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Research thesis

The cloaking effect in metamaterials from the point of view of anomalous localized resonance

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The cloaking effect in metamaterials from the point of view of anomalous localised resonance

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Abstract: We summarize here some basic facts about history and manufacturing the first metamaterial. Some possible applications for metamaterials are mentioned but the main concern of this work is on the metamaterial cloaking. We present some of the many theoretical approaches especially so called cloaking due to anomalous localized resonance (CALR). This concept is then used in this work. Inspired by work of Bouchitt and Schweizer in 2 dimensions calculation of the CALR for spherical symmetric setting dielectricum-metamaterial-dielectricum in 3 and higher dimensions is made . We prove here that CALR does not occur in such case.

Key words: metamaterials, Maxwell's equations, invisibility cloaking, anomalous localized resonance, localization index

Název práce:
Efekt neviditelnosti v metamateriálech z pohledu anomální lokalisované rezonance

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Abstrakt: Seznámíme se zde stručně s historií a výrobou prvních metamateriálů. Zmíníme některé možné aplikace, hlavní pozornost je však věnována maskování pomocí metamateriálů. Předvedeme některé z mnoha teoretických přístupů, které se dají použít k popisu neviditelnosti, zejména se zaměříme na maskování pomocí tzv. anomální lokalizované rezonance. Pomocí tohoto nástroje, inspirováni prací Bouchitté a Schweizera ve dvou dimenzích, dokážeme, že ve speciálním rotačně symetrickém uspořádání dielektrikum-metamateriál-dielektrikum ve třech a vyšších dimenzích nedochází k maskování pomocí anomální lokalizované rezonance.

Klíčová slova: metamateriály, Maxwellovy rovnice, maskování, anomální lokalizovaná rezonance, lokalizační index

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Chapter 1

Introduction

This thesis reviews existing progress in manufacturing metamaterials and theory about cloaking with use of this material. Metamaterials are artificial materials whose structure is designed so they have properties which may not be found in conventional materials. We will focus here on such substances which has simultaneously negative electrical permittivity ϵ and magnetic permeabily μ which results in negative refractive index n [39]. It turns out that these materials provide unprecedented control over electromagnetic fields. This property promises many amazing possibilities for applications.

Here we want to explore one specific use of metamaterials - concept of invisibility or metamaterial cloaking. There is an exponential growth of publications in theory as well in practise in this area since the first metamaterial was manufactured [37]. There are many theoretical ways how to treat this problem. Some of them are shown in Chapter 2. Our attention is mainly focused on the concept of anomalous localized resonance (we will refer to it as ALR, similarly we will call cloaking due to anomalous localized resonance as CALR) which we use in our thesis.

Before we proceed to the definition of ALR let us state the problem first. For this we will now use notation from [28] which is the foundation of the concept of anomalous localized resonance. In this thesis there will be used more than one notation for usually the same things. This might seem a little bit confusing but it is not since every notation is used only with the connection to the particular article which we will be speaking about. Although this can be slightly impractical in our thesis the reader can find it very useful because if he would like to check those cited articles he will not need to get used to new notation.

We will consider quasistatic approximation here. It is quite useful since in this case the electric and magnetic problems decouple so we think about metamaterials with only negative permittivity and positive permeability. Therefore we can use only electric part of Maxwell's equations

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{1.1}$$

$$\vec{\nabla} \times \vec{E} = 0, \tag{1.2}$$

where $\vec{D}$ is a displacement field related to the electric field $\vec{E}$ by relation $\vec{D} = \epsilon\vec{E}$ and ρ is electric charge density. The second equation means that the electrostatic field is potential and that is why we can introduce potential V by relation $\vec{E} = \nabla V$. We use this equation together with relation $\vec{D} = \epsilon\vec{E}$ to (1.1) and then we obtain

$$\vec{\nabla} \cdot (\epsilon\vec{\nabla} V) = \rho. \tag{1.3}$$

The electric charge density is here considered zero so we have an equation

$$\vec{\nabla} \cdot (\epsilon\vec{\nabla} V) = 0 \tag{1.4}$$

that must be satisfied by the complex potential $V(z)$.

Now we have the equation to solve but we need to apply it to some geometry (for now only in two dimensions). Let us have a medium with permittivity ϵ_m in which there is placed a body containing a so called coated cylinder. Coated cylinder is made of cylindrical core and cylindrical shell surrounding the core and having permittivity ϵ_c and ϵ_s respectively. The coated cylinder is then characterized by these dielectric parameters and by the core radius r_c, the shell radius r_s and by the radius r_m around the cylinder centre where the medium has permittivity ϵ_m. Very important for the anomalous localized resonance is that the shell permittivity ϵ_s is a complex number with negative real part and small non-negative imaginary part, we can choose it then for example as $\epsilon_s = -1 + i\delta$. Here δ is modelling losses in the material caused due to the electrical resistance of it (we know that electrical conductivity is calculated as $\sigma = \delta\omega$ where ω is a frequency of radiation). The crucial moment for ALR comes as δ tends to zero and we are investigating what happens in this limit.

Let us proceed to the promised definition of ALR. We will follow the definition by Milton *et al.* stated for example in [23]: Inhomogeneous body exhibits ALR if as the loss goes to zero (or as the system of equations lose ellipticity) the field magnitude diverges to infinity throughout a specific region with sharp boundaries not defined by any discontinuities in the moduli, but converges to a smooth field outside that region.

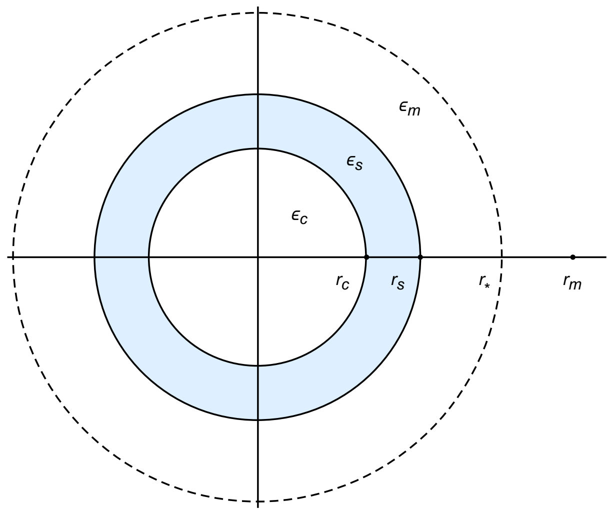

Figure 1.1: A coated cylinder in two dimensions with parameters $(\epsilon_c, r_c), (\epsilon_s, r_s), (\epsilon_m, r_m)$ describing core, shell and surrounding medium respectively. The radius $r_* = \sqrt{r_s^3/r_c}$ characterizes area in which ALR occurs.

There is also another probably way more important reason why we add small positive imaginary part into the permittivity. It is because we cannot treat equation (1.4) by standard methods if there is a sign changing factor ϵ in it. Adding δ helps us evade this problem and we deal with it in another way. Nevertheless there is an effort to overcome this difficulty without such perturbation in permittivity. Probably the best reference is [5] by Behrndt, Krejčiřík where they studied the non-elliptic differential expression $\nabla \cdot \operatorname{sgn} \nabla$ on a rectangle (in two dimensions).

This thesis is organised as follows. In the chapter 2 we review the most important moments of the history of metamaterials. In 2.1 we show how they were fabricated for the first time and then we show some approaches to the theory of metamaterials and cloaking phenomenon. In the second part of this chapter in Section 2.2 we write about the concept of CALR. We devided this section into smaller subsections dedicated to several groups of authors interested in the area of anomalous localized resonance. However the list of the authors and articles is far from complete, these are only the ones related most to this thesis. In the last Chapter 3 we use a simple procedure used already by Bouchitté and Schweizer in [7] in two dimensions (let us note that most of the problems related to CALR is examined only in two dimensions). We apply it for the ball in dimension $d \geq 3$ and prove that there is no cloaking in such case.

Chapter 2

History of metamaterial cloaking

2.1 From Veselago to the 21$^{\text{th}}$ century

The journey to the research of metamaterial cloaking has begun in 1967 when Russian physicist Victor Veselago had the visionary idea of material with negative refractive index [39]. Until then he studied so-called magnetic semiconductors in order to slow down electromagnetic waves (more information about Veselago's life and studies can be found for example in [34]). The wave velocity depends on the refractive index n by relation

$$v = \frac{c}{n} \tag{2.1}$$

where c is the speed of light. The refractive index is given as a square root of electrical permittivity ϵ and magnetic permeability μ

$$n = \sqrt{\epsilon\mu}. \tag{2.2}$$

To achieve slow velocities Veselago wanted to obtain higher values of n so he tried to increase both ϵ and μ in the magnetic semiconductor. But there is an issue that the high values of these quantities could not be realised simultaneously at any frequency and often one of the ϵ or μ became negative, so the wave could not propagate (in such medium). Then he realised very important question: What would happen if both permittivity and permeability were simultaneously negative? Apparently the refractive index stays the same and real so it is obvious to ask whether there will be any difference against materials found in nature. Or whether such substances can even exist. Actually these metamaterials (materials with simultaneous negative values of ϵ and μ) do not contradict any fundamental laws of nature so they are in principle possible and it is shown in Veselago's paper [39] that these materials have many extraordinary properties. For example Poyinting vector $\vec{S}$ and the wave vector $\vec{k}$ are in the opposite direction according to conventional materials. The consequences of these are that Doppler effect, Cerenkov effect and Snell's law would be reversed, that the light pressure is replaced by light attraction, and the greatest attention is focused on the refraction of light in unusual way. The light is refracted in the opposite way than in conventional materials (see Figure 2.1). Note that reflected ray has always the same direction independently of

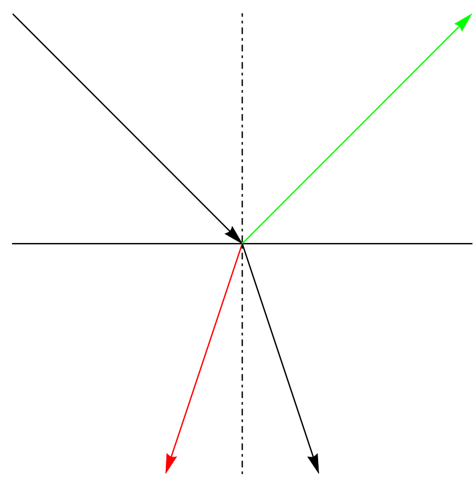

Figure 2.1: Propagation of a ray through the boundary between materials with positive (upper half plane) and negative refractive index (lower half plane). The green ray is the reflected one. The rays in the lower half plane respond to the cases when permittivity is negative (the red ray) and positive (the black ray).

the refractive index. But in a special case when permittivity and permeability of the metamaterial is exactly the opposite to the ϵ and μ of the surrounded medium there is no reflected ray (refraction still occurs). This allows to design very interesting refraction system called Veselago's lens. It is a planar lens (but in fact it is not a lens in usual sense, because it does not focus at a point a bundle of rays coming from infinity) and it is easy to see that if a radiation point is located at a distance shorter than a thickness of plate then such radiation is focused at a point (see Figure 2.2). It is also obvious that usual convex and concave lenses change places if they are made of metamaterial. It means that the rays passing through convex lens diverge and if they pass through concave lens they converge.

This inovative Veselago's article [39] has already more than 4 000 citations

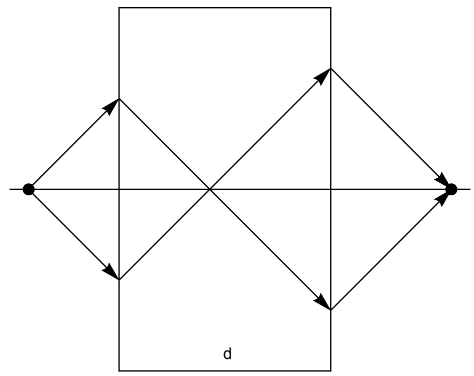

Figure 2.2: Propagation of two rays in Veselago's lens which is a metamaterial plate of thickness d

according to Cross ref and more than 9 000 according to Google Scholar. But only very few of them are from the same century as the Veselago's origin article because until 2000 no such metamaterial existed and whole concept was just on paper.

The material with negative permittivity was achieved by Pendry *et al.* in 1996 (see [32]). It was a periodic structure of very thin infinite wires with permittivity given by a relation

$$\epsilon_{\text{eff}}(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega} \tag{2.3}$$

where ω is a frequency of electromagnetic radiation, ω_p is an effective plasma frequency given by configuration of the wire field, i is an complex unit and γ is a coefficient of attenuation. From this it can be easily seen that for frequencies lower than plasma frequency the real part of permittivity is negative.

Then also permeability μ had to go under zero but it was much greater obstacle than creation of negative permittivity material. The innovative idea for this comes again from Pendry *et al.*. They stated in [31] (1999) that every material is composite - its smaller parts can be even atoms and molecules. And because permittivity and permeability only present an homogeneous view to the electromagnetic properties of a material, we can simply replace the atoms with some unit cells of characteristic dimensions which has to be much smaller than the wave length of the electromagnetic radiation. These cells are set in a periodic structure and their contents define the effective response of the system. The calculated dependence of the permeability is

$$\mu_{\text{eff}}(\omega) = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 - i\omega\Gamma} \tag{2.4}$$

where F, ω_0 and Γ are constants related to the geometry of the system. That periodic structure proposed by Pendry was field of so called split rings. These are flat concentric disks separated from each other by a small distance and these rings are both divided on the exactly opposite sides (see Figure 2.3).

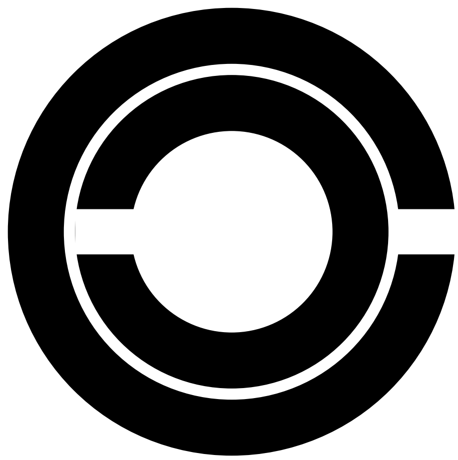

Figure 2.3: Split ring made of two concentric rings. This setting is the original proposed by Pendry but there are also other forms possible. Concept of two divided rings however must stay preserved.

Therefore in 1999 negative permittivity and permeability were finally achieved and after only one year in [37] Smith *et al.* demonstrated a composite medium which combined both previous structures. So after 33 years Viktor Veselago could see his theoretical ideas in practical experiments.

Invention of such extraordinary and promising material caused huge interest of many scientists in this field. Many ways how to use metamaterials were suggested and plenty of them are still being investigated. Let us focus on some of them. In 2000, just after manufacturing the first metamaterial, Pendry [30] suggested that Veselago's lens on Figure 2.2 might act as a superlens. This means that such lens would provide a perfect image which is not possible for conventional lenses because their maximum resolution in the image can never be grater than

$$\Delta \approx \frac{2\pi}{k_{max}} = \frac{2\pi c}{\omega} = \lambda \tag{2.5}$$

where ω is a frequency of an infinitesimal dipole placed in front of the lens. Pendry proves here that the metamaterial amplify evanescent waves and so that both propagating and evanescent waves contribute to the resolution of the image.

This Pendry's proposal that the imaged object would have the same evanescent fields decaying exponentially away from it (as the real object) has been subject to controversy (see [24] or introduction in [23] for more details about this debate). Despite all the published articles dedicated to this problem there still has not been made mathematical proof for the superlensing until 2005 when Milton *et al.* accomplished it due to concept of anomalous localized resonance [23]. This approach will be discussed more precisely in section 2.2).

Except superlenses there are also other possible applications for metamaterials as for example metamaterial antennas [29], metamaterial absorber [14], metamaterials sensors [12], terahertz detectors and of course metamaterial cloak which is the inspiration for this project.

Manufacturing the first negatively refracting substance rapidly increased interest in the area of invisibility due to the metamaterial with negative refractive index n. In 2006 the first real metamaterial cloak for microwave frequencies was calculated [33] and created [35]. The approach used in [33] and also [36] is based on the fact that metamaterials provide a freedom in their design so they can be used to control electromagnetic fields. They use here a coordinate transformation between orthogonal Cartesian mesh x, y, z and the distorted mesh $u(x, y, z), v(x, y, z), w(x, y, z)$ where u, v, w is the location of the new poitnt with respect to the x, y, z axes (see Figure 2.4). Maxwell's equations have exactly the

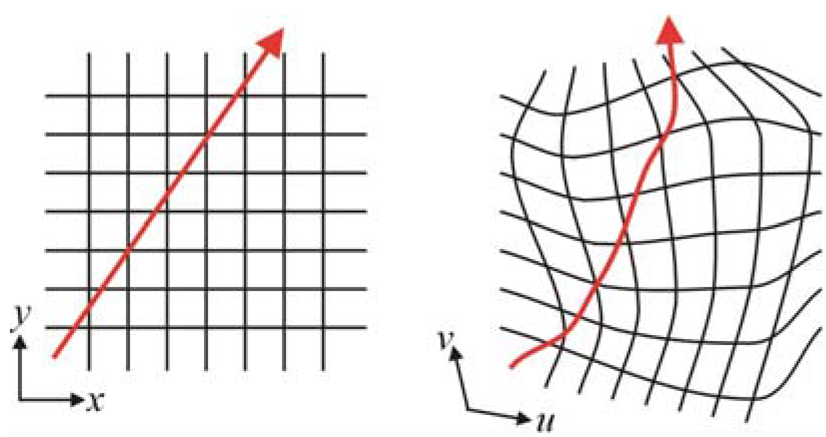

Figure 2.4: A field line in space with Cartesian coordinate system (left) and with the distorted system (right) [33]

same form in any coordinate system but for the permittivity and permeability we must use their renormalized values:

$$\begin{aligned}\epsilon_j' &= \epsilon_j \frac{Q_u Q_v Q_w}{Q_j^2} \\ \mu_j' &= \mu_j \frac{Q_u Q_v Q_w}{Q_j^2}\end{aligned} \tag{2.6}$$

$$\begin{aligned}E_j' &= Q_j E_j \\ H_j' &= Q_j H_j\end{aligned} \tag{2.7}$$

where $j \in \{u, v, w\}$ and

$$\begin{aligned}Q_u^2 &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ Q_v^2 &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\ Q_w^2 &= \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2\end{aligned} \tag{2.8}$$

Purpose of that article was to hide an arbitrary object in space in a way that external observers would have no idea that there is any object hidden from them. This is supposed to be achieved by metamaterial that would guide rays around the object and return them to their original trajectory. In other words we want the object to be invisible. Usually we think of the hiding object as a circle or sphere of radius R_c (core radius) and the cloaking region (where the metamaterial is) as an annulus $R_c < r < R_s$ (shell radius). For the object to be invisible we must find transformation that compress all fields in the region $r < R_s$ into the region $R_c < r < R_s$. The sketch of this transformation can be seen in Figure 2.5. However there are some issues that must be solved. For example there is a singularity that can be seen if we consider a ray heading directly towards the centre of the circle (sphere). Rays near this singularity are bent very close around the inner circle and very tightly to each other. This implies that there must be very rapid changes in permittivity and permeability given by the metamaterial. In practice the big problem is also to achieve very small or very large values of ϵ' and μ'. Cloaking can still occur but will be imperfect. From the ray picture 2.5 it can be easily seen that the cloaking is specific to a single frequency. Despite all these difficulties this theory gave birth to the first such metamaterial cloak working in the microwave frequencies [35].

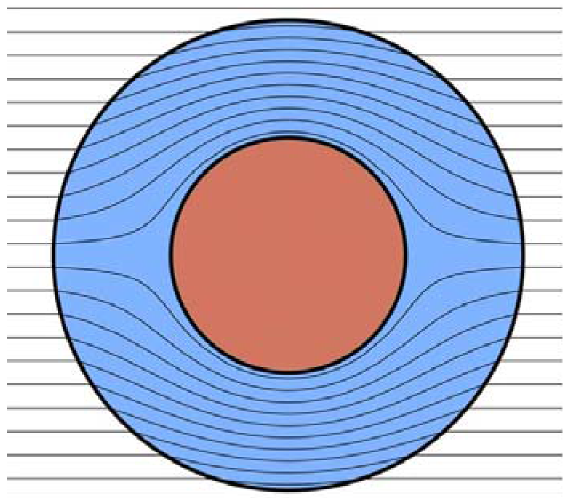

Figure 2.5: The electromagnetic field is made to avoid objects and flow around them and then returning undisturbed to its original trajectories [33]

Simultaneously with Pendry, Smith, Schurig *et al.* also Leonhardt was working on introduction of metamaterial cloak where invisibility should be again preserved by flowing electromagnetic fields around concealed object as if nothing was there. His tool for this theory was optical conformal mapping [16]. Such map preserves angles between the coordinate lines. Advantage of this method is that it is general so it can be used also for other forms of wave propagation like the sound waves. He considered here light propagating from infinity in two-dimensional case which is appropriate for complex valued functions and conformal mapping. It turns out that the main imperfections of invisibility are caused by reflections and time delays. But while reflections can be made exponentially small, the time delays are unavoidable. This delay was calculated by Leonhardt in [15].

The time delay of electromagnetic waves indicates that the general relativity could find some usage in this field of invisibility. In fact it is shown in [17] that it is not only usable for cloaking but that the general relativity unifies the whole theory behind controlling electromagnetic fields. Therefore when a desired function is given, we can calculate material properties for the device that turns this function into fact. This use of general relativity may be here quite surprising but it is not if we realize that the design concept of invisibility, superlenses and other applications is based on Fermat's principle that light rays follow the shortest optical path in media. These paths are effective geodetics and general relativity provides tools for curved geometries. However the metamaterials can also find some use in the physics of gravitation because as shown in [17] they may be applied for laboratory analogues of artificial black holes.

2.2 Cloaking due to anomalous localized resonance

In the previous section there were shown some various views on metamaterial cloaking. The special attention will be now focused on anomalous localized resonance since this concept is used also in our own calculation in Chapter 3.

2.2.1 Milton, Nicorovici, McPhedran

Beginnings of anomalous localized resonance come even before the first metamaterial was manufactured. In 1994 Nicorovici, McPhedran and Milton analyzed the two-dimensional potential around a coated cylinder [28]. We briefly summarize it here.

We consider the coated cylinder described in Chapter 1. We recall that such medium is made of two concentric cylinders (core and shell) with permitivities ϵ_c, ϵ_s and radii r_c,r_s respectively and this medium is placed in a space with permittivity ϵ_m at least in some radius r_m around the cylinder centre. As shown in the Introduction we can get from the electrical part of Maxwell's equations (1.1) and (1.2) the equation for complex potential V when charge density is zero (1.4). Since $V(z)$ is an analytic function of z we can find expansions for each region in our setting

$$V_e(z) = A_0 + \sum_{l=1}^{+\infty}(A_l z^l + B_l z^{-l}) \qquad \text{for } r_s \le r \le r_m \tag{2.9}$$

$$V_s(z) = C_0 + \sum_{l=1}^{+\infty}(C_l z^l + D_l z^{-l}) \qquad \text{for } r_c \le r \le r_s \tag{2.10}$$

$$V_c(z) = E_0 + \sum_{l=1}^{+\infty} E_l z^l \qquad \text{for } r \le r_c \tag{2.11}$$

Conditions of continuity for potential and for the normal component of the electrical displacement on the boundary (i.e. for r_s and r_c) allow us to express coefficients B_l, C_l, D_l, E_l in terms of the A_l

$$B_l = \frac{A_l}{\Delta}[\eta_{ms} + \eta_{sc}(\frac{r_c}{r_s})^{2l}]r_s^{2l}, \tag{2.12}$$

$$C_l = \frac{A_l}{\Delta}(1 + \eta_{ms}), \tag{2.13}$$

$$D_l = \frac{A_l}{\Delta}\eta_{sc}(1 + \eta_{ms})r_c^{2l}, \tag{2.14}$$

$$E_l = \frac{A_l}{\Delta}(1 + \eta_{ms})(1 + \eta_{sc}), \tag{2.15}$$

where $\Delta = 1 + \eta_{ms}\eta_{sc}(\frac{r_c}{r_s})^{2l}$ and parameters

$$\eta_{ms} = \frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_s}, \qquad \eta_{sc} = \frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}, \tag{2.16}$$

characterize the jumps in permittivities across the boundary between the external medium and the shell and between the shell and the core respectively.

The purpose of that article was to determine when the coated cylinder is equivalent to a solid cylinder. Apparently it occurs when the relation between A_l and B_l is exactly the same as for a solid cylinder because this relationship defines the response of a coated cylinder to an external field. It is stated there that there are six special cases that such equivalence occurs. But two of them are even more special than the others:

$$\epsilon_s + \epsilon_c = 0, \tag{2.17}$$

$$\epsilon_s + \epsilon_m = 0. \tag{2.18}$$

Using the first equation, we can calculate the limit

$$B_l = \lim_{\epsilon_s \to -\epsilon_c} \frac{\frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_c} + \frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}(\frac{r_c}{r_s})^{2l}}{1 + \frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_s}\frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}(\frac{r_c}{r_s})^{2l}} r_s^{2l} A_l = \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} r_s^{2l} A_l \tag{2.19}$$

This means we can replace the coated cylinder by a solid cylinder with radius r_s and permittivity ϵ_c without changing the external potential V_e therefore the core properties are extended up to the outer boundary of the shell. Limit for the second equation is calculated in a similar way

$$B_l = \lim_{\epsilon_s \to -\epsilon_m} \frac{\frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_c} + \frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}(\frac{r_c}{r_s})^{2l}}{1 + \frac{\epsilon_m - \epsilon_s}{\epsilon_m + \epsilon_s}\frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}(\frac{r_c}{r_s})^{2l}} r_s^{2l} A_l = \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} a^{2l} A_l \tag{2.20}$$

where we denoted $a = \frac{r_s^2}{r_c}$. This relation tells us that the core properties are now extended up even beyond the shell ($a = \frac{r_s^2}{r_c} > r_s$) without disturbing the potential outside the radius a. Thus equivalent solid cylinder has radius a and permittivity ϵ_c. If both equations 2.17 and 2.18 hold then the coated cylinder can be replaced by the medium material without altering the external field.

These extraordinary properties are in that article called partial resonance. For one of the two special cases we place a line dipole at point z_0 on the positive half of the x axis (see Figure 2.6). The magnitude of that dipole is chosen in a way that

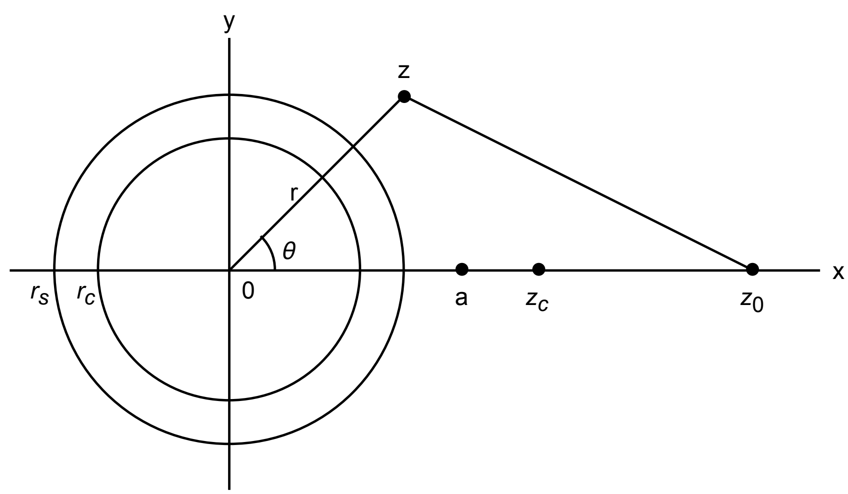

Figure 2.6: A coated cylinder in the electric field of a dipole placed at the point z_0 [28]

its potential is $\frac{1}{z-z_0}$ and may be expanded in z with coefficients $A_l = -(\frac{1}{z_0})^{l+1}$. Then for case $z_0 > z_c = \frac{a^2}{r_s} = \frac{r_s^3}{r_c^2}$

$$\tilde{V}_e(z) = \frac{1}{z - z_0} - \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} \frac{\frac{a^2}{z_0^2}}{z - \frac{a^2}{z_0}} \quad \text{for } |z| \geq r_s, \tag{2.21}$$

$$\tilde{V}_s(z) = -\frac{2\epsilon_c}{\epsilon_m + \epsilon_c} \frac{1}{z_0} + \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} \frac{\frac{r_s^2}{a^2}}{z - z_0 \frac{r_s^2}{a^2}} - \frac{\frac{r_s^2}{z_0^2}}{z - \frac{r_s^2}{z_0}} \quad \text{for } r_c \leq |z| \leq r_s, \tag{2.22}$$

$$\tilde{V}_e(z) = \frac{\epsilon_m - \epsilon_c}{\epsilon_m + \epsilon_c} \frac{1}{z_0} + \frac{2\epsilon_m}{\epsilon_m + \epsilon_c} \frac{\frac{r_s^2}{a^2}}{z - z_0 \frac{r_s^2}{a^2}} \quad \text{for } 0 \leq |z| \leq r_c \tag{2.23}$$

But if we take $z_0 < z_c$ then the ratio test shows that the series 2.9 does not converge for $r_s < r < \frac{a^2}{z_0}$ as well as the series 2.10 for $z_0 \frac{r_s^2}{a^2} < r < r_s$. There also appears two image dipoles (in later works they are called ghost sources) one in the medium at $\frac{a^2}{z_0} = r_{g_1}$ and one in the shell at $z_0 \frac{r_s^2}{a^2} = r_{g_2}$ where they produce unphysical singularities of V_e and V_s respectively. (A ghost source can appear not only in the medium and shell but also in the core as pointed out by authors later in [23]) An unpleasant consequence of this is that we have no physical solution of this problem when $\epsilon_s = -\epsilon_m$ and $z_0 < z_c$. This can be avoided if we add small imaginary part to the permittivity of the shell. In a physical way of speech the shell becomes lossy. Usually we take permittivity of the shell as

$$\epsilon_s = (-1 + i\delta)\epsilon_m \tag{2.24}$$

where $|\delta| \ll 1$. In this case the potential stays very close to the values give by 2.21 - 2.23 besides annulus between the two ghost sources $r_{g_1} \leq r \leq r_{g_2}$. The field outside this radius r_{g_2} converges to the field outside the equivalent solid cylinder. When we approach the ghost source from outside this radius, it looks like a true line source (in the limit when $\delta \to 0$). The last mentioned fact was discovered later in 2005 [23] as the first example of superlensing (see further in this section).

Of course with the existence of actual metamaterials this theory acquire more importance. In 2002 Milton wrote a book [20] about theory of composites where he mentioned in one small section this concept of localized resonance.

As mentioned in Section 2.1 Pendry's article [30] about superlenses started debate about whether such device is in principle possible to be created. Some numerical simulations and calculations were suggesting that it is not possible to achieve this this effect of perfect lenses for any dispersive lossy lens. However many other people suggest that the slab lenses could work when their thickness d is much smaller than wavelength of radiation in free space λ_0. The first mathematically correct proof of superlensing was given in 2005 in [23]. There was again used a quasistatic limit and the principle of anomalous localized resonance (this full name was used here for the first time) since permittivity was considered $\epsilon = -1 + i\epsilon''$ the imaginary part ϵ'' tend to zero.

Their result for a dipole line source being outside a coated cylinder is as follows: In the case when $\epsilon_m \neq \epsilon_c$ there is no ALR for $r_0 > r_{crit}$ in the limit $\delta \to 0$ but for smaller r_0 the potential becomes anomalously locally resonant in an annulus between two ghost sources one in the medium at r_{g_1} and the second one at r_{g_2} which can be in the shell (for r_0 between r_{crit} and a) or in the core (for r_0 between a and r_s), see Figure 2.7. In the case when $\epsilon_m = \epsilon_c$ there is no ALR for $r_0 > r_*$ in the limit $\delta \to 0$ but for r_0 between r_* and r_s the potential becomes anomalously locally resonant in two sometimes overlapping annuli. Outside the anomalously locally resonant regions the potential V converges to $\tilde{V}$ (for more detailed version of this theorem see [23]).

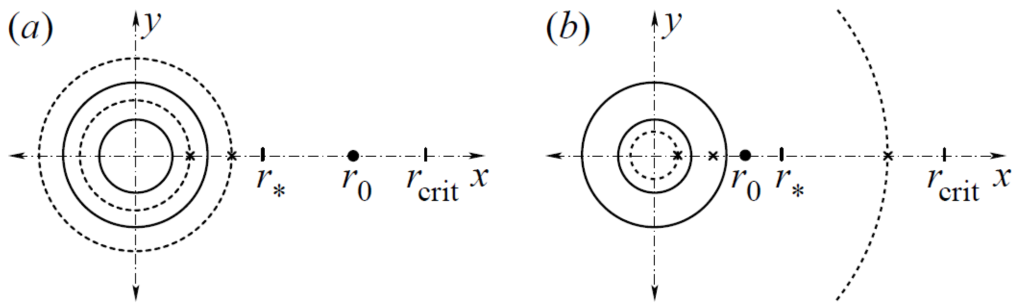

Figure 2.7: The location of ghost sources (marked by the crosses) depends on the position of r_0, namely when (a) r_0 is between r_* and r_{crit} or (b) when r_0 is between r_s and r_*. The solid circles represent the core and shell of the coated cylinder. Anomalously locally resonant region is marked by the dashed lines. [23]

They also considered another geometry than the one with the coated cylinder (now called cylindrical superlens), a slab lens. They had $\epsilon_m = \epsilon_c = 1$ and let r_s, r_c and r_0 tend to infinity while $d = r_s - r_c$ and $d_0 = r_0 - r_s$ were kept fixed. They came to results that there is no resonance as $\delta \to 0$ for $d_0 > d$ and the potential converges to the one that satisfies the properties of a superlens. But for $d_0 < d$ there occurs ALR in two sometimes overlapping layers of thickness $2(d - d_0)$. From that and private communication with Alexei Efros they deduce that in a simple case when $\epsilon_s = -1 + i\epsilon_s''$ the loss scales as $|\epsilon_s''|^{2(d_0/d)-1}|\log \epsilon_s''|$ which goes to zero when $d_0 > \frac{d}{2}$ but diverges to infinity when $d_0 < \frac{d}{2}$. This divergence occurs because in this case the source lie in the ALR region and they have to do increasing amount of work against the locally resonant field here.

For our project is more important article by Milton and Nicorovici from 2006 ([21]) about cloaking effects associated with anomalous localized resonance. They fluently continue in what they have done in [23] and use the results from there not for superlensing but for the invisibility effect. A quasistatic transverse magnetic (TM) field is considered where polarizable line with polarizability $\boldsymbol{\alpha}$ is placed. This TM field surrounds again a coated cylinder and the polarizable line is placed along $x = r_0 > r_s$, $y = 0$. Permittiivities of this setting are chosen to be $\epsilon_s \approx -\epsilon_m \approx -\epsilon_c$ where ϵ_m is assumed to be fixed, real and positive, ϵ_c should also remain fixed but can be complex with non-negative imaginary part and ϵ_s approaches $-\epsilon_m$ along the trajectory in the upper half of the complex plane as $\delta \to 0$, and this number δ is now taken from denotion

$$\frac{(\epsilon_s + \epsilon_c)(\epsilon_m + \epsilon_s)}{(\epsilon_s - \epsilon_c)(\epsilon_m - \epsilon_s)} = \delta e^{i\phi} \tag{2.25}$$

They introduced here two quantities: effective polarizability tenzor and effective source terms

$$\boldsymbol{\alpha}_* = [\boldsymbol{\alpha}^{-1} - c(\delta)\boldsymbol{I}]^{-1}, \quad \begin{pmatrix} k^e_* \\ -k^o_* \end{pmatrix} = [\boldsymbol{I} - c(\delta)\boldsymbol{\alpha}]^{-1} \begin{pmatrix} k^e \\ -k^o \end{pmatrix} \tag{2.26}$$

where $c(\delta)$ is exactly calculated in that article and it is shown that for $\delta \to 0$ this $c(\delta)$ tends to infinity. The quantities k^e and k^o are dipole moments of the polarizable line (k^e gives the amplitude of the dipole component with even symmetry about the x-axis and k^o the amplitude of the component with odd symmetry about the x-axis). If $|c(\delta)|$ is very large we have for the expressions (2.26) these estimates

$$\boldsymbol{\alpha}_* \approx \frac{-\boldsymbol{I}}{c(\delta)}, \quad \begin{pmatrix} k^e_* \\ -k^o_* \end{pmatrix} \approx \frac{-\boldsymbol{\alpha}^{-1}}{c(\delta)} \begin{pmatrix} k^e \\ -k^o \end{pmatrix} \tag{2.27}$$

Therefore both expression tend to zero as $\delta \to 0$ and that according to [21] explains why cloaking occurs.

Even for the slab lens the cloaking is shown. In fact if we have again lens of thickness d than it is proved here that a polarizable line dipole located less than $\frac{d}{2}$ from the lens would be cloaked. This is due to the presence of a resonant field in front of the lens which was shown already in [23].

These three articles became a starting point of the anomalous localized resonance and inspired many other sciences in this area. Let us mention some other work by these three authors, for example [27] with some nice simulations of cloaking which clarify its physical mechanism, or [22] where the core radius r_c of the coated cylinder is supposed to be bigger than the shell radius r_s. They call such setting a folded geometry and give it a physical meaning by transforming it to an equivalent problem in unfolded geometry. Milton also collaborated with Ammari, Ciraolo, Kang and Lee when they wrote [2], [3] and [1] to which Subsection 2.2.3 is dedicated.

2.2.2 Bouchitté, Schweizer

Another important moment for anomalous localized resonance come with [7]. It is one of the first articles where theory of operators was used. The operators investigated there are

$$\mathcal{L}^\eta = \nabla \cdot (a^\eta \nabla) \tag{2.28}$$

and interest of this article is in the solutions u^η of equation $\mathcal{L}^\eta u^\eta = 0$. The coefficient a^η is here for permittivity and it is defined as

$$a^\eta(x) = \begin{cases} -1 + i_0\eta & \text{for } x \in \Sigma \\ +1 & \text{for } x \in \mathbb{R}^2 \setminus \Sigma \end{cases} \tag{2.29}$$

Here $x \in \mathbb{R}^2$, $\eta > 0$, $i_0 \in \mathbb{C}, |i_0| = 1$ and $\operatorname{Im} i_0 > 0$. The set Σ denotes the ring $\Sigma = B_R(0) \setminus B_1(0)$, $R > 1$. We see that this setting is quite simplified than for example in [21]. Although the results will not be so general for the way how we treat this problem and the results we get is this simplification appropriate.

The theorems proved here simply say that there is the cloaking radius $R^* = \sqrt{\frac{R^3}{1}} = R^{3/2} > R$ (in [21] it was denoted as $r_\#$) which they are very closely related to. These are that if we choose a small ball $B_\epsilon(x_0)$ around a point $x_0 \in \mathbb{R}^2$ than we can get two results: First, if $x_0 \notin B_{R^*}(0)$ then a measurement of the whole setting shows there is no ring present (it is invisible). Second, if $x_0 \in B_{R^*}(0)$ then the measurement does not detect the ring but $B_\epsilon(x_0)$ either. For this theorems they introduce two numbers which express a measure for the visibility of dipole inclusion

$$\mathcal{M}_q^\eta = \left(\int_{\partial B_q(0)} |\partial_n v^\eta|^2 \right)^{1/2} \tag{2.30}$$

and a measure of how much the true solution differs from the comparision solution

$$\mathcal{N}_q^\eta(f) = \left(\int_{\partial B_q(0)} |\partial_n u^\eta - \partial_n u^*|^2 \right)^{1/2} \tag{2.31}$$

The results for these quantities depend strongly on the position of x_0 or better whether x_0 lies in the B_{R^*} or not.

Without these coefficients and any Dirichlet-to-Neumann maps we can simply study one limit which will indicate whether cloaking of the $B_\epsilon(x_0)$ occurs or not. In [7] they took the equation

$$\mathcal{L}^\eta u^\eta = 0 \tag{2.32}$$

whose solution in two dimensions (and in our radial geometry) can be found for fixed $k \in \mathbb{N}_0$ with the ansatz

$$u^\eta(x) = U(r)e^{ik\theta} \tag{2.33}$$

where $U : (0, \infty) \to \mathbb{C}$. To find solution of (2.32) (which are harmonic functions) they made the following ansatz for complex numbers $a, b, \alpha, \beta \in \mathbb{C}$

$$U_k(r) = \begin{cases} r^k & \text{for } r \leq 1 \\ ar^k + br^{-k} & \text{for } 1 < r \leq R \\ \alpha r^k + \beta r^{-k} & \text{for } R < r \end{cases} \tag{2.34}$$

In the case when $r \leq 1$ is no complex coefficient because we require the solution to be bounded in $x = 0$ (there would be a problem with term r^{-k}) and also we require normalizing to a unit monomial around 0. The four unkown coefficients a, b, α, β can be determined explicitly from conditions on continuity for u^η and $a^\eta \partial_r u^\eta$ in $r = 1$ and $r = R$. We then get four equations

$$1 = a + b \tag{2.35}$$

$$1 = Aa - Ab \tag{2.36}$$

$$aR^k + bR^{-k} = \alpha R^k + \beta R^{-k} \tag{2.37}$$

$$aAR^k + bAR^{-k} = \alpha R^k - \beta R^{-k} \tag{2.38}$$

Here they denoted $A = -1 + i_0\eta \in \mathbb{C}$. From these equations we express the wanted coefficients

$$a = \frac{A+1}{2A} \tag{2.39}$$

$$b = \frac{A-1}{2A} \tag{2.40}$$

$$\alpha = \frac{1}{4} R^{-k} \left[\frac{(1+A)^2}{A} R^k - \frac{(1-A)^2}{A} R^{-k} \right] \tag{2.41}$$

$$\beta = \frac{1}{4} R^{k} \left[\frac{1-A^2}{A} R^k - \frac{1-A^2}{A} R^{-k} \right] \tag{2.42}$$

They wanted to decide which term becomes dominant whether it is αr^k or βr^{-k}. For this they introduced another very important number called localization index

$$P_k^\eta = \frac{\beta}{\alpha} \in \mathbb{C} \tag{2.43}$$

It is now straightforward to calculate this localization index when he have exact forms of coefficients α, β

$$P_k^\eta = R^{2k} \frac{(1-A^2)(R^k - R^{-k})}{(1+A)^2 R^k - (1-A)^2 R^{-k}} \tag{2.44}$$

We can consider also negative k. The solution will be then given by the same equations and formulae but k will be replaced by $|k|$. To see the importance of cloaking radius R^* we need to create it somewhere in this expression. Also because we want to find out whether the term αr^k or βr^{-k} is dominant we take look at the number P_k^η / r^{2k}

$$\frac{P_k^\eta}{r^{2|k|}} = \left(\frac{R^*}{r}\right)^{2|k|} \frac{(2i_0\eta - i_0^2\eta^2)(1 - R^{-2|k|})}{i_0^2\eta^2 R|k| - (2 - i_0\eta)^2 R^{-|k|}} \tag{2.45}$$

The loss parameter is here η so we want to explore what happens if it tends to zero. In this case we get

$$\max_k \frac{|P_k^\eta|}{r^{2|k|}} \to 0 \quad \text{if } r > R^* \tag{2.46}$$

$$\max_k \frac{|P_k^\eta|}{r^{2|k|}} \to \infty \quad \text{if } r < R^* \tag{2.47}$$

The anomalous localized resonance is related to the fact that P_k^η can become very large. That is exactly what (2.47) says and then the true is that there is the dominance of the term β over the term α in case when $r < R^*$.

This localization index can be translated into a Dirichlet-to-Neumann operator. We recall the definition from [7] for fixed $r > R$ and the boundary $\Gamma = \partial_n u^\eta|_\Gamma$

$$N^{r,\eta} : H^{1/2}(\Gamma, \mathbb{C}) \to H^{-1/2}(\Gamma, \mathbb{C}), u^\eta \mapsto \partial_n u^\eta|_\Gamma \tag{2.48}$$

Here u^η is the solution of equation (2.32) in $B_r(0)$ and $n(x)$ is the exterior normal to $B_r(0)$. We know that $(e^{ik\theta})_{k\in\mathbb{Z}}$ is a basis of both $H^{\pm 1/2}(\Gamma, \mathbb{C})$. Hence we can describe $N^{r,\eta}$ with its Fourier components

$$N^{r,\eta}(e^{ik\theta}) = N_k^{r,\eta} \cdot e^{ik\theta} \tag{2.49}$$

The solution u^η is known to us from (2.33) and (2.34) so for $r > R$ it has the form $u^\eta = c(\alpha r^{|k|} + \beta r^{-|k|})e^{ik\theta}$. Therefore we can find connection between the localization index P_k^η and the Dirichlet-to-Neumann operator $N^{r,\eta}$ by (2.49) and

$$N_k^{r,\eta} = \frac{\partial_r u^\eta}{u^\eta}|_{\partial B_r(0)} = \frac{|k|}{r} \frac{1 - P_k^\eta r^{-2|k|}}{1 + P_k^\eta r^{-2|k|}} \tag{2.50}$$

2.2.3 Ammari, Ciraolo, Kang, Lee and others

Now we will mention some more recent articles dedicated to anomalous localized resonance. From these we would like to point out especially work by Ammari, Ciraolo, Kang, Lee and Milton who we are familiar with from Subsection 2.2.1. In [2] they considered a bounded domain $\Omega \in \mathbb{R}^2$ and then D is a domain whose closure is contained in Ω. With loss parameter δ they defined permittivity in $\mathbb{R}^2$ as

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus \bar{\Omega} \\ -1 + i\delta & \text{in } \Omega \setminus \bar{D} \\ 1 & \text{in } D \end{cases} \tag{2.51}$$

It is obvious that this geometry is more general than what we could see in Subsections 2.2.1 and 2.2.2 earlier in this thesis. But the setting with two concentric disks is also examined here as a special case of this more general one. Still we can think about D as a core with permittivity 1 surrounded by the shell $\Omega \setminus \bar{D}$ with permittivity $-1 + i\delta$.

The dielectric problem in $\mathbb{R}^2$ they dealt with was

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = \alpha f \tag{2.52}$$

Here αf is a source term where function f compactly supported in $\mathbb{R}^2$ satisfies physical condition of conservation of charge

$$\int_{\mathbb{R}^2} f dx = 0 \tag{2.53}$$

and V_δ fulfils the decay condition

$$\lim_{|x| \to \infty} V_\delta(x) = 0 \tag{2.54}$$

Now we should find those functions f in such a way that (when $\alpha = 1$)

$$E_\delta = \int_{\Omega \setminus \bar{D}} \delta |\nabla V_\delta|^2 dx \to \infty \quad \text{as } \delta \to 0 \tag{2.55}$$

$$|V_\delta(x)| < C \quad \text{when } |x| > a \tag{2.56}$$

The quantity E_δ from (2.55) is proportional to the electromagnetic power dissipated into heat. The second equation (2.56) tells that V_δ remains bounded by some constant C outside some radius a independent of δ. There is apparently unphysical situation because it results from (2.55) that amount of energy dissipated per unit time is infinite (in the limit $\delta \to 0$). Therefore we can choose $\alpha = 1/\sqrt{E_\delta}$ then the source αf produce the same power independent of α. The new associated solution of (2.52) is then $V_\delta/\sqrt{E_\delta}$ and it will approach zero outside the radius a. Such described case means that CALR occurs.

Now when the problem is stated let us mention the method used in [3] to deal with it. Their goal was to give a necessary and sufficient condition on the source term so the blow-up (2.55) takes place. For this they used techniques of layer potentials to reduce the dielectric problem to a singularly perturbed system of integral equations which is non-self-adjoint. The integral operators in this article are sometimes called Neumann-Poincaré operators (see [2] for details). A generalization of Calderón's identity is used to the non-self-adjoint system so they could express the solution in terms of the eigenfunctions of a self-adjoint compact operator.

Then they investigated well known case of an annulus (D and Ω are two concentric disks with radii r_i, r_e respectively) and found that there exists a cloaking radius $r_* = \sqrt{r_e^3 r_i^{-1}}$ such that any dipole sources places in the annulus $B_{r_*} \setminus \bar{B}_e$ is cloaked.

In the following article [3] they considered the same problem a little more generally and defined the permittivity distribution as

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^d \setminus \bar{\Omega} \\ -\epsilon_s + i\delta & \text{in } \Omega \setminus \bar{D} \\ \epsilon_c & \text{in } D \end{cases} \tag{2.57}$$

And what more they did not restrict themselves only to two-dimension problem but they examined what happens in three dimensions for radially symmetric structure as well. Using the same procedure as in [2] they proved that CALR occurs in two dimensions only if $\epsilon_s = -1$ (assuming that $\epsilon_c = -1$). For other values of ϵ_s CALR does not occur. In three dimensions CALR does not occur whatever ϵ_s and ϵ_c are.

In fact this non-occurrence of CALR in three dimensions is related to the fact that ϵ_s is constant. It was discovered in [1] that if they use a shell with a specially designed anisotropic permittivity then the CALR occurs for D and Ω two concentric balls in $\mathbb{R}^3$ with radii r_i and r_e respectively and chosse r_0 in the way that

$r_0 > r_e$. For a given loss parameter $\delta > 0$ they defined

$$\boldsymbol{\epsilon}_\delta(\boldsymbol{x}) = \begin{cases} \boldsymbol{I} & |\boldsymbol{x}| > r_e \\ (\epsilon_s + i\delta)a^{-1}(\boldsymbol{I} + \frac{b(b-2|\boldsymbol{x}|)}{|\boldsymbol{x}|^2}\hat{\boldsymbol{x}} \otimes \hat{\boldsymbol{x}}) & r_i < |\boldsymbol{x}| < r_e \\ \epsilon_c\sqrt{\frac{r_0}{r_i}}\boldsymbol{I} & |\boldsymbol{x}| < r_i \end{cases} \tag{2.58}$$

where $\boldsymbol{I}$ is the 3×3 identity matrix, ϵ_s, ϵ_c are constants, $\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$, $a = \frac{r_e - r_i}{r_0 - r_e} > 0$ and $b = (1+a)r_e$. Such $\boldsymbol{\epsilon}_d$ is anisotropic and variable in the shell and it is designed by unfolding a folded geometry (this concept was briefly introduced in Subsection 2.2.1).

There are much more interesting articles about cloaking via anomalous localized resonance but it is not possible to talk about all of them. Let us mention some of the most recent ones like for example [8] where they use the spectral analysis of the Neumann-Poincaré type operator on confocal ellipses to prove that CALR takes place in such setting. Another spectral analysis of Neumann-Poincaré operator is made in [4]. Resonance at eigenvalues and at the essential spectrum is investigated there. It is shown that the resonance at the essential spectrum is weaker than at the eigenvalues. Proof is made here that anomalou localized resonance occurs at the essential spectrum on ellipses (in $\mathbb{R}^2$) but does not occur on three dimensional balls. Among articles dealing with this three dimensional problem we can mention for example [18], [25], [26] that all are very recent (2015).

Chapter 3

ALR on the ball in three and higher dimensions

As was mentioned in Chapter 2 there has already been given a proof that CALR does not occur for three-dimensional ball in [3]. Nevertheless in the geometry with metamaterial slab lens it is known that CALR occurs in three dimensions with a single dipolar source [21].

3.1 A proof of non-occurrence of ALR

Now we use the procedure from [7] described in Subsection 2.2.2 to give our own proof of non-occurrence of ALR on ball in three dimensions and after that this will be generalized for higher dimensions.

Let us rewrite once again equation (1.4)

$$\vec{\nabla} \cdot (\epsilon_\delta(x) \vec{\nabla} \Psi(x)) = 0 \tag{3.1}$$

with permittivity

$$\epsilon_\delta(x) = \begin{cases} +1, & \text{for } x \in B_1(0) \\ -1 + i\delta, & \text{for } x \in \Sigma \\ +1, & \text{for } x \in \mathbb{R}^d \setminus B_R(0) \end{cases} \tag{3.2}$$

where we denoted Σ as an annulus $B_R(0) \setminus B_1(0)$ for $R > 1$. d is here for the dimension of space and $\delta > 0$ as before. Since the permittivity (3.2) is constant on the three regions we are interested in the solution of Laplace equation

$$\Delta\Psi(x) = 0 \tag{3.3}$$

If we suppose that we can write solution of this equation in separated form as

$$\Psi(x) = \psi(r)\phi(\Omega) \tag{3.4}$$

where r denotes radial coordinate and Ω is for angular coordinates then we can write down boundary conditions for the solution as

$$\begin{aligned} \psi(1^-) &= \psi(1^+) \\ \psi(R^-) &= \psi(R^+) \\ \epsilon_\delta(1^-)\frac{d}{dr}\psi(1^-) &= \epsilon_\delta(1^+)\frac{d}{dr}\psi(1^+) \\ \epsilon_\delta(R^-)\frac{d}{dr}\psi(R^-) &= \epsilon_\delta(R^+)\frac{d}{dr}\psi(R^+) \end{aligned} \tag{3.5}$$

To get Laplace equation (3.3) in a radial form we at first take a look at Laplace operator $-\Delta$ which can be written in d dimensions as

$$-\Delta = -\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\Delta_{S^{d-1}} \tag{3.6}$$

where $-\Delta_{S^{d-1}}$ is Laplace-Beltrami operator. Its spectrum is well known $\sigma(-\Delta_{S^{d-1}}) = \{l(d-2+l)\}_{l=0}^{\infty}$ and therefore we can write Laplace operator in a form

$$-\Delta = \bigoplus_{l=0}^{\infty}\bigoplus_{k=-l}^{l}\left(-\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) + \frac{l(d-2+l)}{r^2}\right) \tag{3.7}$$

Hence Laplace equation (3.3) in radial form is

$$-\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\psi(r)\right) + \frac{l(d-2+l)}{r^2}\psi(r) = 0 \tag{3.8}$$

It is easy to check that functions $\psi_1(r) = r^l$ and $\psi_2(r) = r^{-(d-2+l)}$ are solutions to equation (3.8). In case of ψ_1 we have

$$\begin{aligned} -\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\psi_1(r)\right) + r^{d-3}l(d-2+l)\psi_1(r) &= \\ = -\frac{d}{dr}\left(r^{d-1}lr^{l-1}\right) + r^{d-3}l(d-2+l)r^l &= \\ = -l(d-2+l)r^{d+3+l} + l(d-2+l)r^{d-3+l} &= 0 \end{aligned}$$

and in case of ψ_2 we get in the same way

$$\begin{aligned}-\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\psi_2(r)\right)+r^{d-3}l(d-2+l)\psi_2(r)=\\=-\frac{d}{dr}\left(r^{d-1}(-d+2-l)r^{-d+1-l}\right)+r^{d-3}l(d-2+l)r^{-d+2-l}=\\=-l(d-2+l)r^{-l-1}+l(d-2+l)r^{-l-1}=0\end{aligned}$$

Therefore we can express the solution of our equation (3.8) as a superposition of ψ_1 and ψ_2

$$\psi(r)=\sum_{l=0}^{+\infty}\left(A_l(r)r^l+B_l(r)r^{-l+2-d}\right) \tag{3.9}$$

Let us note here that the coefficients are constants in each of the three regions $B_1(0), \Sigma$ and $\mathbb{R}^d \setminus B_R(0)$. For fixed $l \in \mathbb{N}_0$ we can denote one addend of the sum above as

$$\psi_l(r)=\begin{cases}r^l & \text{for } r\leq 1\\ a_lr^l+b_lr^{-l+2-d} & \text{for } 1<r\leq R\\ \alpha_lr^l+\beta_lr^{-l+2-d} & \text{for } R<r\end{cases} \tag{3.10}$$

We will be focused now on the case when $d=3$. Using boundary conditions (3.5) we get four equations for four coefficients $a_l, b_l, \alpha_l, \beta_l$

$$\begin{aligned}1&=a_l+b_l\\ a_lR^l+b_lR^{-l-1}&=\alpha_lR^l+\beta_lR^{-l-1}\\ l&=A_\delta(a_ll+b_l(-l-1))\\ A_\delta(a_llR^{l-1}+b_l(-l-1)R^{-l-2})&=\alpha_llR^{l-1}+\beta_l(-l-1)R^{-l-2}\end{aligned} \tag{3.11}$$

here we denoted $A_\delta=-1+i\delta$ similarly as in the [7]. From this system of equation it is possible to achieve the coefficients

$$\begin{aligned}a_l&=\frac{A_\delta l+A_\delta+l}{A_\delta(2l+1)}\\ b_l&=\frac{l(A_\delta-1)}{A_\delta(2l+1)}\\ \alpha_l&=\frac{[l^2(A_\delta+1)^2+l(A_\delta+1)^2+A_\delta]R^{2l+1}-l(l+1)(A_\delta-1)^2}{A_\delta(2l+1)^2R^{2l+1}}\\ \beta_l&=\frac{l(A_\delta+l-A_\delta^2l-A_\delta^2)(R^{2l+1}-1)}{A_\delta(2l+1)^2}\end{aligned} \tag{3.12}$$

As in [7] we want to find which term in our solution (in case $r > R$) is dominant. Therefore we are interested in a ratio

$$\frac{\beta_l r^{-l-1}}{\alpha_l r^l} = \frac{1}{r^{2l+1}} \frac{\beta_l}{\alpha_l} = \frac{P_l^\delta}{r^{2l+1}} \tag{3.13}$$

where we denoted localization index again as β_l/α_l. Since we know there will be no cloaking for $r > R$ we are interested in the calculation of expression

$$\lim_{\delta\to 0} \max_l \frac{|P_l^\delta|}{r^{2l+1}} \tag{3.14}$$

and we expect this limit to be 0.

To find maximum of $|P_\delta^l|/r^{2l+1}$ in l we need to examine the behaviour of such function (we denote it $g(l)$ for short). The explicit formula for this can be calculated

$$g(l) = \left(\frac{R}{r}\right)^{2l+1} \frac{l\sqrt{4+\delta^2}\sqrt{1+\delta^2(1+l)^2}(R^{2l+1}-1)}{\sqrt{\begin{aligned}&16l^2+32l^3+16l^4+8l^2R^{2l+1}+R^{4l+2}+8lR^{2l+1}+8\delta^2l^2+\\&16\delta^2l^3+8\delta^2l^4+6\delta^2lR^{2l+1}+14\delta^2l^2R^{2l+1}+16\delta^2l^3R^{2l+1}+\\&8\delta^2l^4R^{2l+1}+\delta^2R^{4l+2}+2\delta^2lR^{4l+2}+2\delta^2l^2R^{4l+2}+\delta^4l^2+\\&2\delta^4l^3-\delta^4l^4+2\delta^4l^2R^{2l+1}-4\delta^4l^3R^{2l+1}+2\delta^4l^4R^{2l+1}+\\&\delta^4l^2R^{4l+2}+2\delta^4l^3R^{4l+2}+\delta^4l^4R^{4l+2}\end{aligned}}} \tag{3.15}$$

Because of l in the numerator it is obvious that $g(0) = 0$ and now we want to see what happens in the infinity

$$\lim_{l\to+\infty} g(l) = \lim_{l\to+\infty} \left(\frac{R}{r}\right)^{2l+1} \frac{l^2R^{2l+1}}{l^2R^{2l+1}} \frac{\sqrt{4+\delta^2}\sqrt{\frac{1}{l^2}+\delta^2\left(\frac{1}{l}+1\right)^2}\left(1-\frac{1}{R^{2l+1}}\right)}{\sqrt{o(1)+\delta^4}} \tag{3.16}$$

and this limit is zero since $r > R$. Because this function is smooth and positive on $\langle 0, +\infty)$ and its values on borders are zero it must have a maximum somewhere in this interval. To find it we will make some estimates on function g.

First we want to enlarge the numerator in (3.15). For this we can estimate $\sqrt{4+\delta^2}\sqrt{1+\delta^2(1+l)^2}$ by $C(1+l)$ and the last bracket easily as $R^{2l+1}-1 < R^{2l+1}$. To achieve maximum values of g we need to reduce its denominator. It is easy to see that the whole square root is greater than $\sqrt{R^{4l+2}(1+\delta^4 l^4)}$ and that is obviously greater than $R^{2l+1}(1+\delta^2 l^2)$. Thus meanwhile we have

$$g(l) \leq \alpha^{2l+1} \frac{lC(1+l)R^{2l+1}}{R^{2l+1}(1+\delta^2 l^2)} = \alpha^{2l+1} \frac{lC(1+l)}{1+\delta^2 l^2} \tag{3.17}$$

Crucial is now behaviour of function α^{2l+1}. For all l greater than some $l_0 > 0$ we can make an estimate that $\alpha^{2l+1} \leq \frac{1}{l^2}$ and then

$$g(l) \leq \frac{1}{l^2} \frac{lC(1+l)}{1+\delta^2 l^2} \leq C \frac{1}{1+\delta^2 l^2} \leq C \frac{1}{1+\delta^2 l_0^2} \tag{3.18}$$

and therefore l_0 is the point in which function g is maximal and its maximum is written above on the right side of estimates (3.18). To include also points in the vicinity of 0 we can take interval $\langle 0, 1\rangle$ where we can estimate α^{2l+1} by constant value 1. Then we have

$$g(l) \leq \frac{lC(1+l)}{1+\delta^2 l^2} \leq \frac{2C}{1+\delta^2} \tag{3.19}$$

so in this case the function g is maximal at point $l = 1$ and its maximal value is written above on the right side of estimates (3.19).

We can see that if δ tends to zero in (3.18) or (3.18) we get zero for the limit (3.14). Therefore we proved that there occurs no ALR on three dimensional ball.

The situation in higher dimensions is more or less the same. This time we are interested in ratio

$$\frac{\beta_l r^{-l+2-d}}{\alpha_l r^l} = \frac{P_l^\delta}{r^{2l+d-2}} \tag{3.20}$$

Calculation of the function $f(l) = |P_l^\delta|/r^{2l+d-2}$ in now a little more tedious but with the same progress as in three dimension we prove again that there occurs no limit on d-dimensional ball, $d > 3$, i.e.

$$\lim_{\delta\to 0} \max_l \frac{|P_l^\delta|}{r^{2l+1}} = 0 \tag{3.21}$$

Chapter 4

Conclusions

In this thesis we have tried to get familiar with history behind metamaterial cloaking. We mentioned some of the pioneers and leader scientists in this area and shortly described their research and results. A special attention was paid to the concept of anomalous localized resonance but there are many other possible ways how to treat with the invisibility due to the devices with negative permittivity and permeability. In the last chapter we applied a calculation from [7] and proved that there is no cloaking on ball in three and higher dimensions.

The proved non-occurrence of CALR in such geometry could be explained in a way that the ball in 2D is not simply connected but in higher dimensions it is. Therefore this implies that cloaking is tightly connected to the topology of the system.

There are still some issues connected to the metamaterial cloaking. The adding small imaginary positive imaginary part into the permittivity of metamaterial is quite useful so we could calculate this problem but we would like to see the solution without this perturbation in permittivity. Until now the only article dealing with cloaking in this way is [5] for rectangle in two dimensions. Natural extension of this could be solving this problem in higher dimensions and so generalize its results.

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