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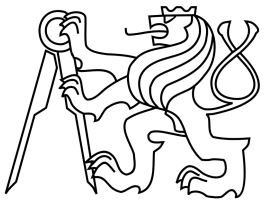

Perkolace a kvantové procházky

Percolations and quantum walks

RESEARCH WORK

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V Praze dne .....

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Acknowledgments

I would like to thank my supervisor, especially for a very interesting subject of the work in a currently developing area of research. I am also grateful for various corrections and recommendations during writing the work.

Bc. Jan Mareš

Contents

Introduction

This work deals with percolation in coined quantum walks. Quantum walks are a quantum counterpart of classical random walks. There is quite a big interest in quantum walks recently as they can be used for example to simulate other physical quantum systems or even be utilised for quantum computing. Great part of the research is done on imperfections in quantum walks. It is important to know, how is the evolution influenced by various external effects.

Here we use the model of percolation known from classical physics as an example of a disturbance in quantum walks. In particular we are interested in asymptotic behaviour.

First we summarize some published results mainly about spreading of a quantum walk on an infinite percolated graph. The main part of this work is than finding possible asymptotic states for a partially percolated quantum walk on two particular finite graphs.

Chapter 1

Percolation

1.1 What is percolation

In this chapter we will introduce the concept of a classical percolation. Percolation is a very simple model of disordered media. To illustrate the concept of percolation on a physical model, one can think of a big block of some porous material submerged in water. Such material contains narrow channels through which water can leak into the block. We assume, that (compared to the size of channels), the block can be considered "infinitely" large. Now we ask, what is the probability of the center of the block getting wet given a microscopic structure of the material. This is actually how the problem of percolation was originally stated.

In modern terms, percolation takes place on some graph $G = (V, E)$, where V is set of vertices and E set of edges. This graph is connected (there is a path between any two vertices) and for simplicity undirected. Standard terminology in percolation theory is using *sites* instead of vertices and *bonds* instead of edges. (In this work we will sometimes use these terms interchangeably.)

In a percolation model we basically create a new graph form G by removing some of it's sites or bonds. Therefore we have two types of percolation: *site percolation* and *bond percolation* if we remove sites or bonds respectively. In both cases there is a given p ($0 \leq p \leq 1$) and each site/bond is independently maintained with the probability p or removed with the probability $1 - p$. It is said, that remaining sites/bonds are *open* and removed sites/bonds are *closed*. The value p is sometimes called *percolation probability*.

We can now ask about the properties of the resulting graph as a function of p. In particular an important question is whether there exists an infinite cluster of connected sites. This takes us back to our motivation example with the block of porous material. The existence of such infinite open cluster containing the centre of the block is equivalent to the center being reached by the water coming from outside.

For a given graph there is a particular critical probability p_c. For $p < p_c$, the probability of the existence of an infinite open cluster is zero and for $p > p_c$ there surely

is exactly one such cluster. Except for several special cases, finding exact values of p_c for different graphs is a very difficult problem. Nevertheless much progress has been done in answering another questions about percolation. In particular, so called *power laws* have been discovered. Those are formulas describing distributions of open clusters in percolated graphs with p close to p_c, which hold for large classes of graphs.

1.2 Classical percolation results

We will use the symbol $\theta(p)$ for the probability of the existence of an open path from the origin to infinity. This $\theta(p)$ is often called *percolation probability*, which we have also used for p. Hopefully, there will be no misunderstanding due to this. We can now define the critical probability as $p_c = \sup\{p|\theta(p) = 0\}$.

Further we denote the open cluster containing the vertex x by $C(x)$ and specifically the open cluster containing the origin as $C(0)$.

If $p < p_c$, all open clusters are finite. On the other hand $p > p_c$ means, that $C(0)$ is infinite with strictly positive probability. If we have a lattice (or another homogeneous graph), the probability of $C(x)$ being infinite is the same for all x. By using Kolmogorov's zero-one law, we obtain the result that there is some infinite cluster with probability 1. It can also be seen intuitively. If $C(0)$ is finite, we just pick some vertex not belonging to $C(0)$ and try again, so we have infinitely many "attempts" with non-zero chance of success.

Exact values of p_c were found just for several types of graphs. For example $p_c = \frac{1}{2}$ for bond percolation on a square 2D lattice and site percolation on a triangular 2D lattice. Some other values of p_c where foud numerically with good precision. For example, for site percolation on a 2D square lattice it is $p_c = 0.59277 \pm 0.00005$ [8].

We will now just very briefly describe, what those earlier mentioned power laws are. Although it has been proven only for some special 2D lattices and higher dimensional graphs (about $d > 19$), it is believed that some functions of p are given by universal formulas independently of a specific graph structure. (Possibility of getting results for high dimensional graphs is given by their similarity with regular trees.) For example the expected number of vertices in $C(0)$ behaves like $(p_c - p)^{-\gamma}$ for some γ as p grows towards p_c. Further $\theta(p)$ behaves like $(p - p_c)^{\beta}$ as we go down to p_c.

These *critical exponents* (γ, β, ...) are believed to depend only on dimensionality of the graph. So for example all 2D lattices should have the same γ.

1.3 Dynamic percolation

So far we have been discussing only cases when the percolation graph is given at the beginning and undergoes no change during the process. When we model our system with some discrete time process like e.g. random walks or coined quantum

walks, we can naturally consider some changes in the graph structure during the process. If these changes are still much slower than the characteristic time step of the process, we call it *static* fluctuations (may lead to Anderson localisation). On the other hand if the typical time of the graph change in comparable to the step time we have *dynamic* fluctuations (may lead to diffusion). An obvious example of the later case is when we generate a new percolated graph in every time step of the process.

Chapter 2

Quantum walks

Here we introduce the concept of a classical random walk and than focus on quantum walks. Even though a quantum walk (QW) is a quantum counterpart of a classical random walk, the difference is significant. While perfect quantum walks rely heavily on coherence and can exhibit specific wave properties, classical random walks are incoherent processes. Nevertheless, when we introduce some disturbance in a QW (partial measurements, classical randomness like percolation,...) the difference may be suppressed or even vanish.

2.1 Classical random walk

Let us first very briefly introduce a classical random walk. Random walk is a discrete stochastic process on a graph. In the simplest scenario, we have an infinite line of discrete uniformly distributed points and a walker with a coin. In discrete time steps, the walker flips a coin and makes one step to the left or to the right according to the coin flip.

Now we can ask, what is the probability of the walker being at a given position x (relative to his starting point) after n steps. In other words, we want to find the probability distribution of his position. Let us assume, that the coin is unbiased (the same probability for heads and tails) and so we get a balanced random walk. It is easy to see, that the walker's position mean is his starting point (for all n). Actually in our case his position follows a binomial distribution (Figure 2.1) and therefore the standard deviation is $\sqrt{n}$.

2.2 Quantum walk on a line

In the quantum version we will also begin with a walk on a line. The underlying graph is the same (line of nodes with edges between nearest neighbours), but now the vertices represent orthonormal states in the *position Hilbert space* of the quantum walker (denoted $\mathcal{H}_x$). The walker has a coin as well. (As in the classical version, we

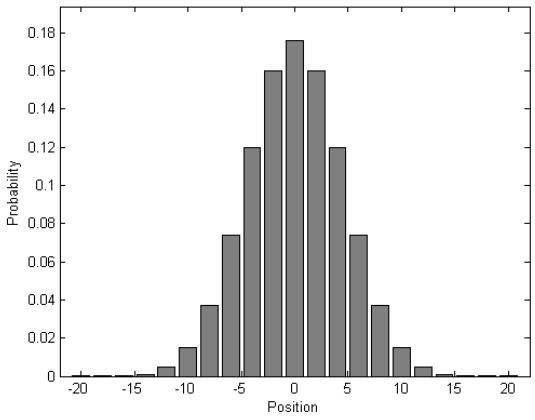


Figure 2.1: The probability distribution of a classical random walk with unbiased coin after 20 steps. If we consider only even positions (odd positions are unoccupied in even steps and vice versa), we get a binomial distribution.

will consider only discrete time quantum walk - a continuous time walk has no coin in either case.) Here the coin is some two level quantum system (spin, polarisation, ...) with the *coin Hilbert space* (denoted $\mathcal{H}_c$). Let us use the symbols $|-1\rangle$ and $|+1\rangle$ for orthonormal bases states in $\mathcal{H}_c$. The entire state of a quantum walker is a vector from the tensor product of the position Hilbert space and the coin space $\mathcal{H}_x \otimes \mathcal{H}_c$. In contrast to classical random walk, the state of a walker can be now an arbitrary superposition of all position and coin states.

The time evolution of a quantum walk is given by the unitary operator $U = SC$. Here $C = (\mathbb{1}_x \otimes C_c)$ is the *coin operator*. C_c acting only on $\mathcal{H}_c$ is extended by unity on position space $\mathcal{H}_x$. S stands for the *step operator* governing the walker's motion on the lattice. The resulting state of the walker starting in the state $|\psi_{in}\rangle$ after t steps is therefore:

$$|\psi_t\rangle = U^t |\psi_{in}\rangle = (SC)^t |\psi_{in}\rangle .$$

In the case of a quantum walk on a line, the operator C is represented by an unitary 2×2 matrix in each vertex. In general it can be written in the form:

$$C_c^{(gen)} = \begin{bmatrix} \cos(\xi) & e^{i\theta}\sin(\xi) \\ e^{i\phi}\sin(\xi) & -e^{i(\theta+\phi)}\cos(\xi) \end{bmatrix}, \tag{2.1}$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$ and $0 \leq \xi \leq 2\pi$. A commonly used example is the Hadamard coin:

$$C_c^{(Hadamard)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The step operator can be written as:

$$S = \sum_x |x+1,+1\rangle \langle x,+1| + |x-1,-1\rangle \langle x,-1| .$$

Here we are using notation $|X\rangle_x \otimes |C\rangle_c \equiv |X,C\rangle$ and the order of symbols is the same for dual bra vectors. So the action of the step operator on bases states is:

$$\begin{aligned} S|x,+1\rangle &= |x+1,+1\rangle , \\ S|x,-1\rangle &= |x-1,-1\rangle . \end{aligned} \tag{2.2}$$

It shifts the position of the walker according to state of the coin and leaves the coin part of the system unchanged. Note, that the action of this operator is the same for all x.

The walker is at the beginning in some state $|\psi_{in}\rangle$, which is typically chosen in the form:

$$|\psi_{in}\rangle = |0\rangle (\alpha |-1\rangle + \beta |+1\rangle).$$

The walker starts at the origin with the coin in some superposition. The evolution "spreads" the walker from the origin by creating superposition of still more distant vertex states.

There are numerous differences between classical random walks and quantum walks. A typical quantum feature is interference, which in a quantum walk fundamentally influences the resulting probability distribution. Figure 2.2 shows the probability distribution of the walker's position (summed over coin states) for a quantum walk on line with the Hadamard coin and the initial state $|\psi_{in}\rangle = |0\rangle \frac{1}{\sqrt{2}}(|-1\rangle + i|+1\rangle)$ after 100 steps. This setting produces a symmetric position distribution of the quantum walker.

As can be seen from the figure, a quantum walk (here we are still talking just about 1D walk) spreads much faster than a classical random walk. It's standard deviation grows linearly with the number of steps. (That is a quadratic speed-up over classical.) This property is not influenced by changing the initial state or the coin operator. An illustration of this is presented on figures 2.3 and 2.4. The figure 2.3 shows the probability distribution of quantum walks with different initial states and 2.4 for various coins. The ballistic spread is obvious in all cases, even if the scaling prefactor depends on the coin choise.

2.3 Quantum walk on a 2D lattice

Let us consider a square two dimensional regular lattice of vertices. (Nearest neighbours are connected with edges.) In the classical case, the walker would need a four sided "coin" (rather a dice) to toss at each step. The same holds for a quantum walk, where this coin is some four level quantum system with basis states

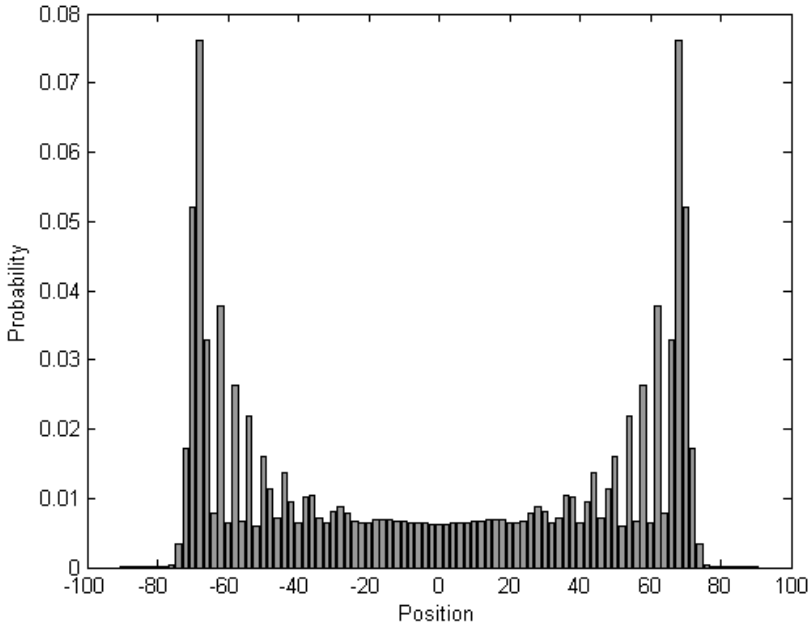


Figure 2.2: Position probability distribution (summed over coin states) for a quantum walk on line with the Hadamard coin and the initial state $|\psi_{in}\rangle = |0\rangle \frac{1}{\sqrt{2}}(|-1\rangle + i\,|+1\rangle)$ after 100 steps. Only even positions are displayed since the probability at odd positions is zero in even time steps.

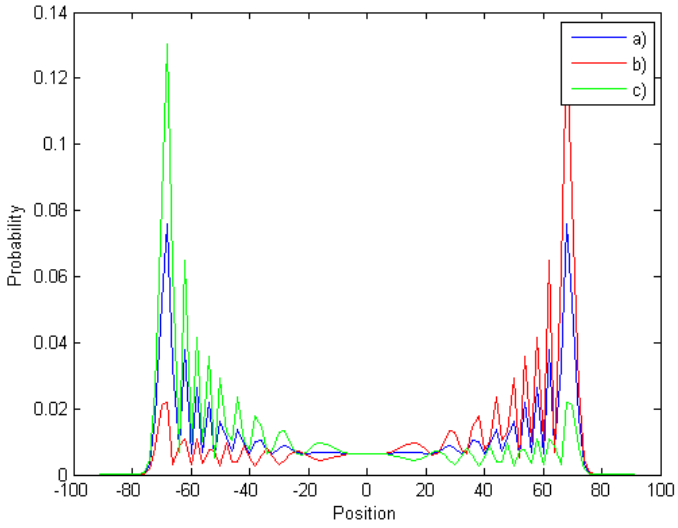


Figure 2.3: Position distribution (summed over coin states) for quantum walk on line with Hadamard coin and the initial states: a) $|\psi_a\rangle = |0\rangle \frac{1}{\sqrt{2}}(|-1\rangle + i\,|+1\rangle)$ b) $|\psi_b\rangle = |0\rangle\,(i\,|+1\rangle)$ and c) $|\psi_a\rangle = |0\rangle\,|-1\rangle$ after 100 steps. Only even positions are displayed since the probability at odd positions is zero in even time steps.

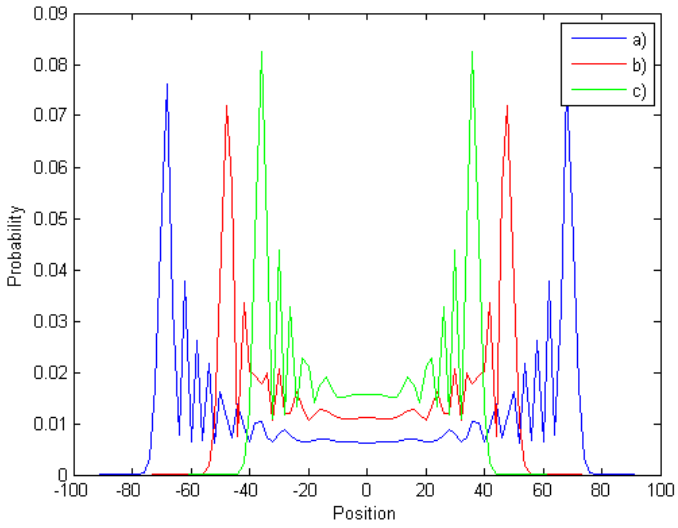


Figure 2.4: Position distribution (summed over coin states) for quantum walk on line with the initial state $|\psi_a\rangle = |0\rangle \frac{1}{\sqrt{2}}(|-1\rangle + i\,|+1\rangle)$ and the coin 2.1 with a) $\xi = \pi/4$ (Hadamard), b)$\xi = \pi/6$ and c) $\xi = \pi/8$ after 100 steps. Even though the speed of the spread differs for different choices of ξ, it is still ballistic (proportional to the number of steps). Only even positions are displayed since the probability at odd positions is zero in even time steps.

$|L\rangle, |R\rangle, |D\rangle, |U\rangle$. In contrast to a 1D walk, the choice of the coin influences significantly the behaviour of a quantum walk. For illustration of a 2D quantum walk, we will use for example so called Grover's coin. In the matrix representation it is written as:

$$C_c^{(Grover)} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}. \tag{2.3}$$

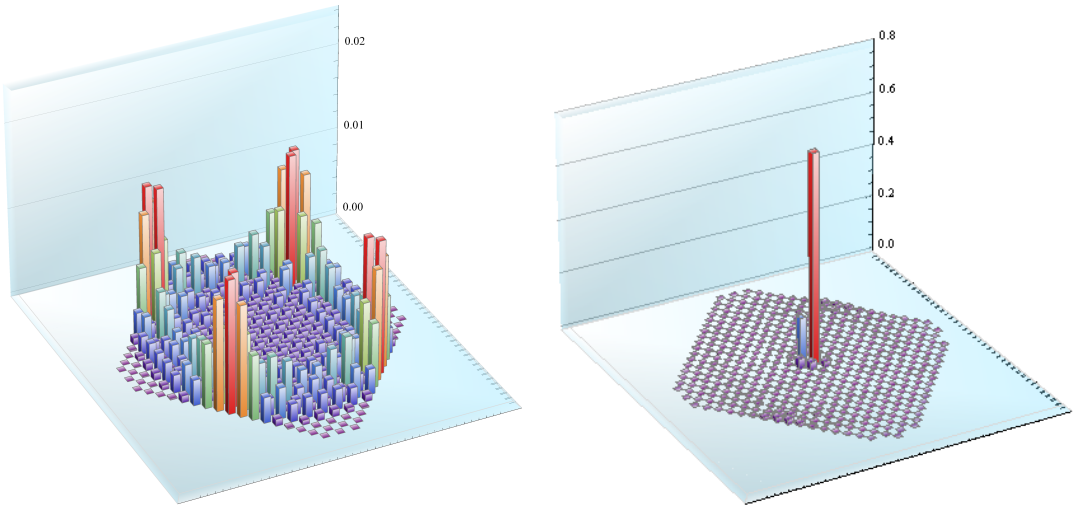
The advantage of the Grover's coin is it's symmetry. (Although is is not the only coin allowing for a symmetric QW.) It treats all directions equally except from a sign flip performed on the direction the walker came from. This choice also requires a modification of the step operator (to preserve unitarity [4]) - it inverts coin states:

$$\begin{aligned} S_4\,|x, y, L\rangle &= |x-1, y, R\rangle\,, \\ S_4\,|x, y, R\rangle &= |x+1, y, L\rangle\,, \\ S_4\,|x, y, D\rangle &= |x, y-1, U\rangle\,, \\ S_4\,|x, y, U\rangle &= |x, y+1, D\rangle\,. \end{aligned} \tag{2.4}$$

A quantum walk on a 2D grid is significantly different from the one dimensional version. While ballistic spread is a robust effect (not dependent on the initial state

nor the coin) in a walk on a line, it is not so here. There is one specific initial state exhibiting spread like 1D walk, but for most initial states, there is a central peak of probability in the origin. Nevertheless the spread is still faster than for a classical random walk. The situation is illustrated on figures 2.5(a) and 2.5(b). Figure 2.5(a) shows the walk with the special initial state mentioned above, specifically $|\psi_{in}\rangle = \frac{1}{2}|0,0\rangle(|L\rangle + |R\rangle + |D\rangle + |U\rangle)$. On the figure 2.5(b) there is probability distribution averaged over 500 walks with randomly chosen initial states.

From this we see, that in 2D quantum walk, it matters which initial state we choose. Therefore in further investigation, we will be careful to indicate whether we use the fast spreading initial state presented above or some other one or possibly some averaging over more initial states.



(a) Random states (b) Random phases

Figure 2.5: Figure 2.5(a) shows a probability distribution of 2D quantum walk with 20 steps, Grover's coin and special initial state $|\psi_{in}\rangle = \frac{1}{2}|0,0\rangle(|L\rangle + |R\rangle + |D\rangle + |U\rangle)$. Figure 2.5(b) shows an average probability distribution for 500 2D quantum walks with 20 steps, Grover's coin and randomly chosen initial states. The random initial states are of the form $|0,0\rangle \otimes$ (rand(-1,1) $|L\rangle$ + rand(-1,1) $|R\rangle$ + rand(-1,1) $|D\rangle$ + rand(-1,1) $|U\rangle$) after normalisation.

Chapter 3

Percolation in QW - Numerical approach

In this chapter we will summarize results by Leung et al [4] to introduce some basic phenomena which can occur in percolated quantum walks on infinite graphs.

3.1 Percolated QW on a line

First let us consider a walk on a line. In the scope of percolation, there is an obvious problem. For any percolation probability $p < 1$, there will certainly be some broken edge both on the left and on the right of the origin. The walker can not pass through those broken links and the walk becomes just a quantum walk on a finite line. Two approaches solving this problem are suggested in [4] - dynamic gaps and quantum tunnelling.

For purposes of this summary, we will stick to the notation of the paper and use slightly differently (but equivalently) parametrized form of a general coin operator:

$$C_c^{(gen)} = \begin{bmatrix} \sqrt{\eta} & e^{i\theta}\sqrt{1-\eta} \\ e^{i\phi}\sqrt{1-\eta} & -e^{i(\theta+\phi)}\sqrt{\eta} \end{bmatrix},$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$ and $0 \leq \eta \leq 1$.

3.1.1 Dynamic gaps

Here we use the concept of dynamic percolation mentioned earlier. We generate a new percolation lattice at every time step. This ensures, that broken sites/edges do not block the walker permanently. We modify the step operator by making it position and time dependent. If the edge in a given direction is present we use operator (2.2). If the edge in plus or minus direction is absent, we use a transformation

$$|x,+1\rangle \to |x,-1\rangle\,, \qquad (3.1)$$
$$|x,-1\rangle \to |x,+1\rangle\,,$$

respectively. The walker is reflected on a missing edge.

While using the Hadamard coin, this type of disruption of a quantum walk leads after some number of steps (scaling as $1/(1-p)$) to a classical spreading of the walk. It has an effect similar to nonunitary evolution caused by random measurements. The speed of such a quantum walk is proportional to $\sqrt{t}$ (with a higher prefactor than for a classical random walk).

3.1.2 Quantum tunnelling

This model considers the well known phenomenon on quantum particles - tunnelling through potential barriers. The quantum walker can now pass broken links, but here we use a biased coin (3.1) with $\eta < 0,5$ on both sides of the gap. This corresponds to a slower spreading over broken links.

Figure 3.1 shows numerical results for a quantum walk with 100, 1000 and 10000 steps with $\eta = 0,25$ and different percolation probabilities. We can see that quantum coherence remains for some time (100 steps), but asymptotically the walk with $0 < p < 1$ spreads like a classical random walk on an unpercolated line. (The scaling prefactor is again higher for a quantum walk.) Long-term quantum coherence lasts only for extreme cases of all edges present or pure tunnelling.

3.2 Percolation in QW on 2D lattice

On percolated lattice we have vertices of different degree, so we must define appropriate coin operators to keep the evolution unitary. In [4] generalized Grover's coin is used which for dimension d has elements $(C_d^{Grover})_{ij} = 2/d - \delta_{ij}$. (Matrix filled with the number $2/d$ and the diagonal reduced by 1.) The coin for particular configuration is a matrix with one-dimensional blocks -1 for broken edges and the rest constructed from corresponding Grover's matrix.

Simulations were done for three settings. For two pure initial states states:

$$|\psi_{max}\rangle = \frac{1}{2}\,|0,0\rangle\,(|L\rangle + |R\rangle + |D\rangle + |U\rangle), \qquad (3.2)$$
$$|\psi_{min}\rangle = \frac{1}{2}\,|0,0\rangle\,(-|L\rangle - |R\rangle + |D\rangle + |U\rangle),$$

with maximal respectively minimal spreading and also for walk averaged over random choices of sign.

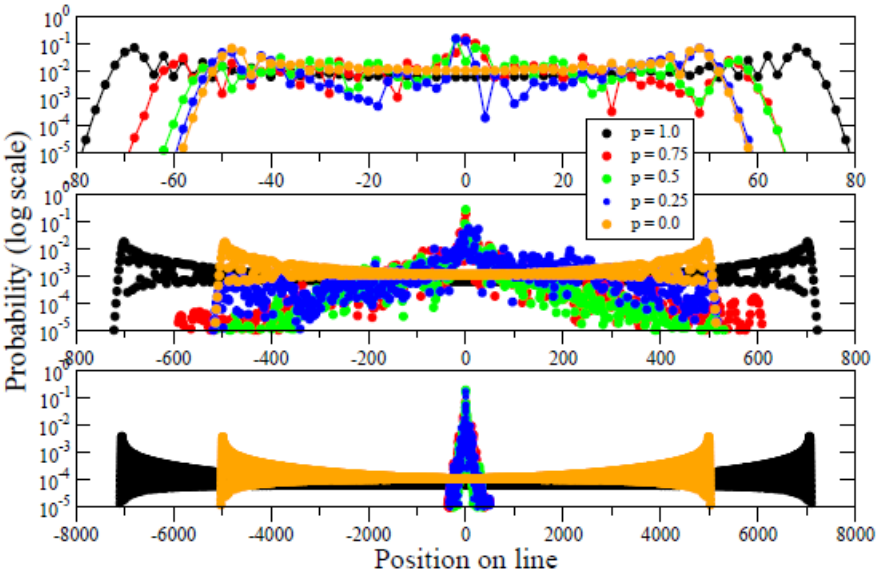


Figure 3.1: Quantum walk with tunnelling ($\eta = 0.25$) with various setting: 100 (top), 1000 (middle) and 10000 (bottom) steps with percolation probability $p = 1$ (black), 0.75 (red), 0.5 (green), 0.25 (blue) and 0 (orange). Only even positions are displayed, since probability at odd positions is zero in even time steps. [4].

As a measure of spreading, the mean distance from the origin:

$$\bar{r} = \sum_{x,y} \sqrt{x^2 + y^2} p(x, y)$$

is used, where $p(x, y)$ is probability of the walker being measured at vertex (x, y) (summed over coin states). Figure 3.2 shows results for the setting with random states.

With very small p, the walker has basically nowhere to go. Than $\bar{r}$ starts growing (before classical percolation probability - 0.5 for bond and 0.59 for site percolation). At some point the growth slows down, when the size of a typical open cluster outgrows the area covered by the walk with given number of steps. At p close to 1 both bond and site percolated walks behave equally. Due to restoring lattice symmetry, the spread grows to perfect-lattice walk values.

We can further study the scaling of $\bar{r}$ with growing t for given p. Figure 3.3 shows log-log plot. We can see, that after over-going initial finite-size effects we get straight lines (neglecting four-steps periodic zig-zag behaviour). It means, that $\bar{r}$ scales as t^{α}. For p around 0.9, the coefficient α reaches the value 0.5 (spreading for classical random walk on perfect lattice) and than rises fast to $\alpha = 1$ for perfect quantum walk. Examining similar data for $|\psi_{max}\rangle$ shows, that spreading in this case has higher α than averaged walks [4], so spreading scales differently.

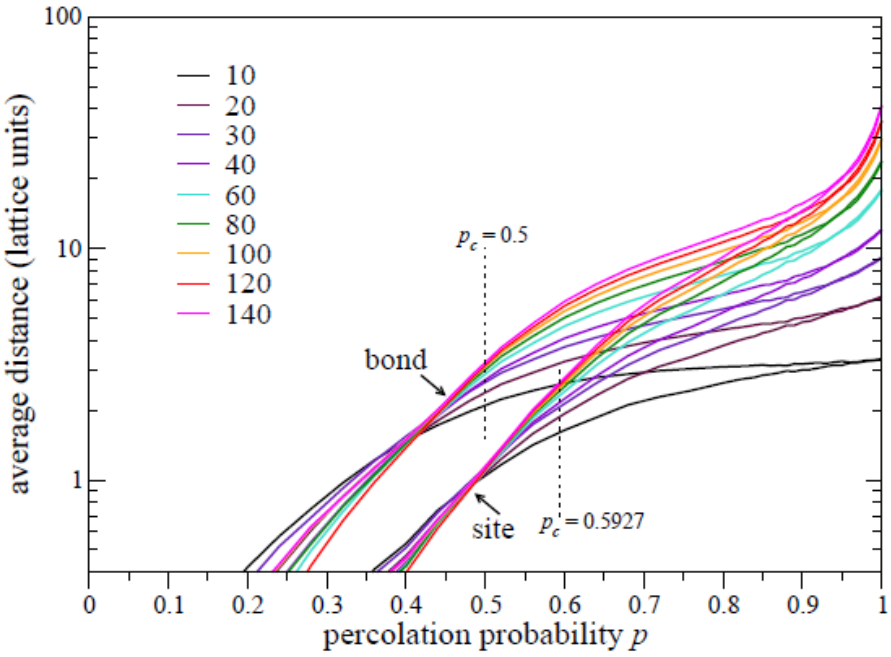


Figure 3.2: Value $\bar{r}$ of 2D quantum walk as a function of percolation probability p for different numbers of steps using initial states with random phases (both, bond and site percolation). [4].

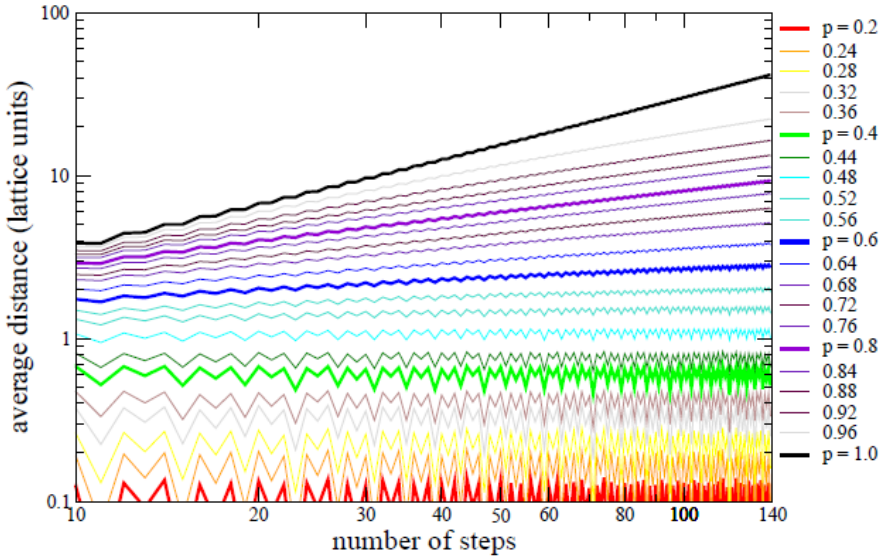


Figure 3.3: Value $\bar{r}$ of 2D quantum walk as a function number of steps t for various percolation probabilities p using initial states with random phases. [4].

Chapter 4

Percolation in QW - Asymptotic dynamics

Applying the results by Novotný et al on asymptotic dynamic of random unitary operations [5], Kollár et al [6, 7] describes asymptotic dynamics of coined quantum walks on finite percolated graphs. We will now introduce these ideas and use them in the next chapter.

4.1 QW on a general percolated graph

First we need to describe quantum walk on an arbitrary bond-percolated graph. Let the graph be $G = (V, E)$, where V is the set o vertices (sites) and E the set of edges (bonds). As in previous cases the walk is driven by an evolution operator U_K. Here $K \subset E$ stands for a subset of edges which remained open, thus (V, K) is a graph given by particular realisation of the percolation process. We can describe U_K as:

$$U_K = S_K C, \tag{4.1}$$

where C is an operator acting on the coin space and S_K is a step operator given by:

$$S_K = \sum_{\substack{a \in V, d \\ (a, a \oplus d) \in K}} |a \oplus d, d\rangle \langle a, d| + \sum_{\substack{a \in V, d \\ (a, a \oplus d) \notin K}} R_a |a, d\rangle \langle a, d| . \tag{4.2}$$

Here d goes over all directions possible at the vertex a, $a \oplus d$ denotes the nearest neighbouring vertex of a in the direction d and R_a is the reflection operator in the vertex a. The operator S_K moves the walker to an appropriate neighbouring site if the edge needed is present in K and leaves him at the current position if this edge is missing. In the case of a missing edge the reflection operator R_a acting on the coin state is applied, which ensures unitarity of S_K.

In the case of a d-regular graph, we may use the same coin operator (C_c) in all vertices and than the resulting coin operator can be written as $C = I_x \otimes C_c$, where again I_x is the identity operator acting on the position space. It is important, that even if the coin in every vertex is different, the evolution operator U_K decomposes into subsequent application of two other unitary operators C and S_K.

Now the problem of investigation is to obtain analytical results about the behaviour of a dynamical percolation on finite graphs. Due to the presence of a classical uncertainty in generating a new percolation graph in every step, we need to describe our walker by a density operator. The initial state is now some ρ_0 and the evolution is given by a superoperator Φ derived from U_K as

$$\rho(n+1) = \Phi(\rho(n)) = \sum_K \pi_K(p) U_K \rho(n) U_K^\dagger, \tag{4.3}$$

where $\pi_K(p)$ is the probability that the percolation process leads to the configuration K if the percolation probability is p.

4.2 The attractor subspace

Equation (4.3) for Φ exactly corresponds with the definition of random unitary operator investigated in [5], therefore we can use methods described there to find the asymptotic dynamics of our quantum walk. First we need to find so called attractors. Those are special operators forming a subspace of the possible asymptotic states of the walker. (They are not necessarily valid density operators.) All other components of the initial state are suppressed after a sufficient number of iterations. Attractors are all possible solutions $X_{\lambda,i}$ of the equation:

$$U_K X_{\lambda,i} U_K^\dagger = \lambda X_{\lambda,i}, \text{ where } |\lambda| = 1, \text{ for } \forall K. \tag{4.4}$$

The index i distinguishes different orthogonal "eigenvectors" (operators solving the equation) of a given eigenvalue λ. We will denote the set of all eigenvalues λ as σ. By $\forall K$ we either mean $\forall K \subset E$ or a subset of 2^E containing all possible configurations, when we consider only partial percolation on some edges. (Therefore some configurations are not allowed by definition of the problem.)

The asymptotic state is given by:

$$\rho(n) = \Phi^n(\rho_0) \overset{n\to\infty}{\longrightarrow} \sum_{\lambda\in\sigma,i} \lambda^n \mathrm{Tr}(X_{\lambda,i}^\dagger \rho_0) X_{\lambda,i}. \tag{4.5}$$

From there we can observe, that the asymptotic behaviour is independent of values $\pi_K(p)$ and therefore independent of p as long as $0 < p < 1$. These probabilities

only determine the speed of convergence of the actual state towards our predicted asymptotic behaviour.

If there is just one attractor, it is the only possible asymptotic state. On the other hand, if there are attractors corresponding to eigenvalues $\lambda \neq 1$, there can be asymptotic cycles in the walker state.

4.2.1 p-attractors

The article [7] gives a method, how to find some attractors more easily by using common eigenvectors of all operators U_K. Let us denote the basis of those common eigenvectors $\{|\phi_{\alpha,i_\alpha}\rangle\}$. Therefore these states fulfil the equation

$$U_K \left|\phi_{\alpha,i_\alpha}\right\rangle = \alpha \left|\phi_{\alpha,i_\alpha}\right\rangle, \text{ for } \forall K. \tag{4.6}$$

Now we can simply construct attractors corresponding to λ from matrices of the type $\left|\phi_{\alpha,i_\alpha}\right\rangle \left\langle\phi_{\beta,i_\beta}\right|$ as:

$$Y_{\lambda,i} = \sum_{\alpha\beta^*=\lambda,i_\alpha,i_\beta} A^{\alpha,i_\alpha}_{\beta,i_\beta} \left|\phi_{\alpha,i_\alpha}\right\rangle \left\langle\phi_{\beta,i_\beta}\right|. \tag{4.7}$$

We will call these attractors *p-attractors*. It is important to know, that p-attractors are not the only attractors of the system.

It is easy to see, that p-attractors satisfy the condition

$$Y_\lambda U_K = \lambda U_{K'} Y_{\lambda,i}, \text{ where } |\lambda| = 1 \text{ for } \forall K, K', \tag{4.8}$$

In fact this relation is equivalent to $Y_{\lambda,i}$ being an p-attractor [7].

4.3 Attractors for QW

4.3.1 General attractors

If we use the specific form of U_K given by (4.1), we can rewrite equation (4.4) into the form

$$CX_{\lambda,i}C^\dagger = \lambda S_K^\dagger X_{\lambda,i} S_K \text{ for } \forall K. \tag{4.9}$$

The left hand side of the equation is independent of chosen $K \subset E$ and therefore:

$$S_K X_{\lambda,i} S_K^\dagger = S_{K'}^\dagger X_{\lambda,i} S_{K'} \text{ for } \forall K, K'. \tag{4.10}$$

We can therefore separately solve

$$C X_{\lambda,i} C^\dagger = \lambda S_K^\dagger X_{\lambda,i} S_K \tag{4.11}$$

for one particular choice of K and than constrain the result by the step conditions 4.10. In comparison to solving fully percolated walk (where all edges undergo percolation), in our case we have a problem. For partial percolation (where some edges are always present) we can not use the convenient configuration with all edges missing (resulting to local conditions). It means, that we really have to solve the equation in the whole Hilbert space (with dimension of the number of all possible directions in all vertices combined).

Hypothetically we could use a brute-force method. In fact the equation 4.12 is equivalent to

$$(SC) \otimes (SC)^* x = \lambda x, \tag{4.12}$$

where x is defined by $\langle a, b \,|\, x \rangle = \langle a \,|\, X \,|\, b \rangle$. This is just the task of finding eigenstates of an unitary operator, but the problem is it's size. For example for our rather small 4-regular graph with 3×6 sites the dimension of the space on which SC acts is 72 and therefore $(SC) \otimes (SC)^*$ is a 5184×5184 matrix. Finding eigenstates for such a matrix has significant demands on computation resources. (It is not reasonably feasible on a 2,6 GHz processor, 6 GB RAM computer using the function Eigensystem[] in Wolfram Mathematica software.)

To overcome the difficulty described above we can use p-attractors. We can use the fact, that p-attractors can be constructed from common eigenstates of the evolution operator. The general attractor equation

$$C X_{\lambda,i} C^\dagger = \lambda S_K^\dagger X_{\lambda,i} S_K \tag{4.13}$$

can be understood as being the p-attractor equation

$$C X_{\lambda,i} C^\dagger = \lambda S_L^\dagger X_{\lambda,i} S_{L'} \tag{4.14}$$

where we have just chosen both L and L' to be K. This will allow us tu find the solution of one configuration (on which we will subsequently apply the step condition 4.10) using pure states requiring just a fraction of computer resources compared to the straightforward approach.

4.3.2 Method summary

The method we will use is the following: Let the dimension of the full Hilbert space be N. First, we find all N eigenstates $\{|\phi_{\alpha,i_\alpha}\rangle\}$ of one chosen operator $U_K = S_K C$ (not common eigenstates). From those we can construct $N \times N$ operators

$$Z_{\alpha,i_\alpha,\beta,i_\beta} = |\phi_{\alpha,i_\alpha}\rangle \langle\phi_{\beta,i_\beta}| . \tag{4.15}$$

Those all solve the equation

$$S_K C Z_{\alpha,i_\alpha,\beta,i_\beta} S_K^\dagger C^\dagger = \alpha\beta^* Z_{\alpha,i_\alpha,\beta,i_\beta}. \tag{4.16}$$

On the other hand we want to solve the equation

$$\begin{aligned} C X_{\lambda,i} C^\dagger &= \lambda S_K^\dagger X_{\lambda,i} S_K, \\ S_K C X_{\lambda,i} (S_K C)^\dagger &= \lambda X_{\lambda,i}. \end{aligned} \tag{4.17}$$

Note, that all $Z_{\alpha,i_\alpha,\beta,i_\beta}$ are solutions of 4.17 with $\lambda = \alpha\beta^*$. As we cannot find more than $N \times N$ mutually orthogonal solutions, we have them all.

The next step is to find restrictions given by the step condition. We do not want to loose any attractors and therefore instead of using the condition for pure states

$$S_K^\dagger |\phi_{\alpha,i_\alpha}\rangle = S_{K'}^\dagger |\phi_{\alpha,i_\alpha}\rangle , \text{ for } \forall K, K', \tag{4.18}$$
$$\tag{4.19}$$

we will instead be follow

$$S_K X_{\lambda,i} S_K^\dagger = S_{K'}^\dagger X_{\lambda,i} S_{K'} \text{ for } \forall K, K'. \tag{4.20}$$

It is important that we do not only test $Z_{\alpha,i_\alpha,\beta,i_\beta}$ on fulfilling the step condition. We must consider all linear combinations within each eigenvalue subspace and find all that fit 4.20.

4.3.3 The step condition

Here we will present some general findings about the step condition 4.10 for partially percolated graphs. Let us have have two operators S_{K_1} and S_{K_2}. (This can be obviously generalised for comparing more operators.) Let us use the notation

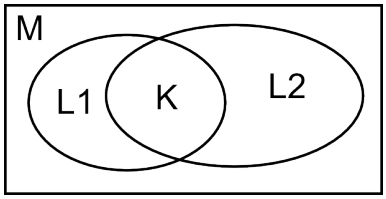


Figure 4.1: Illustration of the notation in solving the step condition.

$K = K_1 \cap K_2$ (edges present in both sets), $L_1 = K_1 \setminus K$ (edges present just in K_1), $L_2 = K_2 \setminus K$ (edges present just in K_2) and $M = E \setminus (L_1 \cup K \cup L2)$ (edges missing in both sets). (See figure 4.1.) Now we can write:

$$\begin{aligned}
S_{K_1} = &\sum_{(a,a\oplus d)\in K} |a\oplus d, d\rangle\langle a, d| + \sum_{(a,a\oplus d)\in L_1} |a\oplus d, d\rangle\langle a, d| + \sum_{(a,a\oplus d)\in L_2} R_a |a, d\rangle\langle a, d| + \\
&+ \sum_{(a,a\oplus d)\in M} R_a |a, d\rangle\langle a, d| \equiv A + B_1 + C_1 + D, \\
S_{K_2} = &\sum_{(a,a\oplus d)\in K} |a\oplus d, d\rangle\langle a, d| + \sum_{(a,a\oplus d)\in L_1} R_a |a, d\rangle\langle a, d| + \sum_{(a,a\oplus d)\in L_2} |a\oplus d, d\rangle\langle a, d| + \\
&+ \sum_{(a,a\oplus d)\in M} R_a |a, d\rangle\langle a, d| \equiv A + B_2 + C_2 + D,
\end{aligned}$$

where A, B_1, B_2, C_1, C_2 and D are just a shorthand for corresponding terms. Now the equation is

$$\begin{aligned}
S_{K_1} X S_{K_1}^\dagger &= S_{K_2} X S_{K_2}^\dagger, \\
(A + B_1 + C_1 + D)X(A + B_1 + C_1 + D)^\dagger &= (A + B_2 + C_2 + D)X(A + B_2 + C_2 + D)^\dagger.
\end{aligned}$$

After removing terms that are the same on both sides we have

$$\begin{aligned}
&(A + D)X(B_1 + C_1)^\dagger + (B_1 + C_1)X(B_1 + C_1)^\dagger + (B_1 + C_1)X(A + D)^\dagger = \\
&= (A + D)X(B_2 + C_2)^\dagger + (B_2 + C_2)X(B_2 + C_2)^\dagger + (B_2 + C_2)X(A + D)^\dagger.
\end{aligned}$$

The operator X can be in general written as $X = \sum_{s_1,c_1,s_2,c_2} X_{s_2,c_2}^{s_1,c_1} |s_1, c_1\rangle\langle s_2, c_2|$. After plugging this form into the above equation we can perform scalar products and we are left with just particular coefficients $X_{s_2,c_2}^{s_1,c_1}$ given by the "bra part" of S and operators given by the "ket part" of S.

Now we should note, that these "ket parts" are the resulting states after applying the step operator with the given configuration. There are two ways how the walker can get into a particular state (position and orientation) - by passing the edge or by reflection. For one given edge configuration only one of those effects may happen

for every initial state (before the application of the step operator). It is often the case, that each state can be reached only from one initial state by applying the step operator. (A typical example is a regular square lattice with the reflection operator that just flips direction.) In such a situation it can not happen, that two terms on one side of the equation would contain the same ket.

Let us now "define" the reflection operator. Let R be such that it does not affect the walker's position and if the walker faces some edge and it happens to be closed, he is reflected to the direction he would have been facing if he had reached the vertex by traversing that edge. In particular we have two cases in mind here. The first one is again the square lattice with the reflection operator that flips directions. The other is a "reversed" situation. Here the reflection operator does nothing (identity operator), but the normal step turns the walker to return. We will use both these settings later.

Let us use the notation that $\tilde{c}$ is the state after reflection if the original state was c, therefore $R\left|s,c\right\rangle = \left|s,\tilde{c}\right\rangle$.

Using our assumptions and because the states are mutually orthogonal, the condition splits into equality of individual terms. The only restriction is that we have to hold together terms associated with each edge, because those swap due to percolation.

Let us for example first examine:

$$AX(B_1)^\dagger = AX(B_2)^\dagger,$$

which is explicitly written as

$$\left(\sum_{(a,a\oplus d)\in K} \left|a\oplus d,d\right\rangle\left\langle a,d\right|\right)\left(\sum_{s_1,c_1,s_2,c_2} X^{s_1,c_1}_{s_2,c_2}\left|s_1,c_1\right\rangle\left\langle s_2,c_2\right|\right)\left(\sum_{(b,b\oplus e)\in L_1}\left|b\oplus b,e\right\rangle\left\langle b,e\right|\right)^\dagger =$$
$$= \left(\sum_{(a,a\oplus d)\in K} \left|a\oplus d,d\right\rangle\left\langle a,d\right|\right)\left(\sum_{s_1,c_1,s_2,c_2} X^{s_1,c_1}_{s_2,c_2}\left|s_1,c_1\right\rangle\left\langle s_2,c_2\right|\right)\left(\sum_{(b,b\oplus e)\in L_1}\left|b,\tilde{e}\right\rangle\left\langle b,e\right|\right)^\dagger.$$

After performing scalar products we have:

$$\sum_{\substack{(a,a\oplus d)\in K\\(b,b\oplus e)\in L_1}} \left|a\oplus d,d\right\rangle X^{a,d}_{b,e}\left\langle b\oplus b,e\right| = \sum_{\substack{(a,a\oplus d)\in K\\(b,b\oplus e)\in L_1}} \left|a\oplus d,d\right\rangle X^{a,d}_{b,e}\left\langle b,\tilde{e}\right|.$$

Now we can make a step similar to the substitution but it really is just renaming the summing index. Let us exchange b for $b\oplus e$ and e for $\tilde{e}$ on the right hand side. Now the equation is:

$$\sum_{\substack{(a,a\oplus d)\in K\\(b,b\oplus e)\in L_1}} |a\oplus d,d\rangle\, X^{a,d}_{b,e}\, \langle b\oplus b,e| = \sum_{\substack{(a,a\oplus d)\in K\\(b\oplus e,b)\in L_1}} |a\oplus d,d\rangle\, X^{a,d}_{b\oplus e,\tilde{e}}\, \langle b\oplus e,e|\,.$$

Finally we just have to realise, that the graph is undirected and therefore $(b, b\oplus e) \in L_1 \Leftrightarrow (b\oplus e, b) \in L_1$, so the conditions in sums are identical. Finally we can compare individual terms in the sum and get the condition

$$X^{a,d}_{b,e} = X^{a,d}_{b\oplus e,\tilde{e}}$$

for all $(a, a\oplus d) \in K$ and $(b, b\oplus e) \in L_1$. By using the other equations with just one "indexed term" (B_1, B_2, C_1 or C_2), we have the overall conditions

$$X^{a,d}_{b,e} = X^{a,d}_{b\oplus e,\tilde{e}} \quad \text{and} \quad X^{b,e}_{a,d} = X^{b\oplus e,\tilde{e}}_{a,d}$$

for all $(a, a\oplus d) \in K \cup M$ and $(b, b\oplus e) \in L_1 \cup L_2$.

Let us now have a look at the equation

$$B_1 X (B_1)^\dagger = B_2 X (B_2)^\dagger.$$

Again the explicit expression is

$$\left(\sum_{(a,a\oplus d)\in L_1} |a\oplus d,d\rangle\,\langle a,d|\right)\left(\sum_{s_1,c_1,s_2,c_2} X^{s_1,c_1}_{s_2,c_2}\,|s_1,c_1\rangle\,\langle s_2,c_2|\right)\left(\sum_{(b,b\oplus e)\in L_1} |b\oplus b,e\rangle\,\langle b,e|\right)^\dagger =$$
$$= \left(\sum_{(a,a\oplus d)\in L_1} \left|a,\tilde{d}\right\rangle\,\langle a,d|\right)\left(\sum_{s_1,c_1,s_2,c_2} X^{s_1,c_1}_{s_2,c_2}\,|s_1,c_1\rangle\,\langle s_2,c_2|\right)\left(\sum_{(b,b\oplus e)\in L_1} |b,\tilde{e}\rangle\,\langle b,e|\right)^\dagger.$$

After performing scalar products we have:

$$\sum_{\substack{(a,a\oplus d)\in L_1\\(b,b\oplus e)\in L_1}} |a\oplus d,d\rangle\, X^{a,d}_{b,e}\, \langle b\oplus e,e| = \sum_{\substack{(a,a\oplus d)\in L_1\\(b,b\oplus e)\in L_1}} \left|a,\tilde{d}\right\rangle X^{a,d}_{b,e}\, \langle b,\tilde{e}|$$

and finally using the same trick as before we have

$$\sum_{\substack{(a,a\oplus d)\in L_1\\(b,b\oplus e)\in L_1}} |a\oplus d,d\rangle\, X^{a,d}_{b,e}\, \langle b\oplus e,e| = \sum_{\substack{(a,a\oplus d)\in L_1\\(b,b\oplus e)\in L_1}} |a\oplus d,d\rangle\, X^{a\oplus d,\tilde{d}}_{b\oplus e,\tilde{e}}\, \langle b\oplus e,e| \,.$$

Together with the three remaining equations the resulting conditions are

$$X^{a,d}_{b,e} = X^{a\oplus d,d}_{b\oplus e,\tilde{e}}$$

for all $(a, a\oplus d) \in L_1$ and $(b, b\oplus e) \in L_1$ or $(a, a\oplus d) \in L_2$ and $(b, b\oplus e) \in L_2$ and further

$$X^{a,d}_{b\oplus e,\tilde{e}} = X^{a\oplus d,d}_{b,e}$$

for all $(a, a\oplus d) \in L_1$ and $(b, b\oplus e) \in L_2$ or $(a, a\oplus d) \in L_2$ and $(b, b\oplus e) \in L_1$.

We will use these results in the next chapter to solve some simple particular cases.

Chapter 5

Fluctuating space

In this chapter we will examine QWs on some specific partially percolated (some edges will always remain present) finite graphs.

The first graph (figure 5.1) can be thought as representing two 1D systems weakly coupled to each other. This can be for instance two molecules forming a weakly bond system. The other possibility is to understand this literally as two coupled one dimensional spaces.

The second graph (figure 5.3) may represent two 2D spaces with a connection that changes it's dimensionality. A one-dimensional connection is still present and the other two connecting edges only open randomly in some time steps. We want to examine the influence of this dimensionality change.

5.1 1D graph example

5.1.1 Settings

Our first graph consists of two 4-loops connected by one edge, see figure (5.1). Only this connecting edge (dashed line) undergoes percolation, all others are still present. This scheme also shows the coin state notation. The vertices with just two neighbours have the coin states in clockwise direction labelled "+" and in counter-clockwise direction labelled "-". Connecting vertices have one additional direction labelled "0".

In analogy to any other discrete quantum walk on a graph, the evolution of our system is given by the unitary operator U, which consists of the coin operator C and the step operator S. We have one fluctuating edge and therefore two different unitary operators $U_1 = CS_1$ for the graph with the connection and $U_2 = CS_2$ for the graph without the connecting edge.

Let us now specify the exact behaviour of our step operator. In vertices $\{1, 2, 3, 6, 7, 8\}$ we just follow the definition given by 4.2, where the second part with the reflection operator is not even needed. In connecting vertices 4 and 5, we will also use this

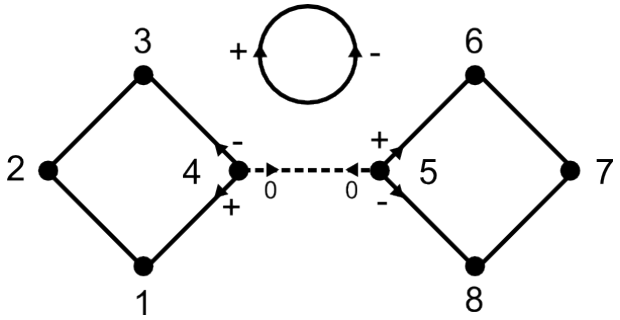


Figure 5.1: The diagram of the investigated graph. The dashed line represents the only percolated edge.

definition with the identity operator as the reflection operator. This choice implies the following behaviour: If the walker comes through the connecting edge, he will face to return. If he faces the connecting edge and it is closed, he just does nothing. In loops the walker circulates unless his direction is changed by the coin operator.

As the coin operator, we will use the Hadamard coin C_2 in vertices of degree 2 and the Grover coin C_3 at vertices of degree 3. In the matrix representation it is:

$$C_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, C_3 = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

The whole coin operator C is therefore a direct sum: $C = C_2 \oplus C_2 \oplus C_2 \oplus C_3 \oplus C_3 \oplus C_2 \oplus C_2 \oplus C_2$.

5.1.2 Perfect graph

Let us first solve the behaviour of the QW on the perfect graph (with the connecting edge present). We will use this result when investigating the percolated version. Here we only need to find eigenstates of the unitary evolution operator $U_1 = CS_1$. It is very straightforward to do, but U_1 is represented by 18×18 matrix. I have done the calculation using Wolfram Mathematica software, see appendix space_1.nb.

5.1.3 Percolated graph

Now we want to restrict our asymptotic states with the step condition to those, that remain even with the percolation of the connecting edge. Here we can use the result from the previous chapter because it is not possible that the reflection would send the walker to a state that he could also reach through some other edge.

We still use the notation S_{K_1} for the step operator for the perfect graph and S_{K_2} for the step operator for the graph with the connecting edge missing. Therefore

$S_{K_1} = E$ and $S_{K_2} = E \setminus (4,5)$. To fit the notation to what we used in the previous chapter we have: $K = E \setminus (4,5), L_1 = (4,5), L_2 = \emptyset, M = \emptyset$.

Let us remind the previous result that we will now use. The attractors must fulfil:

$$X^{a,d}_{b,e} = X^{a,d}_{b\oplus e,\tilde{e}} \quad \text{and} \quad X^{b,e}_{a,d} = X^{b\oplus e,\tilde{e}}_{a,d}$$

for all $(a, a \oplus d) \in K \cup M$ and $(b, b \oplus e) \in L_1 \cup L_2$ and further

$$X^{a,d}_{b,e} = X^{a\oplus d,d}_{b\oplus e,\tilde{e}}$$

for all $(a, a \oplus d) \in L_1$ and $(b, b \oplus e) \in L_1$ or $(a, a \oplus d) \in L_2$ and $(b, b \oplus e) \in L_2$ and

$$X^{a,d}_{b\oplus e,\tilde{e}} = X^{a\oplus d,d}_{b,e}$$

for all $(a, a \oplus d) \in L_1$ and $(b, b \oplus e) \in L_2$ or $(a, a \oplus d) \in L_2$ and $(b, b \oplus e) \in L_1$.

In our particular case that is

$$X^{a,d}_{4,0} = X^{a,d}_{5,0} \quad \text{and} \quad X^{4,0}_{a,d} = X^{5,0}_{a,d},$$

for all $(a, a \oplus d) \in K$ and

$$X^{4,0}_{4,0} = X^{5,0}_{5,0}, \quad X^{4,0}_{5,0} = X^{5,0}_{4,0}.$$

5.1.4 Results

All calculations are to be found in the appendix space_1.nb. The evolution operator for the perfect graph (with the connecting edge) has 18 non-degenerate eigenvalues, so each eigenstate represents a whole subspace. Those give rise to the total number of 82 "operator eigenvalues". The degeneracy of the eigenvalue 1 is 18. The subspace of states corresponding to the eigenvalue 1 is the only one where density matrices can be found.

After introducing the percolation only 41 different eigenvalues remain corresponding in total to 82 attractors. 10 of those attractors are in the subspace of the eigenvalue 1 and can be made density operators. They give 10 different full-state distributions and 6 different distributions summed over coin states. Nevertheless all these distributions follow the symmetry of the graph.

It is important to note, that these distributions are not asymptotic states of some simple initial states (like a state starting at one vertex). It would be very difficult to produce such states and therefore this should be understood just as a visualisation of some building blocks that make up the asymptotic state.

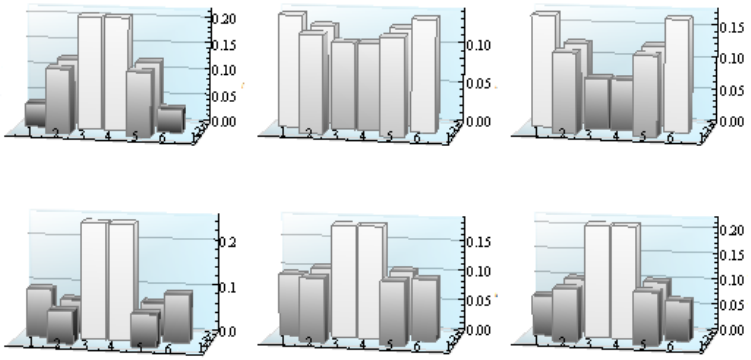


Figure 5.2: Different asymptotic probability distributions of 1D walk that remain after introducing percolation.

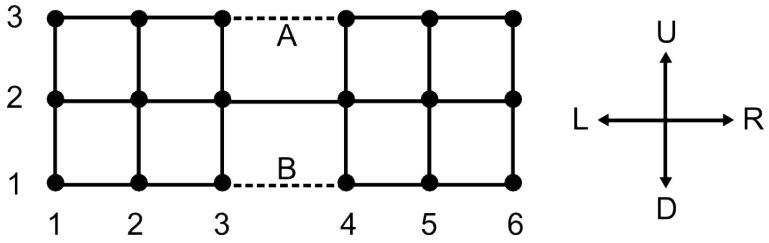


Figure 5.3: Schema of the second investigated graph. Symbols A and B just denote edges $(3, 3-3, 4)$ and $(1, 3-1, 4)$.

5.2 2D graph example

5.2.1 Perfect graph

Our second graph consists of two small two-dimensional square grids connected by three edges (5.3). Two of those edges fluctuate between being simultaneously open and closed. This changes locally the dimensionality of the connection.

In our investigation let us first solve a QW on an unpercolated graph with only the middle connecting edge present. There are no step conditions, we just have to find eigenstates of the evolution operator. We will use the same coin C_c at all vertices and the reflecting boundaries settings. It corresponds to a situation, where there are permanently closed edges pointing out of the graph.

Due to the fact, that the graph we are examining is not homogeneous, there is no obvious simplification for finding eigenstates of the evolution operator U. As the graph has 18 vertices each corresponding to a 4-dimensional subspace, we have in total the operator U acting on a 64-dimensional Hilbert space.

Our QW is fully defined when we choose the coin. Let us follow [7] and examine the Hadamard coin (C_H), the Grover coin(C_G) and the Fourier (C_F) coin:

$$C_H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, C_G = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$
$$C_F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & 1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

When we investigate the eigenstates the symmetry of the graph and the whole walk should be conserved. It is obvious in cases of one-dimensional subspaces of eigenstates corresponding to one eigenvalue. Showing this for example for eigenvalue $\lambda = 1$ (large degeneracy for all three coins) would require finding convenient bases instead of the one given by the numerical solution. That appears to be rather difficult because of possible sign flips in both directions associated with the symmetrical reflection. We will not discuss this question further in this work.

5.2.2 Percolated graph

Let us now examine the QW on the graph shown in the figure 5.3 with the two percolated edges (dashed lines). We have two edges determining the step conditions. In particular the interesting directions are $\phi_{1,3,R}$ with $\phi_{1,4,L}$ and $\phi_{3,3,R}$ with $\phi_{3,4,L}$. We will use the setting where both these edges are closed or open simultaneously. Therefore we again have just two step operators to examine. Here again in the notation of the preceding chapter we have: $K = E \setminus \{(1, 3-1, 4), (3, 3-3, 4)\}, L_1 = \{(1, 3-1, 4), (3, 3-3, 4)\}, L_2 = \emptyset, M = \emptyset$.

Let us once more repeat the general result for the step condition. The attractors must fulfil:

$$X^{a,d}_{b,e} = X^{a,d}_{b\oplus e,\tilde{e}} \quad \text{and} \quad X^{b,e}_{a,d} = X^{b\oplus e,\tilde{e}}_{a,d}$$

for all $(a, a \oplus d) \in K \cup M$ and $(b, b \oplus e) \in L_1 \cup L_2$ and further

$$X^{a,d}_{b,e} = X^{a\oplus d,d}_{b\oplus e,\tilde{e}}$$

for all $(a, a \oplus d) \in L_1$ and $(b, b \oplus e) \in L_1$ or $(a, a \oplus d) \in L_2$ and $(b, b \oplus e) \in L_2$ and

$$X^{a,d}_{b\oplus e,\tilde{e}} = X^{a\oplus d,d}_{b,e}$$

for all $(a, a \oplus d) \in L_1$ and $(b, b \oplus e) \in L_2$ or $(a, a \oplus d) \in L_2$ and $(b, b \oplus e) \in L_1$.

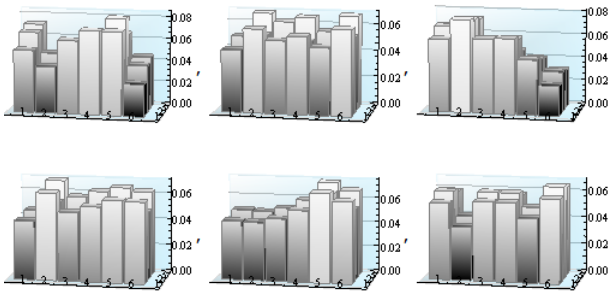


Figure 5.4: Some example probability distributions for 2D percolated walk with the Hadamard coin.

In our particular case that is

$$X^{a,d}_{1,3,R} = X^{a,d}_{1,4,L}, \qquad X^{1,3,R}_{a,d} = X^{1,4,L}_{a,d},$$
$$X^{a,d}_{3,3,R} = X^{a,d}_{3,4,L}, \qquad X^{3,3,R}_{a,d} = X^{3,4,L}_{a,d}$$

for all $(a, a \oplus d) \in K$ and further

$$X^{1,3,R}_{1,3,R} = X^{1,4,L}_{1,4,L}, \quad X^{1,3,R}_{1,4,L} = X^{1,4,L}_{1,3,R},$$
$$X^{3,3,R}_{3,3,R} = X^{3,4,L}_{3,4,L}, \quad X^{3,3,R}_{3,4,L} = X^{3,4,L}_{3,3,R},$$
$$X^{1,3,R}_{3,3,R} = X^{1,4,L}_{3,4,L}, \quad X^{1,3,R}_{3,4,L} = X^{1,4,L}_{3,3,R},$$
$$X^{3,3,R}_{1,3,R} = X^{3,4,L}_{1,4,L}, \quad X^{3,3,R}_{1,4,L} = X^{3,4,L}_{1,3,R}.$$

Due to the fact that here we have more than one-dimensional subspaces, we cannot apply conditions directly on found states and we have to consider linear combinations within particular eigenvalue subspaces.

Hadamard coin

If we use the Hadamard coin, there are in total 36 different eigenvalues of the evolution operator U, and 544 "attractor eigenvalues" for the walk on the perfect graph. For the perfect walk the total degeneracy of the eigenvalue 1 is 280.

After introducing the percolation only 96 eigenvalues remain corresponding in total to 1602 attractors. The most - 150 of them correspond to eigenvalue 1 and therefore are candidates for proper density operators. All 150 attractors corresponding to 1 can be made density operators (Hermitian, trace equal to one, attractor) and give rise to a lot of non-uniform non-symmetrical distributions (even when summed in vertices).

Grover coin

If we use the Grover coin, there are in total 22 different eigenvalues of the evolution operator U (significantly bigger degeneracy of eigenvalue 1 than in the case of the Hadamard coin), and 222 "attractor eigenvalues" for the walk on the perfect graph. For the perfect walk the total degeneracy of the eigenvalue 1 is 996.

After introducing the percolation only 52 eigenvalues remain corresponding in total to 2116 attractors. The most - 808 of them correspond to eigenvalue 1 and therefore are candidates for proper density operators. Unfortunately I was not able to construct proper density matrices from those states. The conditions for hermicity produces too large system of equations due to 808 states in the linear combination (about 3 million equations for 808 variables). The combination of my method and my computer was not capable of such a computation.

Nevertheless it can be presumed that there will again be states with non-uniform probability distributions. Maybe better method or longer computation time will solve this in the future.

Fourier coin

If we use the Fourier coin, all 72 eigenvalues of the evolution operator U are different (no degeneracy), and there are in total 4464 "attractor eigenvalues" for the walk on the perfect graph. For the perfect walk the total degeneracy of the eigenvalue 1 is just 72 (minimum constructed as $\lambda\lambda^*$ for all 72 eigenvalues of U).

After introducing the percolation only the eigenvalue 1 remains corresponding to just 2 attractors. Those can be both made proper density matrices both giving a uniform (full state - not summed in vertices) probability distribution. The only difference is in anti-diagonal terms.

Conclusion

This work has two goals. One is to give some introduction to the fields of random and quantum walks and percolation and also present some previous work on spreading of a percolated quantum walk on an infinite lattice.

The other part deals with asymptotic evolution of quantum walks on finite percolated graphs. Here I have examined two particular finite graphs representing 1D and 2D walks. I have calculated possible asymptotic states for perfect graphs and for partially percolated graphs (graphs consist of two parts connected by percolated edges).

In order to do these calculations I had to examine the step condition for those problems. I consider the general result for the step condition for comparing two step operators to be the main analytical result of my work. It can now be easily used for investigation of various similar graphs.

The two specific example graphs presented here were chosen to represent 1D and 2D walks and in particular the exact form of the percolation of the later graph is supposed to represent a dimensionality change of the connection. This work is just a first step in investigating this phenomenon. For now we only have asymptotic states (the simplest problem we may address) for a few choices of coins in two particular graphs. It is apparent that the coin choice has a fundamental influence on the possible asymptotic states, therefore it is necessary to explore more coins and find some general behaviour that is independent of their choice.

If we really want to focus on the dimensionality changes, we should also vary the graphs and percolation settings. Further it would be desirable to solve larger graphs to eliminate the influence of the boundary conditions. I must admit that using my current methods without further optimization I am not able to solve much larger walks than is the 2D case presented here. Nevertheless some modification can surely be done especially if one would be interested in just some particular property of the asymptotic evolution and some general analytical results and uses of some symmetry or homogeneity may help.

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