \dots, \alpha_{n-1})$ using the method of characteristics. The next step is to pick another equation, say $\hat{X}_\ell$, and find the solutions of the equation

$$\hat{X}_\ell I(\vec{\alpha}) = (\hat{X}_\ell \alpha_i) \frac{\partial}{\partial \alpha_i} I(\vec{\alpha}) \equiv 0. \tag{4.9}$$

If we are able to find $\hat{X}_\ell \alpha_i$ as a function $h_i(\vec{\alpha})$ we can rewrite the equation in the form

$$\hat{X}_\ell I(\vec{\alpha}) = h_i(\vec{\alpha}) \frac{\partial}{\partial \alpha_i} I(\vec{\alpha}), \tag{4.10}$$

which is the same form as that of equation (4.7). Hence, we can solve it by the method of characteristics thus obtaining $n-2$ solutions β_i satisfying both chosen equations. To solve larger systems we simply repeat this process.

We have assumed that the system of PDEs consists of independent equations. Thus assuming that we have ℓ operators $\hat{X}_i$ acting on n-dimensional space we obtain $n - \ell$ solutions of the system. Evidently, the equation are not necessarily independent and the number of functionally independent solutions may be larger than $n - \ell$. The number of independent equations in the system (4.6) is equal to the rank of the following matrix

$$C = \left((x_i c^i_{jk})_{jk}\right), \tag{4.11}$$

where generic values of the coordinates x_i must be assumed. Note that C is antisymmetric, thus its rank is even. The number of functionally independent invariants is then $n - \text{rank } C$. The number of independent equations in all of the cases below is $2(i+1)$, where i is the number of extending elements. Thus, for all of the algebras below, we obtain $2k - 3 - i$ independent invariants.

4.2 Method of Moving Frames

An alternative method of finding the invariants is the method of moving frames. Theoretical background on the general method of moving frames can be found in [4, 5], the application on the computation of the generalized Casimir invariants was introduced in [2]. In order to find the invariants using this method, we first find the local flows $\Psi^{\alpha_k}_{\hat{X}_k}$ of the vector fields (4.6) and compose them into a local action of a Lie group of $\mathfrak{g}$ as

$$\Psi(\vec{\alpha}) \equiv \Psi^{\alpha_n}_{\hat{X}_n} \circ \dots \circ \Psi^{\alpha_1}_{\hat{X}_1}. \tag{4.12}$$

Then we choose a section such that it cuts through the orbits of the action Ψ. For every initial point $p \in \mathfrak{g}^*$, we obtain the intersection of $\Psi(\vec{\alpha})(p)$ with this section. The coordinates of this intersection are by construction invariant w. r. t. the Ad^*-action of G on $\mathfrak{g}$. Technically, this means fixing $r \equiv \text{rank } C$ (C is given above in (4.11)) coordinates of the action $\Psi(\vec{\alpha})(p)$ on a given point $p \in \mathfrak{g}^*$ in a way that yields a system of r independent algebraic equations for the parameters $\alpha_1, \dots, \alpha_n$ that has a solution. This allows to express r of the parameters α_j in terms of the remaining parameters and the coordinates of the point p. Substituting them into the remaining coordinates of the action yields $n - r$ sought invariants.

Some of the flows of the vector fields may be trivial. Then we may omit them from the composition and the action Ψ has a reduced number of the parameters α_j. In all instances below, we have no trivial vector fields at first, but we start by observing that the invariants may not depend on certain coordinates rendering most of the vector fields trivial. For extension by i elements, this effectively reduces the dimension n to $2k - 2$ and the number of nontrivial vector fields to $i+1$, which in turn means we compose only $i+1$ flows, and consequently the action Ψ has only $i + 1$ parameters.

4.3 The Invariants

4.3.1 Invariants of $\mathfrak{r}_{2k-1}$

There are $2k-3$ invariants to be found. The operators (4.6) are given by

$$\begin{aligned}\hat{E}_1 &= 0, \qquad \hat{E}_2 = 0,\\ \hat{E}_\ell &= e_{\ell-2}\frac{\partial}{\partial e_{2k-1}} \qquad \forall \ell \in \{3,\dots,2k-2\},\\ \hat{E}_{2k-1} &= -\sum_{\ell=3}^{2k-2} e_{\ell-2}\frac{\partial}{\partial e_\ell}.\end{aligned}$$

We immediately see that the invariants cannot depend on e_{2k-1} and we are left with only one equation to solve. Using method of characteristics we reduce it to the following system of ODEs

$$\begin{aligned}\dot{\alpha}_\iota(t) &= 0, & \alpha_\iota(0) &= e_\iota & \forall \iota &\in \{1,2\},\\ \dot{\alpha}_\ell(t) &= -\alpha_{\ell-2}(t), & \alpha_\ell(0) &= e_\ell & \forall \ell &\in \{3,\dots,2k-2\}.\end{aligned}$$

The solution is readily found as

$$\begin{aligned}\alpha_\iota(t) &= e_\iota & \forall \iota &\in \{1,2\},\\ \alpha_{2\ell-1}(t) &= e_{2\ell-1} + \sum_{j=1}^{\ell-1}\frac{(-t)^j}{j!}e_{2(\ell-j)-1} & \forall \ell &\in \{2,\dots,k-1\},\\ \alpha_{2\ell}(t) &= e_{2\ell} + \sum_{j=1}^{\ell-1}\frac{(-t)^j}{j!}e_{2(\ell-j)} & \forall \ell &\in \{2,\dots,k-1\}.\end{aligned}$$

Setting $\alpha_3(t) := 0$ we get $t = \frac{e_3}{e_1}$. Substituting this value back to α_i yields the invariants of $\mathfrak{r}_{2k}$ in the form

$$\begin{aligned}\tilde{I}_\iota(\vec{e}) &= e_\iota & \forall \iota &\in \{1,2\},\\ \tilde{I}_{2\ell-1}(\vec{e}) &= e_{2\ell-1} + \sum_{j=1}^{\ell-1}\left(-\frac{e_3}{e_1}\right)^j\frac{1}{j!}e_{2(\ell-j)-1} & \forall \ell &\in \{3,\dots,k-1\},\\ \tilde{I}_{2\ell}(\vec{e}) &= e_{2\ell} + \sum_{j=1}^{\ell-1}\left(-\frac{e_3}{e_1}\right)^j\frac{1}{j!}e_{2(\ell-j)} & \forall \ell &\in \{2,\dots,k-1\}.\end{aligned}$$

Multiplying invariants $\tilde{I}_{2\ell}$ and $\tilde{I}_{2\ell-1}$ by $\tilde{I}_1^{\ell-1}$ for every $\ell \geq 3$, we obtain the polynomial basis of the generalized Casimir invariants of $\mathfrak{r}_{2k-1}$:

$$\begin{aligned}
\mathcal{I}_\iota(\vec{e}) &= e_\iota && \forall \iota \in \{1,2\}, \\
\mathcal{I}_{2\ell-1}(\vec{e}) &= \sum_{j=0}^{\ell-1} \frac{(-e_3)^j e_1^{\ell-2-j}}{j!} e_{2(\ell-j)-1} && \forall \ell \in \{3,\dots,k-1\}, \\
\mathcal{I}_{2\ell}(\vec{e}) &= \sum_{j=0}^{\ell-1} \frac{(-e_3)^j e_1^{\ell-1-j}}{j!} e_{2(\ell-j)} && \forall \ell \in \{2,\dots,k-1\}.
\end{aligned} \tag{4.13}$$

4.3.2 Invariants of Extensions by One Element

In the following the operator corresponding to the extending element f_1 of extension $\mathfrak{s}_{2k,i}$ is denoted by $\hat{F}_1^{(i)}$.

$\mathfrak{s}_{2k,1}(a,c)$

Let us first define $\Bbbk_i$ as

$$\Bbbk_i := (k-1-i)a + c \tag{4.14}$$

to simplify the notation. The differential operators corresponding to the basis $(e_i, f_1)_1^{2k-1}$ are of the form

$$\begin{aligned}
\hat{E}_1 &= (\Bbbk_1 e_1 - e_2)\frac{\partial}{\partial f_1}, \qquad \hat{E}_2 = (e_1 + \Bbbk_1 e_2)\frac{\partial}{\partial f_1}, \\
\hat{E}_{2\ell-1} &= e_{2\ell-3}\frac{\partial}{\partial e_{2k-1}} + (\Bbbk_\ell e_{2\ell-1} - e_{2\ell})\frac{\partial}{\partial f_1} && \forall \ell \in \{2,\dots,k-1\}, \\
\hat{E}_{2\ell} &= e_{2\ell-2}\frac{\partial}{\partial e_{2k-1}} + (e_{2\ell-1} + \Bbbk_\ell e_{2\ell})\frac{\partial}{\partial f_1} && \forall \ell \in \{2,\dots,k-1\}, \\
\hat{E}_{2k-1} &= -\sum_{\ell=3}^{2k-2} e_{\ell-2}\frac{\partial}{\partial e_\ell} + a e_{2k-1}\frac{\partial}{\partial f_1}, \\
-\hat{F}_1^{(1)} &= \sum_{i=1}^{k-1}(\Bbbk_i e_{2i-1} - e_{2i})\frac{\partial}{\partial e_{2i-1}} + (e_{2i-1} + \Bbbk_i e_{2i})\frac{\partial}{\partial e_{2i}} + a e_{2k-1}\frac{\partial}{\partial e_{2k-1}}.
\end{aligned}$$

It is evident that $\frac{\partial}{\partial f_1} I \equiv \frac{\partial}{\partial e_{2k-1}} I \equiv 0$, hence the above operators can be simplified to

$$
\begin{aligned}
\hat{E}_\ell &= 0 \qquad \forall \ell \in \{1, \dots, 2k-2\}, \\
\hat{E}_{2k-1} &= -\sum_{\ell=3}^{2k-2} e_{\ell-2} \frac{\partial}{\partial e_\ell}, \\
\hat{F}_1^{(1)} &= -\sum_{i=1}^{k-1} (\mathbb{k}_i e_{2i-1} - e_{2i}) \frac{\partial}{\partial e_{2i-1}} + (e_{2i-1} + \mathbb{k}_i e_{2i}) \frac{\partial}{\partial e_{2i}}.
\end{aligned}
$$

The corresponding flows are given by

$$
\begin{aligned}
&\forall \ell \in \{1, \dots, k-1\} : \\
&\qquad e^{\#}_{2\ell-1}\left(\Psi^{\alpha}_{\hat{E}_1}(\vec{e}\,)\right) = \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}, \\
&\qquad e^{\#}_{2\ell}\left(\Psi^{\alpha}_{\hat{E}_1}(\vec{e}\,)\right) = \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i}, \\
&\qquad e^{\#}_{2\ell-1}\left(\Psi^{\beta}_{\hat{F}_1^{(1)}}(\vec{e}\,)\right) = \mathrm{e}^{-((k-1-\ell)a+c)\beta}(e_{2\ell-1}\cos\beta + e_{2\ell}\sin\beta), \\
&\qquad e^{\#}_{2\ell}\left(\Psi^{\beta}_{\hat{F}_1^{(1)}}(\vec{e}\,)\right) = \mathrm{e}^{-((k-1-\ell)a+c)\beta}(-e_{2\ell-1}\sin\beta + e_{2\ell}\cos\beta).
\end{aligned}
$$

Composing the flows, we obtain

$$
\begin{aligned}
\Psi &\equiv \Psi^{\beta}_{\hat{F}_1^{(1)}} \circ \Psi^{\alpha}_{\hat{E}_1} \\
e^{\#}_{2\ell-1}(\Psi(\vec{e}\,)) &= \mathrm{e}^{-\mathbb{k}_\ell\beta} \left(\sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}\cos\beta + \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i}\sin\beta \right), \\
e^{\#}_{2\ell}(\Psi(\vec{e}\,)) &= \mathrm{e}^{-\mathbb{k}_\ell\beta} \left(-\sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}\sin\beta + \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i}\cos\beta \right),
\end{aligned}
$$

Choosing the section by setting

$$
e^{\#}_1(\Psi(\vec{e}\,)) = 0, \qquad e^{\#}_4(\Psi(\vec{e}\,)) = 0
$$

yields

$$
\beta = -\arctan\frac{e_1}{e_2}, \qquad \alpha = \frac{e_1 e_3 + e_2 e_4}{e_1^2 + e_2^2}.
$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\forall \ell \in \{2, \dots, k-1\}:$$
$$I_{2\ell-1} = \mathrm{e}^{\Bbbk_\ell \arctan \frac{e_1}{e_2}} \left(e_1^2 + e_2^2\right)^{-\frac{1}{2}} \left(\sum_{i=1}^{\ell} \frac{(-e_1 e_3 - e_2 e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}} (e_2 e_{2i-1} - e_1 e_{2i})\right),$$
$$\forall \ell \in \{1, 3, 4, \dots, k-1\}:$$
$$I_{2\ell} = \mathrm{e}^{\Bbbk_\ell \arctan \frac{e_1}{e_2}} \left(e_1^2 + e_2^2\right)^{-\frac{1}{2}} \left(\sum_{i=1}^{\ell} \frac{(-e_1 e_3 - e_2 e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}} (e_1 e_{2i-1} + e_2 e_{2i})\right).$$

We may replace the invariants $I_{2\ell}$ in the functional basis by rational functions $I'_{2\ell}$ given by

$$I'_{2\ell} \equiv \frac{\sum\limits_{i=1}^{\ell} \frac{(-e_1e_3-e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_1 e_{2i-1} + e_2 e_{2i})}{\sum\limits_{i=1}^{\ell} \frac{(-e_1e_3-e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_2 e_{2i-1} - e_1 e_{2i})} \qquad \forall \ell \in \{1, 3, 4, \dots, k-1\}.$$

$\mathfrak{s}_{2k,2}$

The invariants of these extensions were not found as the equations for the local flows proved to be too complicated in this case.

$\mathfrak{s}_{2k,3}(b, c)$

To simplify the notation, we introduce $\mathbb{b}_i$ and $\mathbb{c}_i$

$$\mathbb{b}_i := (k - 1 - i + b), \qquad \mathbb{c}_i := (k - 1 - i + c). \tag{4.15}$$

The operators (4.6) can be then written as

$$\hat{E}_1 = \mathbb{c}_1 \frac{\partial}{\partial f_1}, \quad \hat{E}_2 = \mathbb{b}_1 \frac{\partial}{\partial f_1},$$
$$\hat{E}_{2\ell-1} = e_{2\ell-3} \frac{\partial}{\partial e_{2k-1}} + \mathbb{c}_\ell e_{2\ell-1} \frac{\partial}{\partial f_1} \qquad \forall \ell \in \{2, \dots, k-1\},$$
$$\hat{E}_{2\ell} = e_{2\ell-2} \frac{\partial}{\partial e_{2k-1}} + \mathbb{b}_\ell e_{2\ell} \frac{\partial}{\partial f_1} \qquad \forall \ell \in \{2, \dots, k-1\},$$
$$\hat{E}_{2k-1} = -\sum_{\ell=3}^{2k-2} e_{\ell-2} \frac{\partial}{\partial e_\ell} + e_{2k-1} \frac{\partial}{\partial f_1},$$
$$\hat{F}_1^{(3)} = -\sum_{i=1}^{k-1} \mathbb{c}_i e_{2i-1} \frac{\partial}{\partial e_{2i-1}} + \mathbb{b}_i e_{2i} \frac{\partial}{\partial e_{2i}} - e_{2k-1} \frac{\partial}{\partial e_{2k-1}}$$

After removing $\frac{\partial}{\partial f_1}$ and $\frac{\partial}{\partial e_{2i-1}}$, the operators simplify to

$$\begin{aligned}
\hat{E}_\ell &= 0 \qquad \forall \ell \in \{1, \dots, 2k-2\}, \\
\hat{E}_{2k-1} &= -\sum_{\ell=3}^{2k-2} e_{\ell-2} \frac{\partial}{\partial e_\ell}, \\
\hat{F}_1^{(3)} &= -\sum_{i=1}^{k-1} \mathbb{c}_i e_{2i-1} \frac{\partial}{\partial e_{2i-1}} + \mathbb{b}_i e_{2i} \frac{\partial}{\partial e_{2i}}.
\end{aligned}$$

Again, the invariants of the nilradical (4.13) are solutions to $\hat{E}_{2k-1} I = 0$. The action of $\hat{F}_1^{(3)}$ on them is given by

$$\begin{aligned}
\hat{F}_1^{(3)} \mathcal{I}_1 &= -\mathbb{c}_1 \mathcal{I}_1, \quad \hat{F}_1^{(3)} \mathcal{I}_2 = -\mathbb{b}_1 \mathcal{I}_2, \\
\hat{F}_1^{(3)} \mathcal{I}_{2\ell} &= -((\ell-1)\mathbb{c}_1 + \mathbb{b}_\ell) \mathcal{I}_{2\ell} && \forall \ell \in \{2, \dots, k-1\}, \\
\hat{F}_1^{(3)} \mathcal{I}_{2\ell-1} &= -(\ell-1)\mathbb{c}_2 \mathcal{I}_{2\ell-1} && \forall \ell \in \{3, \dots, k-1\}.
\end{aligned}$$

The solutions of the corresponding system of ODEs

$$\begin{aligned}
\dot{\beta}_1 &= -\mathbb{c}_1 \beta_1, \quad \dot{\beta}_2 = -\mathbb{b}_1 \beta_2, \\
\dot{\beta}_{2\ell} &= -((\ell-1)\mathbb{c}_1 + \mathbb{b}_\ell) \beta_{2\ell} && \forall \ell \in \{2, \dots, k-1\}, \\
\dot{\beta}_{2\ell-1} &= -(\ell-1)\mathbb{c}_2 \beta_{2\ell-1} && \forall \ell \in \{3, \dots, k-1\}, \\
\beta_\ell(0) &= \mathcal{I}_\ell && \forall \ell \in \{1, 2, 4, 5, 6, \dots, 2k-2\}
\end{aligned}$$

are of the form

$$\begin{aligned}
\beta_1(t) &= \mathcal{I}_1 \mathrm{e}^{-\mathbb{c}_1 t}, \quad \beta_2(t) = \mathcal{I}_2 \mathrm{e}^{-\mathbb{b}_1 t}, \\
\beta_{2\ell}(t) &= \mathcal{I}_{2\ell} \mathrm{e}^{-((\ell-1)\mathbb{c}_1 + \mathbb{b}_\ell) t} && \forall \ell \in \{2, \dots, k-1\}, \\
\beta_{2\ell-1}(t) &= \mathcal{I}_{2\ell-1} \mathrm{e}^{-(\ell-1)\mathbb{c}_2 t} && \forall \ell \in \{3, \dots, k-1\}.
\end{aligned}$$

In case $\mathbb{c}_1 \neq 0$, we set $\beta_1(t) := 1$ to obtain the corresponding value of t, $t = \frac{\ln \mathcal{I}_1}{\mathbb{c}_1}$. Substituting this value back to the remaining β_i, we obtain the invariants of $\mathfrak{s}_{2k,3}$

$$\begin{aligned}
I_2 &= \mathcal{I}_2 \mathcal{I}_1^{-\frac{\mathbb{b}_1}{\mathbb{c}_1}}, \\
I_{2\ell} &= \mathcal{I}_{2\ell} \mathcal{I}_1^{-\ell+1-\frac{\mathbb{b}_\ell}{\mathbb{c}_1}} && \forall \ell \in \{2, \dots, k-1\}, \\
I_{2\ell-1} &= \mathcal{I}_{2\ell-1} \mathcal{I}_1^{-(\ell-1)\frac{\mathbb{c}_2}{\mathbb{c}_1}} && \forall \ell \in \{3, \dots, k-1\}.
\end{aligned}$$

On the other hand, if we have $\mathbb{c}_1 = 0$, that is $c = 2 - k$, we set $\beta_5(t) := 1$ to obtain $t = \frac{\ln \mathcal{I}_5}{2}$ for back substitution. The invariants are then given by

$$\begin{aligned} I_1 &= \mathcal{I}_1, \\ I_{2\ell} &= \mathcal{I}_{2\ell}\mathcal{I}_5^{-\frac{\mathbb{b}_\ell}{2}} && \forall \ell \in \{1, \dots, k-1\}, \\ I_{2\ell-1} &= \mathcal{I}_{2\ell-1}\mathcal{I}_5^{\frac{1-\ell}{2}} && \forall \ell \in \{4, \dots, k-1\}. \end{aligned}$$

$\mathfrak{s}_{2k,4}^{(i)}(c)$

After removing the partial derivatives w. r. t. e_{2k-1} and f_1 the operators $\hat{E}_i$ are same as for the nilradical itself. The operator $\hat{F}_1^{(4)}$ is of the form

$$\begin{aligned} \hat{F}_1^{(4)} &= \hat{F}_1^{(3)}|_{b=i+c} - \sum_{j=i+1}^{k-1} e_{2(j-i)}\frac{\partial}{\partial e_{2j}} = \hat{F}_1^{(3)} + \hat{G}_1^{(4)}, \text{ where} \\ \hat{G}_1^{(4)} &:= -\sum_{j=i+1}^{k-1} e_{2(j-i)}\frac{\partial}{\partial e_{2j}}. \end{aligned}$$

It follows that the solutions $\beta_{2\ell-1}(t)$ for all ℓ and $\beta_{2\ell}(t)$ for $\ell \in \{1, \dots, i\}$ are the same as for $\mathfrak{s}_{2k,3}(i+c, c)$. The action of $\hat{F}_1^{(4)}$ on $\mathcal{I}_{2\ell}$ for $\ell \in \{i+1, \dots, k-1\}$ is given by

$$\begin{aligned} \hat{F}_1^{(4)}\mathcal{I}_{2i+2} &= -(i+1)\mathbb{c}_1\mathcal{I}_{2i+2} - \mathcal{I}_1^{i+1}, \\ \hat{F}_1^{(4)}\mathcal{I}_{2i+4} &= \hat{F}_1^{(3)}\mathcal{I}_{2i+4}, \\ \hat{F}_1^{(4)}\mathcal{I}_{2(i+\ell)} &= ((1-i-\ell)\mathbb{c}_1 - \mathbb{c}_\ell)\mathcal{I}_{2(i+\ell)} - \mathcal{I}_1^{i+1}\mathcal{I}_{2\ell-1} \quad \forall \ell \in \{3, \dots, k-1-i\}, \end{aligned}$$

hence, $\beta_{2i+4}(t)$ remains unchanged as well. The ODEs differing from the preceding case are then given by

$$\begin{aligned} \dot{\beta}_{2i+2} &= -(i+1)\mathbb{c}_1\beta_{2i+2} - \beta_1^{i+1}, \\ \dot{\beta}_{2\ell} &= -((\ell-1)\mathbb{c}_1 + \mathbb{c}_{\ell-i})\beta_{2\ell} - \beta_1^{i+1}\beta_{2(\ell-i)-1} \qquad \forall \ell \in \{i+3, \dots, k-1\}, \end{aligned}$$

which after substituting in the unchanged solutions change to

$$\begin{aligned} &\dot{\beta}_{2i+2} = -(i+1)\mathbb{c}_1\beta_{2i+2} - \mathcal{I}_1^{i+1}\mathrm{e}^{-(i+1)\mathbb{c}_1 t}, \\ &\forall \ell \in \{i+3, \dots, k-1\}: \\ &\quad \dot{\beta}_{2\ell} = -((\ell-1)\mathbb{c}_1 + \mathbb{c}_{\ell-i})\beta_{2\ell} - \mathcal{I}_1^{i+1}\mathrm{e}^{-(i+1)\mathbb{c}_1 t}\mathcal{I}_{2(\ell-i)-1}\mathrm{e}^{-(\ell-i)\mathbb{c}_2 t}, \end{aligned}$$

with solutions changing to

$$\beta_{2i+2}(t) = \left(\mathcal{I}_{2i+2} - t\mathcal{I}_1^{i+1}\right) \mathrm{e}^{-(i+1)\mathbb{c}_1 t},$$

$$\forall \ell \in \{i+3, \dots, k-1\}: \quad \beta_{2\ell}(t) = (\mathcal{I}_{2\ell} - t\mathcal{I}_1^{i+1}\mathcal{I}_{2(\ell-i)-1})\mathrm{e}^{-((\ell-i)\mathbb{c}_1 + \mathbb{c}_{\ell-i})t}.$$

Similarly to the preceding extension, we have to treat the cases $\mathbb{c}_1 \neq 0$ and $\mathbb{c}_1 = 0$ separately. If $\mathbb{c}_1 = k-2+c \neq 0$, then we set $\beta_1(t) := 1$, which yields the invariants in the form

$$\begin{aligned}
I_2 &= \mathcal{I}_2 \mathcal{I}_1^{-\frac{\mathbb{c}_{1-i}}{\mathbb{c}_1}}, \\
I_{2\ell} &= \mathcal{I}_{2\ell} \mathcal{I}_1^{-\ell+1-\frac{\mathbb{c}_{\ell-i}}{\mathbb{c}_1}} && \forall \ell \in \{2, \dots, i\} \cup \{i+2\}, \\
I_{2i+2} &= \mathcal{I}_{2i+2} \mathcal{I}_1^{-i-1} - \frac{\ln \mathcal{I}_1}{\mathbb{c}_1}, \\
I_{2\ell} &= \left(\mathcal{I}_{2\ell} - \frac{\ln \mathcal{I}_1}{\mathbb{c}_1} \mathcal{I}_1^{i+1} \mathcal{I}_{2(\ell-i)-1}\right) \mathcal{I}_1^{-\ell+\frac{\ell-i-1}{\mathbb{c}_1}} && \forall \ell \in \{i+3, \dots, k-1\}, \\
I_{2\ell-1} &= \mathcal{I}_{2\ell-1} \mathcal{I}_1^{-(\ell-1)\frac{\mathbb{c}_2}{\mathbb{c}_1}} && \forall \ell \in \{3, \dots, k-1\}.
\end{aligned}$$

In case $c = 2-k$, we set $\beta_5(t) := 1$ instead and the invariants are of the form

$$\begin{aligned}
I_1 &= \mathcal{I}_1, \\
I_{2\ell} &= \mathcal{I}_{2\ell} \mathcal{I}_5^{\frac{\ell-1-i}{2}} && \forall \ell \in \{1, \dots, i, i+2\}, \\
I_{2i+2} &= \mathcal{I}_{2i+2} - \frac{\ln \mathcal{I}_5}{2}, \\
I_{2\ell} &= \left(\mathcal{I}_{2\ell} - \frac{\ln \mathcal{I}_5}{2} \mathcal{I}_1^{i+1} \mathcal{I}_{2(\ell-i)-1}\right) \mathcal{I}_5^{\frac{\ell-1-i}{2}} && \forall \ell \in \{i+3, \dots, k-1\}, \\
I_{2\ell-1} &= \mathcal{I}_{2\ell-1} \mathcal{I}_5^{\frac{1-\ell}{2}} && \forall \ell \in \{4, \dots, k-1\}.
\end{aligned}$$

$\mathfrak{s}_{2k,5}$

After dropping the derivative by e_{2k-1}, the operator corresponding to the extending element takes up the following form

$$\begin{aligned}
\hat{F}_1^{(5)} = -ce_1 \frac{\partial}{\partial e_1} - e_2 \frac{\partial}{\partial e_2} - (c_{2k-5}e_1 + ce_3) \frac{\partial}{\partial e_3} - e_4 \frac{\partial}{\partial e_4} \\
- \sum_{i=3}^{k-1} \left(\left(ce_{2i-1} + \sum_{j=1}^{i-1} c_{2(k-i+j)-3} e_{2j-1} \right) \frac{\partial}{\partial e_{2i-1}} \right. \\
\left. + \left(e_{2i} + \sum_{j=1}^{i-2} b_{2(k-i-1+j)} e_{2j} \right) \frac{\partial}{\partial e_{2i}} \right),
\end{aligned}$$

its action on the invariants of $\mathfrak{r}$ (4.13) is given by

$$
\begin{aligned}
&\hat{F}_1^{(5)}\mathcal{I}_1 = -c\mathcal{I}_1, \quad \hat{F}_1^{(5)}\mathcal{I}_2 = -\mathcal{I}_2, \quad \hat{F}_1^{(5)}\mathcal{I}_4 = -(1+c)\mathcal{I}_4 + c_{2k-5}\mathcal{I}_1\mathcal{I}_2,\\
&\forall \ell \in \{3, ..., k-1\}:\\
&\hat{F}_1^{(5)}\mathcal{I}_{2\ell-1} = -(\ell-1)c\mathcal{I}_{2\ell-1} - c_{2(k-\ell)-1}\mathcal{I}_1^{\ell-1},\\
&\hat{F}_1^{(5)}\mathcal{I}_{2\ell} = -((\ell-1)c+1)\mathcal{I}_{2\ell} + c_{2(k-\ell)-1}\mathcal{I}_1\mathcal{I}_{2\ell-2} - \sum_{i=2}^{\ell-1} b_{2(k-1-i)}\mathcal{I}_1^i\mathcal{I}_{2(\ell-i)}.
\end{aligned}
$$

This yields the following systems of ODEs

$$
\begin{aligned}
&\dot{\beta}_1 = -c\beta_1, \quad \dot{\beta}_2 = -\beta_2, \quad \dot{\beta}_4 = -(1+c)\beta_4 + c_{2k-5}\beta_1\beta_2,\\
&\forall \ell \in \{3, ..., k-1\}:\\
&\dot{\beta}_{2\ell-1} = -(\ell-1)c\beta_{2\ell-1} - c_{2(k-\ell)-1}\beta_1^{\ell-1},\\
&\dot{\beta}_{2\ell} = -((\ell-1)c+1)\beta_{2\ell} + c_{2(k-\ell)-1}\beta_1\beta_{2\ell-2} - \sum_{i=2}^{\ell-1} b_{2(k-1-i)}\beta_1^i\beta_{2(\ell-i)},\\
&\forall \ell \in \{1, ..., 2k-2\}: \quad \beta_\ell(0) = \mathcal{I}_\ell
\end{aligned}
$$

with the solution

$$
\begin{aligned}
&\beta_1(t) = \mathrm{e}^{-ct}\mathcal{I}_1, \quad \beta_2(t) = \mathrm{e}^{-t}\mathcal{I}_2, \quad \beta_4(t) = \mathrm{e}^{-(1+c)t}(\mathcal{I}_4 - c_{2k-5}\mathcal{I}_1\mathcal{I}_2),\\
&\forall \ell \in \{3, ..., k-1\}:\\
&\beta_{2\ell-1}(t) = \mathrm{e}^{-(\ell-1)ct}(\mathcal{I}_{2\ell-1} - tc_{2(k-\ell)-1}\mathcal{I}_1^{\ell-1}),\\
&\beta_{2\ell}(t) = \mathrm{e}^{-((\ell-1)c+1)t}\Bigg(\mathcal{I}_{2\ell}\\
&\qquad + \sum_{i=1}^{\ell-1}\frac{t^i}{i!}\mathcal{I}_1^i c_{2k-5}^{i-1}\left(c_{2k-5}\mathcal{I}_{2\ell-i} - i\sum_{j=1}^{\ell-1-i} b_{2(k-2-j)}\mathcal{I}_1^j\mathcal{I}_{2(\ell-1-j)}\right)\Bigg).
\end{aligned}
$$

Fixing $\beta_2(t)$ to 1, we get $\ln\mathcal{I}_2$ as a value of t for back-substitution. The invariants are

$$
\begin{aligned}
&I_1 = \mathcal{I}_1\mathcal{I}_2^{-c}, \quad I_4 = \mathcal{I}_2^{-1-c}(\mathcal{I}_4 - c_{2k-5}\mathcal{I}_1\mathcal{I}_2),\\
&\forall \ell \in \{3, \dots, k-1\}:\\
&I_{2\ell-1} = \mathcal{I}_2^{-(\ell-1)c}\left(\mathcal{I}_{2\ell-1} - c_{2(k-\ell)-1}\mathcal{I}_1^{\ell-1}\ln\mathcal{I}_2\right),\\
&I_{2\ell} = \mathcal{I}_2^{-(\ell-1)c-1}\Bigg(\mathcal{I}_{2\ell}\\
&\qquad + \sum_{i=1}^{\ell-1}\frac{(\ln\mathcal{I}_2)^i}{i!}\mathcal{I}_1^i c_{2k-5}^{i-1}\left(c_{2k-5}\mathcal{I}_{2\ell-i} - i\sum_{j=1}^{\ell-1-i} b_{2(k-2-j)}\mathcal{I}_1^j\mathcal{I}_{2(\ell-1-j)}\right)\Bigg),
\end{aligned}
$$

while we can change to a slightly simpler basis with $I'_{2\ell-1}$ instead of $I_{2\ell-1}$, where

$$I'_{2\ell-1} \equiv I_1^{1-\ell} I_{2\ell-1} = \mathcal{I}_1^{1-\ell} \mathcal{I}_{2\ell-1} - c_{2(k-\ell)-1} \ln \mathcal{I}_2$$

for every ℓ in $\{3, \dots, k-1\}$.

$\mathfrak{s}_{2k,6}(c)$

This extension has the same invariants as $\mathfrak{s}_{2k,3}(1, c)$.

$\mathfrak{s}_{2k,7}^{(i)}$

This extension has the same invariants as $\mathfrak{s}_{2k,4}^{(1-i)}(i)$.

$\mathfrak{s}_{2k,8}^{(i)}$

The vector field corresponding to the extending element is after dropping the irrelevant derivatives given by

$$\hat{F}_1^{(8)} = \hat{F}_1^{(3)} - \sum_{j=i}^{k-1} e_{2(j-i+1)} \frac{\partial}{\partial e_{2j-1}}.$$

We immediately see that the action of $\hat{F}_1^{(8)}$ on some of the invariants of $\mathfrak{r}$ is the same as that of $\hat{F}_1^{(3)}$. Namely

$$\hat{F}_1^{(8)} \mathcal{I}_\ell = \hat{F}_1^{(3)} \mathcal{I}_\ell \qquad \forall \ell \in \{1, 5, 7, \dots, 2i-3\} \cup \{2, 4, \dots, k-1\},$$

while the action on the remaining invariants is given by

$$\hat{F}_1^{(8)} \mathcal{I}_{2\ell-1} = \hat{F}_1^{(3)} \mathcal{I}_{2\ell-1} - \mathcal{I}_1^{i-2} \mathcal{I}_{2(\ell-i+1)} \qquad \forall \ell \in \{i, \dots, k-1\}, \tag{4.16}$$

with the corresponding ODEs

$$\left.\begin{aligned} \dot{\beta}_{2\ell-1} &= -(\ell-1)\mathbb{c}_2 \beta_{2\ell-1} - \beta_1^{i-2} \beta_{2(\ell-i+1)} \\ \beta_{2\ell-1}(0) &= \mathcal{I}_{2\ell-1} \end{aligned}\right\} \forall \ell \in \{i, \dots, k-1\}. \tag{4.17}$$

The ODEs corresponding to the unchanged actions remain unchanged as well and since they do not contain the functions for which the differential equations have changed, their solutions remain unchanged as well. After substituting them to the changed equations (4.17) we can find the solutions in the form

$$\beta_{2\ell-1}(t) = \mathrm{e}^{-(\ell-1)(k-3+i)} \left(\mathcal{I}_{2\ell-1} - t \mathcal{I}_1^{i-2} \mathcal{I}_{2(\ell-i+1)} \right) \qquad \forall \ell \in \{i, \dots, k-1\}.$$

The numbers $\mathbb{c}_j$ and $\mathbb{b}_j$ defined in (4.15) simplify to

$$\mathbb{c}_j = k - 1 - j + i, \qquad \mathbb{b}_j = k - j$$

Unlike in the case of $\mathfrak{s}_{2k,3}(b,c)$, it is not possible that $\mathbb{c}_1 = 0$ as $i \geq 3$. Thus we set $\beta_1(t) = 1$ and substitute the value $t = \frac{\ln \mathcal{I}_1}{k-2+i}$ to the remaining β_j to obtain the invariants of $\mathfrak{s}_{2k,8}^{(i)}$

$$\begin{aligned}
I_2 &= \mathcal{I}_2 \mathcal{I}_1^{-\frac{k-2}{k-2+i}}, \\
I_{2\ell} &= \mathcal{I}_{2\ell} \mathcal{I}_1^{-\ell - \frac{\ell-2+i}{k-2+i}} && \forall \ell \in \{2, \dots, k-1\}, \\
I_{2\ell-1} &= \mathcal{I}_{2\ell-1} \mathcal{I}_1^{-(\ell-1)\frac{k-3+i}{k-2+i}} && \forall \ell \in \{3, \dots, i-1\}, \\
I_{2\ell-1} &= \mathcal{I}_1^{-(\ell-1)\frac{k-3+i}{k-2+i}} \left(\mathcal{I}_{2\ell-1} - \frac{\ln \mathcal{I}_1}{k-2+i} \mathcal{I}_{2(\ell-i+1)} \right) && \forall \ell \in \{i, \dots, k-1\}.
\end{aligned}$$

$\mathfrak{s}_{2k,9}$

After dropping the derivative by e_{2k-1}, the operator corresponding to the extending element takes up the following form

$$\begin{aligned}
\hat{F}_1^{(9)} = &-e_1 \frac{\partial}{\partial e_1} - (c_{2k-5} e_1 + e_3) \frac{\partial}{\partial e_3} \\
&- \sum_{i=3}^{k-1} \left(\left(e_{2i-1} + \sum_{j=1}^{i-1} c_{2(k-i+j)-3} e_{2j-1} \right) \frac{\partial}{\partial e_{2i-1}} \right. \\
&\qquad \left. + \left(\sum_{j=1}^{i-2} b_{2(k-i-1+j)} e_{2j} \right) \frac{\partial}{\partial e_{2i}} \right).
\end{aligned}$$

Comparing it to the operator $\hat{F}_1^{(5)}$ with $c = 1$, we see that they only differ in coefficients of derivatives by e_i for i even. Furthermore, since the invariants of the nilradical (4.13) with odd indices do not depend on e_i with i even, the action of $\hat{F}_1^{(9)}$ on them is the same as that of $\hat{F}_1^{(5)}$ and consequently, the corresponding ODEs with the solutions are the same as well. The action of $\hat{F}_1^{(9)}$ on even invariants of the nilradical is given by

$$\begin{aligned}
&\hat{F}_1^{(9)} \mathcal{I}_2 = 0 \\
&\forall \ell \in \{3, ..., k-1\} : \\
&\quad \hat{F}_1^{(5)} \mathcal{I}_{2\ell} = -(\ell - 1)\mathcal{I}_{2\ell} + c_{2(k-\ell)-1} \mathcal{I}_1 \mathcal{I}_{2\ell-2} - \sum_{i=2}^{\ell-1} b_{2(k-1-i)} \mathcal{I}_1^i \mathcal{I}_{2(\ell-i)}.
\end{aligned}$$

with the corresponding changed ODEs

$$
\begin{aligned}
&\dot{\beta}_2 = 0, \quad \dot{\beta}_4 = -\beta_4 + c_{2k-5}\beta_1\beta_2, \\
&\forall \ell \in \{3, ..., k-1\} : \\
&\qquad \dot{\beta}_{2\ell} = -(\ell-1)\beta_{2\ell} + c_{2(k-\ell)-1}\beta_1\beta_{2\ell-2} - \sum_{i=2}^{\ell-1} b_{2(k-1-i)}\beta_1^i\beta_{2(\ell-i)}, \\
&\forall \ell \in \{2, 4, ..., 2k-2\} : \quad \beta_\ell(0) = \mathcal{I}_\ell
\end{aligned}
$$

and the changed solutions

$$
\begin{aligned}
&\beta_2(t) = \mathcal{I}_1, \quad \beta_4(t) = \mathrm{e}^{-t}(\mathcal{I}_4 - c_{2k-5}\mathcal{I}_1\mathcal{I}_2), \\
&\forall \ell \in \{3, ..., k-1\} : \\
&\beta_{2\ell}(t) = \mathrm{e}^{-(\ell-1)t}\Bigg(\mathcal{I}_{2\ell} \\
&\qquad + \sum_{i=1}^{\ell-1} \frac{t^i}{i!}\mathcal{I}_1^i c_{2k-5}^{i-1}\Bigg(c_{2k-5}\mathcal{I}_{2\ell-i} - i\sum_{j=1}^{\ell-1-i} b_{2(k-2-j)}\mathcal{I}_1^j\mathcal{I}_{2(\ell-1-j)}\Bigg)\Bigg).
\end{aligned}
$$

Fixing $\beta_1(t)$ to 1, we get $\ln\mathcal{I}_1$ as a value of t for back-substitution. The invariants of $\mathfrak{s}_{2k,9}$ are

$$
\begin{aligned}
&I_2 = \mathcal{I}_2, \quad I_4 = \mathcal{I}_1^{-1}(\mathcal{I}_4 - c_{2k-5}\mathcal{I}_1\mathcal{I}_2), \\
&\forall \ell \in \{3, \dots, k-1\} : \\
&I_{2\ell-1} = \mathcal{I}_1^{-\ell+1}\left(\mathcal{I}_{2\ell-1} - c_{2(k-\ell)-1}\mathcal{I}_1^{\ell-1}\ln\mathcal{I}_2\right), \\
&\quad I_{2\ell} = \mathcal{I}_1^{-\ell+1}\Bigg(\mathcal{I}_{2\ell} \\
&\qquad + \sum_{i=1}^{\ell-1} \frac{(\ln\mathcal{I}_1)^i}{i!}\mathcal{I}_1^i c_{2k-5}^{i-1}\Bigg(c_{2k-5}\mathcal{I}_{2\ell-i} - i\sum_{j=1}^{\ell-1-i} b_{2(k-2-j)}\mathcal{I}_1^j\mathcal{I}_{2(\ell-1-j)}\Bigg)\Bigg).
\end{aligned}
$$

$\mathfrak{s}_{2k,10}$

The operator corresponding to the extending element is given by

$$
\begin{aligned}
\hat{F}_1^{(10)} &= \hat{F}_1^{(3)}|_{c=2,b=1} + \hat{G}_1^{(10)}, \text{ where} \\
\hat{G}_1^{(10)} &= -\sum_{j=1}^{k-2} e_{2j}\frac{\partial}{\partial e_{2j+1}},
\end{aligned}
$$

its action on the invariants (4.13) is then

$$
\begin{aligned}
\hat{F}_1^{(10)}\mathcal{I}_1 &= -k\mathcal{I}_1,\\
\hat{F}_1^{(10)}\mathcal{I}_2 &= -(k-1)\mathcal{I}_2,\\
\hat{F}_1^{(10)}\mathcal{I}_{2\ell} &= -\ell(k-1)\mathcal{I}_{2\ell} + \mathcal{I}_2\mathcal{I}_{2\ell-2} \qquad \forall \ell \in \{2,\dots,k-1\},\\
\hat{F}_1^{(10)}\mathcal{I}_5 &= -2(k-1)\mathcal{I}_5 - \mathcal{I}_4,\\
\hat{F}_1^{(10)}\mathcal{I}_{2\ell-1} &= -(\ell-1)(k-1)\mathcal{I}_{2\ell-1} - \mathcal{I}_{2\ell-2} + \mathcal{I}_2\mathcal{I}_{2\ell-3} \qquad \forall \ell \in \{4,\dots,k-1\}.
\end{aligned}
$$

The corresponding system of ODEs

$$
\begin{aligned}
\dot{\beta}_1 &= -k\beta_1,\\
\dot{\beta}_2 &= -(k-1)\beta_2,\\
\dot{\beta}_{2\ell} &= -\ell(k-1)\beta_{2\ell} + \beta_2\beta_{2\ell-2} && \forall \ell \in \{2,\dots,k-1\},\\
\dot{\beta}_5 &= -2(k-1)\beta_5 - \beta_4,\\
\dot{\beta}_{2\ell-1} &= -(\ell-1)(k-1)\beta_{2\ell-1} - \beta_{2\ell-2} + \beta_2\beta_{2\ell-3} && \forall \ell \in \{4,\dots,k-1\}
\end{aligned}
$$

with the initial conditions

$$
\beta_\ell(0) = \mathcal{I}_\ell \qquad \forall \ell \in \{1,2\} \ \cup \ \{4,\dots,k-2\}
$$

has the following solution

$$
\begin{aligned}
\beta_1(t) &= \mathrm{e}^{-kt}\mathcal{I}_1,\\
\beta_2(t) &= \mathrm{e}^{-(k-1)t}\mathcal{I}_2,\\
\beta_{2\ell}(t) &= \mathrm{e}^{-\ell(k-1)t}\sum_{j=0}^{\ell-1}\frac{t^j\mathcal{I}_2^j}{j!}\mathcal{I}_{2(\ell-j)} \qquad \forall \ell \in \{2,\dots,k-1\},\\
\beta_5(t) &= \mathrm{e}^{-2(k-1)t}\left(\mathcal{I}_5 - t\mathcal{I}_4 - \frac{t^2}{2}\mathcal{I}_2^2\right),\\
\forall \ell \in \{4,\dots,k-1\}:&\\
\beta_{2\ell-1}(t) &= \mathrm{e}^{-(\ell-1)(k-1)t}\left(\sum_{j=0}^{\ell-3}\frac{t^j\mathcal{I}_2^j}{j!}\mathcal{I}_{2(\ell-j)-1} - \sum_{h=1}^{\ell-2}\sum_{j=h}^{\ell-1}\frac{t^j\mathcal{I}_2^j}{j!}\mathcal{I}_{2(\ell-j)}\right).
\end{aligned}
$$

Fixing $\beta_1(t)$ to 1, we obtain the value $t = \frac{\ln \mathcal{I}_1}{k}$ for substitution into the remaining functions. The invariants are

$$
\begin{aligned}
I_2 &= \mathcal{I}_1^{\frac{1-k}{k}} \mathcal{I}_2, \\
I_{2\ell} &= \mathcal{I}_1^{\ell \frac{1-k}{k}} \sum_{j=0}^{\ell-1} \frac{\ln \mathcal{I}_1{}^j \mathcal{I}_2^j}{k^j j!} \mathcal{I}_{2(\ell-j)} \qquad \forall \ell \in \{2, \dots, k-1\}, \\
I_5 &= \mathcal{I}_1^{2\frac{1-k}{k}} \left(\mathcal{I}_5 - \frac{\ln \mathcal{I}_1}{k} \mathcal{I}_4 - \frac{\ln \mathcal{I}_1{}^2}{2k^2} \mathcal{I}_2^2 \right), \\
I_{2\ell-1} &= \mathcal{I}_1^{\frac{(\ell-1)(1-k)}{k}} \left(\sum_{j=0}^{\ell-3} \frac{\ln \mathcal{I}_1{}^j \mathcal{I}_2^j}{k^j j!} \mathcal{I}_{2(\ell-j)-1} - \sum_{h=1}^{\ell-2} \sum_{j=h}^{\ell-1} \frac{\ln \mathcal{I}_1{}^j \mathcal{I}_2^j}{k^j j!} \mathcal{I}_{2(\ell-j)} \right).
\end{aligned}
$$

$\mathfrak{s}_{2k,11}$

After dropping the derivative by e_{2k-1}, we have the operator

$$
\hat{F}_1^{(11)} = \hat{F}_1^{(3)} - \sum_{i}^{k-1} e_{2i-1} \frac{\partial}{\partial e_{2i}},
$$

with its action on the invariants of the nilradical given by

$$
\begin{aligned}
\hat{F}_1^{(11)} \mathcal{I}_{2\ell-1} &= \hat{F}_1^{(3)} \mathcal{I}_{2\ell-1} & \ell \in \{1, 3, 4, \dots, k-1\}, \\
\hat{F}_1^{(11)} \mathcal{I}_2 &= \hat{F}_1^{(3)} \mathcal{I}_2 - \mathcal{I}_1, \\
\hat{F}_1^{(11)} \mathcal{I}_4 &= \hat{F}_1^{(3)} \mathcal{I}_4, \\
\hat{F}_1^{(11)} \mathcal{I}_{2\ell} &= \hat{F}_1^{(3)} \mathcal{I}_{2\ell} - \mathcal{I}_1 \mathcal{I}_{2\ell-1} & \ell \in \{3, \dots, k-1\}.
\end{aligned}
$$

The solution of the corresponding system of ODEs is following

$$
\begin{aligned}
\beta_1(t) &= \mathrm{e}^{-\mathbb{b}_1 t} \mathcal{I}_1, \quad \beta_2(t) = \mathrm{e}^{-\mathbb{b}_1 t} (\mathcal{I}_2 - t \mathcal{I}_1), \quad \beta_4(t) = \mathrm{e}^{-(2\mathbb{b}_2+1)t} \mathcal{I}_4, \\
\beta_{2\ell-1}(t) &= \mathrm{e}^{-(\ell-1)\mathbb{b}_2 t} \mathcal{I}_{2\ell-1} & \forall \ell \in \{3, \dots, k-1\}, \\
\beta_{2\ell}(t) &= \mathrm{e}^{-(\ell \mathbb{b}_2+1)t} (\mathcal{I}_{2\ell} - t \mathcal{I}_1 \mathcal{I}_{2\ell-1}) & \forall \ell \in \{3, \dots, k-1\}.
\end{aligned}
$$

In case $\mathbb{b}_1 \neq 0$, we set $\beta_1(t) := 1$ to obtain the corresponding value of t, $t = \frac{\ln \mathcal{I}_1}{\mathbb{b}_1}$. Substituting this value back to the remaining β_i, we obtain the invariants of $\mathfrak{s}_{2k,11}(b \neq 2-k)$

$$\begin{aligned} I_2 &= \mathcal{I}_1^{-\frac{\mathbb{b}_2}{\mathbb{b}_1}} \left(\mathcal{I}_2 - \frac{\ln \mathcal{I}_1}{\mathbb{b}_1} \mathcal{I}_1 \right), \\ I_4(t) &= \mathcal{I}_1^{-\frac{2\mathbb{b}_2+1}{\mathbb{b}_1}} \mathcal{I}_4, \\ I_{2\ell-1} &= \mathcal{I}_1^{-(\ell-1)\frac{\mathbb{b}_2}{\mathbb{b}_1}} \mathcal{I}_{2\ell-1} && \forall \ell \in \{3, \dots, k-1\}, \\ I_{2\ell} &= \mathcal{I}_1^{-\frac{\ell \mathbb{b}_2+1}{\mathbb{b}_1}} \left(\mathcal{I}_{2\ell} - \frac{\ln \mathcal{I}_1}{\mathbb{b}_1} \mathcal{I}_1 \mathcal{I}_{2\ell-1} \right) && \forall \ell \in \{3, \dots, k-1\}. \end{aligned}$$

On the other hand, if we have $\mathbb{b}_1 = 0$, that is $b = 2-k$, we set $\beta_5(t) := 1$ to obtain $t = \frac{\ln \mathcal{I}_5}{2}$ for back substitution. The invariants of $\mathfrak{s}_{2k,11}(2-k)$ are given by

$$\begin{aligned} I_1 &= \mathcal{I}_1, \\ I_2 &= \mathcal{I}_2 - \frac{\ln \mathcal{I}_5}{2} \mathcal{I}_1, \\ I_{2\ell-1} &= \mathcal{I}_{2\ell-1} \mathcal{I}_5^{\frac{1-\ell}{2}} && \forall \ell \in \{4, \dots, k-1\}, \\ I_{2\ell} &= \mathcal{I}_5^{-\frac{\ell \mathbb{b}_2+1}{2}} \left(\mathcal{I}_{2\ell} - \frac{\ln \mathcal{I}_5}{2} \mathcal{I}_1 \mathcal{I}_{2\ell-1} \right) && \forall \ell \in \{3, \dots, k-1\}. \end{aligned}$$

$\mathfrak{s}_{2k,12}$

The operator corresponding to the extending element is given by

$$\begin{aligned} \hat{F}_1^{(12)} = \sum_{\ell=1}^{k-1} \Bigg(\sum_{i=1}^{2(\ell-1)} \left(c_{i+2(k-\ell-1)} e_i + c_{2k-4} e_{2\ell-2} + e_{2\ell-1} \right) \frac{\partial}{\partial e_{2\ell-1}} \\ + \left(e_{2\ell-1} + e_{2\ell} \right) \frac{\partial}{\partial e_{2\ell}} \Bigg), \end{aligned}$$

its action on the invariants of the nilradical (4.13) is given by

$$
\begin{aligned}
\hat{F}_1^{(11)}\mathcal{I}_1 &= -\mathcal{I}_1, \\
\hat{F}_1^{(11)}\mathcal{I}_2 &= -\mathcal{I}_1 - \mathcal{I}_2, \\
\hat{F}_1^{(11)}\mathcal{I}_4 &= -2\mathcal{I}_4 + c_{2k-4}\mathcal{I}_2^2, \\
\hat{F}_1^{(11)}\mathcal{I}_5 &= -2\mathcal{I}_5 - c_{2k-4}\mathcal{I}_4 - c_{2k-6}\mathcal{I}_1\mathcal{I}_2 - c_{2k-7}\mathcal{I}_1^2, \\
\hat{F}_1^{(11)}\mathcal{I}_{2\ell} &= -\ell\mathcal{I}_{2\ell} - \mathcal{I}_1\mathcal{I}_{2\ell-1} + c_{2k-4}\mathcal{I}_2\mathcal{I}_{2\ell-2} && \forall \ell \in \{3, \dots, k-1\}, \\
\hat{F}_1^{(11)}\mathcal{I}_{2\ell-1} &= -(\ell-1)\mathcal{I}_{2\ell-1} - c_{2k-4}\mathcal{I}_{2\ell-2} + c_{2k-4}\mathcal{I}_2\mathcal{I}_{2\ell-3} \\
&\quad - \sum_{i=3}^{\ell} c_{2(k-i)}\mathcal{I}_1^{i-2}\mathcal{I}_{2(\ell+1-i)} - c_{2(k-\ell)-1}\mathcal{I}_1^{\ell-1} && \forall \ell \in \{4, \dots, k-1\}. \\
&\quad - \sum_{i=3}^{\ell-2} c_{2(k-i)-1}\mathcal{I}_1^{i-1}\alpha_{2(\ell+1-i)-1}\mathcal{I}_1^{i-1}\mathcal{I}_{2(\ell-i)+1}
\end{aligned}
$$

This leads to the ODEs for characteristics of the form

$$
\begin{aligned}
\dot{\beta}_1 &= -\beta_1, \\
\dot{\beta}_2 &= -\beta_1 - \beta_2, \\
\dot{\beta}_4 &= -2\beta_4 + c_{2k-4}\beta_2^2, \\
\dot{\beta}_5 &= -2\beta_5 - c_{2k-4}\beta_4 - c_{2k-6}\beta_1\beta_2 - c_{2k-7}\beta_1^2, \\
\dot{\beta}_{2\ell} &= -\ell\beta_{2\ell} - \beta_1\beta_{2\ell-1} + c_{2k-4}\beta_2\beta_{2\ell-2} && \forall \ell \in \{3, \dots, k-1\}, \\
\dot{\beta}_{2\ell-1} &= -(\ell-1)\beta_{2\ell-1} - c_{2k-4}\beta_{2\ell-2} + c_{2k-4}\beta_2\beta_{2\ell-3} \\
&\quad - \sum_{i=3}^{\ell} c_{2(k-i)}\beta_1^{i-2}\beta_{2(\ell+1-i)} - c_{2(k-\ell)-1}\beta_1^{\ell-1} && \forall \ell \in \{4, \dots, k-1\}, \\
&\quad - \sum_{i=3}^{\ell-2} c_{2(k-i)-1}\beta_1^{i-1}\alpha_{2(\ell+1-i)-1}\beta_1^{i-1}\beta_{2(\ell-i)+1} \\
\beta_\ell(0) &= \mathcal{I}_\ell && \forall \ell \in \{1, \dots, 2k-2\}.
\end{aligned}
$$

Although we were able to solve these equations for $\ell \in \{1, 2\} \cup \{4, \dots, 11\}$, the solutions turned out to be rather cumbersome and we were unable to deduce a general formula valid for arbitrary ℓ.

$\mathfrak{s}_{2k,13}$, $\mathfrak{s}_{2k,14}$, $\mathfrak{s}_{2k,15}^{(i)}$

Similarly to the preceding case $\mathfrak{s}_{2k,12}$, the invariants of these extensions were not found.

$\mathfrak{s}_{2k,16}$

This extension has the same invariants as $\mathfrak{s}_{2k,11}(1)$.

4.3.3 Invariants of Extension by Two Elements

The vector fields corresponding to the extending elements $f_1,\ f_2$ of the extensions $\mathfrak{s}_{2k+1,i}$ will be denoted $\hat{F}_{1,i},\ \hat{F}_{2,i}$ to differentiate them from the operators $\hat{F}_1^{(j)}$ from the subsection 4.3.2 that will be referenced below. Similarly as in the case of extensions by one element, the vector fields contain derivatives by e_{2k-1} and $f_1,\ f_2$, but it is evident that the invariants can not depend on these. In the following, only the forms stripped off of these derivatives will be used without further notice and the vanishing operators $\{\hat{E}_1, \dots, \hat{E}_{2k-2}\}$ will be omitted.

Invariants of $\mathfrak{s}_{2k+1,1}(C,b)$

The vector fields are following

$$\begin{aligned}
\hat{E}_{2k-1} &= -\sum_{\ell=3}^{2k-2} e_{\ell-2}\frac{\partial}{\partial e_\ell},\\
\hat{F}_{1,1} &= \hat{F}_1^{(1)}|_{c=C,\ a=0},\\
\hat{F}_{2,1} &= -\sum_{i=1}^{k-1}(k-1-i+b)\left(e_{2i-1}\frac{\partial}{\partial e_{2i-1}} + e_{2i}\frac{\partial}{\partial e_{2i}}\right),
\end{aligned}$$

the corresponding flow of the last one of them is given by

$$\left.\begin{aligned}
e_{2\ell}^{\#}\left(\Psi_{\hat{F}_{2,1}}^{\beta_2}(\vec{e})\right) &= \mathrm{e}^{-(k-1-\ell+b)\beta_2}e_{2\ell}\\
e_{2\ell-1}^{\#}\left(\Psi_{\hat{F}_{2,1}}^{\beta_2}(\vec{e})\right) &= \mathrm{e}^{-(k-1-\ell+b)\beta_2}e_{2\ell-1}
\end{aligned}\right.\qquad \forall\ell\{1,\dots,k-1\}.$$

We compose the flows in the following way

$$\Psi \equiv \Psi_{\hat{F}_{2,1}}^{\beta_2} \circ \Psi_{\hat{F}_{1,1}}^{\beta_1} \circ \Psi_{\hat{E}_{2k-1}}^{\alpha},$$

then the components of Ψ are for every $\ell \in \{1,\dots,k-1\}$ given by

$$\begin{aligned}
e_{2\ell-1}^{\#}\left(\Psi(\vec{e})\right) &= \mathrm{e}^{-\mathbb{b}_\ell\beta_2 - C\beta_1}\left(\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i-1}\cos\beta_1 + \sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i}\sin\beta_1\right),\\
e_{2\ell}^{\#}\left(\Psi(\vec{e})\right) &= \mathrm{e}^{-\mathbb{b}_\ell\beta_2 - C\beta_1}\left(-\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i-1}\sin\beta_1 + \sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i}\cos\beta_1\right).
\end{aligned}$$

Choosing the section as

$$e_1^{\#}(\Psi(\vec{e}\,)) = 0, \qquad e_2^{\#}(\Psi(\vec{e}\,)) = 1, \qquad e_4^{\#}(\Psi(\vec{e}\,)) = 0$$

yields

$$\beta_1 = -\arctan\frac{e_1}{e_2}, \qquad \beta_2 = \frac{\frac{1}{2}\ln(e_1^2+e_2^2) + C\arctan\frac{e_1}{e_2}}{k-2+b}, \qquad \alpha = \frac{e_1e_3+e_2e_4}{e_1^2+e_2^2}.$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\forall \ell \in \{2,\dots,k-1\}:$$

$$I_{2\ell-1} = g_\ell(C,b)\left(\sum_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}}(e_2e_{2i-1}-e_1e_{2i})\right),$$

$$\forall \ell \in \{3,\dots,k-1\}:$$

$$I_{2\ell} = g_\ell(C,b)\left(\sum_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}}(e_1e_{2i-1}+e_2e_{2i})\right), \text{ where}$$

$$g_\ell(C,b) = \mathrm{e}^{\frac{\ell-1}{k-2+b}C\arctan\frac{e_1}{e_2}}\left(e_1^2+e_2^2\right)^{\frac{\ell-1}{2(k-2+b)}-1}.$$

Similarly to the simplification of the basis of invariants of $\mathfrak{s}_{2k,1}$ we may replace the invariants $I_{2\ell}$ by rational functions $I'_{2\ell}$ given by

$$I'_{2\ell} \equiv \frac{\sum\limits_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_1e_{2i-1}+e_2e_{2i})}{\sum\limits_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_2e_{2i-1}-e_1e_{2i})} \qquad \forall \ell \in \{3,\dots,k-1\}.$$

Invariants of $\mathfrak{s}_{2k+1,2}$, $\mathfrak{s}_{2k+1,3}$

The invariants of these extensions were not found, since the same vector field that prevented us from finding the invariants of $\mathfrak{s}_{2k,2}$ is involved here as well.

Invariants of $\mathfrak{s}_{2k+1,4}(A,C)$

The vector fields are following

$$\hat{E}_{2k-1} = -\sum_{\ell=3}^{2k-2} e_{\ell-2}\frac{\partial}{\partial e_\ell},$$

$$\hat{F}_{1,4} = \hat{F}_1^{(1)}|_{c=C,\ a=A},$$

$$\hat{F}_{2,4} = -\sum_{i=1}^{k-1} e_{2i-1}\frac{\partial}{\partial e_{2i-1}} + e_{2i}\frac{\partial}{\partial e_{2i}},$$

the corresponding flow of the last one of them is given by

$$\begin{aligned} e^{\#}_{2\ell}\left(\Psi^{\beta_2}_{\hat{F}_{2,4}}(\vec{e})\right) &= \mathrm{e}^{-\beta_2} e_{2\ell} \\ e^{\#}_{2\ell-1}\left(\Psi^{\beta_2}_{\hat{F}_{2,4}}(\vec{e})\right) &= \mathrm{e}^{-\beta_2} e_{2\ell-1} \end{aligned} \qquad \forall \ell \{1, \dots, k-1\}.$$

We compose the flows in the following way

$$\Psi \equiv \Psi^{\beta_2}_{\hat{F}_{2,4}} \circ \Psi^{\beta_1}_{\hat{F}_{1,4}} \circ \Psi^{\alpha}_{\hat{E}_{2k-1}},$$

then the components of Ψ are for every $\ell \in \{1, \dots, k-1\}$ given by

$$e^{\#}_{2\ell-1}\left(\Psi(\vec{e})\right) = \mathrm{e}^{-\beta_2 - \Bbbk_\ell \beta_1}\left(\sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}\cos\beta_1 + \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i}\sin\beta_1\right),$$

$$e^{\#}_{2\ell}\left(\Psi(\vec{e})\right) = \mathrm{e}^{-\beta_2 - \Bbbk_\ell \beta_1}\left(-\sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}\sin\beta_1 + \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i}\cos\beta_1\right).$$

Choosing the section as

$$e^{\#}_1\left(\Psi(\vec{e}\,)\right) = 0, \qquad e^{\#}_2\left(\Psi(\vec{e}\,)\right) = 1, \qquad e^{\#}_4\left(\Psi(\vec{e}\,)\right) = 0$$

yields

$$\beta_1 = -\arctan\frac{e_1}{e_2}, \qquad \alpha = \frac{e_1 e_3 + e_2 e_4}{e_1^2 + e_2^2},$$

$$\beta_2 = \ln\sqrt{e_1^2 + e_2^2} \; + \; ((k-2)A + C)\arctan\frac{e_1}{e_2}.$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$\forall \ell \in \{2, \dots, k-1\}:$

$$I_{2\ell-1} = g_\ell(A)\left(\sum_{i=1}^{\ell} \frac{(-e_1 e_3 - e_2 e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}}(e_2 e_{2i-1} - e_1 e_{2i})\right),$$

$\forall \ell \in \{3, \dots, k-1\}:$

$$I_{2\ell} = g_\ell(A)\left(\sum_{i=1}^{\ell} \frac{(-e_1 e_3 - e_2 e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}}(e_1 e_{2i-1} + e_2 e_{2i})\right), \text{ where}$$

$$g_\ell(A) = \mathrm{e}^{(1-\ell)A\arctan\frac{e_1}{e2}}\left(e_1^2 + e_2^2\right)^{-1}.$$

Again, we may replace the invariants $I_{2\ell}$ by rational functions $I'_{2\ell}$ given by

$$I'_{2\ell} \equiv \frac{\sum\limits_{i=1}^{\ell} \frac{(-e_1e_3 - e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_1 e_{2i-1} + e_2 e_{2i})}{\sum\limits_{i=1}^{\ell} \frac{(-e_1e_3 - e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_2 e_{2i-1} - e_1 e_{2i})} \qquad \forall \ell \in \{3, \dots, k-1\}.$$

Invariants of $\mathfrak{s}_{2k+1,5}(C,c)$

The vector fields corresponding to the extending elements are given by

$$\hat{F}_{1,5} = 2\hat{F}_1^{(3)}|_{c=\frac{C}{2},\ b=0},$$
$$\hat{F}_{2,5} = -\sum_{i=1}^{k-1} ce_{2i-1}\frac{\partial}{\partial e_{2i-1}} + e_{2i}\frac{\partial}{\partial e_{2i}},$$

the components of their vector flows are for every ℓ in $\{1,\dots,k-1\}$

$$e^{\#}_{2\ell-1}\left(\Psi^{\beta_1}_{\hat{F}_{1,5}}(\vec{e})\right) = \mathrm{e}^{-(2(k-1-\ell)+C)\beta_1}e_{2\ell-1}, \qquad e^{\#}_{2\ell-1}\left(\Psi^{\beta_2}_{\hat{F}_{2,5}}(\vec{e})\right) = \mathrm{e}^{-c\beta_2}e_{2\ell-1},$$
$$e^{\#}_{2\ell}\left(\Psi^{\beta_2}_{\hat{F}_{1,5}}(\vec{e})\right) = \mathrm{e}^{-2(k-1-\ell)\beta_1}e_{2\ell}, \qquad e^{\#}_{2\ell}\left(\Psi^{\beta_2}_{\hat{F}_{2,5}}(\vec{e})\right) = \mathrm{e}^{-\beta_2}e_{2\ell}.$$

Composing the flows as

$$\Psi \equiv \Psi^{\beta_2}_{\hat{F}_{2,5}} \circ \Psi^{\beta_1}_{\hat{F}_{1,5}} \circ \Psi^{\alpha}_{\hat{E}_{2k-1}},$$

we obtain the components of Ψ in the form

$$e^{\#}_{2\ell-1}\left(\Psi(\vec{e})\right) = \mathrm{e}^{-c\beta_2-(2(k-1-\ell)+C)\beta_1}\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i-1},$$
$$e^{\#}_{2\ell}\left(\Psi(\vec{e})\right) = \mathrm{e}^{-\beta_2-2(k-1-\ell)\beta_1}\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i}.$$

Choosing the section as

$$e^{\#}_1\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_2\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_3\left(\Psi(\vec{e})\right) = 0$$

yields

$$\beta_1 = \frac{\ln\frac{e_1}{e_2^c}}{2(k-2)(1-c)+C}, \qquad \beta_2 = \frac{\ln\frac{e_2^{2(k-2)+C}}{e_1^{2(k-2)}}}{2(k-2)(1-c)+C}, \qquad \alpha = \frac{e_3}{e_1},$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$\forall \ell \in \{3,\dots,k-1\}$:

$$I_{2\ell-1} = e_1 e_1^{\frac{2(\ell-1)}{2(k-2)(1-c)+C}} e_2^{\frac{2c(1-\ell)}{2(k-2)(1-c)+C}} \sum_{i=1}^{\ell}\frac{1}{(\ell-i)!}\left(-\frac{e_3}{e_1}\right)^{\ell-i} e_{2i-1},$$

$\forall \ell \in \{2,\dots,k-1\}$:

$$I_{2\ell-1} = e_1^{\frac{2(\ell-1)}{2(k-2)(1-c)+C}} e_2^{-1} e_2^{\frac{2c(1-\ell)}{2(k-2)(1-c)+C}} \sum_{i=1}^{\ell}\frac{1}{(\ell-i)!}\left(-\frac{e_3}{e_1}\right)^{\ell-i} e_{2i},$$

Invariants of $\mathfrak{s}'_{2k+1,5}$

The invariants of this extension are the same as the invariants of $\mathfrak{s}_{2k+1,5}(4-2k,0)$.

Invariants of $\mathfrak{s}^{i}_{2k+1,6}(B)$

The operators are given by

$$\hat{F}_{1,6} = 2\hat{F}_1^{(3)}|_{b=\frac{B}{2},\ c=\frac{B}{2}-i},$$
$$\hat{F}_{2,6} = -\sum_{j=1}^{k-1} e_{2j-1}\frac{\partial}{\partial e_{2j-1}} + (e_{2(j-i)-1} + e_{2j}\frac{\partial}{\partial e_{2j}}),$$

where the e_ℓ with $\ell < 1$ are to be understood as zeroes. The components of a compositions of the flows

$$\Psi = \Psi^{\beta_2}_{\hat{F}_{2,6}} \circ \Psi^{\beta_1}_{\hat{F}_{1,6}} \circ \Psi^{\alpha}_{\hat{F}_1^{(2k-1)}}$$

are

$$e^{\#}_{2\ell}(\Psi(\vec{e})) = \mathrm{e}^{-\omega(\beta_1,\beta_2,\ell)}\left(\sum_{j=1}^{\ell}\frac{(-\alpha)^{\ell-j}}{(\ell-j)!}e_{2j} - \beta_2\mathrm{e}^{2i\beta_1}\sum_{j=1}^{\ell-i}\frac{(-\alpha)^{\ell-i-j}}{(\ell-i-j)!}e_{2j-1}\right),$$
$$e^{\#}_{2\ell-1}(\Psi(\vec{e})) = \mathrm{e}^{-\omega(\beta_1,\beta_2,\ell)+2i\beta_1}\sum_{j=1}^{\ell}\frac{(-\alpha)^{\ell-j}}{(\ell-j)!}e_{2j-1}, \quad \text{where}$$
$$\omega(\beta_1,\beta_2,\ell) = \beta_2 + (2(k-1-\ell)+B)\beta_1.$$

Choosing the section as

$$e^{\#}_1(\Psi(\vec{e})) = 1, \qquad e^{\#}_2(\Psi(\vec{e})) = 1, \qquad e^{\#}_3(\Psi(\vec{e})) = 0$$

yields

$$\beta_1 = \frac{1}{2i}\ln\frac{e_2}{e_1}, \qquad \beta_2 = \ln e_1 - \frac{2(k-2-i)+B}{2i}\ln\frac{e_2}{e_1}, \qquad \alpha = \frac{e_3}{e_1},$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\begin{aligned}
&\forall \ell \in \{3, \dots, k-1\}: \\
&\quad I_{2\ell-1} = e_1^{-1} \left(\frac{e_2}{e_1}\right)^{\frac{\ell-1}{i}} \sum_{j=1}^{\ell} \frac{1}{(\ell-j)!} \left(-\frac{e_3}{e_1}\right)^{\ell-j} e_{2j-1}, \\
&\forall \ell \in \{2, \dots, k-1\}: \\
&\quad I_{2\ell} = e_2^{-1} \left(\frac{e_2}{e_1}\right)^{\frac{\ell-1}{i}} \left(\sum_{j=1}^{\ell} \frac{1}{(\ell-j)!} \left(-\frac{e_3}{e_1}\right)^{\ell-j} e_{2j-1} - \frac{e_2}{e_1} \left(\ln e_1 - \right.\right. \\
&\qquad\qquad \left.\left. \frac{2(k-2-i)+B}{2i} \ln \frac{e_2}{e_1} \right) \sum_{j=1}^{\ell-i} \frac{1}{(\ell-i-j)!} \left(-\frac{e_3}{e_1}\right)^{\ell-i-j} e_{2j-1} \right),
\end{aligned}$$

where the sums with the upper limit lower than the lower limit are to be understood as zeroes.

Invariants of $\mathfrak{s}_{2k+1,6}^{\prime(i)}$

The operators are given by

$$\begin{aligned}
\hat{F}_{1,6'} &= -\sum_{j=1}^{k-1} (k-1-i-j) e_{2j-1} \frac{\partial}{\partial e_{2j-1}} + \left(e_{2(j-i)-1} + (k-1-j) e_{2j} \right) \frac{\partial}{\partial e_{2j}}, \\
\hat{F}_{2,6'} &= -\sum_{j=1}^{2k-2} e_j \frac{\partial}{\partial e_j},
\end{aligned}$$

where the e_ℓ with $\ell < 1$ are to be understood as zeroes. The components of a compositions of the flows

$$\Psi = \Psi^{\beta_2}_{\hat{F}_{2,6'}} \circ \Psi^{\beta_1}_{\hat{F}_{1,6'}} \circ \Psi^{\alpha}_{\hat{F}_1^{(2k-1)}}$$

are

$$\begin{aligned}
e^{\#}_{2\ell-1}\left(\Psi(\vec{e})\right) &= \mathrm{e}^{-\beta_2 + (k-1-i-\ell)\beta_1} \sum_{j=1}^{\ell} \frac{(-\alpha)^{\ell-j}}{(\ell-j)!} e_{2j-1}, \\
e^{\#}_{2\ell}\left(\Psi(\vec{e})\right) &= \mathrm{e}^{-\beta_2 + (k-1-\ell)\beta_1} \left(\sum_{j=1}^{\ell} \frac{(-\alpha)^{\ell-j}}{(\ell-j)!} e_{2j} - \beta_1 \sum_{j=1}^{\ell-i} \frac{(-\alpha)^{\ell-i-j}}{(\ell-i-j)!} e_{2j-1} \right).
\end{aligned}$$

Choosing the section as

$$e^{\#}_1\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_2\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_3\left(\Psi(\vec{e})\right) = 0$$

yields

$$\beta_1 = \frac{1}{i}\ln\frac{e_2}{e_1}, \qquad \beta_2 = -\ln\frac{e_2}{e_1^2} - \frac{k-2}{i}\ln\frac{e_2}{e_1}, \qquad \alpha = \frac{e_3}{e_1},$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\begin{aligned}
&\forall \ell \in \{3,\ldots,k-1\}: \\
&\quad I_{2\ell-1} = \frac{e_2^2}{e_1^3}\left(\frac{e_2}{e_1}\right)^{\frac{\ell-1}{i}} \sum_{j=1}^{\ell} \frac{1}{(\ell-j)!}\left(-\frac{e_3}{e_1}\right)^{\ell-j} e_{2j-1}, \\
&\forall \ell \in \{2,\ldots,k-1\}: \\
&\quad I_{2\ell} = \frac{e_2}{e_1^2}\left(\frac{e_2}{e_1}\right)^{\frac{\ell-1}{i}} \left(\sum_{j=1}^{\ell} \frac{1}{(\ell-j)!}\left(-\frac{e_3}{e_1}\right)^{\ell-j} e_{2j-1} - \frac{e_2}{ie_1}\ln\frac{e_2}{e_1} \right. \\
&\qquad\qquad \left. \sum_{j=1}^{\ell-i} \frac{1}{(\ell-i-j)!}\left(-\frac{e_3}{e_1}\right)^{\ell-i-j} e_{2j-1}\right),
\end{aligned}$$

where the sums with the upper limit lower than the lower limit are to be understood as zeroes.

Invariants of $\mathfrak{s}_{2k+1,7}(B)$

The operators are given by

$$\hat{F}_{1,7} = \hat{F}_1^{(3)}|_{c=1,\ b=B}, \qquad \hat{F}_{2,7} = -\sum_{i=1}^{k-1} e_{2i}\frac{\partial}{\partial e_{2i}};$$

the components of a compositions of the flows

$$\Psi = \Psi^{\beta_2}_{\hat{F}_{2,7}} \circ \Psi^{\beta_1}_{\hat{F}_{1,7}} \circ \Psi^{\alpha}_{\hat{F}_1^{(2k-1)}}$$

are

$$e^{\#}_{2\ell-1}(\Psi(\vec{e})) = \mathrm{e}^{-(k-\ell)\beta_1}\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i-1},$$

$$e^{\#}_{2\ell}(\Psi(\vec{e})) = \mathrm{e}^{-\beta_2-2(k-1-\ell+B)\beta_1}\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}e_{2i}.$$

Choosing the section as

$$e^{\#}_1(\Psi(\vec{e})) = 1, \qquad e^{\#}_2(\Psi(\vec{e})) = 1, \qquad e^{\#}_3(\Psi(\vec{e})) = 0$$

yields

$$\beta_1 = \frac{\ln e_1}{k-1}, \qquad \beta_2 = \ln e_2 - \frac{k-2+B}{k-1}\ln e_1, \qquad \alpha = \frac{e_3}{e_1},$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\begin{aligned}
&\forall \ell \in \{3,\dots,k-1\}: \\
&\quad I_{2\ell-1} = e_1^{\frac{\ell-1}{k-1}} e_1^{-1} \sum_{i=1}^{\ell} \frac{1}{(\ell-i)!}\left(-\frac{e_3}{e_1}\right)^{\ell-i} e_{2i-1}, \\
&\forall \ell \in \{2,\dots,k-1\}: \\
&\quad I_{2\ell-1} = e_1^{\frac{\ell-1}{k-1}} e_2^{-1} \sum_{i=1}^{\ell} \frac{1}{(\ell-i)!}\left(-\frac{e_3}{e_1}\right)^{\ell-i} e_{2i},
\end{aligned}$$

Invariants of $\mathfrak{s}_{2k+1,8}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,7}(0)$.

Invariants of $\mathfrak{s}_{2k+1,9}(B)$

The operators are given by

$$\hat{F}_{1,9} = \hat{F}_1^{(3)}|_{b=B,\ c=B}, \qquad \hat{F}_{2,9} = -\sum_{i=1}^{k-1}\left(e_{2i-1}\frac{\partial}{\partial e_{2i-1}} + (e_{2i-1}+e_{2i})\frac{\partial}{\partial e_{2i}}\right);$$

the components of a compositions of the flows

$$\Psi = \Psi^{\beta_2}_{\hat{F}_{2,9}} \circ \Psi^{\beta_1}_{\hat{F}_{1,9}} \circ \Psi^{\alpha}_{\hat{F}_1^{(2k-1)}}$$

are

$$\begin{aligned}
e^{\#}_{2\ell-1}(\Psi(\vec{e})) &= \mathrm{e}^{-\beta_2-(k-1-\ell+B)\beta_1} \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}, \\
e^{\#}_{2\ell}(\Psi(\vec{e})) &= \mathrm{e}^{-\beta_2-(k-1-\ell+B)\beta_1} \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} (e_{2i} - \beta_2 e_{2i-1}).
\end{aligned}$$

Choosing the section as

$$e^{\#}_1(\Psi(\vec{e})) = 1, \qquad e^{\#}_2(\Psi(\vec{e})) = 0, \qquad e^{\#}_3(\Psi(\vec{e})) = 0$$

yields

$$\beta_1 = \frac{\ln e_1}{k-1}, \qquad \beta_2 = \frac{e_2}{e_1}, \qquad \alpha = \frac{e_3}{e_1}.$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\begin{aligned}
&\forall \ell \in \{3, \dots, k-1\}: \\
&\quad I_{2\ell-1} = e_1^{\frac{1-\ell}{k-2+B}} e_1^{-1} \mathrm{e}^{\frac{1-\ell}{k-2+B}\frac{e_2}{e_1}} \sum_{i=1}^{\ell} \frac{1}{(\ell-i)!} \left(-\frac{e_3}{e_1}\right)^{\ell-i} e_{2i-1}, \\
&\forall \ell \in \{2, \dots, k-1\}: \\
&\quad I_{2\ell} = e_1^{\frac{1-\ell}{k-2+B}} e_1^{-1} \mathrm{e}^{\frac{1-\ell}{k-2+B}\frac{e_2}{e_1}} \sum_{i=1}^{\ell} \frac{1}{(\ell-i)!} \left(-\frac{e_3}{e_1}\right)^{\ell-i} \left(e_{2i} - \frac{e_2}{e_1} e_{2i-1}\right),
\end{aligned}$$

Invariants of $\mathfrak{s}_{2k+1,10}$

The operators are given by

$$\begin{aligned}
\hat{F}_{1,10} &= -\sum_{i=1}^{k-1} \left((k-1-\ell) e_{2i-1} \frac{\partial}{\partial e_{2i-1}} + (e_{2i-1} + (k-1-\ell) e_{2i}) \frac{\partial}{\partial e_{2i}} \right), \\
\hat{F}_{2,10} &= -\sum_{i=1}^{2k-2} e_i \frac{\partial}{\partial e_i};
\end{aligned}$$

the components of a compositions of the flows

$$\Psi = \Psi^{\beta_2}_{\hat{F}_{2,10}} \circ \Psi^{\beta_1}_{\hat{F}_{1,10}} \circ \Psi^{\alpha}_{\hat{F}_1^{(2k-1)}}$$

are

$$\begin{aligned}
e^{\#}_{2\ell-1}\left(\Psi(\vec{e})\right) &= \mathrm{e}^{-\beta_2 - (k-1-\ell)\beta_1} \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1}, \\
e^{\#}_{2\ell}\left(\Psi(\vec{e})\right) &= \mathrm{e}^{-\beta_2 - (k-1-\ell)\beta_1} \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} (e_{2i} - \beta_1 e_{2i-1}).
\end{aligned}$$

Choosing the section as

$$e^{\#}_1\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_2\left(\Psi(\vec{e})\right) = 0, \qquad e^{\#}_3\left(\Psi(\vec{e})\right) = 0$$

yields

$$\beta_1 = \frac{e_2}{e_1}, \qquad \beta_2 = \ln e_1 - (k-2)\frac{e_2}{e_1}, \qquad \alpha = \frac{e_3}{e_1}.$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\begin{aligned}
&\forall \ell \in \{3, \dots, k-1\} : \\
&\qquad I_{2\ell-1} = e_1^{-1} \mathrm{e}^{(\ell-1)\frac{e_2}{e_1}} \sum_{i=1}^{\ell} \frac{1}{(\ell-i)!} \left(-\frac{e_3}{e_1}\right)^{\ell-i} e_{2i-1}, \\
&\forall \ell \in \{2, \dots, k-1\} : \\
&\qquad I_{2\ell} = e_1^{-1} \mathrm{e}^{(\ell-1)\frac{e_2}{e_1}} \sum_{i=1}^{\ell} \frac{1}{(\ell-i)!} \left(-\frac{e_3}{e_1}\right)^{\ell-i} \left(e_{2i} - \frac{e_2}{e_1} e_{2i-1}\right),
\end{aligned}$$

Invariants of $\mathfrak{s}_{2k+1,11}$

The invariants of these extensions were not found as finding them would require the knowledge of the invariants of $\mathfrak{s}_{2k,12}$ which we were unable to compute.

4.3.4 Invariants of Extension by Three Elements

Unlike in the preceding two subsections, the operators corresponding to the extending elements f_1, f_2, f_3 are denoted simply $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_3$ for both extensions by three elements as they are present in the respective subsubsections only. We use the results and notation from above here.

Invariants of $\mathfrak{s}_{2k+2,1}$

The vector fields are given by

$$\hat{F}_1 = \hat{F}_{1,1}|_{C=0} = \hat{F}_1^{(1)}|_{c=0,a=0}, \qquad \hat{F}_2 = \hat{F}_{2,1}|_{b=0},$$

$$\hat{F}_3 = -\sum_{i=1}^{2k-2} e_i \frac{\partial}{\partial e_i},$$

the flow of $\hat{F}_3$ is then

$$\Psi_{\hat{F}_3}^{\beta_3}(\vec{e}) = \mathrm{e}^{-\beta_3} \vec{e}.$$

Composing the flows as

$$\Psi = \Psi_{\hat{F}_3}^{\beta_3} \circ \Psi_{\hat{F}_2}^{\beta_2} \circ \Psi_{\hat{F}_1}^{\beta_1} \circ \Psi_{\hat{E}_{2k-1}}^{\alpha},$$

we get the components of Ψ as

$$e^{\#}_{2\ell-1}(\Psi(\vec{e})) = \mathrm{e}^{-(k-1-\ell)\beta_2-\beta_3}\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}\left(e_{2i-1}\cos\beta_1 + e_{2i}\sin\beta_1\right),$$
$$e^{\#}_{2\ell}(\Psi(\vec{e})) = \mathrm{e}^{-(k-1-\ell)\beta_2-\beta_3}\sum_{i=1}^{\ell}\frac{(-\alpha)^{\ell-i}}{(\ell-i)!}\left(-e_{2i-1}\sin\beta_1 + e_{2i}\cos\beta_1\right).$$

Choosing the section as

$$e^{\#}_1(\Psi(\vec{e})) = 0, \qquad e^{\#}_2(\Psi(\vec{e})) = 1, \qquad e^{\#}_3(\Psi(\vec{e})) = 1, \qquad e^{\#}_4(\Psi(\vec{e})) = 0$$

yields

$$\beta_1 = -\arctan\frac{e_1}{e_2}, \qquad \beta_2 = \ln\frac{e_1^2+e_2^2}{e_1e_4-e_2e_3},$$
$$\beta_3 = \ln\sqrt{e_1^2+e_2^2}\,\ln\left(\frac{e_1^2+e_2^2}{e_1e_4-e_2e_3}\right)^{k-2}, \qquad \alpha = \frac{e_1e_3+e_2e_4}{e_1^2+e_2^2}.$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\forall\ell\in\{3,\dots,k-1\}:$$
$$I_{2\ell-1} = \frac{(e_1^2+e_2^2)^{\ell-2}}{(e_1e_4-e_2e_3)^{\ell-1}}\left(\sum_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}}(e_2e_{2i-1}-e_1e_{2i})\right),$$
$$I_{2\ell} = \frac{(e_1^2+e_2^2)^{\ell-2}}{(e_1e_4-e_2e_3)^{\ell-1}}\left(\sum_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}}{(\ell-i)!(e_1^2+e_2^2)^{\ell-i}}(e_1e_{2i-1}+e_2e_{2i})\right).$$

Half of the invariants can be simplified to the rational form

$$I'_{2\ell} \equiv \frac{\sum_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_1e_{2i-1}+e_2e_{2i})}{\sum_{i=1}^{\ell}\frac{(-e_1e_3-e_2e_4)^{\ell-i}(e_1^2+e_2^2)^i}{(\ell-i)!}(e_2e_{2i-1}-e_1e_{2i})} \qquad \forall\ell\in\{3,\dots,k-1\}.$$

Invariants of $\mathfrak{s}_{2k+2,2}$

The vector fields are given by

$$\hat{F}_1 = \frac{1}{2}\hat{F}_{1,5}|_{C=2} = \hat{F}_1^{(1)}|_{c=1,b=0}, \qquad \hat{F}_2 = \hat{F}_{2,5}|_{c=0},$$
$$\hat{F}_3 = -\sum_{i=1}^{2k-2} e_i\frac{\partial}{\partial e_i},$$

the components of the flow of $\hat{F}_3$ are then given by

$$e^{\#}_{2\ell-1}\left(\Psi^{\beta_3}_{\hat{F}_3}(\vec{e})\right) = \mathrm{e}^{-\beta_3} e_{2\ell-1}, \qquad e^{\#}_{2\ell}\left(\Psi^{\beta_3}_{\hat{F}_3}(\vec{e})\right) = e_{2\ell}.$$

Composing the flows as

$$\Psi = \Psi^{\beta_3}_{\hat{F}_3} \circ \Psi^{\beta_2}_{\hat{F}_2} \circ \Psi^{\beta_1}_{\hat{F}_1} \circ \Psi^{\alpha}_{\hat{E}_{2k-1}},$$

we get the components of Ψ as

$$e^{\#}_{2\ell-1}\left(\Psi(\vec{e})\right) = \mathrm{e}^{-\beta_2-(k-\ell)\beta_1} \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i-1},$$
$$e^{\#}_{2\ell}\left(\Psi(\vec{e})\right) = \mathrm{e}^{-\beta_3-(k-1-\ell)\beta_1} \sum_{i=1}^{\ell} \frac{(-\alpha)^{\ell-i}}{(\ell-i)!} e_{2i}.$$

Choosing the section as

$$e^{\#}_1\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_2\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_3\left(\Psi(\vec{e})\right) = 1, \qquad e^{\#}_4\left(\Psi(\vec{e})\right) = 0$$

yields

$$\beta_1 = \ln \frac{e_1}{e_1 e_4 - e_2 e_3} + \ln e_2,$$
$$\beta_2 = \ln e_1 - 2(k-1)\left(\ln \frac{e_1}{e_1 e_4 - e_2 e_3} + \ln e_2\right),$$
$$\beta_3 = (3-k)\ln e_2 + (2-k)\ln \frac{e_1}{e_1 e_4 - e_2 e_3},$$
$$\alpha = \frac{e_3}{e_1}.$$

Substituting these values back to the remaining coordinates, we obtain the invariants in the form

$$\forall \ell \in \{3, \dots, k-1\}:$$
$$I_{2\ell-1} = e_1^k \,(e_1 e_4 - e_2 e_3)^{k-1-\ell} \left(\sum_{i=1}^{\ell} \frac{e_1^{i-2}(-e_3)^{\ell-i}}{(\ell-i)!} e_{2i-1}\right),$$
$$I_{2\ell} = e_2^{\ell-2} \,(e_1 e_4 - e_2 e_3)^{1-\ell} \left(\sum_{i=1}^{\ell} \frac{e_1^{i-1}(-e_3)^{\ell-i}}{(\ell-i)!} e_{2i}\right).$$

Conclusion

The series of nilpotent Lie algebras $(\mathfrak{r}_{2k-1})_4^\infty$ of odd dimension was constructed, basic properties of its elements were found, and all their solvable extensions were classified. Unlike in the case of even-dimensional algebras $\mathfrak{r}_{2k}$ of the similar structure, here, the invariant ideals did not form a flag where codimension of the i-th one in the $(i+1)$-st one was 1. This allowed the automorphism matrices, and thus the derivations matrices, to be nontriangular in general. For the algebras over the field of real numbers, this lead to additional automorphism classes corresponding to the classes of outer derivation with nontriangular matrices, namely, we have obtained one two-parametric family of extensions by one element and another 2k - 5 families with number of parameters ranging from $2k-5$ to 1. Two two-parametric families of extensions by two elements for which one of the elements corresponds to a nontriangular derivation matrix were found, with additional $8(k-3)$ families for which the number of parameters varies. Finally, there was one additional extension by three elements, where one of the elements corresponds to a nontriangular matrix. However, when working over the field of complex numbers, the situation was slightly simplified rather than complicated compared to the even-dimensional case as it was always possible to transform the derivation matrix to the triangular form using the automorphisms and there were additional automorphisms available compared to the even-dimensional case that allowed for merging some of the cases into a single isomorphism class (or family of classes).

Despite relatively simple structure of the nilpotent algebras $\mathfrak{r}_{2k-1}$ and the possibility of identification of its generic outer derivation depending mostly on three parameters only, we have found rather large number of isomorphism classes. However, the structure of the obtained algebras turned out to be too simple to provide new insights or more general results in the field of classification of solvable Lie algebras.

In the last chapter, the invariants of the coadjoint representation, or else, the generalized Casimir invariants, of the constructed algebras were found with the exception of the most degenerate cases, where the calculations turned out to be not feasible. Most of the invariants were found in a nonpolynomial form and in most cases the functional basis of the invariants could not be simplified to a polynomial one. For a fixed element of the series of nilpotent algebras $\mathfrak{r}_{2k-1}$ (of dimension $2k-1$), the number of invariants was $2k-3$ for the nilpotent algebra itself, $2k-4$ for any of the solvable extensions by one element, $2k-5$ for the extensions by two elements, and $2k-6$ for both extensions by three elements.

Bibliography

[1] Ancochea, J., Campoamor-Stursberg, R., and Vergnolle, L. Solvable Lie algebras with naturally graded nilradicals and their invariants. *Journal of Physics A: Mathematical and General*, 39(6):1339–1355, 2006.

[2] Boyko, V., Patera, J., and Popovych, R. Computation of invariants of Lie algebras by means of moving frames. *Journal of Physics A: Mathematical and General*, 39(20):5749–5762, 2006.

[3] Erdmann, K. and Wildon, M. *Introduction to Lie algebras.* Springer, London, 2006.

[4] Fels, M. and Olver, P. Moving coframes: I. a practical algorithm. *Acta Applicandae Mathematicae*, 51(2):161–213, 1998.

[5] Fels, M. and Olver, P. Moving coframes: II. regularization and theoretical foundations. *Acta Applicandae Mathematicae*, 55(2):127–208, 1999.

[6] Jacobson, N. *Lie algebras.* Courier Corporation, 2013.

[7] Morozov, V. V. Classification of nilpotent Lie algebras of sixth order. *Izv. Vyssh Uchebn. Zaved. Mat.*, 4:161–171, 1958.

[8] Mubarakzyanov, G. M. Classification of real structures of Lie algebras of sixth order. *Izv. Vyssh Uchebn. Zaved. Mat.*, 3:99–106, 1963.

[9] Mubarakzyanov, G. M. Classification of solvable Lie algebras of dimension six with one nonnilpotent basis element. *Izv. Vyssh Uchebn. Zaved. Mat.*, 4:104–116, 1963.

[10] Ndogmo, J. and Winternitz, P. Generalized Casimir operators of solvable Lie algebras with abelian nilradicals. *Journal of Physics A: Mathematical and General*, 27(8):2787–2800, 1994.

[11] Ndogmo, J. and Winternitz, P. Solvable Lie algebras with abelian nilradicals. *Journal of Physics A: Mathematical and General*, 27(2):405–423, 1994.

[12] Prokop, J. Classification of classes of solvable Lie algebras with higher dimensional centres. `http://physics.fjfi.cvut.cz/publications/mf/2012/prokop_vu.pdf`. Last visited on 04/05/2016.

[13] Rubin, J. L. and Winternitz, P. Solvable Lie algebras with Heisenberg ideals. *Journal of Physics A: Mathematical and General*, 26(5):1123–1138, 1993.

[14] Safiulina, E. N. Classification of nilpotent Lie algebras of dimension seven. *Math. Methods Phys. Izd. Kazan. Univ.*, 66:66–69, 1964.

[15] Sattinger, D. H. and Weaver, O. L. *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics.* Springer-Verlag, New York, 1986.

[16] Tremblay, S. and Winternitz, P. Solvable Lie algebras with triangular nilradicals. *Journal of Physics A: Mathematical and General*, 31(2):789–806, 1998.

[17] Tremblay, S. and Winternitz, P. Invariants of the nilpotent and solvable triangular Lie algebras. *Journal of Physics A: Mathematical and General*, 34(42):9085–9099, 2001.

[18] Tsagas, G. Classification of nilpotent Lie algebras of dimension eight. *J. Inst. Math. Comput. Sci. Math.*, Ser. 12:179–183, 1999.

[19] Tsagas, G., Kobotis, A., and Koukouvinos, T. Classification of nilpotent Lie algebras of dimension nine whose maximum abelian ideal is of dimension nine. *Int. J. Comput. Math.*, 74:5–28, 2000.

[20] Turkowski, G. M. Low-dimensional Lie algebras. *J. Math. Phys.*, 29:2139–2144, 1988.

[21] Turkowski, G. M. Solvable Lie algebras of dimension six. *J. Math. Phys.*, 31:1344–1350, 1990.

[22] Wang, Y., Lin, J., and Deng, S. Solvable Lie algebras with quasifiliform nilradicals. *Communications in Algebra*, 36(11):4052–4067, 2008.

[23] Šnobl, L. On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras. *J. Phys. A: Math. Theor.*, A43(505202), 2010.

[24] Šnobl, L. Maximal solvable extensions of filiform algebras. *Archivum Mathematicum*, 47:405–414, 2011.

[25] Šnobl, L. and Karásek, D. Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its superalgebras. *Linear Algebra and its Applications*, 432:1836–1850, 2010.

[26] Šnobl, L. and Winternitz, P. A class of solvable Lie algebras and their Casimir invariants. *Journal of Physics A: Mathematical and General*, 38(12):2687–2700, 2005.

[27] Šnobl, L. and Winternitz, P. All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency n - 1. *Journal of Physics A: Mathematical and Theoretical*, 42(10), 2009.

[28] Šnobl, L. and Winternitz, P. Solvable Lie algebras with Borel nilradicals. *J. Phys. A: Math. Theor.*, A45(095202), 2012.

[29] Šnobl, L. and Winternitz, P. *Classification and Identification of Lie Algebras,* CRM Monograph Series, Volume: 33. AMS, Providence, 2014.