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RESEARCH PROJECT

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Crypto-selfadjoint representations of quantum observables

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I declare that I carried out this thesis independently, and only with the cited literature and other professional sources.

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Název práce: **Skrytě samosdružené reprezentace kvantových pozorovatelných**

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Abstrakt: Předmětem práce je analýza skrytě samosdružených operátorů na Hilbertově prostoru: z hlediska analytického jde o nalezení jednoho či více metrických operátorů, které určují fyzikální Hilbertovy prostory daného operátoru. Ve výzkumné části práce jsou analyzovány tři diskrétní modely, pro něž je zkonstruována kompletní množina pseudometrik, určující množinu fyzikálních Hilbertových prostorů.

Klíčová slova: skrytá samosdruženost, metrika, Su-Schrieffer-Heeger, Fano-Anderson, pseudospektrum

Title: **Crypto-selfadjoint representations of quantum observables**

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Abstract: We deal with the concept of crypto-self-adjointness, a way of representing quantum observables by non-self-adjoint operators. We examine a number of crypto-self-adjoint models of discrete character, and construct complete set of pseudometrics, which in turn determines a complete set of Hilbert-space inner products for those models

Keywords: hidden hermicity, metric operator, Su-Schrieffer-Heeger model, Fano-Anderson model, pseudospectrum

Contents

Prologue

This paper encompasses various ideas centered around the field of crypto-hermitian quantum mechanics [1, 2, 3], which has received considerable attention in recent years [4, 5], producing its own conference series and a book of proceedings concerning its mathematical aspects [6]. It is basically a theoretical framework for studying representations of quantum observables by non-hermitian operators. Because the condition of hermicity is Hilbert space-dependent, we might indeed represent quantum observables by formally non-hermitian operators on a certain unphysical Hilbert space $\mathcal{H}^{(F)} = \{\mathcal{V}, (x, y)\}$, which might sometimes prove rewarding in terms of mathematical and computational simplicity. Operators H on $\mathcal{H}^{(F)}$, which are however hermitian in another Hilbert space $\mathcal{H}^{(S)}$, are called quasi-hermitian [7], or in a more novel terminology, crypto-hermitian [8]. Because the inner products (x, y) are in one-to-one correspondence with so-called metric operators Θ (bounded nonsingular positive operators) through the formula

$$(\phi, \psi)_\Theta = (\phi, \Theta\psi) \tag{0.1}$$

we might express the condition of crypto-hermicity in operator form as $H^\dagger\Theta = \Theta H$, which is sometimes called Dieudonné equation. Furthermore, any bounded nonsingular positive Θ can be decomposed as $\Theta = \Omega^\dagger\Omega$ with Ω bounded nonsingular. Inserting this decomposition into the Dieudonné equation yields the condition

$$h := \Omega H \Omega^{-1} = (\Omega H \Omega^{-1})^\dagger = h^\dagger \tag{0.2}$$

which shows that being crypto-hermitian is equivalent to being boundedly diagonalizable with real spectrum. However, the explicit construction of the decomposition $\Theta = \Omega^\dagger\Omega$ may be in general very difficult for a given Θ. Still, this allows us to introduce a third (also physical) Hilbert space $\mathcal{H}^{(T)}$, which has a modified vector space structure instead of the inner product. The whole three-Hilbert-space scheme is expressed in fig. I.

$$\begin{tikzcd}
 & \boxed{\mathcal{H}^{(F)} : \{\psi, (x, y)\}} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\boxed{\mathcal{H}^{(S)} : \{\psi, (x, \Theta y)\}} \arrow[rr, leftrightarrow] & & \boxed{\mathcal{H}^{(P)} : \{\Omega\psi, (x, y)\}}
\end{tikzcd}$$

Figure I: Three-Hilbert-space formulation of crypto-hermitian quantum mechanics

The present paper is divided into two chapters, which have, respectively, a review and research character. In the (first) review chapter, we provide quick introduction into generalized convergence, spectral measures of non-normality, and basic classes of solvable quantum-mechanical model, namely infinite Toeplitz operators and constant-coefficient differential operators. In the (second) research part, we adress closely three discrete models, the material being based mosly on the recent studies [9, 10, 11]. We shall put emphasis on various diverse possibilities of finite-dimensional crypto-hermitian model building, as well as the possibility of explicit construction of a complete metric families for the given models.

Chapter 1

Selected chapters from spectral theory

In this review chapter, we present a number of mathematical topics from theory of non-hermitian operators, which have direct relevance to crypto-hermitian quantum mechanics. Main sources of theoretical material in the study of non-hermitian operators include [12, 13]. while the latest research trends are outlined in the recent book of proceedings [6].

1.1 Generalized convergence

One aspect of operator perturbation theory concerns the very definition of a "small" perturbation, when dealing with unbounded operators. Here we follow the approach of [13], which defines the operator distance (and hence also convergence) in terms of distance of their graphs. Since the operator graphs are closed subspaces of the product space $\mathfrak{X} \times \mathfrak{Y}$, this is equivalent to defining a metric on the set of closed subspaces of Banach spaces. All the theorem numbers in this section refer to [13].

1.1.1 Graph measure

We aim to introduce a metric on the set of operators $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ in terms of metric of the corresponding graphs. Since the general Banach space theory is not much more complicated that the one for Hilbert space, we formulate all theorems in their Banach space form. For any two linear manifolds M and N in a Banach space $\mathfrak{X}$, we define

$$\delta(M,N) = \sup_{\|u\|=1} \rho(u,N), \qquad \hat{\delta}(M,N) = \max[\delta(M,N), \delta(N,M)] \tag{1.1}$$

By definition, $\hat{\delta}$ is clearly positive and symmetric. If $\mathfrak{X}$ is a Hilbert space, it can be shown that $\hat{\delta}(M,N) = \|P - Q\|$, with P and Q denoting orthogonal projections on M and N. Then $\hat{\delta}$ satisfies also the triangle inequality, making it a genuine metric. This is not the case for general Banach spaces, but then we might use instead the function $\hat{d}(M,N) = \max[d(M,N), d(N,M)]$ with $d(M,N) = \sup \rho(u, S_N)$, when S_N denotes the unit circle. This function can be shown to satisfy the axioms of metric, and it is morover equivalent to $\hat{\delta}(M,N)$ in the sense that

$$\hat{\delta}(M,N) \leq \hat{d}(M,N) \leq 2\hat{\delta}(M,N) \tag{1.2}$$

so that the topologies induced by $\hat{\delta}$ and $\hat{d}$ coincide. Specially, the condition $M_n \to M$ defined as $\hat{d}(M_n, M) \to 0$ is equivalent to $\hat{\delta}(M_n, M) \to 0$. Since the function $\hat{\delta}$ is generally simpler to work with, it shall be used is the following text instead of $\hat{d}$. Some basic properties of $\hat{\delta}$ are summarized below.

Theorem (thm IV.2.7/2.9)**.** *For two closed linear manifolds M and N, we have $0 \leq \hat{\delta}(M,N) \leq 1$. If $\hat{\delta}(M,N) < 1$, then $\dim M = \dim N$. For $M^{\perp} = \{f \in \mathfrak{X}^* \mid f(x) = 0 \;\; \forall x \in M\}$, we have $\hat{\delta}(M,N) = \hat{\delta}(M^{\perp}, N^{\perp})$.*

Let M, N be closed linear manifolds in $\mathfrak{X}$. Then $M \cap N$ is also closed, while the manifold $M + N$ is generally not. We define the nullity, deficiency and index of the pair M, N as $\mathrm{nul}(M,N) = \dim(M \cup N)$, $\mathrm{def}(M,N) = \mathrm{codim}(M + N)$ and $\mathrm{ind}(M,N) = \mathrm{nul}(M,N) - \mathrm{def}(M,N)$ (the latter being defined only if nul or def is finite). The pair M, N is called Fredholm (semi-Fredholm) if both (at least one of) the quantities $\mathrm{nul}(M,N)$ and $\mathrm{def}(M,N)$ are finite. Finally, we define the quantity called the minimum gap between M and N as

$$\gamma(M,N) = \inf_{u \in M \setminus N} \frac{\rho(u,N)}{\rho(u, M \cup N)}, \qquad \hat{\gamma}(M,N) = \min[\gamma(M,N), \gamma(N,M)] \tag{1.3}$$

which is defined only when $M \not\subset N$, while we set $\gamma(M,N) = 1$ for the case $M \subset N$. It can be shown (IV.4.14) that the manifold $M + N$ is closed if and only if $\gamma(M,N) > 0$. We now make the transition from linear manifolds to linear operators. Let $T, S \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. Then their graphs $\mathcal{G}(T)$ and $\mathcal{G}(S)$ are closed linear manifolds on the space $\mathfrak{X} \times \mathfrak{Y}$. We define $\hat{\delta}(T,S) = \hat{\delta}(\mathcal{G}(T), \mathcal{G}(S))$. The operator distance defined in this way behaves nicely with respect to conjugation and inversion.

Theorem (thm IV.2.18/2.20)**.** *Let $T, S \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ be densely defined, Then $\hat{\delta}(T,S) = \hat{\delta}(T^*, S^*)$. If S, T are invertible, then $\hat{\delta}(T,S) = \hat{\delta}(T^{-1}, S^{-1})$.*

The common textbook approach to operator converenge usually discusses merely two simple notions: that of norm convergence and of norm-resolvent convergence. An encouraging result is that the graph measure convergence coindices with those earlier notions, as long as they are defined properly, making it a very general concept.

Theorem (thm. IV.2.23)**.** *Let $T, T_n \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. If $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, then $\hat{\delta}(T_n, T) \to 0$ if and only if $T_n \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ for sufficiently large n, and $\|T_n - T\| \to 0$. If $\rho(T) \neq \emptyset$, then $\hat{\delta}(T_n, T) \to 0$ if and only if $\|R(\lambda, T_n) - R(\lambda, T)\| \to 0$ for some (and hence for all) $\lambda \in \rho(T)$.*

1.1.2 Stability theorems

By stability we mean the preservation of some particular property of a linear operator, when subject to a sufficiently small perturbation. In this section, we state several such stability theorems, most of the which are well known for bounded or relatively bounded perturbations. The notion of graph measure of linear operators, however, enables us to state much more general results. The first is about stability of boundness.

Theorem (thm. IV.2.14)**.** *Let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and $S \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$, such that $\hat{\delta}(T,S) < \|T\|^{-1}$. Then $S \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.*

Recall that $T \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is called nonsingular if it's invertible and $T^{-1} \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The following theorem deals with stability of operator invertibility and regularity, and expresses the fact, that both the sets of nonsingular and invertible operators are open in $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$.

Theorem (thm IV.2.21/2.27)**.** *If $T \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is invertible, and $S \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ such that $\hat{\delta}(S,T) < \|T^{-1}\|^{-1}$, then S is invertible. If T is in addition nonsingular, then S is nonsingular as well.*

Let $T \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$. Its nullity, $\operatorname{nul} T$, is defined as $\dim \operatorname{Ker} T$. Its defficiency, $\operatorname{def} T$, is defined as $\operatorname{codim} \operatorname{Ran} T$, and as before $\operatorname{ind} T = \operatorname{nul} T - \operatorname{def} T$. Although this definition has no apparent relevance to the nullities and defects of closed linear manifolds, it can be shown that $\operatorname{nul} T = \operatorname{nul}(\mathcal{G}(T), \mathfrak{X})$ and $\operatorname{def} T = \operatorname{def}(\mathcal{G}(T), \mathfrak{X})$. The stability of nonsingularity established above is equivalent to the stability of $\operatorname{nul} T = \operatorname{def} T = 0$. We can however generalize this theorem to other values of nullity and defficiency under some additional conditions. An operator $T \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ is (semi)-Fredholm, if $\operatorname{Ran} T$ is closed, and both (at least one of) $\operatorname{nul} T$ and $\operatorname{def} T$ are finite. The next theorem adresses stability of (semi)-Fredholmness,

Theorem (thm. IV.5.17)**.** *Let $T, S \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ and let T be (semi)-Fredholm. If $\hat{\delta}(S,T) < 1$, then S is (semi)-Fredholm and $\operatorname{nul} S \leq \operatorname{nul} T, \operatorname{def} S \leq \operatorname{def} T$, and finally $\operatorname{ind} S = \operatorname{ind} T$.*

In the general case, we can prove only the non-increasing property of nullity and decifiency (it is not easy to establish precise conditions under which these quantities are conserved exactly). Still, this general theorem encompasses most of the results known as stability theorems in the literature. One result not contained in the above concerns the stability of self-adjointness, which is very relevant in crypto-hermitian considerations.

Theorem (thm. IV.4.4)**.** *Let $T \in \mathcal{H}$ be self-adjoint. Then there is $\delta > 0$, such that any closed symmetric S with $\hat{\delta}(T,S) < \delta$ is self-adjoint.*

Finally, we adress the question of stability of spectrum under a small Banach space perturbation. The following theorem establishes the upper-semicontinuity of spectrum in a rigorous way (upper-semicontinuity in the sense that the spectrum cannot suddenly expand in an way uncontrolled by the perturbation).

Theorem (thm. IV.3.1)**.** *Let $T \in \mathcal{C}(\mathfrak{X})$ and $\Gamma \subset \rho(T)$ be compact. Then there is a $\delta > 0$, such that $\Gamma \subset \rho(S)$ for any $S \in \mathcal{C}(\mathfrak{X})$, such that $\hat{\delta}(T,S) < \delta$.*

On the other hand, the spectrum is not lower-semicontinuous in general (an arbitrary small perturbation can make the spectrum shrink infinitely). One can consult [13] for illuminating examples, where is also demonstrated, that the lower-semicontinuous property can be established for operators having only isolated eigenvalues.

1.2 Measures of non-normality

The very essence of spectral theory is, that in many cases a linear operator can be effectivelly characterized by a set of complex numbers. This is particularly the case of normal operators, which are determined by their spectral measure (supported on the spectrum). For non-normal operators, other characteristics have to be examined to understand the operator behavior. We introduce two common measures of non-normality: the numerical range and pseudospectrum. The numerical range of a Hilbert space operator $H \in \mathcal{C}(\mathcal{H})$ is defined as $W(H) = \overline{\{(H\psi, \psi) \mid \psi \in \mathfrak{D}(H), \|\psi\| = 1\}}$ (note that in some textbooks define the numerical range without the closure sign).

Theorem ([14], thm. 9.3.2/9.3.4)**.** *For any $H \in \mathcal{C}(\mathcal{H})$, $W(H)$ is non-empty and convex. If in addition $H \in \mathcal{B}(\mathcal{H})$ then $\sigma(H) \subset W(H)$. An operator H is normal, if and only if $W(H)$ is the convex hull of $\sigma(H)$.*

The first part of the above theorem is a classic result due to Toeplitz and Hausdorff. For unbounded operators, the spectrum is indeed not necessarily a part of $W(H)$, even if the operator under question is normal or self-adjoint. Unlike the spectrum, the numerical range is stable to perturbations and truncations, as expressed in following theorem.

Theorem ([14], thm. 9.3.14)**.** *Let $T, S \in \mathcal{C}(\mathcal{H})$, such that $\hat{\delta}(T, S) < \varepsilon$. Then $W(T) \subseteq \{\lambda \mid \operatorname{dist}(\lambda, W(S)) < \varepsilon\}$ and vice versa. Let te sequence $\mathcal{H}_n$ be an increasing with union dense in $\mathcal{H}$, and let H_n be a truncation of H to $\mathcal{H}_n$. Then the union of $W(H_n)$ is dense in $W(H)$.*

An important consequence of the above statements is following: if the spectrum of a closed operator is real, we may not conclude much about the nature of this operator. However, if the numerical range of such an operator is real, it does already establish its self-adjointness. A simple corollary of the above theorem concerns the convergence of numerical range. Let $H, H_n \in \mathcal{C}(\mathcal{H})$, such that $\hat{\delta}(H_n, H) \to 0$. Then $\rho(W(H_n), W(H)) \to 0$. The last assertion is understood in the Hausdorff distance

$$\rho(X, Y) = \max\left[\sup_{x\in X}\inf_{y\in Y}\rho(x, y), \sup_{y\in Y}\inf_{x\in X}\rho(x, y)\right] \tag{1.4}$$

1.2.1 Pseudospectrum

Here we examine a more general quantity, the ε-pseudospectrum, which can be (unlike the numerical range) defined for general Banach space operators. It is a subset of complex plane defined as $\sigma_\varepsilon(H) = \left\{\lambda \in \rho(H) \mid \|(H-\lambda)^{-1}\| > \varepsilon^{-1}\right\}$ with the convention $\|(H-\lambda)^{-1}\| = \infty$ for $\lambda \in \sigma(H)$. In addition to the definition in terms of resolvent norm, three equivalent definitions of the ε-pseudospectra may be introduced.

Theorem ([15], thm. 4.3)**.** *Let $H \in \mathcal{C}(\mathcal{H})$. Thene the ε-pseudospectrum coincides with the subsets of complex plane $\{\lambda \in \mathbb{C} \mid \sigma_{min}(T-\lambda) \leq \varepsilon\}$, $\{\lambda \in \mathbb{C} \mid \|(A-\lambda)v\| \leq \varepsilon, \|v\| = 1\}$ and $\{\lambda \in \mathbb{C} \mid \lambda \in \sigma(T+A), \|A\| < \varepsilon\}$.*

The definition in terms of singular values is suitable for computation purposes, and lies at the core of EigTool [16], a MATLAB suite for pseudospectral computation. The v in the second definition is commonly referred to as pseudoeigenvector, and moreover the second definition shows that there is no analogue to continous spectrum for the ε-pseudospectrum. The third definition in terms of operator perturbations can be also employed for computation, as shown in fig. II. The basic pseudospectral behavior is the subject to the following theorem.

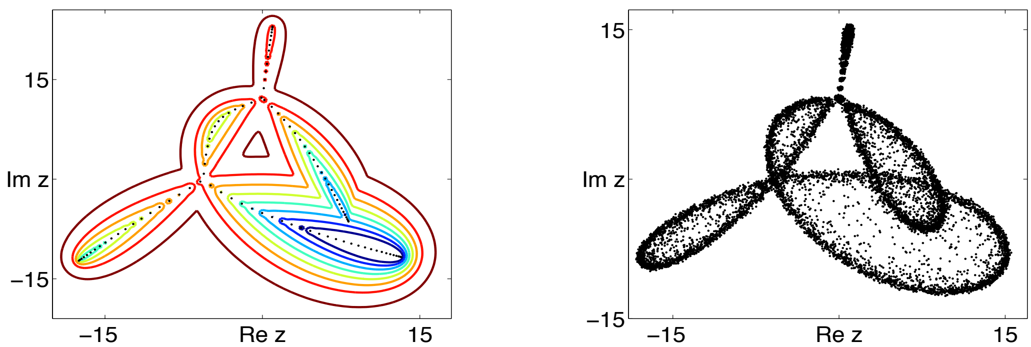


Figure II: Pseudospectra of a Hilbert space operator computed using two definitions above.

Theorem ([15], thm. 13.6)**.** *Let $\Delta_\varepsilon(\Gamma)$ denote the ε-neighborhood of $\Gamma \subset \mathbb{C}$ in the usual Euclidean distance. Let $H \in \mathcal{B}(\mathcal{H})$. Then $\Delta_\varepsilon(\sigma(H)) \subseteq \sigma_\varepsilon(H) \subseteq \Delta_\varepsilon(W(H))$.*

We have already demonstrated the upper-continuity of the spectrum and general continuity of the numerical range. Here we address the same question for the pseudospectra. The corresponding theorm contains a more or less technical assumption of the resolvent norm not being constant on an open set, which is the theme of an interesting discussion, see e.g. [17].

Theorem ([18], thm 5.3)**.** *Let $H, H_n \in \mathcal{C}(\mathcal{H})$, such that $\hat{\delta}(H_n, H) \to 0$. Suppose that $\|(H-\lambda)^{-1}\|$ is not constant on any open subset of $\mathbb{C}$. Then for each $\varepsilon > 0$ and each compact $K \subset \mathbb{C}$, such that $K^o \cap \sigma_\varepsilon(H) \neq 0$, we have*

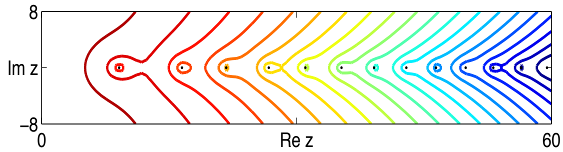
$$\rho\left\{\overline{\sigma_\varepsilon(H_n)} \cap K, \overline{\sigma_\varepsilon(H)} \cap K\right\} \to 0 \tag{1.5}$$

1.2.2 Pseudospectra of crypto-hermitian operators

Recall that crypto-hermitian operators with a metric operator Θ are similar to a hermitian operators through a non-unitary similarity transformation Ω satisfying $\Theta = \Omega^*\Omega$. The following considerations depend crucially on the framework which we choose: if we work with crypto-hermitian operators admitting a bounded nonsingular Θ, we can efficiently use the following theorem (originally due to Bauer and Fike) to characterize the pseudospectra.

Theorem ([15], thm. 3.4)**.** *Let $H \in \mathcal{C}(\mathcal{H})$. Then $h = \Omega H \Omega^{-1}$ with $\Omega \in \mathcal{B}(\mathcal{H})$ if and only if $\sigma_\varepsilon(h) \subseteq \sigma_{\kappa(\Omega)\varepsilon}(H)$ with $\kappa(\Omega) = \|\Omega\|\|\Omega^{-1}\|$.*

For normal operators, this shows that $\sigma_\varepsilon(H) = \Delta_\varepsilon(\sigma(H))$ for all $n \in \mathbb{N}_0$. For crypto-hermitian (or more generally crypto-normal) operators with Θ is bounded nonsingular, Ω is also bounded nonsingular and the pseudospectrum lies in the κ-neighboorhood of the spectrum.



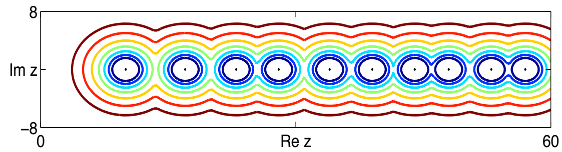


Figure III: Pseudospectra of (1.6) computed in $\mathcal{H}^{(F)}$ and $\mathcal{H}^{(S)}$.

The situation gets more complicated when we consider crypto-hermitian operators with Θ unbounded and/or singular. Then, by the Bauer-Fike theorem, the ε-pseudospectrum does not lie in any κ-neighborhood of $\sigma(H)$. A principal example of such an operator is the imaginary cubic oscillator of Bender et al. [4]

$$(H\psi)(x) = -\psi''(x) + ix^3\psi(x) \qquad \mathfrak{D}(H) = \left\{\psi \in L^2(\mathbb{R}) \mid x^3\psi(x) \in L^2(\mathbb{R})\right\} \tag{1.6}$$

Although the pseudospectrum cannot be determined by a closed formula in all but the simplest cases, we might use a very efficient technique in the case of semiclassical differential operators (with possibly non-constant coefficients). Let $H \in \mathcal{C}(\mathcal{H})$ be such a semiclassical differential operator $(H_h\psi)(x) = \sum a_j(x)h^j\psi^{(j)}(x)$ acting on $L^2\mathbb{R}$ as an extension from $C^\infty(\mathbb{R})$. We define its semiclassical pseudospectrum as

$$\Lambda = \left\{f(x,\xi) \mid (x,\xi) \in \mathbb{R}^2, \frac{1}{2i}\left\{f, \overline{f}\right\}(x,\xi) > 0\right\} \tag{1.7}$$

In the case of second-order Schr'odinger operator with analytic potential $V(x)$, this condition reduces to $\operatorname{Im} V'(x) \neq 0$, because the sign of ξ can be chosen freely. Consequently, the semiclassical pseudospectrum becomes the set $\Lambda = \left\{\xi^2 + V(a) \mid \xi \in \mathbb{R}, \operatorname{Im} V'(a) \neq 0\right\}$.

Theorem ([19])**.** *Let H_h be a semiclassical differential operator, Λ be its semiclassical pseudospectrum. Let $z \in \Lambda$. Then there exists $c > 1$, such that $z \in \sigma_\varepsilon(H_h)$ for $\varepsilon \geq c^{-1/h}$.*

This theorem can be directly applied to ordinary differential operators using the correspondence between the semi-classical and the high-energy limits. For the imaginary cubic oscillator H, we may write $UHU^\dagger = \tau^3 H_h$. where

$(U\psi)(x) = \tau^{1/2}\psi(\tau x)$, where $H_h = -h^2\Delta + ix^3$, where $h = \tau^{-5/2}$, and the semiclassical pseudospectrum of H_h becomes

$$\left\{ z \in \mathbb{C} \;\middle|\; |\arg z| < \frac{\pi}{2} - \delta, |z| \geq \max\left\{c_1, c_2 (\log \varepsilon)^{6/5}\right\} \right\} \subseteq \sigma_\varepsilon(H)$$

It is readily seen that the semiclassical pseudospectrum extends infinitely far into right complex half-plane, demonstrating the nonexistence of a bounded nonsingular metric. The schematic drawing of the semiclassical, as well as the boundaries of classical pseudospectra, are shown in fig. IV.

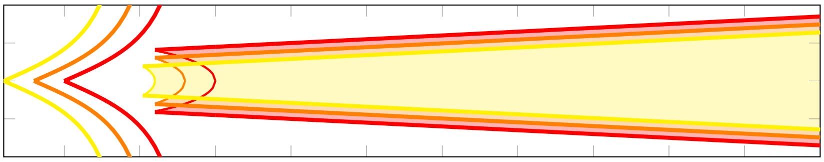


Figure IV: Schematic drawing of the semiclassical and actual pseudospectra for the model (1.6)

1.3 Solvable models in quantum mechanics

The notion of solvability in quantum mechanics is not unambiguous throughout the literature, as different sources vary in the precise formulation. Usually, it is required that the corresponding eigenvalue equation coincides with a certain class (e.g. hypergeometric) of differential equations. We do not pursue the general case, but rather go into detail regarding a class of ℓ^2, respectively L^2 operators: infinite Toeplitz operators and constant-coefficient differential operators (which are incidentally intimately related to each other).

1.3.1 Toeplitz operators

Toeplitz operators form a long studies field, More thorough elaboration of the topics can be found in [20]. A finite Toeplitz matrix $a \in \mathbb{C}^{n,n}$ is any matrix satisfying $a_{k,j} = a_{k+1,j+1}$, while a circulant matrix is a Toeplitz matrix with $a_{k-n} = a_k$. Their infinite-dimensional analogues are defined as operators on $\ell^2(\mathbb{Z})$, resp. $\ell^2(\mathbb{N})$ satisfying $a_{k,j} = a_{k+1,j+1}$ (an infinite circulant matrix is also called a Laurent operator). This definition raises the question, whether an infinite matrix defines an operator on $\ell^2(\mathbb{N})$, respectively $\ell^2(\mathbb{Z})$, and what is such an operator's domain. As long as we restrict attention to bounded operators, we might formulate the following elegant result.

Theorem ([20], thm. 1.1/1.9)**.** *An infinite Toeplitz (Laurent) matrix defines a bounded operator on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$ through the relation $a_{ij} = \langle\psi_i|A|\psi_j\rangle$, if and only if there is a function $a \in L^\infty(\mathbb{T})$, such that*

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}d\theta \tag{1.8}$$

Moreover, the norm of such operator in $\|T(a)\| = \|a\|_\infty$, respectively $\|L(a)\| = \|a\|_\infty$. The original function can be expressed from the coefficients as

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k \tag{1.9}$$

Every (finite or infinite) circulant matrix is normal. To see this, it suffices to write action of such operators as a (semi)-discrete convolution $Av = a \star v$, and recall that Fourier transform brings convolution into multiplication. Thus, circulant matrices are unitarily equivalent to multiplication operators, and therefore normal. On the other hand, general Toeplitz matrices are not normal (normal Toeplitz matrices can be explicitly classified [21]). The spectra of infinite Toeplitz and Laurent matrices can be to some extent characterized in terms of $f(z)$, and the same holds for finite circulant matrices. In the following, $\mathbb{T}$ denotes the unit circle and $\mathbb{T}_n$ the set of n-th roots of unity.

Theorem ([20], thm. 1.2)**.** *Let H be an infinite circulant matrix with associated function $a \in L^\infty(\mathbb{Z})$. Then*

$$\sigma(H(a)) = \mathcal{R}(a) = \{\lambda \in \mathbb{C} \mid \mu(t \in \mathbb{R}, |a(t) - \lambda| < \varepsilon) > 0\} \tag{1.10}$$

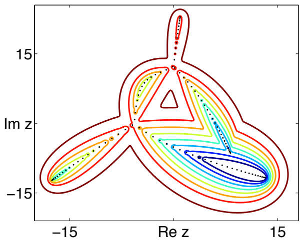
Specially for $a(t)$ continuous, this reduces to $\sigma(H(a)) = a(\mathbb{T})$. Moreover, for finite circulant matrices $\sigma(H(a)) = a(\mathbb{T}_n)$

The statements about circulant matrices can be proven similarly to their normality using discrete convolution. For infinite Toeplitz matrices, there is no such general and simple characterization of spectrum. We can however restrict attention to continuous associated functions a, which allows us to state the following theorem.

Theorem ([20], thm. 1.17)**.** *Let H be an infinite Toeplitz matrix with associated function $a \in C^0(\mathbb{Z})$. Then $\sigma(H) = f(\mathbb{T})$ together with $\lambda \in \mathbb{C}$, such that $I(\lambda, f) \neq 0$.*

In contrast, no simple characterization of spectrum exists for finite Toeplitz matrices, although it is known that they cluster along lines in complex plane. This is the moment when pseudospectra can provide illuminating information about the operator behavior. Although the pseudospectrum cannot be possible characterized explicitly through the associated function, we can still use numerical methods to obtain revealing conclusions. The graphics summary of these two theorems is provided in fig. V. For finite matrices of both kind, we plotted the pseudospectra in addition to the spectrum.

(a) finite Toeplitz



(b) finite circulant

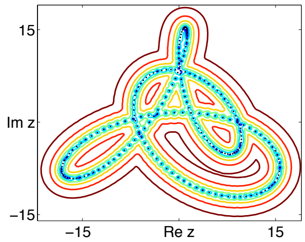


Figure V: Spectra of matrices associated with the symbol $f(z) = \mathrm{i}z^{-3} - 5z^{-2} + 6z^{-1} - 3z^2 - 8\mathrm{i}z^3$

The similarity of pseudospectra for finite circulant and Toeplitz matrices has a general character, which can be formalized through following theorem (which, in short, proves the exponential growth of the resolvent norm inside the curve $a(z)$, while proving norm-boundedness outside the said curve).

Theorem ([15], thm. 7.2)**.** *Let $\{H_n\}$ be a sequence of semi-banded Toeplitz matrices associated to $f(z)$, and let $\lambda \in \mathbb{C}$, such that $I(f, \lambda) \neq 0$. Then exists $c > 1$, such that $\|(H_n - \lambda)^{-1}\| \geq c^n$ for sufficiently large n.*

This also serves as a concrete example for the earlier results concerning spectral and pseudospectral convergence. While the spectra of a sequence of circulant matrices associated with $f(z)$ converge to the spectrum of corresponding Laurent operator, the same does not hold for general Toeplitz matrices. On the other hand, the pseudospectra converge for both cases (in the Hausdorff metric) as predicted by theorem (1.2.1).

1.3.2 Constant-coefficient differential operators

The results of this section will largely mimic the previous one, demonstrating the close correspondence between Toeplitz matrices and constant-coefficient differential operators. We define a general constant-coefficient differential operator as a formal expression

$$(H\psi)(x) = \sum a_j \psi^{(j)}(x) \tag{1.11}$$

with its domain to be discussed shortly. We now specify the domains of our purely formal differential operators, which are in direct correspondence with the classes of matrices in preceding section. All operators of degree d are acting on the subset of AC^{d-1}, functions with absolutely continuous $(d-1)$-th derivative. The domains and their corresponding matrix analogues are summarized as follows.

operator domain	**discrete analogue**
$AC^d[0, \kappa]$ with	finite Toepliz
$AC^d[0, \kappa]$ with periodic conditions	finite circulant
$AC^d(\mathbb{R}^+)$ with β homogenous conditions at $x = 0$	infinite Toeplitz
$AC^d(\mathbb{R})$	infinite circulant

We again associate a symbol to each constant-coefficient differential operator H, this time having the form $a(k) = \sum a_j(-ik)^j$. As in the discrete case, three of these four classes admit elegant spectral, at least when the symbol is a continuous function of k.

Theorem ([15], thm. 10.1)**.** *Let H be a constant-coefficient differential operator of degree d, and let its associated symbol $a(z)$ be continuous. On $L^2[0,\kappa]$ with periodic boundary conditions, we have $\sigma(H) = f(2\pi\mathbb{Z}/a)$. On $L^2(\mathbb{R})$, we have $\sigma(H) = a(\mathbb{R})$. On $L^2[0,\infty)$ with β homogenous boundary conditions at $x = 0$, we have $\sigma(H) = a(\mathbb{R})$ together with all $\lambda \in \mathbb{C}$, such that $I(a,\lambda) \neq d - \beta$.*

The remaning case of finite interval with homogenous conditions admits a characterization directly analogous to the case of finite Toeplitz matrices. To make the analogy complete, we state a theorem concerning the pseudospectral behavior inside and outsice

Theorem ([15], thm. 10.2)**.** *Let (H_n) be a family of constant-coefficient differential operators on $[0,n]$ associated with the symbol $f(z)$, with β homogeneous conditions at $x = 0$ and $d - \beta$ homogeneous conditions at $x = n$. Let $\lambda \in \mathbb{C}$, such that $I(f,\lambda) \neq d - \beta$. Then $\|(H_n - \lambda)^{-1}\| \geq e^{cn}$ for some $c > 0$ and sufficiently large n.*

As in preceding section, the pseudospectra of such an operator can be shown to converge (in the Haudorff metric) to those of the same operator on $[0,\infty)$ with β homogeneous conditions at $x = 0$.

Example. *Consider the second-order advection-diffusion operator associated to the symbol $f(k) = -ik - k^2$ on the interval $[0,a]$ with $\beta = d - \beta = 1$. The differential expression has the form*

$$(H_a\psi)(x) = -\psi''(x) + \psi'(x) \tag{1.12}$$

with $\mathfrak{D}(H_a)$ being the closure of H defined initially on $C_0^\infty(0,a)$. Such an operator is similar to an essentially self-adjoint operator $(\widetilde{H}_a\psi)(x) = -\frac{1}{4}\psi(x) + \psi''(x)$ through the transformation $\Omega = \exp(x/2)$. This similarity transformation is bounded and nonsingular for $a < \infty$, making H_a and $\widetilde{H}_a$ isospectral. Moreover, the eigenvalue equation for $\widetilde{H}_a$ may be solved exactly to yield

$$\sigma_p(H_a) = \left\{ -\frac{1}{4} - \frac{\pi^2 n^2}{a^2} \,\middle|\, n \in \mathbb{N} \right\} \tag{1.13}$$

From the fact that the eigenfunctions of this operator form a Riesz basis, we might conclude, that the original operator H_a is crypto-hermitian and $\sigma_p(H) = \sigma(H)$.

Chapter 2

Crypto-selfadjoint models

The present section is concerned with three distinct finite-dimensional crypto-hermitian model families. The three models in this section are in fact pointing out at three different methods in crypto-hermitian model-building theory. While the Su-Schrieffer-Model uses the correspondence between crypto-hermitian QM and solid state theory, the discrete Robin model uses plain discretization, and finally the dual SSH model uses the correspondence between tridiagonal matrix models and orthogonal polynomials.

2.1 The Su-Schrieffer-Heeger model

The Su-Schrieffer-Heeger (SSH) model was originally introduced in [22] to describe a 1D array of polyacetylene, shown in fig. VI. Thanks to its nontrivial spectral properties, it has later found applications in various models in solid state theory, most recently as one of the simples examples of topological insulators [23].

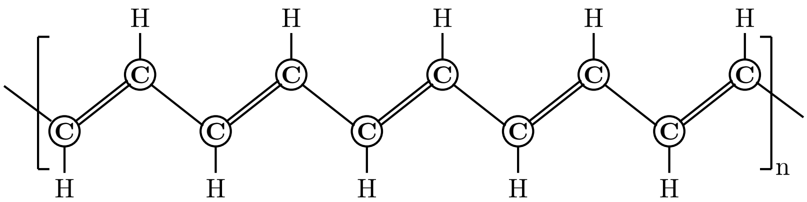


Figure VI: The Su-Schrieffer-Heeger model describing chain of polyacetylene

Besides, it serves as a realistic model for a number of other physical systems, including ultracold fermions [24], and systems exhibiting Zak phase transition [25]. The n-site lattice SSH Hamiltonian is commonly expressed in its second-quantized form

$$H_{SSH}^{(n)} = \sum_{i=1}^{N} \left\{ t(1-\Delta\cos\theta)a_{2i-1}^{\dagger}a_{2i} + t(1+\Delta\cos\theta)a_{2i}^{\dagger}a_{2i+1} + h.c. \right\} \tag{2.1}$$

with $a_i^{\dagger}, a_i$ denoting the i-th site fermionic creation and annihilation operators, and $h.c.$ standing for hermitian conjugate. Since the topological nature of the model is not changed by varying the parameters t and Δ, we set with no loss of generality $t = \Delta = 1$, and denote $\lambda = \cos\theta$. The SSH model shows two topologically distincs phases depending on the value of λ. For $\lambda > 0$, it admits no edge state, while it admits one for $\lambda < 0$, which can be observed on the first picture in fig. VII.

In a recent article [26], its authors suggested to complement this hermitian model with a non-hermitian boundary term $H_I = i\gamma a_1^{\dagger}a_1 - i\gamma a_n^{\dagger}a_n$. Despite the non-hermicity of the resulting Hamiltonian $H = H_{SSH} + H_I$, numerical experiments suggest that the reality of its eigenvalues for γ sufficiently small is conserved. In a canonical matrix

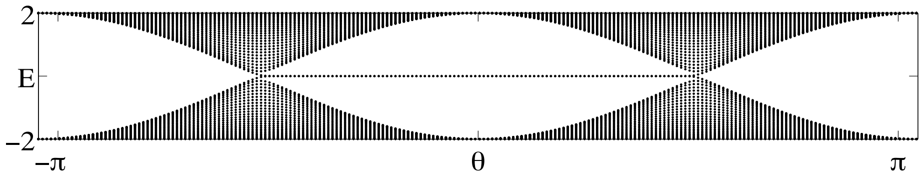


Figure VII: Topological phases of the $n = 50$ hermitian SSH model

representation of the creation and annihilation operators with

$$a_n/a_n^\dagger = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \kappa_i/\kappa_i^\dagger & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \qquad \kappa_n = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \kappa_n^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.2}$$

the model (2.1) becomes a $2n \times 2n$ matrix family, which shall be expressed for simplicity just for $n = 4$ (the extrapolation pattern being clear enough) as

$$H^{(4)} == \begin{bmatrix} \gamma & -1-\lambda & & \\ -1-\lambda & & -1+\lambda & \\ & -1+\lambda & & -1-\lambda \\ & & -1-\lambda & \gamma^* \end{bmatrix} \tag{2.3}$$

The effects of complex γ are illustrated in fig. VIII, with complex parts of eigenvalues being plotted in the bottom row. For every ρ small enough exists a nonempty interval of ω, such that $\sigma(H) \subset \mathbb{R}$, and the size of this interval shrinks with growing ρ.

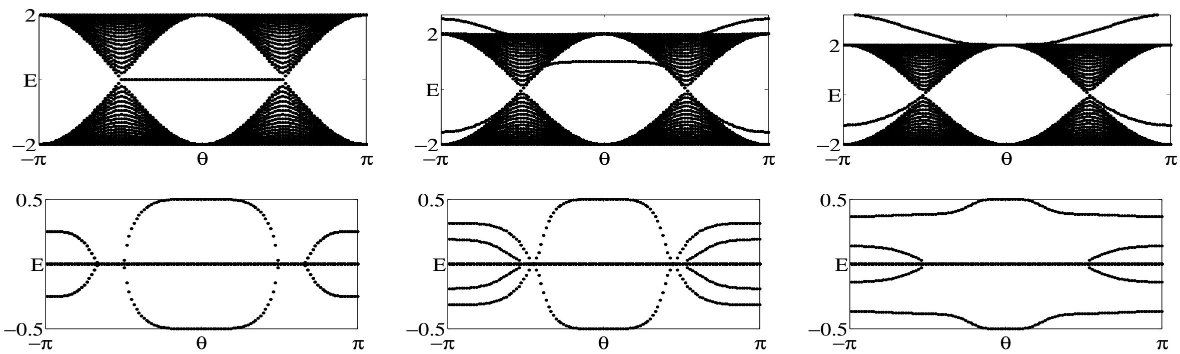


Figure VIII: Topological phases of the non-hermitian SSH model with $\gamma = 0.5i, 0.5i + 1$ and $0.5i + 2$.

Incidentally, the case $\lambda = 0$ has already been studies in crypto-hermitian literature, with the origins of the operator being completely different. Indeed, it comes from a lattice discretization of an operator

$$(H_{\alpha\beta})\psi(x) = -\psi''(x) \qquad \mathfrak{D}(H_{\alpha\beta}) = \left\{\psi \in H^2(-1,1) \mid \psi'(\pm 1) = (\pm i\alpha + \beta)\psi(\pm 1)\right\} \tag{2.4}$$

which has been a subject of a recent article series, withwhich has been subject of a recent article [27]. It turns out, that this particular limit of the SSH model is closely related to a Robin square well [28, 29]. This model, being essentially a Laplacian on a real interval with imaginary Robin boundary conditions, is unique in quasi-hermitian literature, in that it admits construction of a metric operator in a closed form. Applying equidistant discretization on this model [27] results in a family of matrices $H^{(n)}$ of the form

$$H^{(4)} = \begin{bmatrix} \frac{1}{1-\alpha-i\beta} & -1 & & \\ -1 & & -1 & \\ & -1 & & -1 \\ & & -1 & \frac{1}{1-\alpha+i\beta} \end{bmatrix} \tag{2.5}$$

The relationship between the different parametrizations of these models may be written as a nonlinear transformation $\omega = \beta/((1-\alpha)^2+\beta^2)$ and $\rho = (1-\alpha)/((1-\alpha)^2+\beta^2)$. The domains of pseudo-hermicity in both coordinate systems are shown in fig. IX. Their non-shrinking behavior in the limit $n \to \infty$ corresponds with the quasi-hermicity of the limit operator [29].

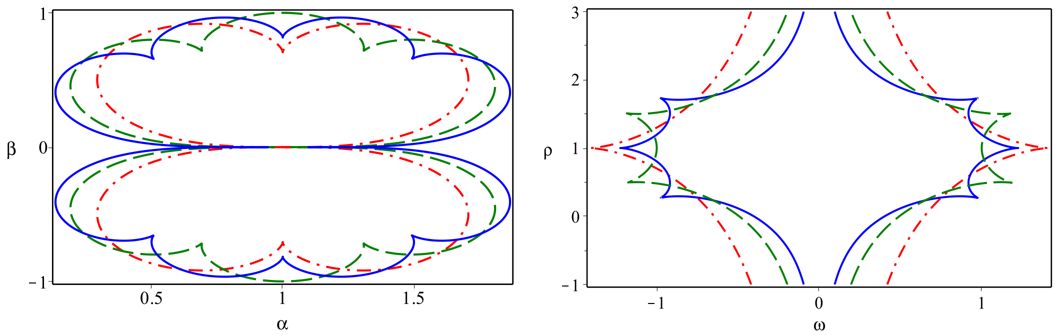


Figure IX: *Domains of observability for model (2.5) for $n = 3, 4, 5$ in the (α, β) and (ρ, ω) coordinates.*

2.1.1 Metric operators

We aim to extend the considerations of [27] and construct a complete family of metric operators for some special cases of (2.1) instead of a single one. By a complete metric family we mean a metric operator depending upon n free parameters, where n is the dimension of the underlying (finite-dimensional) Hilbert space. It is well-known that a general metric operator on an n-dimensional space may be written as

$$\Theta = \sum_{k=1}^{n} \kappa_n |n\rangle\langle n| \tag{2.6}$$

where $|n\rangle$ are the eigenvectors of $H^\dagger$ and $\kappa_n \in \mathbb{R}$. We try, however, to find Θ as a linear combination of matrices satisfying the Dieudonné equation, which have a form extrapolating to arbitrary dimension. A price to pay for such feature is, that we drop the condition of positivity for individual elements $\mathcal{P}^{(k)}$, to which we refer to as pseudometrics. In [27], the authors appreciated the existence of a particular Θ, which served also as a starting point for our considerations. Symbolic manipulations in MAPLE yield the $n = 4$ result

$$\mathcal{P}^{(1)} = \begin{bmatrix} 1 & -\mathrm{i}\omega & -\mathrm{i}\omega\xi & -\mathrm{i}\omega\xi^2 \\ \mathrm{i}\omega & 1 & -\mathrm{i}\omega & -\mathrm{i}\omega\xi \\ \mathrm{i}\omega\xi^* & \mathrm{i}\omega & 1 & -\mathrm{i}\omega \\ \mathrm{i}\omega\xi^{*2} & \mathrm{i}\omega\xi^* & \mathrm{i}\omega & 1 \end{bmatrix} \quad \mathcal{P}^{(2)} = \begin{bmatrix} & 1 & -\mathrm{i}\omega & -\mathrm{i}\omega\xi \\ 1 & \rho & 1 & -\mathrm{i}\omega \\ \mathrm{i}\omega & 1 & \rho & 1 \\ \mathrm{i}\omega\xi^* & \mathrm{i}\omega & 1 & \end{bmatrix} \quad \mathcal{P}^{(3)} = \begin{bmatrix} & & 1 & -\mathrm{i}\omega \\ & 1 & \rho & 1 \\ 1 & \rho & 1 & \\ \mathrm{i}\omega & 1 & & \end{bmatrix} \quad \mathcal{P}^{(4)} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \tag{2.7}$$

where we denoted $\xi = \rho - \mathrm{i}\omega$. Note that $\mathcal{P}^{(1)}$ coincides with Θ obtained in [27], while $\mathcal{P}^{(4)}$ realizes a discrete operator of parity. The two-site case suggests an extrapolation pattern with the k-th pseudometric having $2(n-k)+1$ nonzero antidiagonals. The nonzero elements of such metrics are given by

element	**position**
$-\mathrm{i}\omega\xi^{(i-j-k)}$	$i - j \geq k$
$\mathrm{i}\omega\xi^{*(i-j-k)}$	$j - i \geq k$
ρ	$\lvert i-j\rvert < k,\ i+j-k$ even
1	$\lvert i-j\rvert < k,\ i+j-k$ odd

which can be proven by double induction $n \to n+1$ and $k \to k+1$, analogically to prop. 2 in [27]. The positivity of the resulting metric as a function of ε_i must be then verified separately. Another completely solvable case is $\gamma = 0$ with λ arbitrary (hermitian models can be of course treated as effective crypto-hermitian in another Hilbert space). As long as we denote $\pm = 1 \pm \lambda$, we may write

$$\mathcal{P}^{(1)} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \mathcal{P}^{(2)} = \begin{bmatrix} & - & & \\ + & & + & \\ & - & & - \\ & & + & \end{bmatrix} \quad \mathcal{P}^{(3)} = \begin{bmatrix} & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & \end{bmatrix} \quad \mathcal{P}^{(4)} = \begin{bmatrix} & & & - \\ & & + & \\ & - & & \\ + & & & \end{bmatrix} \tag{2.8}$$

Extrapolation formulae for general $n \in \mathbb{N}$ are clear from these matrices, again with the k-th pseudometric having $2(n-k)+1$ nonzero antidiagonals. Finally, we address (2.1) in full generality. This time, we denote $\pm$ to stand for $(1 \pm \lambda)w$, where w is the corresponding element of the corresponding pseudometric (2.7). The hope of combining the patterns of (2.7) and (2.8) is however not fullfilled, and the general metric of the non-hermitian SSH model appears unavailable in closed form. However, three families of pseudometrics (corresponding to the last three families in the above considerations, which have the least number of nonzero diagonals) appear to be available explicitly in the form

$$\mathcal{P}^{(2)} = \begin{bmatrix} & +- & + & 1 \\ +- & - & -^2 & + \\ + & -^2 & - & +- \\ 1 & + & +- & \end{bmatrix} \qquad \mathcal{P}^{(3)} = \begin{bmatrix} & & 1 & - \\ & 1 & + & 1 \\ 1 & - & 1 & \\ + & 1 & & \end{bmatrix} \qquad \mathcal{P}^{(4)} = \begin{bmatrix} & & & - \\ & & + & \\ & - & & \\ + & & & \end{bmatrix} \tag{2.9}$$

We close this section with the pseudospectral plot fig. X, which shows that the non-hermicity of the model in $\mathcal{H}^{(F)}$ is mild, meaning that the norm of the similarity transformation Ω is in the order of 1 (and we know from before that it does not rise with growing dimension).

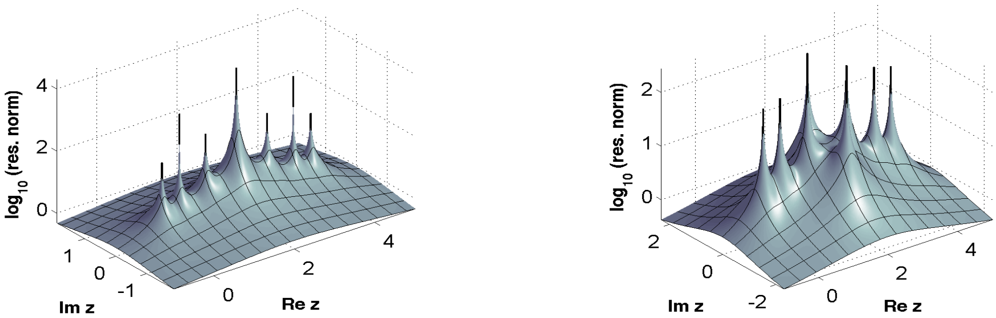


Figure X: Pseudospectra of the $n = 8$ model (2.5) for $\alpha = \beta = 1$ and $\alpha = 1, \beta = 0.7$

2.2 The Fano-Anderson model

Encouraged by the fruitful correspondence between lattice solid state and finite-dimensional crypto-hermitian models, we apply our metric operator considerations on another model known in solid state physics coupled with a non-hermitian gain-loss term. This time, it is the Fano-Anderson model [30, 31], which has the form

$$H^{(n)} = -J \sum_{i=1}^{n} a_{i-1}^{\dagger} a_i + d_1 a_0 + a_0 d_2 + d_1 a_1 + d_2 a_1 + h.c. \tag{2.10}$$

In [32], it was suggested to complement this system with a gain-loss term, this time localized not on the boundary of the lattice, but in its center, explicitly $H_I = i\gamma d_1^{\dagger} d_1 - i\gamma d_2^{\dagger} d_2$. As with the SSH model, we express this non-hermitian Fano-Anderson Hamiltonian in a matrix form using the representation of creation/annihilation operators (2.2). Due to presence of nontrivial central pattern, we express all the matrices for $n = 6$, the extrapolation pattern being again clear for any higher $n \in N$.

$$H^{(6)} = \begin{bmatrix} & -1 & & & & \\ -1 & & -g & -g & & \\ & -g & i\gamma & & -g & \\ & -g & & -i\gamma & -g & \\ & & -g & -g & & -1 \\ & & & & -1 & \end{bmatrix} \tag{2.11}$$

Surprisingly, the special case of this model $\gamma = 0$ again coincides with a Hamiltonian already studied in crypto-hermitian framework. This time, it is a quantum graph of the articles [33, 34], expressed for $n = 6$ as

$$H^{(6)} = \begin{bmatrix} & -1 & & & & \\ -1 & & -1+g & -1+g & & \\ & -1-g & & & -1-g & \\ & -1-g & & & -1-g & \\ & & -1+g & -1+g & & -1 \\ & & & & -1 & \end{bmatrix} \tag{2.12}$$

2.2.1 Metric operators

Although the approach to pseudometric construction for this model remains in principle the same, the results are considerably more complicated than in the SSH case. When $H^{(n)}$ is made into a $2n\times 2n$ matrix in the creation/annihilation operator representation, the set of $2n$ linearly independent pseudometrics splits into three different parts. The first part contains two trivial pseudometrics, the identity and the parity operator $\mathcal{P}^{(4)}$ of (2.7). Next, we get n pseudometrics independent on g, which express the trivial behavior of the Hamiltonian outside the bubble neighborhood of the origin. These metrics contain nonzero entries solely in the $n \times n$ corner blocks of the final matrices, in our case

$$\mathcal{P}_k = \begin{bmatrix} k & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k \end{bmatrix} \qquad k_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad k_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.13}$$

Finally, we have got nontrivial n pseudometric depending on g explicitly, which are to some extent compatible with the bubble shape of the original Hamiltonian. In our case, the matrices have the form

$$\mathcal{P}_4 = \begin{bmatrix} & 1 & & & 1 & \\ 1 & & 2\,g & 2\,g & & 1 \\ & 2\,g & & & 2\,g & \\ & 2\,g & & & 2\,g & \\ 1 & & 2\,g & 2\,g & & 1 \\ & 1 & & & 1 & \end{bmatrix}, \qquad \mathcal{P}_5 = \begin{bmatrix} 1-4\,g^2 & & 2\,g & 2\,g & & 1-4\,g^2 \\ & 1 & & & 1 & \\ 2\,g & & & & & 2\,g \\ 2\,g & & & & & 2\,g \\ & 1 & & & 1 & \\ 1-4\,g^2 & & 2\,g & 2\,g & & 1-4\,g^2 \end{bmatrix} \tag{2.14}$$

The pattern gets more involved for $\gamma \neq 0$, but the pseudometrics still retain an accessible and sparse structure: new nonzero elements appear only inside the "bubbles", the outer elements remain unchanged. Explicitly for $n = 6$, we have

$$\mathcal{P}_4 = \begin{bmatrix} & 1 & & & 1 & \\ 1 & & 2g & 2g & & 1 \\ & 2g & & -2i\gamma & 2g & \\ & 2g & 2i\gamma & & 2g & \\ 1 & & 2g & 2g & & 1 \\ & 1 & & & 1 & \end{bmatrix} \qquad \mathcal{P}_5 = \begin{bmatrix} 1-4g^2 & & 2g & 2\,g & & 1-4g^2 \\ & 1 & 2i\gamma g & -2i\gamma g & 1 & \\ 2g & -2i\gamma g & & -2\gamma^2 & -2i\gamma g & 2g \\ 2g & 2i\gamma g & -2\gamma^2 & & 2i\gamma g & 2g \\ & 1 & 2i\gamma g & -2i\gamma g & 1 & \\ 1-4g^2 & & 2g & 2g & & 1-4g^2 \end{bmatrix} \tag{2.15}$$

The search for all pseudometrics is however not yet complete, it remains to reveal the form of the one metric, which degenerates to identity for $\gamma = 0$. This may be currently perceived as an open problem. The same approach may be employed for the discrete graph case (2.12). The only differences appeach in the third family of g-dependent pseudometrics, giving for $n = 6$

$$\mathcal{P}_5 = \begin{bmatrix} & 1 & & & 1 & \\ 1 & & 2+2\,g & 2+2\,g & & 1 \\ & 2+2\,g & & & 2+2\,g & \\ & 2+2\,g & & & 2+2\,g & \\ 1 & & 2+2\,g & 2+2\,g & & 1 \\ & 1 & & & 1 & \end{bmatrix}, \qquad \mathcal{P}_6 = \begin{bmatrix} 4\,g^2-3 & & 2+2\,g & 2+2\,g & & 4\,g^2-3 \\ & 1 & & & 1 & \\ 2+2\,g & & & & & 2+2\,g \\ 2+2\,g & & & & & 2+2\,g \\ & 1 & & & 1 & \\ 4\,g^2-3 & & 2+2\,g & 2+2\,g & & 4\,g^2-3 \end{bmatrix} \tag{2.16}$$

2.3 Orthogonal polynomials

The final model of this section is intimatelly connected with the theory of orthogonal polynomials, which is a fruitful model-building scheme for finite-dimensional models having the property of crypto-selfadjointness [35, 36, 37, 38, 39, 40, 41]. The essence of these considerations is, that any polynomial sequence orthogonal with respect to a measure μ obeys a recurrent relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \tag{2.17}$$

with $b_n \in \mathbb{R}$ and $a_n c_n > 0$. Moreover, the coefficients a_n, b_n, c_n are determined uniquelly by the measure μ. Conversely, any polynomial sequence satisfying (2.17) is orthogonal with respect to some measure μ (which however needs not to be unique). The roots of polynomials orthogonal on a subset of real line are real, which provides the main motivation to use them in crypto-selfadjoint model building. Indeed, it suffices to see the zeros of $p_n(x)$ from (2.17) are precisely the eigenvalues of

$$H_n = \begin{bmatrix} a_1 & b_1 & 0 & \dots \\ c_1 & a_2 & b_2 & \dots \\ 0 & c_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.18}$$

This class of tridiagonal models admits a simple recurrent construction of a complete set of metric operators. As shown in [40], we can use the ansatz of $\mathcal{P}^{(k)}$ having only $(2k+1)$ nonzero diagonals, and obeying the set of recurrences

$$\sum_{j=0}^{k-1} a_{(k+j)(k)} \mathcal{P}^{(k)}_{(k+j)(k+1)} = \sum_{j=0}^{k-1} a_{(k+j)(k+1)} \mathcal{P}^{(k)}_{(k)(k+j)} \tag{2.19}$$

The model studied in this section is a simple variation on the SSH model (2.1) with $\gamma = 0$. In its matrix form and for $n = 4$, we may write it as

$$H^{(4)}_{dSSH} = \begin{bmatrix} & -1-\lambda & & \\ -1+\lambda & & -1-\lambda & \\ & -1+\lambda & & -1-\lambda \\ & & -1+\lambda & \end{bmatrix} \tag{2.20}$$

In the following final section, we shall study this particular case as well as the more general model with $\gamma \neq 0$, which shall be referred to as dual non-hermitian SSH model.

2.3.1 Metric operators

The symbolic manipulation approach to low-dimension pseudometric construction works equivalently well as for the SSH model, and it is not surprising that the pattern revealed in the pseudometrics is similar. The difference is, that now the odd pseudometrics exhibit varying entries, while the nonzero entries of even pseudometrics consist entirely of ones. For $n = 4$, we might again write

$$\mathcal{P}^{(1)} = \begin{bmatrix} + & & & \\ & - & & \\ & & + & \\ & & & - \end{bmatrix} \qquad \mathcal{P}^{(2)} = \begin{bmatrix} & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{bmatrix} \qquad \mathcal{P}^{(3)} = \begin{bmatrix} & & + & \\ & - & & - \\ + & & + & \\ & - & & \end{bmatrix} \qquad \mathcal{P}^{(4)} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \tag{2.21}$$

where we have again denoted $\pm = (1 \pm \lambda)$. These formulae once again indicate a chessboard extrapolation pattern with k-th pseudometric having $2k+1$ nonzero diagonals and $2(n-k)+1$ nonzero antidiagonals. In order to provide comparison with the pseudometrics of the $\gamma = 0$ SSH model (2.8), we have summarized the nonzero entries of the pseudometrics in the following table

SSH		**dual SSH**	
$+$	k even, i odd	$+$	k odd, i odd
$-$	k even, i even	$-$	k odd, i even
1	k odd	1	k even

Similarly, in the general case of $\gamma \neq 0$, we are able to build merely three families of pseudometrics, this time having the form

$$\mathcal{P}^{(2)}_{dSSH} = \begin{bmatrix} & -^2 & - & 1 \\ -^2 & - & +- & + \\ - & +- & + & +^2 \\ 1 & + & +^2 & \end{bmatrix} \qquad \mathcal{P}^{(3)}_{dSSH} = \begin{bmatrix} & & - & 1 \\ & - & 1 & + \\ - & 1 & + & \\ 1 & + & & \end{bmatrix} \qquad \mathcal{P}^{(4)}_{dSSH} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \tag{2.22}$$

Epilogue

Currently, there are three main directions to pursue in crypto-hermitian quantum mechanics. The first one lies in clarifying the fundamental phenomenological foundations of such a theory, and the case of time-dependent metrics, which, although explored by Znojil in [43, 44, 45], is to this date not very well understood. The second task is the proper mathematical foundation of crypto-hermitian theory for operators with unbounded and/or singular metrics. This is the topic discussed by various scientists all over the world [46, 47, 48, 49], with a nice summary provided in [6]. Finally, the third direction lies in finding proper solvable models, which make crypto-hermitian QM a useful theory.

We pursued the third direction in this paper, and tried to make it a bit more understood. We focused on finite-dimensional quantum-mechanical models, whose understanding forms the crucial basis for further development of the field. In the model-building scheme for finite-dimensional models, there are three directions currently known, that yield some results. They are summarized in the following picture.

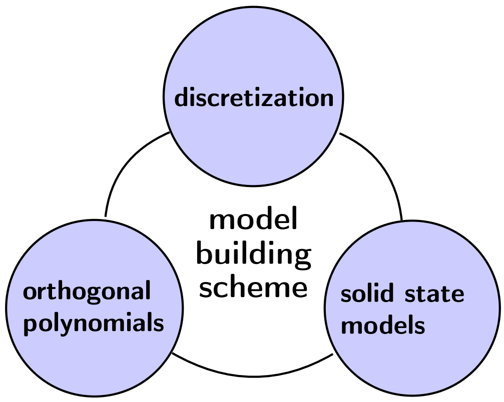


Figure XI: Model-building scheme

The solid state physics correspondence may be of course explored further than this paper did to yield more interesting models to study. Some studies have already been performed, like the one of the non-hermitian Bose-Hubbard model [50] and non-hermitian Aubry-Andre model [51]. However, the task of constructing and classifying the metric operators for these models appears currently open for discussion.

On the other hand, the correspondence between orthogonal polynomials and finite-dimensional non-hermitian models is long and well-known [52], and the task of constructing the metric operators has been established completely, in principle, through the recurrent formula (2.17). The third recipe to construct finite-dimensional models has been seen to lie in discretization of infinite-dimensional models. Because this procedure is complicated and unreliable, to our best knowledge only the non-hermitian Robin model has been succesfully shown to possess such an interesting finite-dimensional counterpart.

Finally, an open task for the present models might lie in exploring the proper infinite-dimensional limits of these discrete models. The same correspondence as between the models (2.4) and (2.5) could probably be established for more general non-hermitian SSH model, and could start the whole new interest in infinite-dimensional models of this kind.

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