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Bases for Representations of Lie Algebras

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Abstract

This thesis is devoted to the construction of the Chevalley basis for a simple Lie algebra, the first task in the construction of bases for representations of simple Lie algebras. First, the fundamentals of Lie algebras theory necessary for the construction are introduced and second, the construction itself is demonstrated. As a result of our work, we present a program for computation of the Chevalley bases for both classical and exceptional simple Lie algebras of an arbitrary type. The Maple 16 source code of our program and the computed bases for all simple Lie algebras up to the rank 4 are attached in appendices.

Keywords: simple Lie algebra, Chevalley basis, root systems, structure constants

Abstrakt

V této práci se zabýváme konstrukcí Chevalleyovy báze prosté Lieovy algebry. Vyřešení tohoto problému je prvním krokem komplexnější úlohy konstrukce bází pro reprezentace prostých Lieových algeber. V první části práce vyložíme potřebné základy teorie Lieových algeber a jejich reprezentací, v části druhé pak představíme samotnou konstrukci Chevalleyovy báze. Výsledkem našeho snažení je program, jež dokáže napočítat Chevalleyovu bázi pro libovolnou (klasickou i výjimečnou) prostou Lieovu algebru. K práci je přiložen zdrojový kód implementace našeho programu do systému počítačové algebry Maple 16 a uvedeny jsou též napočtené báze pro všechny prosté Lieovy algebry až do hodnosti 4.

Klíčová slova: prostá Lieova algebra, Chevalleyova báze, kořenové systémy, strukturní konstanty

Contents

List of Notations

$\mathbb{1}$	identity map
$\mathbb{C}$	field of complex numbers
$\mathbb{C}[t]$	vector space of all complex polynomials in t
dim	dimension
$GL(\mathsf{E})$	general linear group of a vector space E
Ker	kernel of a map
max	the largest element of a set
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of natural numbers with zero
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{R}_{\geq 0}$	set of non-negative real numbers
Ran	range of a map
sgn	signum function
Span	linear span
$\mathrm{Span}_{\mathbb{R}}$	real linear span
Tr	trace form
$\mathbb{Z}$	set of integers
$\mathbb{Z}_{\geq 0}$	set of non-negative integers (same as $\mathbb{N}_0$)
$\mathbb{Z}_{\leq 0}$	set of non-positive integers
$\mathbb{Z}^{m,m}$	set of $m \times m$ matrices with integral entries
δ_{ij}	Kronecker delta of i and j
$\sigma(f)$	spectrum of a linear operator f
$\widehat{n}$	$\{1, 2, \ldots, n\}$
$\underline{\widehat{n}}$	$\{0, 1, 2, \ldots, n\}$
$\overline{z}$	complex conjugate of a complex number z
V^*	dual space of a vector space V
V/W	quotient (factor) space of V with respect to W
$\widetilde{x} \equiv (x)^\sim$	element (class of isomorphism) of a quotient space V/W containing $x \in V$
$\emptyset$	empty set
0	trivial vector space $\{0\}$ (when used for vector spaces)
A^H	Hermitian conjugate of a matrix A
${}^{\mathcal{B}}f$	matrix of a linear map f with respect to basis $\mathcal{B}$
$f\|_V$	restriction of a linear map f to V

Introduction

Lie algebras and their representations occur naturally in many areas of both mathematics and physics and therefore it is not surprising that Lie theory became one of the major points of interest of modern mathematics. One of the problems to concern with in this field is the construction of bases for representations (more precisely, representation spaces) of the so-called simple Lie algebras. First, several different kinds of these bases are known and hence an interesting task is to investigate whether there are any relations among them. Second, some bases are defined just for certain types of simple Lie algebras and the question is whether and how they can be generalized for each simple Lie algebra. Last but not least, one aims to develop the algorithms for the proper computing the bases.

The construction of any basis for a representation of a given simple Lie algebra consists of two basic steps. At first one establishes the set of basis vectors and then one defines how the Lie algebra acts on these vectors. This general action is fully determined by the action of basis vectors of the Lie algebra. Consequently, the very first challenge is to construct a basis of a simple Lie algebra and this is the main topic of this thesis actually. The original goal was to study the bases for representations as well however the rigorous analysis of the construction of bases for simple Lie algebras turned out to be the most convenient to start the work with.

The thesis is organized as follows. In the first chapter we introduce the very fundamentals of Lie algebras theory. Further we define semisimple and simple Lie algebras and we show the relation between them. In the second chapter we present an introduction to representation theory. Predominantly, we introduce the results needed for the classification of semisimple Lie algebras. In the third chapter we define the so-called roots and root systems, we classify them and then we use the obtained results to categorize all semisimple Lie algebras. Finally, the fourth chapter is devoted to the construction of a basis for a simple Lie algebra. First, the constructed basis is introduced and second, we present the algorithms for computation of the structure constants. These algorithms were implemented into several Maple 16 procedures to provide the universal program for computing the Chevalley basis for a simple Lie algebra of an arbitrary type. The source code is attached in the first appendix. In the second appendix some examples of the computed bases are presented as an illustration.

Chapter 1

Lie Algebras

1.1 Basic Definitions and Properties

1.1.1 Definition of Lie Algebras

At the very beginning we introduce the definition of Lie algebras and basic related terms. We give also a few common examples as an illustration (cf. [9], p. 1, 2).

Definition 1.1. Let F be a field. A *Lie algebra* over F is an F-vector space together with a bilinear map $[\ ,\ ] : L \times L \to L$ fulfilling for any x, y, z from L the two following conditions:

$$[x, x] = 0, \tag{1.1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{1.2}$$

The map $[\ ,\ ]$ is called the *Lie bracket* and the vector $[x, y]$ is often called the *commutator* of x and y. The condition (1.2) is known as the *Jacobi identity*. Sometimes we use the notation $(L; [\ ,\ ])$ to specify which Lie bracket on the vector space L we consider.

Throughout this work we study entirely Lie algebras over the field of complex numbers. Such Lie algebras are said to be *complex*. Moreover, we do not deal with infinite-dimensional Lie algebras at all. From now on, "*Lie algebra*" will always mean a finite-dimensional Lie algebra over $\mathbb{C}$ and, similarly, "*vector space*" without any further specification will always denote a finite-dimensional complex vector space.

Remark 1.1. Working over $\mathbb{C}$, one can show *anticommutativity* of the Lie bracket. Let L be a (complex) Lie algebra and take arbitrary $x, y \in L$. Putting $x = y + z$ in (1.1), we obtain

$$0 = [y + z, y + z] = [y, y] + [y, z] + [z, y] + [z, z] = [y, z] + [z, y]$$

and hence $[y, z] = -[z, y]$.

Definition 1.2. Let L be a Lie algebra. A *basis* for the Lie algebra L is a basis for the vector space L. Similarly, by the *dimension* of the Lie algebra we mean the dimension of the underlying vector space.

Contrary to a vector space without any further structure, basis alone is not sufficient to determine the corresponding Lie algebra. Because of the Lie-bracket structure, we also need to establish commutation relations of basis elements. Due to bilinearity of the Lie bracket, we may then uncover the commutators of all vector pairs.

Definition 1.3. Let L be a Lie algebra and let $\mathcal{B} = (x_1, \ldots, x_n)$ be a basis for L. The *structure constants* of L with respect to the basis $\mathcal{B}$ are complex numbers a_{ij}^k such that for all $i, j \in \widehat{n}$ we can write

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k. \tag{1.3}$$

Example 1.2. Suppose that V is a vector space. By $\mathsf{gl}(V)$ we denote the set of all linear maps from V to V. It is well-known that $\mathsf{gl}(V)$ is a vector space as well (cf. [11]). Furthermore, it becomes a Lie algebra with the Lie bracket $[\ ,\ ]$ defined for all $x, y \in \mathsf{gl}(V)$ by

$$[x, y] := x \circ y - y \circ x. \tag{1.4}$$

Clearly, $[\ ,\ ]$ is bilinear, it maps into $\mathsf{gl}(V)$ and for all $x \in \mathsf{gl}(V)$ we have

$$[x, x] = x \circ x - x \circ x = 0,$$

thus it remains to check that the Jacobi identity holds. Suppose arbitrary $x, y, z \in \mathsf{gl}(V)$, substituting by (1.4), we obtain

$$\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= x \circ y \circ z - x \circ z \circ y - y \circ z \circ x + z \circ y \circ x \\
&\quad + y \circ z \circ x - y \circ x \circ z - z \circ x \circ y + x \circ z \circ y \\
&\quad + z \circ x \circ y - z \circ y \circ x - x \circ y \circ z + y \circ x \circ z \\
&= 0.
\end{aligned}$$

Hence $(\mathsf{gl}(V); [\ ,\ ])$ is a Lie algebra indeed. This Lie algebra is called the *general linear algebra*.

Example 1.3. Let $n \in \mathbb{N}$. By $\mathsf{gl}(n, \mathbb{C})$ we denote the set of all $n \times n$ matrices with entries from $\mathbb{C}$. This is again a vector space. To upgrade $\mathsf{gl}(n, \mathbb{C})$ to the Lie algebra, we define the Lie bracket $[\ ,\ ]$ for all $x, y \in \mathsf{gl}(n, \mathbb{C})$ by

$$[x, y] := xy - yx. \tag{1.5}$$

Replacing the map composition "$\circ$" by the multiplication of matrices, we can iterate the procedure from Example 1.2 to check that also in this case $[\ ,\ ]$ is the Lie bracket and hence $(\mathsf{gl}(n, \mathbb{C}); [\ ,\ ])$ is a Lie algebra.

Definition 1.4. The Lie algebra L is said to be *abelian* if for any $x, y \in L$ it holds true that

$$[x, y] = 0. \tag{1.6}$$

1.1.2 Subalgebras and Ideals

Given a Lie algebra L, one often concerns with the subspaces of L which have the same algebraic structure as L. Such subspaces are called *subalgebras* (cf. [7], Sec. 1.3).

Definition 1.5. Let L be a Lie algebra. A *Lie subalgebra* of L is a vector subspace $K \subset\subset L$ such that for all $x, y \in K$ it is satisfied that

$$[x, y] \in K. \tag{1.7}$$

Remark 1.4. It can be easily seen that K (a Lie subalgebra of $(L; [\ ,\ ])$) becomes a Lie algebra in its own right with the Lie bracket $[\ ,\ ]_K$ defined as follows:

$$[\ ,\ ]_K := [\ ,\ ]|_{K \times K}. \tag{1.8}$$

Example 1.5. Let $\mathsf{n}(n, \mathbb{C})$ denote the subset of $\mathsf{gl}(n, \mathbb{C})$ consisting of all strictly upper triangular $n \times n$ matrices over $\mathbb{C}$. Since $\mathsf{n}(n, \mathbb{C})$ is closed under addition and scalar multiplication, $\mathsf{n}(n, \mathbb{C}) \subset\subset \mathsf{gl}(n, \mathbb{C})$. In addition, because matrix product of strictly upper triangular matrices is a strictly upper triangular matrix (see Example 1.17 where we prove even stronger result), $\mathsf{n}(n, \mathbb{C})$ is a Lie subalgebra of $\mathsf{gl}(n, \mathbb{C})$. In addition, by Remark 1.4, $(\mathsf{n}(n, \mathbb{C}), [\ ,\ ])$ with the Lie bracket $[\ ,\ ]$ defined by (1.5) creates a Lie algebra again.

Definition 1.6. Let L be a Lie algebra. An *ideal* of L is a vector subspace $I \subset\subset L$ such that for all $x \in L$ and $y \in I$ it is satisfied that

$$[x, y] \in I. \tag{1.9}$$

Obviously, any ideal is a subalgebra as well. Now, we illustrate how the definition property (1.9) of an ideal is mainly used, we give one useful example of an ideal and then we introduce three different ways of construction with ideals (cf. [7], Chap. 2).

Remark 1.6. Suppose that L is a Lie algebra and $L_1, \ldots, L_k$ are ideals of L such that, as vector spaces, $L = L_1 \oplus \cdots \oplus L_k$. The fact that L can be written as a direct sum of its ideals simplifies the Lie bracket on L. Any $x, y \in L$ can be uniquely decomposed as $x = x_i + \cdots + x_k$ and $y = y_i + \cdots + y_k$ where $x_i, y_i \in L_i$ for all $i \in \widehat{k}$. The commutator of x, y is as follows:

$$[x, y] = [\sum_{i=1}^{k} x_i, \sum_{j=1}^{k} y_j] = \sum_{i,j=1}^{k} [x_i, y_j].$$

Now, for all $i \neq j$ we have $[x_i, y_j] = 0$ because, as both L_i and L_j are ideals, $[x_i, y_j] \in L_i \cap L_j = 0$. After this simplification we may write

$$[x, y] = \sum_{i=1}^{k} [x_i, y_i]. \tag{1.10}$$

Example 1.7. Let L be a Lie algebra. We define the *center* of L by

$$Z(L) := \left\{ x \in L \;\middle|\; \text{for all } y \in L, [x, y] = 0 \right\}. \tag{1.11}$$

Since $0 \in Z(L)$, it is clear that Z(L) is an ideal of L.

Proposition 1.7. *Suppose that L is a Lie algebra and $L_1, \ldots, L_k$ are ideals of L such that $L = L_1 \oplus \cdots \oplus L_k$. Then $Z(L) = Z(L_1) \oplus \cdots \oplus Z(L_k)$.*

Proof. According to (1.10), for an arbitrary $x = \sum_{i=1}^{k} x_i \in L$, where $x_i \in L_i$, $i \in \widehat{k}$, we have: $x \in Z(L)$ if and only if $[x, y] = \sum_{i=1}^{k} [x_i, y_i] = 0$ holds true for all $y = \sum_{i=1}^{k} y_i \in L$, where $y_i \in L_i$, $i \in \widehat{k}$. This arises if and only if $x_i \in Z(L_i)$ for all $i \in \widehat{k}$ because $[x_i, y_i] \in L_i$ for all $i \in \widehat{k}$ and $L = L_1 \oplus \cdots \oplus L_k$. Uniqueness of the decomposition of $x \in Z(L)$ into $\sum_{i=1}^{k} x_i$, $x_i \in Z(L_i)$, is obvious since $Z(L) \subset\subset L$, $Z(L_i) \subset\subset L_i$ for all $i \in \widehat{k}$ and $L = L_1 \oplus \cdots \oplus L_k$. $\square$

Definition 1.8. Let I and J be ideals of a Lie algebra L. We define

$$[I, J] := \operatorname{Span} \left\{ [x, y] \;\middle|\; x \in I, \; y \in J \right\}. \tag{1.12}$$

In particular, $L' := [L, L]$ denotes the so-called *derived algebra* of L.

Proposition 1.9. *Let I and J be ideals of a Lie algebra L. Then $[I, J]$ is an ideal of L as well.*

Proof. First, $[I, J]$ is a subspace by its definition. Second, we have to show that the condition (1.9) is satisfied. Given $x \in L$ and $y \in [I, J]$, there exist $n \in \mathbb{N}$; $u_1, \ldots, u_n \in I$; $v_1, \ldots, v_n \in J$ and complex numbers $\alpha_1, \ldots, \alpha_1$ such that $y = \sum_{i=1}^{n} \alpha_i [u_i, v_i]$. Then bilinearity, Jacobi identity and anticommutativity for each summand give

$$\begin{aligned} [x, y] &= [x, \sum_{i=1}^{n} \alpha_i [u_i, v_i]] = \sum_{i=1}^{n} \alpha_i [x, [u_i, v_i]] = -\sum_{i=1}^{n} \alpha_i [u_i, [v_i, x]] - \sum_{i=1}^{n} \alpha_i [v_i, [x, u_i]] \\ &= \sum_{i=1}^{n} \alpha_i [u_i, [x, v_i]] + \sum_{i=1}^{n} \alpha_i [[x, u_i], v_i]. \end{aligned}$$

Since I and J are ideals, for all $i \in \widehat{n}$ we have $[x, v_i] \in J$ and $[x, u_i] \in I$. Thus, for all $i \in \widehat{n}$ it holds true that $[u_i, [x, v_i]] \in [I, J]$ and $[[x, u_i], v_i] \in [I, J]$ and, because $[I, J]$ is a subspace, $[x, y]$ also lies in $[I, J]$. $\square$

Definition 1.10. Let $(L; [\ ,\ ])$ be a Lie algebra and let I be an ideal of L. The quotient vector space L/I together with a map $[\ ,\ ]_q : L/I \times L/I \to L/I$ defined for all $\widetilde{x}, \widetilde{y} \in L/I$ by

$$[\widetilde{x}, \widetilde{y}]_q := \widetilde{[x, y]}, \ x \in \widetilde{x} \text{ and } y \in \widetilde{y}, \tag{1.13}$$

is called the *quotient Lie algebra* of L by I.

Remark 1.8. We have to check that the map $[\ ,\ ]_q$ from Definition 1.10 is well-defined i.e. $[\widetilde{x}, \widetilde{y}]_q$ does not depend on the choice of representatives $x \in \widetilde{x}$ and $y \in \widetilde{y}$. Suppose $x, x' \in \widetilde{x}$, $x \neq x'$ and $y, y' \in \widetilde{y}$, $y \neq y'$. Then there exist $u, v \in I \subset L$ such that $x - x' = u$ and $y - y' = v$. Using bilinearity of $[\ ,\ ]$, we obtain

$$[x, y] = [x' + u, y' + v] = [x', y'] + [x', v] + [u, y'] + [u, v],$$

where $[x', v] + [u, y'] + [u, v] =: z \in I$ because I is an ideal. Then we can write

$$[\widetilde{x}, \widetilde{y}]_q = \widetilde{[x, y]} = ([x', y'] + z)^{\sim} = \widetilde{[x', y']} = [\widetilde{x'}, \widetilde{y'}]_q.$$

Proposition 1.11. *Let L be a Lie algebra and let I be an ideal of L. The quotient Lie algebra of L by I is a Lie algebra.*

Proof. L/I is a vector space thus it remains to show that $[\ ,\ ]_q$ is the Lie bracket. Suppose $\widetilde{x}, \widetilde{y}, \widetilde{z} \in L/I$, $x \in \widetilde{x}$, $y \in \widetilde{y}$ and $z \in \widetilde{z}$. First, bilinearity is obviously consequence by bilinearity of the Lie bracket on L. Second,

$$[\widetilde{x}, \widetilde{x}]_q = \widetilde{[x, x]} = \widetilde{0} = 0 \in L/I.$$

Third,

$$\begin{aligned}
[\widetilde{x}, [\widetilde{y}, \widetilde{z}]_q]_q + [\widetilde{y}, [\widetilde{z}, \widetilde{x}]_q]_q + [\widetilde{z}, [\widetilde{x}, \widetilde{y}]_q]_q &= [\widetilde{x}, \widetilde{[y, z]}]_q + [\widetilde{y}, \widetilde{[z, x]}]_q + [\widetilde{z}, \widetilde{[x, y]}]_q \\
&= \widetilde{[x, [y, z]]} + \widetilde{[y, [z, x]]} + \widetilde{[z, [x, y]]} \\
&= ([x, [y, z]] + [y, [z, x]] + [z, [x, y]])^{\sim} \\
&= \widetilde{0} = 0 \in L/I.
\end{aligned}$$

$\square$

At the very end of this part we introduce two examples of subalgebras which we shall need later (cf. [9], p. 7).

Definition 1.12. Let L be a Lie algebra.

(a) Let A be a subalgebra of L. The *normalizer* of A in L is defined by

$$N_L(A) := \left\{ x \in L \mid \text{for all } a \in A, [x, a] \in A \right\}. \tag{1.14}$$

(b) Let X be a subset of L. The *centralizer* of X in L is defined by

$$C_L(X) := \left\{ x \in L \mid \text{for all } y \in X, [x, y] = 0 \right\}. \tag{1.15}$$

Proposition 1.13. *Let L be a Lie algebra.*

(a) Let A be a subalgebra of L. $N_L(A)$ is also a subalgebra of L.

(b) Let X be a subset of L. $C_L(X)$ is a subalgebra of L.

Proof. We use the Jacobi identity.

(a) Take arbitrary $x, y \in N_L(A)$ and $a \in A$. Then

$$[[x,y],a] = -[a,[x,y]] = [x,[y,a]] + [y[a,x]] = [x,[y,a]] - [y[x,a]] \in A$$

and hence $[x,y] \in N_L(A)$.

(b) Analogous to (a). □

1.1.3 Lie Algebra Homomorphisms

One might wonder whether two Lie algebras are "similar" in some sense i.e. whether there exists a map between them, preserving their algebraic structure. Like in other areas of algebra, such map is called a *homomorphism* or an *isomorphism*, if it is "one-to-one", and in case it exists, the respective Lie algebras are said to be *homomorphic* or *isomorphic*, eventually. Hereinafter, we state the precise definition, we give two examples and in the end we prove two important relations, the so-called *First* and *Second isomorphism theorems* (cf. [7]).

Definition 1.14. Let $(L_1; [\ ,\ ]_1)$ and $(L_2; [\ ,\ ]_2)$ be Lie algebras. A linear map $\varphi : L_1 \to L_2$ is called a *homomorphism* if for all $x, y \in L_1$ it is satisfied that

$$\varphi([x,y]_1) = [\varphi(x), \varphi(x)]_2. \tag{1.16}$$

We say that φ is an *isomorphism* when it is also bijective.

Remark 1.9. Lie algebras L_1 and L_2 are said to be *isomorphic* if there exists an isomorphism $\varphi : L_1 \to L_2$. We denote this relation $L_1 \cong L_2$.

Remark 1.10. Let L_1, L_2 be Lie algebras and let I and J be ideals of L_1. It is clear from Definition 1.14 that if $\varphi : L_1 \to L_2$ is a homomorphism, then $\varphi([I,J]) = [\varphi(I), \varphi(J)]$, in particular $\varphi(I') = (\varphi(I))'$.

Example 1.11. Let V be an n-dimensional vector space. Let $\mathsf{gl}(V)$ and $\mathsf{gl}(n,\mathbb{C})$ be the Lie algebras defined in Examples 1.2 and 1.3, respectively. When we fix a basis of V, we can define a map $\phi : \mathsf{gl}(V) \to \mathsf{gl}(n,\mathbb{C})$ sending a linear transformation of V to its transformation matrix with respect to the fixed basis. It is well-known from linear algebra that ϕ is a linear bijection and also that the transformation matrix of composition of two maps is the product of the transformation matrices of single maps. From the second fact it is easily seen that the Lie brackets (1.4) and (1.5) are "compatible" with each other and ϕ is an isomorphism. Hence $\mathsf{gl}(V) \cong \mathsf{gl}(n,\mathbb{C})$.

Example 1.12. Let $(L; [\ ,\ ])$ be a Lie algebra. For all $x, y \in L$ we define the *adjoint homomorphism* $\operatorname{ad} : L \to \mathsf{gl}(L)$ as follows:

$$(\operatorname{ad} x)(y) := [x,y]. \tag{1.17}$$

Bilinearity of the Lie bracket implies both linearity of ad and linearity of $\operatorname{ad} x$ for all $x \in L$. Hence, to prove that ad is a homomorphism indeed, we have to check only that

(1.16) holds with the Lie bracket on $\mathfrak{gl}(L)$ defined by (1.4). Using the Jacobi identity, bilinearity and anticommutativity of the Lie bracket on L, for any $x, y, z \in L$ we have

$$\begin{aligned}\operatorname{ad}([x,y])(z) &= [[x,y],z] = -[z,[x,y]] = [x,[y,z]] + [y,[z,x]] = [x,[y,z]] - [y,[x,z]] \\ &= \operatorname{ad} x(\operatorname{ad} y(z)) - \operatorname{ad} y(\operatorname{ad} x(z)) = (\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x)(z),\end{aligned}$$

as required.

If we define the adjoint homomorphism as $\operatorname{ad} : L \to \operatorname{ad}(L)$, then it is always surjective. Since $\operatorname{Ker}(\operatorname{ad}) = Z(L)$, as one can see from

$$x \in \operatorname{Ker}(\operatorname{ad}) \iff \operatorname{ad} x = 0 \iff \text{for all } y \in L,\ \operatorname{ad} x(y) = [x,y] = 0 \iff x \in Z(L),$$

homomorphism $\operatorname{ad} : L \to \operatorname{ad}(L)$ is bijective, and hence it becomes an isomorphism, if and only if $Z(L) = 0$.

Lemma 1.15 (First Isomorphism Theorem)**.** *Let $\varphi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then* $\operatorname{Ker} \varphi$ *is an ideal of L_1 and* $\operatorname{Ran} \varphi$ *is a subalgebra of L_2. Moreover,*

$$L_1 / \operatorname{Ker} \varphi \cong \operatorname{Ran} \varphi.$$

Proof. First, suppose arbitrary $x \in L_1$ and $y \in \operatorname{Ker} \varphi$. Then

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] = [\varphi(x), 0] = 0.$$

Hence $[x,y] \in \operatorname{Ker} \varphi$ and, since kernel of a linear map is always a subspace, $\operatorname{Ker} \varphi$ is an ideal. Second, given $u, v \in \operatorname{Ran} \varphi$, there exist $x, y \in L_1$ such that $u = \varphi(x)$ and $v = \varphi(y)$. Then

$$[u,v] = [\varphi(x), \varphi(y)] = \varphi([x,y])$$

and thus $[u,v] \in \operatorname{Ran} \varphi$. Since range of a linear map is a subspace again, $\operatorname{Ran} \varphi$ is a subalgebra. Finally, we prove that $L_1 / \operatorname{Ker} \varphi$ and $\operatorname{Ran} \varphi$ are isomorphic. For all $\widetilde{x} \in L_1 / \operatorname{Ker} \varphi$ we define a map $\psi : L_1 / \operatorname{Ker} \varphi \to \operatorname{Ran} \varphi$ by

$$\psi(\widetilde{x}) := \varphi(x),\ x \in \widetilde{x}.$$

Because it holds for any $x, x' \in \widetilde{x}$ that $x - x' \in \operatorname{Ker} \varphi$ and $\varphi(x) = \varphi(x')$ consequently, ψ is well-defined. Since $L_1 / \operatorname{Ker} \varphi$ is a linear space and φ is a linear map, for all complex α and $\widetilde{x}, \widetilde{y} \in L_1 / \operatorname{Ker} \varphi$ we can write

$$\psi(\alpha\widetilde{x} + \widetilde{y}) = \psi(\widetilde{\alpha x + y}) = \varphi(\alpha x + y) = \alpha\varphi(x) + \varphi(y) = \alpha\psi(\widetilde{x}) + \psi(\widetilde{y}).$$

This proves linearity of ψ. Injectivity of ψ results from the following implications:

$$\psi(\widetilde{x}) = 0 \iff \varphi(x) = 0 \iff x \in \operatorname{Ker} \varphi \iff \widetilde{x} = \widetilde{0}.$$

Next, given $y \in \operatorname{Ran} \varphi$, there exists $x \in L_1$ such that $\varphi(x) = y$. It suffices to put $\psi^{(-1)}(y) = \widetilde{x}$ to prove surjectivity of ψ. Finally, considering φ is a homomorphism, we show that ψ is a homomorphism as well. Given any $\widetilde{x}, \widetilde{y} \in L_1 / \operatorname{Ker} \varphi$, we can write

$$\psi([\widetilde{x}, \widetilde{y}]) = \psi(\widetilde{[x,y]}) = \varphi([x,y]) = [\varphi(x), \varphi(y)] = [\psi(\widetilde{x}), \psi(\widetilde{y})].$$

Hence ψ is the required isomorphism between $L_1 / \operatorname{Ker} \varphi$ and $\operatorname{Ran} \varphi$. □

Corollary 1.16 (Second Isomorphism Theorem)**.** *Let I an J be ideals of a Lie algebra L. Then*

$$I/(I \cap J) \cong (I + J)/J.$$

Proof. For all $x \in I$ we define a map $\varphi\colon I \to (I+J)/J$ by

$$\varphi(x) := \widetilde{x}.$$

First, we explore the kernel of φ. For any $x \in I$ we have

$$\varphi(x) = \widetilde{0} \in (I+J)/J \iff x \in J$$

and hence $\operatorname{Ker}\varphi = I \cap J$. Second, we show surjectivity of φ. Take any $\widetilde{y} \in (I+J)/I$. There exist $a \in I$ and $b \in J$ such that $y = a + b \in I + J$ and we claim that one might put $\varphi^{(-1)}(\widetilde{y}) = a$. Indeed, $\varphi(a) = \widetilde{a} = \widetilde{a+b} = \widetilde{y}$, and therefore $\operatorname{Ran}\varphi = (I+J)/J$. After all, we prove that φ is a homomorphism. For arbitrary $x, y \in I$ we have

$$\varphi([x,y]) = \widetilde{[x,y]} = [\widetilde{x},\widetilde{y}] = [\varphi(x),\varphi(y)].$$

Thus φ is a homomorphism with $\operatorname{Ker}\varphi = I \cap J$ and $\operatorname{Ran}\varphi = (I+J)/J$. Now we can apply Lemma 1.15 in order to get the statement. □

1.1.4 $\mathsf{sl}(n,\mathbb{C})$ and $\mathsf{sl}(V)$

In this final part of the first section we describe important examples of subalgebras of two Lie algebras that we have already introduced. We shall use the following examples later, mainly in Chapter 2.

Example 1.13. Suppose $n \in \mathbb{N}$. Let $\mathsf{sl}(n,\mathbb{C})$ denote the subset of $\mathsf{gl}(n,\mathbb{C})$ consisting of all $n \times n$ complex matrices of trace 0. Clearly, since the trace form is linear (cf. [1], p. 127), $\mathsf{sl}(n,\mathbb{C}) \subset\subset \mathsf{gl}(n,\mathbb{C})$. Further, for any $A, B \in \mathsf{gl}(n,\mathbb{C})$ we have

$$\operatorname{Tr}([A,B]) = \sum_{i=1}^{n}(AB-BA)_{ii} = \sum_{i,j=1}^{n} A_{ij}B_{ji} - \sum_{i,j=1}^{n} B_{ij}A_{ji} = 0. \tag{1.18}$$

In particular, when we take A, B from $\mathsf{sl}(n,\mathbb{C})$ only, (1.18) implies that $\mathsf{sl}(n,\mathbb{C})$ is a subalgebra of $\mathsf{gl}(n,\mathbb{C})$.

Moreover, (1.18) means that $(\mathsf{gl}(n,\mathbb{C}))' \subset \mathsf{sl}(n,\mathbb{C})$. We claim that even $\mathsf{gl}(n,\mathbb{C})' = \mathsf{sl}(n,\mathbb{C})$. To prove this statement, we first apply the rank-nullity theorem (cf. [11], page 61) to the trace map $\operatorname{Tr} : \mathsf{gl}(n,\mathbb{C}) \to \mathbb{C}$. Obviously, $\operatorname{Ker}\operatorname{Tr} = \mathsf{sl}(n,\mathbb{C})$ and $\operatorname{Ran}\operatorname{Tr} = \mathbb{C}$. Hence we obtain

$$\dim \mathsf{sl}(n,\mathbb{C}) = \dim(\operatorname{Ker}\operatorname{Tr}) = \dim \mathsf{gl}(n,\mathbb{C}) - \dim \operatorname{Ran}\operatorname{Tr} = n^2 - 1.$$

Now, let us consider the standard basis of $\mathsf{gl}(n,\mathbb{C})$:

$$\left\{ E_{ij} \in \mathsf{gl}(n,\mathbb{C}) \;\middle|\; i,j \in \widehat{n} \text{ and for all } k,l \in \widehat{n},\, \left(E_{ij}\right)_{kl} = \delta_{ik}\delta_{jl} \right\}.$$

For all $i, j, k, l, a, b \in \widehat{n}$ we have

$$\begin{aligned}
[E_{ij}, E_{kl}]_{ab} &= \sum_{m=1}^{n} (E_{ij})_{am}(E_{kl})_{mb} - \sum_{m=1}^{n} (E_{kl})_{am}(E_{ij})_{mb} \\
&= \sum_{m=1}^{n} \delta_{ia}\delta_{jm}\delta_{km}\delta_{lb} - \sum_{m=1}^{n} \delta_{ka}\delta_{lm}\delta_{im}\delta_{jb} \\
&= \delta_{ia}\delta_{jk}\delta_{lb} - \delta_{ka}\delta_{il}\delta_{jb} \\
&= \delta_{jk}(E_{il})_{ab} - \delta_{il}(E_{kj})_{ab}
\end{aligned}$$

and hence for all $i,j,k,l \in \widehat{n}$ it holds true that $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$. Particularly, for any $i,j \in \widehat{n}$ such that $i \neq j$ we have $[E_{ij}, E_{jj}] = E_{ij} \in \mathsf{gl}(n,\mathbb{C})'$ and further for any $i \in \widehat{n-1}$ we have $[E_{i,i+1}, E_{i+1,i}] = E_{i,i} - E_{i+1,i+1} \in \mathsf{gl}(n,\mathbb{C})'$. In this way we have found $(n^2-n)+(n-1) = n^2-1$ linearly independent vectors from $\mathsf{gl}(n,\mathbb{C})'$. Now, since $\mathsf{gl}(n,\mathbb{C})' \subset \mathsf{sl}(n,\mathbb{C})$ and $\dim \mathsf{sl}(n,\mathbb{C}) = n^2-1$, it is clear that $\mathsf{gl}(n,\mathbb{C})' = \mathsf{sl}(n,\mathbb{C})$, as desired.

Example 1.14. Suppose V is a vector space. Similarly as in Example 1.13, let $\mathsf{sl}(V)$ denote the subset of $\mathsf{gl}(V)$ consisting of those linear transformations of V whose trace is zero. We use the previous example together with the fact that $\mathsf{gl}(V) \cong \mathsf{gl}(n,\mathbb{C})$, where $n = \dim V$ (cf. Example 1.11), to show also that $\mathsf{gl}(V)' = \mathsf{sl}(V)$. Fix a basis of V and consider the isomorphism $\phi : \mathsf{gl}(n,\mathbb{C}) \to \mathsf{gl}(V)$ sending a matrix to the linear map which the matrix stands for (with respect to the fixed basis). Note that ϕ is the inverse of the isomorphism considered in Example 1.11. According to Remark 1.10, we then may write

$$\mathsf{gl}(V)' = \phi(\mathsf{gl}(n,\mathbb{C}))' = \phi(\mathsf{gl}(n,\mathbb{C})') = \phi(\mathsf{sl}(n,\mathbb{C})).$$

Now, it suffices to realize that a linear operator has trace 0 precisely when its matrix (with respect to an arbitrary basis) has the zero trace as well (cf. [14], p. 386) and thus $\mathsf{gl}(V)' = \phi(\mathsf{sl}(n,\mathbb{C})) = \mathsf{sl}(V)$, as anticipated.

1.2 Decomposition of Lie Algebras

In this section we introduce three particular classes of Lie algebras, the so-called *nilpotent*, *solvable* and *semisimple* Lie algebras. Then we show that each Lie algebra has the unique biggest (in the sense of inclusion) solvable ideal and the quotient of the Lie algebra by this ideal is always semisimple (cf. [7], Chap. 4). Note that nilpotent Lie algebras are in a close relationship with Lie algebras of two other types as discussed in Section 1.3 below.

Definition 1.17. Let L be a Lie algebra. The *lower central series* of L is the sequence $\{L^n\}_{n=0}^{\infty}$ with terms

$$\begin{aligned} L^0 &= L \text{ and} \\ L^n &= [L, L^{n-1}] \text{ for } n \geq 1. \end{aligned} \tag{1.19}$$

Definition 1.18. Let L be a Lie algebra. The *derived series* of L is the sequence $\left\{L^{(n)}\right\}_{n=0}^{\infty}$ with terms

$$\begin{aligned} L^{(0)} &= L \text{ and} \\ L^{(n)} &= [L^{(n-1)}, L^{(n-1)}] \text{ for } n \geq 1. \end{aligned} \tag{1.20}$$

Remark 1.15. According to Proposition 1.9, $L^{(n)}$ and L^n are ideals of L for all $n \in \mathbb{N}_0$.

Remark 1.16. Notice that $L^1 = L^{(1)} = L'$.

Lemma 1.19. *Let L_1 and L_2 be Lie algebras and let $\varphi : L_1 \to L_2$ be a surjective homomorphism. Then for all $n \in \mathbb{N}_0$ it holds that*

(a) $\varphi(L_1^n) = L_2^n$,

(b) $\varphi(L_1^{(n)}) = L_2^{(n)}$.

Proof. We use induction on n. If $n = 0$, we have $L_1^0 = L_1^{(0)} = L_1$ and $L_2^0 = L_2^{(0)} = L_2$. It follows directly from the definition of surjection that $\varphi(L_1^0) = L_2^0$ and $\varphi(L_1^{(0)}) = L_2^{(0)}$.

(a) Now suppose that $\varphi(L_1^{n-1}) = L_2^{n-1}$. Then, since φ is a homomorphism,

$$\varphi(L_1^n) = \varphi([L_1, L_1^{n-1}]) = [\varphi(L_1), \varphi(L_1^{n-1})] = [L_2, L_2^{n-1}] = L_2^n.$$

(b) Analogous to (a). Suppose that $\varphi(L_1^{(n-1)}) = L_2^{(n-1)}$, then

$$\varphi(L_1^{(n)}) = \varphi([L_1^{(n-1)}, L_1^{(n-1)}]) = [\varphi(L_1^{(n-1)}), \varphi(L_1^{(n-1)})] = [L_2^{(n-1)}, L_2^{(n-1)}] = L_2^{(n)}.$$

□

Definition 1.20. The Lie algebra L is said to be *nilpotent* if there exists $n \in \mathbb{N}$ such that $L^n = 0$.

Example 1.17. We claim that $\mathsf{n}(n, \mathbb{C})$ (defined in Example 1.5) is a nilpotent Lie algebra. To prove this, we first show that for all $A_1, \ldots, A_k \in \mathsf{n}(n, \mathbb{C})$, where k is an arbitrary number from $\widehat{n}$, it holds true that

$$(\prod_{i=1}^{k} A_i)_{ab} = 0 \tag{1.21}$$

whenever $a > b - k$. We use incomplete induction on $k \in \widehat{n}$. If $k = 1$, we have only one single strictly upper triangular matrix and our claim follows from its definition. For the inductive step assume that $a > b - k + 1$ implies $B_{ab} := (\prod_{i=1}^{k-1} A_i)_{ab} = 0$. Then, when we realize which summands are zero, we can write

$$(\prod_{i=1}^{k} A_i)_{ab} = \sum_{c=1}^{n} B_{ac}(A_k)_{cb} = \sum_{c=a+k-1}^{b-1} B_{ac}(A_k)_{cb}$$

which clearly equals to zero when $a > b - k$. Thus we have proved (1.21) for all $k \in \widehat{n}$.

Particularly, putting $k = n$, for all $a, b \in \widehat{n}$ we obtain $(\prod_{i=1}^{n} A_i)_{ab} = 0$ and hence product of n strictly upper triangular matrices is always the zero matrix. Now, considering how the respective Lie bracket is defined, one can see easily that $\mathsf{n}(n, \mathbb{C})$ is nilpotent, indeed.

Lemma 1.21. *Let L be a Lie algebra.*

(a) If L is nilpotent, then every subalgebra and every homomorphic image of L are nilpotent.

(b) If $L/Z(L)$ is nilpotent, then L is nilpotent.

Proof.

(a) Suppose L_1 is a subalgebra of L. Clearly, for all $k \in \mathbb{N}_0$ it holds $L_1^k \subset L^k$, thus if $L^m = 0$, then also $L_1^m = 0$. For the second part, let us consider a homomorphism $\varphi : L \to \varphi(L)$ which is obviously surjective. Then, according to Lemma 1.19 (a), $L^m = 0$ implies $(\varphi(L))^m = 0$.

(b) For all $x \in L$ we define a map $\varphi : L \to (L + Z(L))/Z(L)$ by $\varphi(x) := \widetilde{x}$. By exactly the same arguments as in the proof of Corollary 1.16, φ is a surjective homomorphism and hence we may use part (a) of Lemma 1.19 for all $k \in \mathbb{N}_0$ to obtain

$$(L/Z(L))^k = ((L + Z(L))/Z(L))^k = (\varphi(L))^k = \varphi(L^k) = (L^k + Z(L))/Z(L).$$

Then, since there exists $m \in \mathbb{N}_0$ such that $(L/Z(L))^m = 0$, $L^m \subset Z(L)$ and hence $L^{m+1} = [L, L^m] \subset [L, Z(L)] = 0$. □

Definition 1.22. The Lie algebra L is said to be *solvable* if there exists $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

Lemma 1.23. *Let L be a Lie algebra.*

(a) If L is solvable, then every subalgebra and every homomorphic image of L are solvable.

(b) If I is an ideal of L such that both I and L/I are solvable, then L is solvable.

(c) If I and J are solvable ideals of L, then $I + J$ is a solvable ideal of L.

Proof.

(a) Let L_1 be a subalgebra of L. Since for all $k \in \mathbb{N}_0$ it holds $L_1^{(k)} \subset L^{(k)}$, if $L^{(m)} = 0$, then also $L_1^{(m)} = 0$. For the second part, suppose $\varphi : L \to \varphi(L)$ is a homomorphism. φ is surjective by definition and hence, according to Lemma 1.19 (b), $(\varphi(L))^{(m)} = 0$ results from $L^{(m)} = 0$.

(b) For all $x \in L$ we define a map $\varphi : L \to (L + I)/I$ by $\varphi(x) := \tilde{x}$. Again, exactly repeating the proof of Corollary 1.16, one can show that φ is a surjective homomorphism and when we use part (b) of Lemma 1.19, for all $k \in \mathbb{N}_0$ we obtain

$$(L/I)^{(k)} = ((L + I)/I)^{(k)} = (\varphi(L))^{(k)} = \varphi(L^{(k)}) = (L^{(k)} + I)/I.$$

Now, as $(L/I)^{(m)} = 0$ for some m, $L^{(m)} \subset I$ and hence, because $I^{(n)} = 0$ for some n, $L^{(m+n)} = (L^{(m)})^{(n)} \subset I^{(n)} = 0$.

(c) Again, for all $x \in I$ we define a map $\varphi : I \to (I + J)/J$ by $\varphi(x) := \tilde{x}$ and again it is a surjective homomorphism by the proof of Corollary 1.16. Then, since I is solvable, $(I + J)/J$ is also solvable by part (a) of this lemma and finally, since J is solvable, part (b) implies solvability of $I + J$. □

Corollary 1.24. *For each Lie algebra L there exists a unique solvable ideal of L containing every solvable ideal of L.*

Proof. Let R be a solvable ideal of L such that any other solvable ideal L does not have larger dimension than R (recall that we assume L to be finite-dimensional). Let I be one of the other solvable ideals. By part (c) of Lemma 1.23, $R + I$ is a solvable ideal again and clearly $\dim(R + I) \geq \dim R$. In addition, by our assumption, $\dim R \geq \dim(R + I)$ and hence $\dim R = \dim(R + I)$ and $I \subset R$. □

Definition 1.25. Let L be a Lie algebra. The largest solvable ideal of L from Corollary 1.24 is said to be the *radical* of L and it is denoted by $\operatorname{rad} L$.

Definition 1.26. The Lie algebra L is said to be *semisimple* if $\operatorname{rad} L = 0$.

Proposition 1.27. *Let L be a semisimple Lie algebra. Then:*

(a) every isomorphic image of L is semisimple as well;

(b) $Z(L) = 0$.

Proof.

(a) Suppose that $\varphi : L \to \varphi(L)$ is an isomorphism of Lie algebras such that $\operatorname{rad} \varphi(L) \neq 0$ and hence $\varphi(L)$ has a non-trivial solvable ideal, say J. But part (a) of Lemma 1.23 implies that $\varphi^{-1}(J)$ is a non-trivial solvable ideal of L, a contradiction.

(b) Given a Lie algebra L, its center $Z(L)$ is, from its definition (see Example 1.7), a solvable ideal of L: $[Z(L), Z(L)] \subset [L, Z(L)] = 0$. As L is semisimple, $Z(L) = 0$. □

Lemma 1.28. *Let L be a Lie algebra. Then $L/\operatorname{rad} L$ is semisimple.*

Proof. Let I be a solvable ideal of $L/\operatorname{rad} L$. We define $J := \{x \in L \mid \widetilde{x} \in I\}$ and we show that it is an ideal of L. For any $x, y \in J$ we have $\widetilde{[x,y]} = [\widetilde{x}, \widetilde{y}] \in I$ and hence, by the definition of J, $[x,y] \in J$, indeed. In addition, it is easily seen from the definition of J that $I = J/\operatorname{rad} L$. Both $\operatorname{rad} L$ and $J/\operatorname{rad} L$ are solvable and part (b) of Lemma 1.23 then implies solvability of J. As a solvable ideal of L, J is contained in $\operatorname{rad} L$, thus $I = J/\operatorname{rad} L = 0$. This holds for each solvable ideal I of $L/\operatorname{rad} L$ and hence $\operatorname{rad}(L/\operatorname{rad} L) = 0$. □

1.3 Semisimple Lie Algebras

1.3.1 Theorems of Engel and Lie

At the beginning of this section we state two theorems that are crucial in the theory of semisimple Lie algebras, namely *Engel's Theorem* and *Lie's Theorem*. The proofs of both these theorems are quite long and technical, so we omit them. They can be found e.g. in [10].

Theorem 1.29 (Engel)**.** *Let V be a vector space and let L be a Lie subalgebra of $\mathfrak{gl}(V)$ such that each $x \in L$ is a nilpotent linear map. Then there exists a basis of V in which each $x \in L$ is represented by a strictly upper triangular matrix.*

Corollary 1.30. *The Lie algebra L is nilpotent if and only if for any $x \in L$ $(\operatorname{ad} x)\colon L \to L$ is a nilpotent linear map.*

Proof. To prove the "only if" direction, we suppose that L is a nilpotent Lie algebra. In other words

$$[[\dots[[\overbrace{L,L]L],\dots,L],L]}^{m \text{ times}} = 0 \Longleftrightarrow \overbrace{[L,[L,\dots,[L,[L,}^{m \text{ times}}L]]\dots]] = 0.$$

Hence for all $x_1, x_2, \dots, x_{m+1} \in L$ we have $[x_1, [x_2, \dots, [x_{m-1}, [x_m, x_{m+1}]]\dots]] = 0$, particularly for all $x, y \in L$ it holds true that

$$(\operatorname{ad} x)^m(y) = \overbrace{[x,[x,\dots,[x,}^{m \text{ times}}y]\dots]] = 0.$$

For the second direction, we assume nilpotency of $\operatorname{ad} x$ for any $x \in L$. By Engel's Theorem, there exists a basis $\mathcal{B}$ of L such that, with respect to $\mathcal{B}$, for all $x \in L$, $\operatorname{ad} x$ is represented by a strictly upper triangular matrix. As $\operatorname{ad}\colon L \to \mathsf{gl}(L)$ is a homomorphism, by Lemma 1.15, $\operatorname{ad}(L) = \operatorname{Ran}(\operatorname{ad})$ is a subalgebra of $\mathsf{gl}(L)$ which is isomorphic to $\mathsf{gl}(\dim L, \mathbb{C})$ according to Example 1.11. In agreement with this example we consider an isomorphism ϕ sending a linear transformation of L to its transformation matrix with respect to $\mathcal{B}$. Obviously, the image of $\operatorname{ad}(L)$ under this isomorphism is also a subalgebra (of $\mathsf{gl}(\dim L, \mathbb{C})$). Moreover, $\phi(\operatorname{ad}(L)) \subset \mathsf{n}(\dim L, \mathbb{C})$ and hence $\phi(\operatorname{ad}(L))$ is nilpotent. Contrariwise, as ϕ^{-1} is a homomorphism as well, part (a) of Lemma 1.21 gives that $\operatorname{ad}(L)$ is also nilpotent. In addition, Lemma 1.15 together with Example 1.12 imply that $\operatorname{ad}(L) \cong L/Z(L)$, hence $L/Z(L)$ is also nilpotent and finally, by Lemma 1.21 (b), L is nilpotent as well. □

Theorem 1.31 (Lie)**.** *Let V be a vector space and let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists a basis of V in which each $x \in L$ is represented by an upper triangular matrix.*

1.3.2 Jordan Decomposition

In this subsection we introduce, without proof, a useful result from linear algebra concerning the so-called *Jordan decomposition* of a linear map (cf. [2], p. 53, 54). Then, we may apply it to variant subalgebras of general linear algebra which contains right linear maps (cf. [9], Sec. 4.2).

Lemma 1.32. *Let V be a vector space and let $x : V \to V$ be a linear map. Then there exist unique linear maps $d : V \to V$ and $n : V \to V$ such that d is diagonalisable, n is nilpotent, d and n commute and*

$$x = d + n. \tag{1.22}$$

Definition 1.33. Let x be a linear transformation of the vector space V. Equation (1.22), where d and n are the maps from the previous lemma, is called the *Jordan decomposition* of x.

Remark 1.18. From now on, we will use the following convention. Talking about the Jordan decomposition of a linear map, we will always keep to the same order of the two summands in the decomposition. The order will be such that the first summand is diagonalisable and the second one is nilpotent. To avoid an ambiguity, even if any summand is zero, we will never omit it.

Lemma 1.34. *Let V be a vector space. Suppose that $x \in gl(V)$ has Jordan decomposition $d + n$. Then $\operatorname{ad} x : gl(V) \to gl(V)$ has Jordan decomposition $\operatorname{ad} d + \operatorname{ad} n$.*

Proof. First of all, linearity of ad implies $\operatorname{ad}(d+n)$=$\operatorname{ad} d + \operatorname{ad} n$. Since $\operatorname{ad} x$ is a linear map, it suffices to show that if d is diagonalisable, n is nilpotent and d and n commute, then $\operatorname{ad} d$ is diagonalisable, $\operatorname{ad} n$ is nilpotent and $\operatorname{ad} d$ and $\operatorname{ad} n$ commute. Then, uniqueness of the Jordan decomposition already proves the lemma.

Suppose that $d \in \mathrm{gl}(V)$ is a diagonalisable map. There exists a basis $\mathcal{B} = (b_1, \ldots, b_m)$ of V consisting of eigenvectors for d entirely. Let $(\lambda_i)_{i=1}^m$ denote the respective eigenvalues, so for all $i \in \widehat{m}$ we have $d(b_i) = \lambda_i b_i$. Remark that each $b \in V$ can be uniquely decomposed as $b = \sum_{i=1}^m \alpha_i b_i$. Now, let us denote

$$\mathcal{A} := \left\{ A_{ij} \in \mathrm{gl}(V) \;\middle|\; i,j \in \widehat{m} \text{ and for all } k,l \in \widehat{m}, \left({}^{\mathcal{B}}A_{ij}\right)_{kl} = \delta_{ik}\delta_{jl} \right\}.$$

It is not hard to see that $\mathcal{A}$ is a basis for $\mathrm{gl}(V)$. For arbitrary $A_{ij} \in \mathcal{A}$ and $b = \sum_{i=1}^m \alpha_i b_i \in V$ we can write

$$\begin{aligned}(\operatorname{ad} d(A_{ij}))(b) &= [d, A_{ij}](b) = (d \circ A_{ij} - A_{ij} \circ d)(\sum_{k=1}^m \alpha_k b_k) = d(\alpha_j b_i) - A_{ij}(\sum_{k=1}^m \alpha_k \lambda_k b_k) \\ &= \alpha_j \lambda_i b_i - \alpha_j \lambda_j b_i = (\lambda_i - \lambda_j)\alpha_j b_i = (\lambda_i - \lambda_j) A_{ij}(\sum_{k=1}^m \alpha_k \lambda_k b_k) \\ &= (\lambda_i - \lambda_j) A_{ij}(b)\end{aligned}$$

to show that $\mathcal{A}$ consists of eigenvectors for $\operatorname{ad} d$ and thus $\operatorname{ad} d$ is diagonalisable.

For the second part suppose that there exists $r \in \mathbb{N}$ such that $n^r = 0$. For an arbitrary $y \in \mathrm{gl}(V)$, if we write out $(\operatorname{ad} n)^{2r}(y) = \overbrace{[n,[n,\ldots,[n,}^{2r \text{ times}}y]\ldots]]$ by the definition of the Lie bracket on $\mathrm{gl}(V)$, we obtain a sum of linear maps from $\mathrm{gl}(V)$ such that each summand is of the form $n^j y n^{2r-j}$ for some $j \in \widehat{\underline{2r}}$ and thus, since $n^r = 0$, it equals to zero. Overall, $(\operatorname{ad} n)^{2r}$ is the zero map and hence $\operatorname{ad} n$ is nilpotent.

Finally, as ad is a homomorphism and d and n commute, we have

$$[\operatorname{ad} d, \operatorname{ad} n] = \operatorname{ad}[d,n] = 0.$$

□

Remark 1.19. It results directly from the previous proof that if a diagonalisable map $d \in \mathrm{gl}(V)$ has eigenvalues $(\lambda_i)_{i=1}^m$, then $\mathrm{ad}\, d$ has eigenvalues $\left(\lambda_i - \lambda_j\right)_{i,j=1}^m$.

At the end of this subsection we state one more lemma related to the Jordan decomposition that we shall need later. Again we omit the proof which is based only on results from linear algebra and which can be found in [7], Appendix A.

Lemma 1.35. *Let x be a linear transformation of the vector space V. Suppose that $x = d + n$ is the Jordan decomposition of x.*

(a) There is a polynomial $p(t) \in \mathbb{C}[t]$ such that $p(x) = d$.

(b) Let $\mathcal{B}$ be a basis of V in which $D := {}^{\mathcal{B}}d$ is a diagonal matrix. Let $\overline{d}$ be a linear map such that ${}^{\mathcal{B}}\overline{d} = D^H$. Then there is a polynomial $q(t) \in \mathbb{C}[t]$ so that $q(x) = \overline{d}$.

Remark 1.20. A simple corollary of part (a) of the previous lemma is that there also exists a polynomial $\widehat{p}(t) \in \mathbb{C}[t]$ such that $\widehat{p}(x) = n$. Obviously, this is satisfied for $\widehat{p}(t) = t - p(t)$, where $p(t)$ is the polynomial from the lemma.

1.3.3 Decomposition of Semisimple Lie Algebras

In this final part of the first chapter, using previous results, we show that each semisimple Lie algebra is further decomposed into the direct sum of the so-called *simple* Lie algebras. As the titles suggest, the structure of simple Lie algebras is even simpler than the structure of semisimple ones; a simple Lie algebra, has even no ideals except itself and the zero subspace (cf. [7], Chap. 9).

Definition 1.36. Let L be a Lie algebra. The *Killing form* on L is the form $\kappa : L \times L \to \mathbb{C}$ defined for all $x, y \in L$ by
$$\kappa(x,y) := \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y). \tag{1.23}$$

Proposition 1.37. *The Killing form on a Lie algebra L is bilinear, symmetric and associative, in the sense that for all $x, y, z \in L$ it is satisfied that*
$$\kappa([x,y],z) = \kappa(x,[y,z]). \tag{1.24}$$

Proof. The trace form $\mathrm{Tr} : \mathrm{gl}(L) \to \mathbb{C}$ is a linear functional satisfying for any maps $A, B \in \mathrm{gl}(L)$ the following identity:
$$\mathrm{Tr}(A \circ B) = \mathrm{Tr}(B \circ A) \tag{1.25}$$
(cf. [1], p. 127 and [14], p. 386). Now, linearity of the Killing form results clearly from linearity of ad, bilinearity of the composition of two maps and linearity of Tr. Next, (1.25) implies symmetry of κ. Finally, using (1.25) again, for arbitrary $x, y, z \in L$ we can write
$$\begin{aligned}
\kappa([x,y],z) &= \mathrm{Tr}(\mathrm{ad}[x,y] \circ \mathrm{ad}\, z) = \mathrm{Tr}([\mathrm{ad}\, x, \mathrm{ad}\, y] \circ \mathrm{ad}\, z) \\
&= \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y \circ \mathrm{ad}\, z - \mathrm{ad}\, y \circ \mathrm{ad}\, x \circ \mathrm{ad}\, z) \\
&= \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y \circ \mathrm{ad}\, z) - \mathrm{Tr}(\mathrm{ad}\, y \circ (\mathrm{ad}\, x \circ \mathrm{ad}\, z)) \\
&= \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y \circ \mathrm{ad}\, z) - \mathrm{Tr}((\mathrm{ad}\, x \circ \mathrm{ad}\, z) \circ \mathrm{ad}\, y) \\
&= \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y \circ \mathrm{ad}\, z - \mathrm{ad}\, x \circ \mathrm{ad}\, z \circ \mathrm{ad}\, y) \\
&= \mathrm{Tr}(\mathrm{ad}\, x \circ [\mathrm{ad}\, y, \mathrm{ad}\, z]) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}[y,z]) \\
&= \kappa(x,[y,z])
\end{aligned}$$

□

Proposition 1.38. *Let L be a Lie algebra with the Killing form κ. Let I be an ideal of L with the Killing form κ_I. Then $\kappa_I = \kappa|_{I\times I}$.*

Proof. Let $\mathcal{A}$ be a basis of I and let $\mathcal{B}$ be the basis $\mathcal{A}$ extended to the basis of L. Since I is an ideal, for any $x, y \in I$ we obtain the following block matrices:

$$^{\mathcal{B}}(\operatorname{ad} x) = \begin{pmatrix} ^{\mathcal{A}}(\operatorname{ad} x) & \bullet \\ 0 & 0 \end{pmatrix} \text{ and } ^{\mathcal{B}}(\operatorname{ad} y) = \begin{pmatrix} ^{\mathcal{A}}(\operatorname{ad} y) & \bullet \\ 0 & 0 \end{pmatrix},$$

where "$\bullet$" denotes an unspecified insignificant block. Hence

$$^{\mathcal{B}}(\operatorname{ad} x \circ \operatorname{ad} y) = {}^{\mathcal{B}}(\operatorname{ad} x) \cdot {}^{\mathcal{B}}(\operatorname{ad} y) = \begin{pmatrix} ^{\mathcal{A}}(\operatorname{ad} x) \cdot {}^{\mathcal{A}}(\operatorname{ad} y) & \bullet \\ 0 & 0 \end{pmatrix}$$

and finally $\kappa(x, y) = \operatorname{Tr}({}^{\mathcal{B}}(\operatorname{ad} x \circ \operatorname{ad} y)) = \operatorname{Tr}({}^{\mathcal{A}}(\operatorname{ad} x) \cdot {}^{\mathcal{A}}(\operatorname{ad} y)) = \kappa_I(x, y)$. ☐

Lemma 1.39. *Let V be a vector space and let L be a Lie subalgebra of $\mathfrak{gl}(V)$. If $\operatorname{Tr}(x \circ y) = 0$ for all $x, y \in L$, then L is solvable.*

Proof. First, we consider the derived algebra L' and we show that each $x \in L'$ is a nilpotent linear map. Suppose an arbitrary $x \in L'$ with Jordan decomposition $x = d + n$. Our aim is to show that $d = 0$. Putting x into Jordan canonical form (cf. [2], Chap. 5), one can see that we may choose a basis $\mathcal{B}$ of V such that $^{\mathcal{B}}d$ is diagonal and $^{\mathcal{B}}n$ is strictly upper triangular. Let $(\lambda_i)_{i=1}^m$, where $m = \dim V$, be the diagonal entries of $^{\mathcal{B}}d$. To prove that d is the zero map, it is necessary and sufficient to show that all λ_i equal to zero.

Consider $\overline{d} \in \mathrm{gl}(V)$ such that $^{\mathcal{B}}\overline{d} = ({}^{\mathcal{B}}d)^H$, hence $^{\mathcal{B}}\overline{d}$ is diagonal with entries $(\overline{\lambda_i})_{i=1}^m$. By Lemma 1.34, the Jordan decomposition of $\operatorname{ad} x$ is $\operatorname{ad} d + \operatorname{ad} n$ and part (b) of Lemma 1.35 then implies existence of a polynomial $q(t)$ such that $q(\operatorname{ad} x) = \overline{\operatorname{ad} d}$, where $^{\mathcal{A}}\overline{\operatorname{ad} d} = ({}^{\mathcal{A}} \operatorname{ad} d)^H$ for the basis $\mathcal{A}$ of $\mathrm{gl}(V)$ from the proof of Lemma 1.34. Moreover, it results from that proof that $\overline{\operatorname{ad} d} = \operatorname{ad} \overline{d}$. Now, as $q(\operatorname{ad} x)$ clearly maps L into L, $\operatorname{ad} \overline{d}$ does so. Since x is from L', there exist $u, v \in L$ such that $x = [u, v]$ and we have $\operatorname{ad} \overline{d}(u) = [\overline{d}, u] \in L$ so we may apply our assumption to conclude $\operatorname{Tr}([\overline{d}, u] \circ v) = 0$. Using the properties of the trace form discussed in the proof of Proposition 1.37, we may continue as follows:

$$\begin{aligned} 0 &= \operatorname{Tr}([\overline{d}, u] \circ v) = \operatorname{Tr}(\overline{d} \circ u \circ v - u \circ \overline{d} \circ v) = \operatorname{Tr}(\overline{d} \circ u \circ v) - \operatorname{Tr}(u \circ (\overline{d} \circ v)) \\ &= \operatorname{Tr}(\overline{d} \circ u \circ v) - \operatorname{Tr}((\overline{d} \circ v) \circ u) = \operatorname{Tr}(\overline{d} \circ u \circ v - \overline{d} \circ v \circ u) = \operatorname{Tr}(\overline{d} \circ [u, v]) \\ &= \operatorname{Tr}(\overline{d} \circ x) = \operatorname{Tr}(\overline{d} \circ (d + n)) = \operatorname{Tr}(\overline{d} \circ d) + \operatorname{Tr}(\overline{d} \circ n) \\ &= \operatorname{Tr}({}^{\mathcal{B}}\overline{d} \cdot {}^{\mathcal{B}}d) + \operatorname{Tr}({}^{\mathcal{B}}\overline{d} \cdot {}^{\mathcal{B}}n). \end{aligned}$$

We claim that the second term is zero. Indeed, for all $i \in \widehat{m}$ we have

$$({}^{\mathcal{B}}\overline{d} \cdot {}^{\mathcal{B}}n)_{ii} = \sum_{j=1}^{m} ({}^{\mathcal{B}}\overline{d})_{ij} ({}^{\mathcal{B}}n)_{ij} = ({}^{\mathcal{B}}\overline{d})_{ii} ({}^{\mathcal{B}}n)_{ii} = 0$$

because $^{\mathcal{B}}n$ is strictly upper triangular. Thus,

$$0 = \operatorname{Tr}({}^{\mathcal{B}}\overline{d} \cdot {}^{\mathcal{B}}d) = \sum_{i=1}^{m} \overline{\lambda_i} \lambda_i = \sum_{i=1}^{m} |\lambda_i|^2$$

and hence $\lambda_i = 0$ for all $i \in \widehat{m}$, as required.

Now, when we know that L' contains only nilpotent maps, we obtain nilpotency of L' from Corollary 1.30. Finally, if L' is nilpotent, then L' is obviously solvable and hence L is solvable as well. This completes the proof. ☐

At this time, as we have defined the *Killing form* and we have showed its properties, we may use it together with the previous lemma to determination of solvability and semisimplicity of Lie algebras; we introduce so-called *Cartan's criteria* (cf. [7]).

Theorem 1.40 (Cartan's Criterion for Solvability)**.** *The Lie algebra L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L$ and $y \in L'$.*

Proof. For the "only if" direction let us suppose that L is solvable and $\dim L = m$. Then part (a) of Lemma 1.23 implies that $\mathrm{ad}(L)$, as a homomorphic image of L, is a solvable subalgebra of $\mathsf{gl}(L)$ and hence, by Lie's Theorem, there exists a basis $\mathcal{B}$ of L such that, for any $x \in L$, $A_x := {}^{\mathcal{B}}(\mathrm{ad}\, x)$ is an upper triangular matrix. Now, given an arbitrary $y \in L'$, there exist $u, v \in L$ such that $y = [u, v]$. Since ad and $\phi\colon A \in \mathsf{gl}(L) \longmapsto {}^{\mathcal{B}}A \in \mathsf{gl}(m, \mathbb{C})$ are both homomorphisms, we can write

$$A_y = {}^{\mathcal{B}}(\mathrm{ad}[u, v]) = {}^{\mathcal{B}}([\mathrm{ad}\, u, \mathrm{ad}\, v]) = [{}^{\mathcal{B}}(\mathrm{ad}\, u), {}^{\mathcal{B}}(\mathrm{ad}\, v)] = [A_u, A_v] = A_u A_v - A_v A_u$$

whereas A_u and A_v are upper triangular matrices. We claim that A_y is even strictly upper triangular. Because the product and the difference of two upper triangular matrices is an upper triangular matrix again, it suffices to focus on diagonal entries of A_y. Considering which entries in A_u and A_v are zero for sure, for all $i \in \widehat{m}$ we have

$$\begin{aligned}(A_y)_{ii} &= \sum_{j=1}^{m} (A_u)_{ij}(A_v)_{ji} - \sum_{j=1}^{m} (A_v)_{ij}(A_u)_{ji} = \sum_{j=i}^{i} (A_u)_{ij}(A_v)_{ji} - \sum_{j=i}^{i} (A_v)_{ij}(A_u)_{ji} \\ &= (A_u)_{ii}(A_v)_{ii} - (A_v)_{ii}(A_u)_{ii} = 0\end{aligned}$$

which proves strictly-upper-triangularity of A_y. Further, we claim that for any $x \in L$ and $y \in L'$ the product $A_x A_y$ is also strictly upper triangular. Indeed, in the same way as above, for all $i \in \widehat{m}$ we obtain

$$(A_x A_y)_{ii} = \sum_{j=1}^{m} (A_x)_{ij}(A_y)_{ji} = \sum_{j=i}^{i-1} (A_x)_{ij}(A_y)_{ji} = 0.$$

All in all, using the definition of the trace form, for any $x \in L$ and $y \in' L$ we have

$$\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = \mathrm{Tr}(A_x A_y) = 0.$$

Conversely, we assume that for all $x \in L$ and $y \in L'$ and thus for all $x, y \in L'$ it holds true that $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = 0$. Equivalently, we may say that this holds for all $\mathrm{ad}\, x, \mathrm{ad}\, y \in \mathrm{ad}(L')$ and from Lemma 1.39 we obtain solvability of $\mathrm{ad}(L')$. According to Lemma 1.15, we have $\mathrm{ad}(L') \cong L'/Z(L')$ and part (a) of Lemma 1.23 then implies that $L'/Z(L')$, as a homomorphic image of solvable $\mathrm{ad}(L')$, is solvable as well. Finally, since $Z(L')$ is obviously solvable, part (b) of Lemma 1.23 proves solvability of L' which results clearly in solvability of L. □

Definition 1.41. Let I be an ideal of a Lie algebra L and let κ be the Killing form on L. We define

$$I^{\perp} := \left\{ x \in L \mid \text{for all } y \in I, \kappa(x, y) = 0 \right\}. \tag{1.26}$$

Proposition 1.42. *Let I be an ideal of a Lie algebra L. Then $I^{\perp}$ is also an ideal of L.*

Proof. First, it is clear from bilinearity of κ that $I^{\perp}$ is a subspace of L. Second, using associativity of κ, for arbitrary $x, y \in I^{\perp}$ and $z \in I$ we have $\kappa([x, y], z) = \kappa(x, [y, z]) = 0$ because $[y, z] \in I$ since I is an ideal. Hence, $[x, y] \in I^{\perp}$. □

Theorem 1.43 (Cartan's Criterion for Semisimplicity). *The Lie algebra L is semisimple if and only if the Killing form κ on L is non-degenerate.*

Proof. We start with the "only if" direction. Assume that L is semisimple. By Proposition 1.42, $L^{\perp}$ is an ideal of L. Hence for all $x \in L^{\perp}$ and $y \in (L^{\perp})' \subset L$ we have $\kappa(x, y) = 0$. Now, Theorem 1.40 implies solvability of $L^{\perp}$ which must be trivial considering L is semisimple. Thus, for all non-zero $x \in L$ there exists $y \in L$ such that $\kappa(x, y) \neq 0$ and κ is non-degenerate.

For the "if" direction, suppose that L is not semisimple and hence $\operatorname{rad} L \neq 0$. According to the definition of radical, there exists $m \in \mathbb{N}$ such that $(\operatorname{rad} L)^{(m)} = 0$. Denote $j := \max\left\{i \in \widehat{m-1} \mid (\operatorname{rad} L)^{(i)} \neq 0\right\}$ and $A := (\operatorname{rad} L)^{(j)}$. It is clear from the definition of j that A is an abelian ideal of L. For any $a \in A$ and $x, y \in L$ we have

$$(\operatorname{ad} a \circ \operatorname{ad} x \circ \operatorname{ad} a \circ \operatorname{ad} x)(y) = [a, [x, [a, [x, y]]]] = 0$$

because $[x, y] \in L$, $[a, [x, y]] \in A$, $[x, [a, [x, y]]] \in A$ and A is abelian. Consequently, $\operatorname{ad} a \circ \operatorname{ad} x \circ \operatorname{ad} a \circ \operatorname{ad} x = (\operatorname{ad} a \circ \operatorname{ad} x)^2$ is the zero map, hence $\operatorname{ad} a \circ \operatorname{ad} x$ is nilpotent and there is a basis of L in which $\operatorname{ad} a \circ \operatorname{ad} x$ is represented by a strictly upper triangular matrix. Therefore, $\kappa(a, x) = \operatorname{Tr}(\operatorname{ad} a \circ \operatorname{ad} x) = 0$. It means that there exists a non-zero $a \in A \subset L$ such that for each $x \in L$ it is satisfied $\kappa(a, x) = 0$ and thus κ is degenerate, a contradiction. ☐

In the very last part of the first chapter we introduce *simple* Lie algebras and show that each (finite dimensional) semisimple Lie algebra falls uniquely into finite many simple ones.

Definition 1.44. The Lie algebra L is said to be *simple* if it is not abelian and its only ideals are 0 and L.

Remark 1.21. Naturally, any simple Lie algebra is semisimple. To verify this assertion, consider a Lie algebra L which is simple but not semisimple. From the definition of semisimplicity, L has a non-trivial solvable ideal. In other words, as L is simple, L itself is solvable. But because $[L, L]$ is an ideal of L, either $[L, L] = L$ contrary to solvability of L, or $[L, L] = 0$ and thus L is abelian, a contradiction again.

Proposition 1.45. *Let L be a simple Lie algebra. Then:*

(a) $Z(L) = 0$;

(b) $[L, L] = L$.

Proof.

(a) Results trivially from Proposition 1.27 (b) and Remark 1.21.

(b) By Remark 1.15, $[L, L]$ is an ideal of L, hence it could be either 0 or L. But if $[L, L] = 0$, L would be abelian, a contradiction. ☐

Lemma 1.46. *Let I be an ideal of a semisimple Lie algebra L. Then $L = I \oplus I^{\perp}$ and moreover both I and $I^{\perp}$ are semisimple.*

Proof. Let κ be the Killing form on L. Consider a map $\varphi : L \to I^*$ defined for all $x \in L$ and $y \in I$ by $\varphi(x)(y) := \kappa(x, y)$. As φ is linear, we may apply the rank-nullity theorem:

$$\dim L = \dim(\operatorname{Ran} \varphi) + \dim(\operatorname{Ker} \varphi)$$

(cf. [11], page 61). First, we explore $\operatorname{Ker}\varphi$:

$$x \in \operatorname{Ker}\varphi \iff \varphi(x) = 0 \in I^* \iff \text{for all } y \in I,\ \varphi(x)(y) = \kappa(x,y) = 0 \iff x \in I^\perp.$$

Second, we claim that $\operatorname{Ran}\varphi = I^*$. φ may be regarded as the composition of two maps $\varphi = \varphi_2 \circ \varphi_1$, where $\varphi_1 : L \to L^*$ is defined for all $x \in L$ and $y \in L$ by $\varphi_1(x)(y) := \kappa(x,y)$ and $\varphi_2 : L^* \to I^*$ is the restriction to I defined for all $f \in L^*$ as $\varphi_2(f) := f|_I$. In the same way as for φ, we discover that $\operatorname{Ker}\varphi_1 = L^\perp$ and, since κ is non-degenerate, $\operatorname{Ker}\varphi_1 = L^\perp = 0$. Moreover, the rank-nullity theorem used for φ_1 gives that $\operatorname{Ran}\varphi_1 = L^*$ and hence φ_1 is a bijection. For φ_2, we claim that $\operatorname{Ran}\varphi_2 = I^*$. Suppose $\mathcal{B} = (x_i)_{i=1}^{\dim I}$ is a basis for I (notice that $\mathcal{B} = \emptyset$ in case $I = 0$) and let us extend it to $\mathcal{A} = (x_i)_{i=1}^{\dim L}$, a basis for L. Now for an arbitrary $f \in I^*$ we define $g \in L^*$ as follows: $g(x_i) := f(x_i)$, for $i = 1, \ldots, \dim I$, and $g(x_i) := 0$, for $i = \dim I + 1, \ldots, \dim L$. In this way we have found $\varphi_2^{(-1)} = g$ and hence we have proved surjectivity of φ_2. Altogether, $\operatorname{Ran}\varphi = \operatorname{Ran}(\varphi_2 \circ \varphi_1) = \operatorname{Ran}\varphi_2 = I^*$ and, by the rank-nullity theorem,

$$\dim L = \dim I^* + \dim I^\perp = \dim I + \dim I^\perp.$$

Further, by Proposition 1.42, $I^\perp$ is an ideal of L. Clearly, the intersection of two ideals is an ideal as well. In our case, this applies to $I \cap I^\perp$. Moreover, the restriction of the Killing form to $(I \cap I^\perp) \times (I \cap I^\perp)$ is identically zero and hence, according to Proposition 1.38 and Theorem 1.40, $I \cap I^\perp$ is a solvable ideal of L. Then, since L is semisimple, $I \cap I^\perp = 0$ and therefore $L = I \oplus I^\perp$.

For the second part of the lemma assume that I is not semisimple. By Theorem 1.43, the Killing form on I is degenerate which implies existence of $x \in I$ such that $\kappa(x,y) = 0$ for all $y \in I$. By Proposition 1.38 and because $L = I \oplus I^\perp$, $\kappa(x,y) = 0$ even for all $y \in L$. But this means that the Killing form on L is degenerate, contrary to semisimplicity of L. For $I^\perp$, the procedure is analogous, it suffices to realize that $(I^\perp)^\perp = I$. □

Theorem 1.47. *Let L be a Lie algebra. Then L is semisimple if and only if there are simple ideals $I_1, \ldots, I_k$ of L such that $L = I_1 \oplus \cdots \oplus I_k$. In addition, the decomposition is unique up to the labelling of the simple ideals.*

Proof. We prove the "only if" direction proceeding the following algorithm whereby we construct a finite sequence $(L_i)_{i=1}^k$ of semisimple Lie algebras. First, $L_1 := L$. Second, we denote an ideal of L_i of the smallest possible positive dimension by I_i. We use Lemma 1.46 to obtain $L_i = I_i \oplus I_i^\perp$. I_i does not have any non-trivial ideals because if it does, then this ideal of smaller dimension than $\dim I_i$ would be an ideal of L_i as well, contrary to the choice of I_i, hence I_i is a simple ideal of L_i. By Lemma 1.46, $I_i^\perp$ is semisimple and $\dim I_i^\perp < \dim L_i$ additionally. We put $L_{i+1} := I_i^\perp$. The process ends when $I_i = L_i$. Denote the final step as k-th. Since $\dim L_i < \dim L_{i+1}$ for all $i \in \widehat{k-1}$ and $L_1 = L$ is finite-dimensional, it is clear that our procedure has to come to the end. In this way we obtain the decomposition $L = I_1 \oplus \cdots \oplus I_k$, where for all $i \in \widehat{k}$ it holds true that I_i is a simple ideal of L_i. Finally, using incomplete induction on $i \in \widehat{k}$, we show that $I_1, \ldots, I_k$ are also simple ideals of L. I_1 is a simple ideal of L by construction. For the inductive step we assume that $I_1, \ldots, I_{i-1}$ are simple ideals of L. Then

$$\begin{aligned}[L, I_i] &= [I_1 \oplus \cdots \oplus I_{i-1} \oplus I_{i-1}^\perp, I_i] = [I_1 \oplus \cdots \oplus I_{i-1} \oplus L_i, I_i] \\ &= [I_1, I_i] \oplus \cdots \oplus [I_{i-1}, I_i] \oplus [L_i, I_i] \subset [I_1, L_i] \oplus \cdots \oplus [I_{i-1}, L_i] \oplus [L_i, I_i] \subset I_i\end{aligned}$$

because L_i is an ideal of L as well. Hence all I_i are simple ideals of L, as desired.

For the second direction suppose that $L = I_1 \oplus \cdots \oplus I_k$, where $I_1, \ldots, I_k$ are simple ideals of L. Let R be the radical of L. Then for all $i \in \widehat{k}$ it holds true that $[R, I_i] \subset (R \cap I_i)$ and hence $[R, I_i]$ is a solvable ideal of I_i. As I_i is simple, $[R, I_i]$ could be either whole I_i, or trivial. If $[R, I_i] = I_i$, then $(I_i)^{(m)} = I_i$ for an arbitrary $m \in \mathbb{N}$ and it would not be solvable. Thus, for all $i \in \widehat{k}$ we have $[R, I_i] = 0$ and

$$[R, L] = [R, I_1 \oplus \cdots \oplus I_k] = [R, I_1] \oplus \cdots \oplus [R, I_k] = 0.$$

Hence $R \subset Z(L) = Z(I_1) \oplus \cdots \oplus Z(I_k)$, according to Proposition 1.7. Finally, by Proposition 1.45 (a), $Z(I_i) = 0$ for all $i \in \widehat{k}$, so $R = 0$ and L is semisimple.

Finally, let $L = I_1 \oplus I_2 \oplus \cdots \oplus I_k$ and $L = J_1 \oplus J_2 \oplus \cdots \oplus J_l$ be two different decompositions of L into simple ideals (different in the sense $\{I_1, I_2, \ldots, I_k\} \neq \{J_1, J_2, \ldots, J_l\}$). Consequently, there exist $i \in \widehat{k}$ and $j \in \widehat{l}$ such that at once $I_i \cap J_j \neq 0$ and $I_i \neq J_j$. Clearly, $I_i \cap J_j$ is an ideal of L and hence of both I_i and J_j as well. Moreover, as $I_i \neq J_j$, $I_i \cap J_j$ is a proper ideal of at least one I_i or J_j, contradicting their simplicity. □

Corollary 1.48. *If L is a semisimple Lie algebra, then $[L, L] = L$.*

Proof. As L is semisimple, Theorem 1.47 implies existence of simple ideals $I_1, \ldots, I_k$ of L such that $L = I_1 \oplus \cdots \oplus I_k$. Using this fact together with Remark 1.6 and Proposition 1.45 (b), we may write

$$[L, L] = [I_1 \oplus \cdots \oplus I_k, I_1 \oplus \cdots \oplus I_k] = [I_1, I_1] \oplus \cdots \oplus [I_k, I_k] = I_1 \oplus \cdots \oplus I_k = L.$$

□

Proposition 1.49. *If L is a semisimple Lie algebra and I is an ideal of L, then L/I is semisimple.*

Proof. We use the fact that, according to part (a) of Remark 1.27, an isomorphic image of a semisimple Lie algebra is semisimple as well. By Lemma 1.46, $L = I \oplus I^\perp$ and $I^\perp$ is semisimple subsequently. But by Corollary 1.16,

$$I^\perp = I^\perp/0 = I^\perp/(I^\perp \cap I) \cong (I^\perp + I)/I = L/I$$

and hence L/I is semisimple as well. □

Proposition 1.50. *Let I be a semisimple ideal of a Lie algebra L. Then $L = I \oplus I^\perp$.*

Proof. The procedure is analogous to the first part of the proof of Lemma 1.46. Let κ be the Killing form on L. We consider the same map $\varphi : L \to I^*$ as in that proof, thus $\operatorname{Ker} \varphi = I^\perp$. Again we view φ as the composition of two maps $\varphi = \varphi_2 \circ \varphi_1$, but now φ_1 is the projection of L on $I \subset\subset L$ and $\varphi_2 : I \to I^*$ is defined for all $x \in I$ and $y \in I$ by $\varphi_2(x)(y) := \kappa(x, y)$. Obviously, $\operatorname{Ran} \varphi_1 = I$. Further, since the Killing form $\kappa_I = \kappa|_{I \times I}$ on I is non-degenerate, $\operatorname{Ker} \varphi_2 = I^{\perp_I} = 0$ (where $\perp_I$ is taken with respect to κ_I) and hence, according to the rank-nullity theorem, φ_2 is a bijection. All in all, $\operatorname{Ran} \varphi = \varphi_2(\operatorname{Ran} \varphi_1) = \operatorname{Ran} \varphi_2 = I^*$ and the rank-nullity theorem gives

$$\dim L = \dim I^* + \dim I^\perp = \dim I + \dim I^\perp.$$

Further, it suffices to realize that if $I \cap I^\perp$ is a solvable ideal of L (as showed in the proof of Lemma 1.46), then it is a solvable ideal of I as well. But, as I is semisimple, then necessarily $I \cap I^\perp = 0$ and hence $L = I \oplus I^\perp$. □

Chapter 2

Representations of Lie Algebras

2.1 Basic Representation Theory

At the beginning of the second chapter we explain fundamentals of the representation theory. As the title "representation" suggests, this area of Lie algebras theory studies how an abstract Lie algebra can be *represented* by more illustrative Lie algebra of linear maps or matrices, respectively. However, the original purpose of studying representations is *not* to represent Lie algebras in this way. In fact, the most common Lie algebras occur naturally as Lie algebras of matrices. The primary reason for research on the representation theory is to investigate the action of a Lie algebra on a vector space (so-called *module*). Such actions arise in many areas of mathematics as well as in physics.

2.1.1 Representations and Modules

We introduce both *representations* and *modules* and then we show that they are just two different points of view on the same thing. Consequently, we may choose which way is more preferable for us to use in a particular situation (cf. [7], Chap. 7).

Definition 2.1. Suppose that L is a Lie algebra and V is a vector space. A *representation* of L is a Lie algebra homomorphism $\varphi : L \to \mathsf{gl}(V)$.

Definition 2.2. Let L be a Lie algebra. An *L-module* is a vector space V together with a bilinear map $L \times V \to V : (x, y) \mapsto x \cdot v$ fulfilling for all $x, y \in L$ and $v \in V$ the following condition:

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v). \tag{2.1}$$

Instead of "an *L-module*" one can say also "a *Lie module for L*" or briefly "a *module for L*" (cf. [7]). Similarly as in the case of Lie algebras, talking about a module, we usually mention only the vector space alone and we assume automatically that the respective bilinear map is defined as well. Sometimes two or more different module maps may occur in one expression, but it is always clear from the context which module does a map relate to. Recall that, as for any other vector spaces, we consider entirely finite-dimensional representation spaces and modules.

Remark 2.1. Let L be a Lie algebra. Given a representation $\varphi : L \to \mathsf{gl}(V)$, we may always construct an L-module in this way: V is the vector space and the module map is defined for all $x \in L$ and $v \in V$ by

$$x \cdot v := \varphi(x)(v). \tag{2.2}$$

Clearly, $x \cdot v \in V$. Bilinearity results from linearity of φ and linearity of $\varphi(x) \in \mathfrak{gl}(V)$ for an arbitrary $x \in L$. Hence, it remains to verify that the condition (2.1) is fulfilled. Given any $x, y \in L$ and $v \in V$, we have

$$\begin{aligned}[x,y] \cdot v &= \varphi([x,y])(v) = [\varphi(x), \varphi(y)](v) = \varphi(x)(\varphi(y)(v)) - \varphi(y)(\varphi(x)(v)) \\ &= x \cdot (y \cdot v) - y \cdot (x \cdot v),\end{aligned}$$

as required.

Contrariwise, if we have an L-module, we use (2.2) again to define required representation homomorphism φ. Obviously, $\varphi : L \to \mathfrak{gl}(V)$ and hence we must only check that it is a Lie algebra homomorphism. Indeed, we take any $x, y \in L$ and $v \in V$ again and we obtain

$$\begin{aligned}\varphi([x,y])(v) &= [x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) = \varphi(x)(\varphi(y)(v)) - \varphi(y)(\varphi(x)(v)) \\ &= [\varphi(x), \varphi(y)](v).\end{aligned}$$

Evidently, there is no ambiguity in relation (2.2) and thus the construction in both directions (representation $\to$ module; module $\to$ representation) is unique.

Definition 2.3. Let L be a Lie algebra. The representation φ of L is said to be *faithful* when $\varphi : L \to \varphi(L) \subset \mathfrak{gl}(V)$ is a bijection.

It is clear that *faithfulness* is a very important property of a representation. Working with a non-faithful representation, we generally lose some information about the Lie algebra structure. Hence we are interested mainly in faithful representations which tell us as much as possible about the represented Lie algebra. Now, we give several examples of representations as an illustration (cf. [7], Sec. 7.1).

Example 2.2. Given an arbitrary Lie algebra L, one representation of L is always available: when we put $V = L$ and $\varphi = \mathrm{ad}$, we obtain the so-called *adjoint* representation. By Example 1.12, $\mathrm{Ker}(\mathrm{ad}) = Z(L)$ and therefore this representation is faithful if and only if $Z(L) = 0$.

Example 2.3. Another example of representation that can be defined for any Lie algebra is the *trivial* representation. In this case φ is the zero map and V may be chosen arbitrarily. However, this representation is never faithful unless L is trivial.

Example 2.4. Let L be a subalgebra of $\mathfrak{gl}(V)$. Then we may define the *natural* representation as the restriction of the identity on $\mathfrak{gl}(V)$ to L. It is obvious that this representation is always faithful.

2.1.2 Schur's Lemma

As in the case of Lie algebras, it also makes sense to concern with the subspaces of modules which satisfy themselves the definition of module. And, similarly as for Lie algebras, we can examine whether two modules have the same algebraic structure i.e. whether there exists a homomorphism between them. We state precise definitions and at the end of this subsection we introduce a very important lemma describing the homomorphisms from a Lie module into itself (cf. [7], Chap. 7).

Definition 2.4. Let V be an L-module. An *L-submodule* of V is a vector subspace $W \subset\subset V$ such that for all $x \in L$ and $w \in W$ it is satisfied that

$$x \cdot w \in W, \tag{2.3}$$

where "$\cdot$" denotes the bilinear map appertaining to L-module V.

Definition 2.5. Let L be a Lie algebra. The L-module V is said to be *irreducible* if it has precisely two different submodules.

Remark 2.5. Let V be a module for a Lie algebra L. Certainly, both V and the zero subspace of V satisfy the definition of submodule. If V is a non-zero vector space, then the "precisely two different submodules" in the previous definition are 0 and V. In case that $V = 0$, the only subspace of V is the zero subspace and thus the L-module $V = 0$ is *not* irreducible by our definition.

Definition 2.6. Let V and W be modules for a Lie algebra L. A linear map $\phi\colon V \to W$ is called an *L-module homomorphism* if for all $v \in V$ and $x \in L$ it is satisfied that

$$\phi(x \cdot v) = x \cdot \phi(v). \tag{2.4}$$

If ϕ is bijective in addition, we say that ϕ is an *L-module isomorphism*.

Lemma 2.7 (Schur)**.** *Let V be an irreducible module for a Lie algebra L. A map $\phi\colon V \to V$ is an L-module homomorphism if and only if there exists $\lambda \in \mathbb{C}$ such that $\phi = \lambda \mathbb{1}_V$.*

Proof. The "if" direction is clear since for any $x \in L$ and $v \in V$ we can write

$$\phi(x \cdot v) = \lambda(x \cdot v) = x \cdot (\lambda v) = x \cdot \phi(v).$$

For the second direction suppose that $\phi : V \to V$ is an L-module homomorphism. ϕ has, as a linear map between complex vector spaces, an eigenvalue. We denote this eigenvalue λ and the respective eigenvector v_λ. In addition we denote $\psi_\lambda := \phi - \lambda \mathbb{1}_V$. Now, for any $v \in V$ and $x \in L$ we have

$$\psi_\lambda(x \cdot v) = \phi(x \cdot v) - \lambda(x \cdot v) = x \cdot (\phi(v) - \lambda v) = x \cdot \psi_\lambda(v)$$

and hence ψ_λ is an L-module homomorphism. In particular, when we take $v \in \operatorname{Ker} \psi_\lambda$, we can see that $\psi_\lambda(x \cdot v) = x \cdot \psi_\lambda(v) = x \cdot 0 = 0$ for any $x \in L$ which means that $\operatorname{Ker} \psi_\lambda$ is a submodule of V. Moreover, since $v_\lambda \in \operatorname{Ker} \psi_\lambda$, this submodule is non-trivial and, as V is irreducible, necessarily $\operatorname{Ker} \psi_\lambda = V$. But this is equivalent to $0 \equiv \psi_\lambda = \phi - \lambda \mathbb{1}_V$ which completes the proof. ☐

2.1.3 Weyl's Theorem

In the final part of this section we state *Weyl's Theorem*, an extremely important tool in the representation theory. Its proof is fairly long and technical so we omit it. It can be found for example in [9], Sec. 6.3.

Definition 2.8. Let L be a Lie algebra. The L-module V is said to be *completely reducible* if there exist irreducible L-modules $V_1, V_2, \ldots, V_k$ such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$.

Remark 2.6. We define the terms "*irreducible*" and "*completely reducible*" also for representations. Naturally, the representation is *irreducible* or *completely reducible*, respectively, precisely when the respective module (in the sense of Remark 2.1) is.

Lemma 2.9. *Let L be a semisimple Lie algebra and let $\varphi : L \to \mathsf{gl}(V)$ be its representation. Then $\varphi(L) \subset \mathsf{sl}(V)$.*

Proof. First, by Corollary 1.48, $[L, L] = L$. Next, considering Remark 1.10 and Example 1.14, we have

$$\varphi(L) = \varphi([L, L]) = [\varphi(L), \varphi(L)] \subset [\mathsf{gl}(V), \mathsf{gl}(V)] = \mathsf{sl}(V).$$

☐

Theorem 2.10 (Weyl)**.** *Let L be a semisimple Lie algebra. If φ is a representation of L, then φ is completely reducible.*

Proposition 2.11. *Let V be a module for a semisimple Lie algebra L. The decomposition of V into irreducible modules (according to Weyl's Theorem) is unique, up to the order of terms in the direct sum.*

Proof. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_l$ be two different decompositions of V into irreducible submodules (different in the sense as above i.e. $\{V_1, V_2, \ldots, V_k\} \neq \{W_1, W_2, \ldots, W_l\}$). Consequently, there exist $i \in \widehat{k}$ and $j \in \widehat{l}$ such that at once $V_i \cap W_j \neq \{0\}$ and $V_i \neq W_j$. Clearly $V_i \cap W_j$ is a subspace of both V_i and W_j, moreover, as $V_i \neq W_j$, $V_i \cap W_j$ is a proper subspace of at least one of V_i and W_j. Without loss of generality, suppose that $V_i \cap W_j$ is a proper subspace of V_i. But, because both V_i and W_j are closed under the action of L, the same has to be true for $V_i \cap W_j$ which is therefore a non-trivial proper submodule of V_i, contradicting the irreducibility of V_i. □

2.2 Generalization of Jordan Decomposition

This section is based on the results introduced in Subsection 1.3.2. In that part of the first chapter we have concerned ourselves with the *Jordan decomposition* of linear maps. Now, when we already know how to "represent" an arbitrary Lie algebra with the Lie algebra of linear maps, it is time to generalize the Jordan decomposition for elements of any abstract Lie algebra. Unfortunately, the generalization is not very straightforward, we have to look at the set of so-called *derivations* of a Lie algebra at first (cf. [9], Sec. 5.3).

2.2.1 Derivations

Definition 2.12. Let L be a Lie algebra. A linear map $D\colon L \to L$ is called a *derivation* of L if it satisfies for all $x, y \in L$

$$D([x,y]) = [D(x), y] + [x, D(y)]. \tag{2.5}$$

We denote the set of all derivations of L by $\operatorname{Der} L$.

Remark 2.7. Let L be a Lie algebra. Clearly, $\operatorname{Der} L \subset \mathrm{gl}(L)$. Further, bilinearity of the Lie bracket implies that $\operatorname{Der} L \subset\subset \mathrm{gl}(L)$. Additionally, one can show that $\operatorname{Der} L$ is even a subalgebra of $\mathrm{gl}(L)$. Indeed, for all $D, E \in \operatorname{Der} L$ and $x, y \in L$ we have

$$\begin{aligned}
[D,E]([x,y]) &= (D \circ E)([x,y]) - (E \circ D)([x,y]) \\
&= D([E(x),y] + [x,E(y)]) - E([D(x),y] + [x,D(y)]) \\
&= [D(E(x)),y] + [E(x),D(y)] + [D(x),E(y)] + [x,D(E(y))] \\
&\qquad - [E(D(x)),y] - [D(x),E(y)] - [E(x),D(y)] - [x,E(D(y))] \\
&= [D(E(x)) - E(D(x)),y] + [x,D(E(y)) - E(D(y))] \\
&= [[D,E](x),y] + [x,[D,E](y)]
\end{aligned}$$

and thus $[D, E] \in \operatorname{Der} L$. Notice that we do not discriminate between the Lie brackets on L and $\mathrm{gl}(L)$ here but it is easily seen from the context which one we mean at the moment.

Proposition 2.13. *If L is a semisimple Lie algebra, then $\operatorname{ad}(L) = \operatorname{Der} L$.*

Proof. First, given arbitrary $x, y, z \in L$, Jacobi identity implies that $\operatorname{ad} x \in \operatorname{Der} L$:

$$\begin{aligned}(\operatorname{ad} x)([y,z]) &= [x,[y,z]] = -[y,[z,x]] - [z,[x,y]] = [[x,y],z] + [y,[x,z]] \\ &= [(\operatorname{ad} x)(y), z] + [y, (\operatorname{ad} x)(z)].\end{aligned}$$

Moreover, for any $x, y \in L$ and $D \in \operatorname{Der} L$ we have

$$\begin{aligned}[D, \operatorname{ad} x](y) &= D([x,y]) - \operatorname{ad} x(D(y)) = [D(x), y] + [x, D(y)] - [x, D(y)] \\ &= \operatorname{ad}(D(x))(y),\end{aligned}$$

therefore $[D, \operatorname{ad} x] \in \operatorname{ad}(L)$ and hence $\operatorname{ad}(L)$ is an ideal of $\operatorname{Der} L$. Further, as L is semisimple, part (b) of Proposition 1.27 gives $Z(L) = \operatorname{Ker}(\operatorname{ad}) = 0$, thus $\operatorname{ad}: L \to \operatorname{ad}(L)$ is a Lie algebra isomorphism and, by Proposition 1.27 (a), $\operatorname{ad}(L)$ is semisimple as well. Now we can apply Proposition 1.50 to obtain $\operatorname{Der} L = \operatorname{ad}(L) \oplus (\operatorname{ad}(L))^{\perp}$. By Proposition 1.42, $(\operatorname{ad}(L))^{\perp}$ is an ideal of $\operatorname{Der} L$ as well and hence $[\operatorname{ad}(L), (\operatorname{ad}(L))^{\perp}] \subset \operatorname{ad}(L) \cap (\operatorname{ad}(L))^{\perp} = 0$. Consequently, for any $D \in (\operatorname{ad}(L))^{\perp}$ and $x \in L$ we have $0 = [D, \operatorname{ad} x] = \operatorname{ad}(D(x))$ and thus, since ad is an isomorphism here, $D(x) = 0$ for all $x \in L$. This is equivalent to $D \equiv 0$. All in all, $(\operatorname{ad}(L))^{\perp} = 0$ and $\operatorname{Der} L = \operatorname{ad}(L)$, as desired. □

2.2.2 Abstract Jordan Decomposition

At this place we introduce another useful result from linear algebra, the so-called *Primary Decomposition Theorem*. The proof can be found in [2], Chap. 3. Before we state the theorem, we need to define what the so-called *minimum polynomial* is (cf. [1], Chap. 10).

Definition 2.14. Let V be a vector space and let A be a linear transformation of V. The *minimum polynomial* of A is the polynomial $p(t) \in \mathbb{C}[t]$ of least degree such that $p(A) = 0$.

Remark 2.8. Let A be a linear transformation of a vector space V and let

$$p(t) = (t - \lambda_1)^{m_1} \dots (t - \lambda_r)^{m_r},$$

where $\lambda_1, \dots, \lambda_r$ are pairwise distinct, be the minimum polynomial of A. It can be shown (cf. [1], Chap. 10, and [13], Chap. 7) that

(a) the minimum polynomial is unique,

(b) $\sigma(A) = \{\lambda_1, \dots, \lambda_r\}$,

(c) if $q(t) \in \mathbb{C}[t]$ is a polynomial such that $q(A) = 0$, then $p(t)$ divides $q(t)$,

(d) for all $i \in \hat{r}$ it holds true that $m_i = index(\lambda_i)$, where $index(\lambda_i)$ is the smallest positive integer k_i such that $\operatorname{Ker}(A - \lambda_i \mathbb{1})^{k_i} = \operatorname{Ker}(A - \lambda_i \mathbb{1})^{k_i+1}$,

(e) A is diagonalisable if and only if $m_1 = m_2 = \dots = m_r = 1$.

Lemma 2.15 (Primary Decomposition Theorem)**.** *Suppose a vector space V and $A \in gl(V)$. Let the minimum polynomial of A be*

$$p(t) = (t - \lambda_1)^{m_1} \dots (t - \lambda_r)^{m_r},$$

where $\lambda_1, \dots, \lambda_r$ are pairwise distinct. For all $i \in \hat{r}$ let us denote $V_i := \operatorname{Ker}(A - \lambda_i \mathbb{1})^{m_i}$. Then for any $i \in \hat{r}$ it is satisfied that $A(V_i) \subset V_i$ (each subspace V_i is "A-invariant") and moreover $V = V_1 \oplus \dots \oplus V_r$.

Now we use the introduced results from linear algebra to define the *abstract* Jordan decomposition, the Jordan decomposition for elements of an arbitrary semisimple Lie algebra (cf. [7], Sec. 9.6). First, we need to study the (usual) Jordan decomposition of derivations.

Lemma 2.16. *Let L be a Lie algebra. Suppose that $X \in \operatorname{Der} L$ has Jordan decomposition $X = D + N$. Then both $D, N \in \operatorname{Der} L$.*

Proof. For all $\lambda \in \mathbb{C}$, let us denote

$$L_\lambda := \left\{ x \in L \;\middle|\; \text{there exists } m \in \mathbb{N}, (X - \lambda \mathbb{1})^m x = 0 \right\}.$$

Obviously, if $\lambda \notin \sigma(X)$, then L_λ is trivial. By Lemma 2.15 and Remark 2.8, $L = \bigoplus_{\lambda \in \sigma(X)} L_\lambda$. Now, for all $\lambda \in \mathbb{C}$, let us denote

$$\tilde{L}_\lambda := \left\{ x \in L \;\middle|\; \text{there exists } m \in \mathbb{N}, (D - \lambda \mathbb{1})^m x = 0 \right\}.$$

We claim that $L_\lambda = \tilde{L}_\lambda$ for each $\lambda \in \mathbb{C}$. Indeed. As N is nilpotent, there exists $\hat{m} \in \mathbb{N}$ such that $N^{\hat{m}} = 0$. Take any $\lambda \in \mathbb{C}$ and $x \in L_\lambda$. There exists $m \in \mathbb{N}$ such that $(X - \lambda \mathbb{1})^m x = 0$. We search $\tilde{m} \in \mathbb{N}$ such that $m \in \mathbb{N}$ such that $(D - \lambda \mathbb{1})^{\tilde{m}} x = 0$. Since N commutes with D (see Lemma 1.32) and hence also with $X - \lambda \mathbb{1} = D + N - \lambda \mathbb{1}$, we may write

$$(D - \lambda \mathbb{1})^{\tilde{m}} x = (-N + (X - \lambda \mathbb{1}))^{\tilde{m}} x = \sum_{k=0}^{\tilde{m}} \binom{\tilde{m}}{k} (-N)^{\tilde{m}-k} (X - \lambda \mathbb{1})^k x.$$

Thus it suffices to put $\tilde{m} := 2 \max \{m, \hat{m}\}$ to prove that $L_\lambda \subset \tilde{L}_\lambda$. The second inclusion is analogous. In addition, as D is diagonalisable, part (e) of Remark 2.8 permits us to write

$$L = \bigoplus_{\lambda \in \sigma(D)} L_\lambda, \text{ where } L_\lambda := \left\{ x \in L \;\middle|\; (D - \lambda \mathbb{1}) x = 0 \right\}.$$

Further, using the fact that X is a derivation, we show by induction on n that for any $\lambda, \mu \in \mathbb{C}$, $x, y \in L$ and $n \in \mathbb{N}_0$ it holds true that

$$(X - (\lambda + \mu)\mathbb{1})^n [x, y] = \sum_{k=0}^{n} \binom{n}{k} [(X - \lambda \mathbb{1})^k x, (X - \mu \mathbb{1})^{n-k} y]. \tag{2.6}$$

When $n = 0$, we have $\mathbb{1}[x, y] = [\mathbb{1}x, \mathbb{1}y]$, so there is nothing to prove. For the inductive step suppose that (2.6) holds for $n - 1$. Then we can write

$$\begin{aligned}
(X - (\lambda + \mu)\mathbb{1})^n [x, y] &= (X - (\lambda + \mu)\mathbb{1}) (\sum_{k=0}^{n-1} \binom{n-1}{k} [(X - \lambda \mathbb{1})^k x, (X - \mu \mathbb{1})^{n-1-k} y]) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} ([X((X - \lambda \mathbb{1})^k x), (X - \mu \mathbb{1})^{n-1-k} y] \\
&\quad + \sum_{k=0}^{n-1} \binom{n-1}{k} ([(X - \lambda \mathbb{1})^k x, X((X - \mu \mathbb{1})^{n-1-k} y)] \\
&\quad + \sum_{k=0}^{n-1} \binom{n-1}{k} ([-\lambda((X - \lambda \mathbb{1})^k x), (X - \mu \mathbb{1})^{n-1-k} y] \\
&\quad + \sum_{k=0}^{n-1} \binom{n-1}{k} ([(X - \lambda \mathbb{1})^k x, -\mu((X - \mu \mathbb{1})^{n-1-k} y)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \binom{n-1}{k} ([(X-\lambda\mathbb{1})^{k+1}x, (X-\mu\mathbb{1})^{n-1-k}y] \\
&\quad + \sum_{k=0}^{n-1} \binom{n-1}{k} ([(X-\lambda\mathbb{1})^{k}x, (X-\mu\mathbb{1})^{n-k}y] \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} ([(X-\lambda\mathbb{1})^{k}x, (X-\mu\mathbb{1})^{n-k}y] \\
&\quad + \sum_{k=0}^{n-1} \binom{n-1}{k} ([(X-\lambda\mathbb{1})^{k}x, (X-\mu\mathbb{1})^{n-k}y] \\
&= \sum_{k=1}^{n-1} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) [(X-\lambda\mathbb{1})^{k}x, (X-\mu\mathbb{1})^{n-k}y] \\
&\quad + [(X-\lambda\mathbb{1})^{n}x, y] + [x, (X-\mu\mathbb{1})^{n}y] \\
&= \sum_{k=0}^{n} \binom{n}{k} [(X-\lambda\mathbb{1})^{k}x, (X-\mu\mathbb{1})^{n-k}y],
\end{aligned}
$$

as required. From (2.6) we can conclude that for any $\lambda, \mu \in \mathbb{C}$ it holds true $[L_\lambda, L_\mu] \subset L_{\lambda+\mu}$. Indeed. Taken any $x \in L_\lambda$ and $y \in L_\mu$, there are integers m_x and m_y such that $(X-\lambda\mathbb{1})^{m_x}x = (X-\mu\mathbb{1})^{m_y}y = 0$. As (2.6) holds, it can be easily seen that for $m := 2\max\{m_x, m_y\}$ it is $(X-(\lambda+\mu)\mathbb{1})^m[x,y] = 0$ and hence $[x,y] \in L_{\lambda+\mu}$.

Finally, given any $x, y \in L$, there exist λ and μ, eigenvalues of D, such that $x \in L_\lambda$ and $y \in L_\mu$, herewith $[x,y] \in L_{\lambda+\mu}$. As discussed above, since D is diagonalisable, it holds true that $D(x) = \lambda x$, $D(y) = \mu y$ and $D([x,y]) = (\lambda+\mu)[x,y]$. Altogether, we have

$$D([x,y]) = (\lambda+\mu)[x,y] = \lambda[x,y] + \mu[x,y] = [\lambda x, y] + [x, \mu y] = [D(x), y] + [x, D(y)]$$

and thus $D \in \operatorname{Der} L$. Because $\operatorname{Der} L$ is a subspace, then also $N = X - D \in \operatorname{Der} L$. □

Theorem 2.17. *Let L be a semisimple Lie algebra and let $x \in L$. Then there exist unique $d, n \in L$ such that $\operatorname{ad} d$ is diagonalisable, $\operatorname{ad} n$ is nilpotent, $[d,n] = 0$ and*

$$x = d + n. \tag{2.7}$$

Proof. Since $\operatorname{ad} x \in \operatorname{ad}(L)$, Proposition 2.13 implies that $\operatorname{ad} x \in \operatorname{Der} L$. Let $\operatorname{ad} x = D + N$ be the Jordan decomposition of $\operatorname{ad} x$. Notice that D and N are unique. By Lemma 2.16, $D, N \in \operatorname{Der} L = \operatorname{ad}(L)$, hence there exist $d, n \in L$ such that $\operatorname{ad} d = D$ is diagonalisable and $\operatorname{ad} n = N$ is nilpotent. Since L is semisimple, part (b) of Proposition 1.27 implies injectivity of $\operatorname{ad} : L \to \operatorname{ad}(L)$, and thus d and n are, as preimages of unique D and N under (injective) adjoint homomorphism, also unique. All in all, we have

$$\operatorname{ad} x = D + N = \operatorname{ad} d + \operatorname{ad} n = \operatorname{ad}(d+n)$$

and thus, as ad is injective, $x = d + n$. Finally, $0 = [\operatorname{ad} d, \operatorname{ad} n] = \operatorname{ad}([d,n])$ and injectivity of ad imply $[d,n] = 0$. □

Definition 2.18. Let L be a semisimple Lie algebra and let $x \in L$. The decomposition (2.7), where d and n are the elements of L from the previous theorem, is called the *abstract Jordan decomposition* of x. We call d and n the *semisimple* and *nilpotent*, respectively, part of x.

Remark 2.9. Similarly as in the case of the usual Jordan decomposition, also for the abstract one we will use the same notation as established in Remark 1.18 i.e. the first term in the decomposition will always be the semisimple part and the second one will be the nilpotent one.

Proposition 2.19. *Let $L = L_1 \oplus \cdots \oplus L_k$ be the decomposition of a semisimple Lie algebra into its simple ideals and let $x = \sum_{i=1}^k x_i \in L$, where $x_i \in L_i$, $i \in \widehat{k}$. The semisimple and nilpotent part of x are $\sum_{i=1}^k d_i$ and $\sum_{i=1}^k n_i$, respectively, where for all $i \in \widehat{k}$ the abstract Jordan decomposition of $x_i \in L_i$ is $x_i = d_i + n_i$.*

Proof. First, $x = d + n$ obviously. Second, take any $i \in \widehat{k}$ and any eigenvector $y \in L_i$ of $\operatorname{ad} d_i$. Then $\operatorname{ad} d(y) = [d, y] = [d_i, y]$ and hence y is an eigenvector for $\operatorname{ad} d$ as well. Consequently, the union of single bases for L_i consisting of eigenvectors for $\operatorname{ad} d_i$, $i \in \widehat{k}$, is a basis for L consisting of eigenvectors for $\operatorname{ad} d$. In other words, d is diagonalisable. Third, for any $m \in \mathbb{N}$ and $\sum_{i=1}^k z_i$, $z_i \in L_i$, we can write

$$\begin{aligned}(\operatorname{ad} n)^m(z) &= (\operatorname{ad} n)^{m-1}([n, z]) = (\operatorname{ad} n)^{m-1}(\sum_{i=1}^k [n_i, z_i]) \\ &\vdots \\ &= \sum_{i=1}^k (\operatorname{ad} n_i)^m(z_i),\end{aligned}$$

from where it is easily seen that n is nilpotent. Finally,

$$[d, n] = [\sum_{i=1}^k d_i, \sum_{j=1}^k n_j] = \sum_{i,j=1}^k [d_i, n_j] = \sum_{i,j=1}^k [n_i, d_j] = [\sum_{i=1}^k n_i, \sum_{j=1}^k d_j] = [n, d].$$

□

Suppose that V is a vector space and L is a semisimple subalgebra of $\mathsf{gl}(V)$. Then for any $x \in L$ we may consider both "usual" and abstract Jordan decompositions. Fortunately, as we shall show below, the two compositions agree (cf. [6], Sec. 4.6).

Lemma 2.20. *Let V be a vector space and let L be a semisimple subalgebra of $\mathsf{gl}(V)$. Suppose that $x \in L$ has usual Jordan decomposition $x = d + n$. Then both $d, n \in L$.*

Proof. V may be regarded as an L-module, the L-module appertaining to the natural representation representation of L. For any W, an L-submodule of V, we define

$$L_W := \left\{ y \in \mathsf{gl}(V) \mid y(W) \subset W \text{ and } \operatorname{Tr}(y|_W) = 0 \right\}.$$

Further, let $\mathcal{M}$ denote the set of all L-submodules of V and let

$$\tilde{L} := N_{\mathsf{gl}(V)}(L) \cap (\bigcap_{W \in \mathcal{M}} L_W).$$

Suppose W is an L-module, for any $y \in L$ and $w \in W$ it holds true that $y(w) = y \cdot w \in W$. Further, we claim that the restriction of elements from L to W is a representation of L (into $\mathsf{gl}(W)$). Indeed. The previous property of elements from L permits us for any $y, z \in L$ to write

$$[y, z]|_W = (y \circ z)|_W - (z \circ y)|_W = y \circ z|_W - z \circ y|_W = y|_W \circ z|_W - z|_W \circ y|_W = [y|_W, z|_W].$$

When we use Lemma 2.9 now, for any $y \in L$ we obtain $y|_W \in \mathsf{sl}(W)$ and consequently $\operatorname{Tr}(y|_W) = 0$. So both conditions are fulfilled and $L \subset L_W$ for all $W \in \mathcal{M}$. Clearly, since L is a subalgebra, $L \subset N_{\mathsf{gl}(V)}(L)$ as well and hence $L \subset \tilde{L}$. In addition, because $\tilde{L}$ is contained in $N_{\mathsf{gl}(V)}(L)$, we have $[\tilde{L}, L] \subset [N_{\mathsf{gl}(V)}(L), L] \subset L$ which means that L is an ideal of $\tilde{L}$. Then, by Proposition 1.50, $\tilde{L} = L \oplus L^{\perp}$, where, by Proposition 1.42, $L^{\perp}$ is also an ideal of $\tilde{L}$, and therefore $[L, L^{\perp}] \subset L \cap L^{\perp} = 0$. Suppose that W is an irreducible submodule of V and take arbitrary $y \in L \subset L_W$, $z \in L^{\perp} \subset L_W$ and $w \in W$. According to above, it holds true that $[y, z] = y \circ z - z \circ y = 0$ and hence we may write

$$z(y \cdot w) = (z \circ y)(w) = (y \circ z)(w) = y \cdot z(w)$$

to show that $z\colon W \to W$ is an L-module homomorphism. It now follows from Schur's Lemma that $z|_W$ is a scalar multiple of the identity on W, but $\mathrm{Tr}(z|_W) = 0$ and thus $z|_W \equiv 0$. Finally, Weyl's Theorem implies that $z \equiv 0$ on whole V, hence $L^{\perp} = 0$ and therefore $\tilde{L} = L$, as required.

Since the (usual) Jordan decomposition of $\operatorname{ad} x$ is $\operatorname{ad} d + \operatorname{ad} n$, there exist complex polynomials $p(t)$ and $q(t)$, such that $p(\operatorname{ad} x) = \operatorname{ad} d$ and $q(\operatorname{ad} x) = \operatorname{ad} n$ (cf. Section 1.3.2). Remark that ad denotes the adjoint homomorphism on $\mathsf{gl}(V)$ here. Clearly, because L is a subalgebra, $\operatorname{ad} x(L) \subset L$ and hence $\operatorname{ad} d(L) = (p(\operatorname{ad} x))(L) \subset L$ and $\operatorname{ad} n(L) = (q(\operatorname{ad} x))(L) \subset L$, in other words $\operatorname{ad} d, \operatorname{ad} n \in N_{\mathsf{gl}(V)}(L)$. Moreover, let W be an L-submodule of V, as $x \in L = \tilde{L}$, $x \in L_W$ as well. From linearity of the trace form we obtain that $\mathrm{Tr}(d|_W) = \mathrm{Tr}(x|_W) = 0$ because the trace of a nilpotent map is always zero, in particular $\mathrm{Tr}(n|_W) = 0$. Finally, since both d and n may be also expressed as polynomials in x (cf. Section 1.3.2), we have $d(W) \subset W$ and $n(W) \subset W$ and hence $d, n \in L_W$ (for any L-submodule W of V). All in all, $d, n \in \tilde{L} = L$, as desired. □

Corollary 2.21. *Let V be a vector space and let L be a semisimple subalgebra of $\mathsf{gl}(V)$. Suppose that $x \in L$ has usual Jordan decomposition $x = d + n$ and abstract Jordan decomposition $x = \widetilde{d} + \widetilde{n}$. Then $d = \widetilde{d}$ and $n = \widetilde{n}$.*

Proof. It results directly from the previous lemma and from uniqueness of both types of the Jordan decomposition. □

We end this section by presenting another fortunate property of (abstract) Jordan decomposition, namely that the decomposition is preserved by any representation of the respective semisimple Lie algebra.

Theorem 2.22. *Let L be a semisimple Lie algebra and let $\phi\colon L \to \mathsf{gl}(V)$ be its representation. Suppose that $x \in L$ has abstract Jordan decomposition $x = d + n$. Then the usual Jordan decomposition of $\phi(x)$ is $\phi(x) = \phi(d) + \phi(n)$.*

Proof. By Corollary 2.21, it suffices to find the abstract Jordan decomposition of $\phi(x)$, then the usual one is the same. However, the abstract Jordan decomposition is defined entirely for elements of a semisimple Lie algebra, thus we must begin with verification of semisimplicity of $\phi(L)$. By Lemma 1.15, $\operatorname{Ker}\phi$ is an ideal of L and $L/\operatorname{Ker}\phi \cong \operatorname{Ran}\phi = \phi(L)$. Moreover, Proposition 1.49 implies that $L/\operatorname{Ker}\phi$ is semisimple, hence $\phi(L)$, is semisimple as well.

First, from the definition of abstract Jordan decomposition, $\operatorname{ad} d$ is diagonalisable and thus there exist vectors $b_1, \ldots, b_m \in L$ such that $L = \operatorname{Span}\{b_1, \ldots, b_m\}$ and for each $i \in \widehat{m}$ there is a complex number λ_i such that $\operatorname{ad} d(b_i) = \lambda_i b_i$. As $\phi(L)$ is a homomorphism, we have $\phi(L) = \operatorname{Span}\{\phi(b_1), \ldots, \phi(b_m)\}$ and, for all $i \in \widehat{m}$,

$$(\operatorname{ad}\phi(d))(\phi(b_i)) = [\phi(d), \phi(b_i)] = \phi([d, b_i]) = \phi(\operatorname{ad} d(b_i)) = \phi(\lambda_i b_i) = \lambda_i \phi(b_i).$$

Now it is easily seen that we can choose a basis of $\phi(L)$ consisting of eigenvectors for $\operatorname{ad}\phi(d)$ which is thereby diagonalisable.

Second, by the definition of abstract Jordan decomposition again, $\operatorname{ad} n$ is nilpotent and hence there exists $r \in \mathbb{N}$ such that $(\operatorname{ad} n)^r = 0$. As ϕ is a homomorphism, for an arbitrary $\phi(y) \in \phi(L)$ we have

$$\begin{aligned}(\operatorname{ad}\phi(n))^r(\phi(y)) &= \overbrace{[\phi(n),[\phi(n),\ldots,[\phi(n),}^{r\text{ times}}\phi(y)]\ldots]] = \phi(\overbrace{[n,[n,\ldots,[n,}^{r\text{ times}}y]\ldots]])\\ &= \phi(0) = 0\end{aligned}$$

and hence $\operatorname{ad}\phi(n)$ is nilpotent.

Third, as $\operatorname{ad} d$ commutes with $\operatorname{ad} n$, for an arbitrary $\phi(y) \in \phi(L)$ we may write

$$\begin{aligned}(\operatorname{ad}\phi(n)\circ\operatorname{ad}\phi(d))(\phi(y)) &= [\phi(n),[\phi(d),\phi(y)]] = \phi([n,[d,y]]) = \phi((\operatorname{ad} n\circ\operatorname{ad} d)(y))\\ &= \phi((\operatorname{ad} d\circ\operatorname{ad} n)(y)) = \phi([d,[n,y]]) = [\phi(d),[\phi(n),\phi(y)]]\\ &= (\operatorname{ad}\phi(d)\circ\operatorname{ad}\phi(n))(\phi(y))\end{aligned}$$

to show that $\operatorname{ad}\phi(n)$ and $\operatorname{ad}\phi(d)$ commute as well.

Finally, $\operatorname{ad}\phi(x) = \operatorname{ad}\phi(d) + \operatorname{ad}\phi(n)$ and thus it follows from the proof of Theorem 2.17 that $\phi(x) = \phi(d) + \phi(n)$ is the required abstract (and hence usual as well) Jordan decomposition of $\phi(x)$. ☐

2.3 Representations of $\mathfrak{sl}(2,\mathbb{C})$

In this final section of the second chapter we investigate the irreducible modules for $\mathfrak{sl}(2,\mathbb{C})$, the Lie algebra consists of all 2×2 complex matrices with zero trace. As showed in Example 1.13 (where this Lie algebra was defined and where some its properties were discussed), $\dim\mathfrak{sl}(2,\mathbb{C}) = 3$. Throughout this section we shall consider the standard basis of $\mathfrak{sl}(2,\mathbb{C})$ (cf. [7]):

$$e = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad f = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad h = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.8}$$

Remark 2.10. We determine the commutation relations among the basis elements.

$$\begin{aligned}[e,f] &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = h,\\ [h,e] &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix} = 2e,\\ [h,f] &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = -2f.\end{aligned}$$

2.3.1 Classification of Irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules

The main purpose of this subsection is to show that (up to isomorphism) there exist only countable many irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules and moreover that different (non-isomorphic) modules have different dimensions (cf. [9], Chap. 7). Remind that we still consider entirely finite-dimensional vector spaces, in particular modules.

Definition 2.23. Let V be an $\mathfrak{sl}(2,\mathbb{C})$-module. For all $\lambda\in\mathbb{C}$ let us denote

$$V_\lambda := \{v\in V \mid h\cdot v = \lambda v\} = \operatorname{Ker}(h-\lambda\mathbb{1}). \tag{2.9}$$

Whenever V_λ is non-trivial, we call λ a *weight* of h in V and we call V_λ a *weight space* associated to the weight λ.

Proposition 2.24. *Let V be an $\mathfrak{sl}(2,\mathbb{C})$-module and let $\lambda \in \mathbb{C}$. If $v \in V_\lambda$, then*

(a) $e \cdot v \in V_{\lambda+2}$,

(b) $f \cdot v \in V_{\lambda-2}$.

Proof. We prove (a), (b) can be proved analogically. Using Remark 2.10, we have

$$h \cdot (e \cdot v) = e \cdot (h \cdot v) + [h, e] \cdot v = e \cdot (\lambda v) + 2e \cdot v = (\lambda + 2)e \cdot v.$$

□

Remark 2.11. Let V be an $\mathrm{sl}(2,\mathbb{C})$-module. According to Remark 2.8 and Lemma 2.15,

$$V = \bigoplus_{\lambda \in \sigma(h)} V_\lambda,$$

where

$$\sigma(h) = \left\{\lambda \in \mathbb{C} \;\middle|\; \text{there exists } v \in V,\ v \neq 0 \text{ and } h \cdot v = \lambda v\right\} = \left\{\lambda \in \mathbb{C} \;\middle|\; V_\lambda \neq \emptyset\right\}.$$

As $\dim V < +\infty$, there must exist finite $\mu := \max \sigma(h)$. Such μ is called the *maximal weight* of V and any vector $v \in V_\mu$ is said to be a *maximal* or *highest-weight vector*.

Remark 2.12. Let v be any vector from an $\mathrm{sl}(2,\mathbb{C})$-module and let $k \in \mathbb{N}$. From now on, we will use the following notation:

$$e^k \cdot v = \overbrace{e \cdot (e \cdot \ldots (e}^{k \text{ times}} \cdot v) \ldots)$$

and identically for f.

Lemma 2.25. *Let V be an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module. Let $v_0 \in V_\mu$ be a maximal vector of V. For all $k \in \mathbb{N}$ let us denote $v_k := \frac{1}{k!} f^k \cdot v_0$ and $v_{-1} := 0$. Then for all $k \in \mathbb{N}_0$ it holds true that*

(a) $f \cdot v_k = (k+1)v_{k+1}$,

(b) $h \cdot v_k = (\mu - 2k)v_k$,

(c) $e \cdot v_k = (\mu - k + 1)v_{k-1}$.

Proof.

(a) For any $k \in \mathbb{N}_0$ we have

$$f \cdot v_k = f \cdot \left(\frac{1}{k!} f^k \cdot v_0\right) = \frac{1}{k!} f^{k+1} \cdot v_0 = \frac{k+1}{(k+1)!} f^{k+1} \cdot v_0 = (k+1)v_{k+1}.$$

(b) We use induction on k. For $k = 0$ the equation is fulfilled from the choice of v_0. For the inductive step, according to Proposition 2.24 (b), we may write

$$\begin{aligned} h \cdot v_k &= \frac{1}{k} h \cdot (f \cdot v_{k-1}) = \frac{1}{k}(f \cdot (h \cdot v_{k-1}) + [h, f] \cdot v_{k-1}) \\ &= \frac{1}{k}((\mu - 2(k-1)) f \cdot v_{k-1} - 2f \cdot v_{k-1}) = (\mu - 2k + 2 - 2)\frac{1}{k} f \cdot v_{k-1} \\ &= (\mu - 2k)v_k. \end{aligned}$$

(c) Again, we use induction on k. For $k = 0$, $e \cdot v_k \in V_{\mu+2} = 0$. For the inductive step, we may write

$$\begin{aligned}
e \cdot v_k &= \frac{1}{k} e \cdot (f \cdot v_{k-1}) = \frac{1}{k}(f \cdot (e \cdot v_{k-1}) + [e, f] \cdot v_{k-1}) \\
&= \frac{1}{k}((\mu - (k-1) + 1) f \cdot v_{k-2} + h \cdot v_{k-1}) \\
&= \frac{1}{k}((\mu - k + 2)(k-1) v_{k-1} + (\mu - 2(k-1)) v_{k-1}) \\
&= \frac{1}{k}(\mu k - k^2 + 2k - \mu + k - 2 + \mu - 2k + 2) v_{k-1} \\
&= (\mu - k + 1) v_{k-1}.
\end{aligned}$$

□

Theorem 2.26. *Let V be an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module. Let $m := \dim V - 1$. Then*

$$V = \bigoplus_{\lambda \in \sigma(h)} V_\lambda, \text{ where } \sigma(h) = \{m, m-2, \ldots, -(m-2), -m\},$$

and $\dim V_\lambda = 1$ *for each $\lambda \in \sigma(h)$.*

Proof. We keep the notation from the previous lemma. Let $(v_k)_{k \in \mathbb{N}_0}$ be the vectors defined in that lemma, in particular, $v_0 \in V_\mu$ is a maximal vector. Necessarily, all non-zero v_k are linearly independent, since being eigenvectors of h with distinct eigenvalues. Hence, as $0 < \dim V < +\infty$, there exists a positive integer, let us denote it by m, such that $v_m \neq 0$, while $v_{m+1} = 0$. Then, clearly, $v_{m+j} = 0$ for any $j \in \mathbb{N}$ and all $v_0, \ldots, v_m$ are non-zero. Moreover, formulas (a)-(c) of the previous lemma imply that $\operatorname{Span}\{v_0, \ldots, v_m\}$ is closed under the action of all f, h and e and thus it is a non-trivial $\mathfrak{sl}(2, \mathbb{C})$-submodule of V. But as V is irreducible, it holds true that $V = \operatorname{Span}\{v_0, \ldots, v_m\}$.

Further, part (c) of the previous lemma for $k = m + 1$ implies

$$0 = e \cdot 0 = e \cdot v_{m+1} = (\mu - m) v_m$$

and therefore, because $v_m \neq 0$, $\mu = m$. By part (b) of the lemma, for all $k \in \underline{\widehat{m}}$ we have $v_k \in V_{\mu - 2k} = V_{m-2k}$, which, together with Remark 2.11, gives the required decomposition.

Finally, as we have decomposed V as the direct sum of $m + 1$ non-trivial subspaces and $\dim V = m + 1$, all summands in the direct sum must be one-dimensional. □

Corollary 2.27.

(a) Each irreducible $\mathfrak{sl}(2, \mathbb{C})$-module V has, up to non-zero scalar multiples, a unique maximal vector. Its weight (maximal weight of V) is equal to $\dim V - 1$.

(b) Suppose $m \in \mathbb{N}_0$. Up to isomorphism, there exists at most one irreducible $\mathfrak{sl}(2, \mathbb{C})$-module of dimension $m + 1$.

Proof. For (a), there is nothing to prove. For (b), suppose that

$$V = \bigoplus_{\lambda \in \sigma(h)} V_\lambda \text{ and } U = \bigoplus_{\lambda \in \sigma(h)} U_\lambda, \text{ where } \sigma(h) = \{m, m-2, \ldots, -(m-2), -m\},$$

are two $(m + 1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules. According to part (a), we may choose maximal vectors $v_0 \in V_m$ and $u_0 \in U_m$ and then construct the bases $(v_k)_{k=0}^{m}$

and $(u_k)_{k=0}^m$ for V and U, respectively, defined in Lemma 2.25 and specified in Theorem 2.26. It is obvious from the proof of the previous theorem that the map $\phi : V \to U$ defined for all $v = \sum_{j=0}^m \alpha_j v_j \in V$ by $\phi(v) := \sum_{j=0}^m \alpha_j u_j$ is an $\mathsf{sl}(2,\mathbb{C})$-module isomorphism. □

Proposition 2.28. *Let V be an* $\mathsf{sl}(2,\mathbb{C})$*-module and let* $\nu \in \mathbb{C}$*. If*

$$\nu \in \sigma(h) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \text{there exists } v \in V,\ v \neq 0 \text{ and } h \cdot v = \lambda v \right\},$$

then $\nu \in \mathbb{Z}$ *and* $-\nu \in \sigma(h)$*.*

Proof. The case when $V = 0$ is trivial. Otherwise, according to Weyl's Theorem, there exist irreducible $\mathsf{sl}(2,\mathbb{C})$-submodules $V^1, \dots, V^n$ of V such that $V = V^1 \oplus \dots \oplus V^n$. By Theorem 2.26, each submodule V^i can be further decomposed as $V^i = \bigoplus_{\lambda \in \sigma_i(h)} V^i_\lambda$, where $\sigma_i(h) = \left\{ \dim V^i - 1, \dim V^i - 3, \dots, -\dim V^i + 1 \right\}$. Altogether, whole V can be written as

$$V = \bigoplus_{\lambda \in \sigma(h)} V_\lambda, \text{ where } \sigma(h) = \bigcup_{i=1}^n \sigma_i(h).$$

For clarity, we may besides write

$$\sigma(h) = \{M-1, M-3, \dots, -M+1\} \cup \{N-1, N-3, \dots, -N+1\},$$

where $M = \max\left\{ \dim V^i \;\middle|\; \dim V^i \text{ is odd} \right\}$ and $N = \max\left\{ \dim V^i \;\middle|\; \dim V^i \text{ is even} \right\}$. The proposition is now obvious. □

Remark 2.13. Keeping the notation from the proof of the previous proposition, it is clear from the considerations performed in that proof that $n = \dim V_0 + \dim V_1$.

2.3.2 The Modules $W(m)$

Suppose a positive integer, say n. We have proved in the previous subsection that, up to isomorphism, there was at most one irreducible $\mathsf{sl}(2,\mathbb{C})$-module of dimension n. However, the question is whether there exists an n-dimensional irreducible $\mathsf{sl}(2,\mathbb{C})$-module for each $n \in \mathbb{N}$. In the final part of the second chapter we answer this question by defining such the module explicitly (cf. [7], Chap. 8).

Example 2.14. Consider the vector space $\mathbb{C}[s,t]$ of all complex polynomials in two variables s and t. For each $m \in \mathbb{N}_0$ let us denote $W(m) := \operatorname{Span}\left\{ s^m, s^{m-1}t, \dots, st^{m-1}, t^m \right\}$, the subspace of $\mathbb{C}[s,t]$ containing all homogeneous polynomials in s and t of degree m. Obviously, $\dim W(m) = m + 1$.

Given $m \in \mathbb{N}_0$, we define a linear map $\phi \colon \mathsf{sl}(2,\mathbb{C}) \to \mathsf{gl}(W(m))$:

$$\phi(e) := s\frac{\partial}{\partial t}, \quad \phi(f) := t\frac{\partial}{\partial s}, \quad \phi(h) := s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t},$$

where $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ mean the partial derivative with respect to s and t, respectively. It is obvious that all $\phi(e)$, $\phi(f)$ and $\phi(h)$ are linear and preserve the degree of polynomial, hence ϕ maps $\mathsf{sl}(2,\mathbb{C})$ into $\mathsf{gl}(W(m))$. We claim that ϕ is a representation of $\mathsf{sl}(2,\mathbb{C})$. To verify this assertion, it only remains to check that ϕ preserves the Lie bracket i.e. that

$$[\phi(h), \phi(f)] = -2\phi(f),\ [\phi(h), \phi(e)] = 2\phi(e) \text{ and } [\phi(e), \phi(f)] = \phi(h).$$

We only sketch the procedure since the rest is completely analogous. Thus, suppose integers $a \geq 3$ and $b \geq 2$.

$$\begin{aligned}
[\phi(h),\phi(f)](s^a t^b) &= (s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t})(t\frac{\partial}{\partial s}(s^a t^b)) - t\frac{\partial}{\partial s}((s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t})(s^a t^b)) \\
&= s\frac{\partial}{\partial s}(as^{a-1}t^{b+1}) - t\frac{\partial}{\partial t}(as^{a-1}t^{b+1}) - t\frac{\partial}{\partial s}(as^a t^b) + t\frac{\partial}{\partial s}(bs^a t^b) \\
&= a(a-1)s^{a-1}t^{b+1} - a(b+1)s^{a-1}t^{b+1} - a^2 s^{a-1}t^{b+1} + abs^{a-1}t^{b+1} \\
&= (a^2 - a - ab - a - a^2 + ab)s^{a-1}t^{b+1} = -2as^{a-1}t^{b+1} = -2t\frac{\partial}{\partial s}(s^a t^b) \\
&= -2\phi(f)(s^a t^b)
\end{aligned}$$

In this way, one could verify all the other cases (when $a \in \{0,1,2\}$ or $b \in \{0,1\}$) for the first commutator. For two other commutators, the procedure also consists of several cases, which have to be verified separately.

All in all, for each $m \in \mathbb{N}_0$, $W(m)$ is an $(m+1)$-dimensional $\mathsf{sl}(2,\mathbb{C})$-module.

Proposition 2.29. *Let $m \in \mathbb{N}_0$. The $\mathsf{sl}(2,\mathbb{C})$-module $W(m)$ is irreducible.*

Proof. At first, we look at the action of e, f and h on basis monomials from $W(m)$. Let a and b be positive integers such that $a+b=m$. Then

(a) $e \cdot (s^a t^b) = s\frac{\partial}{\partial t}(s^a t^b) = bs^{a+1}t^{b-1}, \quad e \cdot (s^m) = 0, \quad e \cdot (t^m) = mst^{m-1};$

(b) $f \cdot (s^a t^b) = t\frac{\partial}{\partial s}(s^a t^b) = as^{a-1}t^{b+1}, \quad f \cdot (s^m) = ms^{m-1}t, \quad f \cdot (t^m) = 0;$

(c) $h \cdot (s^a t^b) = (s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t})(s^a t^b) = (a-b)s^a t^b, \quad h \cdot (s^m) = ms^m, \quad h \cdot (t^m) = mt^m.$

It is clear from (c) that the (standard) basis of $W(m)$ established in Example 2.14 is the basis of eigenvectors for h, hence d is diagonalisable on $W(m)$.

Now, suppose U is a non-zero $\mathsf{sl}(2,\mathbb{C})$-submodule of $W(m)$. Being a submodule, U has to be closed under the action of h. It results from Remark 2.8 that h is diagonalisable on U as well: the minimum polynomial $p(t)$ of h is the product of distinct linear factors and herewith $p(h)$ equals identically zero on U, hence the minimum polynomial $p_U(t)$ of $h|_U : U \to U$ divides $p(t)$ and thus it is the product of distinct linear factors as well which means that $h|_U$ is diagonalisable indeed. This fact implies that at least one basis vector of $W(m)$, say $s^a t^b$, lies in U. Now assume that there is a different basis vector of $W(m)$, say $s^c t^d$, which does not lie in U. But this assumption leads to a contradiction, because U is, as an submodule, closed under the action of e and f and hence if $a < c$, then $e^{c-a} \cdot (s^a t^b) \in U$ is a non-zero scalar multiple of $s^c t^d$. Similarly, if $a > c$, then $f^{d-b} \cdot (s^a t^b) \in U$ is a non-zero scalar multiple of $s^c t^d$. Therefore all basis vectors of $W(m)$ lie in U and thus $U = W(m)$. □

Chapter 3

Classification of Semisimple Lie Algebras

The purpose of this chapter is to classify all isomorphism classes of complex semisimple Lie algebras via the so-called *roots* and *root systems*. In fact, it is enough to classify only simple Lie algebras since they are the unique "building blocks" for the semisimple ones (cf. Theorem 1.47). We shall see that the root systems decompose into simpler subsystems exactly in the same way as a semisimple Lie algebras decompose into simple ideals. The very essential and fundamental property of this analog is that each simple subalgebra corresponds precisely to the one of these subsystems and vice versa.

3.1 Root Space Decomposition

3.1.1 Cartan Subalgebras

At the very beginning, we need a little linear algebra again (cf. [7], Subsec. 16.3.2).

Lemma 3.1. *Let $x_1, \dots, x_k$ be diagonalisable linear transformations of a vector space V. Then there is a basis of V consisting of simultaneous eigenvectors for all $x_1, \dots, x_k$ (we say that $x_1, \dots, x_k$ are simultaneously diagonalisable) if and only $x_i \circ x_j = x_j \circ x_i$ for all $i, j \in \widehat{k}$.*

Proof. The "only if" direction is easy. Matrices of all $x_1, \dots, x_k$ with respect to the basis of their simultaneous eigenvectors are diagonal. Clearly, diagonal matrices commute and hence $x_1, \dots, x_k$ commute with each other.

For the "if" direction, we proceed by induction on $k \in \mathbb{N}$. The case when $k = 1$ is trivial. Hence assume that the implication holds for $k-1$ and $x_1, \dots, x_k$ commute with each other. First, it follows from Primary Decomposition Theorem that V decomposes into the direct sum of eigenspaces for x_k: $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}$, where $(\lambda_i)_{i=1}^r$ are the distinct eigenvalues of x_k. Choose any $i \in \widehat{r}$. For all $v \in V_{\lambda_i}$ and $j \in \widehat{k-1}$ we may write

$$x_k(x_j(v)) = x_j(x_k(v)) = x_j(\lambda_i v) = \lambda_i(x_j(v))$$

and hence $x_j|_{V_{\lambda_i}} : V_{\lambda_i} \to V_{\lambda_i}$ (for all $j \in \widehat{k-1}$). Then, since $x_1|_{V_{\lambda_i}}, \dots, x_{k-1}|_{V_{\lambda_i}}$ also commute and, as shown in the proof of Proposition 2.29, they are also diagonalisable, there exists a basis for V_{λ_i} consisting of common eigenvectors for $x_1, \dots, x_{k-1}$. These basis vectors are obviously eigenvectors for x_k as well. In this way, we obtain bases of simultaneous eigenvectors of $x_1, \dots, x_k$ for all $(V_{\lambda_i})_{i=1}^r$ and, finally, the union of these bases is the required basis for V. □

Corollary 3.2. *Let A be an arbitrary set of linear transformations of a vector space V such that for all $x, y \in A$ it holds true that $x \circ y = y \circ x$. Then there is a basis for V consisting of simultaneous eigenvectors for all maps from A.*

Proof. As V is finite dimensional, the same holds for $\operatorname{Span} A$. Hence we may choose $x_1, \ldots, x_m$ from A such that $\operatorname{Span} A = \operatorname{Span}\{x_1, \ldots, x_m\}$. According to Lemma 3.1, there exists a basis for V consisting of simultaneous eigenvectors for $x_1, \ldots, x_m$. In other words, all $x_1, \ldots, x_m$ are represented by diagonal matrices with respect to this basis and this must be true also for any $x \in A \subset \operatorname{Span} A = \operatorname{Span}\{x_1, \ldots, x_m\}$. □

The first step in identification of the "roots" of a semisimple Lie algebra L is to establish a distinguished abelian subalgebra of L (cf. [7], Sec. 10.1).

Definition 3.3. Let L be a semisimple Lie algebra. A Lie subalgebra H of L is called a *Cartan subalgebra* or *CSA* if it is abelian, it contains entirely semisimple elements and it is maximal with this properties.

Remark 3.1. In fact, it is not necessary to require Cartan subalgebra to be abelian. It can be shown (cf. [9], p. 35) that every subalgebra of a semisimple Lie algebra consisting of semisimple elements (such subalgebra is said to be *toral*) is abelian.

Proposition 3.4. *Let L be a non-trivial semisimple a Lie algebra. A CSA of L is also non-trivial.*

Proof. If L consisted entirely of nilpotent elements, it would be, according to Corollary 1.30, nilpotent and hence solvable. But a non-trivial Lie algebra cannot be simultaneously semisimple and solvable. □

Definition 3.5. Let H be a CSA of a semisimple Lie algebra L. For all $\alpha \in H^*$ let us denote

$$L_\alpha := \left\{ x \in L \;\middle|\; \text{for all } h \in H,\ [h, x] = \alpha(h)x \right\}. \tag{3.1}$$

Whenever α is not the zero functional and L_α is non-trivial, we say that α is a *root* of L and L_α is the associated *root space*. We denote the set of all roots by Φ. Further we denote $\Phi^0 := \Phi \cup \{0\}$.

Lemma 3.6. *Suppose the L is a semisimple Lie algebra, H is its CSA and Φ is the set of all roots of L with respect to H. Then Φ is finite and*

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha. \tag{3.2}$$

Proof. First, for any $x, y \in H$ and $z \in L$ we have

$$(\operatorname{ad} x \circ \operatorname{ad} y)(z) = [x, [y, z]] = -[y, [z, x]] - [z, [x, y]] = [y, [x, z]] = (\operatorname{ad} y \circ \operatorname{ad} x)(z),$$

hence, by Corollary 3.2, there is a basis for L, say $(b_1, \ldots, b_n)$, consisting of simultaneous eigenvectors for all elements of $\operatorname{ad}(H)$. Now we define n functionals $\alpha_1, \ldots, \alpha_n$ on H: let

$$\operatorname{ad} h(b_i) = \alpha_i(h) b_i$$

be satisfied for all $i \in \widehat{n}$ and for all $h \in H$. Clearly, all α_i are well defined functionals on H. Moreover, it follows from linearity of ad that $\alpha_i \in H^*$ for all $i \in \widehat{n}$. Since $0 \neq b_i \in L_{\alpha_i}$ for each $i \in \widehat{n}$, it is clear that all non-zero α_i are roots of L (not necessarily distinct). We claim that even $\Phi \subset \{\alpha_1, \dots, \alpha_n\}$ i.e. each root of L is one of the α_i. Indeed. Take $\alpha \in \Phi$, $h \in H$ and $x = \sum_{j=1}^{n} \beta_j b_j \in L_\alpha$ such that $x \neq 0$, then

$$[h, x] = \operatorname{ad} h(x) = \operatorname{ad} h \left(\sum_{j=1}^{n} \beta_j b_j \right) = \sum_{j=1}^{n} \beta_j \operatorname{ad} h(b_j) = \sum_{j=1}^{n} \beta_j \alpha_j(h) b_j$$

and

$$[h, x] = \alpha(h)x = \alpha(h) \left(\sum_{j=1}^{n} \beta_j b_j \right) = \sum_{j=1}^{n} \beta_j \alpha(h) b_j.$$

Hence, as $(b_1, \dots, b_n)$ is a basis, for all $j \in \widehat{n}$ we have $\beta_j(\alpha_j(h) - \alpha(h)) = 0$. And thus, because $x \neq 0$, there exists $j_0 \in \widehat{n}$ such that, for any $h \in H$, $\alpha_{j_0}(h) = \alpha(h)$. Now it is obvious that Φ is finite.

For the second part, given any non-zero $x = \sum_{j=1}^{n} \beta_j b_j \in L$, for all $j \in \widehat{n}$ we define $x_j := \beta_j b_j$. Clearly $x = \sum_{j=1}^{n} x_j$ and additionally

$$[h, x_j] = \operatorname{ad} h(x_j) = \operatorname{ad} h(\beta_j b_j) = \beta_j \alpha_j(h) b_j,$$

hence $x_j \in L_{\alpha_j}$. Altogether, we may write

$$L = \sum_{j=1}^{n} L_{\alpha_j} = L_0 + \sum_{\alpha \in \Phi} L_\alpha. \tag{3.3}$$

Finally, take any $\alpha, \mu \in \Phi^0$ such that $\alpha \neq \mu$ and any $x \in L_\alpha \cap L_\mu$. Then for $h \in H$ it has to be fulfilled that $\alpha(h)x = [h, x] = \mu(h)x$ or equivalently $(\alpha(h) - \mu(h))x = 0$. But, since $\alpha \neq \mu$, there exists $h_0 \in H$ such that $\alpha(h_0) \neq \mu(h_0)$, thus $x = 0$ and the sum (3.3) is direct. □

Remark 3.2. Let L be a semisimple Lie algebra and let H be its CSA. It can be easily seen that $L_0 = C_L(H)$.

Proposition 3.7. *Let L be a semisimple Lie algebra and let H be its CSA. Let $\alpha, \beta \in H^*$.*

(a) $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.

(b) If $x \in L_\alpha$ and $\alpha \neq 0$, then $\operatorname{ad} x$ *is nilpotent.*

(c) If $\alpha + \beta \neq 0$, then for all $x \in L_\alpha$ and $y \in L_\beta$ it holds true that $\kappa(x, y)$=0.

(d) The restriction of κ to $C_L(H)$ is non-degenerate.

Proof.

(a) For any $x \in L_\alpha$, $y \in L_\beta$ and $h \in H$ we have

$$\begin{aligned}[h, [x, y]] &= -[x, [y, h]] - [y, [h, x]] = [x, [h, y]] + [[h, x], y] = \beta(h)[x, y] + \alpha(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y].\end{aligned}$$

(b) It follows directly from (a) and from the fact that there are only finitely many $\mu \in H^*$ such that L_μ is non-trivial.

(c) Suppose $x \in L_\alpha$, $y \in L_\beta$ and $h \in H$ such that $(\alpha + \beta)(h) \neq 0$. According to Proposition 1.37, we may write

$$\begin{aligned}(\alpha(h) + \beta(h))\kappa(x, y) &= \kappa(\alpha(h)x, y) + \kappa(x, \beta(h)y) = \kappa([h, x], y) + \kappa(x, [h, y]) \\ &= -\kappa([x, h], y) + \kappa([x, h], y) = 0\end{aligned}$$

and hence, as $\alpha(h) + \beta(h) \neq 0$, $\kappa(x, y) = 0$.

(d) Suppose $y \in C_L(H) = L_0 \subset L$ and that $\kappa(y, x_0) = 0$ for any $x_0 \in L_0 = C_L(H)$. Using (c), this is equivalent to $\kappa(y, x) = 0$ for any $x = x_0 + \sum_{\mu \in \Phi} x_\mu \in L$, where $x_\mu \in L_\mu$, $\mu \in \Phi^0$. But, since L is semisimple, κ is non-degenerate on L (cf. Theorem 1.43) and hence $y = 0$. □

We state the following important lemma without proof. Remark only that it divides into several steps in which one explores the centralizer of H successively to identify it with H in the end. One can found the proof in [9], Sec.8.2.

Lemma 3.8. *Let H be a CSA of a semisimple Lie algebra L. Then $C_L(H) = H$.*

Corollary 3.9. *The restriction of κ to H is non-degenerate.*

Proof. Trivial consequence of Proposition 3.7 (d) and the previous lemma. □

Now we may substitute into (3.2) and introduce the pivotal definition (cf. [7], Sec. 10.3).

Definition 3.10. Suppose that L is a semisimple Lie algebra, H is a CSA of L and Φ is the set of all roots of L with respect to H. Then the relation

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \tag{3.4}$$

is called the *root space decomposition* of L.

The fact that has to be mentioned here is that the Cartan subalgebra is not determined uniquely by its definition. Consequently, the set of roots and the root space decomposition are not unique. However, we shall see later (in the last section of this chapter) that there is no ambiguity in the classification for this reason and that the classificatory tool which we shall use does not depend on the particular choice of the Cartan subalgebra.

3.1.2 Properties of Roots and Root Spaces

Now, we are going to uncover further features and characteristics of roots and the respective root spaces (cf. [6], Sec. 4.9, and [9], Sec. 8.2,3). For brevity, we will keep the following convention in the rest of this section: we will suppose that L is a semisimple Lie algebra (hence the Killing form on L is non-degenerate), that H is a Cartan subalgebra of L and that Φ is the set of roots of L with respect to H. In addition, talking about H or Φ, we will always assume that the respective Lie algebra is given automatically.

Proposition 3.11. $\operatorname{Span} \Phi = H^*$.

Proof. Clearly, $\operatorname{Span} \Phi \subset H^*$. Contrariwise, suppose that $\dim(\operatorname{Span} \Phi) < \dim H^*$. We claim that then there exists $h \in H$ such that $h \neq 0$ and $\alpha(h) = 0$ for all $\alpha \in \Phi$. Indeed. Let $(h_1, \ldots, h_{\dim H})$ be a basis for H and let $\left(\alpha_1, \ldots, \alpha_{\dim(\operatorname{Span} \Phi)}\right)$ be a basis for $\operatorname{Span} \Phi$. We search a non-zero vector $h = \sum_{i=1}^{\dim H} \lambda_i h_i$ such that $0 = \alpha_j(h) = \sum_{i=1}^{\dim H} \alpha_j(h_i)\lambda_i$ for all $j \in \{1, 2, \ldots, \dim(\operatorname{Span} \Phi)\}$. Hence, we have a system of $\dim(\operatorname{Span} \Phi)$ homogeneous linear equations for $\dim H = \dim H^* > \dim(\operatorname{Span} \Phi)$ unknowns. Certainly, such a system has always a non-trivial solution and thus the desired vector h exists.

But for arbitrary $\alpha \in \Phi$ and $x \in L_\alpha$ we have $[h, x] = \alpha(h)x = 0$ and additionally, since H is abelian, for all $g \in H$ it is true that $[h, g] = 0$. All in all, considering the root space decomposition of L, $[h, x] = 0$ for all $x \in L$ and hence $h \in Z(L) = 0$, a contradiction. □

Remark 3.3. Similarly as in the proof of Lemma 1.46, let us consider a map $\phi\colon H \to H^*$ defined for all $x, y \in H$ by $\phi(x)(y) := \kappa(x, y)$. Clearly $\phi(H) \subset H^*$ and further, by Corollary 3.9, $\operatorname{Ker}\phi = 0$. Now, the rank-nullity formula (cf. [11], p. 61) implies

$$\dim H^* = \dim H = \dim(\operatorname{Ran}\phi) + \dim(\operatorname{Ker}\phi) = \dim(\operatorname{Ran}\phi)$$

and hence $\phi(H) = H^*$. All in all, ϕ is a bijection between H and H^*. This allows us to identify H with H^*: for each $\varphi \in H^*$ there exists a unique vector $t_\varphi \in H$ such that $\varphi = \phi(t_\varphi)$. For this vector and for all $h \in H$ the following holds:

$$\varphi(h) = \phi(t_\varphi)(h) = \kappa(t_\varphi, h). \tag{3.5}$$

Note that the assignment $\varphi \mapsto t_\varphi$ is linear since for all $\varphi_1, \varphi_2 \in H^*$, $c \in \mathbb{C}$ and $h \in H$ we have

$$\kappa(t_{\varphi_1+c\varphi_2}, h) = (\varphi_1 + c\varphi_2)(h) = \varphi_1(h) + c\varphi_2(h) = \kappa(t_{\varphi_1}, h) + c\kappa(t_{\varphi_2}, h) = \kappa(t_{\varphi_1} + ct_{\varphi_2}, h)$$

and hence the non-degeneracy of $\kappa|_{H\times H}$ implies $t_{\varphi_1+c\varphi_2} = t_{\varphi_1} + ct_{\varphi_2}$. In particular, the zero functional $0 \in H^*$ corresponds to the zero vector $0 \in H$ obviously.

We will keep also this labelling in the rest of this section: t_φ will always be the vector from H related (in the sense described above) to $\varphi \in H^*$.

Proposition 3.12. *Suppose that $\alpha \in \Phi$.*

(a) $-\alpha \in \Phi$.

(b) Let $x \in L_\alpha$ and $y \in L_{-\alpha}$. Then $[x, y] = \kappa(x, y)t_\alpha$.

(c) $[L_\alpha, L_{-\alpha}]$ is one-dimensional and $[L_\alpha, L_{-\alpha}] = \operatorname{Span}\{t_\alpha\}$.

(d) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

Proof.

(a) Suppose that $-\alpha \notin \Phi$. Then for all $\beta \in \Phi^0$ it holds true that $\alpha + \beta \neq 0$ and hence, according to part (c) of Proposition 3.7 and to the root space decomposition of L, $\kappa(x, y) = 0$ for all $x \in L_\alpha$ and $y \in L$. In particular, since α is a root, this holds for a non-zero vector from $L_\alpha \subset L$, contradicting the non-degeneracy of κ.

(b) First, by Proposition 3.7 (a), $[x, y] \in H$ and hence $[x, y] - \kappa(x, y)t_\alpha \in H$. For an arbitrary $h \in H$ we have

$$\begin{aligned}\kappa(h, [x, y] - \kappa(x, y)t_\alpha) &= \kappa(h, [x, y]) - \kappa(h, \kappa(x, y)t_\alpha) = \kappa([h, x], y) - \kappa(x, y)\kappa(h, t_\alpha)\\ &= \alpha(h)\kappa(x, y) - \kappa(x, y)\alpha(h) = 0\end{aligned}$$

and consequently, by Corollary 3.9, $[x, y] - \kappa(x, y)t_\alpha = 0$.

(c) By the previous item, since $t_\alpha \neq 0$, it suffices to show that there exist $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$. Take any non-zero $x \in L_\alpha$ and suppose that $\kappa(x, y) = 0$ for all $y \in L_{-\alpha}$. Then, according to Proposition 3.7 (c), $\kappa(x, y) = 0$ even for all $y \in L$, but this is again a contradiction with non-degeneracy of κ.

(d) The first equality is just the definition of t_α. Suppose that $\alpha(t_\alpha) = 0$. Then for any $x \in L_\alpha$ and $y \in L_{-\alpha}$ we have $[t_\alpha, x] = [t_\alpha, y] = 0$. As in the previous item, we can find $\tilde{x} \in L_\alpha$ and $\tilde{y} \in L_{-\alpha}$ such that $\kappa(\tilde{x}, \tilde{y}) \neq 0$. Then, if we put $\hat{x} := \frac{\tilde{x}}{\kappa(\tilde{x}, \tilde{y})}$, it holds $[\hat{x}, \tilde{y}] = t_\alpha$ and $S := \operatorname{Span}\{\hat{x}, \tilde{y}, t_\alpha\}$ is a solvable subalgebra of L, obviously. Since L is semisimple, the adjoint representation of L is faithful, hence $S \cong \operatorname{ad}(S) \subset$

gl(L). By Theorem 1.31 and by the proof of Theorem 1.40 (the commutator of upper triangular matrices is a strictly upper triangular matrix), we know that all elements of $[\mathrm{ad}(S), \mathrm{ad}(S)] = \mathrm{ad}([S,S])$ are nilpotent. In particular this holds for $\mathrm{ad}\, t_\alpha$, but $\mathrm{ad}\, t_\alpha$ is semisimple as well because $t_\alpha \in H$. Altogether, $\mathrm{ad}\, t_\alpha = 0$ and hence $t_\alpha \in Z(L) = 0$, a contradiction ($0 \notin \Phi \ni \alpha$). □

Proposition 3.13. *Suppose $\alpha \in \Phi$ and $e_\alpha \in L_\alpha$, $e_\alpha \neq 0$. Then there exists $f_\alpha \in L_{-\alpha}$ such that* Span $\{e_\alpha, f_\alpha, h_\alpha\}$, *where $h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$, is a subalgebra of L isomorphic to sl$(2,\mathbb{C})$.*

Proof. Take any $\tilde{f}_\alpha \in L_{-\alpha}$ such that $\kappa(e_\alpha, \tilde{f}_\alpha) \neq 0$ (this is possible, according to the proof of part (c) of Proposition 3.12) and set $f_\alpha := \frac{2\tilde{f}_\alpha}{\kappa(e_\alpha,\tilde{f}_\alpha)\kappa(t_\alpha,t_\alpha)}$. Then

$$
\begin{aligned}
[e_\alpha, f_\alpha] &= \frac{2}{\kappa(e_\alpha,\tilde{f}_\alpha)\kappa(t_\alpha,t_\alpha)}[e_\alpha,\tilde{f}_\alpha] = \frac{2}{\kappa(e_\alpha,\tilde{f}_\alpha)\kappa(t_\alpha,t_\alpha)}\kappa(e_\alpha,\tilde{f}_\alpha)t_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)} = h_\alpha,\\
[h_\alpha, e_\alpha] &= \frac{2}{\kappa(t_\alpha,t_\alpha)}[t_\alpha,e_\alpha] = \frac{2}{\kappa(t_\alpha,t_\alpha)}\alpha(t_\alpha)e_\alpha = 2e_\alpha,\\
[h_\alpha, f_\alpha] &= \frac{2}{\kappa(t_\alpha,t_\alpha)}[t_\alpha,f_\alpha] = \frac{2}{\kappa(t_\alpha,t_\alpha)}(-\alpha(t_\alpha))f_\alpha = -2f_\alpha.
\end{aligned}
$$

Hence we can see, that e_α, f_α and h_α (it follows from the root space decomposition of L, that they are linearly independent and hence form a basis of Span $\{e_\alpha, f_\alpha, h_\alpha\}$) satisfy the same commutation relations as the basis vectors of sl$(2,\mathbb{C})$ e, f an h, respectively (cf. Sec. 2.3) and thus the linear map ψ: Span $\{e_\alpha, f_\alpha, h_\alpha\} \to$ sl$(2,\mathbb{C})$ defined by

$$\psi(e_\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi(f_\alpha) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \psi(h_\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is, as an obvious bijection, a Lie algebra isomorphism between Span $\{e_\alpha, f_\alpha, h_\alpha\}$ and sl$(2,\mathbb{C})$. □

Corollary 3.14. *Suppose $\alpha \in \Phi$ and h_α from the previous proposition. Then $\alpha(h_\alpha) = 2$.*

Proof. $\alpha(h_\alpha) = \alpha(\frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}) = \frac{2}{\kappa(t_\alpha,t_\alpha)}\alpha(t_\alpha) = \frac{2}{\kappa(t_\alpha,t_\alpha)}\kappa(t_\alpha,t_\alpha) = 2.$ □

Remark 3.4. We will keep the notation from the previous proposition: for the rest of this section for all $\alpha \in \Phi$ we denote $h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$. Moreover, sl$(\alpha) :=$ Span $\{e_\alpha, f_\alpha, h_\alpha\}$.

One might ask whether the subalgebra sl(α) is independent of the choice of e_α and f_α and hence unique. As we show below, the answer is "yes". However, the proof is not very straightforward; we have to look at the modules of sl(α) at first (cf. [7], Sec. 10.5). This is exactly the reason why we have studied the representation theory of sl$(2,\mathbb{C})$ earlier. Remark that the sets of Lie modules for two isomorphic Lie algebras are in one-to-one correspondence with each other, obviously.

Remark 3.5. Given $\alpha \in \Phi$, L may be regarded as an sl(α)-module via the restriction of the adjoint representation of L to sl(α): for all $a \in$ sl(α) and $x \in L$ we define

$$a \cdot x := \mathrm{ad}\, a(x) = [a, x]. \tag{3.6}$$

Definition 3.15. Let $\alpha \in \Phi$ and $\beta \in \Phi$. The set

$$\bigoplus_{\substack{k\in\mathbb{Z}\\ \beta+k\alpha\in\Phi}} L_{\beta+k\alpha} \tag{3.7}$$

is called the *α-root string through β*.

Proposition 3.16. *Let $\alpha \in \Phi$ and $\beta \in \Phi$. The α-root string through β is an $\mathsf{sl}(\alpha)$-submodule of L.*

Proof. First, let us denote the α-root string through β by S. Clearly, S is a subspace of L. Now, given any $k \in \mathbb{Z}$ such that $\beta + k\alpha \in \Phi$ and $x \in L_{\beta+k\alpha}$, it follows from part (a) of Proposition 3.7 that $e_\alpha \cdot x = [e_\alpha, x] \in L_{\beta+(k+1)\alpha}$. If $\beta + (k+1)\alpha \in \Phi$, then clearly $a \cdot x \in S$. Otherwise, $a \cdot x = 0 \in S$. Similarly for f_α. Finally, $h_\alpha \cdot x = (\beta(h) + k\alpha(h))x \in S$. □

Lemma 3.17. *Let $\alpha \in \Phi$.*

(a) $\dim L_\alpha = \dim L_{-\alpha} = 1$.

(b) If c is a complex number such that $c\alpha \in \Phi$, then $c = \pm 1$.

Proof. First, we show that $H + \mathsf{sl}(\alpha) = \operatorname{Ker}\alpha \oplus \mathsf{sl}(\alpha)$. Clearly, since $h_\alpha \in H \cap \mathsf{sl}(\alpha)$, $\dim(H + \mathsf{sl}(\alpha)) \leq \dim H + 2$. Further, according to the root space decomposition of L and to the fact that $\alpha(h_\alpha) = 2 \neq 0$, $\operatorname{Ker}\alpha \cap \operatorname{Span}\{e_\alpha, f_\alpha, h_\alpha\} = 0$, therefore $\dim(\operatorname{Ker}\alpha \oplus \mathsf{sl}(\alpha)) = (\dim H - 1) + 3 = \dim H + 2$. Concurrently, because $\operatorname{Ker}\alpha \subset H$, it is clear that $\operatorname{Ker}\alpha \oplus \mathsf{sl}(\alpha) \subset H + \mathsf{sl}(\alpha)$. Comparing the dimensions, we obtain the desired equality.

In addition, both $\operatorname{Ker}\alpha$ and $\mathsf{sl}(\alpha)$ are $\mathsf{sl}(\alpha)$-submodules of L. Indeed, $\mathsf{sl}(\alpha)$ is a submodule because of being a subalgebra and $\operatorname{Ker}\alpha$ is a submodule because for an arbitrary $h \in \operatorname{Ker}\alpha$ we have

$$\begin{aligned} e_\alpha \cdot h &= [e_\alpha, h] = -[h, e_\alpha] = -\alpha(h)e_\alpha = 0, \\ f_\alpha \cdot h &= [f_\alpha, h] = -[h, f_\alpha] = \alpha(h)f_\alpha = 0, \\ h_\alpha \cdot h &= [h_\alpha, h] = 0. \end{aligned}$$

Now, let us denote

$$S := \bigoplus_{\substack{c\in\mathbb{C} \\ c\alpha\in\Phi^0}} L_{c\alpha} = H \oplus L_\alpha \oplus L_{-\alpha} \oplus \bigoplus_{\substack{c\in\mathbb{C}/\{\pm1\} \\ c\alpha\in\Phi}} L_{c\alpha} = \operatorname{Ker}\alpha \oplus \mathsf{sl}(\alpha) \oplus \bigoplus_{\substack{c\in\mathbb{C}/\{\pm1\} \\ c\alpha\in\Phi}} L_{c\alpha}.$$

Exactly as in the proof of Proposition 3.16, it can be shown that S is an $\mathsf{sl}(\alpha)$-submodule of L. Likewise, let us denote

$$W := \bigoplus_{\substack{c\in\mathbb{C}/\{\pm1\} \\ c\alpha\in\Phi}} L_{c\alpha}$$

which has to be an $\mathsf{sl}(\alpha)$-submodule of L as well resulting from Weyl's Theorem and Proposition 2.11.

Suppose that at least one of the assertions of the lemma does not hold, then W is non-trivial and hence it has an irreducible submodule V of dimension m. We use the classification of irreducible $\mathsf{sl}(2, \mathbb{C})$ discussed in Section 2.3 now.

In case that m is odd, there exists $v \in V \subset W$, an eigenvector for h_α with the zero eigenvalue. In other words, there is $c \in \mathbb{C}/\{-1, 0, 1\}$ such that

$$0 = h_\alpha \cdot v = [h_\alpha, v] = c\alpha(h_\alpha)v = 2cv,$$

a contradiction. Moreover, suppose that $2\alpha \in \Phi$. Then there exists a non-zero $v \in L_{2\alpha}$ such that $h_\alpha \cdot v = [h_\alpha, v] = 2\alpha(h_\alpha)v = 4v$ and hence W has an irreducible submodule of odd dimension. But this is impossible, therefore if $\alpha \in \Phi$, then $2\alpha \notin \Phi$.

Now suppose that m is even. There exists $v \in V \subset W$, an eigenvector for h_α with 1 as an eigenvalue. In other words there is $c \in \mathbb{C}/\{-1, 0, 1\}$ such that

$$v = h_\alpha \cdot v = [h_\alpha, v] = c\alpha(h_\alpha)v = 2cv,$$

thus $c = \frac{1}{2}$ and therefore both $\frac{\alpha}{2}$ and α are roots, a contradiction again. □

Now it is seen that the only freedom of the choice of e_α lies in the non-zero scalar multiplication and that for each $e_\alpha \in L_\alpha$ there exists a unique $f_\alpha \in L_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ whereas the direction of f_α is independent of the particular choice of e_α. Lemma 3.17 establishes this choice up to the scalar multiple and part (b) of Proposition 3.12 then determines the scalar. Hence each subalgebra $\mathsf{sl}(\alpha)$ is determined uniquely.

Proposition 3.18. *Let $\alpha, \beta \in \Phi$.*

(a) $\beta(h_\alpha) \in \mathbb{Z}$.

(b) If $\alpha \neq \pm\beta$, there are $p, q \in \mathbb{N}_0$ such that for all $k \in \mathbb{Z}$ it holds true that $\beta + k\alpha \in \Phi$ if and only if $-p \leq k \leq q$. Moreover, $p - q = \beta(h_\alpha)$.

(c) $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(d) If $\alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

Proof.

(a) For any non-zero $v \in L_\beta$ we have $\beta(h_\alpha)v = [h_\alpha, v] = h_\alpha \cdot v$, hence $\beta(h_\alpha)$ is an eigenvalue for h_α acting on $L_\beta \subset L$. The assertion now follows from Remark 3.5 and Proposition 2.28.

(b) Let S be the α-root string through β. We know that S is an $\mathsf{sl}(\alpha)$-submodule of L (cf. Proposition 3.16). According to the proof of Proposition 2.28, there exist irreducible submodules $S^1, \ldots, S^n$ of S such that $S = S^1 \oplus \cdots \oplus S^n$. But, as all the eigenvalues for h_α acting on S are of the form $(\beta + l\alpha)(h_\alpha) = \beta(h_\alpha) + 2l$, $l \in \mathbb{Z}$, so they are either all odd or all even and additionally, by Lemma 3.17 (a), all corresponding eigenspaces are one-dimensional, Remark 2.13 gives us that $n = 1$ in our case, in other words that S is itself irreducible. According to Theorem 2.26, this fact means that the set of all roots occurring in the string S has to be precisely of the form $\{m, m-2, \ldots, -m\}$, where $m = \dim S - 1$. Thus, if we denote $p := \max\{k \in \mathbb{Z} \mid \beta - k\alpha \in \Phi\}$ and $q := \max\{k \in \mathbb{Z} \mid \beta + k\alpha \in \Phi\}$, we obtain the first part of the proposition.

For the second one, it is obvious that in our case $m = \beta(h_\alpha) + 2q$ and $-m = \beta(h_\alpha) - 2p$ which, added together and divided by 2, gives us the required relation.

(c) Let p and q be the integers from the previous item. Then

$$\beta - \beta(h_\alpha)\alpha = \beta - (p - q)\alpha \in \Phi$$

because $-p \leq -p + q \leq q$.

(d) The inclusion $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ has been already proved (cf. Proposition 3.7), thus it suffices to show that each vector from $L_{\alpha+\beta}$ can be expressed as the commutator of vectors from L_α and L_β, respectively. Consider the same irreducible submodule S as in part (b) and keep the denotation of p and q as well. Take any non-zero $e_\beta \in L_\beta$, $e_\alpha \in L_\alpha$ and $e_{\alpha+\beta} \in L_{\alpha+\beta}$. By Lemma 2.25 and because all root spaces are one-dimensional, we have

$$[e_\alpha, e_\beta] = e_\alpha \cdot e_\beta = (\mu - l + 1)ce_{\alpha+\beta},$$

where $\mu = \beta(h_\alpha) + 2q$, $l \in \widehat{\mu}$ and $c \in \mathbb{C}/\{0\}$. Obviously, $d := (\mu - l + 1)c \neq 0$ and hence we may write $[e_\alpha, \frac{1}{d}e_\beta] = e_{\alpha+\beta}$ in order to complete the proof. □

Remark 3.6. Sometimes we shall use the term "α-root string through β" also for the set of roots $\{\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + (q-1)\alpha, \beta + q\alpha\} \subset \Phi$, where $p, q \in \mathbb{N}_0$ are those from part (b) of the previous proposition. It should be always clear from the context which "root string" we just mean.

3.1.3 Inner Product of Roots

Non-degeneracy of the Killing form on H (cf. Corollary 3.9) allows us to define a real-valued inner product on the real linear span of the set of roots (cf. [7], Sec. 10.6).

Proposition 3.19. *If $\alpha, \beta \in \Phi$, then*

(a) $\kappa(t_\alpha, t_\alpha)\kappa(h_\alpha, h_\alpha) = 4$,

(b) $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$,

(c) $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.

Proof.

(a) $\kappa(t_\alpha, t_\alpha)\kappa(h_\alpha, h_\alpha) = \kappa(t_\alpha, t_\alpha)\kappa(\frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}, \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}) = \kappa(t_\alpha, t_\alpha)\frac{2}{\kappa(t_\alpha,t_\alpha)}\frac{2}{\kappa(t_\alpha,t_\alpha)}\kappa(t_\alpha, t_\alpha) = 4$.

(b) $\kappa(h_\alpha, h_\beta) = \mathrm{Tr}(\mathrm{ad}\, h_\alpha \circ \mathrm{ad}\, h_\beta) = \sum_{\gamma\in\Phi} \mathrm{Tr}((\mathrm{ad}\, h_\alpha \circ \mathrm{ad}\, h_\beta)|_{L_\gamma}) = \sum_{\gamma\in\Phi} \gamma(h_\alpha)\gamma(h_\beta)$, which is integral, according to Proposition 3.18 (a).

(c) $\kappa(t_\alpha, t_\beta) = \kappa(\frac{\kappa(t_\alpha,t_\alpha)}{2}h_\alpha, \frac{\kappa(t_\beta,t_\beta)}{2}h_\beta) = \frac{\kappa(t_\alpha,t_\alpha)}{2}\frac{\kappa(t_\beta,t_\beta)}{2}\kappa(h_\alpha, h_\beta) = \frac{4\kappa(h_\alpha,h_\beta)}{\kappa(h_\alpha,h_\alpha)\kappa(h_\beta,h_\beta)} \in \mathbb{Q}$. □

Definition 3.20. We define a form $(\ ,\ ) : H^* \times H^* \to \mathbb{C}$ for all $\theta, \varphi \in H^*$ as follows:

$$(\theta, \varphi) := \kappa(t_\theta, t_\varphi). \tag{3.8}$$

Remark 3.7. $(\ ,\ )$ is obviously bilinear, since κ is bilinear and the map $H^* \ni \varphi \mapsto t_\varphi \in H$ is linear (cf. Remark 3.3). In addition, it is symmetric and non-degenerate as well since $\kappa|_{H\times H}$ is. Further, according to Proposition 3.19, $(\alpha, \beta) \in \mathbb{Q}$ if $\alpha, \beta \in \Phi$.

Proposition 3.21. *Let $\alpha, \beta \in \Phi$. Then $\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.*

Proof. $\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \kappa(t_\beta, \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}) = \frac{2\kappa(t_\beta,t_\alpha)}{\kappa(t_\alpha,t_\alpha)} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. □

Remark 3.8. It results from Proposition 3.11 that there exists a basis for H^* consisting of roots.

Lemma 3.22. *Let $\mathcal{A} = (\alpha_1, \dots, \alpha_l)$ be a basis of H^* such that $\alpha_i \in \Phi$ for all $i \in \widehat{l}$. Consider $\beta = \sum_{i=1}^{l} c_i\alpha_i \in \Phi$. Then for all $i \in \widehat{l}$ it holds true that $c_i \in \mathbb{Q}$.*

Proof. For all $j \in \widehat{l}$ we have $(\beta, \alpha_j) = (\sum_{i=1}^{l} c_i\alpha_i, \alpha_j) = \sum_{i=1}^{l} c_i(\alpha_i, \alpha_j)$. In others words, we have a system of l non-homogeneous linear equations for l unknowns $c_1, \dots, c_l$. We may write this system in a compact form $\mathbb{A}\vec{c} = \vec{b}$, where $(\mathbb{A})_{ji} = (\mathbb{A})_{ij} = (\alpha_i, \alpha_j)$, $(\vec{c})_i = c_i$ and $(\vec{b})_j = (\beta, \alpha_j)$ for all $i, j \in \widehat{l}$. In fact, $\mathbb{A}$ is the matrix of a non-degenerate bilinear form (with respect to $\mathcal{A}$) and hence it is invertible (cf. [8], Chap. 10). Moreover, since all entries of $\mathbb{A}$ are rational, the same holds true for $\mathbb{A}^{-1}$. The solution of our system is $\vec{c} = \mathbb{A}^{-1}\vec{b}$ and, because all entries of $\vec{b}$ are rational as well, for all $i \in \widehat{l}$ we have $c_i = (\vec{c})_i \in \mathbb{Q}$, as desired. □

Remark 3.9. Let $(\alpha_1, \dots, \alpha_l)$ be a basis of H^* such that $\alpha_i \in \Phi$ for all $i \in \widehat{l}$. Let us denote

$$\mathsf{E} := \mathrm{Span}_{\mathbb{R}} \{\alpha_1, \dots, \alpha_l\}. \tag{3.9}$$

Thanks to the previous lemma, all roots are contained in E, hence E does not depend on the particular choice of the basis.

Proposition 3.23. *The restriction of the form $(\ ,\ )$ to E is an inner product on E.*

Proof. Symmetry and linearity have been already proved (cf. Remark 3.7). Hence it remains to verify that $(\ ,\ )$ maps into $\mathbb{R}$ and that it is positive-definite. Let $(\alpha_1, \dots, \alpha_l)$ be a basis for E consisting of roots. First, for any $\alpha = \sum_{i=1}^{l} a_i\alpha_i$ and $\beta = \sum_{j=1}^{l} b_j\alpha_j$ from E we have

$$(\alpha, \beta) = \left(\sum_{i=1}^{l} a_i\alpha_i, \sum_{j=1}^{l} b_j\alpha_j \right) = \sum_{i,j=1}^{l} a_i b_j (\alpha_i, \alpha_j) \in \mathbb{R}.$$

Second, for any $\alpha \in \mathsf{E}$ and the corresponding t_α we have

$$\begin{aligned}(\alpha, \alpha) &= \kappa(t_\alpha, t_\alpha) = \mathrm{Tr}(\mathrm{ad}\, t_\alpha \circ \mathrm{ad}\, t_\alpha) = \sum_{\beta\in\Phi} \mathrm{Tr}((\mathrm{ad}\, t_\alpha \circ \mathrm{ad}\, t_\alpha)|_{L_\beta}) \\ &= \sum_{\beta\in\Phi} \beta(t_\alpha)^2 = \sum_{\beta\in\Phi} \kappa(t_\beta, t_\alpha)^2 = \sum_{\beta\in\Phi} (\beta, \alpha)^2 \\ &\geq \sum_{j=1}^{l} (\alpha_j, \alpha)^2 \geq 0.\end{aligned}$$

In addition, if $(\alpha, \alpha) = 0$, then for all $j \in \widehat{l}$ we have $0 = (\alpha_j, \alpha) = \sum_{i=1}^{l} a_i(\alpha_j, \alpha_i)$, thus we have again a system of l homogeneous linear equations. But we have already seen the matrix of this system in the proof Lemma 3.22 and we know that it is non-singular. Hence the only solution of our system is the trivial one, consequently $(\alpha, \alpha) = 0$ implies that $\alpha = 0$. □

All in all, the pair $(\mathsf{E}; (\ ,\ ))$ is a finite-dimensional real inner-product vector space.

3.2 Root Systems

Throughout this section, E will always denote a finite-dimensional real vector space with an inner product $(\ ,\ )$. We will also use the norm induced by this inner product i.e. for any $\alpha \in \mathsf{E}$ we put $\|\alpha\| := \sqrt{(\alpha, \alpha)}$ and we say that α has *length* $\|\alpha\|$.

3.2.1 Definition of Root Systems

Now we are going to take the most important properties of the set of roots and axiomatize them. In this way we define the so-called "*root systems*" (cf. [7], Chap. 11). We also introduce some related terms necessary for the classification of root systems.

Remark 3.10. We assign to each non-zero $\alpha \in \mathsf{E}$ a linear map $s_\alpha : \mathsf{E} \to \mathsf{E}$ which is defined for all $\beta \in \mathsf{E}$ by

$$s_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha. \tag{3.10}$$

Obviously, s_α corresponds precisely to the reflection in the hyperplane normal to α: the "part" of β which is perpendicular to this hyperplane is $(\beta, \frac{\alpha}{\|\alpha\|})\frac{\alpha}{\|\alpha\|} = \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. For brevity, for all non-zero $\alpha \in \mathsf{E}$ and $\beta \in \mathsf{E}$ we denote $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. With this notation, we can rewrite (3.10) as

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha. \tag{3.11}$$

Proposition 3.24. *Suppose $\alpha \in E$, $\alpha \neq 0$. Then the reflection s_α preserves the inner product.*

Proof. For any $\beta, \gamma \in \mathsf{E}$ we have

$$(s_\alpha(\beta), s_\alpha(\gamma)) = \left(\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha, \gamma - \frac{2(\gamma, \alpha)}{(\alpha, \alpha)}\alpha \right)$$

$$
\begin{aligned}
&= (\beta,\gamma) - \frac{2(\gamma,\alpha)(\beta,\alpha)}{(\alpha,\alpha)} - \frac{2(\beta,\alpha)(\alpha,\gamma)}{(\alpha,\alpha)} + \frac{4(\beta,\alpha)(\gamma,\alpha)}{(\alpha,\alpha)} \\
&= (\beta,\gamma).
\end{aligned}
$$

□

Corollary 3.25. *Let $\alpha, \beta, \gamma \in E$ such that $\gamma \neq 0$. Then $\langle s_\alpha(\beta), s_\alpha(\gamma)\rangle = \langle \beta, \gamma\rangle$.*

Proof. According to the previous proposition, we can write

$$\langle s_\alpha(\beta), s_\alpha(\gamma)\rangle = \frac{2(s_\alpha(\beta), s_\alpha(\gamma))}{(s_\alpha(\gamma), s_\alpha(\gamma))} = \frac{2(\beta,\gamma)}{(\gamma,\gamma)} = \langle \beta, \gamma\rangle.$$

□

Definition 3.26. A subset Φ of E is called a *root system* if the four following axioms are satisfied:

(R1) Φ is finite, $\operatorname{Span}\Phi = \mathsf{E}$ and $0 \notin \Phi$;

(R2) for all $\alpha \in \Phi$ and $c \in \mathbb{R}$ it holds true that if $c\alpha \in \Phi$, then $c = \pm 1$;

(R3) if $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) \in \Phi$;

(R4) if $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha\rangle \in \mathbb{Z}$.

The elements of Φ are called *roots* and the dimension of E is called the *rank* of Φ.

We extend the notation used in this section: from now on, Φ will always denote a root system in E.

Remark 3.11. Take $\alpha \in \Phi$. Since $s_\alpha(\alpha) = \alpha - \frac{2(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = -\alpha$, (R3) implies that $-\alpha \in \Phi$.

Example 3.12. Naturally, one example of a root system is the set of roots of a semisimple Lie algebra introduced above. The respective real inner-product space was defined in Remark 3.9. However, we must verify that all four axioms in Definition 3.26 are satisfied. Indeed, (R1) holds by Lemma 3.6, by the definition of E and by the definition of Φ (we mean the definitions in the previous section), respectively, (R2) corresponds precisely to part (b) of Lemma 3.17, (R3) is fulfilled because of Proposition 3.21 and Proposition 3.18 (c) and (R4) follows from Proposition 3.21 and part (a) of Proposition 3.18.

We shall use the term "root systems" in this context later. Talking about "root system of a semisimple Lie algebra", we will always mean the set of roots of a given Lie algebra.

Proposition 3.27. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Then $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle \in \{0, 1, 2, 3\}$.*

Proof. Let us denote

$$A(\alpha,\beta) := \langle \alpha, \beta\rangle\langle \beta, \alpha\rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 4\left(\frac{\alpha}{\|\alpha\|}, \frac{\beta}{\|\beta\|}\right)\left(\frac{\beta}{\|\beta\|}, \frac{\alpha}{\|\alpha\|}\right) = 4\cos^2\theta(\alpha,\beta),$$

where $\theta(\alpha,\beta)$ is the angle between α and β. It is clear that $0 \leq A(\alpha,\beta) \leq 4$. Likewise, according to (R4), we have $A(\alpha,\beta) \in \mathbb{Z}$. Altogether, $A(\alpha,\beta) \in \{0,1,2,3,4\}$, but if $A(\alpha,\beta) = 4$, then $\cos\theta(\alpha,\beta) = \pm 1$ and hence $\alpha = \pm\beta$, which is impossible. □

Remark 3.13. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Keeping the notation from the previous proposition, we know that $A(\alpha,\beta) \in \{0,1,2,3\}$, moreover, by (R4), $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle \in \mathbb{Z}$ and hence there are just a few possible values of $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$. Further, as $A(\alpha, \beta) = 4\cos^2\theta(\alpha,\beta)$, $A(\alpha,\beta)$ determines $\cos\theta(\alpha,\beta)$ up to the sign but this sign agrees with the sign of $\langle \alpha, \beta \rangle$. Altogether, we are able to explore all possibilities of the relative position of two linear independent roots. Moreover, if α and β are not perpendicular, we can also determine the ratio of their norms:

$$\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \beta \rangle} = \frac{\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}{\frac{2(\alpha,\beta)}{(\beta,\beta)}} = \frac{(\beta,\beta)}{(\alpha,\alpha)} = \frac{\|\beta\|^2}{\|\alpha\|^2}. \tag{3.12}$$

The summary of these ideas is captured in Table 3.1. Without loss of generality, we assume that $\|\beta\| \geq \|\alpha\|$ which is equivalent, as seen from (3.12), to $|\langle \beta, \alpha \rangle| \geq |\langle \alpha, \beta \rangle|$.

$A(\alpha,\beta)$	$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\cos\theta(\alpha,\beta)$	$\theta(\alpha,\beta)$	$\|\beta\|^2 / \|\alpha\|^2$
0	0	0	0	$\pi/2$	undetermined
1	1	1	$1/2$	$\pi/3$	1
1	-1	-1	$-1/2$	$2\pi/3$	1
2	1	2	$\sqrt{2}/2$	$\pi/4$	2
2	-1	-2	$-\sqrt{2}/2$	$3\pi/4$	2
3	1	3	$\sqrt{3}/2$	$\pi/6$	3
3	-1	-3	$-\sqrt{3}/2$	$5\pi/6$	3

Table 3.1: All possible relative positions of linearly independent roots $\alpha, \beta \in \Phi$

Proposition 3.28. *Let $\alpha, \beta \in \Phi$. Let $\theta(\alpha,\beta)$ denote the angle between α and β.*

(a) If $0 < \theta(\alpha,\beta) < \frac{\pi}{2}$, then $\alpha - \beta \in \Phi$.

(b) If $\frac{\pi}{2} < \theta(\alpha,\beta) < \pi$, then $\alpha + \beta \in \Phi$.

Proof.

(a) If $\|\alpha\| \geq \|\beta\|$, then, according to Table 3.1, $\langle \beta, \alpha \rangle = 1$ and hence, by (R3), $\beta - \alpha = \beta - \langle \beta, \alpha \rangle \alpha = s_\alpha(\beta) \in \Phi$. Consequently, $\alpha - \beta = -(\beta - \alpha) \in \Phi$. In case $\|\beta\| \geq \|\alpha\|$ we obtain in the same way directly $\alpha - \beta \in \Phi$.

(b) Here we may assume, without loss of generality, that $\|\beta\| \geq \|\alpha\|$. Then $\langle \alpha, \beta \rangle = -1$ and hence $\alpha + \beta = \alpha - \langle \alpha, \beta \rangle \beta = s_\beta(\alpha) \in \Phi$. □

Definition 3.29. A subset Δ of Φ is called a *base* for Φ if the two following axioms are satisfied:

(B1) Δ is a basis for E;

(B2) for each β there exist integers $(k_\alpha)_{\alpha\in\Delta}$ such that $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ and either $k_\alpha \geq 0$ for all $\alpha \in \Delta$ or $k_\alpha \leq 0$ for all $\alpha \in \Delta$.

The roots in Δ are said to be *simple*.

Proposition 3.30. *Let Δ be a base for Φ. For any distinct $\alpha, \beta \in \Delta$ it holds true that the angle between α and β is at least $\pi/2$.*

Proof. If the angle is zero, then α and β are not linearly independent, contradicting (B1). In case the angle is strictly acute, Proposition 3.28 implies that the *root* $\alpha - \beta$ has, with respect to the basis Δ, simultaneously one positive and one negative coefficient and hence we reach a contradiction (with (B2) now) again. □

Definition 3.31. Suppose that there exists a base for Φ and denote it by Δ. For each $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ we define the *height* of β as follows:

$$\operatorname{ht}\beta := \sum_{\alpha\in\Delta} k_\alpha. \tag{3.13}$$

The root $\beta \in \Phi$ is said to be *positive* if $\operatorname{ht}\beta > 0$ or *negative* if $\operatorname{ht}\beta < 0$. The set of all positive roots in Φ is denoted by Φ^+ and the set of all negative ones by Φ^-.

Remark 3.14. Suppose that we have a base Δ for Φ. (B1) guarantees that for each $\beta \in \Phi$ the integers $(k_\alpha)_{\alpha\in\Delta}$ are unique, hence the height is well-defined. Moreover, from (B2) it is clear that each root is either positive or negative. Indeed, the only remaining case ($k_\alpha = 0$ for all $\alpha \in \Delta$) corresponds to the zero vector which is not a root. Equivalently, $\Phi = \Phi^+ \cup \Phi^-$. In addition, note that $\Delta \subset \Phi^+$.

The question that still remains is whether a base has to exist for each root system. It can be shown (cf. [9], Theorem 10.1) that each root system has a base indeed. This fact permits us to extend our notation and denote a base for Φ by Δ for the rest of this section.

In the following series of statements we introduce a way how to discover whole set of roots from its base (cf. [7], Sec. 11.3). We shall see that each root can be obtained as a product of finitely many reflections of a simple root. However, the procedure using in computation is different. The algorithm is presented in the next chapter.

Definition 3.32. The subgroup of $GL(\mathsf{E})$ generated by the reflections $(s_\alpha)_{\alpha\in\Phi}$ is called the *Weyl group* of Φ and it is denoted by $\mathcal{W}(\Phi)$.

Remark 3.15. For all $\alpha, \beta \in \Phi$ we have

$$s_\alpha^2(\beta) = s_\alpha(\beta - \langle\beta,\alpha\rangle\alpha) = \beta - \langle\beta,\alpha\rangle\alpha + \langle\beta,\alpha\rangle\alpha = \beta,$$

consequently for all $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n, \beta \in \Phi$ it holds true that

$$\begin{aligned}((s_{\alpha_1} \circ \cdots \circ s_{\alpha_n}) \circ (s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}))(\beta) &= (s_{\alpha_1} \circ \cdots \circ s_{\alpha_n} \circ s_{\alpha_n} \circ \cdots \circ s_{\alpha_1})(\beta)\\ &= (s_{\alpha_1} \circ \cdots \circ s_{\alpha_{n-1}} \circ s_{\alpha_{n-1}} \circ \cdots \circ s_{\alpha_1})(\beta)\\ &\;\;\vdots\\ &= (s_{\alpha_1} \circ s_{\alpha_1})(\beta) = \beta\end{aligned}$$

and hence $(s_{\alpha_1} \circ \cdots \circ s_{\alpha_n})^{-1} = (s_{\alpha_n} \circ \cdots \circ s_{\alpha_1}) \in \mathcal{W}(\Phi)$ and $\mathcal{W}(\Phi)$ is a group indeed. From now on, we will denote the Weyl group $\mathcal{W}(\Phi)$ of Φ simply by $\mathcal{W}$.

Proposition 3.33. *$\mathcal{W}$ is finite.*

Proof. Take any $\alpha \in \Phi$. By (R3), $s_\alpha(\Phi) \subset \Phi$. Likewise, we have seen that s_α was invertible and hence injective. In other words, s_α permutes Φ. But, as Φ spans E, each $s_\alpha \in \mathcal{W}$ is uniquely determined by images $(s_\alpha(\beta))_{\beta\in\Phi}$. Consequently, the number of different projections s_α cannot be larger that the number of all permutations of Φ, which is finite since the same holds true for Φ, by axiom (R1). □

Proposition 3.34. *Let $\alpha \in \Delta$. Then s_α permutes the set $\Phi^+ / \{\alpha\}$.*

Proof. We have seen in the proof of Proposition 3.33 that s_α permuted Φ, therefore it is enough to show that s_α maps $\Phi^+ / \{\alpha\}$ into itself. Suppose that $\beta \in \Phi^+ / \{\alpha\}$, then there are non-negative integers $(k_\gamma)_{\gamma\in\Delta}$ such that $\beta = \sum_{\gamma\in\Delta} k_\gamma \gamma$. Similarly, there are integers $(l_\gamma)_{\gamma\in\Delta}$ (either all non-positive or all non-negative) such that $s_\alpha(\beta) = \sum_{\gamma\in\Delta} l_\gamma \gamma$. Since $\beta \in \Phi^+$, some $\gamma_0 \in \Delta$ such that $\gamma_0 \neq \alpha$ and $k_{\gamma_0} > 0$ have to exist to satisfy $\beta \neq \alpha$. But from $s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$ it is obvious that $l_{\gamma_0} = k_{\gamma_0} > 0$ and hence all $(l_\gamma)_{\gamma\in\Delta}$ are non-negative and $s_\alpha(\beta) \in \Phi^+$. □

Lemma 3.35. *Let $\mathcal{W}_0$ denote the subgroup of $\mathcal{W}$ generated by $(s_\alpha)_{\alpha\in\Delta}$. Then for each $\beta \in \Phi$ there are $s \in \mathcal{W}_0$ and $\alpha \in \Delta$ such that $\beta = s(\alpha)$.*

Proof. First, suppose that $\beta \in \Phi^+$. There are non-negative integers $(k_\gamma)_{\gamma\in\Delta}$ such that $\beta = \sum_{\gamma\in\Delta} k_\gamma \gamma$. For proof, we use induction on height of β. If $\operatorname{ht}\beta = 1$, then $\beta \in \Delta$, hence it suffices to put $\alpha := \beta$ and $s := \mathbb{1}$. For the inductive step, suppose that $\operatorname{ht}\beta > 1$. We claim that there is $\gamma_0 \in \Delta$ such that $(\beta, \gamma_0) > 0$. Indeed. If not, then, because $\beta \neq 0$,

$$0 < \|\beta\|^2 = (\beta, \beta) = (\beta, \sum_{\gamma\in\Delta} k_\gamma \gamma) = \sum_{\gamma\in\Delta} k_\gamma (\beta, \gamma) \leq 0,$$

a contradiction. According to Proposition 3.34, for this γ_0 we have

$$0 < \operatorname{ht} s_{\gamma_0}(\beta) = \operatorname{ht}\beta - \frac{2(\beta, \gamma_0)}{(\gamma_0, \gamma_0)} \operatorname{ht}\gamma_0 < \operatorname{ht}\beta$$

and thus we may apply the inductive hypothesis to $s_{\gamma_0}(\beta)$ to obtain existence of $\tilde{s} \in \mathcal{W}_0$ and $\tilde{\alpha} \in \Delta$ such that $s_{\gamma_0}(\beta) = \tilde{s}(\tilde{\alpha})$. Now, because $(s_{\gamma_0})^{-1} = s_{\gamma_0}$, it suffices to put $\alpha := \tilde{\alpha}$ and $s := s_{\gamma_0} \circ \tilde{s}$ in order to get the statement.

Second, if $\beta \in \Phi^-$, the previous part holds for $-\beta \in \Phi^+$; there exist $\tilde{s} \in \mathcal{W}_0$ and $\tilde{\alpha} \in \Delta$ such that $-\beta = \tilde{s}(\tilde{\alpha})$. Obviously, since $s_{\tilde{\alpha}}(\tilde{\alpha}) = -\tilde{\alpha}$, it suffices to put $\alpha := \tilde{\alpha}$ and $s := \tilde{s} \circ s_{\tilde{\alpha}}$ to finish the proof completely. □

Remark 3.16. The reflections $(s_\alpha)_{\alpha\in\Delta}$ are said to be *simple*.

Proposition 3.36. *Keeping the notation from the previous proposition, we have $\mathcal{W}_0 = \mathcal{W}$.*

Proof. It is enough to show that the generating elements $(s_\alpha)_{\alpha\in\Phi}$ of $\mathcal{W}$ all lie in $\mathcal{W}_0$. Then each element of $\mathcal{W}$ can be clearly "build up" of the generators $(s_\beta)_{\beta\in\Delta}$ of $\mathcal{W}_0$. Let $\alpha \in \Phi$. According to Lemma 3.35, there exist $\gamma \in \Delta$ and $s \in \mathcal{W}_0$ such that $\alpha = s(\gamma)$. We claim that $s_\alpha = s \circ s_\gamma \circ s^{-1}$ and hence that $s_\alpha \in \mathcal{W}_0$. Indeed. For any $\delta \in \Delta$ we have

$$\begin{aligned}(s \circ s_\gamma)(\delta) &= s(\delta - \langle \delta, \gamma \rangle \gamma) = s(\delta) - \langle \delta, \gamma \rangle s(\gamma) = s(\delta) - \langle s(\delta), s(\gamma) \rangle s(\gamma) \\ &= s(\delta) - \langle s(\delta), \alpha \rangle \alpha = s_\alpha(s(\delta)) = (s_\alpha \circ s)(\delta)\end{aligned}$$

and $s \circ s_\gamma = s_\alpha \circ s$ or equivalently $s \circ s_\gamma \circ s^{-1} = s_\alpha$ since Δ spans E. □

At the very end of this subsection we state an important theorem telling us that any two bases for one root system in E have in fact the same geometric proportions; at most, they can be mutually reflected in some hyperplane in E. See [7], Appendix D, for the proof.

Theorem 3.37. *If $s \in \mathcal{W}$, then the set $\{s(\alpha) \mid \alpha \in \Delta\}$ is a base for Φ. Contrariwise, if Δ' is another base for Φ, then there exists $s' \in \mathcal{W}$ such that $\Delta' = \{s'(\alpha) \mid \alpha \in \Delta\}$.*

3.2.2 Cartan Matrices and Dynkin Diagrams

In this subsection we assign two types of invariants to each root system, namely a matrix and a graph (cf. [9], Sec. 11). This step proves to be very useful for classification of root systems since those objects (more specifically the graphs) can be easily classified. Recall that we still keep the meaning of E, Φ, Δ, $\mathcal{W}$ and $\mathcal{W}_0$.

Definition 3.38. Suppose that $(\alpha_1, \ldots, \alpha_l)$ is a base for Φ. The *Cartan matrix* of Φ is defined to be the $l \times l$ matrix with i, j-th entry $\langle \alpha_i, \alpha_j \rangle$. The entries of the Cartan matrix are called *Cartan integers*.

Remark 3.17. By axiom (R4), Cartan integers are integers indeed and moreover, it is obvious directly from the definition of $\langle \ , \ \rangle$ that all diagonal entries of the Cartan matrix are equal to 2.

Remark 3.18. It results immediately from Theorem 3.37 and Corollary 3.25 that the Cartan matrix is defined unambiguously up to the order of chosen simple roots and it does not depend on the particular choice of base. In other words, if A and B are two Cartan matrices for one root system Φ, then there exists a permutation π of $\widehat{l}$ such that for all $i, j \in \widehat{l}$ we have $A_{i,j} = B_{\pi(i),\pi(j)}$. In particular, the set of Cartan integers is unique for each root system.

Now we introduce an even more illustrative invariant of a root system.

Definition 3.39. Let $(\alpha_1, \ldots, \alpha_l)$ be a base for Φ. The *Dynkin diagram* of Φ is a graph Γ with the following properties:

(a) Γ has l vertices labeled by $(\alpha_1, \ldots, \alpha_l)$;

(b) for all $i, k \in \widehat{l}$ such that $i \neq k$ the vertices α_i and α_k are connected by $d_{ik} := \langle \alpha_i, \alpha_k \rangle \langle \alpha_k, \alpha_i \rangle$ edges;

(c) if $d_{ik} > 1$, then an arrow pointing from the longer root of $\{\alpha_i, \alpha_k\}$ to the shorter one is putted between the respective roots.

Remark 3.19. Keep the notation from the previous definition. According to Table 3.1, for all $i, k \in \widehat{l}$ the only possible values of d_{ik} are 0, 1, 2 or 3. Moreover, if we take into consideration that the angle between two simple roots is always obtuse (cf. Proposition 3.30), in case that $d_{ik} \neq 0$ it holds true that at least one of the numbers $\langle \alpha_i, \alpha_k \rangle$ and $\langle \alpha_k, \alpha_i \rangle$ equals -1. Without loss of generality, we may assume that this is satisfied for $\langle \alpha_i, \alpha_k \rangle$. Then we can write

$$\|\alpha_i\|^2 = (\alpha_i, \alpha_i) = \frac{2(\alpha_i, \alpha_k)}{\langle \alpha_i, \alpha_k \rangle} = d_{ik} \frac{2(\alpha_k, \alpha_i)}{\langle \alpha_k, \alpha_i \rangle} = d_{ik}(\alpha_k, \alpha_k) = d_{ik} \|\alpha_k\|^2 .$$

Remark 3.20. According to the same arguments used in Remark 3.18, one can see that the Dynkin diagram of Φ is unique up to the labelling of vertices. Further, notice that the knowledge of the Dynkin diagram is equivalent to the knowledge of the Cartan matrix. The determination of the diagram from the matrix is obvious. For the other direction, the remaining information (diagonal entries and the sign of non-diagonal entries) is provided by Remark 3.17 and Proposition 3.30, respectively.

Definition 3.40. Let E and E' be real inner-product spaces and let Φ and Φ' be root systems in E and E', respectively. Φ and Φ' are *isomorphic* if there exists a vector space isomorphism $\varphi \colon \mathsf{E} \to \mathsf{E}'$ fulfilling the two following conditions:

(a) $\varphi(\Phi) = \Phi'$;

(b) for any $\alpha, \beta \in \Phi$ it holds true that $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$.

Theorem 3.41. *Let E and E' be real inner-product spaces and let Φ and Φ' be root systems in E and E', respectively. Φ and Φ' are isomorphic if and only if their Dynkin diagrams are the same.*

Proof. The "only if" direction is obvious. For the "if" direction choose a base $\Delta = (\alpha_1, \ldots, \alpha_l)$ for Φ and a base $\Delta' = (\alpha'_1, \ldots, \alpha'_l)$ for Φ' such that for all $i, j \in \widehat{l}$ one has

$\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$. Since Δ and Δ' are bases for E and E', respectively, the map $\varphi\colon \mathsf{E} \to \mathsf{E}'$ defined for any $\alpha = \sum_{i=1}^n k_i \alpha_i \in \mathsf{E}$ by

$$\varphi(\alpha) := \sum_{i=1}^n k_i \alpha'_i$$

is a well-defined vector space isomorphism. Now, for any $i, k \in \widehat{l}$ we have

$$\begin{aligned} s_{\varphi(\alpha_i)}(\varphi(\alpha_k)) &= s_{\alpha'_i}(\alpha'_k) = \alpha'_k - \langle \alpha'_k, \alpha'_i \rangle \alpha'_i = \varphi(\alpha_k) - \langle \alpha_k, \alpha_i \rangle \varphi(\alpha_i) = \varphi(\alpha_k - \langle \alpha_k, \alpha_i \rangle \alpha_i) \\ &= \varphi(s_{\alpha_i}(\alpha_k)). \end{aligned}$$

In other words, for all $i \in \widehat{l}$ it is true that $s_{\varphi(\alpha_i)} \circ \varphi = \varphi \circ s_{\alpha_i}$ or equivalently $s_{\alpha'_i} = \varphi \circ s_{\alpha_i} \circ \varphi^{-1}$. As $(s_{\alpha_i})_{i \in \widehat{l}}$ and $(s_{\alpha'_i})_{i \in \widehat{l}}$ generate the corresponding Weyl groups $\mathcal{W}$ and $\mathcal{W}'$ of Φ and Φ', respectively, it is clear that the map $\mathcal{W} \ni s \mapsto \varphi \circ s \circ \varphi^{-1} \in \mathcal{W}'$ is a group isomorphism sending s_{α_i} to $s_{\alpha'_i}$, $i \in \widehat{l}$, additionally. Further, take an arbitrary $\beta \in \Phi$. By Lemma 3.35, there exist $s \in \mathcal{W}_0$ and $i_0 \in \widehat{l}$ such that $\beta = s(\alpha_{i_0})$. But this fact implies that

$$\varphi(\beta) = \varphi(s(\alpha_{i_0})) = (\varphi \circ s \circ \varphi^{-1})(\varphi(\alpha_{i_0})) = (\varphi \circ s \circ \varphi^{-1})(\alpha'_{i_0}) \in \Phi'.$$

In the same way one can show that for any $\beta' \in \Phi'$ it holds true $\varphi^{-1}(\beta') \in \Phi$. Altogether, $\varphi(\Phi) = \Phi'$. It remains to verify that the condition (b) in Definition 3.40 is satisfied as well. For any $\beta, \gamma \in \Phi$ we have

$$(\varphi \circ s_\gamma \circ \varphi^{-1})(\varphi(\beta)) = \varphi(s_\gamma(\beta)) = \varphi(\beta - \langle \beta, \gamma \rangle \gamma) = \varphi(\beta) - \langle \beta, \gamma \rangle \varphi(\gamma)$$

and hence, if β is perpendicular to γ, then $(\varphi \circ s_\gamma \circ \varphi^{-1})(\varphi(\beta)) = \varphi(\beta)$ and moreover in case $\beta = \gamma$ we have

$$(\varphi \circ s_\gamma \circ \varphi^{-1})(\varphi(\gamma)) = \varphi(\gamma) - \langle \gamma, \gamma \rangle \varphi(\gamma) = -\varphi(\gamma).$$

In other words, the map $(\varphi \circ s_\gamma \circ \varphi^{-1})\colon \Phi' \to \Phi'$ corresponds precisely to the reflection in the hyperplane normal to $\varphi(\gamma)$. It follows

$$\begin{aligned} \langle \varphi(\beta), \varphi(\gamma) \rangle \varphi(\gamma) &= \varphi(\beta) - s_{\varphi(\gamma)}(\varphi(\beta)) = \varphi(\beta) - (\varphi \circ s_\gamma \circ \varphi^{-1})(\varphi(\beta)) \\ &= \varphi(\beta) - \varphi(s_\gamma(\beta)) = \varphi(\beta - s_\gamma(\beta)) = \varphi(\langle \beta, \gamma \rangle \gamma) \\ &= \langle \beta, \gamma \rangle \varphi(\gamma) \end{aligned}$$

and finally, since $\Phi' \ni \varphi(\gamma) \neq 0$, we obtain $\langle \varphi(\beta), \varphi(\gamma) \rangle = \langle \beta, \gamma \rangle$. □

Definition 3.42. A root system Φ is said to be *irreducible* if there do not exist any non-empty subsets Φ_1 and Φ_2 of Φ such that $\Phi = \Phi_1 \cup \Phi_2$ and for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ it holds true that $(\alpha, \beta) = 0$.

Lemma 3.43. *Suppose that Φ is a root system in a real inner-product space E. Then there exist pairwise disjoint subsets $\Phi_1, \ldots, \Phi_n$ of Φ such that $\Phi = \Phi_1 \cup \cdots \cup \Phi_n$ and for all $i \in \widehat{n}$ it holds true that Φ_i is an irreducible root system in $E_i := \operatorname{Span} \Phi_i$. In addition, $E = E_1 \oplus \cdots \oplus E_n$.*

Proof. We define an equivalence relation $\sim$ on Φ: for any $\alpha, \beta \in \Phi$ we say that $\alpha \sim \beta$ if there exist $\gamma_1, \ldots, \gamma_s \in \Phi$ such that $\gamma_1 = \alpha$, $\gamma_s = \beta$ and for all $i \in \widehat{s-1}$ it holds true that $(\gamma_i, \gamma_{i+1}) \neq 0$. We claim that the classes of equivalence are the required subsets $\Phi_1, \ldots, \Phi_n$. First, it is clear that $\Phi = \Phi_1 \cup \cdots \cup \Phi_n$ and that $\Phi_i \cap \Phi_j = \emptyset$ for all $i, j \in \widehat{n}$.

Second, we has to convince ourselves that the four axioms are satisfied for each Φ_k. (R1) and (R4) hold trivially, (R2) holds because $(\alpha, -\alpha) = -\|\alpha\|^2 \neq 0$. For (R3), take any $\alpha, \beta \in \Phi_k$. There exist $\gamma_1, \dots, \gamma_s \in \Phi$ such that $\gamma_1 = \alpha$, $\gamma_s = \beta$ and for all $i \in \widehat{s-1}$ it holds true that $(\gamma_i, \gamma_{i+1}) \neq 0$. For each $i \in \widehat{s-1}$ we put $\hat{\gamma}_i := s_\alpha(\gamma_i)$. Obviously, $\hat{\gamma}_i \in \Phi$ for all $i \in \widehat{s}$ and $(\hat{\gamma}_i, \hat{\gamma}_{i+1}) = (s_\alpha(\gamma_i), s_\alpha(\gamma_{i+1})) = (\gamma_i, \gamma_{i+1}) \neq 0$. Moreover $\hat{\gamma}_1 = s_\alpha(\alpha) = -\alpha$ and $\hat{\gamma}_s = s_\alpha(\beta)$. Hence $s_\alpha(\beta) \sim -\alpha \in \Phi_k$. Third, all Φ_k are obviously irreducible.

Clearly, $\mathsf{E} = \mathsf{E}_1 + \dots + \mathsf{E}_n$. To prove the directness, suppose that $0 = \sum_{i=1}^n v_i$, where $v_i \in \mathsf{E}_i$ for each $i \in \widehat{n}$. For all $j \in \widehat{n}$ we have

$$0 = (\sum_{i=1}^n v_i, v_j) = \sum_{i=1}^n (v_i, v_j) = (v_j, v_j) = \|v_j\|^2$$

and hence $v_j = 0$, as desired. □

Remark 3.21. It is easily seen from the proof of the previous lemma that both decompositions considered in that lemma are unique.

Remark 3.22. Obviously, a root system is irreducible precisely when the associated Dynkin diagram is connected (if we assume that two edges may join each other in a vertex entirely). In addition, it is clear that the Dynkin diagram of a general root system Φ consists of mutually unconnected subdiagrams corresponding to the particular irreducible root "subsystems" from the previous lemma.

3.2.3 Classification of Root Systems

According to Theorem 3.41, each root system is determined (up to isomorphism) by its Dynkin diagram. Thus, to classify the root systems it suffices to find all Dynkin diagrams that may occur and this is exactly the purpose of the final part of this section (cf. [7], Chap. 13). By Remark 3.22, it is enough to classify only the connected Dynkin diagrams; as a general root system falls into several irreducible subsystems, its respective Dynkin diagram divides into the same number of mutually disjoint connected Dynkin diagrams.

Definition 3.44. A subset $A = \{\varepsilon_1, \dots, \varepsilon_n\}$ of E is said to be *admissible* if it is linearly independent and the following conditions are fulfilled for all $i, j \in \widehat{n}$ such that $i \neq j$:

(a) $\|\varepsilon_i\| = 1$;

(b) $(\varepsilon_i, \varepsilon_j) \leq 0$;

(c) $4(\varepsilon_i, \varepsilon_j)^2 \in \{0, 1, 2, 3\}$.

Definition 3.45. We assign the following graph Γ_A to any admissible set $A = \{\varepsilon_1, \dots, \varepsilon_n\}$ in E: Γ_A has n vertices labeled by $\varepsilon_1, \dots, \varepsilon_n$ and for all $i, j \in \widehat{n}$ such that $i \neq j$ the vertices ε_i and ε_j are connected by $d_{ij} := 4(\varepsilon_i, \varepsilon_j)^2$ edges.

Remark 3.23. Obviously, any subset $A' \subset A$ is an admissible set in E again. The graph $\Gamma_{A'}$ is obtained from Γ_A by omitting the vertices corresponding to vectors from A/A' as well as the edges incident to these vertices.

For brevity we will always consider an admissible set $A = \{\varepsilon_1, \dots, \varepsilon_n\}$ and the associate graph $\Gamma_A =: \Gamma$ for the rest of this section.

Lemma 3.46. *The number N of pairs of vertices in Γ connected by at least one edge is strictly less then n.*

Proof. Set $\varepsilon := \sum_{i=1}^{n} \varepsilon_i$ and $d_{ij} := 4(\varepsilon_i, \varepsilon_j)^2$. Linear independence of A insures that $\varepsilon \neq 0$. Thus

$$0 < (\varepsilon, \varepsilon) = \sum_{i,j=1}^{n} (\varepsilon_i, \varepsilon_j) = n + 2 \sum_{i<j} (\varepsilon_i, \varepsilon_j) = n - \sum_{i<j} \sqrt{d_{ij}} = n - \sum_{\substack{i<j \\ d_{ij} \geq 1}} \sqrt{d_{ij}} \leq n - N.$$

$\square$

Corollary 3.47. Γ *does not contain any cycles.*

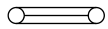
Proof. If it does, then the subset of A corresponding to a cycle is an admissible set. But the number of vertices of the respective graph equals to the number of connected pairs of vertices, a contradiction. $\square$

Lemma 3.48. *At most three edges can be incident to each vertex of* Γ*.*

Proof. Take an arbitrary $\varepsilon \in A$ and let $\{\eta_1, \dots, \eta_k\} \subset A$ be the set of all vertices joined to ε. In view of Corollary 3.47, $(\eta_i, \eta_j) = \delta_{ij}$ for all $i, j \in \widehat{k}$. Since $\varepsilon, \eta_1, \dots, \eta_k$ are linearly independent, we may find $\eta_0 \in A$ such that $(\eta_0, \eta_1, \dots, \eta_k)$ is an orthonormal basis for Span $\{\varepsilon, \eta_1, \dots, \eta_k\}$. It is clear that $(\varepsilon, \eta_0) \neq$ necessarily. Then we can write $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i) \eta_i$ and $1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2$. But as $(\varepsilon, \eta_0)^2 > 0$, we obtain $\sum_{i=1}^{k} (\varepsilon, \eta_i)^2 < 1$ or equivalently

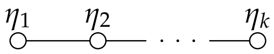
$$4 > \sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 = \sum_{i=1}^{k} d_i,$$

where d_i denotes the number of edges joining the vertices ε and η_i. $\square$

Corollary 3.49. *If* Γ *is connected and it has a triple edge, then* Γ *is of the shape*  .

Proof. It follows immediately from the previous lemma. $\square$

Lemma 3.50. *Suppose that* Γ_A *has a subgraph*



Then the set $A' := (A / \{\eta_1, \dots, \eta_k\}) \cup \{\eta\}$, *where* $\eta := \sum_{i=1}^{k} \eta_i$, *is admissible.*

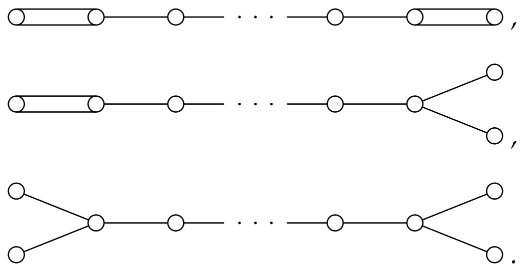
Proof. First, linear independence of A' is obvious. Second, according to the character of the graph, we can write

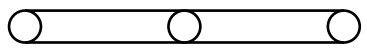
$$\|\eta\|^2 = (\eta, \eta) = \sum_{i,j=1}^{k} (\eta_i, \eta_j) = \sum_{i=1}^{k} (\eta_i, \eta_i) + \sum_{\substack{i,j=1 \\ |i-j|=1}}^{k} (\eta_i, \eta_j) = k + 2(k-1)(-\frac{1}{2}) = 1.$$

Third, for all $\varepsilon \in A' / \{\eta\} = A / \{\eta_1, \dots, \eta_k\}$ we have $(\varepsilon, \eta) = \sum_{i=1}^{k} (\varepsilon, \eta_i) \leq 0$, since the same inequality holds for each single summand. Finally, for each $\varepsilon \in A' / \{\eta\}$ there is at most one $j \in \widehat{k}$ such that $(\varepsilon, \eta_j) \neq 0$, otherwise Γ_A would contain a cycle. Moreover, if such j exists, then $4(\varepsilon, \eta_j) \in \{1, 2, 3\}$. Altogether, for all $\varepsilon \in A' / \{\eta\}$ we have $(\varepsilon, \eta) = \sum_{i=1}^{k} (\varepsilon, \eta_i) \in \{0, 1, 2, 3\}$. $\square$

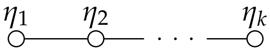
Remark 3.24. It is easily seen from the previous proof that we obtain the graph $\Gamma_{A'}$ by replacing the subgraph of Γ_A corresponding to $\{\eta_1, \dots, \eta_k\}$ by one single vertex η and by joining all the vertices in Γ_A other than $\{\eta_1, \dots, \eta_k\}$ connected in the original graph Γ_A to one of $\{\eta_1, \dots, \eta_k\}$ to this new vertex. For this reason, this lemma is sometimes called "*Shrinking Lemma*" (cf. [7]).

Corollary 3.51. Γ *does not contain any subgraph of the three following shapes:*



Proof. Suppose that Γ contains a subgraph of the first discussed shape. Then, by Lemma 3.50, we may consider some new graph (corresponding to an admissible set) containing a subgraph  , contradicting Lemma 3.48. For the two other shapes, the procedure is identical. □

Lemma 3.52. *Suppose that Γ_A has a subgraph*



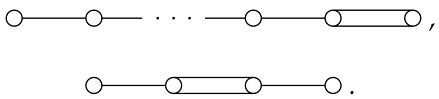
and set $\eta := \sum_{i=1}^{k} i\eta_i$. Then $(\eta, \eta) = \frac{k(k+1)}{2}$.

Proof. Considering how the given subgraph looks like, we may compute

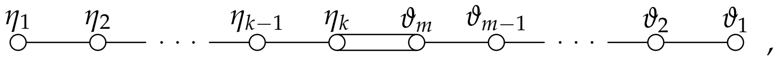
$$\begin{aligned}
(\eta, \eta) &= \sum_{i,j=1}^{k} ij(\eta_i, \eta_j) = \sum_{i=1}^{k} i^2(\eta_i, \eta_i) + \sum_{\substack{i,j=1 \\ |i-j|=1}}^{k} ij(\eta_i, \eta_j) \\
&= \sum_{i=1}^{k} i^2 + \sum_{i=1}^{k-1} i(i+1)(\eta_i, \eta_{i+1}) + \sum_{i=2}^{k} i(i-1)(\eta_i, \eta_{i-1}) \\
&= \sum_{i=1}^{k} i^2 + 2\left(-\frac{1}{2}\sum_{i=1}^{k-1} i(i+1)\right) = k^2 - \sum_{i=1}^{k-1} i = k^2 - \frac{(k-1)k}{2} \\
&= \frac{k(k+1)}{2}.
\end{aligned}$$

□

Lemma 3.53. *If Γ contains a double edge, then it has one of the two following shapes:*



Proof. By Corollary 3.51, Γ may not have more than one double edges as well as both a double edge and a "branch" point, thus it has to be of the form



where $k, m \in \mathbb{N}$. According to the previous lemma, for $\eta := \sum_{i=1}^{k} i\eta_i$ and $\vartheta := \sum_{i=1}^{m} i\vartheta_i$ we have $\|\eta\|^2 = \frac{k(k+1)}{2}$ and $\|\vartheta\|^2 = \frac{m(m+1)}{2}$, respectively. Moreover, according to the shape of our graph

$$(\eta, \vartheta)^2 = \left(\sum_{i=1}^{k}\sum_{j=1}^{m} ij(\eta_i, \vartheta_j)\right)^2 = (km(\eta_k, \vartheta_m))^2 = \frac{k^2m^2}{2}.$$

Since η and ϑ are clearly linearly independent, the Schwarz-Cauchy inequality (cf. [2], p. 13) implies that

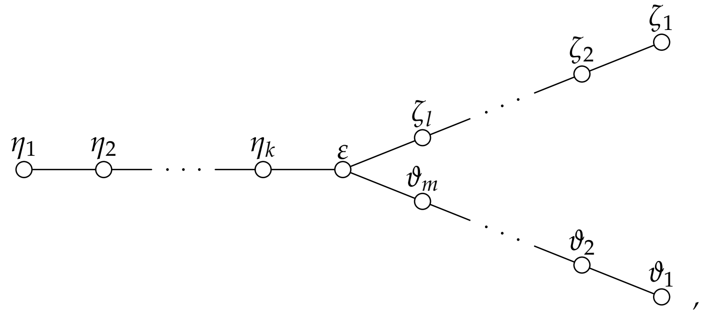
$$\frac{k^2m^2}{2} = (\eta,\vartheta)^2 < \|\eta\|^2 \|\vartheta\|^2 = \frac{k(k+1)m(m+1)}{4},$$

hence $2km < km + k + m + 1$ or equivalently

$$(k-1)(m-1) < 2.$$

Now it is easily seen that either at least one of the positive integers k, m equals to 1 or $k = m = 2$. □

Lemma 3.54. *If Γ has a vertex joined to three different other vertices, then it is of the form*



whereas for the triplet of positive integers (k,l,m) it holds true

$$(k,l,m) \in \{(1,2,2),(1,2,3),(1,2,4)\} \cup \{(1,1,p) \mid p \in \mathbb{N}\}.$$

Proof. First, by Corollary 3.51, Γ may not have more than one "branch" point as well as both a double edge and a "branch", hence it must be of the form as claimed. Second, we uncover all possible combinations of k,l,m. Let us assume, without loss of generality, that $k \le l \le m$. Again we set $\eta := \sum_{i=1}^{k} i\eta_i$, $\zeta := \sum_{i=1}^{l} i\zeta_i$ and $\vartheta := \sum_{i=1}^{m} i\vartheta_i$. There exists $\tilde{\varepsilon} \in \mathsf{E}$ such that the set $\{\tilde{\varepsilon},\tilde{\eta},\tilde{\zeta},\tilde{\vartheta}\}$, where $\tilde{\eta} = \eta/\|\eta\|$, $\tilde{\zeta} = \zeta/\|\zeta\|$ and $\tilde{\vartheta} = \vartheta/\|\vartheta\|$, is an orthonormal basis for Span$\{\varepsilon,\eta,\zeta,\vartheta\}$. In addition, $(\varepsilon,\tilde{\varepsilon}) \neq 0$. Then we may decompose ε as follows:

$$\varepsilon = (\varepsilon,\tilde{\eta})\tilde{\eta} + (\varepsilon,\tilde{\zeta})\tilde{\zeta} + (\varepsilon,\tilde{\vartheta})\tilde{\vartheta} + (\varepsilon,\tilde{\varepsilon})\tilde{\varepsilon}.$$

Consequently, according to Lemma 3.52 and to the considering graph,

$$\begin{aligned}
1 = \|\varepsilon\|^2 &= (\varepsilon,\tilde{\eta})^2 + (\varepsilon,\tilde{\zeta})^2 + (\varepsilon,\tilde{\vartheta})^2 + (\varepsilon,\tilde{\varepsilon})^2 > (\varepsilon,\tilde{\eta})^2 + (\varepsilon,\tilde{\zeta})^2 + (\varepsilon,\tilde{\vartheta})^2 \\
&= \frac{(\varepsilon,\eta)^2}{\|\eta\|^2} + \frac{(\varepsilon,\zeta)^2}{\|\zeta\|^2} + \frac{(\varepsilon,\vartheta)^2}{\|\vartheta\|^2} = \frac{(\varepsilon,k\eta_k)^2}{\|\eta\|^2} + \frac{(\varepsilon,l\zeta_l)^2}{\|\zeta\|^2} + \frac{(\varepsilon,m\vartheta_m)^2}{\|\vartheta\|^2} \\
&= \frac{k}{2(k+1)} + \frac{l}{2(l+1)} + \frac{m}{2(m+1)}.
\end{aligned}$$

From this we have

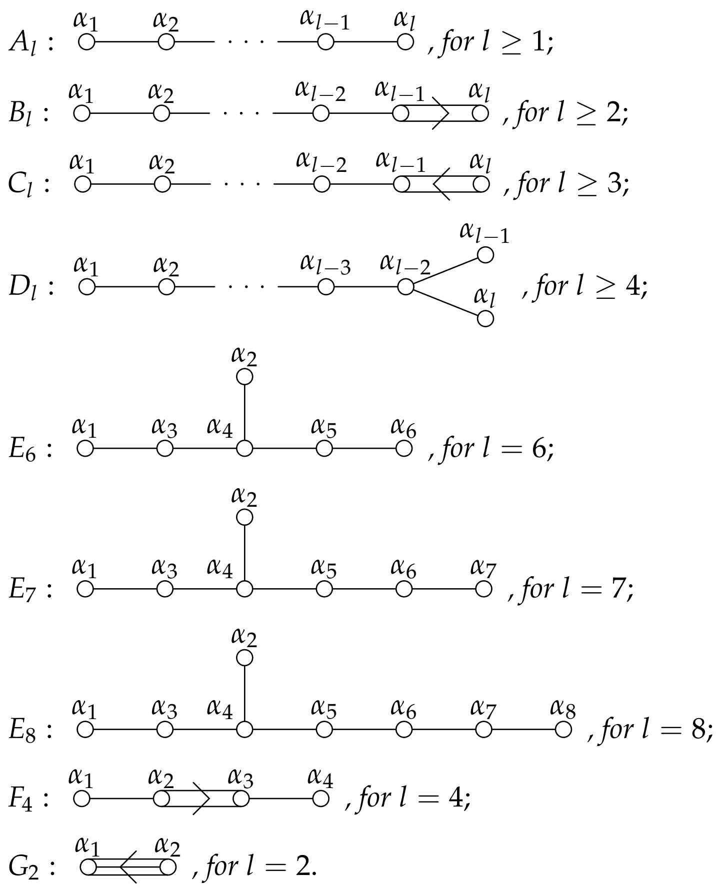
$$2 > \frac{k}{k+1} + \frac{l}{l+1} + \frac{m}{m+1} = 3 - \frac{1}{k+1} - \frac{1}{l+1} - \frac{1}{m+1}$$

or equivalently

$$\frac{1}{k+1} + \frac{1}{l+1} + \frac{1}{m+1} > 1.$$

By our assumption, $\frac{1}{k+1} \ge \frac{1}{l+1} \ge \frac{1}{m+1}$ which implies $1 < \frac{3}{k+1}$. Therefore $k = 1$. Using this fact, we have $\frac{1}{2} < \frac{2}{l+1}$ and hence $l < 3$. If $l = 2$, then $\frac{1}{6} < \frac{1}{m+1}$ resulting in $m < 5$. In case $l = 1$, we have $0 < \frac{1}{m+1}$ and thus m may be arbitrary. This completes the proof. □

Theorem 3.55. *Let Φ be an irreducible root system in E. The unlabeled Dynkin diagram Γ_Φ associated to Φ is, depending on the rank l of Φ, one of those in the following list:*



Remark 3.25. We have used the word "unlabeled" in the statement of the theorem although we have labeled the diagrams above. Clarify that the theorem holds for the Dynkin diagrams without labels. We added the labelling just to establish the standard notation (cf. [3]) that we shall use later.

Proof. Take any base $\Delta = \{\beta_1, \beta_2, \dots, \beta_l\}$ of Φ. We have known that the unlabeled Dynkin diagram for Φ is independent of the particular choice of this base (cf. Theorem 3.41). Further, the set $\tilde{\Delta} := \{\tilde{\beta}_1, \dots, \tilde{\beta}_l\}$, where $\tilde{\beta}_i := \frac{\beta_i}{\|\beta_i\|}$ for all $i \in \widehat{l}$, is admissible. In addition, its respective graph is precisely the Dynkin diagram of Φ with omitted arrows (such graph is said to be *Coxeter* (cf. [6], Sec. 5.9)) because for all $i, j \in \widehat{l}$ we have

$$\langle \beta_i, \beta_j \rangle \langle \beta_j, \beta_i \rangle = \frac{2(\beta_i, \beta_j)}{(\beta_j, \beta_j)} \frac{2(\beta_j, \beta_i)}{(\beta_i, \beta_i)} = 4(\tilde{\beta}_i, \tilde{\beta}_j)^2.$$

Hence we may use the preceding auxiliary propositions and lemmas and apply all the restrictions to the Dynkin diagram associated to Φ.

According to Corollary 3.47, Γ_Φ has no cycles and also, by Lemma 3.48, four or more edges cannot incident to each vertex.

First, consider Γ_Φ without any "branch" point. If Γ_Φ contains a triple edge, it must be of type G_2 (cf. Corollary 3.49). If any double edge occurs, then, by Lemma 3.53, either Γ_Φ is the F_4 or it is a member of the series B_l or C_l; if $l = 2$ then we say that Γ_Φ is of the type B_2 and in case $l > 2$, if there are more longer roots in Δ than the shorter ones (obviously, precisely two different lengths of roots occur (cf. Remark 3.19)), then we say Γ_Φ to be of type B_l, C_l otherwise. The only remaining allowed graph without a "branch" point is the single-edges line, type A_l.

Second, suppose that Γ_Φ has a "branch" point. By Corollary 3.51, this point is precisely one in number. Moreover, by the same corollary, no double edge can occur simultaneously. Thus, the only possible shapes are those investigated in Lemma 3.54 and the statement of that lemma completes the list of possible diagrams obviously. □

Remark 3.26. To finish the classification of Dynkin diagrams completely we should show that for each diagram from the list in Theorem 3.55 there exists an irreducible root system indeed but we omit this part here. The respective root systems for each Dynkin diagram are constructed for example in [6], Sec. 5.10.

With the last theorem we have finished the categorization of root systems into isomorphism classes whereas each class is uniquely determined by the Dynkin diagram. This allows us to adopt the notation of Dynkin diagrams also for the root systems. In irreducible case, we talk about a root system of type T_l, where $l \in \mathbb{N}$ is the rank of the root system and $T \in \{A, B, C, D, E, F, G\}$ is the type of its (connected) Dynkin diagram. If the root system consists of several irreducible subsystems, we say that it is of type $T_{l_1} \times T_{l_2} \times \cdots \times T_{l_m}$, where $T_{l_1}, T_{l_2}, \ldots, T_{l_m}$ are types of the irreducible subsystems.

3.3 Correspondence between Semisimple Lie Algebras and Root Systems

In this very last part of the third chapter we investigate the correspondence between semisimple Lie algebras and root systems. Then we use the previous results concerning classification of root systems to give complete classification of semisimple Lie algebras.

3.3.1 Uniqueness of the Root System of a Semisimple Lie Algebra

First, we clarify the relationship between the decomposition of a semisimple Lie algebra into simple ideals and the decomposition of a root system into irreducible subsystems (cf. [9], Sec. 14.1). As anticipated, this conjugacy is very natural and convenient.

Recall that Example 3.12 permits us to use the term "root system of a semisimple Lie algebra" since the set of roots of such a Lie algebra satisfies the four axioms of root system in Definition 3.26.

Lemma 3.56. *Suppose that L is a simple Lie algebra, H is a Cartan subalgebra of L and Φ is the root system of L with respect to H. Then Φ is irreducible.*

Proof. Suppose that it is not. Then, by Lemma 3.43, Φ can be written as a disjoint union of two mutually perpendicular root systems Φ_1 and Φ_2. Let K be the subalgebra of L generated by a set $\{e_\alpha \in L_\alpha \mid \alpha \in \Phi_1\}$, where, for all $\alpha \in \Phi_1$, $e_\alpha \neq 0$. We claim that K is a non-zero proper ideal of L.

First, K is obviously non-trivial.

Second, suppose that $K = L$. Take any $\beta \in \Phi_2$, non-zero $x \in L_\beta$ and $y \in L$. Then there exists $\alpha \in \Phi_1$ such that $L_\alpha \ni y$. Further, we claim that $\alpha + \beta \notin \Phi$. Indeed, we have $(\alpha, \alpha + \beta) = (\alpha, \alpha) \neq 0$ and hence $\alpha + \beta \notin \Phi_2$. Similarly, since $(\beta, \alpha + \beta) = (\beta, \beta) \neq 0$, $\alpha + \beta \notin \Phi_1$. Consequently, the commutator $[x, y] \in L_{\alpha+\beta}$ has to be zero and, as y can be taken arbitrarily, it follows that $x \in Z(L) = 0$ (cf. Proposition 1.27 (b)), a contradiction.

Finally, we show that K is an ideal. Take an arbitrary generator e_α of K. First, for any $x \in H = L_0$ we have $[x, e_\alpha] = \alpha(x) e_\alpha \in L_\alpha \subset K$. Second, take any $\beta \in \Phi_1$ and $y \in L_\beta$. If $\alpha + \beta \in \Phi_1$, then $[e_\alpha, y] \in L_{\alpha+\beta} \subset K$. Otherwise, if $\alpha + \beta \notin \Phi_1$, we have $(\alpha + \beta, \alpha) = (\alpha + \beta, \beta) = 0$ and accordingly $\alpha + \beta = 0 \notin \Phi$. Then $[e_\alpha, y] = 0 \in K$. Third, we have already seen that for any $\gamma \in \Phi_2$ it holds true $\alpha + \gamma \notin \Phi$ and therefore for each $z \in L_\gamma$, $\gamma \in \Phi_2$, it is satisfied $[e_\alpha, z] = 0 \in K$.

All in all, K is a non-trivial proper ideal of a simple Lie algebra L and hence we reached a contradiction. □

Corollary 3.57. *Suppose that L is a semisimple Lie algebra, H is a Cartan subalgebra of L and Φ is the root system of L with respect to H. Let $L = L_1 \oplus \cdots \oplus L_n$ be the decomposition of L into its simple ideals. Then for all $i \in \hat{n}$ it holds true that $H_i := H \cap L_i$ is a Cartan subalgebra of L_i. Moreover, if for all $i \in \hat{n}$ we define $\tilde{\Phi}_i := \left\{ \alpha \in H^* \mid \alpha|_{H_i} \in \Phi_i \text{ and for all } j \neq i, \alpha|_{H_j} \equiv 0 \right\}$, where Φ_i is the root system of L_i, then $\Phi = \tilde{\Phi}_1 \cup \cdots \cup \tilde{\Phi}_n$ is the decomposition of Φ into irreducible subsystems.*

Proof. Take an arbitrary $i \in \hat{n}$. First, H_i is obviously abelian since H is. Second, by Proposition 2.19, any $h_i \in H_i \subset H$ is semisimple, regarded as an element of L_i. Third, we claim that H_i is maximal with this properties. Indeed, if there exists a semisimple element $x \in L_i / H_i$ such that $[x, h] = 0$ for all $h \in H_i$, then x is semisimple also as an element of L, again by Proposition 2.19. Moreover, according to Remark 1.6, $[x, h] = 0$ even for all $h \in H$, but $x \notin H$, which contradicts the maximality of H.

For the root systems, given any $i \in \hat{n}$ and $\alpha \in \tilde{\Phi}_i$, according to the definition of $\tilde{\Phi}_i$, α is a linear functional on H. Moreover, α is a root of L with a (non-trivial) root space associated to $\alpha|_{H_i} \in \Phi_i$ originally. It is also obvious that $\tilde{\Phi}_i$ is irreducible when Φ_i, as a root system of a simple Lie algebra, is. Contrariwise, for each $\alpha \in \Phi$ there are $j \in \hat{n}$ and $h_j \in H_j$ such that $\alpha(h_j) \neq 0$. If not, then, because $H = H_1 \oplus \cdots \oplus H_n$, $\alpha \equiv 0$ but this is impossible since α is a root. Also there is $y \in L$ such that $y \neq 0$ and $[h_j, y] = \alpha(h_j)y$ or, in other words, $y \in L_j$. Further, for all $k \in \hat{n}$ such that $k \neq j$ and for all $h_k \in H_k \subset L_k$ we have $\alpha(h_k)y = [h_k, y] = 0$ and consequently $\alpha(h_k) = 0$. This implies that $\alpha \in \tilde{\Phi}_j$ and the proof is now complete. □

The problem that still remains is whether the root system depends on the particular choice of Cartan subalgebra. The following theorem, whose proof can be found in Chapter 9 of [10], provides a solution.

Theorem 3.58. *Let H and H' be Cartan subalgebras of a semisimple Lie algebra L. Then there exists a Lie algebra isomorphism $\phi\colon L \to L$ such that $\phi(H) = H'$.*

Remark 3.27. The isomorphism from a Lie algebra onto itself is called *automorphism*.

Corollary 3.59.

(a) The root system of a semisimple Lie algebra is unique, up to isomorphism.

(b) Two isomorphic semisimple Lie algebras have isomorphic root systems.

Proof.

(a) Let L be a Lie algebra and let H and H' be CSA's of L with associated root systems Φ and Φ', respectively. By the previous theorem, there exists an automorphism of L, say ϕ, such that $H' = \phi(H)$. For all $h \in H$ and $\alpha \in H^*$ we define $\phi^*\colon H^* \to H'^*$ by

$$(\phi^*(\alpha))(\phi(h)) := \alpha(h).$$

Clearly, ϕ^* is a bijection. Further, for all $\alpha \in \Phi$, $x \in L_\alpha$ and $h \in H$ let us denote $h' := \phi(h)$, $\alpha' := \phi^*(\alpha)$ and $x' := \phi(x)$. Then we can write

$$[h', x'] = [\phi(h), \phi(x)] = \phi([h, x]) = \phi(\alpha(h)x) = \alpha(h)\phi(x) = (\phi^*(\alpha))(\phi(h))\phi(x)$$
$$= \alpha'(h')x'$$

which implies that $\phi(\Phi) = \Phi'$ and consequently that the restriction of ϕ^* to $\mathsf{E} := \mathrm{Span}_{\mathbb{R}} \Phi$ is a vector space isomorphism sending E to $\mathsf{E}' := \mathrm{Span}_{\mathbb{R}} \Phi'$.

Now, we use the fact that, for all $x, y \in L$, the maps $\mathrm{ad}\, x \circ \mathrm{ad}\, y$ and $\mathrm{ad}\, \phi(x) \circ \mathrm{ad}\, \phi(y)$ have the same eigenvalues. This is obvious when we apply them to any $z \in L$ and $\phi(z) \in \phi(L)$, respectively, and when we realize that ϕ is an isomorphism. Let us consider the vectors $t_\alpha \in H$ and $t_{\alpha'} \in H'$ established in Remark 3.3. For all $\alpha \in \Phi$ and $h \in H$ we have

$$\kappa(\phi(t_\alpha), \phi(h)) = \mathrm{Tr}(\mathrm{ad}\, \phi(t_\alpha) \circ \mathrm{ad}\, \phi(h)) = \mathrm{Tr}(\mathrm{ad}\, t_\alpha \circ \mathrm{ad}\, h) = \kappa(t_\alpha, h) = \alpha(h)$$
$$= (\phi^*(\alpha))(\phi(h)) = \kappa(t_{\phi^*(\alpha)}, \phi(h))$$

and from non-degeneracy of κ (cf. Theorem 1.43) we conclude that $t_{\phi^*(\alpha)} = \phi(t_\alpha)$. Then for arbitrary $\alpha, \beta \in \Phi$ it follows

$$(\phi^*(\alpha), \phi^*(\beta)) = \kappa(t_{\phi^*(\alpha)}, t_{\phi^*(\beta)}) = \kappa(\phi(t_\alpha), \phi(t_\beta)) = \kappa(t_\alpha, t_\beta) = (\alpha, \beta),$$

in particular $\langle \phi^*(\alpha), \phi^*(\beta) \rangle = \langle \alpha, \beta \rangle$.

All in all, Φ and Φ' are isomorphic root systems.

(b) Let $\phi : L \to L'$ be an isomorphism of Lie algebras L and L'. Let H be a CSA of L. We claim that $H' := \phi(H)$ is a CSA of L'. Indeed. First, H' is obviously abelian. Second, for any $\phi(h) \in H'$ it holds true that $\mathrm{ad}\, \phi(h)$ is diagonalisable if and only if $\mathrm{ad}\, h$ is and thus $\phi(h)$ is semisimple as well as h is. Finally, H' has to be maximal with this properties. If it was contained in any larger $G' \supset H'$, $G' \neq H'$, then $G := \phi^{-1}(G') \supset H$, $G \neq H$ would be also abelian and it would contain entirely semisimple elements, contradicting the maximality of H.

Now we may repeat the proof of part (a) exactly (notice that we did not use the fact that $\phi(L) = L$ there at all) in order to show that the root systems associated to H and H', respectively, are isomorphic to each other. Finally, according to (a), any other two root systems are isomorphic as well. □

To avoid an ambiguity, remark that the apostrophes above L and H do not mean the "derived algebra" here.

3.3.2 Existence and Uniqueness Theorems

We have already assigned to each isomorphism class of semisimple Lie algebras exactly one type (or class of isomorphism, equivalently) of root systems. The only question remained is whether this assignment may be reversed i.e. whether for each type of root systems there exists, up to isomorphism, at most one semisimple Lie algebra. The answer is "yes" and in addition more is true. In fact there is one-to-one correspondence between isomorphism classes of semisimple Lie algebras and isomorphism classes of all root systems that may occur. At first, we state two lemmas introducing the "compact" set of generators for a semisimple Lie algebra (cf. [7], Sec. 14.1).

Lemma 3.60. *Let L be a semisimple Lie algebra, let H be a CSA of L and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a base for the root system Φ associated to H. Then L is generated by $e_1, \ldots, e_l, f_1, \ldots, f_l$, where, for all $i \in \widehat{l}$, $(e_i \equiv e_{\alpha_i}, f_i \equiv f_{\alpha_i}, h_i \equiv h_{\alpha_i})$ is a basis for $\mathfrak{sl}(\alpha_i)$ introduced in Proposition 3.13.*

Proof. Considering the root space decomposition of L, it is enough to show that the CSA and all root spaces are generated in this way.

For the CSA, we know that $[e_i, f_i] = h_i$ for all $i \in \widehat{l}$ hence it suffices to show that $h_i, \dots, h_l$ form a basis of H. This is in fact so. It follows from Remark 3.3 that $(t_{\alpha_i}, \dots, t_{\alpha_l})$, where t_{α_i}, $i \in \widehat{l}$, are those from the remark, is a basis for H. By the definition, each h_i is a non-zero multiple of t_{α_i} and hence $(h_i, \dots, h_l)$ is a basis too.

Now, take any $\beta \in \Phi$ and let K denote the subalgebra of L generated by $e_1, \dots, e_l$ and $f_1, \dots, f_l$. We want to show that $L_\beta \subset K$. By Lemma 3.35, there exist $s \in \mathcal{W}_0$ and $j \in \widehat{l}$ such that $\beta = s(\alpha_j)$. As this s is composed of finitely many simple reflections and $L_{\alpha_i} \subset K$ for any $i \in \widehat{l}$ obviously, it suffices to show that if $\beta = s_{\alpha_i}(\gamma)$ for some $i \in \widehat{l}$ and $\gamma \in \Phi$, then $L_\gamma \subset K$ implies $L_\beta \subset K$.

Thus, let M denote the α_i-root string through γ. We have known (cf. proof of part (b) of Proposition 3.18) that M is an irreducible $\mathsf{sl}(\alpha_i)$-module. Moreover, the structure of this module is known as well (from the classification in the second chapter): for any non-zero $e_\gamma \in L_\gamma$ and $k \in \mathbb{N}$ we have $0 \neq e_i^k \cdot e_\gamma = (\operatorname{ad} e_i)^k(e_\gamma) \in L_{\gamma + k\alpha_i}$, whenever $\gamma + k\alpha_i \in \Phi$, and via the action of f_i we may obtain $0 \neq f_i^l \cdot e_\gamma = (\operatorname{ad} f_i)^k(e_\gamma) \in L_{\gamma - l\alpha_i}$, $l \in \mathbb{N}$ and $\gamma - l\alpha_i \in \Phi$. Finally, it suffices to realize that $\beta = s_{\alpha_i}(\gamma) = \gamma - \langle \gamma, \alpha_i \rangle \alpha_i$ to prove $L_\beta \subset K$. ☐

Lemma 3.61. *Suppose that L is a semisimple Lie algebra, H is a CSA of L and $\Delta = \{\alpha_1, \dots, \alpha_l\}$ is a base for the root system associated to H. For each $i \in \widehat{l}$ let (e_i, f_i, h_i) be the basis for $\mathsf{sl}(\alpha_i)$ as before. For all $i, j \in \widehat{l}$ the following relations are fulfilled:*

(S1) $[h_i, h_j] = 0$;

(S2) $[h_i, e_j] = \langle \alpha_j, \alpha_i \rangle e_j$ *and* $[h_i, f_j] = -\langle \alpha_j, \alpha_i \rangle f_j$;

(S3) $[e_i, f_j] = \delta_{ij} h_i$;

(S4) *if* $i \neq j$, *then* $(\operatorname{ad} e_i)^{1 - \langle \alpha_j, \alpha_i \rangle}(e_j) = 0$;

(S5) *if* $i \neq j$, *then* $(\operatorname{ad} f_i)^{1 - \langle \alpha_j, \alpha_i \rangle}(f_j) = 0$.

Proof. First, (S1) holds since H is abelian. Second, by Proposition 3.21, we have

$$[h_i, e_j] = \alpha_j(h_i) e_j = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} e_j = \langle \alpha_j, \alpha_i \rangle e_j,$$

thus the first part of (S2) holds. For the second part, it suffices to realize that each $f_i \in L_{-\alpha_i}$, $i \in \widehat{l}$. For (S3), if $i = j$, then the statement holds directly from Proposition 3.13, where the basis was established. In case i and j are distinct, we have $[e_i, f_j] \in L_{\alpha_i - \alpha_j}$ but, from the definition of a base for a root system, $\alpha_i - \alpha_j$ cannot be a root and hence $L_{\alpha_i - \alpha_j} = 0$. Finally, by the previous part, the α_i-root string through α_j is of the form $\{\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + q\alpha_i\}$, where $q = -\alpha_j(h_i) = -\langle \alpha_j, \alpha_i \rangle$ (cf. part (b) of Proposition 3.18). Now the statement is clear because $(\operatorname{ad} e_i)^{1 - \langle \alpha_j, \alpha_i \rangle}(e_j) \in L_{\alpha_j + (q+1)\alpha_i}$ and $\alpha_j + (q+1)\alpha_i \notin \Phi$. (S5) is analogous to (S4). ☐

At this place we state an important theorem telling us that the five relations in the previous lemma are sufficient to define a semisimple Lie algebra backwards. We omit the proof, one can found it in Section 18 of [9], however we prove two immediate consequences that complete the whole classification of semisimple Lie algebras.

Theorem 3.62 (Serre). *Suppose that Φ is a root system and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ is its base. Let L be the complex Lie algebra generated by a set of abstract vectors $\left\{e_i, f_i, h_i \mid i \in \widehat{l}\right\}$ satisfying the relations (S1), (S2), (S3), (S4), (S5). Then L is a (finite-dimensional) semisimple Lie algebra with Cartan subalgebra $H := \operatorname{Span}\left\{h_i \mid i \in \widehat{l}\right\}$ and with root system Φ.*

Remark 3.28. Let L and L' be two semisimple Lie algebras with respective CSA's H and H' and root systems Φ and Φ'. Suppose that Φ and Φ' are isomorphic i.e. that there exists a vector space isomorphism $\varphi\colon \operatorname{Span}_{\mathbb{R}} \Phi \to \operatorname{Span}_{\mathbb{R}} \Phi'$ such that $\varphi(\Phi) = \Phi'$. We claim that φ induces a Lie algebras isomorphism $\phi\colon H \to H'$. To show this, we use the coupling $H \leftrightarrow H^*$ and $H' \leftrightarrow H'^*$ via the Killing form introduced in Remark 3.3.

First, let $\Delta := \{\alpha_1, \ldots, \alpha_l\}$ be as base of Φ. Then $\Delta' := \left\{\varphi(\alpha_i) \mid i \in \widehat{l}\right\}$ is a base of Φ' since φ is a bijection and since for any $\alpha' \in \Phi'$ there exists $\alpha = \sum_{i=1}^{l} c_i\alpha_i \in \Phi$, where the coefficients $(c_i)_{i\in\widehat{l}}$ are either all non-negative or all non-positive, such that $\alpha' = \varphi(\alpha)$. Then

$$\alpha' = \varphi(\alpha) = \varphi\left(\sum_{i=1}^{l} c_i\alpha_i\right) = \sum_{i=1}^{l} c_i\varphi(\alpha_i).$$

and thus we are able to express any root from Φ' as a linear combination of simple roots from Δ' with either completely non-negative or completely non-positive coefficients.

Now, for all $i \in \widehat{l}$ let $t_i \equiv t_{\alpha_i} \in H$ and $t_i' \equiv t_{\varphi(\alpha_i)} \in H'$, respectively, be the vectors from Remark 3.3. It results from the remark that $(t_1, \ldots, t_l)$ is a basis for H and similarly $(t_1', \ldots, t_l')$ is a basis for H'. We define a map $\phi\colon H \to H'$ as follows: for all $i \in \widehat{l}$ we put

$$\phi(t_i) := \frac{(\alpha_i, \alpha_i)}{(\varphi(\alpha_i), \varphi(\alpha_i))} t_i' \tag{3.14}$$

and for any other $h \in H$ we define $\phi(h)$ as linear extension of (3.14). Clearly, ϕ is a linear bijection. Moreover, since both H and H' are abelian, ϕ also preserves the Lie bracket and thus it is the required isomorphism.

Remark 3.29. Let Φ and Φ' be two isomorphic root systems and let $\varphi : \mathsf{E} \to \mathsf{E}'$, where $\mathsf{E} := \operatorname{Span}_{\mathbb{R}} \Phi$ and $\mathsf{E}' := \operatorname{Span}_{\mathbb{R}} \Phi'$, be a vector space isomorphism such that $\varphi(\Phi) = \Phi'$. Consider the decomposition $\Phi = \Phi_1 \cup \cdots \cup \Phi_n$ into irreducible root systems and take any $j \in \widehat{n}$. Further, take any $\alpha, \beta \in \Phi_j$ such that $(\alpha, \beta) \neq 0$. There exists non-zero $k_j \in \mathbb{R}$ such that $(\varphi(\alpha), \varphi(\beta)) = k_j(\alpha, \beta)$. Now, take any other $\gamma, \delta \in \Phi_j$ such that $(\gamma, \delta) \neq 0$. According to the proof of Lemma 3.43, there exist $\gamma_1, \ldots, \gamma_s \in \Phi$ such that $\gamma_1 = \beta$, $\gamma_s = \gamma$ and for all $i \in \widehat{s-1}$ it holds true that $(\gamma_i, \gamma_{i+1}) \neq 0$. Moreover, if we denote $\gamma_0 := \alpha$ and $\gamma_{s+1} := \delta$, then $(\gamma_i, \gamma_{i+1}) \neq 0$ even for all $i \in \widehat{\underline{s}}$.

Using incomplete induction on i, we show that $(\varphi(\gamma_i), \varphi(\gamma_{i+1})) = k_j(\gamma_i, \gamma_{i+1})$ for all $i \in \widehat{\underline{s}}$. The case $i = 0$ is just our definition of k_j. For the inductive step assume that $(\varphi(\gamma_{i-1}), \varphi(\gamma_i)) = k_j(\gamma_{i-1}, \gamma_i)$. Consequently, since $\langle\varphi(\gamma_{i-1}), \varphi(\gamma_i)\rangle = \langle\gamma_{i-1}, \gamma_i\rangle$, also $(\varphi(\gamma_i), \varphi(\gamma_i)) = k_j(\gamma_i, \gamma_i)$ and further, as $\langle\varphi(\gamma_{i+1}), \varphi(\gamma_i)\rangle = \langle\gamma_{i+1}, \gamma_i\rangle$, we have

$$(\varphi(\gamma_i), \varphi(\gamma_{i+1})) = (\varphi(\gamma_{i+1}), \varphi(\gamma_i)) = k_j(\gamma_{i+1}, \gamma_i) = k_j(\gamma_i, \gamma_{i+1}),$$

as desired.

In particular, this proves that $(\varphi(\gamma), \varphi(\delta)) = k_j(\gamma, \delta)$ for all $\gamma, \delta \in \Phi_j$ (the case when $(\gamma, \delta) = 0$ is contained trivially).

Now, we define a new inner product $(\ ,\ )_{new}$ on E as follows: for $\alpha \in \mathsf{E}_i := \operatorname{Span}_{\mathbb{R}} \Phi_i$ and $\beta \in \mathsf{E}_j := \operatorname{Span}_{\mathbb{R}} \Phi_j$, $i, j \in \widehat{n}$, we put

$$(\alpha, \beta)_{new} := \delta_{ij}k_j(\alpha, \beta), \tag{3.15}$$

where k_j is as above. One can easily see that (3.15) establishes an inner-product indeed. Moreover, all four axioms of root system are still satisfied after this change obviously, thus Φ is a root system also in real inner-product space $(\mathsf{E}; (\ ,\ )_{new})$ and, finally, the new inner product is preserved by φ (in the sense that for all $\alpha, \beta \in \Phi$ we have $(\varphi(\alpha), \varphi(\beta)) = (\alpha, \beta)_{new}$).

All in all, considering an isomorphism between two root systems, we may assume, without loss of generality, that it preserves the inner product.

Corollary 3.63 (Existence and Uniqueness Theorems)**.**

(a) Let Φ be a root system. Then there exists a semisimple Lie algebra having Φ as its root system.

(b) Let L and L' be semisimple Lie algebras with respective CSA's H and H' and root systems Φ and Φ'. Suppose that Φ and Φ' are isomorphic and let φ denote the respective isomorphism ($\varphi(\Phi) = \Phi'$). Further, let $\{\alpha_1, \ldots, \alpha_l\}$ be a base of Φ and let $\phi : H \to H'$ be the induced isomorphism introduced in Remark 3.28. For each $i \in \widehat{l}$ select arbitrary non-zero $e_i \in L_{\alpha_i}$ and $e'_i \in L'_{\varphi(\alpha_i)}$. Then there exists a unique Lie algebras isomorphism $\psi : L \to L'$ such that $\psi|_H = \phi$ and for all $i \in \widehat{l}$ it holds true that $\psi(e_i) = e'_i$.

Proof.

(a) It suffices to take any set of $3l$ linearly independent vectors and define the Lie bracket so that all relations (S1) - (S5) are satisfied.

(b) Let $(t_1, \ldots, t_l)$ and $(t'_1, \ldots, t'_l)$ be the bases for H and H', respectively, from Remark 3.28. Obviously, $(h_1, \ldots, h_l)$ and $(h'_1, \ldots, h'_l)$, where $h_i = \frac{2t_i}{(\alpha_i, \alpha_i)}$ and $h'_i = \frac{2t'_i}{(\varphi(\alpha_i), \varphi(\alpha_i))}$, $i \in \widehat{l}$, are bases as well. For all $i \in \widehat{l}$, let $f_i \in L_{-\alpha_i}$ be such that $[e_i, f_i] = h_i$ and let $f'_i \in L'_{-\varphi(\alpha_i)}$ be such that $[e'_i, f'_i] = h'_i$. We define

$$\psi(h_i) := h'_i, \quad \psi(e_i) := e'_i, \quad \psi(f_i) := f'_i,$$

$i \in \widehat{l}$. We must verify that ψ extends ϕ. Indeed. According to Remark 3.29, we may assume that φ preserves the inner product. Then for all $i \in \widehat{l}$ we have

$$\phi(h_i) = \phi(\frac{2t_i}{(\alpha_i, \alpha_i)}) = \frac{2\phi(t_i)}{(\alpha_i, \alpha_i)} = \frac{2t'_i}{(\varphi(\alpha_i), \varphi(\alpha_i))} = h'_i.$$

Finally, since $\left\{e'_i, f'_i, h'_i \mid i \in \widehat{l}\right\}$ generate the semisimple Lie algebra L' where the same commutation relations hold as in the semisimple Lie algebra L generated by $\left\{e_i, f_i, h_i \mid i \in \widehat{l}\right\}$, it is clear that $\psi : L \to L'$ is a Lie algebras homomorphism. Moreover, the same holds true for the map $\psi' : L' \to L$ defined for all $i \in \widehat{l}$ by

$$\psi'(h'_i) := h_i, \quad \psi'(e'_i) := e_i, \quad \psi'(f'_i) := f_i.$$

Obviously, ψ and ψ' are mutually inverse and hence ψ is the desired isomorphism of Lie algebras. Notice that we had no freedom how to define this isomorphism except the selection in the statement of the theorem. □

Existence and uniqueness theorems complete the process of classification of semisimple Lie algebras. Let us summarize the classification at this place.

We have discovered that there exists precisely one class of isomorphism of semisimple Lie algebras for each type of the root system. This fact allows us to accept the notation of root systems also for semisimple Lie algebras. Thus, talking about a semisimple

Lie algebra $T_{l_1} \times T_{l_2} \times \cdots \times T_{l_m}$, we will always mean a semisimple Lie algebra, say L, having the root system of type $T_{l_1} \times T_{l_2} \times \cdots \times T_{l_m}$. Any other semisimple Lie algebra with the root system of the same type has to be isomorphic to L. Moreover, L is composed of m simple ideals (in sense of Theorem 1.47) and the root systems of these ideals are exactly of those types $T_{l_1}, T_{l_2}, \ldots, T_{l_m}$, respectively.

Notice that the list of all possible Dynkin diagrams in Theorem 3.55 corresponds precisely to the list of isomorphic classes of all simple Lie algebras that may occur. Using the standard terminology, the simple Lie algebras A_l, B_l, C_l and D_l are called *classical* and the others, E_6, E_7, E_8, F_4 and G_2, are said to be *exceptional*. Remark that the classical Lie algebras occur naturally as certain subalgebras of Lie algebras $\mathsf{gl}(n, \mathbb{C})$ for suitable $n \in \mathbb{N}$. One of these subalgebras is the Lie algebra $\mathsf{sl}(n, \mathbb{C})$ introduced in Subsection 1.1.4. It can be shown that the root system of $\mathsf{sl}(l+1, \mathbb{C})$ has type A_l, in particular $\mathsf{sl}(l+1, \mathbb{C})$ is simple (cf. [7]).

Finally, the correspondence described above permits us to define the *rank* of a semisimple Lie algebra L to be the rank of the root system of L.

Chapter 4

Construction of Simple Lie Algebras

In this last chapter we show how to construct a simple Lie algebra of an arbitrary type. The generalization to the semisimple case is straightforward since each semisimple Lie algebra falls uniquely into the direct sum of simple ideals.

The problem of construction of the Lie algebra consists in giving a basis and in establishing commutation relations among the basis elements. We need to do this in such a way, that the constructed simple Lie algebra has the desired root system (corresponding to the type of the Lie algebra). Naturally, one aims to choose an advisable basis in order to simplify the commutation relations as much as possible. In the first section of this chapter we introduce such a basis and in the second section we present the algorithms for computing the structure constants with respect to this convenient basis.

4.1 Chevalley Basis

Remark 4.1. Let L be a (finite-dimensional complex) semisimple Lie algebra. Consider the root space decomposition of L (3.4):

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

where H is a Cartan subalgebra of L and Φ is the root system associated to H. In Proposition 3.13 we assigned a three-dimensional subalgebra $\mathsf{sl}(\alpha)$ to each $\alpha \in \Phi$. This subalgebra was spanned by the vectors e_α, f_α and h_α, whereas e_α was chosen arbitrarily to fulfill only $e_\alpha \neq 0$, $h_\alpha \in H$ was given by $h_\alpha = \frac{2t_\alpha}{\|\alpha\|^2}$, where $0 \neq t_\alpha \in H$ was defined in Remark 3.3, and $f_\alpha \in L_{-\alpha}$ was determined (uniquely) by $[e_\alpha, f_\alpha] = h_\alpha$. Notice that $f_\alpha \neq 0$ clearly.

Now we choose Δ, a base for Φ, and denote the set of all positive roots with respect to this base by Φ^+. We saw (for example in the proof of Lemma 3.60) that $\{h_\alpha \mid \alpha \in \Delta\}$ forms a basis of H. Remark that the vectors h_α are also defined for $\alpha = \sum_{\beta \in \Delta} c_\beta \beta \in \Phi / \Delta$ and by Remark 3.3 it holds true that

$$h_\alpha = \frac{2}{\|\alpha\|^2} t_\alpha = \frac{2}{\|\alpha\|^2} \sum_{\beta \in \Delta} c_\beta t_\beta = \sum_{\beta \in \Delta} \left(\frac{\|\beta\|^2}{\|\alpha\|^2} c_\beta \right) h_\beta. \tag{4.1}$$

In fact, the relation (4.1) holds for all $\alpha \in \Phi$ but it is trivial for $\alpha \in \Delta$.

Further, let us choose the non-zero vectors $e_\alpha \in L_\alpha$ but only for $\alpha \in \Phi^+$. For the negative roots $\alpha \in \Phi^-$ we put $e_\alpha := f_{-\alpha}$, where $f_{-\alpha}$ is as above. From part (a) of Lemma 3.17 it is clear that for all $\alpha \in \Phi$ it holds true that $L_\alpha = \operatorname{Span}\{e_\alpha\}$.

Altogether, the set $\mathcal{B}_0 := \{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ forms a basis for L.

In the following text, we will keep the notation from the previous remark. In particular, we will assume that the convenient basis $\mathcal{B}_0$ of L introduced there is given. Now, we summarize known commutation relations among the basis vectors.

Proposition 4.1. *With the notation above, the following relations hold:*

(a) if $\alpha, \beta \in \Delta$, then $[h_\alpha, h_\beta] = 0$;

(b) if $\alpha \in \Delta$ and $\beta \in \Phi$, then $[h_\alpha, e_\beta] = \langle \beta, \alpha \rangle e_\beta$;

(c) if $\alpha \in \Phi$, then $[e_\alpha, e_{-\alpha}] = h_\alpha$;

(d) if $\alpha, \beta \in \Phi$ such that $\alpha \neq -\beta$ and $\alpha + \beta \notin \Phi$, then $[e_\alpha, e_\beta] = 0$.

Proof. First, (a) is obvious since H is abelian. For (b), since $e_\gamma \in L_\gamma$ and $h_\alpha \in H$, we may write

$$[h_\alpha, e_\beta] = \beta(h_\alpha) e_\beta = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} e_\beta = \langle \beta, \alpha \rangle e_\beta.$$

Third, (c) holds from the choice of e_α directly and finally, (d) is implied by Proposition 3.7 (a) together with the fact that $\alpha + \beta \notin \Phi$. □

Remark 4.2. The list of the structure constants of L in the previous proposition is incomplete. It remains to determine the constants $N_{\alpha,\beta}$ defined by

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}, \tag{4.2}$$

$\alpha, \beta, \alpha + \beta \in \Phi$. For the correctness of the definition remark that, according to Proposition 3.7 (a), $[e_\alpha, e_\beta] \in L_{\alpha+\beta}$. Moreover, part (d) of the previous proposition allows us to define $N_{\alpha,\beta}$ for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \neq 0$: we put $N_{\alpha,\beta} := 0$ whenever $\alpha + \beta \notin \Phi$. Just for convenience, we put $N_{\alpha,\beta} := 0$ for all non zero $\alpha, \beta \in H^*$ such that at least one of α and β is not a root.

To specify those $N_{\alpha,\beta}$, $\alpha + \beta \in \Phi$, we have to do more work. We start with two auxiliary assertions (cf. [9], Sec. 25.1) and then we use them to present the relations among the structure constants $N_{\alpha,\beta}$ (cf. [4], Sec. 4.1).

Proposition 4.2. *Let $\alpha, \beta \in \Phi$. At most two different lengths of roots occur in the α-root string through β.*

Proof. Let us denote $\Phi' := \Phi \cap \{a\alpha + b\beta \mid a, b \in \mathbb{Z}\}$. Obviously, Φ' satisfies the four axioms for the root system in $\mathsf{E}' := \mathrm{Span}_{\mathbb{R}} \{\alpha, \beta\}$: (R1), (R2) and (R4) are satisfied trivially; for (R3), for any $a, b, c, d \in \mathbb{Z}$ such that $a\alpha + b\beta, c\alpha + d\beta \in \Phi'$ we have

$$\begin{aligned} s_{a\alpha+b\beta}(c\alpha + d\beta) &= c\alpha + d\beta + \langle c\alpha + d\beta, a\alpha + b\beta \rangle (a\alpha + b\beta) \\ &= (c + a\langle c\alpha + d\beta, a\alpha + b\beta \rangle)\alpha + (d + b\langle c\alpha + d\beta, a\alpha + b\beta \rangle)\beta \in \Phi' \end{aligned}$$

by (R3) for Φ and because the coefficients are obviously integral. Further, the rank of Φ' is at most 2 thus there are obviously at most two different lengths of simple roots in Φ. But the lemmas and propositions on the Weyl group discussed in Section 3.2 imply that each root can be generated from a simple root by finitely many simple reflections which preserve the inner product and hence the norm as well. □

Lemma 4.3. *Suppose that $\alpha, \beta, \alpha + \beta \in \Phi$. Let $\{\beta - p\alpha, \ldots, \beta + q\alpha\}$ be the α-root string through β. Then*

$$p + 1 = q \frac{(\alpha + \beta, \alpha + \beta)}{(\beta, \beta)}. \tag{4.3}$$

Proof. First, we have $p - q = \beta(h_\alpha) = \frac{2}{(\alpha,\alpha)}\beta(t_\alpha) = 2\frac{\kappa(t_\beta,t_\alpha)}{(\alpha,\alpha)} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \langle\beta,\alpha\rangle$. To prove the lemma we must prove that

$$\begin{aligned} A := p + 1 - q\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)} &= p + 1 - q\left(\frac{(\alpha,\alpha)}{(\beta,\beta)} + 2\frac{(\beta,\alpha)}{(\beta,\beta)} + 1\right) \\ = \langle\beta,\alpha\rangle + 1 - q(\frac{\|\alpha\|^2}{\|\beta\|^2} + \langle\alpha,\beta\rangle) &= (\langle\beta,\alpha\rangle + 1)\left(1 - q\frac{\|\alpha\|^2}{\|\beta\|^2}\right) \end{aligned}$$

equals zero.

First, assume that $\|\alpha\| \geq \|\beta\|$. This implies $|\langle\beta,\alpha\rangle| \leq |\langle\alpha,\beta\rangle|$ and hence, by Table 3.1, $\langle\beta,\alpha\rangle \in \{-1,0,1\}$. If $\langle\beta,\alpha\rangle = -1$, then $A = 0$. Otherwise $(\beta,\alpha) \geq 0$ and hence $\|\alpha+\beta\|^2 = (\alpha+\beta,\alpha+\beta) = \|\alpha\|^2 + 2(\alpha,\beta) + \|\beta\|^2 \geq \|\alpha\|^2 + \|\beta\|^2$. Therefore $\alpha+\beta \in \Phi$ is strictly longer then both α and β and, by Proposition 4.2, $\|\alpha\| = \|\beta\|$. Similarly, $\|\beta + 2\alpha\| > \|\alpha+\beta\|$ and thus $\beta + 2\alpha \notin \Phi$. Consequently $q = 1$ and hence $A = 0$.

Now, suppose that $\|\alpha\| < \|\beta\|$. Then either α or β must have the same length as $\alpha+\beta$ and therefore $\|\alpha+\beta\|^2 = \|\alpha\|^2 + 2(\alpha,\beta) + \|\beta\|^2$ implies that $(\alpha,\beta) < 0$. This fact results in $\|\beta-\alpha\|^2 = (\beta-\alpha,\beta-\alpha) = \|\beta\|^2 - 2(\beta,\alpha) + \|\alpha\|^2 > \|\beta\|^2 > \|\alpha\|^2$ and hence $\beta - \alpha \notin \Phi$ or equivalently $p = 0$. As $(\alpha,\beta) < 0$ and $\|\alpha\| < \|\beta\|$, there remains only these two possibilities in Table 3.1: $\langle\alpha,\beta\rangle = -1$ and $\langle\beta,\alpha\rangle = -2$ or $\langle\alpha,\beta\rangle = -1$ and $\langle\beta,\alpha\rangle = -3$. In either case $q = p - \langle\beta,\alpha\rangle = 0 + \frac{\langle\beta,\alpha\rangle}{(-1)} = \frac{\langle\beta,\alpha\rangle}{\langle\alpha,\beta\rangle} = \frac{\|\beta\|^2}{\|\alpha\|^2}$ and hence $A = 0$ again. ☐

Theorem 4.4. *The structure constants $N_{\alpha,\beta}$ defined by* (4.2) *satisfy the following relations:*

(a) if $\alpha,\beta \in \Phi$, then

$$N_{\alpha,\beta} = -N_{\beta,\alpha}. \tag{4.4}$$

(b) if $\alpha,\beta,\gamma \in \Phi$ such that $\alpha+\beta+\gamma = 0$, then

$$\frac{N_{\alpha,\beta}}{\|\gamma\|^2} = \frac{N_{\beta,\gamma}}{\|\alpha\|^2} = \frac{N_{\gamma,\alpha}}{\|\beta\|^2}; \tag{4.5}$$

(c) if $\alpha,\beta \in \Phi$ such that $\alpha \neq \pm\beta$, then

$$N_{\alpha,\beta}N_{-\alpha,-\beta} = -(p+1)^2, \tag{4.6}$$

where p is the largest (non-negative) integer such that $\beta - p\alpha \in \Phi$;

(d) if $\alpha,\beta,\gamma,\delta \in \Phi$ such that $\alpha+\beta+\gamma+\delta = 0$ and no pair is opposite, then

$$\frac{N_{\alpha,\beta}N_{\gamma,\delta}}{\|\alpha+\beta\|^2} + \frac{N_{\beta,\gamma}N_{\alpha,\delta}}{\|\beta+\gamma\|^2} + \frac{N_{\gamma,\alpha}N_{\beta,\delta}}{\|\gamma+\alpha\|^2} = 0. \tag{4.7}$$

Proof.

(a) It follows from $[e_\alpha, e_\beta] = -[e_\beta, e_\alpha]$.

(b) The Jacobi identity takes the following form:

$$[e_\alpha,[e_\beta,e_\gamma]] + [e_\beta,[e_\gamma,e_\alpha]] + [e_\gamma,[e_\alpha,e_\beta]] = 0.$$

Since $\alpha+\beta+\gamma = 0$, then also $t_\alpha + t_\beta + t_\gamma = t_{\alpha+\beta+\gamma} = 0$ and hence we may write

$$0 = N_{\beta,\gamma}[e_\alpha,e_{-\alpha}] + N_{\gamma,\alpha}[e_\beta,e_{-\beta}] + N_{\alpha,\beta}[e_\gamma,e_{-\gamma}]$$

$$
\begin{aligned}
&= N_{\beta,\gamma} h_\alpha + N_{\gamma,\alpha} h_\beta + N_{\alpha,\beta} h_\gamma \\
&= N_{\beta,\gamma} \frac{2t_\alpha}{\|\alpha\|^2} + N_{\gamma,\alpha} \frac{2t_\beta}{\|\beta\|^2} + N_{\alpha,\beta} \frac{2t_\gamma}{\|\gamma\|^2} \\
&= 2\left(\frac{N_{\beta,\gamma}}{\|\alpha\|^2} - \frac{N_{\alpha,\beta}}{\|\gamma\|^2} \right) t_\alpha + 2\left(\frac{N_{\gamma,\alpha}}{\|\beta\|^2} - \frac{N_{\alpha,\beta}}{\|\gamma\|^2} \right) t_\beta.
\end{aligned}
$$

Now, we claim that α and β are linearly independent. Indeed, if not, $\alpha = \pm\beta$ and thus either $\gamma = 0$ or $\gamma = -2\alpha$, a contradiction. As α and β are linearly independent, t_α and t_β are linearly independent as well and therefore both coefficients in our linear combination equal to zero.

(c) Again we start with the Jacobi identity (we put $e_\delta := 0$, whenever $\delta \in H^*/\Phi$):

$$
\begin{aligned}
0 &= [e_\alpha, [e_{-\alpha}, e_\beta]] + [e_{-\alpha}, [e_\beta, e_\alpha]] + [e_\beta, [e_\alpha, e_{-\alpha}]] \\
&= N_{-\alpha,\beta}[e_\alpha, e_{-\alpha+\beta}] + N_{\beta,\alpha}[e_{-\alpha}, e_{\alpha+\beta}] - [h_\alpha, e_\beta] \\
&= N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta} e_\beta + N_{\beta,\alpha} N_{-\alpha,\alpha+\beta} e_\beta - \beta(h_\alpha) e_\beta \\
&= (N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta} + N_{\beta,\alpha} N_{-\alpha,\alpha+\beta} - \langle \beta, \alpha \rangle) e_\beta.
\end{aligned}
$$

Now, realizing that $e_\beta \neq 0$ and using item (a) additionally, we obtain

$$
\begin{aligned}
\langle \beta, \alpha \rangle &= N_{\beta,\alpha} N_{-\alpha,\alpha+\beta} + N_{-\alpha,\beta} N_{\alpha,-\alpha+\beta} \\
&= N_{\alpha,\beta} N_{\alpha+\beta,-\alpha} - N_{\alpha,-\alpha+\beta} N_{\beta,-\alpha}.
\end{aligned}
$$

At this place we use item (b), where we put $\tilde{\alpha} = \alpha + \beta$, $\tilde{\beta} = -\alpha$, $\tilde{\gamma} = -\beta$ and $\tilde{\alpha} = \beta$, $\tilde{\beta} = -\alpha$, $\tilde{\gamma} = \alpha - \beta$, respectively. We have

$$
\langle \beta, \alpha \rangle = N_{\alpha,\beta} N_{-\alpha,-\beta} \frac{\|-\beta\|^2}{\|\alpha+\beta\|^2} - N_{\alpha,-\alpha+\beta} N_{-\alpha,\alpha-\beta} \frac{\|\alpha-\beta\|^2}{\|\beta\|^2}. \tag{4.8}
$$

If we denote $M_{\alpha,\beta} := N_{\alpha,\beta} N_{-\alpha,-\beta} \frac{\|\beta\|^2}{\|\alpha+\beta\|^2}$ for all non-zero $\alpha, \beta \in H^*$ such that $\alpha + \beta \neq 0$, we can rewrite (4.8) as

$$
\langle \beta, \alpha \rangle = M_{\alpha,\beta} - M_{\alpha,-\alpha+\beta}. \tag{4.9}
$$

Let $\{\beta - p\alpha, \dots, \beta + q\alpha\}$ be the α-root string through β. We can repeat the procedure above for all pairs of roots $\{\alpha, \beta - j\alpha\}$, $j \in \widehat{p}$. In this way we obtain $p + 1$ equations parallel to (4.9):

$$
\begin{aligned}
M_{\alpha,\beta} - M_{\alpha,-\alpha+\beta} &= \langle \beta, \alpha \rangle \\
M_{\alpha,-\alpha+\beta} - M_{\alpha,-2\alpha+\beta} &= \langle -\alpha + \beta, \alpha \rangle = -\langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle = -2 + \langle \beta, \alpha \rangle \\
&\vdots \\
M_{\alpha,-j\alpha+\beta} - M_{\alpha,-(j+1)\alpha+\beta} &= \langle -j\alpha + \beta, \alpha \rangle = -j\langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle = -2j + \langle \beta, \alpha \rangle \\
&\vdots \\
M_{\alpha,-p\alpha+\beta} - M_{\alpha,-(p+1)\alpha+\beta} &= \langle -p\alpha + \beta, \alpha \rangle = -p\langle \alpha, \alpha \rangle + \langle \beta, \alpha \rangle = -2p + \langle \beta, \alpha \rangle.
\end{aligned}
$$

Adding this equations together and realizing $M_{\alpha,-(p+1)\alpha+\beta} = 0$ (because it is true that $-(p+1)\alpha + \beta \notin \Phi$), we obtain

$$
M_{\alpha,\beta} = (p+1)\langle \beta, \alpha \rangle - \sum_{j=1}^{p} 2j = (p+1)\langle \beta, \alpha \rangle - p(p+1).
$$

Now, substituting for $M_{\alpha,\beta}$ and for $\langle\beta,\alpha\rangle$ (for the second substitution, cf. part (b) of Proposition 3.18), we have

$$N_{\alpha,\beta}N_{-\alpha,-\beta}\frac{\|\beta\|^2}{\|\alpha+\beta\|^2} = M_{\alpha,\beta} = (p+1)\langle\beta,\alpha\rangle - p(p+1) = (p+1)(p-q) - p(p+1) = -q(p+1)$$

and finally, according to Lemma 4.3,

$$N_{\alpha,\beta}N_{-\alpha,-\beta} = -q(p+1)\frac{\|\alpha+\beta\|^2}{\|\beta\|^2} = -(p+1)^2.$$

(d) Again from the Jacobi identity we obtain

$$\begin{aligned}0 &= [e_\alpha,[e_\beta,e_\gamma]] + [e_\beta,[e_\gamma,e_\alpha]] + [e_\gamma,[e_\alpha,e_\beta]]\\ &= (N_{\beta,\gamma}N_{\alpha,\beta+\gamma} + N_{\gamma,\alpha}N_{\beta,\gamma+\alpha} + N_{\alpha,\beta}N_{\gamma,\alpha+\beta})e_{\alpha+\beta+\gamma}\end{aligned}$$

and consequently, since $\alpha+\beta+\gamma = -\delta \in \Phi$ and thus $e_{\alpha+\beta+\gamma} \neq 0$,

$$\begin{aligned}0 &= N_{\beta,\gamma}N_{\alpha,\beta+\gamma} + N_{\gamma,\alpha}N_{\beta,\gamma+\alpha} + N_{\alpha,\beta}N_{\gamma,\alpha+\beta}\\ &= N_{\beta,\gamma}N_{\delta,\alpha}\frac{\|\delta\|^2}{\|\beta+\gamma\|^2} + N_{\gamma,\alpha}N_{\delta,\beta}\frac{\|\delta\|^2}{\|\gamma+\alpha\|^2} + N_{\alpha,\beta}N_{\delta,\gamma}\frac{\|\delta\|^2}{\|\alpha+\beta\|^2}\\ &= -\|\delta\|^2\left(\frac{N_{\alpha,\beta}N_{\gamma,\delta}}{\|\alpha+\beta\|^2} + \frac{N_{\beta,\gamma}N_{\alpha,\delta}}{\|\beta+\gamma\|^2} + \frac{N_{\gamma,\alpha}N_{\beta,\delta}}{\|\gamma+\alpha\|^2}\right).\end{aligned}$$

Since $-\|\delta\|^2$ is obviously non-zero, we get the desired identity. Remark that we have used results (b) and (a), respectively. □

The relations presented in the previous theorem hold for any choice of basis $\mathcal{B}_0$ introduced in Remark 4.1. However, it is possible to choose the basis vectors in such a way that the relations among the structure constants are even more convenient (cf. [4], Sec. 4.2).

Proposition 4.5. *Suppose $\alpha \in \Phi^+$. Then there exist (not necessarily distinct) simple roots $\beta_1,\dots,\beta_m \in \Delta$ such that $\alpha = \sum_{i=1}^m \beta_i$ and for all $k \in \hat{m}$ it holds true that $\sum_{i=1}^k \beta_i \in \Phi^+$.*

Proof. We use induction on height of α. Notice that $\operatorname{ht}\alpha > 0$. If $\operatorname{ht}\alpha = 1$, then $\alpha \in \Delta$ and we are done. For the inductive step suppose that the proposition holds for positive roots of height $m \geq 0$ and take any $\alpha \in \Phi^+$ such that $\operatorname{ht}\alpha = m+1$. We know that $\alpha = \sum_{\beta\in\Delta} k_\beta\beta$, whereas $k_\beta \geq 0$ for all $\beta \in \Delta$. Moreover

$$0 < (\alpha,\alpha) = (\alpha, \sum_{\beta\in\Delta} k_\beta\beta) = \sum_{\beta\in\Delta} k_\beta(\alpha,\beta)$$

and hence there exists $\tilde\beta \in \Delta$ such that $k_{\tilde\beta}(\alpha,\tilde\beta) > 0$ and $(\alpha,\tilde\beta) > 0$, consequently. α and $\tilde\beta$ are obviously linearly independent and thus the angle between them is non-zero and strictly acute. But, by Proposition 3.28 (a), $\tilde\alpha := \alpha - \tilde\beta \in \Phi$. In addition, it holds true that

$$\tilde\alpha = \sum_{\substack{\beta\in\Delta\\ \beta\neq\tilde\beta}} k_\beta\beta + (k_{\tilde\beta}-1)\tilde\beta$$

and hence $\tilde\alpha$ is a positive root of height m which our induction hypothesis can be applied on. Since $\alpha = \tilde\alpha + \tilde\beta$, the statement is now obvious. □

Lemma 4.6. *One can choose the basis $\mathcal{B}_0$ in such a way that for all $\alpha, \beta \in \Phi$ it holds true that*

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}. \tag{4.10}$$

Proof. We define an automorphism $\psi : L \to L$. According to Corollary 3.63 (b), such the automorphism is fully determined by an isomorphism $\varphi : \Phi \to \Phi'$ between root systems and by the choice of $\{\psi(e_\alpha) \mid \alpha \in \Delta\}$. Thus, for $\alpha \in \Phi$ we define

$$\varphi(\alpha) := -\alpha.$$

Obviously, $\Phi' := \varphi(\Phi)$ is a root system of L with the base $\Delta' := \varphi(\Delta)$ and φ is an isomorphism between the root systems. Then for all $\alpha \in \Delta$ we have

$$\psi(h_\alpha) = h_{\varphi(\alpha)} = h_{-\alpha} = -h_\alpha.$$

Further, for all $\alpha \in \Delta$ we define

$$\psi(e_\alpha) := -e_{-\alpha}.$$

As seen from the proof of Corollary 3.63, the images $\psi(e_{-\alpha}) \in L_{-\varphi(\alpha)} = L_\alpha$, $\alpha \in \Delta$, have to satisfy $[\psi(e_\alpha), \psi(e_{-\alpha})] = \psi(h_\alpha)$. Clearly, this condition is fulfilled for

$$\psi(e_{-\alpha}) := -e_\alpha.$$

Now, consider $\alpha \in \Phi$ such that $\pm\alpha \notin \Delta$. If $\alpha \in \Phi^+$, then Proposition 4.5 implies existence of $\alpha_1, \ldots, \alpha_m \in \Delta$ such that $\alpha = \alpha_1 + \cdots + \alpha_m$ and for all $k \in \hat{m}$ it holds true that $\alpha_1 + \cdots + \alpha_k \in \Phi$. The generalization for negative roots is obvious. Proposition 3.18 (d) then implies existence of non-zero $\lambda_\alpha \in \mathbb{C}$ such that

$$[[\ldots[[e_{\alpha_1}, e_{\alpha_2}], e_{\alpha_3}], \ldots, e_{\alpha_{m-1}}], e_{\alpha_m}] = \lambda_\alpha e_\alpha.$$

Applying the isomorphism ψ we obtain

$$[[\ldots[[e_{-\alpha_1}, e_{-\alpha_2}], e_{-\alpha_3}], \ldots, e_{-\alpha_{m-1}}], e_{-\alpha_m}] = (-1)^m \lambda_\alpha \psi(e_\alpha).$$

As in the previous case, there exists a non-zero $\lambda'_\alpha \in \mathbb{C}$ such that

$$[[\ldots[[e_{-\alpha_1}, e_{-\alpha_2}], e_{-\alpha_3}], \ldots, e_{-\alpha_{m-1}}], e_{-\alpha_m}] = \lambda'_\alpha e_{-\alpha}.$$

Altogether, for all $\alpha \in \Phi / \{\pm\beta \mid \beta \in \Delta\}$ there is a non-zero $\Lambda_\alpha \equiv (-1)^m \frac{\lambda'_\alpha}{\lambda_\alpha} \in \mathbb{C}$ such that $\psi(e_\alpha) = \Lambda_\alpha e_{-\alpha}$. Our aim is to set new basis vectors $\tilde{e}_\alpha \in L_\alpha$, $\alpha \in \Phi$, in such a way that $\psi(\tilde{e}_\alpha) = -\tilde{e}_{-\alpha}$, for all $\alpha \in \Phi$. Then, applying ψ on relation $[\tilde{e}_\alpha, \tilde{e}_\beta] = N_{\alpha,\beta}\tilde{e}_{\alpha+\beta}$, where $\alpha, \beta, \alpha + \beta \in \Phi$, we would have

$$\begin{aligned} -N_{\alpha,\beta}\tilde{e}_{-\alpha-\beta} &= \psi(N_{\alpha,\beta}\tilde{e}_{\alpha+\beta}) = \psi([\tilde{e}_\alpha, \tilde{e}_\beta]) = [\psi(\tilde{e}_\alpha), \psi(\tilde{e}_\beta)] = [-\tilde{e}_{-\alpha}, -\tilde{e}_{-\beta}] \\ &= N_{-\alpha,-\beta}\tilde{e}_{-\alpha-\beta} \end{aligned}$$

and thus $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$.

The question is whether such a choice can be made. For the vectors corresponding to the simple roots and their negatives, we are already done; we let them without change ($\tilde{e}_{\pm\alpha} := e_{\pm\alpha}$, $\alpha \in \Delta$). Regarding the others, the only freedom in choosing vector $\tilde{e}_\alpha \in L_\alpha$ is in (non-zero) scalar multiplication. Hence we search for convenient non-zero constants $\mu_\alpha \in \mathbb{C}$, $\alpha \in \Phi / \{\pm\beta \mid \beta \in \Delta\}$, such that for all the new basis vectors $\tilde{e}_\alpha = \mu_\alpha e_\alpha$ the condition $\psi(\tilde{e}_\alpha) = -\tilde{e}_{-\alpha}$ is fulfilled. First, from

$$[e_\alpha, e_{-\alpha}] = h_\alpha = [\tilde{e}_\alpha, \tilde{e}_{-\alpha}] = [\mu_\alpha e_\alpha, \mu_{-\alpha} e_{-\alpha}] = \mu_\alpha \mu_{-\alpha} [e_\alpha, e_{-\alpha}]$$

we obtain the condition $\mu_\alpha = \mu_{-\alpha}^{-1}$. Second, we apply our isomorphism:

$$\psi(\tilde{e}_\alpha) = \psi(\mu_\alpha e_\alpha) = \mu_\alpha \psi(e_\alpha) = \mu_\alpha \Lambda_\alpha e_{-\alpha} = \mu_\alpha \Lambda_\alpha \mu_\alpha \mu_{-\alpha} e_{-\alpha} = \mu_\alpha^2 \Lambda_\alpha \tilde{e}_{-\alpha}.$$

One can see that to satisfy our desire for $\psi(\tilde{e}_\alpha) = -\tilde{e}_{-\alpha}$ it is necessary to choose μ_α such that $\mu_\alpha^2 \Lambda_\alpha = -1$. However, this is certainly possible since we are working over $\mathbb{C}$. $\square$

Remark 4.3. Given a root system Φ in a real inner-product space E, one can show that the set $\Phi^\# := \left\{\alpha^\# \equiv \frac{2\alpha}{\|\alpha\|^2} \;\middle|\; \alpha \in \Phi\right\}$, the so-called *dual* root system of Φ, is a root system in E as well. Indeed. Axioms (R1) and (R2) are satisfied obviously. Further, for any $\alpha^\#, \beta^\# \in \Phi^\#$, we have

$$s_{\alpha^\#}(\beta^\#) = \beta^\# - \frac{2(\beta^\#,\alpha^\#)}{(\alpha^\#,\alpha^\#)}\alpha^\# = \frac{2\beta}{\|\beta\|^2} - \frac{2\left(\frac{2\beta}{\|\beta\|^2}, \frac{2\alpha}{\|\alpha\|^2}\right)}{\left(\frac{2\alpha}{\|\alpha\|^2}, \frac{2\alpha}{\|\alpha\|^2}\right)}\frac{2\alpha}{\|\alpha\|^2} = \frac{2}{\|\beta\|^2}\left(\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right)$$

and hence, because $\left\|\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right\|^2 = \|\beta\|^2$, (R3) holds as well. Finally, for (R4) we have

$$\langle\alpha^\#,\beta^\#\rangle = \frac{2(\alpha^\#,\beta^\#)}{(\beta^\#,\beta^\#)} = \frac{2\left(\frac{2\alpha}{\|\alpha\|^2}, \frac{2\beta}{\|\beta\|^2}\right)}{\left(\frac{2\beta}{\|\beta\|^2}, \frac{2\beta}{\|\beta\|^2}\right)} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \langle\beta,\alpha\rangle \in \mathbb{Z}.$$

Additionally, it can be shown that if Δ is a base for Φ, then $\Delta^\# := \left\{\alpha^\# \;\middle|\; \alpha \in \Delta\right\}$ is a base for $\Phi^\#$. This fact follows immediately from the proof of Theorem 10.1 in [9] ("Each root system has a base") which we did not prove here. We only presented the fact that there is a base for an arbitrary root system in Section 3.2.

Remark 4.4. Given a semisimple Lie algebra L, its Cartan subalgebra H and the root system Φ incident to H, one can show that the set $\left\{t_\alpha \;\middle|\; \alpha \in \Phi\right\}$ is a root system in E^*, where the inner product is defined for all $h_1, h_2 \in \mathsf{E}^* \subset H$ naturally as

$$(h_1,h_2) := \kappa(h_1,h_2). \tag{4.11}$$

For verification of the four axioms of root system, it suffices to realize that $(\alpha,\beta) = (t_\alpha, t_\beta)$ for any $\alpha,\beta \in \Phi$ and that the assignment $\alpha \mapsto t_\alpha$ is linear (cf. Remark 3.3).

Theorem 4.7 (Chevalley)**.** *The basis $\mathcal{B}_0 := \left\{h_\alpha \;\middle|\; \alpha \in \Delta\right\} \cup \left\{e_\alpha \;\middle|\; \alpha \in \Phi\right\}$ of L introduced above can be chosen in such a way that the commutation relations of basis vectors are as follows:*

(a) if $\alpha,\beta \in \Delta$, then
$$[h_\alpha, h_\beta] = 0; \tag{4.12}$$

(b) if $\alpha \in \Delta$ and $\beta \in \Phi$, then
$$[h_\alpha, e_\beta] = \langle\beta,\alpha\rangle e_\beta; \tag{4.13}$$

(c) if $\alpha \in \Phi$, then
$$[e_\alpha, e_{-\alpha}] = h_\alpha; \tag{4.14}$$

(d) if $\alpha,\beta \in \Phi$ such that $\alpha \neq -\beta$ and $\alpha+\beta \notin \Phi$, then
$$[e_\alpha, e_\beta] = 0; \tag{4.15}$$

(e) if $\alpha,\beta,\alpha+\beta \in \Phi$, then
$$[e_\alpha, e_\beta] = \pm(p+1)e_{\alpha+\beta}, \tag{4.16}$$
where p is the largest integer such that $\beta - p\alpha \in \Phi$.

In particular, if such a basis is chosen, then all the structure constants of L with respect to this basis are integral.

Proof. The first four items are precisely the statement of Proposition 4.1. For (e), by (4.2) we have $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_\alpha$. Further, according to Theorem 4.4 (c), $-N_{\alpha,\beta}N_{-\alpha,-\beta} = (p+1)^2$, where p is the largest non-negative integer such that $\beta - p\alpha \in \Phi$. But we saw in Lemma 4.6 that we might choose the basis $\mathcal{B}_0$ such that $-N_{-\alpha,-\beta} = N_{\alpha,\beta}$. Altogether, we have

$$N^2_{\alpha,\beta} = (p+1)^2$$

and hence, because $(p+1) \in \mathbb{R}_{\geq 0}$ obviously, $N_{\alpha,\beta} = \pm(p+1)$.

For integrality of the the structure constants, notice first that it suffices to verify integrality of the non-zero structure constants appearing in items (a) - (e) of this theorem. Any other remaining structure constant is either zero or (-1)-multiple of some constant already appeared. First, $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$ by the definition of root system (axiom (R4)). Second, according to (4.1), for each $\alpha = \sum_{\beta \in \Delta} k_\beta \beta \in \Phi$ we have

$$h_\alpha = \sum_{\beta \in \Delta} \left(\frac{\|\beta\|^2}{\|\alpha\|^2} c_\beta \right) h_\beta.$$

Moreover, Remarks 4.3 and 4.4 together imply that the set $\{h_\alpha \mid \alpha \in \Phi\}$ forms a root system and the set $\{h_\alpha \mid \alpha \in \Delta\}$ is its base. Hence, we obtain $\frac{\|\beta\|^2}{\|\alpha\|^2} c_\beta \in \mathbb{Z}$ for each $\beta \in \Delta$ from the definition of base for a root system. Finally, it is clear that $\pm(p+1) \in \mathbb{Z}$ since $p \in \mathbb{Z}_{\geq 0}$. □

Definition 4.8. A basis satisfying the conditions from Chevalley's Theorem is called a *Chevalley basis* for L. We shall denote such a basis by $\mathcal{B}$.

4.2 Construction of a Simple Lie Algebra

Now we introduce the algorithms used for construction of a simple Lie algebra with the root system of an arbitrary type. These algorithms were implemented into several procedures in Maple 16 computer algebra system and they together provide the program that computes a Chevalley basis of a simple Lie algebra of any type. The whole Maple code is attached in Appendix A.

4.2.1 Determination of the Root System from its Cartan Matrix

The very first step in construction of simple Lie algebra corresponding to a given type of root system (which is determined by its Dynkin diagram or Cartan matrix, equivalently) is to determine which integral combinations of simple roots are in the root system. We saw in the previous chapter that, in principle, it was possible to recover whole root system by finitely many simple reflections of simple roots. However, this way is inconvenient for computation. The algorithm that is usually used and that we present here is taken from [6]. We assume E to be an l-dimensional real vector space again.

Algorithm 1 (RootSystem)**.**

Input: Cartan matrix $C \in \mathbb{Z}^{l,l}$ of type T_l and a set $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of linearly independent vectors from E satisfying $\langle \alpha_i, \alpha_j \rangle = C_{ij}$.

Step 1. Set $\Phi^+ := \Delta$ and $n := 1$.

Step 2. For all $\alpha \in \Phi^+$ such that $\operatorname{ht} \alpha = n$ and for all $\alpha_j \in \Delta$ do the following:

Substep 2.1. Write

$$\alpha = \sum_{i=1}^{l} k_i \alpha_i.$$

Substep 2.2. Determine the largest integer p such that $\alpha - p\alpha_j \in \Phi^+$.

Substep 2.3. Set

$$q := p - \sum_{i=1}^{l} k_i C_{ij}.$$

Substep 2.4. If $q > 0$, then set $\Phi^+ := \Phi^+ \cup \{\alpha + \alpha_j\}$.

Step 3. If Φ^+ has changed throughout Step 2, then set $n := n + 1$ and repeat Step 2. Otherwise return $\Phi^+ \cup \{\beta \in \mathsf{E} \mid -\beta \in \Phi^+\}$.

Output: the root system Φ in E that has Δ as its base and is of the type T_l.

Remark 4.5. It is very useful to work with l-tuples $(k_1, k_2, \dots, k_l)$ of coefficients in base Δ within a computation with roots or with some constants having roots as indices. Note that our computations were conducted in this way as well.

Proposition 4.9. *Let Φ be a root system and let Δ be a base for Φ. Suppose that C is the Cartan matrix of Φ with respect to Δ. Then Algorithm 1 returns the set Φ indeed.*

Proof. It is enough to show that Φ^+ contains all positive roots of Φ after the final repetition of Step 2. Then it is clear that in the final step we obtain whole Φ. We will proceed by incomplete induction on the (positive) height of positive roots contained in Φ^+. First, the only positive roots of height one are obviously the simple ones and these are all included in Φ^+ by Step 1. Second, suppose that Φ^+ contains all positive roots of height at most n and take any $\alpha \in \Phi$, $\operatorname{ht}\alpha = n + 1$. According to Proposition 4.5, we may write

$$\alpha = \beta + \alpha_j,$$

where β is a positive root of height n and $\alpha_j \in \Delta$. Thus, Step 2 takes into consideration the pair (β, α_1) and therefore it suffices to show that, with the notation of the algorithm, the number p can be determined and the number q is positive. Let $\beta - p'\alpha_j, \dots, \beta + q'\alpha_j$ be the α_j-root string through β. Since $\beta + \alpha_j \in \Phi$, certainly $\beta \neq \alpha_j$. Consequently, if we decompose β as $\beta = \sum_{i=1}^{l} k_i \alpha_i$, then there is $i \in \hat{l}$ such that $i \neq j$ and $k_i > 0$. This coefficient stays unchanged even if we add or subtract some multiples of α_j, hence all roots in the α_j-root string through β are positive and the desired number p is the p' from the string. Finally, since $\beta + 1 \cdot \alpha_j = \alpha \in \Phi$, we have

$$1 \leq q' = p' - \beta(h_{\alpha_j}) = p - \langle \beta, \alpha_j \rangle = p - \langle \sum_{i=1}^{l} k_i \alpha_i, \alpha_j \rangle = p - \sum_{i=1}^{l} k_i C_{ij} = q.$$

□

We did not have to implement whole of the previous algorithm in Maple 16 because the major part of it was already contained as PositiveRoots procedure. As the title suggest, this procedure computes the positive roots, hence we had to only add the set of negative roots which was not difficult apparently. Remark that the procedure PositiveRoots is based precisely on the algorithm 1, as one can find in Maple 16 documentation. Another advantage of the used computer algebra system is that all Cartan matrices are included as well. One can call the appropriate matrix by command CartanMatrix. Both PositiveRoots and CartanMatrix are parts of LieAlgebras Maple package which is a subpackage of DifferentialGeometry package.

4.2.2 Special and Extraspecial Pairs of Roots

Now, when we already know a whole root system, we may consider an abstract Chevalley basis introduced in the first section of this chapter. To determine the Lie algebra structure, we need to define commutators of the basis vectors. We saw in Chevalley's Theorem that if our basis ought to be "Chevalley", then the commutators were already "almost" determined by the definition of this basis. The only potential degrees of freedom were in the signs of the structure constants $N_{\alpha,\beta} = \pm(p+1)$ in (4.16). On the other hand one can see that we may not choose all signs arbitrarily; we have to bear in mind that the Jacobi identity has to be still fulfilled. In the following subsection, we establish exactly which signs can be chosen arbitrarily and we show that the others are uniquely determined (cf. [4], Sec. 4.2). Firstly, we shall need the ordering of positive roots.

Definition 4.10. Let Φ be a root system and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a base of Φ. For any positive roots $\alpha = \sum_{i=1}^{l} a_i\alpha_i$ and $\beta = \sum_{i=1}^{l} b_i\alpha_i$ such that $\alpha \neq \beta$ we put

$$\alpha \prec \beta$$

whenever $\operatorname{ht}\alpha < \operatorname{ht}\beta$ or if $\operatorname{ht}\alpha = \operatorname{ht}\beta$ and $a_m > b_m$, where $m = \min\left\{i \in \widehat{l} \;\middle|\; a_i \neq b_i\right\}$. For any $\alpha, \beta \in \Phi^+$ we denote

$$\alpha \preceq \beta$$

if either $\alpha \prec \beta$ or $\alpha = \beta$.

Remark 4.6. Obviously, the relation $\prec$ is a total order of Φ^+ (cf. [12]).

One can easily see that the part of Algorithm 1 that computes the set of positive roots works precisely with respect to the ordering just defined providing that we take the roots $\alpha \in \Phi^+$ by our ordering and the simple roots α_j by j increasing from 1 to l in Step 2. As one can convince herself by command

```
showstat(PositiveRoots::PositiveRootsFromCartanMatrix),
```

Maple 16 works exactly in this way and hence the positive roots are computed in accordance with our ordering.

Definition 4.11. An ordered pair (α, β) of positive roots is called a *special pair* if $\alpha \prec \beta$ and $\alpha + \beta \in \Phi$. A special pair (α, β) is said to be *extraspecial* if for all special pairs (α', β') such that $\alpha' + \beta' = \alpha + \beta$ it holds true that $\alpha \preceq \alpha'$.

Remark 4.7. Let $\alpha \in \Phi^+$. In case $\alpha \in \Delta$ we have $\operatorname{ht}\alpha = 1$ and α cannot be expressed as the sum of two positive roots forming an extraspecial pair (not even a special one) obviously. Suppose now that $\operatorname{ht}\alpha > 1$. Then Proposition 4.5 guarantees the existence of a special pair (β, γ) such that $\alpha = \beta + \gamma$. Since Φ is finite, the number of such the pairs is finite as well and hence we may find the extraspecial one among them. Notice that the extraspecial pair is unique because the order of positive roots is total. To summarize, the set of all extraspecial pairs is in one-to-one correspondence with the set Φ^+/Δ.

Proposition 4.12. *The signs of the structure constants $N_{\alpha,\beta}$ may be chosen arbitrarily for extraspecial pairs (α, β). Furthermore, the signs of $N_{\alpha,\beta}$ for all other pairs are uniquely determined by this choice.*

Proof. We saw in the proof of Lemma 4.6 how to transform any basis $\mathcal{B}_0$ defined in Remark 4.1 into a Chevalley one. Remind that the transformation consists in multiplication of basis vectors e_α, $\alpha \in \Phi/\left\{\pm\beta \;\middle|\; \beta \in \Delta\right\}$, by a number $\mu_\alpha \in \mathbb{C}$ which satisfies $\mu_\alpha^2\Lambda_\alpha = -1$ for given non-zero $\Lambda_\alpha \in \mathbb{C}$. Obviously, for each α we have precisely two possibilities for the choice of μ_α, namely $\pm\sqrt{\frac{-1}{\Lambda_\alpha}}$. Moreover the relation $\mu_\alpha = \mu_{-\alpha}^{-1}$ must hold true. Altogether, we may choose arbitrarily the signs of μ_α, and thus of e_α consequently, for $\alpha \in \Phi^+/\Delta$. If we modify the signs of these vectors, we obtain a Chevalley basis again. But this modification agrees precisely with the alteration of the structure constants $N_{\beta,\gamma}$ for extraspecial pairs (β,γ). Indeed, the extraspecial pairs are in bijective correspondence with the non-simple positive roots and, considering an extraspecial pair, the left hand side in (4.16) stays unchanged and thus the change of the sign of vector on the right hand side must be succeeded by the change of the sign of respective structure constant.

For the second part, consider two Chevalley bases $\mathcal{B} = \{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\beta \mid \beta \in \Phi\}$ and $\mathcal{B}' = \{h_\alpha \mid \alpha \in \Delta\} \cup \left\{e'_\beta \mid \beta \in \Phi\right\}$. Let $N_{\alpha,\beta}$ and $N'_{\alpha,\beta}$ be the structure constants as before corresponding to $\mathcal{B}$ and $\mathcal{B}'$, respectively. For all $\alpha,\beta \in \Phi$ it holds true that $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$ and $[e'_\alpha, e'_\beta] = N'_{\alpha,\beta}e'_{\alpha+\beta}$ whenever $\alpha+\beta \in \Phi$. We know that, for all $\alpha \in \Phi$, e_α and e'_α have to be linearly dependent and non-zero, thus there is a non-zero $\lambda_\alpha \in \mathbb{C}$ such that $e'_\alpha = \lambda_\alpha e_\alpha$. Then for $\alpha+\beta \in \Phi$ we have $\lambda_\alpha\lambda_\beta N_{\alpha,\beta} = \lambda_{\alpha+\beta}N'_{\alpha,\beta}$. Now suppose that $N_{\alpha,\beta} = N'_{\alpha,\beta}$ for all extraspecial pairs (α,β). Since the structure constants $N_{\alpha,\beta}$ are non-zero, it follows that $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ for extraspecial pairs. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Taken any $\alpha \in \Phi^+$ we may decompose it as $\alpha = \sum_{i=1}^{l} k_i\alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$, $i \in \widehat{l}$. We claim that the following relation holds true:

$$\lambda_\alpha = \lambda_{\alpha_1}^{k_1}\lambda_{\alpha_2}^{k_2}\ldots\lambda_{\alpha_l}^{k_l}. \tag{4.17}$$

We prove this assertion by induction on $\operatorname{ht}\alpha$. If $\operatorname{ht}\alpha = 1$, then $\alpha \in \Delta$ and hence (4.17) obviously holds. For the inductive step, suppose that $\operatorname{ht}\alpha = n > 1$ and that (4.17) holds for all positive roots of height less then n. As $\alpha \in \Phi^+/\Delta$, there exists an extraspecial pair (β,γ) such that $\beta+\gamma = \alpha$ and hence $\lambda_\alpha = \lambda_{\beta+\gamma} = \lambda_\beta\lambda_\gamma$. Now, since $\operatorname{ht}\beta, \operatorname{ht}\gamma < \operatorname{ht}\alpha = n$, we may apply (4.17) on β and γ to obtain the decomposition (4.17) for α as well.

Further, for all $\alpha \in \Phi^+$, it must be true that $[e_\alpha, e_{-\alpha}] = h_\alpha = [e'_\alpha, e'_{-\alpha}]$ and consequently $\lambda_\alpha\lambda_{-\alpha} = 1$. Thus for all $\alpha = \sum_{i=1}^{l} k_i\alpha_i \in \Phi^-$ (notice that $k_i \in \mathbb{Z}_{\leq 0}$, $i \in \widehat{l}$, now) we may write

$$\lambda_\alpha = (\lambda_{-\alpha})^{-1} = (\lambda_{\alpha_1}^{|k_1|}\lambda_{\alpha_2}^{|k_2|}\ldots\lambda_{\alpha_l}^{|k_l|})^{-1} = \lambda_{\alpha_1}^{k_1}\lambda_{\alpha_2}^{k_2}\ldots\lambda_{\alpha_l}^{k_l}$$

in order to extend validity of (4.17) also to the negative roots.

Finally, since (4.17) holds true for all $\alpha \in \Phi$, we have $\lambda_\beta\lambda_\gamma = \lambda_{\beta+\gamma} \neq 0$ and consequently $N_{\beta,\gamma} = N'_{\beta,\gamma}$ for all $\beta,\gamma,\beta+\gamma \in \Phi$. Thus, we have no freedom in choice of the signs of constants $N_{\alpha,\beta}$ for pairs (α,β) that are not extraspecial, when the signs of those constants are given for the extraspecial pairs. This proves the second part of the proposition. □

According to the previous proposition, the next task of our construction is to find the extraspecial pairs among all pairs of positive roots. Concerning this problem, we present the following algorithm. We use the fact that the positive roots are computed with respect to their order introduced above and hence we do not have to browse on all special pairs intricately and determine whether a pair is extraspecial or not.

Algorithm 2 (ExtraspecialPairs)**.**

Input: ordered set of positive roots $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$, $\alpha_i \prec \alpha_j$ when $1 \leq i < j \leq m$.

Step 1. Set $i := 1$, $j := 1$ and $ESP := \emptyset$.

Step 2. Substep 2.1. If $\alpha_i + \alpha_j \in \Phi$ and if for any $(\alpha, \beta) \in ESP$ it holds true that $\alpha + \beta \neq \alpha_i + \alpha_j$, then set $ESP := ESP \cup \{(\alpha_i, \alpha_j)\}$.

Substep 2.2. Set $j := j + 1$.

Step 3. If $j \leq m$, then repeat Step 2. Otherwise set $i := i + 1$.

Step 4. If $i \leq m$, then set $j := i$ and go back to Step 2. Otherwise return ESP.

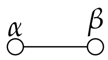
Output: the set ESP of all extraspecial pairs of roots in the root system Φ corresponding to the set of positive roots Φ^+.

Remark 4.8. Obviously, Algorithm 2 takes into consideration all special pairs and in fact even more pairs of roots, namely the pairs of type (α, α) are inspected as well. Clearly, these pairs can never satisfy the condition in Substep 2.1 (twice a root is never a root) but, writing the algorithm in this way, we do not have to investigate the case $l = 1$ separately.

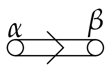
The question that still remains is whether the conditions in Substep 2.1 are adequate to explore all extraspecial pairs. Indeed. If (α, β) is an extraspecial pair, then $\alpha + \beta \in \Phi$ and for all other pairs (γ, δ) such that $\gamma + \delta = \alpha + \beta$ it holds true that $\gamma \succ \alpha$. Thus both conditions in Substep 2.1 are fulfilled and (α, β) is added to ESP. Conversely, if $\alpha + \beta \notin \Phi$, then (α, β) is clearly not extraspecial. Otherwise, if $\alpha + \beta \in \Phi$ and some (γ, δ), such that $\gamma + \delta = \alpha + \beta$, is already contained in ESP, then it must be $\gamma \prec \alpha$ and thus (α, β) is not extraspecial again.

4.2.3 Determination of the Norms of Roots

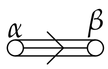
Later we shall need to compute the squares of (relative) norms of roots. Obviously, the absolute norms are indeterminable for an abstract root system, because if Φ is a root system, then, for any $k \in \mathbb{R}/\{0\}$, $\Phi' := \{k\alpha \mid \alpha \in \Phi\}$ is a root system of the same type as Φ. Thus, we will accept the following convention: given a root system Φ of rank l and its base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, we will assume that the simple roots α_i, $i \in \widehat{l}$, correspond precisely to the labeled Dynkin diagrams in Theorem 3.55 (this is exactly the reason why we have labeled the diagrams there) and further that $\|\alpha_1\|^2 = 2$. Accepting this convention, one can easily determine the squares of norms of all other simple roots from the shape of respective Dynkin diagram by virtue of (cf. Remark 3.19):



$$\Longrightarrow \|\alpha\|^2 = \|\beta\|^2,$$



$$\Longrightarrow \|\alpha\|^2 = 2\,\|\beta\|^2,$$



$$\Longrightarrow \|\alpha\|^2 = 3\,\|\beta\|^2.$$

Recall that we construct a simple Lie algebra and hence we consider an irreducible root system whose Dynkin diagram is connected.

The square of norm of any other (non-simple) root $\gamma = \sum_{i=1}^{l} k_i \alpha_i$ can be then determined as follows:

$$\|\gamma\|^2 = (\gamma, \gamma) = \sum_{i,j=1}^{l} k_i k_j (\alpha_i, \alpha_j) = \sum_{i,j=1}^{l} \frac{k_i k_j}{2} (\alpha_j, \alpha_j) \langle \alpha_i, \alpha_j \rangle = \sum_{i,j=1}^{l} \frac{k_i k_j}{2} \|\alpha_j\|^2 C_{ij},$$

where C_{ij} is the i,j-th entry of the corresponding Cartan matrix.

4.2.4 Computation of the Structure Constants $N_{\alpha,\beta}$

At first, according to the definition of Chevalley basis, for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$ we have $|N_{\alpha,\beta}| = p + 1$, where p is the largest integer such that $\beta - p\alpha \in \Phi$. We know that the roots such as $\beta - p\alpha$, $p \in \mathbb{Z}$, form a string (cf. part (b) of Proposition 3.18) thus it is enough to find an integer p such that $\beta - p\alpha \in \Phi$ and $\beta - (p+1)\alpha \notin \Phi$.

The remaining problem is how to establish the signs of these constants. Recall that, according to Proposition 4.12, we may choose the signs arbitrarily for those $N_{\alpha,\beta}$ which correspond to the extraspecial pairs (α, β) and the others are then determined uniquely. In the following text we present the derivation of the signs of any $N_{\alpha,\beta}$ from the values of the structure constants corresponding to the extraspecial pairs. We proceed in two steps. First, we determine the signs corresponding to those pairs of positive roots whose sum is a root again and second, we compute all remaining signs. Both the parts are based on algorithms introduced in [5].

Remark at this place that we have found two mistakes in the original paper [5]. First, there is an ambiguity in the algorithm corresponding to the first part: in that paper, this first step consists in computation of the structure constants corresponding *only* to the special pairs of roots. However, we uncovered that it was not possible to compute precisely this set of structure constants in general. One needs to use also the structure constants corresponding to the swapped special pairs. Thus, it is necessary to determine the structure constants for *all* pairs of positive roots such that their sum is a (positive) root again. The second incorrectness has occurred in the algorithm corresponding to the second part. We have verified that the computation including this mistake did not produce a Lie algebra at all because the Jacobi identity failed for some vectors. This defect is specified below.

In spite of the first mistake, we have to determine the special pairs of roots first of all. On this problem, we present the following simple algorithm computing the special pairs (α, β) in the following order: first by $\mathrm{ht}(\alpha + \beta)$ and second by the order of α with respect to the total order of positive roots introduced above. Notice that we use the fact that Algorithm 1 produces the positive roots with respect to the total order again.

Algorithm 3 (SpecialPairs)**.**

Input: ordered set of positive roots $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$, $\alpha_i \prec \alpha_j$ when $1 \leq i < j \leq m$.

Step 1. Set $i := 1$ and $SP := \emptyset$.

Step 2. Substep 2.1. For all integers k, j such that $1 \leq k \leq j \leq i$ do the following: if $\alpha_k + \alpha_j = \alpha_i$, then set $SP := SP \cup \{(\alpha_k, \alpha_j)\}$.

Substep 2.2. Set $i := i + 1$.

Step 3. If $i \leq m$, then repeat Step 2. Otherwise return SP.

Output: the set SP of all special pairs of roots in the root system Φ corresponding to the set of positive roots Φ^+.

Remark 4.9. Clearly, if the sum of two positive roots is a root, then this sum is a positive root as well. Now it is obvious that Algorithm 3 produces precisely all special pairs of roots indeed. Remark once again that the set of special pairs computed by the algorithm is ordered (first by the height of the sum of roots and then by the order of the first root), we shall use this fact immediately.

Now, we may approach to the algorithm for computation of the structure constants $N_{\alpha,\beta}$, $\alpha, \beta, \alpha + \beta \in \Phi^+$, based originally on "ALGORITHM 1" from the source paper [5]. Considering a root system, We assume that some base for this system is given as well and thus it makes sense to talk about the set of positive roots, special pairs, etc.

Algorithm 4 (StructureConstantsForPositivePairs).

Input: root system Φ, the set $SP = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ of special pairs of roots from Φ, such that $\mathrm{ht}(\alpha_i + \beta_i) \leq \mathrm{ht}(\alpha_j + \beta_j)$ or $\alpha_i \prec \alpha_j$ in case of equality for all $1 \leq i < j \leq m$, the set ESP of extraspecial pairs of roots from Φ and the structure constants $(N_{\alpha,\beta})_{(\alpha,\beta)\in ESP}$.

Step 1. For all $(\alpha, \beta) \in ESP$ set $N_{\beta,\alpha} := -N_{\alpha,\beta}$.

Step 2. Set $i := 1$.

Step 3. Substep 3.1. Set $\alpha := \alpha_i$ and $\beta := \beta_i$.

Substep 3.2. If $(\alpha, \beta) \in ESP$, then skip to Substep 3.7. Otherwise find $(\alpha', \beta') \in ESP$ such that $\alpha + \beta = \alpha' + \beta'$.

Substep 3.3. If $\beta - \alpha' \in \Phi$, then set

$$t_1 := \frac{\|\beta - \alpha'\|^2}{\|\beta\|^2} N_{\alpha',\beta-\alpha'} N_{\alpha,\beta'-\alpha}.$$

Otherwise set $t_1 := 0$.

Substep 3.4. If $\alpha - \alpha' \in \Phi$, then set

$$t_2 := \frac{\|\alpha - \alpha'\|^2}{\|\alpha\|^2} N_{\alpha',\alpha-\alpha'} N_{\beta,\beta'-\beta}.$$

Otherwise set $t_2 := 0$.

Substep 3.5. Determine the integer p such that $\beta - p\alpha \in \Phi$ and $\beta - (p+1)\alpha \notin \Phi$.

Substep 3.6. Return

$$N_{\alpha_i,\beta_i} \equiv N_{\alpha,\beta} := \mathrm{sgn}(N_{\alpha',\beta'}) \, \mathrm{sgn}(t_1 - t_2) \cdot (p+1)$$

and

$$N_{\beta_i,\alpha_i} \equiv N_{\beta,\alpha} := -N_{\alpha,\beta}.$$

Substep 3.7. Set $i := i + 1$.

Step 4. If $i \leq m$, then repeat Step 3. Otherwise end.

Output: the structure constants $N_{\alpha,\beta}$ for all $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi^+$.

Lemma 4.13. *Let Φ be a root system, let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be its base and let $\alpha, \beta \in \Phi^+$, where Φ^+ is taken with respect to Δ. If $\alpha \prec \beta$ and $\beta - \alpha \in \Phi$, then* $\mathrm{ht}\,\alpha < \mathrm{ht}\,\beta$.

Proof. According to the definition of the ordering of positive roots, $\mathrm{ht}\,\alpha \leq \mathrm{ht}\,\beta$. Suppose for contradiction that $\mathrm{ht}\,\alpha = \mathrm{ht}\,\beta$. We decompose α and β as $\alpha = \sum_{i=1}^{l} k_i \alpha_i$ and $\beta = \sum_{i=1}^{l} m_i \alpha_i$, respectively. Consequently, $\beta - \alpha = \sum_{i=1}^{l} (k_i - m_i)\alpha_i$. Since $\mathrm{ht}\,\alpha = \mathrm{ht}\,\beta$, then $\sum_{i=1}^{l} k_i = \sum_{i=1}^{l} m_i$. Further, because $\alpha \prec \beta$, there exists $i_0 \in \widehat{l}$ such that $k_{i_0} < m_{i_0}$. On the other hand, as heights of α and β are equal, there must exist $i_1 \in \widehat{l}$ such that $k_{i_1} > m_{i_1}$. Yet, according to the definition of a base for a root system, $\beta - \alpha$ cannot be a root, hence we reached a contradiction. ☐

Proposition 4.14. *Let Φ be a root system with a given base. Let $SP = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ be the set of special pairs of roots from Φ ordered such that for all $1 \leq i < j \leq m$ we have $\mathrm{ht}(\alpha_i + \beta_i) \leq \mathrm{ht}(\alpha_j + \beta_j)$ or, in case of equality, $\alpha_i \prec \alpha_j$. Let ESP be the set of extraspecial pairs of roots from Φ. Suppose that the structure constants $(N_{\alpha,\beta})_{(\alpha,\beta)\in ESP}$ are given. Then Algorithm 4 determines the structure constants $N_{\alpha,\beta}$ for $\alpha, \beta, \alpha + \beta \in \Phi^+$ correctly.*

Proof. First, notice that the algorithm leaves the structure constants $N_{\alpha,\beta}$ corresponding to extraspecial pairs (α, β) unchanged. Second, according to (4.4), Step 1 works properly. Now, take any special pair (α, β) which is not extraspecial and take the extraspecial pair (α', β') such that $\alpha + \beta = \alpha' + \beta'$. Then $\alpha + \beta - \alpha' - \beta' = 0$. Moreover, the following inequalities certainly hold: $\alpha \prec \beta$, $\alpha' \prec \beta'$ and $\alpha' \prec \alpha$. Consequently $\alpha' \prec \beta$ and hence no pair among α, β, α' and β' is antipodal and thus, according to (4.7), we have

$$\frac{N_{\alpha,\beta}N_{-\alpha',-\beta'}}{\|\alpha+\beta\|^2} + \frac{N_{\beta,-\alpha'}N_{\alpha,-\beta'}}{\|\beta-\alpha'\|^2} + \frac{N_{-\alpha',\alpha}N_{\beta,-\beta'}}{\|\alpha-\alpha'\|^2} = 0.$$

Using (4.10), the property of Chevalley basis, we have

$$N_{\alpha,\beta} = \frac{\|\alpha+\beta\|^2}{N_{\alpha',\beta'}}\left(\frac{N_{\beta,-\alpha'}N_{\alpha,-\beta'}}{\|\beta-\alpha'\|^2} + \frac{N_{-\alpha',\alpha}N_{\beta,-\beta'}}{\|\alpha-\alpha'\|^2}\right). \tag{4.18}$$

Assuming both $\beta - \alpha'$ and $\alpha - \alpha'$ are roots, then $\beta' - \alpha$ and $\beta' - \beta'$ are roots as well and we may use (4.5) and (4.4) to obtain

$$\begin{aligned}
N_{\alpha,\beta} &= \frac{\|\alpha+\beta\|^2}{N_{\alpha',\beta'}}\left(\frac{N_{-\alpha',-\beta+\alpha'}}{\|\beta\|^2}N_{\alpha,-\beta'} + \frac{N_{\alpha'-\alpha,-\alpha'}}{\|\alpha\|^2}N_{\beta,-\beta'}\right)\\
&= \frac{\|\alpha+\beta\|^2}{N_{\alpha',\beta'}}\left(\frac{N_{-\alpha',-\beta+\alpha'}}{\|\beta\|^2}\frac{\|\beta'-\alpha\|^2}{\|\beta'\|^2}N_{\beta'-\alpha,\alpha} + \frac{N_{\alpha'-\alpha,-\alpha'}}{\|\alpha\|^2}\frac{\|\beta'-\beta\|^2}{\|\beta'\|^2}N_{\beta'-\beta,\beta}\right).
\end{aligned}$$

Now, using properties (4.10) and (4.4) together with the fact that $\alpha + \beta = \alpha' + \beta'$, we may continue as

$$\begin{aligned}
N_{\alpha,\beta} &= \frac{\|\alpha+\beta\|^2}{N_{\alpha',\beta'}\|\beta'\|^2}\left(\frac{\|\beta-\alpha'\|^2}{\|\beta\|^2}N_{\alpha',\beta-\alpha'}N_{\alpha,\beta'-\alpha} - \frac{\|\alpha-\alpha'\|^2}{\|\alpha\|^2}N_{\alpha',\alpha-\alpha'}N_{\beta,\beta'-\beta}\right)\\
&= \frac{\|\alpha+\beta\|^2}{N_{\alpha',\beta'}\|\beta'\|^2}(t_1 - t_2).
\end{aligned}$$

If $\beta - \alpha' \notin \Phi$, then $N_{\beta,-\alpha'} = 0$ and therefore the first summand in the bracket on the right-hand side in (4.18) equals zero. Similarly, if $\alpha - \alpha' \notin \Phi$, then the same holds for the second one. All in all, the last relation holds generally, also if $\beta - \alpha' \notin \Phi$ or $\alpha - \alpha' \notin \Phi$. Now it is clear that $\operatorname{sgn}(N_{\alpha,\beta}) = \operatorname{sgn}(N_{\alpha',\beta'})\operatorname{sgn}(t_1 - t_2)$. Since the absolute value of $N_{\alpha,\beta}$ is known to be $p + 1$, where p is the largest integer such that $\beta - p\alpha \in \Phi$, correctness of the computed structure constants is verified.

The remaining question is whether the constants $N_{\alpha',\beta-\alpha'}$, $N_{\alpha,\beta'-\alpha}$, $N_{\alpha',\alpha-\alpha'}$, $N_{\beta,\beta'-\beta}$ are among those already known and they can be used for computation of $N_{\alpha,\beta}$. In other words, we must check that for each of these constants of the form $N_{\gamma,\delta}$ it holds true $\gamma, \delta, \gamma + \delta \in \Phi^+$ and $\operatorname{ht}(\gamma + \delta) \le \operatorname{ht}(\alpha + \beta)$ or $\gamma \prec \alpha$, in case of equality between heights. We show this in the first two cases (corresponding to the constant t_1), for the remaining two pairs (corresponding to t_2), the procedure is completely analogous. First, $\alpha', \alpha, \beta, \beta' \in \Phi^+$. Second, we assume that $\beta - \alpha', \beta' - \alpha \in \Phi$ and that $\alpha' \prec \beta$. Consequently, $\alpha = \alpha' + \beta' - \beta \prec \beta'$. According to Lemma 4.13, we may write

$$\begin{aligned}
\operatorname{ht}(\beta - \alpha') &= \operatorname{ht}(\beta) - \operatorname{ht}(\alpha') > 0,\\
\operatorname{ht}(\beta' - \alpha) &= \operatorname{ht}(\beta') - \operatorname{ht}(\alpha) > 0
\end{aligned}$$

and therefore both roots $\beta - \alpha'$ and $\beta' - \alpha$ are positive. Finally, we have

$$\operatorname{ht}(\alpha' + (\beta - \alpha')) = \operatorname{ht}\beta < \operatorname{ht}(\alpha + \beta),$$

$$\mathrm{ht}(\alpha + (\beta' - \alpha)) = \mathrm{ht}\,\beta' < \mathrm{ht}(\alpha' + \beta') = \mathrm{ht}(\alpha + \beta).$$

□

Now we are able to determine the structure constants $N_{\alpha,\beta}$ for an arbitrary pair of roots, using "ALGORITHM 2" from [5]. Remind that an incorrectness has occurred in the original algorithm, we specify it below.

Algorithm 5 (StructureConstantForAnyPair)**.**

Input: root system Φ, pair of roots (α, β) from Φ and the structure constants $N_{\gamma,\delta}$ for $\gamma, \delta, \gamma + \delta \in \Phi^+$.

Step 1. If $\alpha + \beta \notin \Phi$, then return $N_{\alpha,\beta} := 0$ and end. Otherwise set $m := 1$ and $(\gamma, \delta) := (\alpha, \beta)$.

Step 2. If $\delta \in \Phi^-$, then set $(\gamma, \delta) := (-\gamma, -\delta)$ and $m := -m$.

Step 3. If $\gamma \in \Phi^-$, then do the following:

Substep 3.1. If $-\gamma \prec \delta$, then set $(\gamma, \delta) := (\gamma + \delta, -\gamma)$ and $m := -m$[1]. Otherwise set $(\gamma, \delta) := (\delta, -\gamma - \delta)$.

Step 4. Determine the integer p such that $\beta - p\alpha \in \Phi$ and $\beta - (p+1)\alpha \notin \Phi$.

Step 5. Return $N_{\alpha,\beta} := m \cdot \mathrm{sgn}(N_{\gamma,\delta}) \cdot (p+1)$.

Output: the structure constant $N_{\alpha,\beta}$.

Proposition 4.15. *Let Φ be a root system and let $\alpha, \beta \in \Phi$. Then Algorithm 5 determines the structure constant $N_{\alpha,\beta}$ correctly.*

Proof. First, it is our convention to put $N_{\alpha,\beta} = 0$ when $\alpha + \beta \notin \Phi$. Second, suppose that $\alpha + \beta \in \Phi$. The absolute value of $N_{\alpha,\beta}$ is already prescribed and one can see from Steps 4 and 5 that it is computed correctly. It remains to verify the sign of $N_{\alpha,\beta}$. We keep the notation from the algorithm. Obviously, each change of m corresponds to the change of the sign of $N_{\gamma,\delta}$ within the algorithm. As $N_{\gamma,\delta} = N_{\alpha,\beta}$ in the beginning, the sign of $N_{\alpha,\beta}$ must equal to $m \cdot \mathrm{sgn}(N_{\gamma,\delta})$, where $N_{\gamma,\delta}$ is the final one. This agrees with Step 5. The question is whether the (final) constant $N_{\gamma,\delta}$ is among those $N_{\gamma,\delta}$ given in input of the algorithm.

Step 2 arranges the pair (γ, δ) in such a way that $\delta \in \Phi^+$, however by (4.10) we have $N_{\gamma,\delta} = -N_{-\gamma,-\delta}$, the sign changes here and hence we must set $m := -m$. Further, we claim that we have $\gamma, \delta \in \Phi^+$ after Step 3. Indeed. Suppose that $\gamma \in \Phi^-$ before this step, otherwise we are done. First, if $-\gamma \prec \delta$, then $-\gamma$ is a positive root obviously and, according to Lemma 4.13, $\gamma + \delta = \delta - (-\gamma) \in \Phi^+$ as well. For the sign alteration, (4.5) and (4.10) imply $\mathrm{sgn}(N_{\gamma,\delta}) = \mathrm{sgn}(N_{-\gamma-\delta,\gamma}) = -\mathrm{sgn}(N_{\gamma+\delta,-\gamma})$ and thus we must set $m := -m$. Second, if $-\gamma \succ \delta$, $\delta \in \Phi^+$ thanks to Step 2 and $-\gamma - \delta \in \Phi^+$ is a consequence of Lemma 4.13. In this case m stays unchanged since we have $\mathrm{sgn}(N_{\gamma,\delta}) = \mathrm{sgn}(N_{\delta,-\gamma-\delta})$ by (4.5). □

Remark 4.10. There is one more step in the original algorithm in [5] in comparison to Algorithm 5. Namely, it is the command putting the pair (γ, δ) of positive roots such that $\gamma + \delta$ is a positive root as well into the right order to be the special one. We could omit this step since we have computed the structure constants $N_{\alpha,\beta}$ for all $\alpha, \beta, \alpha + \beta \in \Phi^+$ by Algorithm 4.

[1] At this place the incorrectness has occurred in the original paper [5]. In ALGORITHM 1 on page 1482 of that paper the command $m := -m$ is missing in this step.

Conclusion

In this thesis we have focused on the construction of the Chevalley basis for a simple Lie algebra, the first step in the construction of various bases for representations of simple Lie algebras.

In the first three chapters we have summarized the fundamentals of Lie algebras theory needed for the construction. Especially, we have defined semisimple and simple Lie algebras and we have presented their complete classification.

In the first part of the fourth chapter we have introduced the Chevalley basis for a semisimple Lie algebra. We have established the set of basis vectors and we have uncovered the structure constants with respect to this basis. Furthermore, we have shown that such the basis existed for each algebra. In the second part of the final chapter we have introduced the series of algorithms for practical computation of the Chevalley basis and the respective structure constants. Most of the structure constants were already prescribed by the definition of the Chevalley basis, in fact many of them were trivial. The only problem was to discover the signs for certain subset of the structure constants (we have denoted these constants by $N_{\alpha,\beta}$ in the main text). We have shown which of these signs could be chosen arbitrarily and that the others were then uniquely determined. We have chosen the optional signs to be all equal to +1. Determination of the remaining signs was the pivotal step of the construction. For this task, we have used two algorithms introduced originally in paper [5]. However we have found an incorrectness in each of them. We have specified the mistakes and we have proposed our algorithms avoiding this inaccuracy.

Subsequently, the algorithms presented in the fourth chapter were implemented into several Maple 16 procedures providing together the program for computation of the Chevalley basis for a simple Lie of an arbitrary type (for both classical and exceptional simple Lie algebras). More precisely, our program writes out all the non-zero commutators of basis vectors from the Chevalley basis. Additionally, the option of writing out the list of vectors contained in the Chevalley basis is available as well. The whole Maple source code is attached in the first appendix. As an example, we have presented the computed Chevalley bases for all simple Lie algebras up to the rank 4 in the second appendix.

As mentioned, the construction of a basis for a simple Lie algebra is the first task in the construction of bases for representations. In our following work (research work, diploma thesis) we are going to use the results presented in this thesis to study the problems concerning the construction of bases for representations of simple Lie algebras as well as the comparison of the different kinds of bases.

Appendix A

Maple 16 Source Code

In the first appendix, we present several Maple 16 procedures based on the algorithms introduced in Chapter 4 which together provide a program for computing of Chevalley basis for a simple Lie algebra of an arbitrary type. Before stating the proper code, we give a brief description of the particular procedures.

A.1 Specification

First of all, one has to call packages DifferentialGeometry and LieAlgebras to keep the Cartan matrices and positive roots at disposition.

The procedure RootSystem produces the table of n-tuples of coefficients of all roots in root system of type T_n with respect to the standard basis (established by the labeling of Dynkin diagrams in Theorem 3.55). This is based on the final step of Algorithm 1. The preceding steps of that algorithm are already implemented in PositiveRoots Maple procedure as discussed in Subsection 4.2.1.

The procedure sqNorm calculates the square of norm of an arbitrary root r from the squares of norms of the simple roots ($SsqN$).

The next procedure, MaxPab takes two roots a, b and determines the largest integer p such that $b - p \cdot a$ is a root.

The procedure StructureConstants computes the table of structure constants with respect to a Chevalley basis in a simple Lie algebra of type T_n. This procedure is based on algorithms and methods introduced in Subsections 4.2.2, 4.2.3 and 4.2.4 completely. Notice that not all the structure constants are computed; we determine only such constants that are needed in the pivotal procedure WriteCommutators. Within the procedure StructureConstants, the choice of the signs of the structure constants $N_{\alpha,\beta}$ has to be made for the extraspecial pairs (α, β) (cf. Proposition 4.12). We chose these signs to be all $+1$.

Further, the procedure Direction gives the basis vectors occurring non-trivially in the decomposition of vector $[a, b]$ into the Chevalley basis. Then, the linear combination of these vectors with the respective structure constants as coefficients is the required commutator $[a, b]$.

The semifinal procedure WriteBasis writes out the set of vectors forming a Chevalley basis in a simple Lie algebra of type T_n.

The last procedure WriteCommutators is the pivotal one. After compiling this and all the previous procedures (except WriteBasis, that is not needed), the command

```
WriteCommutators(T,n)
```

writes out all the non-zero commutators of all pairs of basis vectors from the Chevalley basis (with the signs of $N_{\alpha,\beta}$ chosen for the extraspecial pairs (α, β) as above) for a simple Lie algebra of type T_n.

A.2 Proper Code

```
> with(DifferentialGeometry): with(LieAlgebras):

> RootSystem:=proc(T,n)::table;
  local CM,PR,i;
  global RS;
  if T=A then CM:=CartanMatrix("A",n);
  elif T=B then CM:=CartanMatrix("B",n);
  elif T=C then CM:=CartanMatrix("C",n);
  elif T=D then CM:=CartanMatrix("D",n);
  elif T=E then CM:=CartanMatrix("E",n,version ="I");
  elif T=F then CM:=CartanMatrix("F",n);
  elif T=G then CM:=CartanMatrix("G",n);
  end if;
  PR:=PositiveRoots(CM);
  RS:=table( );
  for i from 1 to numelems(PR) do
     RS[i]:=[seq(PR[i][j],j=1..n)];
  end do;
  for i from 1 to numelems(PR) do
     RS[i+numelems(PR)]:=[seq(-PR[i][j],j=1..n)];
  end do;
  RS;
  end proc:

> sqNorm:=proc(r::list,SsqN::table,CM::Matrix)
  local i,j,n;
  global N;
  n:=numelems([entries(SsqN)]);
  N:=0;
  for i from 1 to n do
     for j from 1 to n do
        N:=N+(1/2)*r[i]*r[j]*CM[i][j]*SsqN[j];
     end do;
  end do;
  end proc:

> MaxPab:=proc(RS::table,a::list,b::list)
  local j;
  global p;
  p:=0;
  j:=1;
  while p<j do
     if -j*a+b in [entries(RS,'nolist')] then
        p:=j;
        j:=j+1;
     else j:=j-1;
     end if;
  end do;
  p;
  end proc:

> StructureConstants:=proc(T,n)::table;
  local CM,RS,i,j,k,l,m,count1,SsqN,SP,ESP,Nab,t1,t2,N,a,b,c;
  global SC;
  if T=A then CM:=CartanMatrix("A",n);
  elif T=B then CM:=CartanMatrix("B",n);
  elif T=C then CM:=CartanMatrix("C",n);
  elif T=D then CM:=CartanMatrix("D",n);
  elif T=E then CM:=CartanMatrix("E",n,version="I");
  elif T=F then CM:=CartanMatrix("F",n);
  elif T=G then CM:=CartanMatrix("G",n);
```

```
end if;
count1:=0;
RS:=table([ ]);
RS:=RootSystem(T,n);
N:=(1/2)*numelems(RS);

SsqN:=table([ ]);
if T=B then
    for i from 1 to n-1 do SsqN[i]:=2 end do;
    SsqN[n]:=1;
elif T=C then
    for i from 1 to n-1 do SsqN[i]:=2 end do;
    SsqN[n]:=4;
elif T=F then
    SsqN[1]:=2;
    SsqN[2]:=2;
    SsqN[3]:=1;
    SsqN[4]:=1;
elif T=G then
    SsqN[1]:=2;
    SsqN[2]:=6;
else for i from 1 to n do SsqN[i]:=2 end do;
end if;

ESP:=table([ ]);
for i from 1 to N do
    for j from i+1 to N do
        if RS[i]+RS[j] in [entries(RS,'nolist')] then
            if not RS[i]+RS[j] in [indices(ESP,'nolist')] then
                ESP[RS[i]+RS[j]]:=[RS[i],RS[j]];
            end if;
        end if;
    end do;
end do;

SP:=table([ ]);
l:=1;
for i from 1 to N do
    for j from 1 to i do
        for k from j to i do
            if RS[j]+RS[k]=RS[i] then
                SP[l]:=[RS[j],RS[k]];
                l:=l+1;
            end if;
        end do;
    end do;
end do;

Nab:=table([ ]);
for i from 1 to N do
    for j from i+1 to N do
        if [RS[i],RS[j]] in [entries(ESP,'nolist')] then
            Nab[RS[i],RS[j]]:=MaxPab(RS,RS[i],RS[j])+1;
        end if;
    end do;
end do;
for i from 1 to N do
    for j from i+1 to N do
        if [RS[i],RS[j]] in [entries(ESP,'nolist')] then
            Nab[RS[j],RS[i]]:=-Nab[RS[i],RS[j]];
        end if;
    end do;
end do;
```

```
for i from 1 to numelems([entries(SP)]) do
    if not [op(SP[i])] in [entries(ESP,'nolist')] then
        if SP[i][2]-ESP[SP[i][1]+SP[i][2]][1] in [entries(RS,'nolist')] then
            t1:=(sqNorm(SP[i][2]-ESP[SP[i][1]+SP[i][2]][1],SsqN,CM))/(sqNorm(SP[i][2],
                 SsqN,CM))*Nab[ESP[SP[i][1]+SP[i][2]][1],SP[i][2]-ESP[SP[i][1]+SP[i
                 ][2]][1]]*Nab[SP[i][1],ESP[SP[i][1]+SP[i][2]][2]-SP[i][1]]
        else t1:=0;
        end if;
        if SP[i][1]-ESP[SP[i][1]+SP[i][2]][1] in [entries(RS,'nolist')] then
            t2:=(sqNorm(SP[i][1]-ESP[SP[i][1]+SP[i][2]][1],SsqN,CM))/(sqNorm(SP[i][1],
                 SsqN,CM))*Nab[ESP[SP[i][1]+SP[i][2]][1],SP[i][1]-ESP[SP[i][1]+SP[i
                 ][2]][1]]*Nab[SP[i][2],ESP[SP[i][1]+SP[i][2]][2]-SP[i][2]]
        else t2:=0;
        end if;
        Nab[op(SP[i])]:=sign(t1-t2)*sign(Nab[op(ESP[SP[i][1]+SP[i][2]])])*(MaxPab(RS,op
             (SP[i]))+1);
        Nab[op(SP[i])[2], op(SP[i])[1]]:=-Nab[op(SP[i])];
    end if;
end do;

SC:=table([ ]);
for i from 1 to n do
    for j from 1 to 2*N do
        for k from 1 to n do
            count1:=count1+RS[j][k]*CM[k][i];
        end do;
        SC[[h,RS[i]],[e,RS[j]]]:=[count1];
        count1:=0;
    end do;
end do;
for i from 1 to 2*N do
    for j from i+1 to 2*N do
        if RS[i]=-RS[j] then
            SC[[e,RS[i]],[e,RS[j]]]:=[ ];
            for k from 1 to n do
                SC[[e,RS[i]],[e,RS[j]]]:=[op(SC[[e,RS[i]],[e,RS[j]]]),RS[i][k]*(SsqN[k])
                     /(sqNorm(RS[i],SsqN,CM))];
            end do;
        elif not RS[i]+RS[j] in [entries(RS,'nolist')] then
            SC[[e,RS[i]],[e,RS[j]]]:=[0];
        elif[RS[i],RS[j]] in [entries(SP,'nolist')] then
            SC[[e,RS[i]],[e,RS[j]]]:=[Nab[RS[i],RS[j]]];
        else
            m:=1;
            k:=i;
            l:=j;
            a:=RS[k];
            b:=RS[l];
            if l>N then
                l:=l-N;
                m:=-m;
                if k<=N then k:=k+N;
                else k:=k-N;
                end if;
                a:=RS[k];
                b:=RS[l];
            end if;
            if k>N then
                if k-N<l then
                    c:=-RS[k];
                    a:=RS[k]+RS[l];
                    b:=c;
                    m:=-m;
```

```
                else
                    c:=-RS[k]-RS[l];
                    a:=RS[l];
                    b:=c;
                end if;
            end if;
            SC[[e,RS[i]],[e,RS[j]]]:=[m*sign(Nab[a,b])*(MaxPab(RS,RS[i],RS[j])+1)];
        end if;
    end do;
end do;

SC;
end proc:
```

```
> Direction:=proc(a::list,b::list,RS::table,n)::list;
global c;
if a[1]=b[1]=e then
    if a[2]=-b[2] then
        c:=[seq([h,RS[j]],j=1..n)];
    elif a[2]+b[2] in [entries(RS,'nolist')] then
        c:=[[e,a[2]+b[2]]];
    end if;
elif a[1]=h and b[1]=e then
    c:=[b];
end if;
end proc:
```

```
> WriteBasis:=proc(T,n)
local i,RS;
RS:=RootSystem(T,n);
for i from 1 to n do
    print(h(op(RS[i])));
end do;
for i from 1 to numelems([entries(RS)]) do
    print(e(op(RS[i])));
end do;
end proc:
```

```
> WriteCommutators:=proc(T,n)
local i,j,k,SC,RS,count;
SC:=StructureConstants(T,n);
RS:=RootSystem(T,n);
for i from 1 to n do
    for j from 1 to numelems(RS) do
        if not SC[[h,RS[i]],[e,RS[j]]]=[0] then
            print([h(op(RS[i])),e(op(RS[j]))]=SC[[h,RS[i]],[e,RS[j]]][1]*Direction([h,
                RS[i]],[e,RS[j]],RS,n)[1][1](op(Direction([h,RS[i]],[e,RS[j]],RS,n)
                [1][2])));
        end if;
    end do;
end do;
for i from 1 to numelems(RS) do
    for j from i+1 to numelems(RS) do
        if not SC[[e,RS[i]],[e,RS[j]]]=[0] then
            count:=0;
            for k from 1 to numelems(SC[[e,RS[i]],[e,RS[j]]]) do
                count:=count+SC[[e,RS[i]],[e,RS[j]]][k]*Direction([e,RS[i]],[e,RS[j]],RS
                    ,n)[k][1](op(Direction([e,RS[i]],[e,RS[j]],RS,n)[k][2]));
            end do;
            print([e(op(RS[i])),e(op(RS[j]))]=count);
        end if;
    end do;
end do;
end proc:
```

Appendix B

Examples of Computed Bases

In the second appendix, we present Chevalley bases for all simple Lie algebras up to the rank 4. The bases were calculated exactly in the way introduced in Appendix A.

A_1

```
[h(1),e(1)]=2e(1)
[h(1),e(-1)]=-2e(-1)
[e(1),e(-1)]=h(1)
```

A_2

```
[h(1,0),e(1,0)]=2e(1,0)
[h(1,0),e(0,1)]=-e(0,1)
[h(1,0),e(1,1)]=e(1,1)
[h(1,0),e(-1,0)]=-2e(-1,0)
[h(1,0),e(0,-1)]=e(0,-1)
[h(1,0),e(-1,-1)]=-e(-1,-1)
[h(0,1),e(1,0)]=-e(1,0)
[h(0,1),e(0,1)]=2e(0,1)
[h(0,1),e(1,1)]=e(1,1)
[h(0,1),e(-1,0)]=e(-1,0)
[h(0,1),e(0,-1)]=-2e(0,-1)
[h(0,1),e(-1,-1)]=-e(-1,-1)
[e(1,0),e(0,1)]=e(1,1)
[e(1,0),e(-1,0)]=h(1,0)
[e(1,0),e(-1,-1)]=-e(0,-1)
[e(0,1),e(0,-1)]=h(0,1)
[e(0,1),e(-1,-1)]=e(-1,0)
[e(1,1),e(-1,0)]=-e(0,1)
[e(1,1),e(0,-1)]=e(1,0)
[e(1,1),e(-1,-1)]=h(1,0)+h(0,1)
[e(-1,0),e(0,-1)]=-e(-1,-1)
```

B_2

```
[h(1,0),e(1,0)]=2e(1,0)
[h(1,0),e(0,1)]=-e(0,1)
[h(1,0),e(1,1)]=e(1,1)
[h(1,0),e(-1,0)]=-2e(-1,0)
[h(1,0),e(0,-1)]=e(0,-1)
[h(1,0),e(-1,-1)]=-e(-1,-1)
[h(0,1),e(1,0)]=-2e(1,0)
[h(0,1),e(0,1)]=2e(0,1)
[h(0,1),e(1,2)]=2e(1,2)
[h(0,1),e(-1,0)]=2e(-1,0)
[h(0,1),e(0,-1)]=-2e(0,-1)
[h(0,1),e(-1,-2)]=-2e(-1,-2)
[e(1,0),e(0,1)]=e(1,1)
[e(1,0),e(-1,0)]=h(1,0)
[e(1,0),e(-1,-1)]=-e(0,-1)
[e(0,1),e(1,1)]=2e(1,2)
[e(0,1),e(0,-1)]=h(0,1)
[e(0,1),e(-1,-1)]=2e(-1,0)
[e(0,1),e(-1,-2)]=-e(-1,-1)
[e(1,1),e(-1,0)]=-e(0,1)
[e(1,1),e(0,-1)]=2e(1,0)
[e(1,1),e(-1,-1)]=2h(1,0)+h(0,1)
[e(1,1),e(-1,-2)]=e(0,-1)
[e(1,2),e(0,-1)]=-e(1,1)
[e(1,2),e(-1,-1)]=e(0,1)
[e(1,2),e(-1,-2)]=h(1,0)+h(0,1)
[e(-1,0),e(0,-1)]=-e(-1,-1)
[e(0,-1),e(-1,-1)]=-2e(-1,-2)
```

G_2

```
[h(1,0),e(1,0)]=2e(1,0)
[h(1,0),e(0,1)]=-3e(0,1)
[h(1,0),e(1,1)]=-e(1,1)
[h(1,0),e(2,1)]=e(2,1)
[h(1,0),e(3,1)]=3e(3,1)
[h(1,0),e(-1,0)]=-2e(-1,0)
[h(1,0),e(0,-1)]=3e(0,-1)
[h(1,0),e(-1,-1)]=e(-1,-1)
[h(1,0),e(-2,-1)]=-e(-2,-1)
[h(1,0),e(-3,-1)]=-3e(-3,-1)
[h(0,1),e(1,0)]=-e(1,0)
[h(0,1),e(0,1)]=2e(0,1)
[h(0,1),e(1,1)]=e(1,1)
[h(0,1),e(3,1)]=-e(3,1)
[h(0,1),e(3,2)]=e(3,2)
[h(0,1),e(-1,0)]=e(-1,0)
[h(0,1),e(0,-1)]=-2e(0,-1)
[h(0,1),e(-1,-1)]=-e(-1,-1)
[h(0,1),e(-3,-1)]=e(-3,-1)
[h(0,1),e(-3,-2)]=-e(-3,-2)
[e(1,0),e(0,1)]=e(1,1)
[e(1,0),e(1,1)]=2e(2,1)
[e(1,0),e(2,1)]=3e(3,1)
[e(1,0),e(-1,0)]=h(1,0)
[e(1,0),e(-1,-1)]=-3e(0,-1)
[e(1,0),e(-2,-1)]=-2e(-1,-1)
[e(1,0),e(-3,-1)]=-e(-2,-1)
[e(0,1),e(3,1)]=e(3,2)
[e(0,1),e(0,-1)]=h(0,1)
[e(0,1),e(-1,-1)]=e(-1,0)
[e(0,1),e(-3,-2)]=-e(-3,-1)
[e(1,1),e(2,1)]=-3e(3,2)
[e(1,1),e(-1,0)]=-3e(0,1)
[e(1,1),e(0,-1)]=e(1,0)
[e(1,1),e(-1,-1)]=h(1,0)+3h(0,1)
[e(1,1),e(-2,-1)]=2e(-1,0)
[e(1,1),e(-3,-2)]=e(-2,-1)
```

```
[e(2,1),e(-1,0)]=-2e(1,1)
[e(2,1),e(-1,-1)]=2e(1,0)
[e(2,1),e(-2,-1)]=2h(1,0)+3h(0,1)
[e(2,1),e(-3,-1)]=e(-1,0)
[e(2,1),e(-3,-2)]=-e(-1,-1)
[e(3,1),e(-1,0)]=-e(2,1)
[e(3,1),e(-2,-1)]=e(1,0)
[e(3,1),e(-3,-1)]=h(1,0)+h(0,1)
[e(3,1),e(-3,-2)]=e(0,-1)
[e(3,2),e(0,-1)]=-e(3,1)
[e(3,2),e(-1,-1)]=e(2,1)
[e(3,2),e(-2,-1)]=-e(1,1)
[e(3,2),e(-3,-1)]=e(0,1)
[e(3,2),e(-3,-2)]=h(1,0)+2h(0,1)
[e(-1,0),e(0,-1)]=-e(-1,-1)
[e(-1,0),e(-1,-1)]=-2e(-2,-1)
[e(-1,0),e(-2,-1)]=-3e(-3,-1)
[e(0,-1),e(-3,-1)]=-e(-3,-2)
[e(-1,-1),e(-2,-1)]=3e(-3,-2)
```

A_3

```
[h(1,0,0),e(1,0,0)]=2e(1,0,0)
[h(1,0,0),e(0,1,0)]=-e(0,1,0)
[h(1,0,0),e(1,1,0)]=e(1,1,0)
[h(1,0,0),e(0,1,1)]=-e(0,1,1)
[h(1,0,0),e(1,1,1)]=e(1,1,1)
[h(1,0,0),e(-1,0,0)]=-2e(-1,0,0)
[h(1,0,0),e(0,-1,0)]=e(0,-1,0)
[h(1,0,0),e(-1,-1,0)]=-e(-1,-1,0)
[h(1,0,0),e(0,-1,-1)]=e(0,-1,-1)
[h(1,0,0),e(-1,-1,-1)]=-e(-1,-1,-1)
[h(0,1,0),e(1,0,0)]=-e(1,0,0)
[h(0,1,0),e(0,1,0)]=2e(0,1,0)
[h(0,1,0),e(0,0,1)]=-e(0,0,1)
[h(0,1,0),e(1,1,0)]=e(1,1,0)
[h(0,1,0),e(0,1,1)]=e(0,1,1)
[h(0,1,0),e(-1,0,0)]=e(-1,0,0)
[h(0,1,0),e(0,-1,0)]=-2e(0,-1,0)
[h(0,1,0),e(0,0,-1)]=e(0,0,-1)
[h(0,1,0),e(-1,-1,0)]=-e(-1,-1,0)
[h(0,1,0),e(0,-1,-1)]=-e(0,-1,-1)
[h(0,0,1),e(0,1,0)]=-e(0,1,0)
[h(0,0,1),e(0,0,1)]=2e(0,0,1)
[h(0,0,1),e(1,1,0)]=-e(1,1,0)
[h(0,0,1),e(0,1,1)]=e(0,1,1)
[h(0,0,1),e(1,1,1)]=e(1,1,1)
[h(0,0,1),e(0,-1,0)]=e(0,-1,0)
[h(0,0,1),e(0,0,-1)]=-2e(0,0,-1)
[h(0,0,1),e(-1,-1,0)]=e(-1,-1,0)
[h(0,0,1),e(0,-1,-1)]=-e(0,-1,-1)
[h(0,0,1),e(-1,-1,-1)]=-e(-1,-1,-1)
[e(1,0,0),e(0,1,0)]=e(1,1,0)
[e(1,0,0),e(0,1,1)]=e(1,1,1)
[e(1,0,0),e(-1,0,0)]=h(1,0,0)
[e(1,0,0),e(-1,-1,0)]=-e(0,-1,0)
[e(1,0,0),e(-1,-1,-1)]=-e(0,-1,-1)
[e(0,1,0),e(0,0,1)]=e(0,1,1)
[e(0,1,0),e(0,-1,0)]=h(0,1,0)
[e(0,1,0),e(-1,-1,0)]=e(-1,0,0)
[e(0,1,0),e(0,-1,-1)]=-e(0,0,-1)
[e(0,0,1),e(1,1,0)]=-e(1,1,1)
[e(0,0,1),e(0,0,-1)]=h(0,0,1)
[e(0,0,1),e(0,-1,-1)]=e(0,-1,0)
[e(0,0,1),e(-1,-1,-1)]=e(-1,-1,0)
[e(1,1,0),e(-1,0,0)]=-e(0,1,0)
[e(1,1,0),e(0,-1,0)]=e(1,0,0)
[e(1,1,0),e(-1,-1,0)]=h(1,0,0)+h(0,1,0)
[e(1,1,0),e(-1,-1,-1)]=-e(0,0,-1)
[e(0,1,1),e(0,-1,0)]=-e(0,0,1)
[e(0,1,1),e(0,0,-1)]=e(0,1,0)
[e(0,1,1),e(0,-1,-1)]=h(0,1,0)+h(0,0,1)
[e(0,1,1),e(-1,-1,-1)]=e(-1,0,0)
[e(1,1,1),e(-1,0,0)]=-e(0,1,1)
[e(1,1,1),e(0,0,-1)]=e(1,1,0)
[e(1,1,1),e(-1,-1,0)]=-e(0,0,1)
[e(1,1,1),e(0,-1,-1)]=e(1,0,0)
[e(1,1,1),e(-1,-1,-1)]=
=h(1,0,0)+h(0,1,0)+h(0,0,1)
[e(-1,0,0),e(0,-1,0)]=-e(-1,-1,0)
[e(-1,0,0),e(0,-1,-1)]=-e(-1,-1,-1)
[e(0,-1,0),e(0,0,-1)]=-e(0,-1,-1)
[e(0,0,-1),e(-1,-1,0)]=e(-1,-1,-1)
```

B_3

```
[h(1,0,0),e(1,0,0)]=2e(1,0,0)
[h(1,0,0),e(0,1,0)]=-e(0,1,0)
[h(1,0,0),e(1,1,0)]=e(1,1,0)
[h(1,0,0),e(0,1,1)]=-e(0,1,1)
[h(1,0,0),e(1,1,1)]=e(1,1,1)
[h(1,0,0),e(0,1,2)]=-e(0,1,2)
[h(1,0,0),e(1,1,2)]=e(1,1,2)
[h(1,0,0),e(-1,0,0)]=-2e(-1,0,0)
[h(1,0,0),e(0,-1,0)]=e(0,-1,0)
[h(1,0,0),e(-1,-1,0)]=-e(-1,-1,0)
[h(1,0,0),e(0,-1,-1)]=e(0,-1,-1)
[h(1,0,0),e(-1,-1,-1)]=-e(-1,-1,-1)
[h(1,0,0),e(0,-1,-2)]=e(0,-1,-2)
[h(1,0,0),e(-1,-1,-2)]=-e(-1,-1,-2)
[h(0,1,0),e(1,0,0)]=-e(1,0,0)
[h(0,1,0),e(0,1,0)]=2e(0,1,0)
[h(0,1,0),e(0,0,1)]=-e(0,0,1)
[h(0,1,0),e(1,1,0)]=e(1,1,0)
[h(0,1,0),e(0,1,1)]=e(0,1,1)
[h(0,1,0),e(1,1,2)]=-e(1,1,2)
[h(0,1,0),e(1,2,2)]=e(1,2,2)
[h(0,1,0),e(-1,0,0)]=e(-1,0,0)
[h(0,1,0),e(0,-1,0)]=-2e(0,-1,0)
[h(0,1,0),e(0,0,-1)]=e(0,0,-1)
[h(0,1,0),e(-1,-1,0)]=-e(-1,-1,0)
[h(0,1,0),e(0,-1,-1)]=-e(0,-1,-1)
[h(0,1,0),e(-1,-1,-2)]=e(-1,-1,-2)
[h(0,1,0),e(-1,-2,-2)]=-e(-1,-2,-2)
[h(0,0,1),e(0,1,0)]=-2e(0,1,0)
[h(0,0,1),e(0,0,1)]=2e(0,0,1)
[h(0,0,1),e(1,1,0)]=-2e(1,1,0)
[h(0,0,1),e(0,1,2)]=2e(0,1,2)
[h(0,0,1),e(1,1,2)]=2e(1,1,2)
[h(0,0,1),e(0,-1,0)]=2e(0,-1,0)
[h(0,0,1),e(0,0,-1)]=-2e(0,0,-1)
[h(0,0,1),e(-1,-1,0)]=2e(-1,-1,0)
[h(0,0,1),e(0,-1,-2)]=-2e(0,-1,-2)
[h(0,0,1),e(-1,-1,-2)]=-2e(-1,-1,-2)
[e(1,0,0),e(0,1,0)]=e(1,1,0)
[e(1,0,0),e(0,1,1)]=e(1,1,1)
[e(1,0,0),e(0,1,2)]=e(1,1,2)
[e(1,0,0),e(-1,0,0)]=h(1,0,0)
[e(1,0,0),e(-1,-1,0)]=-e(0,-1,0)
[e(1,0,0),e(-1,-1,-1)]=-e(0,-1,-1)
[e(1,0,0),e(-1,-1,-2)]=-e(0,-1,-2)
[e(0,1,0),e(0,0,1)]=e(0,1,1)
[e(0,1,0),e(1,1,2)]=e(1,2,2)
[e(0,1,0),e(0,-1,0)]=h(0,1,0)
[e(0,1,0),e(-1,-1,0)]=e(-1,0,0)
[e(0,1,0),e(0,-1,-1)]=-e(0,0,-1)
[e(0,1,0),e(-1,-2,-2)]=-e(-1,-1,-2)
[e(0,0,1),e(1,1,0)]=-e(1,1,1)
[e(0,0,1),e(0,1,1)]=2e(0,1,2)
[e(0,0,1),e(1,1,1)]=2e(1,1,2)
[e(0,0,1),e(0,0,-1)]=h(0,0,1)
[e(0,0,1),e(0,-1,-1)]=2e(0,-1,0)
[e(0,0,1),e(-1,-1,-1)]=2e(-1,-1,0)
[e(0,0,1),e(0,-1,-2)]=-e(0,-1,-1)
[e(0,0,1),e(-1,-1,-2)]=-e(-1,-1,-1)
[e(1,1,0),e(0,1,2)]=-e(1,2,2)
```

```
[e(1,1,0),e(-1,0,0)]=-e(0,1,0)
[e(1,1,0),e(0,-1,0)]=e(1,0,0)
[e(1,1,0),e(-1,-1,0)]=h(1,0,0)+h(0,1,0)
[e(1,1,0),e(-1,-1,-1)]=-e(0,0,-1)
[e(1,1,0),e(-1,-2,-2)]=e(0,-1,-2)
[e(0,1,1),e(1,1,1)]=2e(1,2,2)
[e(0,1,1),e(0,-1,0)]=-e(0,0,1)
[e(0,1,1),e(0,0,-1)]=2e(0,1,0)
[e(0,1,1),e(0,-1,-1)]=2h(0,1,0)+h(0,0,1)
[e(0,1,1),e(-1,-1,-1)]=2e(-1,0,0)
[e(0,1,1),e(0,-1,-2)]=e(0,0,-1)
[e(0,1,1),e(-1,-2,-2)]=-e(-1,-1,-1)
[e(1,1,1),e(-1,0,0)]=-e(0,1,1)
[e(1,1,1),e(0,0,-1)]=2e(1,1,0)
[e(1,1,1),e(-1,-1,0)]=-e(0,0,1)
[e(1,1,1),e(0,-1,-1)]=2e(1,0,0)
[e(1,1,1),e(-1,-1,-1)]=
=2h(1,0,0)+2h(0,1,0)+h(0,0,1)
[e(1,1,1),e(-1,-1,-2)]=e(0,0,-1)
[e(1,1,1),e(-1,-2,-2)]=e(0,-1,-1)
[e(0,1,2),e(0,0,-1)]=-e(0,1,1)
[e(0,1,2),e(0,-1,-1)]=e(0,0,1)
[e(0,1,2),e(0,-1,-2)]=h(0,1,0)+h(0,0,1)
[e(0,1,2),e(-1,-1,-2)]=e(-1,0,0)
[e(0,1,2),e(-1,-2,-2)]=-e(-1,-1,0)
[e(1,1,2),e(-1,0,0)]=-e(0,1,2)
[e(1,1,2),e(0,0,-1)]=-e(1,1,1)
[e(1,1,2),e(-1,-1,-1)]=e(0,0,1)
[e(1,1,2),e(0,-1,-2)]=e(1,0,0)
[e(1,1,2),e(-1,-1,-2)]=
=h(1,0,0)+h(0,1,0)+h(0,0,1)
[e(1,1,2),e(-1,-2,-2)]=e(0,-1,0)
[e(1,2,2),e(0,-1,0)]=-e(1,1,2)
[e(1,2,2),e(-1,-1,0)]=e(0,1,2)
[e(1,2,2),e(0,-1,-1)]=-e(1,1,1)
[e(1,2,2),e(-1,-1,-1)]=e(0,1,1)
[e(1,2,2),e(0,-1,-2)]=-e(1,1,0)
[e(1,2,2),e(-1,-1,-2)]=e(0,1,0)
[e(1,2,2),e(-1,-2,-2)]=
=h(1,0,0)+2h(0,1,0)+h(0,0,1)
[e(-1,0,0),e(0,-1,0)]=-e(-1,-1,0)
[e(-1,0,0),e(0,-1,-1)]=-e(-1,-1,-1)
[e(-1,0,0),e(0,-1,-2)]=-e(-1,-1,-2)
[e(0,-1,0),e(0,0,-1)]=-e(0,-1,-1)
[e(0,-1,0),e(-1,-1,-2)]=-e(-1,-2,-2)
[e(0,0,-1),e(-1,-1,0)]=e(-1,-1,-1)
[e(0,0,-1),e(0,-1,-1)]=-2e(0,-1,-2)
[e(0,0,-1),e(-1,-1,-1)]=-2e(-1,-1,-2)
[e(-1,-1,0),e(0,-1,-2)]=e(-1,-2,-2)
[e(0,-1,-1),e(-1,-1,-1)]=-2e(-1,-2,-2)
```

C_3

```
[h(1,0,0),e(1,0,0)]=2e(1,0,0)
[h(1,0,0),e(0,1,0)]=-e(0,1,0)
[h(1,0,0),e(1,1,0)]=e(1,1,0)
[h(1,0,0),e(0,1,1)]=-e(0,1,1)
[h(1,0,0),e(1,1,1)]=e(1,1,1)
[h(1,0,0),e(0,2,1)]=-2e(0,2,1)
[h(1,0,0),e(2,2,1)]=2e(2,2,1)
[h(1,0,0),e(-1,0,0)]=-2e(-1,0,0)
[h(1,0,0),e(0,-1,0)]=e(0,-1,0)
[h(1,0,0),e(-1,-1,0)]=-e(-1,-1,0)
[h(1,0,0),e(0,-1,-1)]=e(0,-1,-1)
[h(1,0,0),e(-1,-1,-1)]=-e(-1,-1,-1)
[h(1,0,0),e(0,-2,-1)]=2e(0,-2,-1)
[h(1,0,0),e(-2,-2,-1)]=-2e(-2,-2,-1)
[h(0,1,0),e(1,0,0)]=-e(1,0,0)
[h(0,1,0),e(0,1,0)]=2e(0,1,0)
[h(0,1,0),e(0,0,1)]=-2e(0,0,1)
[h(0,1,0),e(1,1,0)]=e(1,1,0)
[h(0,1,0),e(1,1,1)]=-e(1,1,1)
[h(0,1,0),e(0,2,1)]=2e(0,2,1)
[h(0,1,0),e(1,2,1)]=e(1,2,1)
[h(0,1,0),e(-1,0,0)]=e(-1,0,0)
[h(0,1,0),e(0,-1,0)]=-2e(0,-1,0)
[h(0,1,0),e(0,0,-1)]=2e(0,0,-1)
[h(0,1,0),e(-1,-1,0)]=-e(-1,-1,0)
[h(0,1,0),e(-1,-1,-1)]=e(-1,-1,-1)
[h(0,1,0),e(0,-2,-1)]=-2e(0,-2,-1)
[h(0,1,0),e(-1,-2,-1)]=-e(-1,-2,-1)
[h(0,0,1),e(0,1,0)]=-e(0,1,0)
[h(0,0,1),e(0,0,1)]=2e(0,0,1)
[h(0,0,1),e(1,1,0)]=-e(1,1,0)
[h(0,0,1),e(0,1,1)]=e(0,1,1)
[h(0,0,1),e(1,1,1)]=e(1,1,1)
[h(0,0,1),e(0,-1,0)]=e(0,-1,0)
[h(0,0,1),e(0,0,-1)]=-2e(0,0,-1)
[h(0,0,1),e(-1,-1,0)]=e(-1,-1,0)
[h(0,0,1),e(0,-1,-1)]=-e(0,-1,-1)
[h(0,0,1),e(-1,-1,-1)]=-e(-1,-1,-1)
[e(1,0,0),e(0,1,0)]=e(1,1,0)
[e(1,0,0),e(0,1,1)]=e(1,1,1)
[e(1,0,0),e(0,2,1)]=e(1,2,1)
[e(1,0,0),e(1,2,1)]=2e(2,2,1)
[e(1,0,0),e(-1,0,0)]=h(1,0,0)
[e(1,0,0),e(-1,-1,0)]=-e(0,-1,0)
[e(1,0,0),e(-1,-1,-1)]=-e(0,-1,-1)
[e(1,0,0),e(-1,-2,-1)]=-2e(0,-2,-1)
[e(1,0,0),e(-2,-2,-1)]=-e(-1,-2,-1)
[e(0,1,0),e(0,0,1)]=e(0,1,1)
[e(0,1,0),e(0,1,1)]=2e(0,2,1)
[e(0,1,0),e(1,1,1)]=e(1,2,1)
[e(0,1,0),e(0,-1,0)]=h(0,1,0)
[e(0,1,0),e(-1,-1,0)]=e(-1,0,0)
[e(0,1,0),e(0,-1,-1)]=-2e(0,0,-1)
[e(0,1,0),e(0,-2,-1)]=-e(0,-1,-1)
[e(0,1,0),e(-1,-2,-1)]=-e(-1,-1,-1)
[e(0,0,1),e(1,1,0)]=-e(1,1,1)
[e(0,0,1),e(0,0,-1)]=h(0,0,1)
[e(0,0,1),e(0,-1,-1)]=e(0,-1,0)
[e(0,0,1),e(-1,-1,-1)]=e(-1,-1,0)
[e(1,1,0),e(0,1,1)]=e(1,2,1)
[e(1,1,0),e(1,1,1)]=2e(2,2,1)
[e(1,1,0),e(-1,0,0)]=-e(0,1,0)
[e(1,1,0),e(0,-1,0)]=e(1,0,0)
[e(1,1,0),e(-1,-1,0)]=h(1,0,0)+h(0,1,0)
[e(1,1,0),e(-1,-1,-1)]=-2e(0,0,-1)
[e(1,1,0),e(-1,-2,-1)]=-e(0,-1,-1)
[e(1,1,0),e(-2,-2,-1)]=-e(-1,-1,-1)
[e(0,1,1),e(0,-1,0)]=-2e(0,0,1)
[e(0,1,1),e(0,0,-1)]=e(0,1,0)
[e(0,1,1),e(0,-1,-1)]=h(0,1,0)+2h(0,0,1)
[e(0,1,1),e(-1,-1,-1)]=e(-1,0,0)
[e(0,1,1),e(0,-2,-1)]=e(0,-1,0)
[e(0,1,1),e(-1,-2,-1)]=e(-1,-1,0)
[e(1,1,1),e(-1,0,0)]=-e(0,1,1)
[e(1,1,1),e(0,0,-1)]=e(1,1,0)
[e(1,1,1),e(-1,-1,0)]=-2e(0,0,1)
[e(1,1,1),e(0,-1,-1)]=e(1,0,0)
[e(1,1,1),e(-1,-1,-1)]=
=h(1,0,0)+h(0,1,0)+2h(0,0,1)
[e(1,1,1),e(-1,-2,-1)]=e(0,-1,0)
[e(1,1,1),e(-2,-2,-1)]=e(-1,-1,0)
[e(0,2,1),e(0,-1,0)]=-e(0,1,1)
[e(0,2,1),e(0,-1,-1)]=e(0,1,0)
[e(0,2,1),e(0,-2,-1)]=h(0,1,0)+h(0,0,1)
[e(0,2,1),e(-1,-2,-1)]=e(-1,0,0)
[e(1,2,1),e(-1,0,0)]=-2e(0,2,1)
[e(1,2,1),e(0,-1,0)]=-e(1,1,1)
[e(1,2,1),e(-1,-1,0)]=-e(0,1,1)
[e(1,2,1),e(0,-1,-1)]=e(1,1,0)
[e(1,2,1),e(-1,-1,-1)]=e(0,1,0)
[e(1,2,1),e(0,-2,-1)]=e(1,0,0)
[e(1,2,1),e(-1,-2,-1)]=
=h(1,0,0)+2h(0,1,0)+2h(0,0,1)
```

```
[e(1,2,1),e(-2,-2,-1)]=e(-1,0,0)
[e(2,2,1),e(-1,0,0)]=-e(1,2,1)
[e(2,2,1),e(-1,-1,0)]=-e(1,1,1)
[e(2,2,1),e(-1,-1,-1)]=e(1,1,0)
[e(2,2,1),e(-1,-2,-1)]=e(1,0,0)
[e(2,2,1),e(-2,-2,-1)]=
=h(1,0,0)+h(0,1,0)+h(0,0,1)
[e(-1,0,0),e(0,-1,0)]=-e(-1,-1,0)
[e(-1,0,0),e(0,-1,-1)]=-e(-1,-1,-1)
[e(-1,0,0),e(0,-2,-1)]=-e(-1,-2,-1)
[e(-1,0,0),e(-1,-2,-1)]=-2e(-2,-2,-1)
[e(0,-1,0),e(0,0,-1)]=-e(0,-1,-1)
[e(0,-1,0),e(0,-1,-1)]=-2e(0,-2,-1)
[e(0,-1,0),e(-1,-1,-1)]=-e(-1,-2,-1)
[e(0,0,-1),e(-1,-1,0)]=e(-1,-1,-1)
[e(-1,-1,0),e(0,-1,-1)]=-e(-1,-2,-1)
[e(-1,-1,0),e(-1,-1,-1)]=-2e(-2,-2,-1)
```

A_4

```
[h(1,0,0,0),e(1,0,0,0)]=2e(1,0,0,0)
[h(1,0,0,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(1,0,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(1,0,0,0),e(0,1,1,0)]=-e(0,1,1,0)
[h(1,0,0,0),e(1,1,1,0)]=e(1,1,1,0)
[h(1,0,0,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(1,0,0,0),e(1,1,1,1)]=e(1,1,1,1)
[h(1,0,0,0),e(-1,0,0,0)]=-2e(-1,0,0,0)
[h(1,0,0,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(1,0,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(1,0,0,0),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(1,0,0,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(1,0,0,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(1,0,0,0),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(0,1,0,0),e(1,0,0,0)]=-e(1,0,0,0)
[h(0,1,0,0),e(0,1,0,0)]=2e(0,1,0,0)
[h(0,1,0,0),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,1,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(0,1,0,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,1,0,0),e(0,0,1,1)]=-e(0,0,1,1)
[h(0,1,0,0),e(0,1,1,1)]=e(0,1,1,1)
[h(0,1,0,0),e(-1,0,0,0)]=e(-1,0,0,0)
[h(0,1,0,0),e(0,-1,0,0)]=-2e(0,-1,0,0)
[h(0,1,0,0),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,1,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(0,1,0,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,1,0,0),e(0,0,-1,-1)]=e(0,0,-1,-1)
[h(0,1,0,0),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,0,1,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(0,0,1,0),e(0,0,1,0)]=2e(0,0,1,0)
[h(0,0,1,0),e(0,0,0,1)]=-e(0,0,0,1)
[h(0,0,1,0),e(1,1,0,0)]=-e(1,1,0,0)
[h(0,0,1,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,0,1,0),e(0,0,1,1)]=e(0,0,1,1)
[h(0,0,1,0),e(1,1,1,0)]=e(1,1,1,0)
[h(0,0,1,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(0,0,1,0),e(0,0,-1,0)]=-2e(0,0,-1,0)
[h(0,0,1,0),e(0,0,0,-1)]=e(0,0,0,-1)
[h(0,0,1,0),e(-1,-1,0,0)]=e(-1,-1,0,0)
[h(0,0,1,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,0,1,0),e(0,0,-1,-1)]=-e(0,0,-1,-1)
[h(0,0,1,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(0,0,0,1),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,0,0,1),e(0,0,0,1)]=2e(0,0,0,1)
[h(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,0)
[h(0,0,0,1),e(0,0,1,1)]=e(0,0,1,1)
[h(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,0)
[h(0,0,0,1),e(0,1,1,1)]=e(0,1,1,1)
[h(0,0,0,1),e(1,1,1,1)]=e(1,1,1,1)
[h(0,0,0,1),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,0,0,1),e(0,0,0,-1)]=-2e(0,0,0,-1)
[h(0,0,0,1),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(0,0,0,1),e(0,0,-1,-1)]=-e(0,0,-1,-1)
[h(0,0,0,1),e(-1,-1,-1,0)]=e(-1,-1,-1,0)
[h(0,0,0,1),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,0,0,1),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(1,0,0,0),e(0,1,0,0)]=e(1,1,0,0)
[e(1,0,0,0),e(0,1,1,0)]=e(1,1,1,0)
[e(1,0,0,0),e(0,1,1,1)]=e(1,1,1,1)
[e(1,0,0,0),e(-1,0,0,0)]=h(1,0,0,0)
[e(1,0,0,0),e(-1,-1,0,0)]=-e(0,-1,0,0)
[e(1,0,0,0),e(-1,-1,-1,0)]=-e(0,-1,-1,0)
[e(1,0,0,0),e(-1,-1,-1,-1)]=-e(0,-1,-1,-1)
[e(0,1,0,0),e(0,0,1,0)]=e(0,1,1,0)
[e(0,1,0,0),e(0,0,1,1)]=e(0,1,1,1)
[e(0,1,0,0),e(0,-1,0,0)]=h(0,1,0,0)
[e(0,1,0,0),e(-1,-1,0,0)]=e(-1,0,0,0)
[e(0,1,0,0),e(0,-1,-1,0)]=-e(0,0,-1,0)
[e(0,1,0,0),e(0,-1,-1,-1)]=-e(0,0,-1,-1)
[e(0,0,1,0),e(0,0,0,1)]=e(0,0,1,1)
[e(0,0,1,0),e(1,1,0,0)]=-e(1,1,1,0)
[e(0,0,1,0),e(0,0,-1,0)]=h(0,0,1,0)
[e(0,0,1,0),e(0,-1,-1,0)]=e(0,-1,0,0)
[e(0,0,1,0),e(0,0,-1,-1)]=-e(0,0,0,-1)
[e(0,0,1,0),e(-1,-1,-1,0)]=e(-1,-1,0,0)
[e(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,1)
[e(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,1)
[e(0,0,0,1),e(0,0,0,-1)]=h(0,0,0,1)
[e(0,0,0,1),e(0,0,-1,-1)]=e(0,0,-1,0)
[e(0,0,0,1),e(0,-1,-1,-1)]=e(0,-1,-1,0)
[e(0,0,0,1),e(-1,-1,-1,-1)]=e(-1,-1,-1,0)
[e(1,1,0,0),e(0,0,1,1)]=e(1,1,1,1)
[e(1,1,0,0),e(-1,0,0,0)]=-e(0,1,0,0)
[e(1,1,0,0),e(0,-1,0,0)]=e(1,0,0,0)
[e(1,1,0,0),e(-1,-1,0,0)]=h(1,0,0,0)+h(0,1,0,0)
[e(1,1,0,0),e(-1,-1,-1,0)]=-e(0,0,-1,0)
[e(1,1,0,0),e(-1,-1,-1,-1)]=-e(0,0,-1,-1)
[e(0,1,1,0),e(0,-1,0,0)]=-e(0,0,1,0)
[e(0,1,1,0),e(0,0,-1,0)]=e(0,1,0,0)
[e(0,1,1,0),e(0,-1,-1,0)]=h(0,1,0,0)+h(0,0,1,0)
[e(0,1,1,0),e(-1,-1,-1,0)]=e(-1,0,0,0)
[e(0,1,1,0),e(0,-1,-1,-1)]=-e(0,0,0,-1)
[e(0,0,1,1),e(0,0,-1,0)]=-e(0,0,0,1)
[e(0,0,1,1),e(0,0,0,-1)]=e(0,0,1,0)
[e(0,0,1,1),e(0,0,-1,-1)]=h(0,0,1,0)+h(0,0,0,1)
[e(0,0,1,1),e(0,-1,-1,-1)]=e(0,-1,0,0)
[e(0,0,1,1),e(-1,-1,-1,-1)]=e(-1,-1,0,0)
[e(1,1,1,0),e(-1,0,0,0)]=-e(0,1,1,0)
[e(1,1,1,0),e(0,0,-1,0)]=e(1,1,0,0)
[e(1,1,1,0),e(-1,-1,0,0)]=-e(0,0,1,0)
[e(1,1,1,0),e(0,-1,-1,0)]=e(1,0,0,0)
[e(1,1,1,0),e(-1,-1,-1,0)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)
[e(1,1,1,0),e(-1,-1,-1,-1)]=-e(0,0,0,-1)
[e(0,1,1,1),e(0,-1,0,0)]=-e(0,0,1,1)
[e(0,1,1,1),e(0,0,0,-1)]=e(0,1,1,0)
[e(0,1,1,1),e(0,-1,-1,0)]=-e(0,0,0,1)
[e(0,1,1,1),e(0,0,-1,-1)]=e(0,1,0,0)
[e(0,1,1,1),e(0,-1,-1,-1)]=
=h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(0,1,1,1),e(-1,-1,-1,-1)]=e(-1,0,0,0)
[e(1,1,1,1),e(-1,0,0,0)]=-e(0,1,1,1)
[e(1,1,1,1),e(0,0,0,-1)]=e(1,1,1,0)
[e(1,1,1,1),e(-1,-1,0,0)]=-e(0,0,1,1)
[e(1,1,1,1),e(0,0,-1,-1)]=e(1,1,0,0)
[e(1,1,1,1),e(-1,-1,-1,0)]=-e(0,0,0,1)
[e(1,1,1,1),e(0,-1,-1,-1)]=e(1,0,0,0)
[e(1,1,1,1),e(-1,-1,-1,-1)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(-1,0,0,0),e(0,-1,0,0)]=-e(-1,-1,0,0)
[e(-1,0,0,0),e(0,-1,-1,0)]=-e(-1,-1,-1,0)
[e(-1,0,0,0),e(0,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(0,-1,0,0),e(0,0,-1,0)]=-e(0,-1,-1,0)
[e(0,-1,0,0),e(0,0,-1,-1)]=-e(0,-1,-1,-1)
[e(0,0,-1,0),e(0,0,0,-1)]=-e(0,0,-1,-1)
[e(0,0,-1,0),e(-1,-1,0,0)]=e(-1,-1,-1,0)
```

```
[e(0,0,0,-1),e(0,-1,-1,0)]=e(0,-1,-1,-1)
[e(0,0,0,-1),e(-1,-1,-1,0)]=e(-1,-1,-1,-1)
[e(-1,-1,0,0),e(0,0,-1,-1)]=-e(-1,-1,-1,-1)
```

B_4

```
[h(1,0,0,0),e(1,0,0,0)]=2e(1,0,0,0)
[h(1,0,0,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(1,0,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(1,0,0,0),e(0,1,1,0)]=-e(0,1,1,0)
[h(1,0,0,0),e(1,1,1,0)]=e(1,1,1,0)
[h(1,0,0,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(1,0,0,0),e(1,1,1,1)]=e(1,1,1,1)
[h(1,0,0,0),e(0,1,1,2)]=-e(0,1,1,2)
[h(1,0,0,0),e(1,1,1,2)]=e(1,1,1,2)
[h(1,0,0,0),e(0,1,2,2)]=-e(0,1,2,2)
[h(1,0,0,0),e(1,1,2,2)]=e(1,1,2,2)
[h(1,0,0,0),e(-1,0,0,0)]=-2e(-1,0,0,0)
[h(1,0,0,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(1,0,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(1,0,0,0),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(1,0,0,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(1,0,0,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(1,0,0,0),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(1,0,0,0),e(0,-1,-1,-2)]=e(0,-1,-1,-2)
[h(1,0,0,0),e(-1,-1,-1,-2)]=-e(-1,-1,-1,-2)
[h(1,0,0,0),e(0,-1,-2,-2)]=e(0,-1,-2,-2)
[h(1,0,0,0),e(-1,-1,-2,-2)]=-e(-1,-1,-2,-2)
[h(0,1,0,0),e(1,0,0,0)]=-e(1,0,0,0)
[h(0,1,0,0),e(0,1,0,0)]=2e(0,1,0,0)
[h(0,1,0,0),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,1,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(0,1,0,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,1,0,0),e(0,0,1,1)]=-e(0,0,1,1)
[h(0,1,0,0),e(0,1,1,1)]=e(0,1,1,1)
[h(0,1,0,0),e(0,0,1,2)]=-e(0,0,1,2)
[h(0,1,0,0),e(0,1,1,2)]=e(0,1,1,2)
[h(0,1,0,0),e(1,1,2,2)]=-e(1,1,2,2)
[h(0,1,0,0),e(1,2,2,2)]=e(1,2,2,2)
[h(0,1,0,0),e(-1,0,0,0)]=e(-1,0,0,0)
[h(0,1,0,0),e(0,-1,0,0)]=-2e(0,-1,0,0)
[h(0,1,0,0),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,1,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(0,1,0,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,1,0,0),e(0,0,-1,-1)]=e(0,0,-1,-1)
[h(0,1,0,0),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,1,0,0),e(0,0,-1,-2)]=e(0,0,-1,-2)
[h(0,1,0,0),e(0,-1,-1,-2)]=-e(0,-1,-1,-2)
[h(0,1,0,0),e(-1,-1,-2,-2)]=e(-1,-1,-2,-2)
[h(0,1,0,0),e(-1,-2,-2,-2)]=-e(-1,-2,-2,-2)
[h(0,0,1,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(0,0,1,0),e(0,0,1,0)]=2e(0,0,1,0)
[h(0,0,1,0),e(0,0,0,1)]=-e(0,0,0,1)
[h(0,0,1,0),e(1,1,0,0)]=-e(1,1,0,0)
[h(0,0,1,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,0,1,0),e(0,0,1,1)]=e(0,0,1,1)
[h(0,0,1,0),e(1,1,1,0)]=e(1,1,1,0)
[h(0,0,1,0),e(0,1,1,2)]=-e(0,1,1,2)
[h(0,0,1,0),e(1,1,1,2)]=-e(1,1,1,2)
[h(0,0,1,0),e(0,1,2,2)]=e(0,1,2,2)
[h(0,0,1,0),e(1,1,2,2)]=e(1,1,2,2)
[h(0,0,1,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(0,0,1,0),e(0,0,-1,0)]=-2e(0,0,-1,0)
[h(0,0,1,0),e(0,0,0,-1)]=e(0,0,0,-1)
[h(0,0,1,0),e(-1,-1,0,0)]=e(-1,-1,0,0)
[h(0,0,1,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,0,1,0),e(0,0,-1,-1)]=-e(0,0,-1,-1)
[h(0,0,1,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(0,0,1,0),e(0,-1,-1,-2)]=e(0,-1,-1,-2)
[h(0,0,1,0),e(-1,-1,-1,-2)]=e(-1,-1,-1,-2)
[h(0,0,1,0),e(0,-1,-2,-2)]=-e(0,-1,-2,-2)
[h(0,0,1,0),e(-1,-1,-2,-2)]=-e(-1,-1,-2,-2)
[h(0,0,0,1),e(0,0,1,0)]=-2e(0,0,1,0)
[h(0,0,0,1),e(0,0,0,1)]=2e(0,0,0,1)
[h(0,0,0,1),e(0,1,1,0)]=-2e(0,1,1,0)
[h(0,0,0,1),e(1,1,1,0)]=-2e(1,1,1,0)
[h(0,0,0,1),e(0,0,1,2)]=2e(0,0,1,2)
[h(0,0,0,1),e(0,1,1,2)]=2e(0,1,1,2)
[h(0,0,0,1),e(1,1,1,2)]=2e(1,1,1,2)
[h(0,0,0,1),e(0,0,-1,0)]=2e(0,0,-1,0)
[h(0,0,0,1),e(0,0,0,-1)]=-2e(0,0,0,-1)
[h(0,0,0,1),e(0,-1,-1,0)]=2e(0,-1,-1,0)
[h(0,0,0,1),e(-1,-1,-1,0)]=2e(-1,-1,-1,0)
[h(0,0,0,1),e(0,0,-1,-2)]=-2e(0,0,-1,-2)
[h(0,0,0,1),e(0,-1,-1,-2)]=-2e(0,-1,-1,-2)
[h(0,0,0,1),e(-1,-1,-1,-2)]=-2e(-1,-1,-1,-2)
[e(1,0,0,0),e(0,1,0,0)]=e(1,1,0,0)
[e(1,0,0,0),e(0,1,1,0)]=e(1,1,1,0)
[e(1,0,0,0),e(0,1,1,1)]=e(1,1,1,1)
[e(1,0,0,0),e(0,1,1,2)]=e(1,1,1,2)
[e(1,0,0,0),e(0,1,2,2)]=e(1,1,2,2)
[e(1,0,0,0),e(-1,0,0,0)]=h(1,0,0,0)
[e(1,0,0,0),e(-1,-1,0,0)]=-e(0,-1,0,0)
[e(1,0,0,0),e(-1,-1,-1,0)]=-e(0,-1,-1,0)
[e(1,0,0,0),e(-1,-1,-1,-1)]=-e(0,-1,-1,-1)
[e(1,0,0,0),e(-1,-1,-1,-2)]=-e(0,-1,-1,-2)
[e(1,0,0,0),e(-1,-1,-2,-2)]=-e(0,-1,-2,-2)
[e(0,1,0,0),e(0,0,1,0)]=e(0,1,1,0)
[e(0,1,0,0),e(0,0,1,1)]=e(0,1,1,1)
[e(0,1,0,0),e(0,0,1,2)]=e(0,1,1,2)
[e(0,1,0,0),e(1,1,2,2)]=e(1,2,2,2)
[e(0,1,0,0),e(0,-1,0,0)]=h(0,1,0,0)
[e(0,1,0,0),e(-1,-1,0,0)]=e(-1,0,0,0)
[e(0,1,0,0),e(0,-1,-1,0)]=-e(0,0,-1,0)
[e(0,1,0,0),e(0,-1,-1,-1)]=-e(0,0,-1,-1)
[e(0,1,0,0),e(0,-1,-1,-2)]=-e(0,0,-1,-2)
[e(0,1,0,0),e(-1,-2,-2,-2)]=-e(-1,-1,-2,-2)
[e(0,0,1,0),e(0,0,0,1)]=e(0,0,1,1)
[e(0,0,1,0),e(1,1,0,0)]=-e(1,1,1,0)
[e(0,0,1,0),e(0,1,1,2)]=e(0,1,2,2)
[e(0,0,1,0),e(1,1,1,2)]=e(1,1,2,2)
[e(0,0,1,0),e(0,0,-1,0)]=h(0,0,1,0)
[e(0,0,1,0),e(0,-1,-1,0)]=e(0,-1,0,0)
[e(0,0,1,0),e(0,0,-1,-1)]=-e(0,0,0,-1)
[e(0,0,1,0),e(-1,-1,-1,0)]=e(-1,-1,0,0)
[e(0,0,1,0),e(0,-1,-2,-2)]=-e(0,-1,-1,-2)
[e(0,0,1,0),e(-1,-1,-2,-2)]=-e(-1,-1,-1,-2)
[e(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,1)
[e(0,0,0,1),e(0,0,1,1)]=2e(0,0,1,2)
[e(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,1)
[e(0,0,0,1),e(0,1,1,1)]=2e(0,1,1,2)
[e(0,0,0,1),e(1,1,1,1)]=2e(1,1,1,2)
[e(0,0,0,1),e(0,0,0,-1)]=h(0,0,0,1)
[e(0,0,0,1),e(0,0,-1,-1)]=2e(0,0,-1,0)
[e(0,0,0,1),e(0,-1,-1,-1)]=2e(0,-1,-1,0)
[e(0,0,0,1),e(0,0,-1,-2)]=-e(0,0,-1,-1)
[e(0,0,0,1),e(-1,-1,-1,-1)]=2e(-1,-1,-1,0)
[e(0,0,0,1),e(0,-1,-1,-2)]=-e(0,-1,-1,-1)
[e(0,0,0,1),e(-1,-1,-1,-2)]=-e(-1,-1,-1,-1)
[e(1,1,0,0),e(0,0,1,1)]=e(1,1,1,1)
[e(1,1,0,0),e(0,0,1,2)]=e(1,1,1,2)
[e(1,1,0,0),e(0,1,2,2)]=-e(1,2,2,2)
[e(1,1,0,0),e(-1,0,0,0)]=-e(0,1,0,0)
[e(1,1,0,0),e(0,-1,0,0)]=e(1,0,0,0)
[e(1,1,0,0),e(-1,-1,0,0)]=h(1,0,0,0)+h(0,1,0,0)
[e(1,1,0,0),e(-1,-1,-1,0)]=-e(0,0,-1,0)
[e(1,1,0,0),e(-1,-1,-1,-1)]=-e(0,0,-1,-1)
[e(1,1,0,0),e(-1,-1,-1,-2)]=-e(0,0,-1,-2)
[e(1,1,0,0),e(-1,-2,-2,-2)]=e(0,-1,-2,-2)
[e(0,1,1,0),e(0,0,1,2)]=-e(0,1,2,2)
[e(0,1,1,0),e(1,1,1,2)]=e(1,2,2,2)
[e(0,1,1,0),e(0,-1,0,0)]=-e(0,0,1,0)
[e(0,1,1,0),e(0,0,-1,0)]=e(0,1,0,0)
[e(0,1,1,0),e(0,-1,-1,0)]=h(0,1,0,0)+h(0,0,1,0)
[e(0,1,1,0),e(-1,-1,-1,0)]=e(-1,0,0,0)
[e(0,1,1,0),e(0,-1,-1,-1)]=-e(0,0,0,-1)
```

```
[e(0,1,1,0),e(0,-1,-2,-2)]=e(0,0,-1,-2)
[e(0,1,1,0),e(-1,-2,-2,-2)]=-e(-1,-1,-1,-2)
[e(0,0,1,1),e(0,1,1,1)]=2e(0,1,2,2)
[e(0,0,1,1),e(1,1,1,1)]=2e(1,1,2,2)
[e(0,0,1,1),e(0,0,-1,0)]=-e(0,0,0,1)
[e(0,0,1,1),e(0,0,0,-1)]=2e(0,0,1,0)
[e(0,0,1,1),e(0,0,-1,-1)]=
=2h(0,0,1,0)+h(0,0,0,1)
[e(0,0,1,1),e(0,-1,-1,-1)]=2e(0,-1,0,0)
[e(0,0,1,1),e(0,0,-1,-2)]=e(0,0,0,-1)
[e(0,0,1,1),e(-1,-1,-1,-1)]=2e(-1,-1,0,0)
[e(0,0,1,1),e(0,-1,-2,-2)]=-e(0,-1,-1,-1)
[e(0,0,1,1),e(-1,-1,-2,-2)]=-e(-1,-1,-1,-1)
[e(1,1,1,0),e(0,0,1,2)]=-e(1,1,2,2)
[e(1,1,1,0),e(0,1,1,2)]=-e(1,2,2,2)
[e(1,1,1,0),e(-1,0,0,0)]=-e(0,1,1,0)
[e(1,1,1,0),e(0,0,-1,0)]=e(1,1,0,0)
[e(1,1,1,0),e(-1,-1,0,0)]=-e(0,0,1,0)
[e(1,1,1,0),e(0,-1,-1,0)]=e(1,0,0,0)
[e(1,1,1,0),e(-1,-1,-1,0)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)
[e(1,1,1,0),e(-1,-1,-1,-1)]=-e(0,0,0,-1)
[e(1,1,1,0),e(-1,-1,-2,-2)]=e(0,0,-1,-2)
[e(1,1,1,0),e(-1,-2,-2,-2)]=e(0,-1,-1,-2)
[e(0,1,1,1),e(1,1,1,1)]=2e(1,2,2,2)
[e(0,1,1,1),e(0,-1,0,0)]=-e(0,0,1,1)
[e(0,1,1,1),e(0,0,0,-1)]=2e(0,1,1,0)
[e(0,1,1,1),e(0,-1,-1,0)]=-e(0,0,0,1)
[e(0,1,1,1),e(0,0,-1,-1)]=2e(0,1,0,0)
[e(0,1,1,1),e(0,-1,-1,-1)]=
=2h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(0,1,1,1),e(-1,-1,-1,-1)]=2e(-1,0,0,0)
[e(0,1,1,1),e(0,-1,-1,-2)]=e(0,0,0,-1)
[e(0,1,1,1),e(0,-1,-2,-2)]=e(0,0,-1,-1)
[e(0,1,1,1),e(-1,-2,-2,-2)]=-e(-1,-1,-1,-1)
[e(0,0,1,2),e(0,0,0,-1)]=-e(0,0,1,1)
[e(0,0,1,2),e(0,0,-1,-1)]=e(0,0,0,1)
[e(0,0,1,2),e(0,0,-1,-2)]=h(0,0,1,0)+h(0,0,0,1)
[e(0,0,1,2),e(0,-1,-1,-2)]=e(0,-1,0,0)
[e(0,0,1,2),e(-1,-1,-1,-2)]=e(-1,-1,0,0)
[e(0,0,1,2),e(0,-1,-2,-2)]=-e(0,-1,-1,0)
[e(0,0,1,2),e(-1,-1,-2,-2)]=-e(-1,-1,-1,0)
[e(1,1,1,1),e(-1,0,0,0)]=-e(0,1,1,1)
[e(1,1,1,1),e(0,0,0,-1)]=2e(1,1,1,0)
[e(1,1,1,1),e(-1,-1,0,0)]=-e(0,0,1,1)
[e(1,1,1,1),e(0,0,-1,-1)]=2e(1,1,0,0)
[e(1,1,1,1),e(-1,-1,-1,0)]=-e(0,0,0,1)
[e(1,1,1,1),e(0,-1,-1,-1)]=2e(1,0,0,0)
[e(1,1,1,1),e(-1,-1,-1,-1)]=
=2h(1,0,0,0)+2h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(1,1,1,1),e(-1,-1,-1,-2)]=e(0,0,0,-1)
[e(1,1,1,1),e(-1,-1,-2,-2)]=e(0,0,-1,-1)
[e(1,1,1,1),e(-1,-2,-2,-2)]=e(0,-1,-1,-1)
[e(0,1,1,2),e(0,-1,0,0)]=-e(0,0,1,2)
[e(0,1,1,2),e(0,0,0,-1)]=-e(0,1,1,1)
[e(0,1,1,2),e(0,-1,-1,-1)]=e(0,0,0,1)
[e(0,1,1,2),e(0,0,-1,-2)]=e(0,1,0,0)
[e(0,1,1,2),e(0,-1,-1,-2)]=
=h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(0,1,1,2),e(-1,-1,-1,-2)]=e(-1,0,0,0)
[e(0,1,1,2),e(0,-1,-2,-2)]=e(0,0,-1,0)
[e(0,1,1,2),e(-1,-2,-2,-2)]=-e(-1,-1,-1,0)
[e(1,1,1,2),e(-1,0,0,0)]=-e(0,1,1,2)
[e(1,1,1,2),e(0,0,0,-1)]=-e(1,1,1,1)
[e(1,1,1,2),e(-1,-1,0,0)]=-e(0,0,1,2)
[e(1,1,1,2),e(0,0,-1,-2)]=e(1,1,0,0)
[e(1,1,1,2),e(-1,-1,-1,-1)]=e(0,0,0,1)
[e(1,1,1,2),e(0,-1,-1,-2)]=e(1,0,0,0)
[e(1,1,1,2),e(-1,-1,-1,-2)]=h(1,0,0,0)
+h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(1,1,1,2),e(-1,-1,-2,-2)]=e(0,0,-1,0)
[e(1,1,1,2),e(-1,-2,-2,-2)]=e(0,-1,-1,0)
[e(0,1,2,2),e(0,0,-1,0)]=-e(0,1,1,2)
[e(0,1,2,2),e(0,-1,-1,0)]=e(0,0,1,2)
[e(0,1,2,2),e(0,0,-1,-1)]=-e(0,1,1,1)
[e(0,1,2,2),e(0,-1,-1,-1)]=e(0,0,1,1)
[e(0,1,2,2),e(0,0,-1,-2)]=-e(0,1,1,0)
[e(0,1,2,2),e(0,-1,-1,-2)]=e(0,0,1,0)
[e(0,1,2,2),e(0,-1,-2,-2)]=
=h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(0,1,2,2),e(-1,-1,-2,-2)]=e(-1,0,0,0)
[e(0,1,2,2),e(-1,-2,-2,-2)]=-e(-1,-1,0,0)
[e(1,1,2,2),e(-1,0,0,0)]=-e(0,1,2,2)
[e(1,1,2,2),e(0,0,-1,0)]=-e(1,1,1,2)
[e(1,1,2,2),e(0,0,-1,-1)]=-e(1,1,1,1)
[e(1,1,2,2),e(-1,-1,-1,0)]=e(0,0,1,2)
[e(1,1,2,2),e(0,0,-1,-2)]=-e(1,1,1,0)
[e(1,1,2,2),e(-1,-1,-1,-1)]=e(0,0,1,1)
[e(1,1,2,2),e(-1,-1,-1,-2)]=e(0,0,1,0)
[e(1,1,2,2),e(0,-1,-2,-2)]=e(1,0,0,0)
[e(1,1,2,2),e(-1,-1,-2,-2)]=
=h(1,0,0,0)+h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(1,1,2,2),e(-1,-2,-2,-2)]=e(0,-1,0,0)
[e(1,2,2,2),e(0,-1,0,0)]=-e(1,1,2,2)
[e(1,2,2,2),e(-1,-1,0,0)]=e(0,1,2,2)
[e(1,2,2,2),e(0,-1,-1,0)]=-e(1,1,1,2)
[e(1,2,2,2),e(-1,-1,-1,0)]=e(0,1,1,2)
[e(1,2,2,2),e(0,-1,-1,-1)]=-e(1,1,1,1)
[e(1,2,2,2),e(-1,-1,-1,-1)]=e(0,1,1,1)
[e(1,2,2,2),e(0,-1,-1,-2)]=-e(1,1,1,0)
[e(1,2,2,2),e(-1,-1,-1,-2)]=e(0,1,1,0)
[e(1,2,2,2),e(0,-1,-2,-2)]=-e(1,1,0,0)
[e(1,2,2,2),e(-1,-1,-2,-2)]=e(0,1,0,0)
[e(1,2,2,2),e(-1,-2,-2,-2)]=
=h(1,0,0,0)+2h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(-1,0,0,0),e(0,-1,0,0)]=-e(-1,-1,0,0)
[e(-1,0,0,0),e(0,-1,-1,0)]=-e(-1,-1,-1,0)
[e(-1,0,0,0),e(0,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(-1,0,0,0),e(0,-1,-1,-2)]=-e(-1,-1,-1,-2)
[e(-1,0,0,0),e(0,-1,-2,-2)]=-e(-1,-1,-2,-2)
[e(0,-1,0,0),e(0,0,-1,0)]=-e(0,-1,-1,0)
[e(0,-1,0,0),e(0,0,-1,-1)]=-e(0,-1,-1,-1)
[e(0,-1,0,0),e(0,0,-1,-2)]=-e(0,-1,-1,-2)
[e(0,-1,0,0),e(-1,-1,-2,-2)]=-e(-1,-2,-2,-2)
[e(0,0,-1,0),e(0,0,0,-1)]=-e(0,0,-1,-1)
[e(0,0,-1,0),e(-1,-1,0,0)]=e(-1,-1,-1,0)
[e(0,0,-1,0),e(0,-1,-1,-2)]=-e(0,-1,-2,-2)
[e(0,0,-1,0),e(-1,-1,-1,-2)]=-e(-1,-1,-2,-2)
[e(0,0,0,-1),e(0,-1,-1,0)]=e(0,-1,-1,-1)
[e(0,0,0,-1),e(0,0,-1,-1)]=-2e(0,0,-1,-2)
[e(0,0,0,-1),e(-1,-1,-1,0)]=e(-1,-1,-1,-1)
[e(0,0,0,-1),e(0,-1,-1,-1)]=-2e(0,-1,-1,-2)
[e(0,0,0,-1),e(-1,-1,-1,-1)]=-2e(-1,-1,-1,-2)
[e(-1,-1,0,0),e(0,0,-1,-1)]=-e(-1,-1,-1,-1)
[e(-1,-1,0,0),e(0,0,-1,-2)]=-e(-1,-1,-1,-2)
[e(-1,-1,0,0),e(0,-1,-2,-2)]=e(-1,-2,-2,-2)
[e(0,-1,-1,0),e(0,0,-1,-2)]=e(0,-1,-2,-2)
[e(0,-1,-1,0),e(-1,-1,-1,-2)]=-e(-1,-2,-2,-2)
[e(0,0,-1,-1),e(0,-1,-1,-1)]=-2e(0,-1,-2,-2)
[e(0,0,-1,-1),e(-1,-1,-1,-1)]=-2e(-1,-1,-2,-2)
[e(-1,-1,-1,0),e(0,0,-1,-2)]=e(-1,-1,-2,-2)
[e(-1,-1,-1,0),e(0,-1,-1,-2)]=e(-1,-2,-2,-2)
[e(0,-1,-1,-1),e(-1,-1,-1,-1)]=-2e(-1,-2,-2,-2)
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C_4

```
[h(1,0,0,0),e(1,0,0,0)]=2e(1,0,0,0)
[h(1,0,0,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(1,0,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(1,0,0,0),e(0,1,1,0)]=-e(0,1,1,0)
[h(1,0,0,0),e(1,1,1,0)]=e(1,1,1,0)
[h(1,0,0,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(1,0,0,0),e(1,1,1,1)]=e(1,1,1,1)
[h(1,0,0,0),e(0,1,2,1)]=-e(0,1,2,1)
[h(1,0,0,0),e(1,1,2,1)]=e(1,1,2,1)
[h(1,0,0,0),e(0,2,2,1)]=-2e(0,2,2,1)
```

```
[h(1,0,0,0),e(2,2,2,1)]=2e(2,2,2,1)
[h(1,0,0,0),e(-1,0,0,0)]=-2e(-1,0,0,0)
[h(1,0,0,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(1,0,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(1,0,0,0),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(1,0,0,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(1,0,0,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(1,0,0,0),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(1,0,0,0),e(0,-1,-2,-1)]=e(0,-1,-2,-1)
[h(1,0,0,0),e(-1,-1,-2,-1)]=-e(-1,-1,-2,-1)
[h(1,0,0,0),e(0,-2,-2,-1)]=2e(0,-2,-2,-1)
[h(1,0,0,0),e(-2,-2,-2,-1)]=-2e(-2,-2,-2,-1)
[h(0,1,0,0),e(1,0,0,0)]=-e(1,0,0,0)
[h(0,1,0,0),e(0,1,0,0)]=2e(0,1,0,0)
[h(0,1,0,0),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,1,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(0,1,0,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,1,0,0),e(0,0,1,1)]=-e(0,0,1,1)
[h(0,1,0,0),e(0,1,1,1)]=e(0,1,1,1)
[h(0,1,0,0),e(0,0,2,1)]=-2e(0,0,2,1)
[h(0,1,0,0),e(1,1,2,1)]=-e(1,1,2,1)
[h(0,1,0,0),e(0,2,2,1)]=2e(0,2,2,1)
[h(0,1,0,0),e(1,2,2,1)]=e(1,2,2,1)
[h(0,1,0,0),e(-1,0,0,0)]=e(-1,0,0,0)
[h(0,1,0,0),e(0,-1,0,0)]=-2e(0,-1,0,0)
[h(0,1,0,0),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,1,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(0,1,0,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,1,0,0),e(0,0,-1,-1)]=e(0,0,-1,-1)
[h(0,1,0,0),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,1,0,0),e(0,0,-2,-1)]=2e(0,0,-2,-1)
[h(0,1,0,0),e(-1,-1,-2,-1)]=e(-1,-1,-2,-1)
[h(0,1,0,0),e(0,-2,-2,-1)]=-2e(0,-2,-2,-1)
[h(0,1,0,0),e(-1,-2,-2,-1)]=-e(-1,-2,-2,-1)
[h(0,0,1,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(0,0,1,0),e(0,0,1,0)]=2e(0,0,1,0)
[h(0,0,1,0),e(0,0,0,1)]=-2e(0,0,0,1)
[h(0,0,1,0),e(1,1,0,0)]=-e(1,1,0,0)
[h(0,0,1,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,0,1,0),e(1,1,1,0)]=e(1,1,1,0)
[h(0,0,1,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(0,0,1,0),e(0,0,2,1)]=2e(0,0,2,1)
[h(0,0,1,0),e(1,1,1,1)]=-e(1,1,1,1)
[h(0,0,1,0),e(0,1,2,1)]=e(0,1,2,1)
[h(0,0,1,0),e(1,1,2,1)]=e(1,1,2,1)
[h(0,0,1,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(0,0,1,0),e(0,0,-1,0)]=-2e(0,0,-1,0)
[h(0,0,1,0),e(0,0,0,-1)]=2e(0,0,0,-1)
[h(0,0,1,0),e(-1,-1,0,0)]=e(-1,-1,0,0)
[h(0,0,1,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,0,1,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(0,0,1,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(0,0,1,0),e(0,0,-2,-1)]=-2e(0,0,-2,-1)
[h(0,0,1,0),e(-1,-1,-1,-1)]=e(-1,-1,-1,-1)
[h(0,0,1,0),e(0,-1,-2,-1)]=-e(0,-1,-2,-1)
[h(0,0,1,0),e(-1,-1,-2,-1)]=-e(-1,-1,-2,-1)
[h(0,0,0,1),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,0,0,1),e(0,0,0,1)]=2e(0,0,0,1)
[h(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,0)
[h(0,0,0,1),e(0,0,1,1)]=e(0,0,1,1)
[h(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,0)
[h(0,0,0,1),e(0,1,1,1)]=e(0,1,1,1)
[h(0,0,0,1),e(1,1,1,1)]=e(1,1,1,1)
[h(0,0,0,1),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,0,0,1),e(0,0,0,-1)]=-2e(0,0,0,-1)
[h(0,0,0,1),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(0,0,0,1),e(0,0,-1,-1)]=-e(0,0,-1,-1)
[h(0,0,0,1),e(-1,-1,-1,0)]=e(-1,-1,-1,0)
[h(0,0,0,1),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,0,0,1),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(1,0,0,0),e(0,1,0,0)]=e(1,1,0,0)
[e(1,0,0,0),e(0,1,1,0)]=e(1,1,1,0)
[e(1,0,0,0),e(0,1,1,1)]=e(1,1,1,1)
[e(1,0,0,0),e(0,1,2,1)]=e(1,1,2,1)
[e(1,0,0,0),e(0,2,2,1)]=e(1,2,2,1)
[e(1,0,0,0),e(1,2,2,1)]=2e(2,2,2,1)
[e(1,0,0,0),e(-1,0,0,0)]=h(1,0,0,0)
[e(1,0,0,0),e(-1,-1,0,0)]=-e(0,-1,0,0)
[e(1,0,0,0),e(-1,-1,-1,0)]=-e(0,-1,-1,0)
[e(1,0,0,0),e(-1,-1,-1,-1)]=-e(0,-1,-1,-1)
[e(1,0,0,0),e(-1,-1,-2,-1)]=-e(0,-1,-2,-1)
[e(1,0,0,0),e(-1,-2,-2,-1)]=-2e(0,-2,-2,-1)
[e(1,0,0,0),e(-2,-2,-2,-1)]=-e(-1,-2,-2,-1)
[e(0,1,0,0),e(0,0,1,0)]=e(0,1,1,0)
[e(0,1,0,0),e(0,0,1,1)]=e(0,1,1,1)
[e(0,1,0,0),e(0,0,2,1)]=e(0,1,2,1)
[e(0,1,0,0),e(0,1,2,1)]=2e(0,2,2,1)
[e(0,1,0,0),e(1,1,2,1)]=e(1,2,2,1)
[e(0,1,0,0),e(0,-1,0,0)]=h(0,1,0,0)
[e(0,1,0,0),e(-1,-1,0,0)]=e(-1,0,0,0)
[e(0,1,0,0),e(0,-1,-1,0)]=-e(0,0,-1,0)
[e(0,1,0,0),e(0,-1,-1,-1)]=-e(0,0,-1,-1)
[e(0,1,0,0),e(0,-1,-2,-1)]=-2e(0,0,-2,-1)
[e(0,1,0,0),e(0,-2,-2,-1)]=-e(0,-1,-2,-1)
[e(0,1,0,0),e(-1,-2,-2,-1)]=-e(-1,-1,-2,-1)
[e(0,0,1,0),e(0,0,0,1)]=e(0,0,1,1)
[e(0,0,1,0),e(1,1,0,0)]=-e(1,1,1,0)
[e(0,0,1,0),e(0,0,1,1)]=2e(0,0,2,1)
[e(0,0,1,0),e(0,1,1,1)]=e(0,1,2,1)
[e(0,0,1,0),e(1,1,1,1)]=e(1,1,2,1)
[e(0,0,1,0),e(0,0,-1,0)]=h(0,0,1,0)
[e(0,0,1,0),e(0,-1,-1,0)]=e(0,-1,0,0)
[e(0,0,1,0),e(0,0,-1,-1)]=-2e(0,0,0,-1)
[e(0,0,1,0),e(-1,-1,-1,0)]=e(-1,-1,0,0)
[e(0,0,1,0),e(0,0,-2,-1)]=-e(0,0,-1,-1)
[e(0,0,1,0),e(0,-1,-2,-1)]=-e(0,-1,-1,-1)
[e(0,0,1,0),e(-1,-1,-2,-1)]=-e(-1,-1,-1,-1)
[e(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,1)
[e(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,1)
[e(0,0,0,1),e(0,0,0,-1)]=h(0,0,0,1)
[e(0,0,0,1),e(0,0,-1,-1)]=e(0,0,-1,0)
[e(0,0,0,1),e(0,-1,-1,-1)]=e(0,-1,-1,0)
[e(0,0,0,1),e(-1,-1,-1,-1)]=e(-1,-1,-1,0)
[e(1,1,0,0),e(0,0,1,1)]=e(1,1,1,1)
[e(1,1,0,0),e(0,0,2,1)]=e(1,1,2,1)
[e(1,1,0,0),e(0,1,2,1)]=e(1,2,2,1)
[e(1,1,0,0),e(1,1,2,1)]=2e(2,2,2,1)
[e(1,1,0,0),e(-1,0,0,0)]=-e(0,1,0,0)
[e(1,1,0,0),e(0,-1,0,0)]=e(1,0,0,0)
[e(1,1,0,0),e(-1,-1,0,0)]=h(1,0,0,0)+h(0,1,0,0)
[e(1,1,0,0),e(-1,-1,-1,0)]=-e(0,0,-1,0)
[e(1,1,0,0),e(-1,-1,-1,-1)]=-e(0,0,-1,-1)
[e(1,1,0,0),e(-1,-1,-2,-1)]=-2e(0,0,-2,-1)
[e(1,1,0,0),e(-1,-2,-2,-1)]=-e(0,-1,-2,-1)
[e(1,1,0,0),e(-2,-2,-2,-1)]=-e(-1,-1,-2,-1)
[e(0,1,1,0),e(0,0,1,1)]=e(0,1,2,1)
[e(0,1,1,0),e(0,1,1,1)]=2e(0,2,2,1)
[e(0,1,1,0),e(1,1,1,1)]=e(1,2,2,1)
[e(0,1,1,0),e(0,-1,0,0)]=-e(0,0,1,0)
[e(0,1,1,0),e(0,0,-1,0)]=e(0,1,0,0)
[e(0,1,1,0),e(0,-1,-1,0)]=h(0,1,0,0)+h(0,0,1,0)
[e(0,1,1,0),e(-1,-1,-1,0)]=e(-1,0,0,0)
[e(0,1,1,0),e(0,-1,-1,-1)]=-2e(0,0,0,-1)
[e(0,1,1,0),e(0,-1,-2,-1)]=-e(0,0,-1,-1)
[e(0,1,1,0),e(0,-2,-2,-1)]=-e(0,-1,-1,-1)
[e(0,1,1,0),e(-1,-2,-2,-1)]=-e(-1,-1,-1,-1)
[e(0,0,1,1),e(1,1,1,0)]=-e(1,1,2,1)
[e(0,0,1,1),e(0,0,-1,0)]=-2e(0,0,0,1)
[e(0,0,1,1),e(0,0,0,-1)]=e(0,0,1,0)
[e(0,0,1,1),e(0,0,-1,-1)]=
=h(0,0,1,0)+2h(0,0,0,1)
[e(0,0,1,1),e(0,-1,-1,-1)]=e(0,-1,0,0)
[e(0,0,1,1),e(0,0,-2,-1)]=e(0,0,-1,0)
[e(0,0,1,1),e(-1,-1,-1,-1)]=e(-1,-1,0,0)
[e(0,0,1,1),e(0,-1,-2,-1)]=e(0,-1,-1,0)
[e(0,0,1,1),e(-1,-1,-2,-1)]=e(-1,-1,-1,0)
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[e(1,1,1,0),e(0,1,1,1)]=e(1,2,2,1)
[e(1,1,1,0),e(1,1,1,1)]=2e(2,2,2,1)
[e(1,1,1,0),e(-1,0,0,0)]=-e(0,1,1,0)
[e(1,1,1,0),e(0,0,-1,0)]=e(1,1,0,0)
[e(1,1,1,0),e(-1,-1,0,0)]=-e(0,0,1,0)
[e(1,1,1,0),e(0,-1,-1,0)]=e(1,0,0,0)
[e(1,1,1,0),e(-1,-1,-1,0)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)
[e(1,1,1,0),e(-1,-1,-1,-1)]=-2e(0,0,0,-1)
[e(1,1,1,0),e(-1,-1,-2,-1)]=-e(0,0,-1,-1)
[e(1,1,1,0),e(-1,-2,-2,-1)]=-e(0,-1,-1,-1)
[e(1,1,1,0),e(-2,-2,-2,-1)]=-e(-1,-1,-1,-1)
[e(0,1,1,1),e(0,-1,0,0)]=-e(0,0,1,1)
[e(0,1,1,1),e(0,0,0,-1)]=e(0,1,1,0)
[e(0,1,1,1),e(0,-1,-1,0)]=-2e(0,0,0,1)
[e(0,1,1,1),e(0,0,-1,-1)]=e(0,1,0,0)
[e(0,1,1,1),e(0,-1,-1,-1)]=
=h(0,1,0,0)+h(0,0,1,0)+2h(0,0,0,1)
[e(0,1,1,1),e(-1,-1,-1,-1)]=e(-1,0,0,0)
[e(0,1,1,1),e(0,-1,-2,-1)]=e(0,0,-1,0)
[e(0,1,1,1),e(0,-2,-2,-1)]=e(0,-1,-1,0)
[e(0,1,1,1),e(-1,-2,-2,-1)]=e(-1,-1,-1,0)
[e(0,0,2,1),e(0,0,-1,0)]=-e(0,0,1,1)
[e(0,0,2,1),e(0,0,-1,-1)]=e(0,0,1,0)
[e(0,0,2,1),e(0,0,-2,-1)]=h(0,0,1,0)+h(0,0,0,1)
[e(0,0,2,1),e(0,-1,-2,-1)]=e(0,-1,0,0)
[e(0,0,2,1),e(-1,-1,-2,-1)]=e(-1,-1,0,0)
[e(1,1,1,1),e(-1,0,0,0)]=-e(0,1,1,1)
[e(1,1,1,1),e(0,0,0,-1)]=e(1,1,1,0)
[e(1,1,1,1),e(-1,-1,0,0)]=-e(0,0,1,1)
[e(1,1,1,1),e(0,0,-1,-1)]=e(1,1,0,0)
[e(1,1,1,1),e(-1,-1,-1,0)]=-2e(0,0,0,1)
[e(1,1,1,1),e(0,-1,-1,-1)]=e(1,0,0,0)
[e(1,1,1,1),e(-1,-1,-1,-1)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)+2h(0,0,0,1)
[e(1,1,1,1),e(-1,-1,-2,-1)]=e(0,0,-1,0)
[e(1,1,1,1),e(-1,-2,-2,-1)]=e(0,-1,-1,0)
[e(1,1,1,1),e(-2,-2,-2,-1)]=e(-1,-1,-1,0)
[e(0,1,2,1),e(0,-1,0,0)]=-2e(0,0,2,1)
[e(0,1,2,1),e(0,0,-1,0)]=-e(0,1,1,1)
[e(0,1,2,1),e(0,-1,-1,0)]=-e(0,0,1,1)
[e(0,1,2,1),e(0,0,-1,-1)]=e(0,1,1,0)
[e(0,1,2,1),e(0,-1,-1,-1)]=e(0,0,1,0)
[e(0,1,2,1),e(0,0,-2,-1)]=e(0,1,0,0)
[e(0,1,2,1),e(0,-1,-2,-1)]=
=h(0,1,0,0)+2h(0,0,1,0)+2h(0,0,0,1)
[e(0,1,2,1),e(-1,-1,-2,-1)]=e(-1,0,0,0)
[e(0,1,2,1),e(0,-2,-2,-1)]=e(0,-1,0,0)
[e(0,1,2,1),e(-1,-2,-2,-1)]=e(-1,-1,0,0)
[e(1,1,2,1),e(-1,0,0,0)]=-e(0,1,2,1)
[e(1,1,2,1),e(0,0,-1,0)]=-e(1,1,1,1)
[e(1,1,2,1),e(-1,-1,0,0)]=-2e(0,0,2,1)
[e(1,1,2,1),e(0,0,-1,-1)]=e(1,1,1,0)
[e(1,1,2,1),e(-1,-1,-1,0)]=-e(0,0,1,1)
[e(1,1,2,1),e(0,0,-2,-1)]=e(1,1,0,0)
[e(1,1,2,1),e(-1,-1,-1,-1)]=e(0,0,1,0)
[e(1,1,2,1),e(0,-1,-2,-1)]=e(1,0,0,0)
[e(1,1,2,1),e(-1,-1,-2,-1)]=
=h(1,0,0,0)+h(0,1,0,0)+2h(0,0,1,0)+2h(0,0,0,1)
[e(1,1,2,1),e(-1,-2,-2,-1)]=e(0,-1,0,0)
[e(1,1,2,1),e(-2,-2,-2,-1)]=e(-1,-1,0,0)
[e(0,2,2,1),e(0,-1,0,0)]=-e(0,1,2,1)
[e(0,2,2,1),e(0,-1,-1,0)]=-e(0,1,1,1)
[e(0,2,2,1),e(0,-1,-1,-1)]=e(0,1,1,0)
[e(0,2,2,1),e(0,-1,-2,-1)]=e(0,1,0,0)
[e(0,2,2,1),e(0,-2,-2,-1)]=
=h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(0,2,2,1),e(-1,-2,-2,-1)]=e(-1,0,0,0)
[e(1,2,2,1),e(-1,0,0,0)]=-2e(0,2,2,1)
[e(1,2,2,1),e(0,-1,0,0)]=-e(1,1,2,1)
[e(1,2,2,1),e(-1,-1,0,0)]=-e(0,1,2,1)
[e(1,2,2,1),e(0,-1,-1,0)]=-e(1,1,1,1)
[e(1,2,2,1),e(-1,-1,-1,0)]=-e(0,1,1,1)
[e(1,2,2,1),e(0,-1,-1,-1)]=e(1,1,1,0)
[e(1,2,2,1),e(-1,-1,-1,-1)]=e(0,1,1,0)
[e(1,2,2,1),e(0,-1,-2,-1)]=e(1,1,0,0)
[e(1,2,2,1),e(-1,-1,-2,-1)]=e(0,1,0,0)
[e(1,2,2,1),e(0,-2,-2,-1)]=e(1,0,0,0)
[e(1,2,2,1),e(-1,-2,-2,-1)]=
=h(1,0,0,0)+2h(0,1,0,0)+2h(0,0,1,0)+2h(0,0,0,1)
[e(1,2,2,1),e(-2,-2,-2,-1)]=e(-1,0,0,0)
[e(2,2,2,1),e(-1,0,0,0)]=-e(1,2,2,1)
[e(2,2,2,1),e(-1,-1,0,0)]=-e(1,1,2,1)
[e(2,2,2,1),e(-1,-1,-1,0)]=-e(1,1,1,1)
[e(2,2,2,1),e(-1,-1,-1,-1)]=e(1,1,1,0)
[e(2,2,2,1),e(-1,-1,-2,-1)]=e(1,1,0,0)
[e(2,2,2,1),e(-1,-2,-2,-1)]=e(1,0,0,0)
[e(2,2,2,1),e(-2,-2,-2,-1)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(-1,0,0,0),e(0,-1,0,0)]=-e(-1,-1,0,0)
[e(-1,0,0,0),e(0,-1,-1,0)]=-e(-1,-1,-1,0)
[e(-1,0,0,0),e(0,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(-1,0,0,0),e(0,-1,-2,-1)]=-e(-1,-1,-2,-1)
[e(-1,0,0,0),e(0,-2,-2,-1)]=-e(-1,-2,-2,-1)
[e(-1,0,0,0),e(-1,-2,-2,-1)]=-2e(-2,-2,-2,-1)
[e(0,-1,0,0),e(0,0,-1,0)]=-e(0,-1,-1,0)
[e(0,-1,0,0),e(0,0,-1,-1)]=-e(0,-1,-1,-1)
[e(0,-1,0,0),e(0,0,-2,-1)]=-e(0,-1,-2,-1)
[e(0,-1,0,0),e(0,-1,-2,-1)]=-2e(0,-2,-2,-1)
[e(0,-1,0,0),e(-1,-1,-2,-1)]=-e(-1,-2,-2,-1)
[e(0,0,-1,0),e(0,0,0,-1)]=-e(0,0,-1,-1)
[e(0,0,-1,0),e(-1,-1,0,0)]=e(-1,-1,-1,0)
[e(0,0,-1,0),e(0,0,-1,-1)]=-2e(0,0,-2,-1)
[e(0,0,-1,0),e(0,-1,-1,-1)]=-e(0,-1,-2,-1)
[e(0,0,-1,0),e(-1,-1,-1,-1)]=-e(-1,-1,-2,-1)
[e(0,0,0,-1),e(0,-1,-1,0)]=e(0,-1,-1,-1)
[e(0,0,0,-1),e(-1,-1,-1,0)]=e(-1,-1,-1,-1)
[e(-1,-1,0,0),e(0,0,-1,-1)]=-e(-1,-1,-1,-1)
[e(-1,-1,0,0),e(0,0,-2,-1)]=-e(-1,-1,-2,-1)
[e(-1,-1,0,0),e(0,-1,-2,-1)]=-e(-1,-2,-2,-1)
[e(-1,-1,0,0),e(-1,-1,-2,-1)]=-2e(-2,-2,-2,-1)
[e(0,-1,-1,0),e(0,0,-1,-1)]=-e(0,-1,-2,-1)
[e(0,-1,-1,0),e(0,-1,-1,-1)]=-2e(0,-2,-2,-1)
[e(0,-1,-1,0),e(-1,-1,-1,-1)]=-e(-1,-2,-2,-1)
[e(0,0,-1,-1),e(-1,-1,-1,0)]=e(-1,-1,-2,-1)
[e(-1,-1,-1,0),e(0,-1,-1,-1)]=-e(-1,-2,-2,-1)
[e(-1,-1,-1,0),e(-1,-1,-1,-1)]=-2e(-2,-2,-2,-1)
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[h(1,0,0,0),e(1,0,0,0)]=2e(1,0,0,0)
[h(1,0,0,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(1,0,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(1,0,0,0),e(0,1,1,0)]=-e(0,1,1,0)
[h(1,0,0,0),e(0,1,0,1)]=-e(0,1,0,1)
[h(1,0,0,0),e(1,1,1,0)]=e(1,1,1,0)
[h(1,0,0,0),e(1,1,0,1)]=e(1,1,0,1)
[h(1,0,0,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(1,0,0,0),e(1,1,1,1)]=e(1,1,1,1)
[h(1,0,0,0),e(-1,0,0,0)]=-2e(-1,0,0,0)
[h(1,0,0,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(1,0,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(1,0,0,0),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(1,0,0,0),e(0,-1,0,-1)]=e(0,-1,0,-1)
[h(1,0,0,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(1,0,0,0),e(-1,-1,0,-1)]=-e(-1,-1,0,-1)
[h(1,0,0,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(1,0,0,0),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(0,1,0,0),e(1,0,0,0)]=-e(1,0,0,0)
[h(0,1,0,0),e(0,1,0,0)]=2e(0,1,0,0)
[h(0,1,0,0),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,1,0,0),e(0,0,0,1)]=-e(0,0,0,1)
[h(0,1,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(0,1,0,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,1,0,0),e(0,1,0,1)]=e(0,1,0,1)
[h(0,1,0,0),e(1,1,1,1)]=-e(1,1,1,1)
```

```
[h(0,1,0,0),e(1,2,1,1)]=e(1,2,1,1)
[h(0,1,0,0),e(-1,0,0,0)]=e(-1,0,0,0)
[h(0,1,0,0),e(0,-1,0,0)]=-2e(0,-1,0,0)
[h(0,1,0,0),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,1,0,0),e(0,0,0,-1)]=e(0,0,0,-1)
[h(0,1,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(0,1,0,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,1,0,0),e(0,-1,0,-1)]=-e(0,-1,0,-1)
[h(0,1,0,0),e(-1,-1,-1,-1)]=e(-1,-1,-1,-1)
[h(0,1,0,0),e(-1,-2,-1,-1)]=-e(-1,-2,-1,-1)
[h(0,0,1,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(0,0,1,0),e(0,0,1,0)]=2e(0,0,1,0)
[h(0,0,1,0),e(1,1,0,0)]=-e(1,1,0,0)
[h(0,0,1,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,0,1,0),e(0,1,0,1)]=-e(0,1,0,1)
[h(0,0,1,0),e(1,1,1,0)]=e(1,1,1,0)
[h(0,0,1,0),e(1,1,0,1)]=-e(1,1,0,1)
[h(0,0,1,0),e(0,1,1,1)]=e(0,1,1,1)
[h(0,0,1,0),e(1,1,1,1)]=e(1,1,1,1)
[h(0,0,1,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(0,0,1,0),e(0,0,-1,0)]=-2e(0,0,-1,0)
[h(0,0,1,0),e(-1,-1,0,0)]=e(-1,-1,0,0)
[h(0,0,1,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,0,1,0),e(0,-1,0,-1)]=e(0,-1,0,-1)
[h(0,0,1,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(0,0,1,0),e(-1,-1,0,-1)]=e(-1,-1,0,-1)
[h(0,0,1,0),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,0,1,0),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(0,0,0,1),e(0,1,0,0)]=-e(0,1,0,0)
[h(0,0,0,1),e(0,0,0,1)]=2e(0,0,0,1)
[h(0,0,0,1),e(1,1,0,0)]=-e(1,1,0,0)
[h(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,0)
[h(0,0,0,1),e(0,1,0,1)]=e(0,1,0,1)
[h(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,0)
[h(0,0,0,1),e(1,1,0,1)]=e(1,1,0,1)
[h(0,0,0,1),e(0,1,1,1)]=e(0,1,1,1)
[h(0,0,0,1),e(1,1,1,1)]=e(1,1,1,1)
[h(0,0,0,1),e(0,-1,0,0)]=e(0,-1,0,0)
[h(0,0,0,1),e(0,0,0,-1)]=-2e(0,0,0,-1)
[h(0,0,0,1),e(-1,-1,0,0)]=e(-1,-1,0,0)
[h(0,0,0,1),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(0,0,0,1),e(0,-1,0,-1)]=-e(0,-1,0,-1)
[h(0,0,0,1),e(-1,-1,-1,0)]=e(-1,-1,-1,0)
[h(0,0,0,1),e(-1,-1,0,-1)]=-e(-1,-1,0,-1)
[h(0,0,0,1),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,0,0,1),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(1,0,0,0),e(0,1,0,0)]=e(1,1,0,0)
[e(1,0,0,0),e(0,1,1,0)]=e(1,1,1,0)
[e(1,0,0,0),e(0,1,0,1)]=e(1,1,0,1)
[e(1,0,0,0),e(0,1,1,1)]=e(1,1,1,1)
[e(1,0,0,0),e(-1,0,0,0)]=h(1,0,0,0)
[e(1,0,0,0),e(-1,-1,0,0)]=-e(0,-1,0,0)
[e(1,0,0,0),e(-1,-1,-1,0)]=-e(0,-1,-1,0)
[e(1,0,0,0),e(-1,-1,0,-1)]=-e(0,-1,0,-1)
[e(1,0,0,0),e(-1,-1,-1,-1)]=-e(0,-1,-1,-1)
[e(0,1,0,0),e(0,0,1,0)]=e(0,1,1,0)
[e(0,1,0,0),e(0,0,0,1)]=e(0,1,0,1)
[e(0,1,0,0),e(1,1,1,1)]=e(1,2,1,1)
[e(0,1,0,0),e(0,-1,0,0)]=h(0,1,0,0)
[e(0,1,0,0),e(-1,-1,0,0)]=e(-1,0,0,0)
[e(0,1,0,0),e(0,-1,-1,0)]=-e(0,0,-1,0)
[e(0,1,0,0),e(0,-1,0,-1)]=-e(0,0,0,-1)
[e(0,1,0,0),e(-1,-2,-1,-1)]=-e(-1,-1,-1,-1)
[e(0,0,1,0),e(1,1,0,0)]=-e(1,1,1,0)
[e(0,0,1,0),e(0,1,0,1)]=e(0,1,1,1)
[e(0,0,1,0),e(1,1,0,1)]=e(1,1,1,1)
[e(0,0,1,0),e(0,0,-1,0)]=h(0,0,1,0)
[e(0,0,1,0),e(0,-1,-1,0)]=e(0,-1,0,0)
[e(0,0,1,0),e(-1,-1,-1,0)]=e(-1,-1,0,0)
[e(0,0,1,0),e(0,-1,-1,-1)]=-e(0,-1,0,-1)
[e(0,0,1,0),e(-1,-1,-1,-1)]=-e(-1,-1,0,-1)
[e(0,0,0,1),e(1,1,0,0)]=-e(1,1,0,1)
[e(0,0,0,1),e(0,1,1,0)]=e(0,1,1,1)
[e(0,0,0,1),e(1,1,1,0)]=e(1,1,1,1)
[e(0,0,0,1),e(0,0,0,-1)]=h(0,0,0,1)
[e(0,0,0,1),e(0,-1,0,-1)]=e(0,-1,0,0)
[e(0,0,0,1),e(-1,-1,0,-1)]=e(-1,-1,0,0)
[e(0,0,0,1),e(0,-1,-1,-1)]=-e(0,-1,-1,0)
[e(0,0,0,1),e(-1,-1,-1,-1)]=-e(-1,-1,-1,0)
[e(1,1,0,0),e(0,1,1,1)]=-e(1,2,1,1)
[e(1,1,0,0),e(-1,0,0,0)]=-e(0,1,0,0)
[e(1,1,0,0),e(0,-1,0,0)]=e(1,0,0,0)
[e(1,1,0,0),e(-1,-1,0,0)]=h(1,0,0,0)+h(0,1,0,0)
[e(1,1,0,0),e(-1,-1,-1,0)]=-e(0,0,-1,0)
[e(1,1,0,0),e(-1,-1,0,-1)]=-e(0,0,0,-1)
[e(1,1,0,0),e(-1,-2,-1,-1)]=e(0,-1,-1,-1)
[e(0,1,1,0),e(1,1,0,1)]=e(1,2,1,1)
[e(0,1,1,0),e(0,-1,0,0)]=-e(0,0,1,0)
[e(0,1,1,0),e(0,0,-1,0)]=e(0,1,0,0)
[e(0,1,1,0),e(0,-1,-1,0)]=h(0,1,0,0)+h(0,0,1,0)
[e(0,1,1,0),e(-1,-1,-1,0)]=e(-1,0,0,0)
[e(0,1,1,0),e(0,-1,-1,-1)]=e(0,0,0,-1)
[e(0,1,1,0),e(-1,-2,-1,-1)]=-e(-1,-1,0,-1)
[e(0,1,0,1),e(1,1,1,0)]=e(1,2,1,1)
[e(0,1,0,1),e(0,-1,0,0)]=-e(0,0,0,1)
[e(0,1,0,1),e(0,0,0,-1)]=e(0,1,0,0)
[e(0,1,0,1),e(0,-1,0,-1)]=h(0,1,0,0)+h(0,0,0,1)
[e(0,1,0,1),e(-1,-1,0,-1)]=e(-1,0,0,0)
[e(0,1,0,1),e(0,-1,-1,-1)]=e(0,0,-1,0)
[e(0,1,0,1),e(-1,-2,-1,-1)]=-e(-1,-1,-1,0)
[e(1,1,1,0),e(-1,0,0,0)]=-e(0,1,1,0)
[e(1,1,1,0),e(0,0,-1,0)]=e(1,1,0,0)
[e(1,1,1,0),e(-1,-1,0,0)]=-e(0,0,1,0)
[e(1,1,1,0),e(0,-1,-1,0)]=e(1,0,0,0)
[e(1,1,1,0),e(-1,-1,-1,0)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)
[e(1,1,1,0),e(-1,-1,-1,-1)]=e(0,0,0,-1)
[e(1,1,1,0),e(-1,-2,-1,-1)]=e(0,-1,0,-1)
[e(1,1,0,1),e(-1,0,0,0)]=-e(0,1,0,1)
[e(1,1,0,1),e(0,0,0,-1)]=e(1,1,0,0)
[e(1,1,0,1),e(-1,-1,0,0)]=-e(0,0,0,1)
[e(1,1,0,1),e(0,-1,0,-1)]=e(1,0,0,0)
[e(1,1,0,1),e(-1,-1,0,-1)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,0,1)
[e(1,1,0,1),e(-1,-1,-1,-1)]=e(0,0,-1,0)
[e(1,1,0,1),e(-1,-2,-1,-1)]=e(0,-1,-1,0)
[e(0,1,1,1),e(0,0,-1,0)]=-e(0,1,0,1)
[e(0,1,1,1),e(0,0,0,-1)]=-e(0,1,1,0)
[e(0,1,1,1),e(0,-1,-1,0)]=e(0,0,0,1)
[e(0,1,1,1),e(0,-1,0,-1)]=e(0,0,1,0)
[e(0,1,1,1),e(0,-1,-1,-1)]=
=h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(0,1,1,1),e(-1,-1,-1,-1)]=e(-1,0,0,0)
[e(0,1,1,1),e(-1,-2,-1,-1)]=-e(-1,-1,0,0)
[e(1,1,1,1),e(-1,0,0,0)]=-e(0,1,1,1)
[e(1,1,1,1),e(0,0,-1,0)]=-e(1,1,0,1)
[e(1,1,1,1),e(0,0,0,-1)]=-e(1,1,1,0)
[e(1,1,1,1),e(-1,-1,-1,0)]=e(0,0,0,1)
[e(1,1,1,1),e(-1,-1,0,-1)]=e(0,0,1,0)
[e(1,1,1,1),e(0,-1,-1,-1)]=e(1,0,0,0)
[e(1,1,1,1),e(-1,-1,-1,-1)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(1,1,1,1),e(-1,-2,-1,-1)]=e(0,-1,0,0)
[e(1,2,1,1),e(0,-1,0,0)]=-e(1,1,1,1)
[e(1,2,1,1),e(-1,-1,0,0)]=e(0,1,1,1)
[e(1,2,1,1),e(0,-1,-1,0)]=-e(1,1,0,1)
[e(1,2,1,1),e(0,-1,0,-1)]=-e(1,1,1,0)
[e(1,2,1,1),e(-1,-1,-1,0)]=e(0,1,0,1)
[e(1,2,1,1),e(-1,-1,0,-1)]=e(0,1,1,0)
[e(1,2,1,1),e(0,-1,-1,-1)]=-e(1,1,0,0)
[e(1,2,1,1),e(-1,-1,-1,-1)]=e(0,1,0,0)
[e(1,2,1,1),e(-1,-2,-1,-1)]=
=h(1,0,0,0)+2h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(-1,0,0,0),e(0,-1,0,0)]=-e(-1,-1,0,0)
[e(-1,0,0,0),e(0,-1,-1,0)]=-e(-1,-1,-1,0)
[e(-1,0,0,0),e(0,-1,0,-1)]=-e(-1,-1,0,-1)
```

```
[e(-1,0,0,0),e(0,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(0,-1,0,0),e(0,0,-1,0)]=-e(0,-1,-1,0)
[e(0,-1,0,0),e(0,0,0,-1)]=-e(0,-1,0,-1)
[e(0,-1,0,0),e(-1,-1,-1,-1)]=-e(-1,-2,-1,-1)
[e(0,0,-1,0),e(-1,-1,0,0)]=e(-1,-1,-1,0)
[e(0,0,-1,0),e(0,-1,0,-1)]=-e(0,-1,-1,-1)
[e(0,0,-1,0),e(-1,-1,0,-1)]=-e(-1,-1,-1,-1)
[e(0,0,0,-1),e(-1,-1,0,0)]=e(-1,-1,0,-1)
[e(0,0,0,-1),e(0,-1,-1,0)]=-e(0,-1,-1,-1)
[e(0,0,0,-1),e(-1,-1,-1,0)]=-e(-1,-1,-1,-1)
[e(-1,-1,0,0),e(0,-1,-1,-1)]=e(-1,-2,-1,-1)
[e(0,-1,-1,0),e(-1,-1,0,-1)]=-e(-1,-2,-1,-1)
[e(0,-1,0,-1),e(-1,-1,-1,0)]=-e(-1,-2,-1,-1)
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[h(1,0,0,0),e(1,0,0,0)]=2e(1,0,0,0)
[h(1,0,0,0),e(0,1,0,0)]=-e(0,1,0,0)
[h(1,0,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(1,0,0,0),e(0,1,1,0)]=-e(0,1,1,0)
[h(1,0,0,0),e(1,1,1,0)]=e(1,1,1,0)
[h(1,0,0,0),e(0,1,2,0)]=-e(0,1,2,0)
[h(1,0,0,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(1,0,0,0),e(1,1,2,0)]=e(1,1,2,0)
[h(1,0,0,0),e(1,1,1,1)]=e(1,1,1,1)
[h(1,0,0,0),e(0,1,2,1)]=-e(0,1,2,1)
[h(1,0,0,0),e(1,1,2,1)]=e(1,1,2,1)
[h(1,0,0,0),e(0,1,2,2)]=-e(0,1,2,2)
[h(1,0,0,0),e(1,1,2,2)]=e(1,1,2,2)
[h(1,0,0,0),e(1,3,4,2)]=-e(1,3,4,2)
[h(1,0,0,0),e(2,3,4,2)]=e(2,3,4,2)
[h(1,0,0,0),e(-1,0,0,0)]=-2e(-1,0,0,0)
[h(1,0,0,0),e(0,-1,0,0)]=e(0,-1,0,0)
[h(1,0,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(1,0,0,0),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(1,0,0,0),e(-1,-1,-1,0)]=-e(-1,-1,-1,0)
[h(1,0,0,0),e(0,-1,-2,0)]=e(0,-1,-2,0)
[h(1,0,0,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(1,0,0,0),e(-1,-1,-2,0)]=-e(-1,-1,-2,0)
[h(1,0,0,0),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(1,0,0,0),e(0,-1,-2,-1)]=e(0,-1,-2,-1)
[h(1,0,0,0),e(-1,-1,-2,-1)]=-e(-1,-1,-2,-1)
[h(1,0,0,0),e(0,-1,-2,-2)]=e(0,-1,-2,-2)
[h(1,0,0,0),e(-1,-1,-2,-2)]=-e(-1,-1,-2,-2)
[h(1,0,0,0),e(-1,-3,-4,-2)]=e(-1,-3,-4,-2)
[h(1,0,0,0),e(-2,-3,-4,-2)]=-e(-2,-3,-4,-2)
[h(0,1,0,0),e(1,0,0,0)]=-e(1,0,0,0)
[h(0,1,0,0),e(0,1,0,0)]=2e(0,1,0,0)
[h(0,1,0,0),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,1,0,0),e(1,1,0,0)]=e(1,1,0,0)
[h(0,1,0,0),e(0,1,1,0)]=e(0,1,1,0)
[h(0,1,0,0),e(0,0,1,1)]=-e(0,0,1,1)
[h(0,1,0,0),e(0,1,1,1)]=e(0,1,1,1)
[h(0,1,0,0),e(1,1,2,0)]=-e(1,1,2,0)
[h(0,1,0,0),e(1,2,2,0)]=e(1,2,2,0)
[h(0,1,0,0),e(1,1,2,1)]=-e(1,1,2,1)
[h(0,1,0,0),e(1,2,2,1)]=e(1,2,2,1)
[h(0,1,0,0),e(1,1,2,2)]=-e(1,1,2,2)
[h(0,1,0,0),e(1,2,2,2)]=e(1,2,2,2)
[h(0,1,0,0),e(1,2,4,2)]=-e(1,2,4,2)
[h(0,1,0,0),e(1,3,4,2)]=e(1,3,4,2)
[h(0,1,0,0),e(-1,0,0,0)]=e(-1,0,0,0)
[h(0,1,0,0),e(0,-1,0,0)]=-2e(0,-1,0,0)
[h(0,1,0,0),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,1,0,0),e(-1,-1,0,0)]=-e(-1,-1,0,0)
[h(0,1,0,0),e(0,-1,-1,0)]=-e(0,-1,-1,0)
[h(0,1,0,0),e(0,0,-1,-1)]=e(0,0,-1,-1)
[h(0,1,0,0),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,1,0,0),e(-1,-1,-2,0)]=e(-1,-1,-2,0)
[h(0,1,0,0),e(-1,-2,-2,0)]=-e(-1,-2,-2,0)
[h(0,1,0,0),e(-1,-1,-2,-1)]=e(-1,-1,-2,-1)
[h(0,1,0,0),e(-1,-2,-2,-1)]=-e(-1,-2,-2,-1)
[h(0,1,0,0),e(-1,-1,-2,-2)]=e(-1,-1,-2,-2)
[h(0,1,0,0),e(-1,-2,-2,-2)]=-e(-1,-2,-2,-2)
[h(0,1,0,0),e(-1,-2,-4,-2)]=e(-1,-2,-4,-2)
[h(0,1,0,0),e(-1,-3,-4,-2)]=-e(-1,-3,-4,-2)
[h(0,0,1,0),e(0,1,0,0)]=-2e(0,1,0,0)
[h(0,0,1,0),e(0,0,1,0)]=2e(0,0,1,0)
[h(0,0,1,0),e(0,0,0,1)]=-e(0,0,0,1)
[h(0,0,1,0),e(1,1,0,0)]=-2e(1,1,0,0)
[h(0,0,1,0),e(0,0,1,1)]=e(0,0,1,1)
[h(0,0,1,0),e(0,1,2,0)]=2e(0,1,2,0)
[h(0,0,1,0),e(0,1,1,1)]=-e(0,1,1,1)
[h(0,0,1,0),e(1,1,2,0)]=2e(1,1,2,0)
[h(0,0,1,0),e(1,1,1,1)]=-e(1,1,1,1)
[h(0,0,1,0),e(0,1,2,1)]=e(0,1,2,1)
[h(0,0,1,0),e(1,1,2,1)]=e(1,1,2,1)
[h(0,0,1,0),e(1,2,2,1)]=-e(1,2,2,1)
[h(0,0,1,0),e(1,2,3,1)]=e(1,2,3,1)
[h(0,0,1,0),e(1,2,2,2)]=-2e(1,2,2,2)
[h(0,0,1,0),e(1,2,4,2)]=2e(1,2,4,2)
[h(0,0,1,0),e(0,-1,0,0)]=2e(0,-1,0,0)
[h(0,0,1,0),e(0,0,-1,0)]=-2e(0,0,-1,0)
[h(0,0,1,0),e(0,0,0,-1)]=e(0,0,0,-1)
[h(0,0,1,0),e(-1,-1,0,0)]=2e(-1,-1,0,0)
[h(0,0,1,0),e(0,0,-1,-1)]=-e(0,0,-1,-1)
[h(0,0,1,0),e(0,-1,-2,0)]=-2e(0,-1,-2,0)
[h(0,0,1,0),e(0,-1,-1,-1)]=e(0,-1,-1,-1)
[h(0,0,1,0),e(-1,-1,-2,0)]=-2e(-1,-1,-2,0)
[h(0,0,1,0),e(-1,-1,-1,-1)]=e(-1,-1,-1,-1)
[h(0,0,1,0),e(0,-1,-2,-1)]=-e(0,-1,-2,-1)
[h(0,0,1,0),e(-1,-1,-2,-1)]=-e(-1,-1,-2,-1)
[h(0,0,1,0),e(-1,-2,-2,-1)]=e(-1,-2,-2,-1)
[h(0,0,1,0),e(-1,-2,-3,-1)]=-e(-1,-2,-3,-1)
[h(0,0,1,0),e(-1,-2,-2,-2)]=2e(-1,-2,-2,-2)
[h(0,0,1,0),e(-1,-2,-4,-2)]=-2e(-1,-2,-4,-2)
[h(0,0,0,1),e(0,0,1,0)]=-e(0,0,1,0)
[h(0,0,0,1),e(0,0,0,1)]=2e(0,0,0,1)
[h(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,0)
[h(0,0,0,1),e(0,0,1,1)]=e(0,0,1,1)
[h(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,0)
[h(0,0,0,1),e(0,1,2,0)]=-2e(0,1,2,0)
[h(0,0,0,1),e(0,1,1,1)]=e(0,1,1,1)
[h(0,0,0,1),e(1,1,2,0)]=-2e(1,1,2,0)
[h(0,0,0,1),e(1,1,1,1)]=e(1,1,1,1)
[h(0,0,0,1),e(1,2,2,0)]=-2e(1,2,2,0)
[h(0,0,0,1),e(0,1,2,2)]=2e(0,1,2,2)
[h(0,0,0,1),e(1,1,2,2)]=2e(1,1,2,2)
[h(0,0,0,1),e(1,2,3,1)]=-e(1,2,3,1)
[h(0,0,0,1),e(1,2,2,2)]=2e(1,2,2,2)
[h(0,0,0,1),e(1,2,3,2)]=e(1,2,3,2)
[h(0,0,0,1),e(0,0,-1,0)]=e(0,0,-1,0)
[h(0,0,0,1),e(0,0,0,-1)]=-2e(0,0,0,-1)
[h(0,0,0,1),e(0,-1,-1,0)]=e(0,-1,-1,0)
[h(0,0,0,1),e(0,0,-1,-1)]=-e(0,0,-1,-1)
[h(0,0,0,1),e(-1,-1,-1,0)]=e(-1,-1,-1,0)
[h(0,0,0,1),e(0,-1,-2,0)]=2e(0,-1,-2,0)
[h(0,0,0,1),e(0,-1,-1,-1)]=-e(0,-1,-1,-1)
[h(0,0,0,1),e(-1,-1,-2,0)]=2e(-1,-1,-2,0)
[h(0,0,0,1),e(-1,-1,-1,-1)]=-e(-1,-1,-1,-1)
[h(0,0,0,1),e(-1,-2,-2,0)]=2e(-1,-2,-2,0)
[h(0,0,0,1),e(0,-1,-2,-2)]=-2e(0,-1,-2,-2)
[h(0,0,0,1),e(-1,-1,-2,-2)]=-2e(-1,-1,-2,-2)
[h(0,0,0,1),e(-1,-2,-3,-1)]=e(-1,-2,-3,-1)
[h(0,0,0,1),e(-1,-2,-2,-2)]=-2e(-1,-2,-2,-2)
[h(0,0,0,1),e(-1,-2,-3,-2)]=-e(-1,-2,-3,-2)
[e(1,0,0,0),e(0,1,0,0)]=e(1,1,0,0)
[e(1,0,0,0),e(0,1,1,0)]=e(1,1,1,0)
[e(1,0,0,0),e(0,1,2,0)]=e(1,1,2,0)
[e(1,0,0,0),e(0,1,1,1)]=e(1,1,1,1)
[e(1,0,0,0),e(0,1,2,1)]=e(1,1,2,1)
[e(1,0,0,0),e(0,1,2,2)]=e(1,1,2,2)
[e(1,0,0,0),e(1,3,4,2)]=e(2,3,4,2)
[e(1,0,0,0),e(-1,0,0,0)]=h(1,0,0,0)
[e(1,0,0,0),e(-1,-1,0,0)]=-e(0,-1,0,0)
[e(1,0,0,0),e(-1,-1,-1,0)]=-e(0,-1,-1,0)
```

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[e(1,0,0,0),e(-1,-1,-2,0)]=-e(0,-1,-2,0)
[e(1,0,0,0),e(-1,-1,-1,-1)]=-e(0,-1,-1,-1)
[e(1,0,0,0),e(-1,-1,-2,-1)]=-e(0,-1,-2,-1)
[e(1,0,0,0),e(-1,-1,-2,-2)]=-e(0,-1,-2,-2)
[e(1,0,0,0),e(-2,-3,-4,-2)]=-e(-1,-3,-4,-2)
[e(0,1,0,0),e(0,0,1,0)]=e(0,1,1,0)
[e(0,1,0,0),e(0,0,1,1)]=e(0,1,1,1)
[e(0,1,0,0),e(1,1,2,0)]=e(1,2,2,0)
[e(0,1,0,0),e(1,1,2,1)]=e(1,2,2,1)
[e(0,1,0,0),e(1,1,2,2)]=e(1,2,2,2)
[e(0,1,0,0),e(1,2,4,2)]=e(1,3,4,2)
[e(0,1,0,0),e(0,-1,0,0)]=h(0,1,0,0)
[e(0,1,0,0),e(-1,-1,0,0)]=e(-1,0,0,0)
[e(0,1,0,0),e(0,-1,-1,0)]=-e(0,0,-1,0)
[e(0,1,0,0),e(0,-1,-1,-1)]=-e(0,0,-1,-1)
[e(0,1,0,0),e(-1,-2,-2,0)]=-e(-1,-1,-2,0)
[e(0,1,0,0),e(-1,-2,-2,-1)]=-e(-1,-1,-2,-1)
[e(0,1,0,0),e(-1,-2,-2,-2)]=-e(-1,-1,-2,-2)
[e(0,1,0,0),e(-1,-3,-4,-2)]=-e(-1,-2,-4,-2)
[e(0,0,1,0),e(0,0,0,1)]=e(0,0,1,1)
[e(0,0,1,0),e(1,1,0,0)]=-e(1,1,1,0)
[e(0,0,1,0),e(0,1,1,0)]=2e(0,1,2,0)
[e(0,0,1,0),e(1,1,1,0)]=2e(1,1,2,0)
[e(0,0,1,0),e(0,1,1,1)]=e(0,1,2,1)
[e(0,0,1,0),e(1,1,1,1)]=e(1,1,2,1)
[e(0,0,1,0),e(1,2,2,1)]=e(1,2,3,1)
[e(0,0,1,0),e(1,2,2,2)]=e(1,2,3,2)
[e(0,0,1,0),e(1,2,3,2)]=2e(1,2,4,2)
[e(0,0,1,0),e(0,0,-1,0)]=h(0,0,1,0)
[e(0,0,1,0),e(0,-1,-1,0)]=2e(0,-1,0,0)
[e(0,0,1,0),e(0,0,-1,-1)]=-e(0,0,0,-1)
[e(0,0,1,0),e(-1,-1,-1,0)]=2e(-1,-1,0,0)
[e(0,0,1,0),e(0,-1,-2,0)]=-e(0,-1,-1,0)
[e(0,0,1,0),e(-1,-1,-2,0)]=-e(-1,-1,-1,0)
[e(0,0,1,0),e(0,-1,-2,-1)]=-e(0,-1,-1,-1)
[e(0,0,1,0),e(-1,-1,-2,-1)]=-e(-1,-1,-1,-1)
[e(0,0,1,0),e(-1,-2,-3,-1)]=-e(-1,-2,-2,-1)
[e(0,0,1,0),e(-1,-2,-3,-2)]=-2e(-1,-2,-2,-2)
[e(0,0,1,0),e(-1,-2,-4,-2)]=-e(-1,-2,-3,-2)
[e(0,0,0,1),e(0,1,1,0)]=-e(0,1,1,1)
[e(0,0,0,1),e(1,1,1,0)]=-e(1,1,1,1)
[e(0,0,0,1),e(0,1,2,0)]=-e(0,1,2,1)
[e(0,0,0,1),e(1,1,2,0)]=-e(1,1,2,1)
[e(0,0,0,1),e(0,1,2,1)]=2e(0,1,2,2)
[e(0,0,0,1),e(1,2,2,0)]=-e(1,2,2,1)
[e(0,0,0,1),e(1,1,2,1)]=2e(1,1,2,2)
[e(0,0,0,1),e(1,2,2,1)]=2e(1,2,2,2)
[e(0,0,0,1),e(1,2,3,1)]=e(1,2,3,2)
[e(0,0,0,1),e(0,0,0,-1)]=h(0,0,0,1)
[e(0,0,0,1),e(0,0,-1,-1)]=e(0,0,-1,0)
[e(0,0,0,1),e(0,-1,-1,-1)]=e(0,-1,-1,0)
[e(0,0,0,1),e(-1,-1,-1,-1)]=e(-1,-1,-1,0)
[e(0,0,0,1),e(0,-1,-2,-1)]=2e(0,-1,-2,0)
[e(0,0,0,1),e(-1,-1,-2,-1)]=2e(-1,-1,-2,0)
[e(0,0,0,1),e(0,-1,-2,-2)]=-e(0,-1,-2,-1)
[e(0,0,0,1),e(-1,-2,-2,-1)]=2e(-1,-2,-2,0)
[e(0,0,0,1),e(-1,-1,-2,-2)]=-e(-1,-1,-2,-1)
[e(0,0,0,1),e(-1,-2,-2,-2)]=-e(-1,-2,-2,-1)
[e(0,0,0,1),e(-1,-2,-3,-2)]=-e(-1,-2,-3,-1)
[e(1,1,0,0),e(0,0,1,1)]=e(1,1,1,1)
[e(1,1,0,0),e(0,1,2,0)]=-e(1,2,2,0)
[e(1,1,0,0),e(0,1,2,1)]=-e(1,2,2,1)
[e(1,1,0,0),e(0,1,2,2)]=-e(1,2,2,2)
[e(1,1,0,0),e(1,2,4,2)]=e(2,3,4,2)
[e(1,1,0,0),e(-1,0,0,0)]=-e(0,1,0,0)
[e(1,1,0,0),e(0,-1,0,0)]=e(1,0,0,0)
[e(1,1,0,0),e(-1,-1,0,0)]=h(1,0,0,0)+h(0,1,0,0)
[e(1,1,0,0),e(-1,-1,-1,0)]=-e(0,0,-1,0)
[e(1,1,0,0),e(-1,-1,-1,-1)]=-e(0,0,-1,-1)
[e(1,1,0,0),e(-1,-2,-2,0)]=e(0,-1,-2,0)
[e(1,1,0,0),e(-1,-2,-2,-1)]=e(0,-1,-2,-1)
[e(1,1,0,0),e(-1,-2,-2,-2)]=e(0,-1,-2,-2)
[e(1,1,0,0),e(-2,-3,-4,-2)]=-e(-1,-2,-4,-2)
[e(0,1,1,0),e(0,0,1,1)]=-e(0,1,2,1)
[e(0,1,1,0),e(1,1,1,0)]=2e(1,2,2,0)
[e(0,1,1,0),e(1,1,1,1)]=e(1,2,2,1)
[e(0,1,1,0),e(1,1,2,1)]=-e(1,2,3,1)
[e(0,1,1,0),e(1,1,2,2)]=-e(1,2,3,2)
[e(0,1,1,0),e(1,2,3,2)]=2e(1,3,4,2)
[e(0,1,1,0),e(0,-1,0,0)]=-e(0,0,1,0)
[e(0,1,1,0),e(0,0,-1,0)]=2e(0,1,0,0)
[e(0,1,1,0),e(0,-1,-1,0)]=
=2h(0,1,0,0)+h(0,0,1,0)
[e(0,1,1,0),e(-1,-1,-1,0)]=2e(-1,0,0,0)
[e(0,1,1,0),e(0,-1,-2,0)]=e(0,0,-1,0)
[e(0,1,1,0),e(0,-1,-1,-1)]=-e(0,0,0,-1)
[e(0,1,1,0),e(0,-1,-2,-1)]=e(0,0,-1,-1)
[e(0,1,1,0),e(-1,-2,-2,0)]=-e(-1,-1,-1,0)
[e(0,1,1,0),e(-1,-2,-2,-1)]=-e(-1,-1,-1,-1)
[e(0,1,1,0),e(-1,-2,-3,-1)]=e(-1,-1,-2,-1)
[e(0,1,1,0),e(-1,-2,-3,-2)]=2e(-1,-1,-2,-2)
[e(0,1,1,0),e(-1,-3,-4,-2)]=-e(-1,-2,-3,-2)
[e(0,0,1,1),e(1,1,1,0)]=e(1,1,2,1)
[e(0,0,1,1),e(0,1,1,1)]=-2e(0,1,2,2)
[e(0,0,1,1),e(1,1,1,1)]=-2e(1,1,2,2)
[e(0,0,1,1),e(1,2,2,0)]=-e(1,2,3,1)
[e(0,0,1,1),e(1,2,2,1)]=e(1,2,3,2)
[e(0,0,1,1),e(1,2,3,1)]=2e(1,2,4,2)
[e(0,0,1,1),e(0,0,-1,0)]=-e(0,0,0,1)
[e(0,0,1,1),e(0,0,0,-1)]=e(0,0,1,0)
[e(0,0,1,1),e(0,0,-1,-1)]=h(0,0,1,0)+h(0,0,0,1)
[e(0,0,1,1),e(0,-1,-1,-1)]=2e(0,-1,0,0)
[e(0,0,1,1),e(-1,-1,-1,-1)]=2e(-1,-1,0,0)
[e(0,0,1,1),e(0,-1,-2,-1)]=-e(0,-1,-1,0)
[e(0,0,1,1),e(-1,-1,-2,-1)]=-e(-1,-1,-1,0)
[e(0,0,1,1),e(0,-1,-2,-2)]=e(0,-1,-1,-1)
[e(0,0,1,1),e(-1,-1,-2,-2)]=e(-1,-1,-1,-1)
[e(0,0,1,1),e(-1,-2,-3,-1)]=2e(-1,-2,-2,0)
[e(0,0,1,1),e(-1,-2,-3,-2)]=-e(-1,-2,-2,-1)
[e(0,0,1,1),e(-1,-2,-4,-2)]=-e(-1,-2,-3,-1)
[e(1,1,1,0),e(0,1,1,1)]=-e(1,2,2,1)
[e(1,1,1,0),e(0,1,2,1)]=e(1,2,3,1)
[e(1,1,1,0),e(0,1,2,2)]=e(1,2,3,2)
[e(1,1,1,0),e(1,2,3,2)]=2e(2,3,4,2)
[e(1,1,1,0),e(-1,0,0,0)]=-e(0,1,1,0)
[e(1,1,1,0),e(0,0,-1,0)]=2e(1,1,0,0)
[e(1,1,1,0),e(-1,-1,0,0)]=-e(0,0,1,0)
[e(1,1,1,0),e(0,-1,-1,0)]=2e(1,0,0,0)
[e(1,1,1,0),e(-1,-1,-1,0)]=
=2h(1,0,0,0)+2h(0,1,0,0)+h(0,0,1,0)
[e(1,1,1,0),e(-1,-1,-2,0)]=e(0,0,-1,0)
[e(1,1,1,0),e(-1,-1,-1,-1)]=-e(0,0,0,-1)
[e(1,1,1,0),e(-1,-2,-2,0)]=e(0,-1,-1,0)
[e(1,1,1,0),e(-1,-1,-2,-1)]=e(0,0,-1,-1)
[e(1,1,1,0),e(-1,-2,-2,-1)]=e(0,-1,-1,-1)
[e(1,1,1,0),e(-1,-2,-3,-1)]=-e(0,-1,-2,-1)
[e(1,1,1,0),e(-1,-2,-3,-2)]=-2e(0,-1,-2,-2)
[e(1,1,1,0),e(-2,-3,-4,-2)]=-e(-1,-2,-3,-2)
[e(0,1,2,0),e(1,1,1,1)]=e(1,2,3,1)
[e(0,1,2,0),e(1,1,2,2)]=-e(1,2,4,2)
[e(0,1,2,0),e(1,2,2,2)]=-e(1,3,4,2)
[e(0,1,2,0),e(0,0,-1,0)]=-e(0,1,1,0)
[e(0,1,2,0),e(0,-1,-1,0)]=e(0,0,1,0)
[e(0,1,2,0),e(0,-1,-2,0)]=h(0,1,0,0)+h(0,0,1,0)
[e(0,1,2,0),e(-1,-1,-2,0)]=e(-1,0,0,0)
[e(0,1,2,0),e(0,-1,-2,-1)]=-e(0,0,0,-1)
[e(0,1,2,0),e(-1,-2,-2,0)]=-e(-1,-1,0,0)
[e(0,1,2,0),e(-1,-2,-3,-1)]=-e(-1,-1,-1,-1)
[e(0,1,2,0),e(-1,-2,-4,-2)]=e(-1,-1,-2,-2)
[e(0,1,2,0),e(-1,-3,-4,-2)]=e(-1,-2,-2,-2)
[e(0,1,1,1),e(1,1,2,0)]=e(1,2,3,1)
[e(0,1,1,1),e(1,1,1,1)]=-2e(1,2,2,2)
[e(0,1,1,1),e(1,1,2,1)]=-e(1,2,3,2)
[e(0,1,1,1),e(1,2,3,1)]=2e(1,3,4,2)
[e(0,1,1,1),e(0,-1,0,0)]=-e(0,0,1,1)
[e(0,1,1,1),e(0,0,0,-1)]=e(0,1,1,0)
```

```
[e(0,1,1,1),e(0,-1,-1,0)]=-e(0,0,0,1)
[e(0,1,1,1),e(0,0,-1,-1)]=2e(0,1,0,0)
[e(0,1,1,1),e(0,-1,-1,-1)]=
=2h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(0,1,1,1),e(-1,-1,-1,-1)]=2e(-1,0,0,0)
[e(0,1,1,1),e(0,-1,-2,-1)]=e(0,0,-1,0)
[e(0,1,1,1),e(0,-1,-2,-2)]=-e(0,0,-1,-1)
[e(0,1,1,1),e(-1,-2,-2,-1)]=-e(-1,-1,-1,0)
[e(0,1,1,1),e(-1,-2,-3,-1)]=-2e(-1,-1,-2,0)
[e(0,1,1,1),e(-1,-2,-2,-2)]=e(-1,-1,-1,-1)
[e(0,1,1,1),e(-1,-2,-3,-2)]=e(-1,-1,-2,-1)
[e(0,1,1,1),e(-1,-3,-4,-2)]=-e(-1,-2,-3,-1)
[e(1,1,2,0),e(0,1,2,2)]=e(1,2,4,2)
[e(1,1,2,0),e(1,2,2,2)]=-e(2,3,4,2)
[e(1,1,2,0),e(-1,0,0,0)]=-e(0,1,2,0)
[e(1,1,2,0),e(0,0,-1,0)]=-e(1,1,1,0)
[e(1,1,2,0),e(-1,-1,-1,0)]=e(0,0,1,0)
[e(1,1,2,0),e(0,-1,-2,0)]=e(1,0,0,0)
[e(1,1,2,0),e(-1,-1,-2,0)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)
[e(1,1,2,0),e(-1,-2,-2,0)]=e(0,-1,0,0)
[e(1,1,2,0),e(-1,-1,-2,-1)]=-e(0,0,0,-1)
[e(1,1,2,0),e(-1,-2,-3,-1)]=e(0,-1,-1,-1)
[e(1,1,2,0),e(-1,-2,-4,-2)]=-e(0,-1,-2,-2)
[e(1,1,2,0),e(-2,-3,-4,-2)]=e(-1,-2,-2,-2)
[e(1,1,1,1),e(0,1,2,1)]=e(1,2,3,2)
[e(1,1,1,1),e(1,2,3,1)]=2e(2,3,4,2)
[e(1,1,1,1),e(-1,0,0,0)]=-e(0,1,1,1)
[e(1,1,1,1),e(0,0,0,-1)]=e(1,1,1,0)
[e(1,1,1,1),e(-1,-1,0,0)]=-e(0,0,1,1)
[e(1,1,1,1),e(0,0,-1,-1)]=2e(1,1,0,0)
[e(1,1,1,1),e(-1,-1,-1,0)]=-e(0,0,0,1)
[e(1,1,1,1),e(0,-1,-1,-1)]=2e(1,0,0,0)
[e(1,1,1,1),e(-1,-1,-1,-1)]=
=2h(1,0,0,0)+2h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(1,1,1,1),e(-1,-1,-2,-1)]=e(0,0,-1,0)
[e(1,1,1,1),e(-1,-2,-2,-1)]=e(0,-1,-1,0)
[e(1,1,1,1),e(-1,-1,-2,-2)]=-e(0,0,-1,-1)
[e(1,1,1,1),e(-1,-2,-3,-1)]=2e(0,-1,-2,0)
[e(1,1,1,1),e(-1,-2,-2,-2)]=-e(0,-1,-1,-1)
[e(1,1,1,1),e(-1,-2,-3,-2)]=-e(0,-1,-2,-1)
[e(1,1,1,1),e(-2,-3,-4,-2)]=-e(-1,-2,-3,-1)
[e(0,1,2,1),e(1,1,2,1)]=-2e(1,2,4,2)
[e(0,1,2,1),e(1,2,2,1)]=-2e(1,3,4,2)
[e(0,1,2,1),e(0,0,-1,0)]=-e(0,1,1,1)
[e(0,1,2,1),e(0,0,0,-1)]=2e(0,1,2,0)
[e(0,1,2,1),e(0,-1,-1,0)]=e(0,0,1,1)
[e(0,1,2,1),e(0,0,-1,-1)]=-e(0,1,1,0)
[e(0,1,2,1),e(0,-1,-2,0)]=-e(0,0,0,1)
[e(0,1,2,1),e(0,-1,-1,-1)]=e(0,0,1,0)
[e(0,1,2,1),e(0,-1,-2,-1)]=
=2h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(0,1,2,1),e(-1,-1,-2,-1)]=2e(-1,0,0,0)
[e(0,1,2,1),e(0,-1,-2,-2)]=e(0,0,0,-1)
[e(0,1,2,1),e(-1,-2,-2,-1)]=-2e(-1,-1,0,0)
[e(0,1,2,1),e(-1,-2,-3,-1)]=e(-1,-1,-1,0)
[e(0,1,2,1),e(-1,-2,-3,-2)]=e(-1,-1,-1,-1)
[e(0,1,2,1),e(-1,-2,-4,-2)]=e(-1,-1,-2,-1)
[e(0,1,2,1),e(-1,-3,-4,-2)]=e(-1,-2,-2,-1)
[e(1,2,2,0),e(0,1,2,2)]=e(1,3,4,2)
[e(1,2,2,0),e(1,1,2,2)]=e(2,3,4,2)
[e(1,2,2,0),e(0,-1,0,0)]=-e(1,1,2,0)
[e(1,2,2,0),e(-1,-1,0,0)]=e(0,1,2,0)
[e(1,2,2,0),e(0,-1,-1,0)]=-e(1,1,1,0)
[e(1,2,2,0),e(-1,-1,-1,0)]=e(0,1,1,0)
[e(1,2,2,0),e(0,-1,-2,0)]=-e(1,1,0,0)
[e(1,2,2,0),e(-1,-1,-2,0)]=e(0,1,0,0)
[e(1,2,2,0),e(-1,-2,-2,0)]=
=h(1,0,0,0)+2h(0,1,0,0)+h(0,0,1,0)
[e(1,2,2,0),e(-1,-2,-2,-1)]=-e(0,0,0,-1)
[e(1,2,2,0),e(-1,-2,-3,-1)]=-e(0,0,-1,-1)
[e(1,2,2,0),e(-1,-3,-4,-2)]=-e(0,-1,-2,-2)
[e(1,2,2,0),e(-2,-3,-4,-2)]=-e(-1,-1,-2,-2)
[e(1,1,2,1),e(1,2,2,1)]=-2e(2,3,4,2)
[e(1,1,2,1),e(-1,0,0,0)]=-e(0,1,2,1)
[e(1,1,2,1),e(0,0,-1,0)]=-e(1,1,1,1)
[e(1,1,2,1),e(0,0,0,-1)]=2e(1,1,2,0)
[e(1,1,2,1),e(0,0,-1,-1)]=-e(1,1,1,0)
[e(1,1,2,1),e(-1,-1,-1,0)]=e(0,0,1,1)
[e(1,1,2,1),e(-1,-1,-2,0)]=-e(0,0,0,1)
[e(1,1,2,1),e(-1,-1,-1,-1)]=e(0,0,1,0)
[e(1,1,2,1),e(0,-1,-2,-1)]=2e(1,0,0,0)
[e(1,1,2,1),e(-1,-1,-2,-1)]=
=2h(1,0,0,0)+2h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(1,1,2,1),e(-1,-2,-2,-1)]=2e(0,-1,0,0)
[e(1,1,2,1),e(-1,-1,-2,-2)]=e(0,0,0,-1)
[e(1,1,2,1),e(-1,-2,-3,-1)]=-e(0,-1,-1,0)
[e(1,1,2,1),e(-1,-2,-3,-2)]=-e(0,-1,-1,-1)
[e(1,1,2,1),e(-1,-2,-4,-2)]=-e(0,-1,-2,-1)
[e(1,1,2,1),e(-2,-3,-4,-2)]=e(-1,-2,-2,-1)
[e(0,1,2,2),e(0,0,0,-1)]=-e(0,1,2,1)
[e(0,1,2,2),e(0,0,-1,-1)]=e(0,1,1,1)
[e(0,1,2,2),e(0,-1,-1,-1)]=-e(0,0,1,1)
[e(0,1,2,2),e(0,-1,-2,-1)]=e(0,0,0,1)
[e(0,1,2,2),e(0,-1,-2,-2)]=
=h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(0,1,2,2),e(-1,-1,-2,-2)]=e(-1,0,0,0)
[e(0,1,2,2),e(-1,-2,-2,-2)]=-e(-1,-1,0,0)
[e(0,1,2,2),e(-1,-2,-3,-2)]=e(-1,-1,-1,0)
[e(0,1,2,2),e(-1,-2,-4,-2)]=e(-1,-1,-2,0)
[e(0,1,2,2),e(-1,-3,-4,-2)]=e(-1,-2,-2,0)
[e(1,2,2,1),e(0,-1,0,0)]=-e(1,1,2,1)
[e(1,2,2,1),e(0,0,0,-1)]=2e(1,2,2,0)
[e(1,2,2,1),e(-1,-1,0,0)]=e(0,1,2,1)
[e(1,2,2,1),e(0,-1,-1,0)]=-e(1,1,1,1)
[e(1,2,2,1),e(-1,-1,-1,0)]=e(0,1,1,1)
[e(1,2,2,1),e(0,-1,-1,-1)]=-e(1,1,1,0)
[e(1,2,2,1),e(-1,-1,-1,-1)]=e(0,1,1,0)
[e(1,2,2,1),e(0,-1,-2,-1)]=-2e(1,1,0,0)
[e(1,2,2,1),e(-1,-2,-2,0)]=-e(0,0,0,1)
[e(1,2,2,1),e(-1,-1,-2,-1)]=2e(0,1,0,0)
[e(1,2,2,1),e(-1,-2,-2,-1)]=
=2h(1,0,0,0)+4h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(1,2,2,1),e(-1,-2,-3,-1)]=e(0,0,-1,0)
[e(1,2,2,1),e(-1,-2,-2,-2)]=e(0,0,0,-1)
[e(1,2,2,1),e(-1,-2,-3,-2)]=e(0,0,-1,-1)
[e(1,2,2,1),e(-1,-3,-4,-2)]=-e(0,-1,-2,-1)
[e(1,2,2,1),e(-2,-3,-4,-2)]=-e(-1,-1,-2,-1)
[e(1,1,2,2),e(-1,0,0,0)]=-e(0,1,2,2)
[e(1,1,2,2),e(0,0,0,-1)]=-e(1,1,2,1)
[e(1,1,2,2),e(0,0,-1,-1)]=e(1,1,1,1)
[e(1,1,2,2),e(-1,-1,-1,-1)]=-e(0,0,1,1)
[e(1,1,2,2),e(-1,-1,-2,-1)]=e(0,0,0,1)
[e(1,1,2,2),e(0,-1,-2,-2)]=e(1,0,0,0)
[e(1,1,2,2),e(-1,-1,-2,-2)]=
=h(1,0,0,0)+h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(1,1,2,2),e(-1,-2,-2,-2)]=e(0,-1,0,0)
[e(1,1,2,2),e(-1,-2,-3,-2)]=-e(0,-1,-1,0)
[e(1,1,2,2),e(-1,-2,-4,-2)]=-e(0,-1,-2,0)
[e(1,1,2,2),e(-2,-3,-4,-2)]=e(-1,-2,-2,0)
[e(1,2,3,1),e(0,0,-1,0)]=-e(1,2,2,1)
[e(1,2,3,1),e(0,-1,-1,0)]=e(1,1,2,1)
[e(1,2,3,1),e(0,0,-1,-1)]=2e(1,2,2,0)
[e(1,2,3,1),e(-1,-1,-1,0)]=-e(0,1,2,1)
[e(1,2,3,1),e(0,-1,-2,0)]=-e(1,1,1,1)
[e(1,2,3,1),e(0,-1,-1,-1)]=-2e(1,1,2,0)
[e(1,2,3,1),e(-1,-1,-2,0)]=e(0,1,1,1)
[e(1,2,3,1),e(-1,-1,-1,-1)]=2e(0,1,2,0)
[e(1,2,3,1),e(0,-1,-2,-1)]=e(1,1,1,0)
[e(1,2,3,1),e(-1,-2,-2,0)]=-e(0,0,1,1)
[e(1,2,3,1),e(-1,-1,-2,-1)]=-e(0,1,1,0)
[e(1,2,3,1),e(-1,-2,-2,-1)]=e(0,0,1,0)
[e(1,2,3,1),e(-1,-2,-3,-1)]=
=2h(1,0,0,0)+4h(0,1,0,0)+3h(0,0,1,0)+h(0,0,0,1)
[e(1,2,3,1),e(-1,-2,-3,-2)]=e(0,0,0,-1)
[e(1,2,3,1),e(-1,-2,-4,-2)]=e(0,0,-1,-1)
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[e(1,2,3,1),e(-1,-3,-4,-2)]=e(0,-1,-1,-1)
[e(1,2,3,1),e(-2,-3,-4,-2)]=e(-1,-1,-1,-1)
[e(1,2,2,2),e(0,-1,0,0)]=-e(1,1,2,2)
[e(1,2,2,2),e(0,0,0,-1)]=-e(1,2,2,1)
[e(1,2,2,2),e(-1,-1,0,0)]=e(0,1,2,2)
[e(1,2,2,2),e(0,-1,-1,-1)]=e(1,1,1,1)
[e(1,2,2,2),e(-1,-1,-1,-1)]=-e(0,1,1,1)
[e(1,2,2,2),e(0,-1,-2,-2)]=-e(1,1,0,0)
[e(1,2,2,2),e(-1,-2,-2,-1)]=e(0,0,0,1)
[e(1,2,2,2),e(-1,-1,-2,-2)]=e(0,1,0,0)
[e(1,2,2,2),e(-1,-2,-2,-2)]=
=h(1,0,0,0)+2h(0,1,0,0)+h(0,0,1,0)+h(0,0,0,1)
[e(1,2,2,2),e(-1,-2,-3,-2)]=e(0,0,-1,0)
[e(1,2,2,2),e(-1,-3,-4,-2)]=-e(0,-1,-2,0)
[e(1,2,2,2),e(-2,-3,-4,-2)]=-e(-1,-1,-2,0)
[e(1,2,3,2),e(0,0,-1,0)]=-2e(1,2,2,2)
[e(1,2,3,2),e(0,0,0,-1)]=-e(1,2,3,1)
[e(1,2,3,2),e(0,-1,-1,0)]=2e(1,1,2,2)
[e(1,2,3,2),e(0,0,-1,-1)]=-e(1,2,2,1)
[e(1,2,3,2),e(-1,-1,-1,0)]=-2e(0,1,2,2)
[e(1,2,3,2),e(0,-1,-1,-1)]=e(1,1,2,1)
[e(1,2,3,2),e(-1,-1,-1,-1)]=-e(0,1,2,1)
[e(1,2,3,2),e(0,-1,-2,-1)]=e(1,1,1,1)
[e(1,2,3,2),e(-1,-1,-2,-1)]=-e(0,1,1,1)
[e(1,2,3,2),e(0,-1,-2,-2)]=e(1,1,1,0)
[e(1,2,3,2),e(-1,-2,-2,-1)]=e(0,0,1,1)
[e(1,2,3,2),e(-1,-1,-2,-2)]=-e(0,1,1,0)
[e(1,2,3,2),e(-1,-2,-3,-1)]=e(0,0,0,1)
[e(1,2,3,2),e(-1,-2,-2,-2)]=e(0,0,1,0)
[e(1,2,3,2),e(-1,-2,-3,-2)]=2h(1,0,0,0)+
+4h(0,1,0,0)+3h(0,0,1,0)+2h(0,0,0,1)
[e(1,2,3,2),e(-1,-2,-4,-2)]=e(0,0,-1,0)
[e(1,2,3,2),e(-1,-3,-4,-2)]=e(0,-1,-1,0)
[e(1,2,3,2),e(-2,-3,-4,-2)]=e(-1,-1,-1,0)
[e(1,2,4,2),e(0,0,-1,0)]=-e(1,2,3,2)
[e(1,2,4,2),e(0,0,-1,-1)]=-e(1,2,3,1)
[e(1,2,4,2),e(0,-1,-2,0)]=e(1,1,2,2)
[e(1,2,4,2),e(-1,-1,-2,0)]=-e(0,1,2,2)
[e(1,2,4,2),e(0,-1,-2,-1)]=e(1,1,2,1)
[e(1,2,4,2),e(-1,-1,-2,-1)]=-e(0,1,2,1)
[e(1,2,4,2),e(0,-1,-2,-2)]=e(1,1,2,0)
[e(1,2,4,2),e(-1,-1,-2,-2)]=-e(0,1,2,0)
[e(1,2,4,2),e(-1,-2,-3,-1)]=e(0,0,1,1)
[e(1,2,4,2),e(-1,-2,-3,-2)]=e(0,0,1,0)
[e(1,2,4,2),e(-1,-2,-4,-2)]=
=h(1,0,0,0)+2h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(1,2,4,2),e(-1,-3,-4,-2)]=e(0,-1,0,0)
[e(1,2,4,2),e(-2,-3,-4,-2)]=e(-1,-1,0,0)
[e(1,3,4,2),e(0,-1,0,0)]=-e(1,2,4,2)
[e(1,3,4,2),e(0,-1,-1,0)]=-e(1,2,3,2)
[e(1,3,4,2),e(0,-1,-2,0)]=e(1,2,2,2)
[e(1,3,4,2),e(0,-1,-1,-1)]=-e(1,2,3,1)
[e(1,3,4,2),e(0,-1,-2,-1)]=e(1,2,2,1)
[e(1,3,4,2),e(-1,-2,-2,0)]=-e(0,1,2,2)
[e(1,3,4,2),e(0,-1,-2,-2)]=e(1,2,2,0)
[e(1,3,4,2),e(-1,-2,-2,-1)]=-e(0,1,2,1)
[e(1,3,4,2),e(-1,-2,-3,-1)]=e(0,1,1,1)
[e(1,3,4,2),e(-1,-2,-2,-2)]=-e(0,1,2,0)
[e(1,3,4,2),e(-1,-2,-3,-2)]=e(0,1,1,0)
[e(1,3,4,2),e(-1,-2,-4,-2)]=e(0,1,0,0)
[e(1,3,4,2),e(-1,-3,-4,-2)]=
=h(1,0,0,0)+3h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(1,3,4,2),e(-2,-3,-4,-2)]=e(-1,0,0,0)
[e(2,3,4,2),e(-1,0,0,0)]=-e(1,3,4,2)
[e(2,3,4,2),e(-1,-1,0,0)]=-e(1,2,4,2)
[e(2,3,4,2),e(-1,-1,-1,0)]=-e(1,2,3,2)
[e(2,3,4,2),e(-1,-1,-2,0)]=e(1,2,2,2)
[e(2,3,4,2),e(-1,-1,-1,-1)]=-e(1,2,3,1)
[e(2,3,4,2),e(-1,-2,-2,0)]=-e(1,1,2,2)
[e(2,3,4,2),e(-1,-1,-2,-1)]=e(1,2,2,1)
[e(2,3,4,2),e(-1,-2,-2,-1)]=-e(1,1,2,1)
[e(2,3,4,2),e(-1,-1,-2,-2)]=e(1,2,2,0)
[e(2,3,4,2),e(-1,-2,-3,-1)]=e(1,1,1,1)
[e(2,3,4,2),e(-1,-2,-2,-2)]=-e(1,1,2,0)
[e(2,3,4,2),e(-1,-2,-3,-2)]=e(1,1,1,0)
[e(2,3,4,2),e(-1,-2,-4,-2)]=e(1,1,0,0)
[e(2,3,4,2),e(-1,-3,-4,-2)]=e(1,0,0,0)
[e(2,3,4,2),e(-2,-3,-4,-2)]=
=2h(1,0,0,0)+3h(0,1,0,0)+2h(0,0,1,0)+h(0,0,0,1)
[e(-1,0,0,0),e(0,-1,0,0)]=-e(-1,-1,0,0)
[e(-1,0,0,0),e(0,-1,-1,0)]=-e(-1,-1,-1,0)
[e(-1,0,0,0),e(0,-1,-2,0)]=-e(-1,-1,-2,0)
[e(-1,0,0,0),e(0,-1,-1,-1)]=-e(-1,-1,-1,-1)
[e(-1,0,0,0),e(0,-1,-2,-1)]=-e(-1,-1,-2,-1)
[e(-1,0,0,0),e(0,-1,-2,-2)]=-e(-1,-1,-2,-2)
[e(-1,0,0,0),e(-1,-3,-4,-2)]=-e(-2,-3,-4,-2)
[e(0,-1,0,0),e(0,0,-1,0)]=-e(0,-1,-1,0)
[e(0,-1,0,0),e(0,0,-1,-1)]=-e(0,-1,-1,-1)
[e(0,-1,0,0),e(-1,-1,-2,0)]=-e(-1,-2,-2,0)
[e(0,-1,0,0),e(-1,-1,-2,-1)]=-e(-1,-2,-2,-1)
[e(0,-1,0,0),e(-1,-1,-2,-2)]=-e(-1,-2,-2,-2)
[e(0,-1,0,0),e(-1,-2,-4,-2)]=-e(-1,-3,-4,-2)
[e(0,0,-1,0),e(0,0,0,-1)]=-e(0,0,-1,-1)
[e(0,0,-1,0),e(-1,-1,0,0)]=e(-1,-1,-1,0)
[e(0,0,-1,0),e(0,-1,-1,0)]=-2e(0,-1,-2,0)
[e(0,0,-1,0),e(-1,-1,-1,0)]=-2e(-1,-1,-2,0)
[e(0,0,-1,0),e(0,-1,-1,-1)]=-e(0,-1,-2,-1)
[e(0,0,-1,0),e(-1,-1,-1,-1)]=-e(-1,-1,-2,-1)
[e(0,0,-1,0),e(-1,-2,-2,-1)]=-e(-1,-2,-3,-1)
[e(0,0,-1,0),e(-1,-2,-2,-2)]=-e(-1,-2,-3,-2)
[e(0,0,-1,0),e(-1,-2,-3,-2)]=-2e(-1,-2,-4,-2)
[e(0,0,0,-1),e(0,-1,-1,0)]=e(0,-1,-1,-1)
[e(0,0,0,-1),e(-1,-1,-1,0)]=e(-1,-1,-1,-1)
[e(0,0,0,-1),e(0,-1,-2,0)]=e(0,-1,-2,-1)
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[e(-1,-2,-2,0),e(0,-1,-2,-2)]=-e(-1,-3,-4,-2)
[e(-1,-2,-2,0),e(-1,-1,-2,-2)]=-e(-2,-3,-4,-2)
[e(-1,-1,-2,-1),e(-1,-2,-2,-1)]=2e(-2,-3,-4,-2)

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