DISCRETIZATION AND ORTHOGONALITY RELATIONS OF ORBIT FUNCTIONS OF WEYL GROUPS

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Abstract. The present work summarizes and further develops properties of orbit functions. Orbit functions, based on sign homomorphism of Weyl groups, are defined. For each Weyl group of two different root lenghts there are four sign homomorphisms, which lead to ten types of orbit functions (E-,S-,C-functions). Proof of their continuous orthogonality and completness relation on the fundamental domain is given. In the key part of the work, we discretize the fundamental domains and prove the discrete orthogonality relation and completness relation of orbit functions on the discretized fundamental domain.

1. Introduction

Orbit functions are special functions corresponding to Weyl groups and their root systems. Orbit functions were first introduced in [20], later called C-functions. In [21], E-functions and S-functions are introduced. In [3] sign homomorphisms i.e. $\sigma : W \to \{\pm 1\}$ are introduced and used to define six types of E-functions. In [15] it is proven that only four sign homomorphisms exist for root systems with two different lengths of roots. So together there is one C-function, three S-functions and six E-functions.

The discretization of C-functions was done in [17], [16]. In [14] this was extended to one type of S-function. The discretization of all types S-functions was done in [5]. The discretization of one type of E-function was done in [6] The method for calculating the of elements of the discretized domain was devised in [7].

In the present work, by using elegant mathematical notations, we unite all ten types orbit functions. For orbit function $\Psi_\lambda^{\sigma,\widetilde{\sigma}}$ corresponding to sign homomorphisms $\sigma, \widetilde{\sigma}$ and weight λ from the modified weight lattice $P^{\sigma,\widetilde{\sigma}}$ we find and define its domain $F^{\sigma,\widetilde{\sigma}}$. We prove that the set orbit functions $\left\{\Psi_\lambda^{\sigma,\widetilde{\sigma}} \mid \lambda \in P^{\sigma,\widetilde{\sigma}}\right\}$ form a orthogonal basis in the space $\mathcal{L}^2(F^{\sigma,\widetilde{\sigma}})$.

In the key part of the work, we discretize the domain $F^{\sigma,\widetilde{\sigma}}$ into the domain grid $F_M^{\sigma,\widetilde{\sigma}}$, where M is an integer defining the density of the grid. We prove that the set of discretized orbit functions defined on the domain grid $F_M^{\sigma,\widetilde{\sigma}}$ labeled by finite modified weight grid $\Lambda_M^{\sigma,\widetilde{\sigma}}$ $\left\{\Psi_\lambda^{\sigma,\widetilde{\sigma}} \mid \lambda \in \Lambda_M^{\sigma,\widetilde{\sigma}}\right\}$ form an orthogonal basis of dimension of number of elements of $\Lambda_M^{\sigma,\widetilde{\sigma}}$. In order to verify the completness of the discretized orbit functions we need to comapre the number of points in the domain grid $F_M^{\sigma,\widetilde{\sigma}}$ and the number of elements of the weight grid $\Lambda_M^{\sigma,\widetilde{\sigma}}$.

In section 2 the properties of Weyl groups and their root systems are reviewed. We study the properties of sign homomorphisms and define even subgroups and affine even subgroups in section 3. In section 4 fundamental domains of even subgroups and affine even subgroups are constructed. The same is applied analogously to the dual Lie algebra in section 5. In section 6 we define the modified weight lattice $P^{\sigma,\widetilde{\sigma}}$ and the domain of orbit function $F^{\sigma,\widetilde{\sigma}}$ and prove that set $\left\{\Psi_\lambda^{\sigma,\widetilde{\sigma}} \mid \lambda \in P^{\sigma,\widetilde{\sigma}}\right\}$ forms an orthogonal basis. The discretization is done in section 6. The explicit formula for the number of elements of the discretized domain is given. The discrete othogonality relation is proven. We show that number of points in the domain grid $F_M^{\sigma,\widetilde{\sigma}}$ and the number of elements of the weight grid $\Lambda_M^{\sigma,\widetilde{\sigma}}$ are equal. Summarization and comments are found in the last section.

Date: September 2, 2014.

2. Properties of simple Lie groups and their Lie algebras

2.1. **Basic Properties.**

Consider the set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of a Lie algebra of a compact simple Lie group of rank n with two different lenghts of roots, spanning the Euclidean space $\mathbb{R}^n$ [1, 8, 22]. The set of simple roots consists of short simple roots Δ_s and long simple roots Δ_l i.e.

$$\Delta = \Delta_s \cup \Delta_l.$$

A number of related quantities and virtually all the properties of the Lie group are determined from Δ. We use the following standard properties.

The highest root ξ of the root system can be written as follows

$$\xi \equiv -\alpha_0 = m_1\alpha_1 + \dots + m_n\alpha_n.$$

The coefficients m_j are known positive integers, also called marks. The Coxeter number m is defined as

$$m = 1 + m_1 + \dots + m_n.$$

The elements of the Cartan matrix C of the Lie algebra are

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \quad i, j \in \{1, \dots, n\}.$$

The order c of the center of the Lie group is the determinant of the Cartan matrix i.e.

$$c = \det C.$$

The positive dual weight lattice $P^{+\vee}$ is defined as

$$P^{+\vee} = \mathbb{Z}_0^+\omega_1^\vee + \dots + \mathbb{Z}_0^+\omega_n^\vee.$$

The root lattice Q is defined as

$$Q = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n.$$

The dual root lattice is defined as

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee, \quad \text{where} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}.$$

2.2. **Weyl group and affine Weyl group.**

The properties of Weyl groups and affine Weyl groups can be found for example in [9, 2]. The finite Weyl group W is generated by n reflections r_α, $\alpha \in \Delta$, in $(n-1)$-dimensional 'mirrors' orthogonal to simple roots intersecting at the origin:

$$r_{\alpha_i}\chi \equiv r_i\chi = \chi - \frac{2\langle \chi, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i, \qquad \chi \in \mathbb{R}^n.$$

The Fundamental Domain D of a Weyl group W is

$$\begin{aligned} D &= \{\chi \in \mathbb{R}^n \mid (\forall i \in \{1, \cdots, n\})(\langle \chi, \alpha_i \rangle \geq 0)\} \\ &= \left\{ y_1\omega_1^\vee + \dots + y_n\omega_n^\vee \mid y_1, \dots, y_n \in \mathbb{R}^{\geq 0} \right\}. \end{aligned} \tag{1}$$

The infinite affine Weyl group W^{aff} is the semidirect product of the Abelian group of translations $Q^\vee$ and of the Weyl group W

$$W^{\text{aff}} = Q^\vee \rtimes W.$$

Thefore

$$(\forall w^{\text{aff}} \in W^{\text{aff}})(\exists!\, w \in W)(\exists!\, q^\vee \in Q^\vee)(\forall \chi \in \mathbb{R}^n)(w^{\text{aff}}\chi = w\chi + q^\vee).$$

We will sometimes denote the elements from W^{aff} as $(w, q^\vee)$ defined as

$$(\forall \chi \in \mathbb{R}^n)((w, q^\vee)\chi = w\chi + q^\vee)$$

We define the retraction homomorphism $\psi : W^{\text{aff}} \mapsto W$ as

$$(\forall (w, q^\vee) \in W^{\text{aff}})(\psi(w, q^\vee) = w).$$

Equivalently, W^{aff} is generated by reflections r_i and reflection r_0, where

$$r_0\alpha = r_\xi\alpha + \frac{2\xi}{\langle \xi, \xi \rangle}, \qquad r_\xi\alpha = \alpha - \frac{2\langle \alpha, \xi\rangle}{\langle \xi, \xi \rangle}\xi, \qquad \alpha \in \mathbb{R}^n.$$

The fundamental region F of W^{aff} is

$$F = \{\chi \in D \mid \langle \xi, \chi \rangle \leq 1\}.$$

Equivalently, F is the convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}$:

$$\begin{aligned} F &= \left\{y_1\omega_1^\vee + \dots + y_n\omega_n^\vee \mid y_0, \dots, y_n \in \mathbb{R}^{\geq 0}, y_0 + y_1 m_1 + \dots + y_n m_n = 1\right\} \\ &= \{\chi \in \mathbb{R}^n \mid \langle \chi, \alpha \rangle \geq 0, \forall \alpha \in \Delta, \langle \chi, \xi \rangle \leq 1\}. \end{aligned} \tag{2}$$

The set of reflections $r_1 \equiv r_{\alpha_1}, \cdots, r_n \equiv r_{\alpha_n}$ is denoted by

$$S = \{r_1, \cdots, r_n\}.$$

The set of reflections together with r_0 is denoted by

$$R = S \cup \{r_0\}.$$

3. Sign homomorphisms, even subgroups, sign coxeter numbers

3.1. **Sign Homomorphisms.**

A homomorphism $\sigma : W \longmapsto \{1, -1\}$, where $\{1, -1\}$ is the multiplicative group containing elements 1 and -1 is a **sign homomorphism**. Two obvious choices of sign homomorphisms are the trivial homomorphism and determinant denoted as

$$\begin{aligned} &(\forall w \in W)(\sigma_e(w) = \det(w)), \\ &(\forall w \in W)(\mathbb{1}(w) = 1). \end{aligned}$$

Consider a Weyl group W with a simple system Δ. Since simple reflections generate W, it is sufficient to define a sign homomorphism on simple reflections S. The set of 'negative' generators of the Weyl group W of the sign homomorphism σ is denoted by S^σ i.e.

$$\sigma(r) = \begin{cases} -1 & \text{if } r \in S^\sigma, \\ 1 & \text{otherwise.} \end{cases}$$

The sign homomorphisms σ_l and σ_s are defined on the set of generators $S = \{r_\alpha \mid \alpha \in \Delta\}$ as

$$\sigma_s(r_\alpha) = \begin{cases} -1 & \text{if } \alpha \in \Delta^s, \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma_l(r_\alpha) = \begin{cases} -1 & \text{if } \alpha \in \Delta^l, \\ 1 & \text{otherwise.} \end{cases}$$

The sets S^s and S^l denote the following

$$\begin{aligned} S^s &\equiv S^{\sigma_s} = \{r_\alpha \mid \alpha \in \Delta_s\}, \\ S^l &\equiv S^{\sigma_l} = \{r_\alpha \mid \alpha \in \Delta_l\}. \end{aligned}$$

The set S^{σ_e} is the same as the set of all simple relfections i.e.

$$S^{\sigma_e} = S.$$

The Weyl group W is isomorphic to a normal subgroup of W^{aff}

$$W \cong (W, \mathbb{0}) \equiv \left\{(w, 0) \in W^{\mathrm{aff}} \mid w \in W\right\} \triangleleft W^{\mathrm{aff}}.$$

We will abuse notation and we will not distinguish these groups. Therefore any sign homomorphism σ is also defined on the subgroup $(W, \mathbb{0})$ of the affine Weyl group. Now we naturally expand the domain of the sign homomophism to the affine Weyl group by defining the sign of the last generator r_0 as the sign of r_ξ. The set of 'negative' generators of the affine Weyl group W^{aff} of the sign homomorphism σ is denoted by R^σ

$$R^\sigma = \begin{cases} S^\sigma & \text{if } \sigma(r_\xi) = 1, \\ S^\sigma \cup \{r_0\} & \text{if } \sigma(r_\xi) = -1. \end{cases}$$

Equivavently, the sign homomorphism can be expanded to the affine Weyl group using the retraction homomorphism ψ

$$(\forall (w, q^\vee) \in W^{\text{aff}})(\sigma(w, q^\vee) \equiv \sigma(\psi(w, q^\vee)) = \sigma(w)).$$

Again, we will abuse notation will not distinguish these isomorphic groups

$$W^{\text{aff}}/\ker\psi \cong \psi(W^{\text{aff}}) \cong (W, \mathbb{O}) \cong W.$$

Therefore we will also not distinguish between the sign homomorphism and its expanded version

$$\sigma \circ \psi \equiv \sigma.$$

For the sign homomophisms σ_e, σ_s, σ_l

$$\sigma_s(r_0) \equiv \sigma_s(r_\xi) = 1,$$
$$\sigma_l(r_0) \equiv \sigma_l(r_\xi) = -1,$$
$$\sigma_e(r_0) \equiv \sigma_e(r_\xi) = -1.$$

The sets R^s and R^l denote the following

$$R^s \equiv R^{\sigma_s} = S^s,$$
$$R^l \equiv R^{\sigma_l} = S^l \cup \{r_0\}.$$

The set R^{σ_e} is the same as the set of all generators

$$R^{\sigma_e} = R.$$

3.2. **Abelian group of sign homomorphisms.**

We define the group of all sign homomorphisms Σ with the operation $\cdot$ defined as

$$(\forall w \in W^{\text{aff}})((\sigma_1 \cdot \sigma_2)(w) = \sigma_1(w)\sigma_2(w)).$$

Evidently the group Σ is abelian and the neutral element in this group is the trivial sign homomorphism $\mathbb{1}$.

Proposition 3.2.1. The set of sign homomorphisms Σ of a Weyl group W with two different lengths of roots contains only sign homomorphisms $\mathbb{1}$, σ_e, σ_s, σ_l,

$$\Sigma = \{\mathbb{1}, \sigma_e, \sigma_s, \sigma_l\}.$$

Proof. (1) B_n :

The simple reflections S_{B_n} hold the following relations:

$$(r_i r_{i+1})^3 = 1 \quad \text{for } i \in \{1, \cdots, n-2\}$$
$$(r_i r_j)^2 = 1 \quad \text{for } i, j \in \{1, \cdots, n\} \text{ and } |i - j| \neq 1$$
$$(r_n r_{n-1})^4 = 1$$

If we apply an arbitrary sign homomorphism σ on these equations we get

$$\sigma(r_i)^3 \sigma(r_{i+1})^3 = 1 \quad \text{for } i \in \{1, \cdots, n-2\}$$
$$\sigma(r_i)^2 \sigma(r_j)^2 = 1 \quad \text{for } i, j \in \{1, \cdots, n\} \text{ and } |i - j| \neq 1$$
$$\sigma(r_n)^4 \sigma(r_{n-1})^4 = 1$$

The last two equations hold trivially for any sign homomorphism. From the first equation we get the four wanted sign homorphisms:

(a) $\sigma(r_1) = \sigma(r_2), \cdots, = \sigma(r_{n-1}) = 1$
  (i) $\sigma(r_n) = 1$ we get $\sigma = \mathbb{1}$,
  (ii) $\sigma(r_n) = -1$ we get $\sigma = \sigma_s$.

   (b) $\sigma(r_1) = \sigma(r_2), \cdots, = \sigma(r_{n-1}) = -1$
      (i) $\sigma(r_n) = 1$ we get $\sigma = \sigma_l$,
      (ii) $\sigma(r_n) = -1$ we get $\sigma = \sigma_e$.

(2) C_n :
Analogous to B_n.

(3) G_2 :
For the simple reflections the following relation holds:

$$(r_1 r_2)^6 = 1.$$

If we apply an arbitrary sign homomorphism σ on this equation we get

$$\sigma(r_1)^6 \sigma(r_2)^6 = 1.$$

This equation holds trivially for any sign homomorphism. To get the four wanted sign homorphisms:
   (a) $\sigma(r_1) = \sigma(r_2) = 1$ then $\sigma = \mathbb{1}$,
   (b) $\sigma(r_1) = \sigma(r_2) = -1$ then $\sigma = \sigma_e$,
   (c) $\sigma(r_1) = -1$ and $\sigma(r_2) = 1$ then $\sigma = \sigma_l$,
   (d) $\sigma(r_1) = 1$ and $\sigma(r_2) = -1$ then $\sigma = \sigma_s$.

(4) F_4 :
For the simple reflections the following relations hold

$$\begin{aligned}
(r_1 r_2)^3 &= 1, \\
(r_2 r_3)^4 &= 1, \\
(r_3 r_4)^3 &= 1, \\
(r_2 r_4)^2 &= 1, \\
(r_1 r_3)^2 &= 1, \\
(r_1 r_4)^2 &= 1.
\end{aligned}$$

If we apply an arbitrary sign homomorphism σ on these equations we get

$$\begin{aligned}
\sigma(r_1)^3 \sigma(r_2)^3 &= 1, \\
\sigma(r_2)^4 \sigma(r_3)^4 &= 1, \\
\sigma(r_3)^3 \sigma(r_4)^3 &= 1, \\
\sigma(r_2)^2 \sigma(r_4)^2 &= 1, \\
\sigma(r_1)^2 \sigma(r_3)^2 &= 1, \\
\sigma(r_1)^2 \sigma(r_4)^2 &= 1.
\end{aligned}$$

Only the first and the third equations are non-trivial for any sign homomorphism. Thus, we get the four wanted sign homorphisms:
   (a) $\sigma(r_1) = \sigma(r_2) = 1$
      (i) $\sigma(r_3) = \sigma(r_4) = 1$ we get $\sigma = \mathbb{1}$,
      (ii) $\sigma(r_3) = \sigma(r_4) = -1$ we get $\sigma = \sigma_s$.
   (b) $\sigma(r_1) = \sigma(r_2) = -1$
      (i) $\sigma(r_3) = \sigma(r_4) = 1$ we get $\sigma = \sigma_l$,
      (ii) $\sigma(r_3) = \sigma(r_4) = -1$ we get $\sigma = \sigma_e$.

□

3.3. **Even subgroups and affine even subgroups.**
Kernels of non-trivial sign homomorphisms of the Weyl group W are **even subgroups**. Denoted as

$$W^\sigma \equiv \{w \in W \mid \sigma(w) = 1\}.$$

The corresponding affine even subgroup is the kernel of the expanded sign homomorphism or equivalently the semidirect product of the group of translation $Q^\vee$ and the even subgroup W^σ i.e.

$$W^{\mathrm{aff}}_\sigma \equiv \left\{w^{\mathrm{aff}} \in W^{\mathrm{aff}} \mid \sigma(w^{\mathrm{aff}}) = 1\right\} = Q^\vee \rtimes W^\sigma.$$

Even subgroups W^e, W^s, W^l are the kernels of sign homomorphisms σ_e, σ_s, σ_l, respectively,

$$W^e = \{w \in W \mid \sigma_e(w) = 1\}, \quad W^s = \{w \in W \mid \sigma_s(w) = 1\}, \quad W^l = \{w \in W \mid \sigma_l(w) = 1\}.$$

Their corresponding affine groups are

$$W^{\mathrm{aff}}_e = Q^\vee \rtimes W^e, \quad W^{\mathrm{aff}}_s = Q^\vee \rtimes W^s, \quad W^{\mathrm{aff}}_l = Q^\vee \rtimes W^l.$$

3.4. **Sign Coxeter number.**
We define sign Coxeter number m^σ as

$$m^\sigma = \sum_{r_i \in R^\sigma} m_i.$$

We define zero index Coxeter number as $m_0 = 1$. The long and short Coxeter numbers are defined as

$$m^s \equiv m^{\sigma_s} = \sum_{r_i \in R^s} m_i,$$
$$m^l \equiv m^{\sigma_l} = \sum_{r_i \in R^l} m_i.$$

Their sum gives the Coxeter number

$$m = m^s + m^l.$$

The even Coxeter number m^{σ_e} is the same as the Coxeter number

$$m^{\sigma_e} = m.$$

4. Fundamental Domain Theorems

4.1. **Known Properties.**
The set D is the fundamental domain of W, i.e.

Proposition 4.1.1.

(1) $WD = \mathbb{R}^n$
(2) $(\forall \chi, \widetilde{\chi} \in D)(\exists w \in W)(\widetilde{\chi} = w\chi) \Rightarrow (\widetilde{\chi} = \chi = w\chi)$
(3) Consider a point $\chi = y_1\omega_1^\vee + \cdots + y_n\omega_n^\vee \in D$, such that $y_1, \ldots, y_n \geq 0$ for isotropy group it holds that

$$\mathrm{Stab}_W(\chi) = \{1\} \Leftrightarrow (\forall i = 1, \ldots, n)(y_i > 0),$$
$$\mathrm{Stab}_W(\chi) = \langle\{r_i \in S \mid (i = 1, \ldots, n) \wedge (y_i = 0)\}\rangle.$$

The set F is a fundamental region of W^{aff} i.e.

Proposition 4.1.2.

(1) $W^{\mathrm{aff}}F = \mathbb{R}^n$
(2) $(\forall \chi, \widetilde{\chi} \in F)(\exists (w, q^\vee) \in W^{\mathrm{aff}})(\widetilde{\chi} = w\chi + q^\vee) \Rightarrow (\widetilde{\chi} = \chi = w\chi + q^\vee)$.
(3) Consider a point $\chi = y_1\omega_1^\vee + \cdots + y_n\omega_n^\vee \in F$, such that $y_0 + y_1 m_1 + \cdots + y_n m_n = 1$ for isotropy group it holds that

$$\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) = \{1\} \Leftrightarrow (\forall i = 0, \ldots, n)(y_i > 0) \Leftrightarrow (\chi \in F^\circ),$$
$$\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) = \langle\{r_i \in R \mid (i = 0, \ldots, n) \wedge (y_i = 0)\}\rangle,$$

where F° denotes the interior of F and $\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)$ is finite.

4.2. **Notations.**

We define the set J^σ needed for the Fundamental domain

$$\begin{aligned} J^\sigma &= \{\chi \in D \mid (\exists r \in S^\sigma)(r\chi = \chi)\}, \\ J^{\sigma_s} \equiv J^s &= \{\chi \in D \mid (\exists r \in S^s)(r\chi = \chi)\}, \\ J^{\sigma_l} \equiv J^l &= \left\{\chi \in D \mid (\exists r \in S^l)(r\chi = \chi)\right\}, \\ J^{\sigma_e} \equiv J^e &= \{\chi \in D \mid (\exists r \in S)(r\chi = \chi)\}. \end{aligned}$$

We define these sets

$$\begin{aligned} H^\sigma &= \{\chi \in F \mid (\exists r \in R^\sigma)(r\chi = \chi)\}, \\ H^{\sigma_s} \equiv H^s &= \{\chi \in F \mid (\exists r \in R^s)(r\chi = \chi)\}, \\ H^{\sigma_l} \equiv H^l &= \left\{\chi \in F \mid (\exists r \in R^l)(r\chi = \chi)\right\}, \\ H^{\sigma_e} \equiv H^e &= \{\chi \in F \mid (\exists r \in R)(r\chi = \chi)\}. \end{aligned}$$

Lemma 4.2.1. For Γ_σ defined as

$$\Gamma_\sigma = \left\{\chi \in F \mid (\exists w \in W)(\sigma(w) = -1)(\exists q^\vee \in Q^\vee)(w\chi + q^\vee = \chi)\right\}$$

it holds that

$$\Gamma_\sigma = H^\sigma.$$

Proof. We will prove two inclusions, The inclusion ($\supset$) is trivial. For the inclusion ($\subset$), consider a point $\chi \in \Gamma_\sigma$ then clearly $\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) \neq \{1\}$. Therefore from (4.1.2 (3)) we get $\chi \notin F^\circ$, now again from (4.1.2 (3)) we get

$$\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) = \langle\{r \in R \mid r\chi = \chi\}\rangle \neq \{1\}$$

thus, there exists $r_\sigma \in R^\sigma$ such that $r_\sigma\chi = \chi$. □

The set $F^{\sigma+} = F \cup r_\sigma(F \setminus H^\sigma)$, where $r_\sigma \in R^\sigma$ is arbitrary, but fixed is the fundamental domain of W^{aff}_σ, meaning:

Proposition 4.2.1.

(1) $W^{\mathrm{aff}}_\sigma F^{\sigma+} = \mathbb{R}^n$
(2) $(\forall \chi, \widetilde{\chi} \in F^{\sigma+})[(\exists((w, q^\vee) \in W^{\mathrm{aff}}_\sigma)(\widetilde{\chi} = w\chi + q^\vee) \Rightarrow (\widetilde{\chi} = \chi = w\chi + q^\vee)]$.
(3) Consider a point $\chi \in F^{\sigma+}$. If $\chi \in F \setminus H^\sigma$ or $\chi \in r_\sigma(F \setminus H^\sigma)$ then

$$\mathrm{Stab}_{W^{\mathrm{aff}}_\sigma}(\chi) = \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi).$$

If $\chi \in H^\sigma$ then

$$|\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)| = 2\left|\mathrm{Stab}_{W^{\mathrm{aff}}_\sigma}(\chi)\right|.$$

Proof. (1) We use this sequence of equalities

$$\begin{aligned} \mathbb{R}^n &= W^{\mathrm{aff}} F = Q^\vee(WF) = Q^\vee((W^\sigma \cup r_\sigma W^\sigma)F) = Q^\vee(W^\sigma(F \cup r_\sigma F)) \\ &= Q^\vee(W^\sigma(F \cup r_\sigma(F \setminus H^\sigma))) = W^{\mathrm{aff}}_\sigma F^{\sigma+}. \end{aligned}$$

The third equality is the decomposition of W into left cosets. The fifth equality is just taking out the elements which are in the intersection. So it is sufficient to prove that

$$F \cap r_\sigma(F \setminus H^\sigma) = \emptyset.$$

Consider a $\chi \in F \cap r_\sigma(F \setminus H^\sigma)$, therefore $\chi \in F \wedge \chi \in r_\sigma(F \setminus H^\sigma)$, therefore there exists $\gamma \in F$ such that $\gamma \notin H^\sigma$ i.e. $(\forall \widetilde{r} \in R^\sigma)(\widetilde{r}\gamma \neq \gamma)$ and $\chi = r_\sigma\gamma$, where $\chi, \gamma \in F$ and from theorem 4.1.2 (2) we get

$$\chi = r_\sigma\gamma = \gamma.$$

Therefore $r_\sigma\gamma = \gamma$ and $(\forall \widetilde{r} \in R^\sigma)(\widetilde{r}\gamma \neq \gamma)$ meaning that the set is empty.

(2) Suppose we have $\chi, \widetilde{\chi} \in F^{\sigma+}$ and $w \in W^\sigma$, $q \in Q^\vee$ such that

$$w\chi + q^\vee = \widetilde{\chi}. \tag{3}$$

Since $F^{\sigma+}$ consists of two disjoint parts F and $r_\sigma(F \setminus H^\sigma)$, we distinguish the following cases:

(a) $\chi, \widetilde{\chi} \in F$. It follows immediately from (4.1.2 (2)) that $\chi = \widetilde{\chi}$.
(b) $\chi, \widetilde{\chi} \in r_\sigma(F \setminus H^\sigma)$. Consider $\gamma, \widetilde{\gamma} \in F \setminus H^\sigma$ such that $\chi = r_\sigma \gamma$ and $\widetilde{\chi} = r_\sigma \widetilde{\gamma}$. Then $\widetilde{\gamma} = r_\sigma w r_\sigma \gamma + r_\sigma q^\vee$ and from (4.1.2 (2)) we obtain $\gamma = \widetilde{\gamma}$, i.e. $\chi = \widetilde{\chi}$.
(c) $\chi \in F$, $\widetilde{\chi} \in r_\sigma(F \setminus H^\sigma)$. We act by r_σ on (3) we get

$$r_\sigma w \chi + r_\sigma q^\vee = r_\sigma \widetilde{\chi}, \tag{4}$$

since $r_\sigma \widetilde{\chi} \in F$ and $\chi \in F$ then from (4.1.2 (2)) we get

$$\chi = r_\sigma \widetilde{\chi} \in (F \setminus H^\sigma). \tag{5}$$

We substitute this in (3)

$$r_\sigma w \chi + r_\sigma q^\vee = \chi.$$

Therefore $\chi \in \Gamma_\sigma$ and from lemma (4.2.1) we get that $\chi \in H^\sigma$, this contradicts with (5), therefore the option (2c) cannot occur.

(3) We have an arbitrary $\chi \in F \setminus H^\sigma$ We know that

$$\chi \in H^\sigma \Leftrightarrow (\exists r \in R^\sigma)(r\chi = \chi).$$

We use this series of equivalencies

$$\chi \in F \setminus H^\sigma \Leftrightarrow \chi \notin H^\sigma \Leftrightarrow (\forall r_\sigma \in R^\sigma)(r_\sigma \chi \neq \chi).$$

Therefore only 'positive' reflections stabilize χ

$$\{r \in R \mid r\chi = \chi\} \subset W_\sigma^{\mathrm{aff}}.$$

For the isotropy group we get

$$\{r \in R \mid r\chi = \chi\} \subset \mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi) \subset \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) = \langle \{r \in R \mid r\chi = \chi\} \rangle.$$

If we act on this by $\langle\rangle$ we get

$$\mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi) = \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi).$$

If $\chi \in r_\sigma(F \setminus H^\sigma)$ we get that

$$r_\sigma \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) r_\sigma = \mathrm{Stab}_{W^{\mathrm{aff}}}(r_\sigma \chi) = \mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(r_\sigma \chi) = r_\sigma \mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi) r_\sigma,$$

therefore

$$\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) = \mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi).$$

If $\chi \in H^\sigma$ therefore $(\exists \widetilde{r} \in R^\sigma)(\widetilde{r}\chi = \chi)$. We define the homomophism τ^σ as the sign homomorphism σ constricted on the domain $\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)$. To prove that τ^σ is surjective we know that $\tau^\sigma(\mathbb{I}) = 1$ and $\tau^\sigma(\widetilde{r}) = -1$. It is obvious that $\ker \tau^\sigma = \mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi) = \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi) \cap W_\sigma^{\mathrm{aff}}$, thus

$$\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)/\mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi) \cong \{\pm 1\},$$

from this we conclude that

$$|\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)| = 2\left|\mathrm{Stab}_{W_\sigma^{\mathrm{aff}}}(\chi)\right|.$$

□

The set $D^{\sigma+} = D \cup r_\sigma(D \setminus J^\sigma)$, where $r_\sigma \in S^\sigma$ is arbitrary, but fixed it holds that

Proposition 4.2.2.
(1) $W^\sigma D^{\sigma+} = \mathbb{R}^n$
(2) $(\forall \chi, \widetilde{\chi} \in D^{\sigma+})(\exists w \in W^\sigma)(\widetilde{\chi} = w\chi) \Rightarrow (\widetilde{\chi} = \chi = w\chi)$.
(3) Consider a point $\chi \in D^{\sigma+}$. If $\chi \in D \setminus J^\sigma$ or $\chi \in r_\sigma(D \setminus J^\sigma)$ then

$$\mathrm{Stab}_{W^\sigma}(\chi) = \mathrm{Stab}_W(\chi).$$

If $\chi \in J^\sigma$ then

$$|\mathrm{Stab}_W(\chi)| = 2\,|\mathrm{Stab}_{W^\sigma}(\chi)|.$$

Proof. Analogous to 4.2.1. □

We will denote the number of elements of $\mathrm{Stab}_{W^\sigma}(\chi)$ as d_χ^σ

$$d_\chi^\sigma \equiv |\mathrm{Stab}_{W^\sigma}(\chi)|.$$

4.3. **Action of W^σ on the maximal torus $\mathbb{R}^n/Q^\vee$.**

If we have two elements $\chi, \widetilde{\chi} \in \mathbb{R}^n$ such that $\widetilde{\chi} - \chi = q^\vee$, with $q^\vee \in Q^\vee$, then for $w \in W^\sigma$ we have $w\chi - w\widetilde{\chi} = wq^\vee \in Q^\vee$, i.e. we have a natural action of W^σ on the torus $\mathbb{R}^n/Q^\vee$. For $\chi \in \mathbb{R}^n/Q^\vee$ we denote the isotropy group and its order by

$$h^\sigma_\chi \equiv |\mathrm{Stab}^\sigma(\chi)|, \quad \mathrm{Stab}^\sigma(\chi) = \{w \in W^\sigma \mid w\chi = \chi\} \tag{6}$$

We denote the orbit and its order by

$$\epsilon^\sigma(\chi) \equiv |W^\sigma \chi|, \quad W^\sigma \chi = \left\{ w\chi \in \mathbb{R}^n/Q^\vee \mid w \in W^\sigma \right\}.$$

Orbit-stabilizer theorem states that

$$\epsilon^\sigma(\chi) = \frac{|W^\sigma|}{h^\sigma_\chi}. \tag{7}$$

On the maximal torus $\mathbb{R}^n/Q^\vee$ it holds that

Proposition 4.3.3.

(1) $(\forall \chi \in \mathbb{R}^n/Q^\vee)(\exists \widetilde{\chi} \in F^{\sigma+} \cap \mathbb{R}^n/Q^\vee)(\exists w \in W^\sigma)(\chi = w\widetilde{\chi})$
(2) $(\forall \chi, \widetilde{\chi} \in F^{\sigma+} \cap \mathbb{R}^n/Q^\vee)(\exists w \in W^\sigma)(\widetilde{\chi} = w\chi) \Rightarrow (\widetilde{\chi} = \chi = w\chi)$
(3) Consider a point $\chi \in F^{\sigma+} \cap \mathbb{R}^n/Q^\vee$ i.e. $\chi = \gamma + Q^\vee$, where $\gamma \in F^\sigma$. For the isotropy group it holds that

$$\mathrm{Stab}^\sigma(\chi) \cong \mathrm{Stab}_{W^{\mathrm{aff}}_\sigma}(\gamma).$$

Proof.

(1) Follows directly from (4.2.1 (1)).
(2) Follows directly from (4.2.1 (2)).
(3) Let us assume that $\gamma = (w, q^\vee)\gamma = w\gamma + q^\vee$ then $\gamma - w\gamma = q^\vee \in Q^\vee$, i.e. $w\chi = \chi$ and vice versa. Therefore $\psi(w, q^\vee) = w \in \mathrm{Stab}^\sigma(\chi)$. Thus,

$$\psi(\mathrm{Stab}_{W^{\mathrm{aff}}_\sigma}(\gamma)) = \mathrm{Stab}^\sigma(\chi).$$

We also have

$$\ker \psi = \left\{ (\mathbb{I}, q^\vee) \mid (\mathbb{I}, q^\vee) \in \mathrm{Stab}^{\mathrm{aff}}_{W_\sigma}(\gamma) \right\} = \{1\}.$$

□

5. Dual Lie algebra

The set of simple dual roots $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ is a system of simple roots of the dual Lie algebra, where $\alpha_i^\vee \equiv \frac{\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$. The system $\Delta^\vee$ also spans Euclidean space $\mathbb{R}^n$. The dual system $\Delta^\vee$ has analogous properties as the root system Δ. The highest dual root η of the dual root system can be written as follows

$$\eta \equiv -\alpha_0^\vee = m_1^\vee \alpha_1^\vee + \cdots + m_n^\vee \alpha_n^\vee$$

. The coefficients $m_j^\vee$ are called dual marks . The elements of the dual Cartan matrix $C^\vee$ are

$$C^\vee_{ij} = \frac{2\langle \alpha_i^\vee, \alpha_j^\vee \rangle}{\langle \alpha_j^\vee, \alpha_j^\vee \rangle} = C_{ji}, \quad i, j \in \{1, \ldots, n\}.$$

The positive weight lattice P^+ is defined as

$$P^+ = Z_0^+ \omega_1 + \cdots + \mathbb{Z}_0^+ \omega_n.$$

5.1. **Dual affine Weyl group and dual affine even subgroup.**

Dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ is a semidirect product of the group of shifts Q and the Weyl group W

$$\widehat{W}^{\mathrm{aff}} = Q \rtimes W. \tag{8}$$

Equivalently, $\widehat{W}^{\mathrm{aff}}$ is generated by reflections r_i and reflection $r_0^\vee$, where

$$r_0^\vee \chi = r_\eta \chi + \frac{2\eta}{\langle \eta, \eta \rangle}, \quad r_\eta \chi = \chi - \frac{2\langle \chi, \eta \rangle}{\langle \eta, \eta \rangle}\eta, \quad \chi \in \mathbb{R}^n.$$

The set of dual simple reflections $R^\vee$ ($r_\alpha^\vee \equiv r_\alpha$)

$$R^\vee = S \cup \{r_0^\vee\}$$

The fundamental region $F^\vee$ of $\widehat{W}^{\mathrm{aff}}$ is the convex hull of the vertices $\left\{0, \frac{\omega_1}{m_1^\vee}, \dots, \frac{\omega_n}{m_n^\vee}\right\}$:

$$\begin{aligned} F^\vee &= \left\{ z_1\omega_1 + \dots + z_n\omega_n \mid z_0, \dots, z_n \in \mathbb{R}^{\geq 0}, z_0 + z_1 m_1^\vee + \dots + z_n m_n^\vee = 1 \right\} \\ &= \{\chi \in D \mid \langle \chi, \eta \rangle \leq 1\} \end{aligned} \tag{9}$$

The expansion of the sign homomorphism σ to the dual affine even subgroup is

$$R^{\vee\sigma} = \begin{cases} S^\sigma & \text{if } \sigma(r_\eta) = 1 \\ S^\sigma \cup \{r_0^\vee\} & \text{if } \sigma(r_\eta) = -1. \end{cases}$$

The sets $R^{\vee s} \equiv R^{\vee \sigma_s}$ and $R^{\vee l} \equiv R^{\vee \sigma_l}$ in their explicit form are

$$\begin{aligned} R^{\vee s} &= \{r_\alpha \mid \alpha \in \Delta_s\} \cup \{r_0^\vee\} \\ R^{\vee l} &= \{r_\alpha \mid \alpha \in \Delta_l\}. \end{aligned}$$

Again, we will not distinguish the sign homomorphism and its expanded version i.e.

$$\sigma \circ \psi^\vee = \sigma,$$

where $\psi^\vee$ is the retraction homomorphism on the dual affine Weyl group. The dual affine even subgroup $\widehat{W}^{\mathrm{aff}}_\sigma$ is the kernel of the expanded sign homomorphism or equivalently the semidirect product of the group of translations Q, and of the even subgroup W^σ i.e.

$$\widehat{W}^{\mathrm{aff}}_\sigma = \left\{ \widehat{w}^{\mathrm{aff}} \in \widehat{W}^{\mathrm{aff}} \mid \sigma(\widehat{w}^{\mathrm{aff}}) = 1 \right\} = Q \rtimes W^\sigma. \tag{10}$$

We define the dual sign Coxeter number $m^{\vee\sigma}$ as

$$m^{\vee\sigma} = \sum_{r_i \in R^{\vee\sigma}} m_i^\vee.$$

We define zero index dual Coxeter number as $m_0^\vee = 1$. The sum of dual long Coxeter number and dual short Coxeter number is the Coxeter number

$$m = m^{\vee s} + m^{\vee l}.$$

The even dual Coxeter number $m^{\vee\sigma_e}$ is the same as the Coxeter number

$$m^{\vee\sigma_e} = m.$$

Now we will prove that $m^\sigma = m^{\vee\sigma}$, but first we need to prove the following lemma.

Lemma 5.1.1. The following holds for these types of Weyl groups

$$\begin{aligned} B_n &: (\exists w \in W)(r_\xi = w r_1 w^{-1}) \\ &\quad (\exists w \in W)(r_\eta = w r_n w^{-1}) \\ C_n &: (\exists w \in W)(r_\xi = w r_n w^{-1}) \\ &\quad (\exists w \in W)(r_\eta = w r_1 w^{-1}) \\ F_4 &: (\exists w \in W)(r_\xi = w r_2 w^{-1}) \\ &\quad (\exists w \in W)(r_\eta = w r_3 w^{-1}) \\ G_2 &: (\exists w \in W)(r_\xi = w r_1 w^{-1}) \\ &\quad (\exists w \in W)(r_\eta = w r_2 w^{-1}) \end{aligned}$$

Proof. (1) $B_n:$ $\alpha_1 = (1,-1,0,\cdots,0)$ and $\xi = (1,1,0,\cdots,0)$ therefore $w = \mathrm{diag}(1,-1,1,\cdots,1) \in W$. $\eta = (1,0,\cdots,0)$ and $\alpha_n = (0,\cdots,0,1)$ therefore $w \in W$ is

$$w = \begin{pmatrix} 0 & \vec{0}^\top & 1 \\ \vec{0} & \mathbb{I} & \vec{0} \\ 1 & \vec{0}^\top & 0 \end{pmatrix},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n-2,n-2}$ and $\vec{0}$ is zero vector in $\mathbb{R}^{n-2}$.

(2) $C_n:$ $\alpha_n = (0,\cdots,0,\sqrt{2})$ and $\xi = (\sqrt{2},0,\cdots,0)$ therefore $w \in W$ is

$$w = \begin{pmatrix} 0 & \vec{0}^\top & 1 \\ \vec{0} & \mathbb{I} & \vec{0} \\ 1 & \vec{0}^\top & 0 \end{pmatrix},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n-2,n-2}$ and $\vec{0}$ is zero vector in $\mathbb{R}^{n-2}$.

$\eta = (\sqrt{2},\sqrt{2},0,\cdots,0)$ and $\alpha_1^\vee = (\sqrt{2},-\sqrt{2},0,\cdots,0)$ therefore $w = \mathrm{diag}(1,-1,1,\cdots,1) \in W$.

(3) $F_4:$ $\alpha_2 = (0,0,1,-1)$ and $\xi = (1,1,0,0)$ therefore $w \in W$ is

$$w = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$\eta = (2,0,0,0)$ and $\alpha_3^\vee = (0,0,0,2)$ therefore therefore $w \in W$ is

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(4) $G_2:$ $\alpha_1 = (\sqrt{2},0)$ and $\xi = (\frac{\sqrt{2}}{2},\frac{\sqrt{6}}{2})$ therefore $w = r_1 \cdot r_2 \in W$ is

$$w = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

$\eta = (0,\sqrt{6})$ and $\alpha_2 = (-\frac{\sqrt{2}}{2},\frac{\sqrt{6}}{6})$ therefore $w = r_2 \cdot r_1 \in W$ is

$$w = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

□

Proposition 5.1.2. Let σ be a sign homomorfism then for the numbers m^σ and $m^{\vee\sigma}$ it holds that

$$m^\sigma = m^{\vee\sigma}. \tag{11}$$

Proof.
To simplify the proof we will use the notation $\vec{m} = (m_1, m_2, \cdots, m_n)$.

(1) B_n, $\vec{m} = (1, 2, \cdots, 2)$, $\vec{m}^\vee = (2, \cdots, 2, 1)$

(a) $\sigma(r_\xi) = -1$ and $\sigma(r_\eta) = -1$, therefore from lemma 5.1.1 we get $\sigma(r_1) = -1$ and $\sigma(r_n) = -1$. The sign Coxeter numbers are

$$m^\sigma = 1 + \sum_{r_i \in R^\sigma, i \in \{1, \cdots, n\}} m_i$$
$$m^{\vee\sigma} = 1 + \sum_{r_i \in R^\sigma, i \in \{1, \cdots, n\}} m_i^\vee.$$

By subtracting these equations we get

$$m^\sigma - m^{\vee\sigma} = \sum_{r_i \in R^\sigma, i \in \{1, n\}} m_i - \sum_{r_i \in R^\sigma, i \in \{1, n\}} m_i^\vee = m_1 + m_n - m_1^\vee - m_n^\vee,$$

since $m_i = m_i^\vee$ for $i \in \{2, 3, \cdots, n-1\}$ and $m_1 = m_n^\vee$, $m_n = m_1^\vee$ we get $m^\sigma - m^{\vee\sigma} = 0$.

(b) $\sigma(r_\xi) = 1$ and $\sigma(r_\eta) = 1$, therefore from lemma 5.1.1 we get $\sigma(r_1) = 1$ and $\sigma(r_n) = 1$. The sign Coxeter numbers are

$$m^\sigma = \sum_{r_i \in R^\sigma, i \in \{1, \cdots, n\}} m_i$$
$$m^{\vee\sigma} = \sum_{r_i \in R^\sigma, i \in \{1, \cdots, n\}} m_i^\vee.$$

By subtracting these equations we get

$$m^\sigma - m^{\vee\sigma} = \sum_{r_i \in R^\sigma, i \in \{1, n\}} m_i - \sum_{r_i \in R^\sigma, i \in \{1, n\}} m_i^\vee = 0,$$

since $r_n \notin R^\sigma$ and $r_1 \notin R^\sigma$.

(c) $\sigma(r_\xi) = 1$ and $\sigma(r_\eta) = -1$, therefore from lemma 5.1.1 we get $\sigma(r_1) = 1$ and $\sigma(r_n) = -1$. The sign Coxeter numbers are

$$m^\sigma = \sum_{r_i \in R^\sigma, i \in \{1, \cdots, n\}} m_i$$
$$m^{\vee\sigma} = 1 + \sum_{r_i \in R^\sigma, i \in \{1, \cdots, n\}} m_i^\vee.$$

By subtracting these equations we get

$$m^\sigma - m^{\vee\sigma} = -1 + \sum_{r_i \in R^\sigma, i \in \{1, n\}} m_i - \sum_{r_i \in R^\sigma, i \in \{1, n\}} m_i^\vee = -1 + m_n - m_n^\vee = -1 + m_n - m_1 = 0.$$

(d) $\sigma(r_\xi) = -1$ and $\sigma(r_\eta) = 1$ is done analogously to (1c).

(2) C_n, $\vec{m} = (2, \cdots, 2, 1)$, $\vec{m}^\vee = (1, 2, \cdots, 2)$ is done analogously to (1)

(3) F_4, $\vec{m} = (2, 3, 4, 2)$, $\vec{m}^\vee = (2, 4, 3, 2)$

(a) $\sigma(r_\xi) = -1$ and $\sigma(r_\eta) = -1$ is done analogously to (1a).

(b) $\sigma(r_\xi) = 1$ and $\sigma(r_\eta) = 1$ is done analogously to (1b).

(c) $\sigma(r_\xi) = 1$ and $\sigma(r_\eta) = -1$, therefore from lemma 5.1.1 we get $\sigma(r_2) = 1$ and $\sigma(r_3) = -1$. The sign Coxeter numbers are

$$m^\sigma = \sum_{r_i \in R^\sigma, i \in \{1, \cdots, 4\}} m_i$$
$$m^{\vee\sigma} = 1 + \sum_{r_i \in R^\sigma, i \in \{1, \cdots, 4\}} m_i^\vee.$$

By subtracting these equations we get

$$m^\sigma - m^{\vee\sigma} = -1 + \sum_{r_i\in R^\sigma,\, i\in\{2,3\}} m_i - \sum_{r_i\in R^\sigma,\, i\in\{2,3\}} m_i^\vee = -1 + m_3 - m_3^\vee = -1 + m_3 - m_2 = 0.$$

(d) $\sigma(r_\xi) = -1$ and $\sigma(r_\eta) = 1$ is done analogously to (3c).

(4) G_2, $\vec{m} = (2,3)$, $\vec{m}^\vee = (3,2)$
  (a) $\sigma(r_\xi) = -1$ and $\sigma(r_\eta) = -1$ is done analogously to (1a).
  (b) $\sigma(r_\xi) = 1$ and $\sigma(r_\eta) = 1$ is done analogously to (1b).
  (c) $\sigma(r_\xi) = 1$ and $\sigma(r_\eta) = -1$, therefore from lemma 5.1.1 we get $\sigma(r_1) = 1$ and $\sigma(r_2) = -1$. The sign Coxeter numbers are

$$m^\sigma = \sum_{r_i\in R^\sigma,\, i\in\{1,2\}} m_i$$
$$m^{\vee\sigma} = 1 + \sum_{r_i\in R^\sigma,\, i\in\{1,2\}} m_i^\vee.$$

By subtracting these equations we get

$$m^\sigma - m^{\vee\sigma} = -1 + \sum_{r_i\in R^\sigma,\, i\in\{1,2\}} m_i - \sum_{r_i\in R^\sigma,\, i\in\{1,2\}} m_i^\vee = -1 + m_2 - m_2^\vee = -1 + m_2 - m_1 = 0.$$

  (d) $\sigma(r_\xi) = -1$ and $\sigma(r_\eta) = 1$ is done analogously to (4c).

□

Analogously we define these sets

$$H^{\vee\sigma} = \left\{\chi \in F^\vee \mid (\exists r \in R^{\sigma\vee})(r\chi = \chi)\right\}$$
$$H^{\vee\sigma_s} \equiv H^{s\vee} = \left\{\chi \in F^\vee \mid (\exists r \in R^{s\vee})(r\chi = \chi)\right\}$$
$$H^{\vee\sigma_l} \equiv H^{l\vee} = \left\{\chi \in F^\vee \mid (\exists r \in R^{l\vee})(r\chi = \chi)\right\}$$
$$H^{\vee\sigma_e} \equiv H^{e\vee} = \left\{\chi \in F^\vee \mid (\exists r \in R^{\vee})(r\chi = \chi)\right\}.$$

We choose some fixed $r_\sigma \in R^{\vee\sigma}$ and define the set $F^{\vee\sigma+}$ by

$$F^{\vee\sigma+} = F^\vee \cup r_\sigma(F^\vee \setminus H^{\vee\sigma}). \tag{12}$$

Analogously to proposition 4.2.1, we obtain that $F^{\vee\sigma+}$ is a fundamental region of the dual affine even subgroup $\widehat{W}_\sigma^{\mathrm{aff}}$.

Proposition 5.1.1.

(1) $\widehat{W}_\sigma^{\mathrm{aff}} F^{\vee\sigma+} = \mathbb{R}^n$.
(2) $(\forall \chi, \widetilde{\chi} \in F^{\vee\sigma+})[(\exists (w,q) \in \widehat{W}_\sigma^{\mathrm{aff}})(\widetilde{\chi} = w\chi + q) \Rightarrow (\widetilde{\chi} = \chi = w\chi + q)]$.
(3) Consider a point $\chi \in F^{\vee\sigma+}$. If $\chi \in F^\vee \setminus H^{\vee\sigma}$ or $\chi \in r_\sigma(F^\vee \setminus H^{\vee\sigma})$ then

$$\mathrm{Stab}_{\widehat{W}_\sigma^{\mathrm{aff}}}(\chi) = \mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(\chi).$$

If $\chi \in H^{\vee\sigma}$ then

$$\left|\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(\chi)\right| = 2\left|\mathrm{Stab}_{\widehat{W}_\sigma^{\mathrm{aff}}}(\chi)\right|.$$

5.2. Action of W^σ on the maximal torus $\mathbb{R}^n/Q$.

If we have two elements $\lambda, \widetilde{\lambda} \in \mathbb{R}^n$ such that $\widetilde{\lambda} - \lambda = q$, with $q \in Q$, then for $w \in W^\sigma$ we have $w\lambda - w\widetilde{\lambda} = wq \in Q$, i.e. we have a natural action of W^σ on the torus $\mathbb{R}^n/Q$. For $\lambda \in \mathbb{R}^n/Q$ we denote the isotropy group and its order by

$$h_\lambda^{\vee\sigma} \equiv |\mathrm{Stab}^{\vee\sigma}(\lambda)|, \quad \mathrm{Stab}^{\vee\sigma}(\lambda) = \{w \in W^\sigma \mid w\lambda = \lambda\} \tag{13}$$

We denote the orbit and its order by

$$\epsilon^{\vee\sigma}(\lambda) \equiv |W^\sigma\lambda|, \quad W^\sigma\lambda = \{w\lambda \in \mathbb{R}^n/Q \mid w \in W^\sigma\}.$$

Orbit-stabilizer theorem states that

$$\epsilon^{\vee\sigma}(\lambda) = \frac{|W^\sigma|}{h_\lambda^{\sigma\vee}}. \tag{14}$$

On the maximal torus $\mathbb{R}^n/Q$ analogously to proposition 4.3.3 it holds that

Proposition 5.2.2.

(1) $(\forall\lambda \in \mathbb{R}^n/Q)(\exists\widetilde{\lambda} \in F^{\vee\sigma+} \cap \mathbb{R}^n/Q)(\exists w \in W^\sigma)(\lambda = w\widetilde{\lambda})$
(2) $(\forall\lambda, \widetilde{\lambda} \in F^{\vee\sigma+} \cap \mathbb{R}^n/Q)[(\exists w \in W^\sigma)(\widetilde{\lambda} = w\lambda) \Rightarrow (\widetilde{\lambda} = \lambda = w\lambda)]$.
(3) Consider a point $\lambda \in F^{\vee\sigma+} \cap \mathbb{R}^n/Q$ i.e. $\lambda = \gamma + Q$, where $\gamma \in F^{\vee\sigma+}$. For the isotropy group it holds that

$$\mathrm{Stab}^{\vee\sigma}(\lambda) \cong \mathrm{Stab}_{\widehat{W}_\sigma^{\mathrm{aff}}}(\gamma).$$

6. Orbit functions

We define orbit function of weight λ, sign homomorphisms $\sigma, \widetilde{\sigma}$ in point χ as

$$\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi) = \sum_{w\in W^\sigma} \widetilde{\sigma}(w)\mathrm{e}^{2\pi\mathrm{i}\langle w\lambda,\chi\rangle}.$$

These functions are $\widetilde{\sigma}$-invariant i.e.

$$(\forall\chi \in \mathbb{R}^n)(\forall w \in W)(\forall q^\vee \in Q^\vee)(\Psi_\lambda^{\sigma,\widetilde{\sigma}}(w\chi + q^\vee) = \widetilde{\sigma}(w)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)). \tag{15}$$

They are also $\widetilde{\sigma}$-invariant in weights i.e.

$$(\forall w \in W)((\Psi_{w\lambda}^{\sigma,\widetilde{\sigma}} = \widetilde{\sigma}(w)\Psi_\lambda^{\sigma,\widetilde{\sigma}}). \tag{16}$$

If $\widetilde{\sigma} = \mathbb{1}$ or $\widetilde{\sigma} = \sigma$ and $\sigma = \sigma_e, \sigma_s, \sigma_l$ then these functions are called E-functions

$$\Xi^{e+} \equiv \Psi^{\sigma_e,\mathbb{1}} = \Psi^{\sigma_e,\sigma_e},$$
$$\Xi^{s+} \equiv \Psi^{\sigma_e,\mathbb{1}} = \Psi^{\sigma_s,\sigma_s},$$
$$\Xi^{l+} \equiv \Psi^{\sigma_l,\mathbb{1}} = \Psi^{\sigma_l,\sigma_l}.$$

If $\widetilde{\sigma} \neq \sigma$ and $\widetilde{\sigma} \neq \mathbb{1}$ then these functions are called mixed E-functions

$$\Xi^{e-} \equiv \Psi^{\sigma_e,\sigma_l} = \Psi^{\sigma_e,\sigma_s},$$
$$\Xi^{s-} \equiv \Psi^{\sigma_s,\sigma_l} = \Psi^{\sigma_s,\sigma_e},$$
$$\Xi^{l-} \equiv \Psi^{\sigma_l,\sigma_s} = \Psi^{\sigma_l,\sigma_e}.$$

If $\sigma = \mathbb{1}$ then these functions are called S-functions

$$\varphi^e \equiv \Psi^{\mathbb{1},\sigma_e},$$
$$\varphi^s \equiv \Psi^{\mathbb{1},\sigma_s},$$
$$\varphi^l \equiv \Psi^{\mathbb{1},\sigma_l},$$

and if both are $\mathbb{1}$ then they are called C-functions

$$\phi \equiv \Psi^{\mathbb{1},\mathbb{1}}.$$

6.1. Domains for orbit functions.

Due to the property 15 the natural domain for χ is $F^{\sigma+}$, but we have to take out special values in which orbit functions are zero. Let us take an arbitrary point $\chi \in F$, 15 means that,

$$\Psi_\lambda^{\sigma,\widetilde{\sigma}}(w\chi + q^\vee) = \widetilde{\sigma}(w)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi).$$

Therefore $\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi) = 0$ if and only if

$$(\exists w^{\mathrm{aff}} \in W_\sigma^{\mathrm{aff}})(w^{\mathrm{aff}}\chi = \chi) \wedge (\widetilde{\sigma}(w^{\mathrm{aff}}) = -1).$$

The set of zero points of $\Psi_\lambda^{\sigma,\widetilde{\sigma}}$ in F is

$$\Gamma_{\sigma,\widetilde{\sigma}} = \left\{\chi \in F \mid (\exists w \in W)(\exists q^\vee \in Q^\vee)(\sigma(w) = 1) \wedge (\widetilde{\sigma}(w) = -1) \wedge (w\chi + q^\vee = \chi)\right\}.$$

Lemma 6.1.1.

$$\Gamma_{\sigma,\widetilde{\sigma}} = H^{\widetilde{\sigma}} \cap H^{\sigma\cdot\widetilde{\sigma}}$$

Proof. The first inclusion ($\subset$) we can see that $\Gamma_{\sigma,\widetilde{\sigma}} \subset \Gamma_{\widetilde{\sigma}}$ and $\Gamma_{\sigma,\widetilde{\sigma}} \subset \Gamma_{\widetilde{\sigma}\cdot\sigma}$, therefore $\Gamma_{\sigma,\widetilde{\sigma}} \subset \Gamma_{\widetilde{\sigma}} \cap \Gamma_{\widetilde{\sigma}\cdot\sigma}$ by using lemma 5.1.1 we get $\Gamma_{\sigma,\widetilde{\sigma}} \subset H^{\widetilde{\sigma}} \cap H^{\sigma\cdot\widetilde{\sigma}}$.

The second inclusion ($\supset$) we will prove by contradiction. Let us assume an arbitrary, but fixed, point $\chi \in F$ such that

(1) $\chi \in H^{\widetilde{\sigma}} \cap H^{\sigma\cdot\widetilde{\sigma}}$
(2) $\chi \notin \Gamma_{\sigma,\widetilde{\sigma}}$

Since $\chi \in H^{\widetilde{\sigma}}$, therefore

$$(\exists((w_1, q_1^\vee) \in \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi))(\widetilde{\sigma}(w_1) = -1)$$

and $\chi \in H^{\sigma\cdot\widetilde{\sigma}}$, therefore

$$(\exists((w_2, q_2^\vee) \in \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi))(\sigma(w_2)\widetilde{\sigma}(w_2) = -1).$$

Since $\chi \notin \Gamma_{\sigma,\widetilde{\sigma}}$, therefore

$$\neg(\exists(w, q^\vee) \in \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi))(\sigma(w) = 1) \wedge (\widetilde{\sigma}(w) = -1) \Leftrightarrow$$
$$(\forall(w, q^\vee) \in \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi))(\sigma(w) \neq 1) \vee (\widetilde{\sigma}(w) \neq -1) \Leftrightarrow$$

$$(\forall(w, q^\vee) \in \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi))(\sigma(w) = -1) \vee (\widetilde{\sigma}(w) = 1) \tag{17}$$

if we apply 17 on $(w_1, q_1^\vee)$ and $(w_2, q_2^\vee)$ we get

$$(\sigma(w_1) = -1 \wedge \widetilde{\sigma}(w_1) = -1) \,\text{and}\, (\sigma(w_2) = -1 \wedge \widetilde{\sigma}(w_2) = 1)$$

By taking $(w_1, q_1^\vee)(w_2, q_2^\vee) = (w_1 w_2, w_1 q_2^\vee + q_1^\vee) \in \mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)$ we get that

$$\sigma(w_1 w_2) = 1 \wedge \widetilde{\sigma}(w_1 w_2) = -1.$$

This contradicts with 17. □

The domain $F^{\sigma,\widetilde{\sigma}}$ for orbit function $\Psi_\lambda^{\sigma,\widetilde{\sigma}}$ is

$$F^{\sigma,\widetilde{\sigma}} = (F \setminus \Gamma_{\sigma,\widetilde{\sigma}}) \cup r_\sigma((F \setminus H^\sigma) \setminus \Gamma_{\sigma,\widetilde{\sigma}}) = (F \setminus (H^{\widetilde{\sigma}} \cap H^{\widetilde{\sigma}\cdot\sigma})) \cup r_\sigma(F \setminus (H^\sigma \cup (H^{\widetilde{\sigma}} \cap H^{\widetilde{\sigma}\cdot\sigma})).$$

6.2. **Weight domain of orbit functions.**

If we want the orbit functions to be invariant to the abelian group of translations $Q^\vee$ and due to property 16 the natural weight domain is

$$P^{\sigma+} = P^+ \cup r_\sigma(P^+ \setminus J^\sigma),$$

but again we have to take out special values of weights in which the orbit function is equal to zero. Let us take an arbitrary point $\lambda \in P^+$, 16 means that,

$$\Psi_{w\lambda}^{\sigma,\widetilde{\sigma}} = \widetilde{\sigma}(w)\Psi_\lambda^{\sigma,\widetilde{\sigma}}.$$

Therefore $\Psi_\lambda^{\sigma,\widetilde{\sigma}} = 0$ if and only if

$$(\exists w \in W^\sigma)(w\lambda = \lambda) \wedge (\widetilde{\sigma}(w) = -1).$$

The set of special values of weight of $\Psi^{\sigma,\widetilde{\sigma}}$ in P^+ is

$$\Pi_{\sigma,\widetilde{\sigma}} = \{\lambda \in P^+ \mid (\exists w \in W)(\sigma(w) = 1) \wedge (\widetilde{\sigma}(w) = -1) \wedge (w\lambda = \lambda)\}.$$

Analogously to lemma 6.1.1 we get that,

$$\Pi_{\sigma,\widetilde{\sigma}} = J^{\widetilde{\sigma}} \cap J^{\widetilde{\sigma}\cdot\sigma} \cap P^+$$

The weight domain $P^{\sigma,\widetilde{\sigma}}$ for orbit functions $\Psi^{\sigma,\widetilde{\sigma}}$is

$$P^{\sigma,\widetilde{\sigma}} = (P^+ \setminus \Pi_{\sigma,\widetilde{\sigma}}) \cup r_\sigma((P^+ \setminus J^\sigma) \setminus \Pi_{\sigma,\widetilde{\sigma}}) = (P^+ \setminus (J^{\widetilde{\sigma}} \cap J^{\widetilde{\sigma}\cdot\sigma})) \cup r_\sigma(P^+ \setminus (J^\sigma \cup (J^{\widetilde{\sigma}} \cap J^{\widetilde{\sigma}\cdot\sigma})).$$

6.3. **Continuous orthogonality of orbit functions.**

To prove the continuous orthogonality relation we need the orthogonality relation of normal exponential functions on the torus $\mathbb{T} = WF$ and $\lambda, \lambda' \in P$

$$\int_{\mathbb{T}} \mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\chi\rangle}\mathrm{e}^{-2\pi\mathrm{i}\langle\lambda',\chi\rangle} = \mu(\mathbb{T})\delta_{\lambda,\lambda'} \tag{18}$$

Lemma 6.3.1. For any integrable function f on the torus $\mathbb{T}$ invariant to the Weyl group it holds that

$$\int_{F^{\sigma,\widetilde{\sigma}}} f = \frac{1}{|W^\sigma|}\int_{\mathbb{T}} f$$

Proof. It is obvious that the average value of f on $F^{\sigma,\widetilde{\sigma}}$ and on $\mathbb{T}$ is the same

$$\frac{1}{\mu(F^{\sigma,\widetilde{\sigma}})}\int_{F^{\sigma,\widetilde{\sigma}}} f = \frac{1}{\mu(\mathbb{T})}\int_{\mathbb{T}} f,$$

where μ is the Lebesgue measure. For the measures $\mu(F^{\sigma,\widetilde{\sigma}}) = \mu(F^{\sigma+})$, since the difference of these sets has a zero measure. Applying $\mu(\mathbb{T}) = |W^\sigma|\mu(F^{\sigma+})$ on the previous equation we get the desired lemma. □

Lemma 6.3.2.

$$\left(\forall\lambda,\lambda' \in P^{\sigma,\widetilde{\sigma}}\right)\left(\sum_{w\in W^\sigma}\widetilde{\sigma}(w)\delta_{w\lambda,\lambda'} = d^\sigma_\lambda\delta_{\lambda,\lambda'}\right)$$

Proof. If $\lambda \neq \lambda'$ then, since $\lambda, \lambda' \in P^{\sigma,\widetilde{\sigma}} \subset D^{\sigma+}$ and 4.2.2, $w\lambda \neq \lambda'$ for all w in W, therefore the whole sum is zero.

If $\lambda = \lambda'$ then the sum is equal to

$$\sum_{w\in W^\sigma}\widetilde{\sigma}(w)\delta_{w\lambda,\lambda} = \sum_{w\in \mathrm{Stab}_{W^\sigma}(\lambda)}\widetilde{\sigma}(w)\delta_{w\lambda,\lambda} = \sum_{w\in \mathrm{Stab}_{W^\sigma}(\lambda)}\widetilde{\sigma}(w).$$

The set $P^{\sigma,\widetilde{\sigma}}$ is defined as $(P^+ \setminus \Pi_{\sigma,\widetilde{\sigma}}) \cup r_\sigma((P^+ \setminus J^\sigma) \setminus \Pi_{\sigma,\widetilde{\sigma}})$ therefore

$$\begin{aligned}\lambda \notin \Pi_{\sigma,\widetilde{\sigma}} \vee r_\sigma\lambda \notin \Pi_{\sigma,\widetilde{\sigma}} \Leftrightarrow {}& [(\forall w \in \mathrm{Stab}_W(\lambda))(\sigma(w) = -1) \vee (\widetilde{\sigma}(w) = 1)] \vee \\ & \vee [(\forall w \in \mathrm{Stab}_W(r_\sigma\lambda))(\sigma(w) = -1) \vee (\widetilde{\sigma}(w) = 1)].\end{aligned}$$

From $\mathrm{Stab}_W(\lambda) \cap W^\sigma = \mathrm{Stab}_{W^\sigma}(\lambda)$ we obtain

$$(\forall w \in \mathrm{Stab}_{W^\sigma}(\lambda))(\widetilde{\sigma}(w) = 1) \vee (\forall w \in \mathrm{Stab}_{W^\sigma}(r_\sigma\lambda))(\widetilde{\sigma}(w) = 1).$$

Since $r_\sigma\mathrm{Stab}_{W^\sigma}(\lambda)r_\sigma = \mathrm{Stab}_{W^\sigma}(r_\sigma\lambda)$, we obtain

$$(\forall w \in \mathrm{Stab}_{W^\sigma}(\lambda))(\widetilde{\sigma}(w) = 1).$$

Lastly for the sum we get

$$\sum_{w\in \mathrm{Stab}_{W^\sigma}(\lambda)}\widetilde{\sigma}(w) = \sum_{w\in \mathrm{Stab}_{W^\sigma}(\lambda)} 1 = d^\sigma_\lambda.$$

□

Proposition 6.3.3. For all λ, λ' in $P^{\sigma,\widetilde{\sigma}}$ it holds that

$$\int_{F^{\sigma,\widetilde{\sigma}}}\Psi^{\sigma,\widetilde{\sigma}}_\lambda(\chi)\overline{\Psi^{\sigma,\widetilde{\sigma}}_{\lambda'}(\chi)}d\chi = |W^\sigma|\mu(F^{\sigma+})d^\sigma_\lambda\delta_{\lambda,\lambda'}.$$

Proof. First we use lemma 6.3.1 and get.

$$\begin{aligned}\int_{F^{\sigma,\widetilde{\sigma}}}\Psi^{\sigma,\widetilde{\sigma}}_\lambda(\chi)\overline{\Psi^{\sigma,\widetilde{\sigma}}_{\lambda'}(\chi)}d\chi &= \frac{1}{|W^\sigma|}\int_{\mathbb{T}}\Psi^{\sigma,\widetilde{\sigma}}_\lambda(\chi)\overline{\Psi^{\sigma,\widetilde{\sigma}}_{\lambda'}(\chi)}d\chi = \\ =&\frac{1}{|W^\sigma|}\sum_{w_2\in W^\sigma}\sum_{w_1\in W^\sigma}\widetilde{\sigma}(w_1)\widetilde{\sigma}(w_2)\int_{\mathbb{T}} e^{2\pi\mathrm{i}\langle w_1\lambda,\chi\rangle}e^{-2\pi\mathrm{i}\langle w_2\lambda',\chi\rangle}d\chi\end{aligned}$$

since $w_1\lambda$, $w_2\lambda' \in P$, we can use the orthogonality relation 18

$$\frac{1}{|W^\sigma|}\sum_{w_2\in W^\sigma}\sum_{w_1\in W^\sigma}\widetilde{\sigma}(w_1)\widetilde{\sigma}(w_2)\int_{\mathbb{T}} e^{2\pi \mathrm{i}\langle w_1\lambda,\chi\rangle}e^{-2\pi \mathrm{i}\langle w_2\lambda,\chi\rangle}d\chi =$$
$$=\frac{1}{|W^\sigma|}\sum_{w_2\in W^\sigma}\sum_{w_1\in W^\sigma}\widetilde{\sigma}(w_1)\widetilde{\sigma}(w_2)\mu(\mathbb{T})\delta_{w_1\lambda,w_2\lambda'} = \frac{|W^\sigma|\mu(F^{\sigma+})}{|W^\sigma|}\sum_{w_1\in W^\sigma}\sum_{w_2\in W^\sigma}\widetilde{\sigma}(w_1^{-1}w_2)\delta_{\lambda,w_1^{-1}w_2\lambda'} =$$
$$=\mu(F^{\sigma+})\sum_{w_1\in W^\sigma}\sum_{w'\in W^\sigma}\widetilde{\sigma}(w')\delta_{\lambda,w'\lambda'} = |W^\sigma|\mu(F^{\sigma+})\sum_{w'\in W^\sigma}\widetilde{\sigma}(w')\delta_{\lambda,w'\lambda'}$$

Using lemma 6.3.2 we obtain

$$|W^\sigma|\mu(F^{\sigma+})\sum_{w'\in W^\sigma}\widetilde{\sigma}(w')\delta_{\lambda,w'\lambda'} = |W^\sigma|\mu(F^{\sigma+})d_\lambda^\sigma\delta_{\lambda,\lambda'}.$$

□

We have just proven that the set $\left\{\Psi_\lambda^{\sigma,\widetilde{\sigma}} \mid \lambda\in P^{\sigma,\widetilde{\sigma}}\right\}$ is orthogonal. Next we prove that the set is complete in the space $\mathcal{L}^2(F^{\sigma,\widetilde{\sigma}})$, for this we need the following lemma.

Lemma 6.3.4. Let g_λ be any expression dependent on λ then it holds that

$$\sum_{\lambda\in P^{\sigma+}}\sum_{\gamma\in W^\sigma\lambda} g_\gamma = \sum_{\lambda\in P} g_\lambda,$$

and

$$\sum_{\gamma\in W^\sigma\lambda} d_\gamma^\sigma g_\gamma = \sum_{w\in W^\sigma} g_{w\lambda}$$

Proof. The first equation is trivial once we know that $W^\sigma P^{\sigma+} = P$. The second equation just uses the orbit stabilizer theorem. □

Proposition 6.3.5.

$$(\forall f\in\mathcal{L}^2(F^{\sigma,\widetilde{\sigma}}))(\forall\lambda\in P^{\sigma,\widetilde{\sigma}})(\exists c_\lambda\in\mathbb{C})(f=\sum_{\lambda\in P^{\sigma,\widetilde{\sigma}}} c_\lambda\Psi_\lambda^{\sigma,\widetilde{\sigma}}),$$

where the infinite sum $\sum_{\lambda\in P^{\sigma,\widetilde{\sigma}}}$ converges in the space $\mathcal{L}^2(F^{\sigma,\widetilde{\sigma}})$ and the coefficients c_λ are

$$c_\lambda = \frac{1}{|W^\sigma|\mu(F^{\sigma+})d_\lambda^\sigma}\langle f,\Psi_\lambda^{\sigma,\widetilde{\sigma}}\rangle_{F^{\sigma,\widetilde{\sigma}}} = \frac{1}{|W^\sigma|\mu(F^{\sigma+})d_\lambda^\sigma}\int_{F^{\sigma,\widetilde{\sigma}}} f(\chi)\overline{\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)}d\chi.$$

Proof. For an arbitrary $f\in\mathcal{L}^2(F^{\sigma,\widetilde{\sigma}})$, we define f on the torus $\mathbb{T}$ by

$$(\forall w\in W)(f\circ w = \widetilde{\sigma}(w)f)$$

The set of problematic points are on the boundary of $F^{\sigma,\widetilde{\sigma}}$, this set has zero measure. Therefore $f\in\mathcal{L}^2(\mathbb{T})$. Now we can write f in the basis $\left\{\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} \mid \lambda\in P\right\}$ of $\mathcal{L}^2(\mathbb{T})$

$$f=\sum_{\lambda\in P} k_\lambda \mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle},$$

where

$$k_\lambda = \frac{1}{\mu(\mathbb{T})}\langle f,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle}\rangle_{\mathbb{T}}.$$

Due to $\widetilde{\sigma}$-invariance

$$\sum_{w\in W^\sigma}\widetilde{\sigma}(w)(f\circ w) = |W^\sigma|f.$$

Therefore

$$\begin{aligned} f &= \frac{1}{|W^\sigma|}\sum_{w\in W^\sigma}\widetilde{\sigma}(w)(f\circ w) = \frac{1}{|W^\sigma|}\sum_{w\in W^\sigma}\widetilde{\sigma}(w)\sum_{\lambda\in P}\frac{1}{\mu(\mathbb{T})}\langle f\circ w, \mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle}\rangle_{\mathbb{T}}\,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} = \\ &= \frac{1}{|W^\sigma|}\sum_{w\in W^\sigma}\widetilde{\sigma}(w)\sum_{\lambda\in P}\frac{1}{\mu(\mathbb{T})}\langle f, \mathrm{e}^{2\pi\mathrm{i}\langle w\lambda,\cdot\rangle}\rangle_{\mathbb{T}}\,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} = \\ &= \frac{1}{|W^\sigma|\mu(\mathbb{T})}\sum_{\lambda\in P}\langle f, \sum_{w\in W^\sigma}\widetilde{\sigma}(w)\mathrm{e}^{2\pi\mathrm{i}\langle w\lambda,\cdot\rangle}\rangle_{\mathbb{T}}\,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} = \\ &= \frac{1}{|W^\sigma|\mu(\mathbb{T})}\sum_{\lambda\in P}\langle f, \Psi_\lambda^{\sigma,\widetilde{\sigma}}\rangle_{\mathbb{T}}\,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} \overbrace{=}^{(6.3.1)} \frac{1}{|W^\sigma|\mu(F^{\sigma+})}\sum_{\lambda\in P}\langle f, \Psi_\lambda^{\sigma,\widetilde{\sigma}}\rangle_{F^{\sigma,\widetilde{\sigma}}}\,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} = \\ &= \sum_{\lambda\in P} d_\lambda^\sigma c_\lambda\,\mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\cdot\rangle} \overbrace{=}^{(6.3.4)} \sum_{\lambda\in P^{\sigma+}}\sum_{\gamma\in W^\sigma\lambda} d_\gamma^\sigma c_\gamma\,\mathrm{e}^{2\pi\mathrm{i}\langle\gamma,\cdot\rangle} \overbrace{=}^{(6.3.4)} \sum_{\lambda\in P^{\sigma+}}\sum_{w\in W^\sigma} c_{w\lambda}\,\mathrm{e}^{2\pi\mathrm{i}\langle w\lambda,\cdot\rangle} = \\ &= \sum_{\lambda\in P^{\sigma+}} c_\lambda \sum_{w\in W^\sigma}\widetilde{\sigma}(w)\,\mathrm{e}^{2\pi\mathrm{i}\langle w\lambda,\cdot\rangle} = \sum_{\lambda\in P^{\sigma+}} c_\lambda\Psi_\lambda^{\sigma,\widetilde{\sigma}} = \sum_{\lambda\in P^{\sigma,\widetilde{\sigma}}} c_\lambda\Psi_\lambda^{\sigma,\widetilde{\sigma}} \end{aligned}$$

From the definition of c_λ we get that $c_{w\lambda} = \widetilde{\sigma}(w)c_\lambda$ and $c_\lambda = 0$ for $\lambda \in P^{\sigma+} \setminus P^{\sigma,\widetilde{\sigma}}$. Therefore the set $\left\{\Psi_\lambda^{\sigma,\widetilde{\sigma}} \mid \lambda \in P^{\sigma,\widetilde{\sigma}}\right\}$ forms an orthogonal basis in $\mathcal{L}^2(F^{\sigma,\widetilde{\sigma}})$. □

7. Discretization of orbit functions

7.1. Grid $F_M^{\sigma+}$.

The grid $F_M^{\sigma+}$ is the finite fragment of the lattice $\frac{1}{M}P^\vee$ which is found inside of $F^{\sigma+}$. Suppose we have a fixed $M \in \mathbb{N}$ and consider the W-invariant group of translations $\frac{1}{M}P^\vee/Q^\vee$. The group $\frac{1}{M}P^\vee/Q^\vee$ is finite with the order

$$\left|\frac{1}{M}P^\vee/Q^\vee\right| = cM^n. \tag{19}$$

We define the grid $F_M^{\sigma+}$ as such cosets from $\frac{1}{M}P^\vee/Q^\vee$ which have a representative element in the fundamental domain $F^{\sigma+}$:

$$F_M^{\sigma+} \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^{\sigma+}.$$

From the relation (4.3.3 (1)), we have that

$$W^\sigma F_M^{\sigma+} = \frac{1}{M}P^\vee/Q^\vee. \tag{20}$$

The grid $F_M^{\sigma+}$ can be viewed as a union of two disjoint grids – the grid $F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F$ and $r_\sigma(F_M \setminus H_M^\sigma)$, where $H_M^\sigma = \frac{1}{M}P^\vee/Q^\vee \cap H^\sigma$,

$$F_M^{\sigma+} = F_M \cup r_\sigma(F_M \setminus H_M^\sigma). \tag{21}$$

We obtain from (2) that the set F_M, or more precisely its representative points, can be identified as

$$F_M = \left\{\frac{u_1}{M}\omega_1^\vee + \cdots + \frac{u_n}{M}\omega_n^\vee \mid u_0, u_1, \ldots, u_n \in \mathbb{Z}_0^+,\ u_0 + \sum_{i=1}^n u_i m_i = M\right\}. \tag{22}$$

7.2. Number of elements of $F_M^{\sigma+}$.

The number of elements of F_M, denoted by $|F_M|$, are calculated in [7] for all simple Lie algebras. Using these results, we derive the number of elements of $F_M^{\sigma+}$. We define the symbols $u_i^\sigma \in \mathbb{R}$, $i = 0, \cdots, n$:

$$\begin{aligned} u_i^\sigma &\in \mathbb{N}, \quad r_i \in R^\sigma, \\ u_i^\sigma &\in \mathbb{Z}_0^+, \quad r_i \in R \setminus R^\sigma. \end{aligned}$$

The explicit form of $F_M \setminus H_M^\sigma$ is :

$$F_M \setminus H_M^\sigma = \left\{ \frac{u_1^\sigma}{M}\omega_1^\vee + \cdots + \frac{u_n^\sigma}{M}\omega_n^\vee \mid u_0^\sigma m_0 + u_1^\sigma m_1 + \cdots + u_n^\sigma m_n = M \right\} \tag{23}$$

Using the following proposition, the number of elements of $F_M^{\sigma+}$ can be obtained from the formulas for $|F_M|$.

Proposition 7.2.1. Let m^σ be the sign Coxeter number. Then

$$|F_M \setminus H_M^\sigma| = \begin{cases} 0 & M < m^\sigma \\ 1 & M = m^\sigma \\ |F_{M-m^\sigma}| & M > m^\sigma \end{cases}$$

Proof. Taking non-negative numbers $u_i \in \mathbb{Z}_0^+$ and substituting the relations $u_i^\sigma = 1 + u_i$ if $r_i \in R^\sigma$ and $u_i^\sigma = u_i$ if $r_i \in R \setminus R^\sigma$ into the defining relation (23), we obtain

$$u_0 m_0 + u_1 m_1 + \cdots + u_n m_n = M - m^\sigma, \quad u_0, \cdots, u_n \in \mathbb{Z}_0^+$$

This equation has one solution $[0, \cdots, 0]$ if $M = m^\sigma$, no solution if $M < m_\sigma$, and is equal to the defining relation (22) of F_{M-m^σ} if $M > m^\sigma$. □

7.3. **Grid $\Lambda_M^{\sigma+}$.**

The W-invariant group of translations P/MQ is isomorhpic to $\frac{1}{M}P^\vee/Q^\vee$. Therefore the group P/MQ is finite with the order

$$|P/MQ| = cM^n. \tag{24}$$

We define the grid $\Lambda_M^{\sigma+}$ as

$$\Lambda_M^{\sigma+} \equiv P/MQ \cap MF^{\vee\sigma+}$$

The grid $\Lambda_M^{\sigma+}$ can be viewed as a union of two disjoint grids – the grid $\Lambda_M \equiv P/MQ \cap MF^\vee$ and $r_\sigma(\Lambda_M \setminus H_M^{\vee\sigma})$, where $H_M^{\vee\sigma} = P/MQ \cap MH^{\vee\sigma}$,

$$\Lambda_M^{\sigma+} = \Lambda_M \cup r_\sigma(\Lambda_M \setminus H_M^{\vee\sigma}). \tag{25}$$

We obtain from (9) that the set Λ_M, or more precisely its representative points, can be identified as

$$\Lambda_M = \left\{ t_1\omega_1 + \cdots + t_n\omega_n \mid t_0, t_1, \ldots, t_n \in \mathbb{Z}_0^+,\ t_0 + \sum_{i=1}^n t_i m_i^\vee = M \right\}. \tag{26}$$

7.4. **Number of elements of $\Lambda_M^{\sigma+}$.**

The number of elements of Λ_M, denoted by $|\Lambda_M|$, are calculated in [7] for all simple Lie algebras. Usining these results, we derive the number of elements of $\Lambda_M^{\sigma+}$. We define the symbols $t_i^{\sigma\vee} \in \mathbb{R}^\vee$, $i = 0, \cdots, n$:

$$\begin{aligned} t_i^{\vee\sigma} &\in \mathbb{N}, & r_i &\in R^{\vee\sigma} \\ t_i^{\vee\sigma} &\in \mathbb{Z}_0^+, & r_i &\in R^\vee \setminus R^{\vee\sigma} \end{aligned}$$

The explicit form of $\Lambda_M \setminus H_M^{\sigma\vee}$ is :

$$\Lambda_M \setminus H_M^{\vee\sigma} = \left\{ t_1^{\vee\sigma}\omega_1 + \cdots + t_n^{\vee\sigma}\omega_n \mid t_0^{\vee\sigma} m_0 + t_1^{\vee\sigma} m_1 + \cdots + t_n^{\vee\sigma} m_n = M \right\} \tag{27}$$

Similarly to Proposition 7.2.1 we obtain the following.

Proposition 7.4.1. Let $m^{\vee\sigma}$ be the dual sign Coxeter number. Then

$$|\Lambda_M \setminus H_M^{\vee\sigma}| = \begin{cases} 0 & M < m^{\sigma\vee} \\ 1 & M = m^{\sigma\vee} \\ |F_{M-m^{\vee\sigma}}| & M > m^{\sigma\vee} \end{cases}$$

Combining Propositions 7.2.1, 7.4.1 and 5.1.2 and taking into account that $|\Lambda_M| = |F_M|$ we obtain the following crucial result.

Proposition 7.4.2. For the numbers of elements of the set $\Lambda_M^{\sigma+}$ it holds that

$$\left|\Lambda_M^{\sigma+}\right| = \left|F_M^{\sigma+}\right|.$$

7.5. Grid $F_M^{\sigma,\widetilde{\sigma}}$.

We define the grid $F_M^{\sigma,\widetilde{\sigma}}$ as such cosets from $\frac{1}{M}P^\vee/Q^\vee$ which have a representative element in the fundamental domain $F^{\sigma,\widetilde{\sigma}}$,

$$F_M^{\sigma,\widetilde{\sigma}} \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^{\sigma,\widetilde{\sigma}}.$$

The grid $F_M^{\sigma,\widetilde{\sigma}}$ can be viewed as a union of two disjoint grids – the grid $F_M \setminus (H_M^{\widetilde{\sigma}} \cap H_M^{\widetilde{\sigma}\cdot\sigma})$ and the grid $r_\sigma(F_M \setminus (H_M^{\sigma} \cup (H_M^{\widetilde{\sigma}} \cap H_M^{\widetilde{\sigma}\cdot\sigma})))$ i.e.

$$F_M^{\sigma,\widetilde{\sigma}} = F_M \setminus (H_M^{\widetilde{\sigma}} \cap H_M^{\widetilde{\sigma}\cdot\sigma}) \cup r_\sigma(F_M \setminus (H_M^{\sigma} \cup (H_M^{\widetilde{\sigma}} \cap H_M^{\widetilde{\sigma}\cdot\sigma}))). \tag{28}$$

7.6. Number of elements of $F_M^{\sigma,\widetilde{\sigma}}$.

Since there are only four sign homorphisms we get the following choices of grids

$$\begin{aligned}
F_M^{e+} &\equiv F_M^{\sigma_e,\sigma_e} = F_M \cup r_\sigma(F_M \setminus H_M^e)\\
F_M^{s+} &\equiv F_M^{\sigma_s,\sigma_s} = F_M \cup r_\sigma(F_M \setminus H_M^s)\\
F_M^{l+} &\equiv F_M^{\sigma_l,\sigma_l} = F_M \cup r_\sigma(F_M \setminus H_M^l)\\
F_M^{e-} &\equiv F_M^{\sigma_e,\sigma_l} = F_M \setminus (H_M^l \cap H_M^s) \cup r_\sigma(F_M \setminus H_M^e)\\
F_M^{s-} &\equiv F_M^{\sigma_s,\sigma_l} = F_M \setminus H_M^l \cup r_\sigma(F_M \setminus H_M^e)\\
F_M^{l-} &\equiv F_M^{\sigma_l,\sigma_s} = F_M \setminus H_M^s \cup r_\sigma(F_M \setminus H_M^e),
\end{aligned}$$

where $H_M^e \equiv H_M^{\sigma_e}$, $H_M^l \equiv H_M^{\sigma_l}$ and $H_M^s \equiv H_M^{\sigma_s}$.

For the number of elements $\left|F_M^{\sigma,\widetilde{\sigma}}\right|$ it holds that

$$\begin{aligned}
\left|F_M^{e+}\right| &= |F_M| + \left|F_M \setminus H_M^e\right|\\
\left|F_M^{s+}\right| &= |F_M| + \left|F_M \setminus H_M^s\right|\\
\left|F_M^{l+}\right| &= |F_M| + \left|F_M \setminus H_M^l\right|\\
\left|F_M^{e-}\right| &= \left|F_M \setminus H_M^l\right| + \left|F_M \setminus H_M^s\right|\\
\left|F_M^{s-}\right| &= \left|F_M \setminus H_M^l\right| + \left|F_M \setminus H_M^e\right|\\
\left|F_M^{l-}\right| &= \left|F_M \setminus H_M^s\right| + \left|F_M \setminus H_M^e\right|.
\end{aligned}$$

For the number of elements $\left|F_M^{e-}\right|$ we use the following sequence of equalities

$$\begin{aligned}
\left|F_M^{e-}\right| &= \left|F_M \setminus (H_M^l \cap H_M^s) \cup r_\sigma(F_M \setminus H_M^e)\right| = \left|F_M \setminus (H_M^l \cap H_M^s)\right| + \left|F_M \setminus H_M^e\right| =\\
&= \left|F_M \setminus H_M^l \cup F_M \setminus H_M^s\right| + \left|F_M \setminus H_M^e\right| =\\
&= \left|F_M \setminus H_M^l\right| + \left|F_M \setminus H_M^s\right| - \left|F_M \setminus H_M^l \cap F_M \setminus H_M^s\right| + \left|F_M \setminus H_M^e\right| =\\
&= \left|F_M \setminus H_M^l\right| + \left|F_M \setminus H_M^s\right| - \left|F_M \setminus (H_M^l \cup H_M^s)\right| + \left|F_M \setminus H_M^e\right| = \left|F_M \setminus H_M^l\right| + \left|F_M \setminus H_M^s\right|,
\end{aligned}$$

where in the third equality we use the inclusion–exclusion principle.

7.7. Grid $\Lambda_M^{\sigma,\widetilde{\sigma}}$.

We define the grid $\Lambda_M^{\sigma,\widetilde{\sigma}}$ as

$$\Lambda_M^{\sigma,\widetilde{\sigma}} \equiv P/MQ \cap MF^{\vee\sigma,\widetilde{\sigma}},$$

where $F^{\vee\sigma,\widetilde{\sigma}} = (F^\vee \setminus (H^{\vee\widetilde{\sigma}} \cap H^{\vee\widetilde{\sigma}\cdot\sigma})) \cup r_\sigma(F^\vee \setminus (H^{\vee\sigma} \cup (H^{\vee\widetilde{\sigma}} \cap H^{\vee\widetilde{\sigma}\cdot\sigma}))$. The grid $\Lambda_M^{\sigma,\widetilde{\sigma}}$ can be viewed as a union of two disjoint grids – the grid $\Lambda_M \setminus (H_M^{\vee\widetilde{\sigma}} \cap H_M^{\vee\widetilde{\sigma}\cdot\sigma})$ and the grid $r_\sigma(\Lambda_M \setminus (H_M^{\vee\sigma} \cup (H_M^{\vee\widetilde{\sigma}} \cap H_M^{\vee\widetilde{\sigma}\cdot\sigma})))$i.e.

$$\Lambda_M^{\sigma,\widetilde{\sigma}} = \Lambda_M \setminus (H_M^{\vee\widetilde{\sigma}} \cap H_M^{\vee\widetilde{\sigma}\cdot\sigma}) \cup r_\sigma(\Lambda_M \setminus (H_M^{\vee\sigma} \cup (H_M^{\vee\widetilde{\sigma}} \cap H_M^{\vee\widetilde{\sigma}\cdot\sigma}))). \tag{29}$$

7.8. **Number of elements of** $\Lambda_M^{\sigma,\widetilde{\sigma}}$**.**
Since there are only four sign homorphisms we get the following choices of grids

$$\begin{aligned}
\Lambda_M^{e+} &\equiv \Lambda_M^{\sigma_e,\sigma_e} = \Lambda_M \cup r_\sigma(\Lambda_M \setminus H_M^{\vee e}) \\
\Lambda_M^{s+} &\equiv \Lambda_M^{\sigma_s,\sigma_s} = \Lambda_M \cup r_\sigma(\Lambda_M \setminus H_M^{\vee s}) \\
\Lambda_M^{l+} &\equiv \Lambda_M^{\sigma_l,\sigma_l} = \Lambda_M \cup r_\sigma(\Lambda_M \setminus H_M^{\vee l}) \\
\Lambda_M^{e-} &\equiv \Lambda_M^{\sigma_e,\sigma_l} = \Lambda_M \setminus (H_M^{\vee l} \cap H_M^{\vee s}) \cup r_\sigma(\Lambda_M \setminus H_M^{\vee e}) \\
\Lambda_M^{s-} &\equiv \Lambda_M^{\sigma_s,\sigma_l} = \Lambda_M \setminus H_M^{\vee l} \cup r_\sigma(\Lambda_M \setminus H_M^{\vee e}) \\
\Lambda_M^{l-} &\equiv \Lambda_M^{\sigma_l,\sigma_s} = \Lambda_M \setminus H_M^{\vee s} \cup r_\sigma(\Lambda_M \setminus H_M^{\vee e}),
\end{aligned}$$

where $H_M^{\vee e} \equiv H_M^{\vee \sigma_e}$, $H_M^{\vee l} \equiv H_M^{\vee \sigma_l}$ and $H_M^{\vee s} \equiv H_M^{\vee \sigma_s}$.
For the number of elements $\left|\Lambda_M^{\sigma,\widetilde{\sigma}}\right|$

$$\begin{aligned}
\left|\Lambda_M^{e+}\right| &= |\Lambda_M| + \left|\Lambda_M \setminus H_M^{\vee e}\right| \\
\left|\Lambda_M^{s+}\right| &= |\Lambda_M| + \left|\Lambda_M \setminus H_M^{\vee s}\right| \\
\left|\Lambda_M^{l+}\right| &= |\Lambda_M| + \left|\Lambda_M \setminus H_M^{\vee l}\right| \\
\left|\Lambda_M^{e-}\right| &= \left|\Lambda_M \setminus H_M^{\vee l}\right| + \left|\Lambda_M \setminus H_M^{\vee s}\right| \\
\left|\Lambda_M^{s-}\right| &= \left|\Lambda_M \setminus H_M^{\vee l}\right| + \left|\Lambda_M \setminus H_M^{\vee e}\right| \\
\left|\Lambda_M^{l-}\right| &= \left|\Lambda_M \setminus H_M^{\vee s}\right| + \left|\Lambda_M \setminus H_M^{\vee e}\right|.
\end{aligned}$$

The following proposition is needed for the proof of the completness relation for discretized orbit functions.

Proposition 7.8.1. For the numbers of elements of the set $\Lambda_M^{\sigma,\widetilde{\sigma}}$ and $F_M^{\sigma,\widetilde{\sigma}}$ it holds that

$$\left|\Lambda_M^{\sigma,\widetilde{\sigma}}\right| = \left|F_M^{\sigma,\widetilde{\sigma}}\right|.$$

7.9. **Number of elements of** $F_M^{e+}, F_M^{s+}, F_M^{l+}, F_M^{e-}, F_M^{s-}, F_M^{l-}$**.**
For Lie algebras B_n, C_n, G_2 and F_4 the number of elements of $F_M^{e+}, F_M^{s+}, F_M^{l+}, F_M^{e-}, F_M^{s-}, F_M^{l-}$ are given by the following relations, which are derived from [5].
Proposition 7.9.1.
(1) C_n, $n \geq 2$,

$$\begin{aligned}
|F_{2k}^{s+}(C_n)| &= \binom{k+n}{n} + \binom{k+n-1}{n} + \binom{k+1}{n} + \binom{k}{n} \\
|F_{2k+1}^{s+}(C_n)| &= 2\binom{k+n}{n} + 2\binom{k+1}{n},
\end{aligned}$$

$$\begin{aligned}
|F_{2k}^{l+}(C_n)| &= \binom{k+n}{n} + \binom{k+n-1}{n} + \binom{n+k-1}{n} + \binom{n+k-2}{n} \\
|F_{2k+1}^{l+}(C_n)| &= 2\binom{k+n}{n} + 2\binom{n+k-1}{n},
\end{aligned}$$

$$\begin{aligned}
|F_{2k}^{e+}(C_n)| &= \binom{k+n}{n} + \binom{k+n-1}{n} + \binom{k}{n} + \binom{k-1}{n} \\
|F_{2k+1}^{e+}(C_n)| &= 2\binom{k+n}{n} + 2\binom{k}{n},
\end{aligned}$$

$$|F^{s-}_{2k}(C_n)| = \binom{n+k-1}{n} + \binom{n+k-2}{n} + \binom{k}{n} + \binom{k-1}{n}$$
$$|F^{s-}_{2k+1}(C_n)| = 2\binom{n+k-1}{n} + 2\binom{k}{n},$$

$$|F^{l-}_{2k}(C_n)| = \binom{k+1}{n} + 2\binom{k}{n} + \binom{k-1}{n}$$
$$|F^{l-}_{2k+1}(C_n)| = 2\binom{k+1}{n} + 2\binom{k}{n},$$

$$|F^{e-}_{2k}(C_n)| = \binom{k+1}{n} + \binom{k}{n} + \binom{n+k-1}{n} + \binom{n+k-2}{n}$$
$$|F^{e-}_{2k+1}(C_n)| = 2\binom{n+k-1}{n} + 2\binom{n+k-1}{n},$$

(2) B_n, $n \geq 3$,

$$|F^{s+}_M(B_n)| = |F^{l+}_M(C_n)|,$$
$$|F^{l+}_M(B_n)| = |F^{s+}_M(C_n)|,$$
$$|F^{e+}_M(B_n)| = |F^{e+}_M(C_n)|,$$
$$|F^{s-}_M(B_n)| = |F^{l-}_M(C_n)|,$$
$$|F^{l-}_M(B_n)| = |F^{s-}_M(C_n)|,$$
$$|F^{e-}_M(B_n)| = |F^{e-}_M(C_n)|,$$

(3) G_2,

$$|F^{s+}_{6k}(G_2)| = 1 + 3k + 6k^2, \qquad |F^{s+}_{6k+1}(G_2)| = 1 + 5k + 6k^2$$
$$|F^{s+}_{6k+2}(G_2)| = 2 + 7k + 6k^2, \qquad |F^{s+}_{6k+3}(G_2)| = 4 + 9k + 6k^2$$
$$|F^{s+}_{6k+4}(G_2)| = 5 + 11k + 6k^2, \qquad |F^{s+}_{6k+5}(G_2)| = 7 + 13k + 6k^2$$
$$|F^{e+}_{6k}(G_2)| = 2 + 6k^2, \qquad |F^{e+}_{6k+1}(G_2)| = 1 + 2k + 6k^2$$
$$|F^{e+}_{6k+2}(G_2)| = 2 + 4k + 6k^2, \qquad |F^{e+}_{6k+3}(G_2)| = 3 + 6k + 6k^2$$
$$|F^{e+}_{6k+4}(G_2)| = 4 + 8k + 6k^2, \qquad |F^{e+}_{6k+5}(G_2)| = 5 + 10k + 6k^2$$
$$|F^{s-}_{6k}(G_2)| = 1 - 3k + 6k^2, \qquad |F^{s-}_{6k+1}(G_2)| = -k + 6k^2$$
$$|F^{s-}_{6k+2}(G_2)| = k + 6k^2, \qquad |F^{s-}_{6k+3}(G_2)| = 1 + 3k + 6k^2$$
$$|F^{s-}_{6k+4}(G_2)| = 1 + 5k + 6k^2, \qquad |F^{s-}_{6k+5}(G_2)| = 2 + 7k + 6k^2$$
$$|F^{e-}_{6k}(G_2)| = 6k^2, \qquad |F^{e-}_{6k+1}(G_2)| = 2k + 6k^2$$
$$|F^{e-}_{6k+2}(G_2)| = 4k + 6k^2, \qquad |F^{e-}_{6k+3}(G_2)| = 2 + 6k + 6k^2$$
$$|F^{e-}_{6k+4}(G_2)| = 2 + 8k + 6k^2, \qquad |F^{e-}_{6k+5}(G_2)| = 4 + 10k + 6k^2,$$

$$|F^{l+}_M(G_2)| = |F^{s+}_M(G_2)|,$$
$$|F^{l-}_M(G_2)| = |F^{s-}_M(G_2)|,$$

(4) F_4,

$|F^{s+}_{12k}(F_4)| = 1+8k+25k^2+36k^3+36k^4,$ $|F^{s+}_{12k+1}(F_4)| = 1+10k+31k^2+48k^3+36k^4$

$|F^{s+}_{12k+2}(F_4)| = 3+20k+49k^2+60k^3+36k^4,$ $|F^{s+}_{12k+3}(F_4)| = 4+25k+61k^2+72k^3+36k^4$

$|F^{s+}_{12k+4}(F_4)| = 8+42k+85k^2+84k^3+36k^4,$ $|F^{s+}_{12k+5}(F_4)| = 10+52k+103k^2+96k^3+36k^4$

$|F^{s+}_{12k+6}(F_4)| = 18+78k+133k^2+108k^3+36k^4,$ $|F^{s+}_{12k+7}(F_4)| = 22+95k+157k^2+78k^3+36k^4$

$|F^{s+}_{12k+8}(F_4)| = 35+132k+193k^2+132k^3+36k^4,$ $|F^{s+}_{12k+9}(F_4)| = 43+158k+223k^2+144k^3+36k^4$

$|F^{s+}_{12k+10}(F_4)| = 63+208k+265k^2+156k^3+36k^4,$ $|F^{s+}_{12k+11}(F_4)| = 76+245k+301k^2+168k^3+36k^4$

$|F^{e+}_{12k}(F_4)| = 2+52k^2+36k^4,$ $|F^{e+}_{12k+1}(F_4)| = 1+8k+49k^2+12k^3+36k^4$

$|F^{e+}_{12k+2}(F_4)| = 3+18k+58k^2+24k^3+36k^4,$ $|F^{e+}_{12k+3}(F_4)| = 4+26k+61k^2+36k^3+36k^4$

$|F^{e+}_{12k+4}(F_4)| = 8+40k+76k^2+48k^3+36k^4,$ $|F^{e+}_{12k+5}(F_4)| = 10+50k+85k^2+60k^3+36k^4$

$|F^{e+}_{12k+6}(F_4)| = 17+70k+106k^2+72k^3+36k^4$ $|F^{e+}_{12k+7}(F_4)| = 21+84k+121k^2+84k^3+36k^4$

$|F^{e+}_{12k+8}(F_4)| = 32+112k+148k^2+96k^3+36k^4,$ $|F^{e+}_{12k+9}(F_4)| = 39+132k+169k^2+108k^3+36k^4$

$|F^{e+}_{12k+10}(F_4)| = 55+170k+202k^2+120k^3+36k^4,$ $|F^{e+}_{12k+11}(F_4)| = 66+198k+229k^2+132k^3+36k^4$

$|F^{s-}_{12k}(F_4)| = 1-8k+25k^2-36k^3+36k^4,$ $|F^{s-}_{12k+1}(F_4)| = -3k+13k^2-24k^3+36k^4$

$|F^{s-}_{12k+2}(F_4)| = -2k+13k^2-12k^3+36k^4,$ $|F^{s-}_{12k+3}(F_4)| = 7k^2+36k^4$

$|F^{s-}_{12k+4}(F_4)| = 2k+13k^2+12k^3+36k^4,$ $|F^{s-}_{12k+5}(F_4)| = 3k+13k^2+24k^3+36k^4$

$|F^{s-}_{12k+6}(F_4)| = 1+8k+25k^2+36k^3+36k^4,$ $|F^{s-}_{12k+7}(F_4)| = 1+10k+31k^2+48k^3+36k^4$

$|F^{s-}_{12k+8}(F_4)| = 3+20k+49k^2+60k^3+36k^4,$ $|F^{s-}_{12k+9}(F_4)| = 4+25k+61k^2+72k^3+36k^4$

$|F^{s-}_{12k+10}(F_4)| = 8+42k+85k^2+84k^3+36k^4,$ $|F^{s-}_{12k+11}(F_4)| = 10+52k+103k^2+96k^3+36k^4$

$|F^{e-}_{12k}(F_4)| = -2k^2+36k^4,$ $|F^{e-}_{12k+1}(F_4)| = -k-5k^2+12k^3+36k^4$

$|F^{e-}_{12k+2}(F_4)| = 4k^2+24k^3+36k^4,$ $|F^{e-}_{12k+3}(F_4)| = -k+7k^2+36k^3+36k^4$

$|F^{e-}_{12k+4}(F_4)| = 4k+22k^2+48k^3+36k^4,$ $|F^{e-}_{12k+5}(F_4)| = 5k+31k^2+60k^3+36k^4$

$|F^{e-}_{12k+6}(F_4)| = 2+16k+52k^2+72k^3+36k^4$ $|F^{e-}_{12k+7}(F_4)| = 2+21k+67k^2+84k^3+36k^4$

$|F^{e-}_{12k+8}(F_4)| = 6+40k+94k^2+96k^3+36k^4,$ $|F^{e-}_{12k+9}(F_4)| = 8+51k+115k^2+108k^3+36k^4$

$|F^{e-}_{12k+10}(F_4)| = 16+80k+148k^2+120k^3+36k^4,$ $|F^{e-}_{12k+11}(F_4)| = 20+99k+175k^2+132k^3+36k^4,$

$$|F^{l+}_{M}(F_4)| = |F^{s+}_{M}(F_4)|,$$
$$|F^{l-}_{M}(F_4)| = |F^{s-}_{M}(F_4)|.$$

7.10. **Discrete orthogonality of orbit functions.**

To prove the discrete orthogonality relation of discretized functions $\left\{\Psi^{\sigma,\widetilde{\sigma}}_{\lambda} \mid \lambda\in\Lambda^{\sigma,\widetilde{\sigma}}_{M}\right\}$ with the finite domain $F^{\sigma,\widetilde{\sigma}}_{M}$, we need the discrete orthogonality relation of normal exponential functions on the discretized torus $\frac{1}{M}P^\vee/Q^\vee$ and $\lambda,\lambda'\in P$ from [7] i.e.

$$\sum_{\chi\in\frac{1}{M}P^\vee/Q^\vee} \mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\chi\rangle}\mathrm{e}^{-2\pi\mathrm{i}\langle\lambda',\chi\rangle} = \left|\frac{1}{M}P^\vee/Q^\vee\right|\delta_{\lambda,\lambda'} = cM^n\delta_{\lambda,\lambda'}. \tag{30}$$

Lemma 7.10.1. Let $f(\chi)$ be an expression dependent on χ and invariant to the even subgroup W^σ, i.e. $f(w\chi)=f(\chi)$ for all $w\in W^\sigma$, then

$$\sum_{\chi\in F^{\sigma+}_M}\epsilon^\sigma(\chi)f(\chi) = \sum_{\chi\in\frac{1}{M}P^\vee/Q^\vee} f(\chi)$$

Proof. Due to invariance

$$f(\chi)=\frac{1}{|W^\sigma|}\sum_{w\in W^\sigma} f(w\chi).$$

therefore

$$\sum_{\chi\in F_M^{\sigma+}} \epsilon^\sigma(\chi)f(\chi)=\frac{1}{|W^\sigma|}\sum_{\chi\in F_M^{\sigma+}}\sum_{w\in W^\sigma}\epsilon^\sigma(\chi)f(w\chi)=\sum_{\chi\in F_M^{\sigma+}}\sum_{w\in W^\sigma}\frac{1}{h_\chi^\sigma}f(w\chi)=$$
$$=\sum_{\chi\in F_M^{\sigma+}}\sum_{\gamma\in W^\sigma\chi} f(\gamma)=\sum_{\chi\in\frac{1}{M}P^\vee/Q^\vee} f(\chi)$$

□

Finally, the discrete orthogonality relation follows.

Proposition 7.10.2. For all λ,λ' in $\Lambda_M^{\sigma,\widetilde{\sigma}}$

$$\sum_{\chi\in F_M^{\sigma,\widetilde{\sigma}}}\epsilon^\sigma(\chi)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)\overline{\Psi_{\lambda'}^{\sigma,\widetilde{\sigma}}(\chi)}=|W^\sigma|cM^n h_\lambda^{\sigma\vee}\delta_{\lambda,\lambda'}$$

Proof. First we use the fact that orbit functions have zero points, then we use this series of equalities,

$$\sum_{\chi\in F_M^{\sigma,\widetilde{\sigma}}}\epsilon^\sigma(\chi)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)\overline{\Psi_{\lambda'}^{\sigma,\widetilde{\sigma}}(\chi)}=\sum_{\chi\in F_M^{\sigma+}}\epsilon^\sigma(\chi)\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)\overline{\Psi_{\lambda'}^{\sigma,\widetilde{\sigma}}(\chi)}=\sum_{\chi\in\frac{1}{M}P^\vee/Q^\vee}\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)\overline{\Psi_{\lambda'}^{\sigma,\widetilde{\sigma}}(\chi)}=$$
$$=\sum_{\chi\in\frac{1}{M}P^\vee/Q^\vee}\sum_{w'\in W^\sigma}\sum_{w\in W^\sigma}\widetilde{\sigma}(w)\widetilde{\sigma}(w')\mathrm{e}^{2\pi\mathrm{i}\langle w\lambda,\chi\rangle}\mathrm{e}^{-2\pi\mathrm{i}\langle w'\lambda',\chi\rangle}=$$
$$=\sum_{w'\in W^\sigma}\sum_{w\in W^\sigma}cM^n\widetilde{\sigma}(w)\widetilde{\sigma}(w')\delta_{w\lambda,w'\lambda'}=\sum_{w'\in W^\sigma}\sum_{w\in W^\sigma}cM^n\widetilde{\sigma}(w'^{-1}w)\delta_{(w'^{-1}w)\lambda,\lambda'}=$$
$$=|W^\sigma|\sum_{w\in W^\sigma}cM^n\widetilde{\sigma}(w)\delta_{w\lambda,\lambda'}=|W^\sigma|cM^n h_\lambda^{\sigma\vee}\delta_{\lambda,\lambda'},$$

where $\sum_{w\in W^\sigma}\widetilde{\sigma}(w)\delta_{w\lambda,\lambda'}=h_\lambda^{\sigma\vee}\delta_{\lambda,\lambda'}$ is analogous to lemma 6.3.2. □

The set $\left\{\Psi_\lambda^{\sigma,\widetilde{\sigma}}\mid\lambda\in\Lambda_M^{\sigma,\widetilde{\sigma}}\right\}$ is an orthogonal basis of a $\left|F_M^{\sigma,\widetilde{\sigma}}\right|$ dimensional space, due to

$$\left|F_M^{\sigma,\widetilde{\sigma}}\right|=\left|\Lambda_M^{\sigma,\widetilde{\sigma}}\right|.$$

Arbitrary function f defined on $F_M^{\sigma,\widetilde{\sigma}}$ can be expressed as

$$f(\chi)=\sum_{\lambda\in\Lambda_M^{\sigma,\widetilde{\sigma}}} b_\lambda\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi),$$

where

$$b_\lambda=\frac{1}{cM^n|W^\sigma|h_\lambda^\vee}\sum_{\chi\in F_M^{\sigma,\widetilde{\sigma}}} f(\chi)\overline{\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)}.$$

Also, the discrete orthogonality of orbit functions allows the interpolation of an arbitrary function g defined on $F^{\sigma,\widetilde{\sigma}}$, into function g_M depending on the density of the grid. This result is crucial for application in data processing.

8. Conclusion

In the present work, we have succeeded in uniting ten types orbit functions. The most important property of orbit function is their discrete orthogonality in its potential utilization in increasing processing speed of digital data.

In addition, the present work raises new questions. It is unknown under which condition do the series of interpolated functions $\{g_M\}_{M=1}^{\infty}$ converge. Another matter which requires further study are the properties of orbit functions indexed by a general point in $\mathbb{R}^n$. Last but not least, a general one to one correspondence between orbit functions and orthogonal polynomials in n variables exist [18]. Most sources study only 2-variable polynomials.

Potentially orbit functions can be applied in proccessing digital data. In informatics, they can be potentially used in data compression and data hiding. In physics, their potential use lies in image recognition.

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