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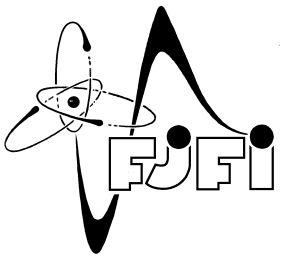

Quantum networks with three body interactions

DIPLOMA THESIS

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V Praze dne Jiří Maryška

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Poděkování

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Název práce: **Kvantové sítě s tříčásticovými interakcemi**

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Abstrakt:

Cílem této práce je prozkoumat tříčásticové interakce v rámci modelu náhodných unitárních operací. Hlavní důraz je kladen nalezení tzv. minimálního atraktorového prostoru a zkoumání vlivu topologie dané interakce na asymptotický vývoj dané náhodné unitární operace. Dále jsou diskutovány vlastnosti asymptotických stavů tříčásticových interakcí a je provedeno jejich srovnání s asymptotickými stavy dvoučásticových interakcí.

Klíčová slova: Otevřený kvantový systém, náhodná unitární operace, asymptotická dynamika, kvantová síť, kvantové počítání, kvantová operace, dvoučásticová interakce, tříčásticová interakce

Title: **Quantum networks with three body interactions**

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Abstract:

This thesis is devoted to the study of three-particle interactions with the help of random unitary operations. We focus primarily on finding the so-called base attractor state and examination how the topology of given interaction affects the resulting asymptotic dynamic of the corresponding random unitary operation. Next we study the properties of the asymptotic states of three-particle interactions and the comparison with asymptotic states of two-particle interaction is made.

Keywords: Open quantum system, random unitary operation, asymptotic dynamic, quantum computing, quantum operation, quantum network, two-qubit interaction, three-qubit interaciton

Contents

List of used symbols

The following list gives an overview of the symbols used through the thesis.

Symbol	**Meaning**
$\mathbb{N}$	The set of all positive integers, i.e., $\{1, 2, 3, \dots\}$
$\hat{N}$	The set $\{1, 2, \dots, N\}$ with $N \in \mathbb{N}$
$\mathscr{H}$	The Hilbert space
$\|x\rangle$	The ket-vector - element of the Hilbert space $\mathscr{H}$
$\langle x\|$	The bra-vector - adjoint of the ket vector $\|x\rangle$
$\hat{A}^\dagger$	The hermitean adjoint of an operator $\hat{A}$
$\mathrm{Tr}\hat{A}$	The trace of an operator $\hat{A}$
$\hat{I}_N$	The identity operator on the space $\mathscr{H} = \mathbb{C}^{2N}$
$U \bigoplus V$	The direct sum of vector spaces U and V
$\mathrm{Ker}(A)$	The subspace of the Hilbert space $\mathscr{H}$ defined as $\{\|x\rangle \in \mathscr{H} \| \hat{A}\|x\rangle = 0\}$
$\mathrm{Ran}(A)$	The subspace of the Hilbert space $\mathscr{H}$ defined as $\{\|x\rangle \| \exists \|y\rangle \in \mathscr{H}, \|z\rangle = \hat{A}\|y\rangle\}$
$\otimes$	The tensor procuct
$\|xy\rangle$	The tensor product of vectors $\|x\rangle$ and $\|y\rangle$, i.e. $\|xy\rangle = \|x\rangle \otimes \|y\rangle$
$\|x\rangle_i$	The ket-vector corresponding to the i-th particle
${}_i\langle x\|$	The bra-vector corresponding to the i-th particle
$\oplus$	The addition modulo 2
$\binom{V}{k}$	The set of all subsets U of the set V, which meet the condition $\|U\| = k$ with $k \leq \|V\|$

Structure of thesis

The aim of this thesis is to investigate the asymptotic evolution of a quantum network consisting of qubits, which are interacting with each other via applications of controlled unitary three-qubit gates with one or two control qubits. These gates are chosen in analogy with previously studied cotrolled unitary two-qubit interactions [1].

This thesis is structured as follows. Chapter 1 is devoted to the short introduction to the description of the time evolution of open quantum systems with stress putted on the model of quantum operations. Next, we describe the most widely used models of quantum computation, the adiabatic quantum computing and the quantum circuit model and we introduce the specific quantum gates, which are important for this thesis.

In Chapter 2, we define random unitary operations, we review their basic properties which are important for the description of asymptotic evolution of given random unitary operation and discuss the use of random unitary operations for describing multi-qubit interaction.

In Chapter 3 basic concepts of graph theory and hypergraphs are defined. Graphs and hypergraphs are going to be instrumental for visual representation of multi-qubit interactions and for the classification of their asymptotic evolution.

In Chapter 4, we review previous results concerning the control unitary two-qubit interactions and we discuss how the concept of control unitary three-qubit interactions is related to this topic. Then we present the main result of this thesis - so-called base attractor spaces of control unitary three-qubit interactions with one or two control qubits

and we present base graphs of these interactions. Using the obtained results we analyze basic properties of asymptotic states of controlled unitary three-qubit interactions. The comparison with basic properties of controlled unitary two-qubit interaction is made.

The results of this thesis are summed up in Chapter 5.

Chapter 1

Introduction

Every real physical system is in contact with its environment. The system (called usually the principal system) interacting with its environment can be made isolated by taking the combined system consisting of the principal system and the environment. The combined system is closed and its time evolution is thus described by an unitary operator. The contact of the environment with the principal system can introduce irreversibilities [2] into the time evolution of the principal system and thus the principal system itself does not have to evolve according to the action of some unitary operator. Systems of this kind are called open systems. If the contact with environment introduces irreversibilities into the time evolution the open quantum system is called the dissipative system.

There are several ways how to approach the problem of time evolution of a general open quantum system. One can derive the time evolution of the composed system (which is closed and thus the time evolution is unitary) and then take trace over the environment, which defines the time evolution on the given system. However, this method is often not very efficient since environment can be large and the mathematical description of its interaction with the principal system can be complicated. We refer to this approach as to the **environment-trace** description.

The other possibility how to model the action of the environment on the principal system is to introduce a **non-Hermitean Hamiltonian** [3] which describes this action. The

non-hermicity of the Hamiltonian results in the exchange of energy between the principal system and its environment. The time evolution of the state of the given system is then described by the non-Hermitean Schrödinger equation. Since the generator of the time evolution is non-Hermitean, the propagator of the principal system is not unitary.

The last model of open quantum systems we mention is the model of **quantum operations** [4, 5] which is widely used in the theory of quantum information. Since our main concern in this thesis is closely related to quantum operations, we will describe the basic concept of quantum operations in the following section.

1.1 Quantum operations

Within the model of quantum operations, we describe the action of the environment on the principal system by a completely positive map $\mathscr{E}$ which acts on the principal system. We require the principal system and its environment to start in the separable state $\rho = \rho^{(S)} \otimes \rho^{(E)}$ and we define the map $\mathscr{E}$ as

$$\mathscr{E}(\rho^{(S)}) = \mathrm{Tr}_E \left[\hat{U}(\rho^{(S)} \otimes \rho^{(E)})\hat{U}^\dagger \right].$$

Although the right-hand side of this equation is identical with the environment-trace description, within the model of quantum operations we do not specify properties of the environment as its Hilbert space $\mathscr{H}_E$, we merely define the map $\mathscr{E}$ and use it to describe the evolution of the principal system. This description is significantly different from the previous one since it describes the evolution of system as a discrete process which takes the initial state of the principal system $\rho^{(S)}$, brings it into the contact with the environment in the initial state $\rho^{(E)}$ and after a fixed finite time T the interaction of the principal system and environment is terminated. The resulting state of the principal system is then $\mathscr{E}(\rho^{(S)})$.

There exists a very useful form of quantum operation called the **operator-sum representation** [4, 5]. Let $|e_k\rangle$ be the orthonormal basis of Hilbert space of the environment and let the initial state of the environment be $|e_0\rangle$. There is no loss of generality by assuming that the environment begins in a state with a well defined state vector, since we

can always use the quantum purification process to do so [5]. Then we define operators on the principal system as

$$\hat{E}_k = \langle e_k|\hat{U}|e_0\rangle\,.$$

With help of these operators we can rewrite the action of a quantum operation as

$$\mathscr{E}(\rho) = \sum_k \hat{E}_k \rho \hat{E}_k^\dagger.$$

Since $\mathscr{E}(\rho)$ must be a density matrix we require for all density matrices ρ

$$1 = \mathrm{Tr}\left[\mathscr{E}(\rho)\right] = \mathrm{Tr}\left[\sum_k \hat{E}_k \rho \hat{E}_k^\dagger\right] = \sum_k \mathrm{Tr}\left[\hat{E}_k^\dagger \hat{E}_k \rho\right] \Rightarrow \sum_k \hat{E}_k^\dagger \hat{E}_k = \hat{I}.$$

This representation has a following physical interpretation. Let ρ_k be the operator

$$\rho_k = \frac{\hat{E}_k \rho \hat{E}_k^\dagger}{Tr[\hat{E}_k \rho \hat{E}_k^\dagger]}.$$

Clearly ρ_k is a density matrix. With help of these density matrices we can rewrite the action of the quantum operation as

$$\mathscr{E}(\rho) = \sum_k P_k \rho_k = \sum_k P_k \frac{\hat{E}_k \rho \hat{E}_k^\dagger}{Tr[\hat{E}_k \rho \hat{E}_k^\dagger]}$$

with $P_k = Tr[\hat{E}_k \rho \hat{E}_k^\dagger]$. We can thus interpret the action of the map $\mathscr{E}$ on the state ρ as follows. $\mathscr{E}$ takes the state ρ and replaces it randomly by the state ρ_k with the probability P_k.

1.2 Quantum computation

The main part of this thesis is concerned with the random unitary operations model. Although this model can be used to solve a very wide range of problems in physics and biology, one of the most useful applications of this model is certainly its application to the quantum computation, where it can describe the contact with the principal system and its environment, which leads to decoherence of the system. As quantum computation is a powerful tool to solve various problems it is of a significant interest of present-day science. To process information, quantum computation uses qubits. Although our main

concern is going to be the quantum circuit model of quantum computation, we mention the adiabatic quantum computing model to stress the fact that there are various paths we can take in the quantum computation.

The model of **adiabatic quantum computing** [6] is based on the adiabatic theorem [7]. First, the complex Hamiltonian, whose ground state gives the solution of the given problem is found. This Hamiltonian has usually a form [8]

$$H = \sum_i h_i Z_i + \sum_i \Delta_i X_i + \sum_{i<j} J^{ij} Z_i \otimes Z_j + \sum_{i<j} K^{ij} X_i \otimes X_j. \tag{1.1}$$

Here X_i represents the Pauli matrix σ_x which acts on the qubit i and Z_i stands for the Pauli matrix σ_z which acts on the qubit i. It was proven that Hamiltonian of the form (1.1) is the simplest Hamiltonian that allows the universal adiabatic quantum computing [8].

In the next step, the system, whose dynamic is governed by a simple Hamiltonian is initialized in its ground state. Finally, the Hamiltonian of the system is slowly (adiabatically) varied towards the complex Hamiltonian H. By adiabatic theorem, the resulting state of the system will be the ground state of the complex Hamiltonian and thus this method leads to the solution of given problem.

The model of quantum computation we are going to be most concerned about is the **quantum circuit model** [4, 9]. This is a direct analog of classical circuit model used in classical computers. Within the quantum circuit model we have at our disposal the set of so-called **quantum gates**. Quantum gates are objects performing operations on input qubits. The idea is to initialize the system in some specific state, then run it through a specific set of quantum gates. The resulting state of the system then encodes the solution of the problem.

There are two basic types of quantum gates - single-qubit gates and multi-qubit gates. As the names suggest, single-qubit gates operate on the single qubit and multi-qubit gates operate on at least two qubits. The important kind of multi-qubit gates are control gates. These gates change the state of one set of qubits (called the target qubits)

conditionally on the state of the second set of qubits (called the control qubits). Because of their importance in this thesis, we will describe two of these gates - the CNOT gate and the Toffoli gate.

The CNOT [4, 5, 9] or control NOT gate is a quantum gate which acts on two qubits. This action is usually described in the so-called **computational basis**. In this basis the CNOT gate can be written as

$$\hat{U}_{12} = |0\rangle_{11}\langle 0| \otimes \hat{I}_2 + |1\rangle_{11}\langle 1| \otimes [|0\rangle_{22}\langle 1| + |1\rangle_{22}\langle 0|] .$$

In this expression $\hat{I}_2$ exceptionally stands for the identity operator on the Hilbert space of the second qubit. The CNOT gate thus changes the state of second qubit only if the first qubit is in the state $|1\rangle$.

The Toffoli gate [4, 5, 9], which is sometimes called the double CNOT gate acts on three qubits by the following rule:

$$\hat{U}_{12,3} = [\hat{I}_{12} - |11\rangle_{12\ 12}\langle 11|] \otimes \hat{I}_3 + |11\rangle_{12\ 12}\langle 11| \otimes [|0\rangle_{33}\langle 1| + |1\rangle_{33}\langle 0|] .$$

In this expression $\hat{I}_{i_1 \ldots i_k}$ exceptionally stands for the identity operator on the Hilbert space of the system consisting of qubits $i_1, \ldots i_k$. Thus the Toffoli gate can be viewed as the CNOT gate with two control qubits. The state of the third qubits is changed only if both first and second qubit are in the state $|1\rangle$.

Any quantum gate can be viewed as the map $\mathscr{E}$ which describes the time evolution of the principal system within the model of the quantum operation. Thus the quantum circuit model of the quantum computation and the quantum operation model of the time evolution of open systems are closely related. As we will see in the next chapter, we can always find the analytical form of the asymptotic dynamic of special case of quantum operations - random unitary operations, which makes it a powerful tool for studying decoherence effects in open quantum systems.

Chapter 2

Random unitary operations

In the previous chapter we described the general concept of quantum operations. Now we turn to the case where the operators in the operator-sum representation $\hat{E}_k$ are unitary. Since the unitarity implies the equation $\hat{E}_k^\dagger \hat{E}_k = I$ the equation $P_k = \mathrm{Tr}[\hat{E}_k^\dagger \hat{E}_k \rho] = \mathrm{Tr}\rho = 1$ holds. Thus operators P_k lose their probabilistic interpretation. Since the relation

$$\sum_k \hat{E}_k^\dagger \hat{E}_k = I$$

must hold, we are forced to introduce the probabilistic nature of the outcome artificially by defining probability distribution p_k. In this manner we arrive at the random unitary operation (RUO), which is going to be denoted as Φ and the unitary operators in the operator-sum representation of Φ are going to be denoted as $\hat{U}_k$.

Before we take a step towards the formal definition of RUO [10], let us name some of the results derived with the help of this model. The most important result related to this thesis is the asymptotic dynamic of the quantum network consisted of qubits interacting in pairs via controlled unitary interactions [1]. From the other results let us name the following. The existence of two kinds of entangled states - robust one and fragile one [11], the asymptotic dynamic of the quantum Markov chain [12] and the effect of graph percolations on the quantum walks [13].

2.1 Basic properties of random unitaery operations

A random unitary operation Φ on a finite-dimensional Hilbert space $\mathscr{H}$ is a completely positive trace-preserving map $\Phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ admitting a convex decomposition of the form [10]

$$\Phi(\rho) = \sum_{k=1}^{n} p_k \hat{U}_k \rho \hat{U}_k^\dagger.$$

Here the $\hat{U}_k$ are unitary operators and $p_k > 0$ fulfill the relation $\sum_k p_k = 1$. The interpretation of action of Φ is the following: The state of the system ρ is by applying the map Φ randomly replaced by the state $\hat{U}_k \rho \hat{U}_k^\dagger$ with the probability p_k. These probabilities are the result of the classical uncertainties which can have different origins. They can be the result of an unknown error mechanism, an unknown unitary evolution involving an additional ancillary system or the result of uncertainty about its degrees of freedom.

The main focus of this thesis is devoted to the asymptotic dynamic of a special kind of RUO in which the certain quantum gates are applied on several qubits. Starting with the initial state of the system $\rho(0)$, we iterate this state by the rule $\rho(n+1) = \Phi(\rho(n))$.

Before we turn our attention to the asymptotic dynamic, we summarize the basic properties of random unitary operations. They belong to the class of unital maps which leave the maximally mixed state invariant:

$$\Phi(\hat{I}) = \sum_{k=1}^{n} p_k \hat{U}_k \hat{I} \hat{U}_k^\dagger = \sum_{k=1}^{n} p_k \hat{I} = \hat{I}.$$

With respect to the Hilbert-Schmidt scalar product [14] the adjoint map $\Phi^\dagger$ is given by

$$\Phi^\dagger(\rho) = \sum_{k=1}^{n} p_k \hat{U}_k^\dagger \rho \hat{U}_k.$$

This equation immediately implies that RUO Φ is generally not unitary, neither Hermitean or even normal and thus it needs not to be diagonalizable. However, it turns out that the Jordan form of Φ^n has in the limit $n \to \infty$ diagonal from. This property is going to be beneficial for the description of the asymptotic dynamic of Φ.

The following properties of random unitary operations are introduced without proof,

which can be found in [10].

The norm of the RUO Φ induced by the Hilbert-Schmidt scalar product fulfils the equation $||\Phi||_{HS} = 1$. This property follows from the unitary invariance of the Hilbert-Schmidt norm and the unitality of the RUO Φ. The most important consequence of this statement is that if λ is an eigenvalue of Φ, then we must have $|\lambda| \leq 1$.

If $X_\lambda \in B(H)$ is a generalized eigenvector corresponding to the eigenvalue λ of Φ, then either $\lambda = 1$ or $\mathrm{Tr}[X_\lambda] = 0$. The proof of this property follows directly from the definition of generalized eigenvector of linear map and the equation $\mathrm{Tr}\left[(\Phi - \lambda I)^n X_\lambda\right] = (1-\lambda)^n \mathrm{Tr}[X_\lambda]$ which can be proved by induction on n.

There exists a subspace of $\mathcal{B}(\mathscr{H})$ which is of extreme importance for the description of asymptotic evolution of RUO Φ. It is called the **attractor space** and it is defined as

$$\mathrm{Atr}(\Phi) = \bigoplus_{|\lambda|\leq 1} \mathrm{Ker}(\Phi - \lambda I).$$

As we will see in the next section, the asymptotic dynamics of RUO Φ is completely determined by its attractor space. The elements of the attractor space are called attractors.

The main result about the structure of the attractor space of RUO Φ is the following theorem.

Theorem 2.1.1. *The eigenspace Ker$(\Phi - \lambda I)$ of RUO Φ with $|\lambda| = 1$ is equal to the set*

$$D_\lambda = \{X \in \mathcal{B}(\mathscr{H}) | \hat{U}_k X = \lambda X \hat{U}_k, \forall k \in \hat{n}\}.$$

The fact that every $X \in D_\lambda$ belongs to the kernel of Φ corresponding to the eigenvalue λ is almost trivial, thus this theorem proves the converse and it gives us the instrument to construct the attractor space of RUO Φ.

From this structure theorem of the attractor space we can easily derive the basic properties of attractors of Φ. These are particularly useful when searching for the explicit form of the attractor space. The straightforward consequence of the structure theorem is, that

if X_{λ_1} and X_{λ_2} are attractors corresponding to eigenvalues λ_1 and λ_2 with $|\lambda_i| = 1$, then $X_{\lambda_1}X_{\lambda_2}$ is an attractor corresponding to eigenvalue $\lambda_1\lambda_2$ or a zero operator. Furthermore $X_{\lambda_i}^\dagger$ is an attractor corresponding to eigenvalue λ_i^*.

2.2 Asymptotic evolution of random unitary operations

The Jordan form of RUO Φ has properties which are useful for the description of asymptotic evolution of Φ. It was shown that if we consider the Jordan form of RUO Φ, then all Jordan blocks corresponding to eigenvalues λ of Φ with $|\lambda| = 1$ are one-dimensional and thus all generalized eigenvectors corresponding to eigenvalues with magnitude equal to one are eigenvectors. This is the simple consequence of the fact, that for eigenvalue λ with $|\lambda| = 1$, we have

$$\mathrm{Ker}[\Phi - \lambda I] \cap \mathrm{Ran}[\Phi - \lambda I] = \{0\}.$$

Next, it is straightforward to show, that all Jordan blocks J_s corresponding to eigenvalues λ_s with $|\lambda_s| < 1$ vanish in the limit of large numbers of iterations [10]:

$$\lim_{n\to\infty} (J_s)^n = 0.$$

Thus Φ^n has a diagonal form in the limit of large n. As a last step, it was shown that the mutually orthogonal subspaces

$$J_0 = \bigoplus_{\lambda\in\sigma_{|1|}} \mathrm{Ker}[\Phi - \lambda I] \quad \text{and} \quad J_1 = \bigcap_{\lambda\in\sigma_{|1|}} \mathrm{Ran}[\Phi - \lambda I]$$

are invariant under map Φ and they fulfill the relation $J_0 \oplus J_1 = \mathcal{B}(\mathscr{H})$ [10]. These properties imply that the asymptotic dynamic of Φ is given by the state

$$\rho_\infty(n) = \sum_{\substack{\lambda\in\sigma_{|1|} \\ i=1}}^{d_\lambda} \lambda^n \mathrm{Tr}[\rho(0)X_{\lambda,i}^\dagger]X_{\lambda,i},$$

in the sense

$$\lim_{n\to\infty} ||\rho(n) - \rho_\infty(n)|| = 0.$$

In this expression, $\sigma_{|1|} = \{\lambda \in \sigma(\Phi), |\lambda| = 1\}$, $\rho(n) = \Phi^n(\rho(0))$ and $\{X_{\lambda,1} \ldots, X_{\lambda,d_\lambda}\}$ is the orthonormal basis of the subspace $\mathrm{Ker}[\Phi - \lambda I]$.

To summarize, the asymptotic dynamic of RUO Φ is completely determined by the attractor space of Φ. Furthermore, the asymptotic dynamic is completely independent on the probabilities p_i. This follows immediately from the independence of the attractor space on these probabilities. Thus two RUOs with the same unitary operators in their convex decomposition have the same asymptotic dynamic. The nonzero probabilities determine only the rate of convergence of the iteration process towards the attractor space.

2.3 Random unitary operations as the model for multi-qubit interaction

The unitarity is the only condition which must be satisfied by operators $\hat{U}_i$ which arise in the convex decomposition of RUO Φ. Because of that RUOs form a broad class of quantum operations with a wide spectrum of behavior. There are two basic cases. In the first one, operators $\hat{U}_i$ act nontrivially on the whole Hilbert space $\mathscr{H}$. The opposite of this case is the situation in which operators $\hat{U}_i$ act nontrivially only on the small fraction of the Hilbert space $\mathscr{H}$. Multi-qubit interactions are an example of such RUO. These are of particular interest in quantum computation since the quantum gate model of quantum computation is based on the concept of quantum gates, which act on one or several qubits. The multi-qubit interaction model introduced in the Chapter 4 can be also viewed as a simple model of particle collisions. Consider a rarefied gas with a short-range interaction and suppose we are not able to track the positions of the molecules, but we are able to track if they are in the ground state or in the excited state. The inability of tracking positions of particles leads to uncertainty about the system and thus introduces random behavior. Since the gas is rarefied, we can decompose the interaction of the molecules into steps which are well separated in time and during each step there is only one interacting pair or triplet of molecules. Since this subsystem is closed at this particular step, it evolves according to some unitary operator. These considerations lead us thus to model of random unitary operations.

Chapter 3

Graphs and hypergraphs

Many physical systems can be visualized by means of graphs. Informally speaking, graph is a set of items with connections (called edges) between them. Graphs are useful in many physical models, they can for instance represent the Internet, social networks, distribution networks or interacting physical systems [15]. There are many different kinds of graphs, which depend on the properties of their vertices and edges. The properties of graphs were extensively investigated during past decades. In recent years, the attention was drawned towards graphs with a large number of vertices. Large graphs are of interest because they represent systems composed of a huge number of elements and they can thus represent the above mentioned systems. With increasing size of the graph the computational complexity of its properties is also increasing, thereby the analysis of even the simple properties of graphs such as its connectivity becomes problematic. One thus turns to the study of large-scale properties of graphs e.g. clustering [16], but as we will see in next chapter, the small-scale properties e.g. connectivity still do play an important role for large graphs. An useful generalization of a graph called a hypergraph accounts for more complex connection between vertices and will be particularly useful in this thesis.

The purpose of this chapter is to introduce the basic concepts of graph theory, which are used in next chapters. We will be interested in both graphs and hypergraphs, but we will define only those properties of these structures, which we are going to use.

3.1 Graphs and their basic properties

We define an **undirected graph** as the ordered triplet $G = (V, E, \varphi)$, where $V = \{v_1, \dots v_N\}$ is the set of vertices, $E = \{e_1 \dots e_m\}$ is the set of edges and $\varphi : E \to \binom{V}{2}$. Such definition of a graph does not include graphs with loops (edges with both ends at same vertex), but it does allow multiple edges (two vertices can be connected by more than one edge).

A finite sequence $P = (e_{i_1}, \dots, e_{i_l})$ of edges, which fulfils the conditions $\varphi(e_{i_j}) \cap \varphi(e_{i_{j+1}}) \neq \emptyset$ and $\varphi(e_{i_j}) \neq \varphi(e_{i_{j+1}})$ for every $j \in \widehat{l-1}$ is called a **path** in G, the vertex $u \in V$, which fulfils $v \cap \varphi(e_{i_2}) = \emptyset$ is called the beginning of path P, the vertex v, which fulfils $v \cap \varphi(e_{i_{l-1}}) = \emptyset$ is called the end of path P and the length of path P is $l-1$. We will call the path with the beginning u and the end v simply as the path between u and v and we will denote it P_{uv}. We say that u and v are **connected** in G. A undirected graph G is called **connected**, if every pair of its vertices is connected.

If $\varphi : E \to V \times V$, we call G a **directed graph**. The elements of a E of directed graph are called directed edges or arrows. If $\varphi(e_i) = (v_j, v_k)$, we call v_j the tail of e_i and v_k the head of e_i.

There are two kinds of connectivity in a directed graph. If a directed graph G is connected

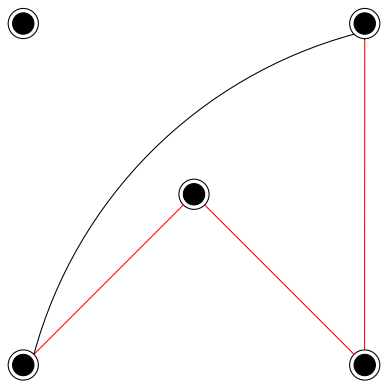


Figure 3.1: An example of an undirected graph which is not connected and an example of a path in an undirected graph (in red).

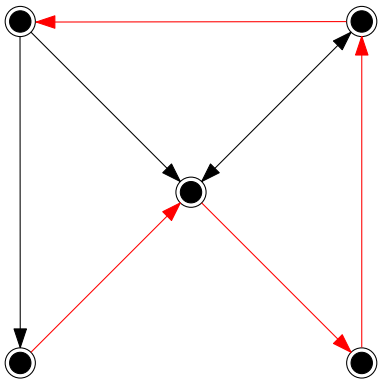
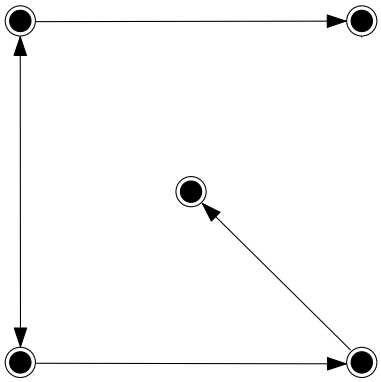


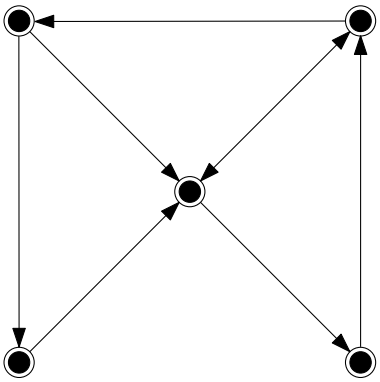
Figure 3.2: An example of a directed graph and an example of a path in a directed graph (in red).

in the sense of an undirected graph, we say that G is **weakly connected**. If every $u, v \in V$ are connected by a path $P_{uv} = (e_{i_1}, \ldots, e_{i_l})$ which fulfils $[\varphi(e_{i_j})]_2 \cap [\varphi(e_{i_j+1})]_1 \neq \emptyset$, where $[\varphi(e_{i_k})]_l$ is the l-th component of $\varphi(e_{i_k})$, we say that G is **strongly connected**. This kind of path is called the **directed path**.

A graph G equipped with the map $C : \{1, \ldots, k\} \to E$ is called **edge-colored graph** and $C(e)$ for $e \in E$ is called as **color of the edge e**. We will call the graph G **m-connected**, if for every pair of m vertices $(u_1, \ldots, u_m), (v_1, \ldots, v_m)$ there are paths $P_1, \ldots P_m$, with $P_i = (e_{i_1}, \ldots e_{i_l})$, where P_i connects u_i with v_i and for every $i, j \in \hat{m}$ and for every $n \in \hat{l}$ the paths P_i and P_j fulfill the relation $C(e_{i_n}) = C(e_{j_n})$.



(a) An example of a weakly connected directed graph.



(b) An example of a strongly connected directed graph.

3.2 Hypergraphs and their basic properties

A hypergraph [17] is a direct generalization of a graph in which we do not restrict the relations between vertices to be the binary ones. Instead we define the **undirected hypergraph** as the ordered pair $H = (V, E)$ where E is an arbitrary subset of set of all subsets of V. The elements of E are called hyperedges. We dropped the map φ in the definition of hypergraph to simplify the notation, since we are not going to need multiple edges while using hypergraphs.

Rather as a sequence of hyperedges, we define a **path** in hypergraph as a sequence of pairs of vertices $P = (u = v_{i_1}v_{i_2}, \ldots, v_{i_{l-1}}v_{i_l} = v)$ for which there is a sequence of hyperedges $(e_{i_1}, \ldots, e_{i_{l-1}})$ with $e_{i_j} = E_j$ such that $v_{i_1} \in E_1$, $v_{i_l} \in E_{l-1}$ and for every $j \in \{2, \ldots, l-1\}$ the condition $v_{i_j} \in E_{j-1} \cap E_j$ holds. We denote this path by P_{uv} and we say that u and v are **connected** in H. A hypergraph H is called connected if every pair of vertices $u, v \in V$ is connected.

A **directed hypergraph** is $H = (V, E)$, where E is arbitrary subset of the set of all pairs of disjoint subsets of V. More precisely, if $e \in E$, then $e = (X, Y)$ with $X, Y \subset V$ and $X \cap Y = \emptyset$. If $u \in X$, then u is called the tail of hyperedge e, if $v \in Y$, then v is called the head of hyperedge e.

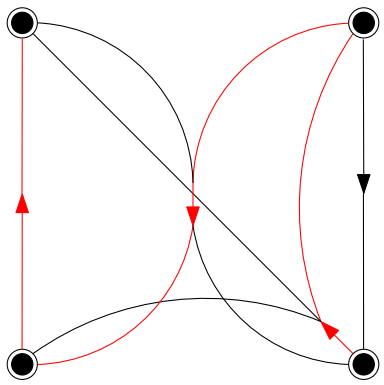
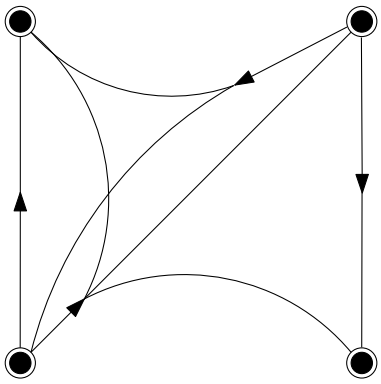
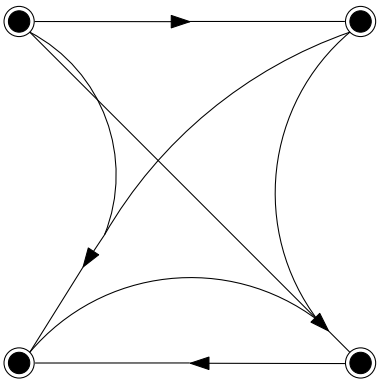


Figure 3.4: An example of a strongly connected directed hypergraph and an example of a path in a directed hypergraph (in red).



(a) An example of a F-graph.



(b) An example of a B-graph.

For a given directed hypergraph $H_d = (V, E)$ we define a corresponding undirected hypergraph $H_u = (V, F)$, where $F = \{(X \cup Y)|(X, Y) \in E\}$. We say that the directed hypergraph H_d is **weakly connected** if the undirected hypergraph H_u is connected. We say that the hypergraph H is **strongly connected**, if for every $u, v \in V$ there is a path $P_{uv} = (u = v_{i_1}v_{i_2}, \ldots, v_{i_{l-1}}v_{i_l} = v)$ such that if v_{i_j} is the tail of hyperedge e then $v_{i_{j+1}}$ is the head of hyperedge e holds for every $j \in \hat{l}$.

There are special types of hyperedges which will be particularly useful for us. Let $e = (X, Y)$ be a hyperedge. If $|X| = 1$, e is called a **F-arc**, if $|Y| = 1$, e is called a **B-arc**. A hypergraph H in which all the hyperedges are F-arcs is called a **F-graph**, a hypergraph H in which all the hyperedges are B-arcs is called a **B-graph**.

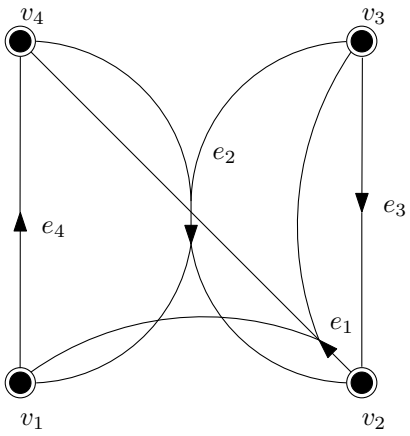
3.3 Incidency matrix

Both graphs/hypergraphs can be represented by the so-called incidency matrix. Let n be number of vertices of a given graph/hypergraph, let m be the number of its edges/hyperedges. For undirected graphs/hypergraphs, the incidency matrix is $A \in \{0, 1\}^{n,m}$ with matrix elements defined as

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

For directed graphs/hypergraphs, the incidency matrix is $A \in \{-1, 0, 1\}^{n,m}$ with matrix elements defined as

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is head of } e_j \\ -1 & \text{if } v_i \text{ is tail of } e_j \\ 0 & \text{otherwise} \end{cases}$$



$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

Figure 3.6: An example of a directed hypergraph G and it's corresponding incidency matrix A_G.

With increasing number of vertices and edges of a given graph/hypergraph, the visual representation tends to be chaotic. The incidency matrix of a given graph/hypergraph can be very useful in this situation since it encodes the whole information about a given graph/hypergraph.

Chapter 4

Three body interactions

The properties of two body interactions were extensively studied within the random unitary operations model [1, 11, 12]. Since we know that the attractor space is independent on the nonzero probabilities which appear in the convex decomposition of a given RUO, the question arises, how three body interactions affect the resulting attractor space. As was already stated in Section 2.3, there are two distinct types of RUOs. The first one is represented for instance by the two body interactions, the operators involved in the convex decomposition of this RUO act nontrivially only on the low-dimensional subspace of given Hilbert space. The other are RUOs, where the operators involved in the convex decomposition act nontrivially on the whole Hilbert space. As an example of this case we can list quantum walks on percolation graphs [13]. Three body interactions then represent the first and the simplest step from the first case towards the second one. Thus it is interesting to look for their attractor space and search for new features, which are not present within the two body interactions. As the three body interaction can be studied from different point of views, we will focus on the two following cases. We will examine the case with a single control qubit and with two control qubits, from which we deduce the influence of number of control qubits on the resulting dynamics.

4.1 Base attractor space and base graphs

A very sophisticated and unifying model of multi-qubit interaction is the model of **quantum networks**. A quantum network in our context is the set of qubits which are coupled with each other by the quantum gates. For sake of simplicity, qubit networks in which qubits interact with each other via controlled unitary multi-qubit interactions will be referred to simply as qubit networks.

When studying multi-qubit interaction, one finds that the resulting attractor space strongly depends on the topology of the interaction. The topology of the interaction is encoded in the so-called **interaction graph** of a given system, the definition of this graph depends on the particular interaction. Generally, there exists a topology of the interaction, corresponding attractor space of which is a subspace of any attractor space of given type of interaction. This attractor space is called the **base attractor space** and the corresponding interaction graphs are called **base graphs**. Because of their importance we study them in following sections.

4.2 Controlled unitary two-qubit interactions

Let us consider a quantum network consisting of $N \geq 2$ qubits interacting with each other by application of controlled unitary two-qubit gates $\hat{U}$, which are applied to the pairs of qubits according to the prescribed probability distribution p_{ij} with $\sum_{i,j} p_{ij} = 1$.

We are interested in the one-parameter family of controlled unitary two-qubit interactions which in the computational basis have the form

$$\hat{U}_{ij}^{(\phi)} = |0\rangle_{ii}\langle 0| \otimes \hat{I}_j + |1\rangle_{ii}\langle 1| \otimes \hat{u}_j^{(\phi)},$$

where

$$\hat{u}_j^{(\phi)} = \cos\phi(|0\rangle_{jj}\langle 0| - |1\rangle_{jj}\langle 1|) + \sin\phi(|0\rangle_{jj}\langle 1| + |1\rangle_{jj}\langle 0|).$$

Thus the considered RUOs have the form

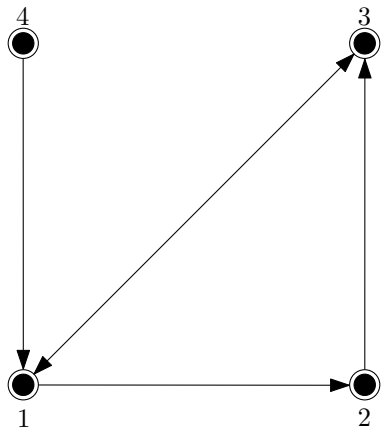


Figure 4.1: An example of an interaction graph G corresponding to the RUO $\Phi_G^{(2)}$

$$\Phi^{(2)}(\rho) = \sum_{e \in E} p_e \hat{U}_e^{(\phi)} \rho \hat{U}_e^{\dagger(\phi)}.$$

Here E is the subset of the set of the ordered pairs (i, j). In the special case $\phi = \frac{\pi}{2}$, the controlled unitary transformation $\hat{U}_{ij}^{(\phi)}$ reduces to a CNOT gate with a control qubit i and a target qubit j. The eigenvalues of the transformation $\hat{U}_{ij}^{(\phi)}$ are given by $\lambda_1 = 1$ and $\lambda_2 = -1$. Because of that, the attractor spectrum $\sigma_{|1|}$ fulfils the relation $\sigma_{|1|} \subset \{-1, 1\}$. For $\phi = 0$ or $\phi = \pi$ the RUO Φ is diagonal in the computational basis.

We begin the description of controlled unitary two-qubit interactions by construction of an interaction graph corresponding to such qubit network. Let us consider a graph $G = (V, E, \varphi)$ with $V = \hat{N}$, where the vertex $i \in V$ corresponds to the qubit i. The vertices i and j are connected in G by a directed edge $e \in E$, whose head is j and whose tail is i, if $p_{ij} > 0$. Since G contains no multiple edges, the map φ is not needed in the description of the graph. Because of this we will use the notation $G = (V, E)$ and $e = (i, j)$. The directed graph $G = (V, E)$ which is constructed in this way is called the interaction graph of the given qubit network.

Concerning the base attractor space and base graphs of this one-parameter family of interactions, the following theorem was proved [1]:

Theorem 4.2.1. *For $\phi \neq 0, \pi$ the base graphs of the considered family of interactions are exactly all strongly connected interaction graphs $G = (V, E)$. For a given number of qubits N with $N > 2$, the associated base attractors are elements of a five-dimensional attractor space $Atr(\Phi_G^{(2)})$. An orthonormal basis system of linear operators in this space of base attractors is given by*

$$\begin{aligned}
\hat{X}_1 &= |0_N\rangle \langle 0_N| , \qquad \hat{X}_2 = |0_N\rangle \langle \psi_N| , \\
\hat{X}_3 &= |\psi_N\rangle \langle 0_N| , \qquad \hat{X}_4 = |\psi_N\rangle \langle \psi_N| , \\
\hat{X}_5 &= \frac{1}{\sqrt{2^N - 2}} \left(\hat{I}_N - |0_N\rangle \langle 0_N| - |\psi_N\rangle \langle \psi_N| \right)
\end{aligned} \tag{4.1}$$

with N-qubit states

$$\begin{aligned}
|0_N\rangle &= |0\rangle^{\otimes N} , \\
|\psi_N\rangle &= \frac{1}{\sqrt{\langle \theta_N | \theta_N \rangle}} |\theta_N\rangle , \\
|\theta_N\rangle &= \sum_{0 \neq z \in I_N} \left(\cos \frac{\phi}{2} \right)^{N - \tau(z)} \left(\sin \frac{\phi}{2} \right)^{\tau(z)} |z\rangle .
\end{aligned}$$

I_N denotes the set of all possible binary N-tuples and $\tau(z)$ is the sum of the bit values of all N qubits of the N-qubit string z. All of these base attractors solve the attractor equations with eigenvalue $\lambda = 1$. For $N > 2$ there are no non-trivial solutions of attractor equations with eigenvalue $\lambda = -1$.

In the special case of two qubits, i.e. $N = 2$, the attractor space $Atr(\Phi_G^{(2)})$ of the base attractors is six-dimensional. There is a five-dimensional subspace associated with eigenvalue $\lambda = 1$ whose orthonormal basis is given by $\{\hat{X}_i | i \in \{1, \ldots, 5\}\}$. But now, also a non-trivial one-dimensional subspace exists that corresponds to the eigenvalue $\lambda = -1$ and that contains the normalized linear operator,

$$\begin{aligned}
\hat{X}_6 = \frac{1}{\sqrt{2 + 2\cos^2 \frac{\phi}{2}}} \Big(& \cos \frac{\phi}{2} \left(|01\rangle \langle 11| - |10\rangle \langle 11| - |11\rangle \langle 01| + |11\rangle |10\rangle \right) - \\
& - \sin \frac{\phi}{2} \left(|01\rangle \langle 10| - |10\rangle \langle 01| \right) \Big).
\end{aligned}$$

Base graphs of the one-parameter family of two-qubit interactions $\Phi^{(2)}$ with $\phi \in (0, \pi)$ for any $N \geq 2$ are thus strongly connected interaction graphs. The asymptotic dynamic of quantum networks with strongly connected interaction graphs for $N > 2$ is stationary

and takes place on the five-dimensional subspace of $\mathcal{B}(\mathscr{H})$ with the basis given by (4.1), which are the solutions of the attractor equations for $\lambda = 1$. This subspace is also the subspace of the attractor space of these particular interactions for $N = 2$, but in this case the asymptotic dynamic can express periodic behavior provided by the solution of attractor equations $\hat{X}_6$ which corresponds to the eigenvalue $\lambda = -1$.

For the generic case $N > 2$ any initial quantum state ρ_{in} approaches by the iterative application of the map $\Phi^{(2)}$ to the state

$$\rho_{\infty}^{(2)} = p^{(2)} \frac{\hat{P}_N^{(2)} \rho_{\text{in}} \hat{P}_N^{(2)}}{p^{(2)}} + (1 - p^{(2)}) \frac{\hat{I}_N - \hat{P}_N^{(2)}}{2^N - 2}, \tag{4.2}$$

with $\hat{P}_N^{(2)} = |0_N\rangle \langle 0_N| + |\psi_N\rangle \langle \psi_N|$ and $p^{(2)} = \text{Tr}[\rho_{\text{in}} \hat{P}_N^{(2)}]$. $\hat{P}_N^{(2)}$ is the projector onto an N-qubit decoherence-free subspace of the quantum states which are not affected by unitary transformations $\hat{U}_{ij}^{(\phi)}$.

4.3 Controlled unitary three-qubit interactions with one control qubit

The controlled unitary three-qubit interactions with one control qubit are the direct analogy of the the two-qubit interactions which were reviewed in the previous section. Let us suppose we are given a quantum network consisting of $N \geq 3$ qubits interacting with each other by application of the controlled unitary two-qubit gates $\hat{U}$, which are applied to the triplets of qubits according to a prescribed probability distribution $p_{i,jk}$, with $\sum_{i,jk} p_{i,jk} = 1$.

In analogy with the two-qubit interactions, we define the one-parameter family of three-qubit interactions with one control qubit, which have in the computational basis the form

$$\hat{U}_{i,jk}^{(\phi)} = |0\rangle_{ii}\langle 0| \otimes \hat{I}_{jk} + |1\rangle_{ii}\langle 1| \otimes \hat{u}_j^{(\phi)} \otimes \hat{u}_k^{(\phi)},$$

where

$$\hat{u}_j^{(\phi)} = \cos\phi(|0\rangle_{jj}\langle 0| - |1\rangle_{jj}\langle 1|) + \sin\phi(|0\rangle_{jj}\langle 1| + |1\rangle_{jj}\langle 0|).$$

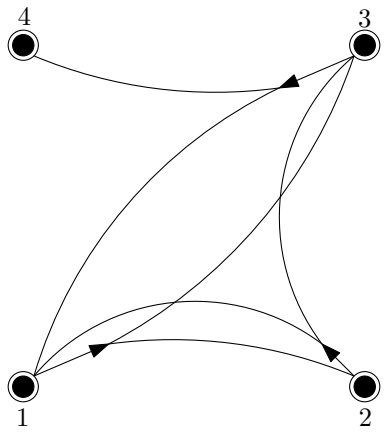


Figure 4.2: An example of an interaction F_1-graph G corresponding to the RUO Φ_G

Thus the considered RUOs have the form

$$\Phi^{(3,1)}(\rho) = \sum_{e \in E} p_e \hat{U}_e^{(\phi)} \rho \hat{U}_e^{\dagger(\phi)}.$$

Here E is the subset of the set of partially ordered triples $(i; j, k) = (i; k, j)$. As in the previous case, we will discard cases $\phi = 0, \pi$ since the RUO $\Phi^{(3,1)}$ is then diagonal in the computational basis. The eigenvalues of the transformation $\hat{U}_{i,jk}^{(\phi)}$ are again given by $\lambda_1 = 1$ and $\lambda_2 = -1$ and thus the attractor spectrum $\sigma_{|1|}$ fulfils the relation $\sigma_{|1|} \subset \{-1, 1\}$.

The hypergraphs are more convenient mathematical object to represent the three-qubit interactions with. More precisely, we are going to describe given RUOs by F-graphs. We define the **interaction F_1-graph** of corresponding RUO $\Phi^{(3,1)}$ as $G = (V, E)$, with $V = \hat{N}$, where the vertex i corresponds to the qubit i. If $p_{i,jk} > 0$ then the vertices i, j and k are connected in G by a F-arc $e \in E$ whose head are the qubits j and k and whose tail is the qubit i.

The base F_1-graphs and the base attractor space is given by the following theorem.

Theorem 4.3.1. *For $\phi \neq 0, \pi$, the base F_1-graphs of the considered family of interactions are exactly all strongly connected interaction F_1-graphs $G = (V, E)$. For the case $\phi \neq \frac{\pi}{2}$ and any number of qubits $N \geq 3$, the associated base attractors are elements of a ten-dimensional attractor space $Atr(\Phi_G^{(3,1)})$. An orthonormal basis system of linear operators*

in this space of base attractors is given by

$$\begin{aligned}
\hat{X}_1 &= |\tilde{0}_N\rangle\langle\tilde{0}_N|, \qquad \hat{X}_2 = |\tilde{0}_N\rangle\langle\varphi_N^+|, \\
\hat{X}_3 &= |\tilde{0}_N\rangle\langle\varphi_N^-|, \qquad \hat{X}_4 = |\varphi_N^+\rangle\langle\tilde{0}_N|, \\
\hat{X}_5 &= |\varphi_N^-\rangle\langle\tilde{0}_N|, \qquad \hat{X}_6 = |\varphi_N^+\rangle\langle\varphi_N^+|, \\
\hat{X}_7 &= |\varphi_N^-\rangle\langle\varphi_N^-|, \qquad \hat{X}_8 = |\varphi_N^-\rangle\langle\varphi_N^+|, \\
\hat{X}_9 &= |\varphi_N^+\rangle\langle\varphi_N^-|,
\end{aligned} \tag{4.3a}$$

$$\hat{X}_{10} = \frac{1}{\sqrt{2^N-3}}\left(\hat{I}_N - |\tilde{0}_N\rangle\langle\tilde{0}_N| - |\varphi_N^+\rangle\langle\varphi_N^+| - |\varphi_N^-\rangle\langle\varphi_N^-|\right), \tag{4.3b}$$

with N-qubits states

$$|\varphi_N^+\rangle = \left(\cos\frac{\phi}{2}|0\rangle + \sin\frac{\phi}{2}|1\rangle\right)^{\otimes N}, \qquad |\varphi_N^-\rangle = \left(\sin\frac{\phi}{2}|0\rangle - \cos\frac{\phi}{2}|1\rangle\right)^{\otimes N},$$

$$|\tilde{0}_N\rangle = \frac{1}{\sqrt{1+\left(\cos\frac{\phi}{2}\right)^{2N}+\left(\sin\frac{\phi}{2}\right)^{2N}}}\left(|0_N\rangle - \left(\cos\frac{\phi}{2}\right)^N|\varphi_N^+\rangle - \left(\sin\frac{\phi}{2}\right)^N|\varphi_N^-\rangle\right).$$

All of these base attractors solve the attractor equations with eigenvalue $\lambda = 1$. Thus for $\phi \neq \frac{\pi}{2}$ there are no non-trivial solutions of attractor equations with eigenvalue $\lambda = -1$.

In the special case $\phi = \frac{\pi}{2}$, for $N > 3$ the base attractor space $Atr(\Phi_G^{(3,1)})$ is eleven-dimensional. The orthonormal basis of the attractor space is given by the operators $\{\hat{X}_i | i \in \hat{9}\}$ and the following two operators:

$$\begin{aligned}
\hat{X}'_{10} &= \frac{1}{\sqrt{2^{N-1}-2}}\left(\hat{I}_{N,E} - |\tilde{0}_N\rangle\langle\tilde{0}_N| - \frac{1}{2}\left(|\varphi_N^+\rangle + |\varphi_N^-\rangle\right)\left(\langle\varphi_N^+| + \langle\varphi_N^-|\right)\right), \\
\hat{X}'_{11} &= \frac{1}{\sqrt{2^{N-1}-1}}\left(\hat{I}_{N,O} - \frac{1}{2}\left(|\varphi_N^+\rangle - |\varphi_N^-\rangle\right)\left(\langle\varphi_N^+| - \langle\varphi_N^-|\right)\right),
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned}
\hat{I}_{N,E} &= \frac{1}{2}\sum_{z\in I_N}\left(1+(-1)^{\tau(z)}\right)|z\rangle\langle z|, \\
\hat{I}_{N,O} &= \frac{1}{2}\sum_{z\in I_N}\left(1-(-1)^{\tau(z)}\right)|z\rangle\langle z|.
\end{aligned}$$

All of these base attractors solve the attractor equations with eigenvalue $\lambda = 1$. As in the previous case, there are no non-trivial solutions of attractor equations with eigenvalue $\lambda = -1$.

For the case $N = 3$, the base attractor space is twelve-dimensional. It is consisting of the eleven-dimensional subspace given by operators $\{\hat{X}_1, \ldots, \hat{X}_9, \hat{X}'_{10}, \hat{X}'_{11}\}$ corresponding to the eigenvalue $\lambda = 1$. The eigenspace corresponding to the value $\lambda = -1$ is one-dimensional. It is given by the operator

$$\hat{X}_{12} = \frac{1}{6}\left(|101\rangle\langle 011| - |110\rangle\langle 011| + |110\rangle\langle 101| - |011\rangle\langle 101| + |011\rangle\langle 110| - |101\rangle\langle 110|\right).$$

The proof of this theorem is given in the appendices A and B.

Similarly to the previous case of the controlled unitary two-qubit interaction, for the generic case $N > 3$ and $\phi \neq \frac{\pi}{2}$, the initial quantum state ρ_{in} approaches by the iterative application of the map $\Phi^{(3,1)}$ to the state

$$\rho_{\infty}^{(3,1)} = p^{(3,1)} \frac{\hat{P}_N^{(3,1)} \rho_{\text{in}} \hat{P}_N^{(3,1)}}{p^{(3,1)}} + (1 - p^{(3,1)}) \frac{\hat{I}_N - \hat{P}_N^{(3,1)}}{2^N - 3}, \tag{4.5}$$

with $\hat{P}_N^{(3,1)} = |\tilde{0}_N\rangle\langle\tilde{0}_N| + |\varphi_N^+\rangle\langle\varphi_N^+| + |\varphi_N^-\rangle\langle\varphi_N^-|$ and $p^{(3,1)} = \text{Tr}[\rho_{\text{in}} \hat{P}_N^{(3,1)}]$. $\hat{P}_N^{(3,1)}$ is the projector onto an N-qubit decoherence-free subspace of the quantum states which are not affected by unitary transformations $\hat{U}_{i,jk}^{(\phi)}$.

4.4 Controlled unitary three-qubit interactions with two control qubits

As we will see in this section, there are significant differences between the asymptotic dynamic of the controlled unitary three-qubit interactions with two control qubits and the previous cases. Let us suppose we are given a quantum network consisting of $N \geq 3$ qubits interacting with each other by the application fo the controlled unitary three-qubit gates $\hat{U}$, which are applied to triplets of qubits according to the prescribed probability distribution $p_{ij,k}$ with $\sum_{ij,k} p_{ij,k} = 1$.

The considered three-qubit interactions belong to the one-parameter family of transformations with two control qubits which have in the computational basis form

$$\hat{U}_{ij,k}^{(\phi)} = \left(\hat{I}_{ij} - |11\rangle_{ij\ ij}\langle 11|\right) \otimes \hat{I}_k + |11\rangle_{ij\ ij}\langle 11| \otimes \hat{u}_k^{(\phi)},$$

where

$$\hat{u}_k^{(\phi)} = \cos\phi(|0\rangle_{kk}\langle 0| - |1\rangle_{kk}\langle 1|) + \sin\phi(|0\rangle_{kk}\langle 1| + |1\rangle_{kk}\langle 0|).$$

The considered RUOs have the form

$$\Phi^{(3,2)}(\rho) = \sum_{e\in E} p_e \hat{U}_e^{(\phi)} \rho \hat{U}_e^{\dagger(\phi)}.$$

Similar to the previous case, E is the subset of the set of partially ordered triples $(i,j;k) = (j,i;k)$. We will discard the cases $\phi = 0, \pi$ since the RUO $\Phi^{(3,2)}$ is then diagonal in the computational basis. The eigenvalues of the transformation $\hat{U}_{ij,k}^{(\phi)}$ are again given by $\lambda_1 = 1$ and $\lambda_2 = -1$ and thus the attractor spectrum $\sigma_{|1|}$ fulfils the relation $\sigma_{|1|} \subset \{-1, 1\}$.

As mentioned in the previous section, hypergraphs are more natural object to describe the topology of three-qubit interactions. However, the useful hypergraph representation of three-qubit interactions with two control qubits is significantly different from the hypergraph representation of three-qubit interactions with one control qubit. If we would construct the interaction hypergraph to these interactions as an analogy to the interaction hypergraph of three-qubit interactions with one control qubit, we would arrive to B-graph. If we look at the nature of the three-qubit interactions with two control qubits,

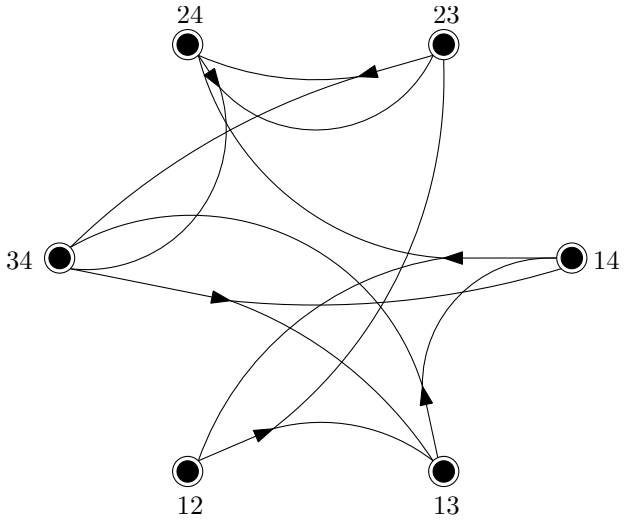


Figure 4.3: An example of an interaction F_2-graph G corresponding to the RUO $\Phi_G^{(3,2)}$. This F_2-graph belongs to the class of the base graphs of RUO $\Phi^{(3,2)}$.

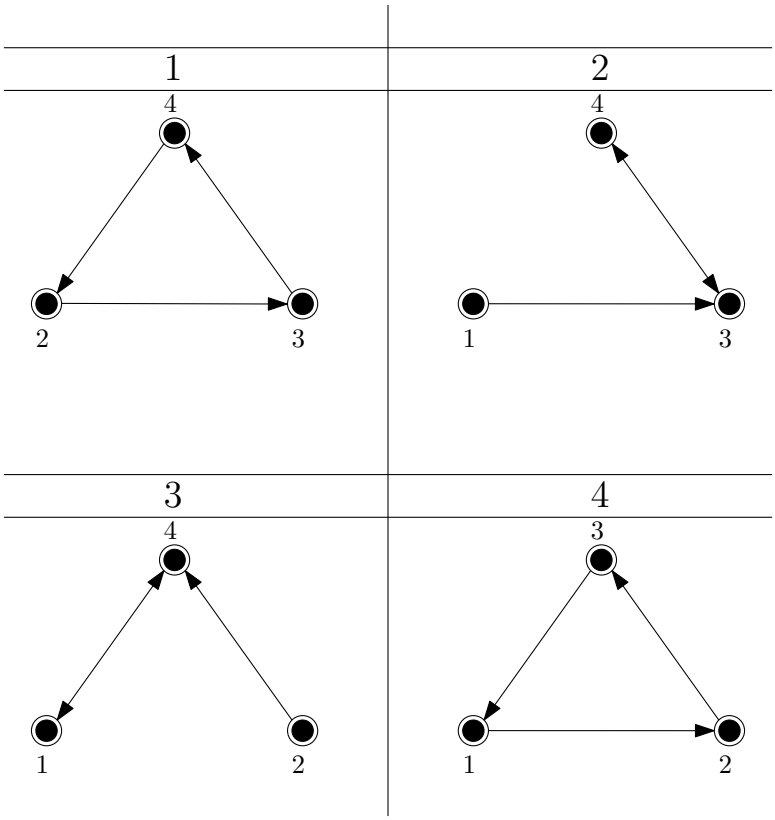


Figure 4.4: An example of a graph of symmetries corresponding to the interaction F-graph G.

we need both control qubits to be in the state $|1\rangle$ to produce a nontrivial action on the target qubit. This cannot be described by paths in B-graphs as each edge in the path connects exactly two vertices of given B-graph. Thus we have to construct a F-graph which encodes the information about qubit couplings.

Let us consider a oriented hypergraph $G = (V, E)$, where $V = \{\{i, j\}|i < j \in \hat{N}$. Thus vertices of V represent pairs of qubits, for instance vertex $\{i, j\}$ represents qubits i and j. Thus $|V| = \binom{N}{2}$. If $p_{ij,k} > 0$, then the vertices $\{i, j\}, \{i, k\}$ and $\{j, k\}$ are joined with a F-arc whose head is the vertex $\{i, j\}$ and whose tail is the vertices $\{i, k\}$ and $\{j, k\}$. We call G the **interaction $\mathbf{F_2}$-graph** of $\Phi^{(3,2)}$.

Another useful representation of controlled unitary three-qubit interactions with two control qubits can be constructed. The interaction F_2-graph of given three-qubit in-

teraction with two control qubits lacks the information about the symmetry of qubit interactions. Thus it is favorable to define so-called **graph of symmetries**. Despite the name it is a set of N graphs, each having $N-1$ vertices. Each vertex represents given qubit, so the vertex i represents the qubit i. The i-th graph in this set of graphs is the graph $G_i = (V_i, E_i)$, where $V_i = \hat{N}\backslash\{i\}$. If $p_{ij,k} > 0$, then $(i,k) \in E_j$ and $(j,k) \in E_i$. We can immediately see that the graph of symmetries is defined in the symmetrical way in the control qubits. The interaction F_2-graph $G = (V, E)$ and the graph of symmetries $\{G_i = (V_i, E_i) | i \in \{1, \ldots, N\}\}$ are related to each other in the following way. The subgraph $H_i \subset G$, where $V(H_i) = \{ij | j \in \{1, \ldots, N\}\backslash\{i\}\}$ is exactly the graph G_i.

The base F_2-graphs and the base attractor space are given by the following theorem.

Theorem 4.4.1. *For $\phi \neq 0, \pi$ the base F_2-graphs of the considered family of interactions are the F-graphs with the following property. The F-graph $G = (V, E)$ is a base graph if every vertex $\{i, j\} \in V$ is connected to a vertex $\{k, l_k\}$ for every possible $k \neq i, j$ and for any $l_k \in \hat{N}$. For any $N \geq 3$ and any $\phi \in (0, \pi)$ the associated base attractors are elements of a $\left((N+2)^2+1\right)$-dimensional attractor space $Atr(\Phi_G^{(3,2)})$. An orthonormal basis system of linear operators in this space of base attractor is given by*

$$
\begin{aligned}
\hat{X}_1 &= |0_N\rangle \langle 0_N| , \\
\hat{X}_{1+i} &= |0_N\rangle \langle \mathbf{1}_i| , i \in \hat{N}, \\
\hat{X}_{N+1+i} &= |\mathbf{1}_i\rangle \langle 0_N| , i \in \hat{N}, \\
\hat{X}_{(1+i)N+j+1} &= |\mathbf{1}_i\rangle \langle \mathbf{1}_j| , i, j \in \hat{N}, \\
\hat{X}_{(N+1)^2+1} &= |0_N\rangle \langle \tilde{\psi}_N| , \\
\hat{X}_{(N+1)^2+1+i} &= |\mathbf{1}_i\rangle \langle \tilde{\psi}_N| , i \in \hat{N}, \\
\hat{X}_{(N+1)^2+N+2} &= |\tilde{\psi}_N\rangle \langle 0_N| , \\
\hat{X}_{(N+1)^2+N+2+i} &= |\tilde{\psi}_N\rangle \langle \mathbf{1}_i| , i \in \hat{N}, \\
\hat{X}_{(N+2)^2} &= |\tilde{\psi}_N\rangle \langle \tilde{\psi}_N| , \\
\hat{X}_{(N+2)^2+1} &= \frac{1}{\sqrt{2^N - N - 2}} \left(\hat{I}_N - \sum_{\tau(z)=1} |z\rangle \langle z| - |\tilde{\psi}_N\rangle \langle \tilde{\psi}_N| \right) .
\end{aligned}
\tag{4.6}
$$

with N-qubits states

$$|\mathbf{1}_i\rangle = |0\rangle \otimes \cdots \otimes \underbrace{|1\rangle}_{i} \otimes \cdots \otimes |0\rangle\,,$$
$$|\tilde{\psi}_N\rangle = \frac{1}{\sqrt{\langle\tilde{\theta}_N|\tilde{\theta}_N\rangle}}\,|\tilde{\theta}_N\rangle\,,$$
$$|\tilde{\theta}_N\rangle = \sum_{z\in I_N,\tau(z)>1} \left(\cos\frac{\phi}{2}\right)^{N-\tau(z)} \left(\sin\frac{\phi}{2}\right)^{\tau(z)} |z\rangle\,.$$

All of these base attractors solve the attractor equations with eigenvalue $\lambda = 1$. There are no non-trivial solutions of attractor equations with eigenvalue $\lambda = -1$.

For the generic case $N \geq 3$ the initial quantum state ρ_{in} approaches by iterative application of the map $\Phi^{(3,2)}$ to the state

$$\rho_\infty^{(3,2)} = p^{(3,2)}\frac{\hat{P}_N^{(3,2)}\rho_{\text{in}}\hat{P}_N^{(3,2)}}{p^{(3,2)}} + (1-p^{(3,2)})\frac{\hat{I}_N - \hat{P}_N^{(3,2)}}{2^N - N - 2}, \tag{4.7}$$

with $\hat{P}_N^{(3,2)} = \sum_{\tau(z)\leq 1, z\in I_N} |z\rangle\langle z| + |\tilde{\psi}_N\rangle\langle\tilde{\psi}_N|$ and $p^{(3,2)} = \text{Tr}[\rho_{\text{in}}\hat{P}_N^{(3,2)}]$. The dimension of the decoherence-free subspace $\hat{P}_N^{(3,2)}\mathscr{H}$ is thus in this case directly dependent on the number of qubits N.

4.5 Properties of asymptotic states of three-qubit interactions

In this section we derive basic properties of asymptotic states of three-qubit interactions with one and two control qubits. We consider the case when the interaction F-graph belongs to the class of base graphs. Then we compare results obtained for controlled unitary three-qubit interactions with results obtained for controlled unitary two-qubit interactions.

From the results of the previous section we can see that the asymptotic states of the considered interaction can be easily compared in the case $N > 3$ and $\phi \neq \frac{\pi}{2}$. They are given by (4.2), (4.5) and (4.7). All of these asymptotic states have the same nature, the only difference is in the size of the decoherence-free subspace of the individual cases. It is clear that the relation $\hat{P}_N^{(2)}\mathscr{H} \subset \hat{P}_N^{(3,i)}\mathscr{H}$ holds with $i = 1, 2$. As we will see, this result has an impact on the value of the von Neumann entropy [4, 5, 18] of the individual cases.

In the case of the controlled unitary three-qubit interactions with one control qubit, we have seen that for $\phi = \frac{\pi}{2}$ there are two independent mixed states given by (4.4), which belong to the attractor space. It is so because operators $\hat{U}_{i,jk}^{(\frac{\pi}{2})}$ preserve the parity of $\tau(z)$ for $z \in I_N$. Let us define $\mathscr{H}_E$ as the subspace of $\mathscr{H}$ with orthonormal basis $\{|z\rangle\,|, z \in I_N, \tau(z) = 2k, k \in \mathbb{N} \cup \{0\}\}$ and $\mathscr{H}_O$ as the subspace of $\mathscr{H}$ with basis $\{|z\rangle\,|z \in I_N, \tau(z) = 2k-1, k \in \mathbb{N}\}$. For sake of simplicity we will call these subspaces the even and the odd part of $\mathscr{H}$. We can say that the even and the odd part of the input changes independently on the other one. It turns out that in this case a more suitable orthonormal basis of the attractor space exists. It is given by substituting the three orthonormal eigenvectors $|\tilde{0}_N\rangle$ and $|\varphi_N^{\pm}\rangle$ for vectors $|0_N\rangle$, $|\chi_N^E\rangle$ and $|\chi_N^O\rangle$ defined as

$$
\begin{aligned}
|\chi_N^E\rangle &= \frac{1}{\sqrt{2\left(1-\frac{1}{2^{N-1}}\right)}}\left(|\varphi_N^+\rangle + |\varphi_N^-\rangle - \frac{2}{\sqrt{2^N}}\,|0_N\rangle\right), \\
|\chi_N^O\rangle &= \frac{1}{\sqrt{2}}(|\varphi_N^+\rangle - |\varphi_N^-\rangle).
\end{aligned}
$$

These vectors form an orthonormal basis of the subspace $\hat{P}_N^{(3,1)}\mathscr{H}$ and furthermore, for $\phi = \frac{\pi}{2}$ vectors $|0_N\rangle$ and $|\chi_N^E\rangle$ form an orthonormal basis of the even part of $\hat{P}_N^{(3,1)}\mathscr{H}$ and the vector χ_N^O forms an orthonormal basis of the odd part of $\hat{P}_N^{(3,1)}\mathscr{H}$. Now we can construct the projectors

$$
\begin{aligned}
\hat{P}_{N,E}^{(3,1)} &= |0_N\rangle\langle 0_N| + |\chi_N^E\rangle\langle\chi_N^E|\,, \\
\hat{P}_{N,E}^{(3,1)} &= |\chi_N^O\rangle\langle\chi_N^O|\,.
\end{aligned}
$$

With these projectors, we can separate the even and odd part of the asymptotic state which can be written as

$$
\begin{aligned}
\rho_\infty^{(3,1)} =& (p_0+p_E)\frac{\hat{P}_{N,E}^{(3,1)}\rho_{\text{in}}\hat{P}_{N,E}^{(3,1)}}{p_0+p_E} + (P_E - p_0 - p_E)\frac{\hat{I}_{N,E} - \hat{P}_{N,E}^{(3,1)}}{2^N-2} + \\
&+ p_O\frac{\hat{P}_{N,O}^{(3,1)}\rho_{\text{in}}\hat{P}_{N,O}^{(3,1)}}{p_O} + (P_O - p_O)\frac{\hat{I}_{N,O} - \hat{P}_{N,O}^{(3,1)}}{2^N-1}.
\end{aligned}
\tag{4.8}
$$

The relation (4.8) is a direct analogy of the asymptotic state (4.5) with $P_E = \mathrm{Tr}[\hat{I}_{N,E}\rho_{\text{in}}]$, $P_O = \mathrm{Tr}[\hat{I}_{N,O}\rho_{\text{in}}]$, $p_0 = \langle 0_N|\rho_{\text{in}}|0_N\rangle$, $p_E = \langle\chi_N^E|\rho_{\text{in}}|\chi_N^E\rangle$ and $p_O = \langle\chi_N^O|\rho_{\text{in}}|\chi_N^O\rangle$.

To compare the von Neumann entropies of asymptotic states $\rho_\infty^{(\cdot)}$ we choose a suitable orthonormal basis of the Hilbert space $\mathscr{H}$. First we choose an orthonormal basis of the subspace $\hat{P}_N^{(2)}\mathscr{H}$. Then we add another vector to get the orthonormal basis of the subspace $\hat{P}_N^{(3,1)}$. In the last step we pick an orthonormal basis of the subspace $\left(\hat{I}_N - \hat{P}_N^{(3,1)}\right)\mathscr{H}$. We have thus constructed an orthonormal basis of the Hilbert space $\mathscr{H}$. In this basis the density matrices (4.2) and (4.5) have the block diagonal form

$$\rho_\infty^{(\cdot)} = \begin{pmatrix} \hat{P}_N^{(\cdot)}\rho_{\mathrm{in}}\hat{P}_N^{(\cdot)}|_{\hat{P}_N^{(\cdot)}\mathscr{H}} & 0 & \cdots & 0 \\ 0 & \frac{1-p^{(\cdot)}}{2^N-\dim(\hat{P}_N^{(\cdot)}\mathscr{H})} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1-p^{(\cdot)}}{2^N-\dim(\hat{P}_N^{(\cdot)}\mathscr{H})} \end{pmatrix}. \tag{4.9}$$

Furthermore, in this basis $\hat{P}_N^{(3,1)}\rho_{\mathrm{in}}\hat{P}_N^{(3,1)}|_{\hat{P}_N^{(2)}\mathscr{H}} = \hat{P}_N^{(2)}\rho_{\mathrm{in}}\hat{P}_N^{(2)}|_{\hat{P}_N^{(2)}\mathscr{H}}$. This relation immediately implies that for the same initial density matrix ρ_{in} the equation

$$S\left(\rho_\infty^{(2)}\right) \geq S\left(\rho_\infty^{(3,1)}\right)$$

holds. An analogous result could be derived for the controlled unitary three-qubit interactions with two control qubits, i.e. $S\left(\rho_\infty^{(2)}\right) \geq S\left(\rho_\infty^{(3,2)}\right)$. The value of the entropy can be written as

$$S\left(\rho_\infty^{(\cdot)}\right) = -\sum_{i=1}^{\dim(\hat{P}_N^{(\cdot)}\mathscr{H})} \lambda_i \log \lambda_i + (1-p^{(\cdot)}) \log\left(\frac{2^N - \dim(\hat{P}_N^{(\cdot)}\mathscr{H})}{1-p^{(\cdot)}}\right),$$

where λ_i are the eigenvalues of the matrix $\hat{P}_N^{(\cdot)}\rho_{\mathrm{in}}\hat{P}_N^{(\cdot)}|_{\hat{P}_N^{(\cdot)}\mathscr{H}}$. The eigenvalues of the matrix $\hat{P}_N^{(\cdot)}\rho_{\mathrm{in}}\hat{P}_N^{(\cdot)}|_{\hat{P}_N^{(\cdot)}\mathscr{H}}$ λ_i satisfy the relation $\sum_{i=1}^{\dim(\hat{P}_N^{(\cdot)}\mathscr{H})} \lambda_i = p^{(\cdot)}$.

Let us examine a K qubit subsystem S of a quantum network consisting of $N+K$ qubits, which are interacting via controlled unitary three-qubit interactions with one control qubit. In analogy with controlled unitary two-qubit interaction, the asymptotic state of the subsystem S can be for $N \gg 1$ written for any $\phi \in (0,\pi)$ as

$$\rho_\infty^{(3,1),(S)} = p_0 \left|0_K\right\rangle \left\langle 0_K\right| + p_+ \left|\varphi_K^+\right\rangle \left\langle \varphi_K^+\right| + p_- \left|\varphi_K^-\right\rangle \left\langle \varphi_K^-\right| + (1-p_0-p_+-p_-)\frac{\hat{I}_K}{2^K}$$

with $p_0 = \langle 0_{N+K}|\rho_{\rm in}|0_{N+K}\rangle$, $p_+ = \langle \varphi^+_{N+K}|\rho_{\rm in}|\varphi^+_{N+K}\rangle$ and $p_- = \langle \varphi^-_{N+K}|\rho_{\rm in}|\varphi^-_{N+K}\rangle$. Similar result could be obtained for controlled unitary three-qubit interactions with two control qubits. This result follows from the fact that for $N \gg 1$ $\lim_{N\to\infty}|\tilde{0}_N\rangle = |0_N\rangle$. Since this density matrix is diagonal, the resulting state is not entangled, i.e. the concurrence [19] of the resulting state is zero:

$$C\left(\rho_\infty^{(3,1),(S)}\right) = 0.$$

The resulting state $\rho_\infty^{(3,1),(S)}$ is symmetric with respect to all possible permutations of the qubits. In analogy with controlled unitary two-qubit interactions [1], the entropy can be extracted from the K-qubit subsystem. This is easily seen from the case where the initial state of the subsystem is given by

$$\rho_{\rm in}^{(S)} = \frac{1}{2^K}\hat{I}_K.$$

The coefficients of $\rho_\infty^{(3,1),(S)}$ are then given by $p_0 = p_+ = p_- = \frac{1}{2^k}$ and thus the equation $S\left(\rho_\infty^{(3,1),(S)}\right) < S\left(\rho_{\rm in}^{(S)}\right)$ holds. The asymptotic state of a single qubit subsystem of a quantum network consisting of $N \gg 1$ qubits can be written as

$$\rho_\infty^{(3,1),(1)} = \frac{1}{2}\begin{pmatrix} 1+p_0+(p_+-p_-)\cos\phi & (p_+-p_-)\sin\phi \\ (p_+-p_-)\sin\phi & 1-p_0-(p_+-p_-)\cos\phi \end{pmatrix},$$

which can be also expressed as $\rho_\infty^{(3,1),(1)} = \frac{1}{2}(\hat{I}_1 + \vec{a}\cdot\vec{\sigma})$, where $\vec{\sigma} = (X, Y, Z)$ is the vector whose elements are Pauli matrices and $\vec{a} = ((p_+ - p_-)\sin\phi,\ 0,\ p_0 + (p_+ - p_-)\cos\phi)$ is

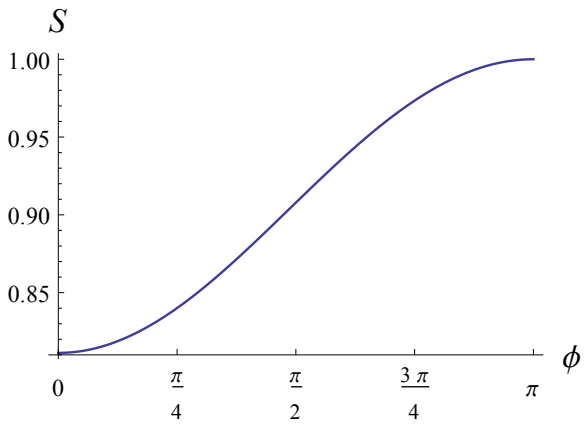


Figure 4.5: The value of the entropy S of a single qubit as the function of the parameter ϕ for the case $p_0 = \frac{1}{4}$, $p_+ - p_- = \frac{1}{4}$.

the Bloch vector [18]. The eigenvalues of the density matrix $\rho_{\infty}^{(3,1),(1)}$ are $\lambda_{\pm} = \frac{1}{2}(1 \pm |\vec{a}|)$. This result gives us some insight on the influence of the value of the parameter ϕ on physical quantities of a single qubit subsystem, e.g. on entropy (fig. 4.5). For $p_0 = 0$ the eigenvalues $\lambda_{\pm}$ are independent on the value of ϕ, but for other cases the eigenvalues $\lambda_{\pm}$ depend on the value of ϕ.

Similar calculations can be done for a subsystem consisting of two qubits. We find that the index of correlation $S(A : B) = S(A) + S(B) - S(A, B)$ [20] of arbitrary two qubits A and B is independent on the size of the quantum network N. The index of correlation thus depends only on values of overlaps p_0 and $p_{\pm}$. These are shown on figures 4.6-4.8 for the special cases. By choosing the initial state ρ_{in} such that $p_{\pm} = \frac{1}{2}$

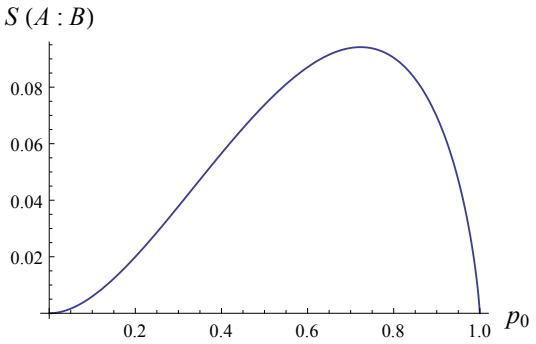


Figure 4.6: The value of the index of correlation of arbitrary two qubits A and B as the function of p_0 for the case $p_+ = p_- = 0$ and $\phi = \frac{\pi}{2}$.

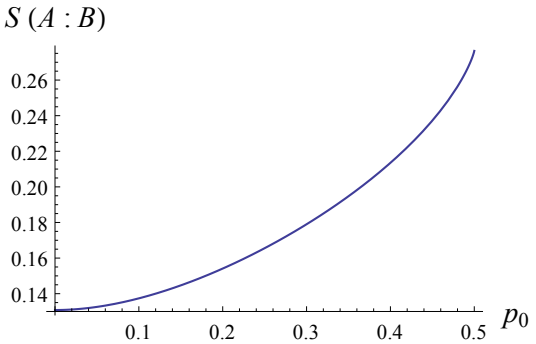


Figure 4.7: The value of the index of correlation of arbitrary two qubits A and B as the function of p_0 for the case $p_+ = p_- = \frac{1}{4}$ and $\phi = \frac{\pi}{2}$. This result shows that the shape of the curve $S(A : B)$ depends for this case on the value of $p = p_+ + p_-$

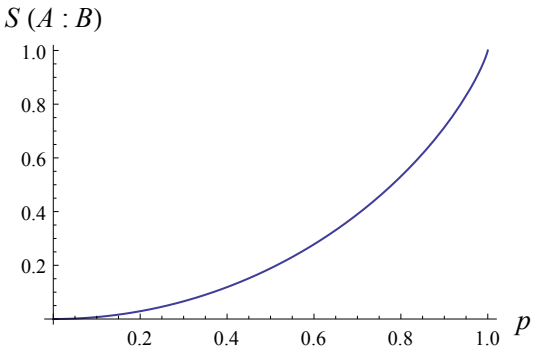


Figure 4.8: The value of the index of correlation of arbitrary two qubits A and B as the function of $p = p_+ + p_-$ for the case $p_+ = p_-$, $p_0 = 0$ and $\phi = \frac{\pi}{2}$.

we get the value $S(A : B) = 1$ for $\phi = \frac{\pi}{2}$, which is the maximal possible value in agreement with the previous result $C\left(\rho_\infty^{(3,1),(S)}\right) = 0$.

We have shown that the entropy of the asymptotic state of quantum network interacting via three-qubit interactions can never exceed the entropy of the asymptotic state of quantum network interacting via two-qubit interactions, i.e., $S\left(\rho_\infty^{(2)}\right) \geq S\left(\rho_\infty^{(3,\cdot)}\right)$. The asymptotic state of a K qubit subsystem of a quantum network consisting of $N+K$ qubits interacting via three-qubit interactions is analogous to the asymptotic state of a K qubit subsystem of a quantum network consisting of $N + K$ qubits interacting via two-qubit interactions. The concurrence of arbitrary two qubits is equal to 0. This result is in an agreement with the calculation of the index of correlation, for which the equation $S(A : B) \leq 1$ holds.

These results imply that the randomness introduced by the probability distribution p_e in the convex decomposition of given RUO $\Phi^{(3,\cdot)}$ tends to destroy the non-classical properties of the two-qubit states for quantum networks with base graph. It seems that three-qubit interactions can lead to stronger correlations of assymptotic states than two-qubit interactions, as there are two-qubit asymptotic states with the value of the index of correlation equal to 1. On the contrary there are no asymptotic states of two-qubit interactions with the value of index of correlation this high.

Chapter 5

Conclusion

The aim of this thesis was to study the controlled unitary three-qubit interactions with one or two control qubits which are defined as an analogy with the two-qubit interactions. The main results are given in theorems 4.3.1 and 4.4.1. These theorems completely determine the asymptotic evolution of the controlled unitary three-qubit interaction with one or two control qubits for certain class of interaction F-graphs. We can easily see that the base attractor space of controlled unitary two-qubit interactions which is given by theorem 4.2.1. is a subspace of the base attractor space of the controlled unitary three-qubit interactions with one or two control qubits. We can thus conclude that allowing the three-qubit interactions within the quantum network interacting via two-qubit interactions with strongly connected interaction graph does not affect the resulting asymptotic state.

Despite the fact that the base attractor space of controlled unitary two-qubit interactions forms a subspace of the base attractor space of controlled unitary three-qubit interactions with one or two control qubits, there are significant differences between the former and the latter. The attractor space of controlled unitary three-qubit interactions with one control qubit is dependent on the value of the parameter ϕ. The dimension of the base attractor space of the controlled unitary three-qubit interactions with two control qubits is strongly dependent on the size of the corresponding quantum network. The topology of the base graphs of controlled unitary three-qubit interactions with two control qubits is significantly different from the topology of base graphs of the controlled

unitary two-qubit interactions and the controlled unitary three-qubit interactions with one control qubit as it is more constraining than the latter. We can easily check that the minimum number of hyperedges needed for an interaction F_2-graph to be a base graph is $\binom{N}{2}$ which is for $N > 3$ larger than the minimum number of edges/hyperedges needed for an interaction graph/F_1-graph, which is equal to N.

Furthermore, from the results we can expect the validity of general rules holding for controlled unitary N-qubit interactions. We can see that increasing of the number of the target qubits of the considered interactions results in the creation of new eigenvectors and thus enlarges the attractor space. Increasing of the number of the control qubits of the considered interactions results in the different topology of the base graphs as well as in the creation of new eigenvectors. The proof of the form of base graphs suggests that two-connectedness of the index graph is the only property which determines the resulting attractor space. Thus we can expect two different controlled unitary n-qubit interactions $\Phi_1^{(n,k)}$ and $\Phi_2^{(n,k)}$ with k control qubits to have the same attractor space if and only if the corresponding index graphs g_1 and g_2 have identical two-connected components.

Appendix A

The base attractor space

In this appendix we derive the form of the base attractor space of control unitary three-qubit interactions with one or two control qubits. Since the nature of these interactions are different, we will treat both cases separately. However, our method of deriving the form of the base attractor spaces will be analogous for both cases.

The base attractor space definitely appears in the case, where every possible triplets are present in the set E of an interaction F-graph $G = (V, E)$. We call this F-graph the maximal interaction F-graph. We can directly form and solve the attractor equations for this interaction F-graph. First we will consider the case $\phi = \frac{\pi}{2}$. After justifying the form of the attractor space for this case, we show that the number of independent solutions of attractor equations for $\phi \neq \frac{\pi}{2}$ must be less or equal to the number of solutions for the case $\phi = \frac{\pi}{2}$. In the final step we solve the attractor equations for the case $\phi \neq \frac{\pi}{2}$ with the help of symmetry of the attractor equations which is resulting from the maximality of the interaction F-graph.

In the following we denote the orthonormal basis states of the computational basis by $|\mathbf{z}\rangle$ with $\mathbf{z} = (z_1, \ldots, z_N) \in \{0,1\}^N = I_N$. Any operator $X \in \mathcal{B}(\mathscr{H})$ can be written in the form

$$X = \sum_{\mathbf{i},\mathbf{j} \in I_N} X^{\mathbf{i}}_{\mathbf{j}} \, |\mathbf{i}\rangle \langle \mathbf{j}| \, .$$

A.1 The base attractor space the of controlled unitary three-qubit interactions with one control qubit

In case of control unitary three-qubit interactions with one control qubit, we can rewrite the attractor equations for $\phi = \frac{\pi}{2}$ as

$$\hat{U}_{l,mn}X = \sum_{\mathbf{i},\mathbf{j}} X^{\mathbf{i}}_{\mathbf{j}} \hat{U}_{l,mn} |\mathbf{i}\rangle \langle \mathbf{j}| = \sum_{\mathbf{i},\mathbf{j}\in I_N} X^{\mathbf{i}}_{\mathbf{j}} |\mathbf{i'}\rangle \langle \mathbf{j}| = \lambda X \hat{U}_{l,mn} =$$
$$= \lambda \sum_{\mathbf{i},\mathbf{j}\in I_N} X^{\mathbf{i}}_{\mathbf{j}} |\mathbf{i}\rangle \langle \mathbf{j}| \hat{U}_{l,mn} = \lambda \sum_{\mathbf{i},\mathbf{j}\in I_N} X^{\mathbf{i}}_{\mathbf{j}} |\mathbf{i}\rangle \langle \mathbf{j'}|$$

Because of the property $\hat{U}^2_{l,mn} = \hat{I}_N$, the following form of this equations hold

$$\sum_{\mathbf{i},\mathbf{j}\in I_N} X^{\mathbf{i'}}_{\mathbf{j}} |\mathbf{i}\rangle \langle \mathbf{j}| = \lambda \sum_{\mathbf{i},\mathbf{j}\in I_N} X^{\mathbf{i}}_{\mathbf{j'}} |\mathbf{i}\rangle \langle \mathbf{j}|$$

and thus can be reduced to equations for matrix elements

$$X^{\mathbf{i'}}_{\mathbf{j}} = \lambda X^{\mathbf{i}}_{\mathbf{j'}},$$

where $\mathbf{i} = (i_1, \ldots, i_m, \ldots, i_n, \ldots, i_N)$, $\mathbf{i'} = (i_1, \ldots, i_m \oplus i_l, \ldots, i_n \oplus i_l, \ldots, i_N)$ and similar relation holds for $\mathbf{j}$.

The set of all pairs of indices $[\mathbf{i}, \mathbf{j}] \in I_N \times I_N$ can be divided into several subsets with the property, that matrix elements corresponding to a particular set are coupled only to matrix elements which correspond to this set. These subsets are given by

$$\begin{aligned}
A_1 &= \{[\mathbf{0}, \mathbf{0}]\}, \qquad A_2 = \{[\mathbf{0}, \mathbf{i}] | \tau(\mathbf{i}) = 2k, k \in \mathbb{N}\}, \\
A_3 &= \{([\mathbf{0}, \mathbf{i}] | \tau(\mathbf{i}) = 2k-1, k \in \mathbb{N}\}, \\
A_4 &= \{[\mathbf{i}, \mathbf{0}] | \tau(\mathbf{i}) = 2k, k \in \mathbb{N}\}, \qquad A_5 = \{[\mathbf{i}, \mathbf{0}] | \tau(\mathbf{i}) = 2k-1, k \in \mathbb{N}\}, \\
A_6 &= \{[\mathbf{i}, \mathbf{i}] | \tau(\mathbf{i}) = 2k, k \in \mathbb{N}\}, \qquad A_7 = \{[\mathbf{i}, \mathbf{i}] | \tau(\mathbf{i}) = 2k-1, k \in \mathbb{N}\}, \\
A_8 &= \{[\mathbf{i}, \mathbf{j}] | \tau(\mathbf{i}) = 2k, \tau(\mathbf{j}) = 2l, \ k, l \in \mathbb{N}\}, \\
A_9 &= \{[\mathbf{i}, \mathbf{j}] | \tau(\mathbf{i}) = 2k-1, \tau(\mathbf{j}) = 2l, \ k, l \in \mathbb{N}\}, \\
A_{10} &= \{[\mathbf{i}, \mathbf{j}] | \tau(\mathbf{i}) = 2k, \tau(\mathbf{j}) = 2l-1, \ k, l \in \mathbb{N}\}, \\
A_{11} &= \{[\mathbf{i}, \mathbf{j}] | \tau(\mathbf{i}) = 2k-1, \tau(\mathbf{j}) = 2l-1, \ k, l \in \mathbb{N}\}.
\end{aligned} \tag{A.1}$$

First, we consider the case $\lambda = 1$. Suppose we have a quantum network consisting of $N \geq 3$ qubits with the maximal interaction F_1-graph. We pick randomly 3 of those qubits. For this moment, we will forget that those qubits are coupled to other qubits and we will consider only the couplings between qubits induced by the operator $\hat{U}^{(\phi)}_{a,bc}$. After that it is rather straightforward to prove that the found attractors solve the attractor equations for maximal interaction F_1-graph. This is due to symmetry of the considered interactions which is a consequence of the maximality of the interaction F_1-graph. To simplify the notation we skip all other indices apart from those which correspond to our picked qubits.

The attractor equations for any $\phi \in (0, \pi)$ can be then written in the following form

$$\begin{aligned} \sin\phi \left(X^{100}_{0ij} - X^{111}_{0ij}\right) &= \cos\phi \left(X^{101}_{0ij} + X^{110}_{0ij}\right), \\ \sin\phi \left(X^{0ij}_{100} - X^{0ij}_{111}\right) &= \cos\phi \left(X^{0ij}_{101} + X^{0ij}_{110}\right), \end{aligned} \tag{A.2}$$

$$\begin{aligned} \sin^2\phi X^{110}_{0ij} + \sin\phi\cos\phi \left(X^{100}_{0ij} - X^{111}_{0ij}\right) - \left(1+\cos^2\phi\right) X^{101}_{0ij} &= 0, \\ \sin^2\phi X^{101}_{0ij} + \sin\phi\cos\phi \left(X^{100}_{0ij} - X^{111}_{0ij}\right) - \left(1+\cos^2\phi\right) X^{111}_{0ij} &= 0, \\ \sin^2\phi X^{0ij}_{110} + \sin\phi\cos\phi \left(X^{0ij}_{100} - X^{0ij}_{111}\right) - \left(1+\cos^2\phi\right) X^{0ij}_{101} &= 0, \\ \sin^2\phi X^{0ij}_{101} + \sin\phi\cos\phi \left(X^{0ij}_{100} - X^{0ij}_{111}\right) - \left(1+\cos^2\phi\right) X^{0ij}_{110} &= 0, \end{aligned}$$

$$\begin{aligned} \sin\phi \left(X^{111}_{100} - X^{100}_{111}\right) &= \cos\phi \left(X^{100}_{101} + X^{100}_{110} - X^{101}_{100} - X^{110}_{100}\right), \\ \sin\phi \left(X^{100}_{100} - X^{111}_{111}\right) &= \cos\phi \left(X^{111}_{101} + X^{111}_{110} + X^{101}_{100} + X^{110}_{100}\right), \\ \sin\phi \left(X^{110}_{101} - X^{101}_{110}\right) &= \cos\phi \left(X^{101}_{100} - X^{101}_{111} + X^{111}_{101} - X^{100}_{101}\right), \\ \sin\phi \left(X^{101}_{101} - X^{110}_{110}\right) &= \cos\phi \left(X^{110}_{100} - X^{110}_{111} + X^{111}_{101} - X^{100}_{101}\right), \\ \sin\phi \left(X^{101}_{101} - X^{110}_{110}\right) &= \cos\phi \left(X^{100}_{110} - X^{111}_{110} + X^{101}_{111} - X^{101}_{111}\right), \\ \sin\phi \left(X^{101}_{101} - X^{110}_{101}\right) &= \cos\phi \left(X^{100}_{110} - X^{111}_{110} + X^{110}_{111} - X^{110}_{100}\right), \\ \sin\phi \left(X^{100}_{100} - X^{111}_{111}\right) &= \cos\phi \left(X^{101}_{111} + X^{110}_{111} + X^{100}_{101} + X^{100}_{110}\right), \\ \sin\phi \left(X^{100}_{111} - X^{111}_{100}\right) &= \cos\phi \left(X^{101}_{111} + X^{110}_{111} - X^{111}_{101} - X^{111}_{110}\right), \end{aligned}$$

$$
\begin{aligned}
&2\cos^2\phi X_{100}^{101} + \sin\phi\cos\phi\left(X_{101}^{101} + X_{110}^{101} + X_{100}^{111} - X_{100}^{100}\right) + \sin^2\phi\left(X_{111}^{101} - X_{100}^{110}\right) = 0,\\
&2\cos^2\phi X_{100}^{110} + \sin\phi\cos\phi\left(X_{110}^{110} + X_{101}^{110} + X_{100}^{111} - X_{100}^{100}\right) + \sin^2\phi\left(X_{111}^{110} - X_{100}^{101}\right) = 0,\\
&2\cos^2\phi X_{101}^{100} + \sin\phi\cos\phi\left(X_{101}^{101} + X_{101}^{110} + X_{110}^{100} - X_{100}^{100}\right) + \sin^2\phi\left(X_{101}^{111} - X_{110}^{100}\right) = 0,\\
&2\cos^2\phi X_{101}^{111} + \sin\phi\cos\phi\left(X_{111}^{111} - X_{100}^{111} - X_{101}^{110} - X_{101}^{101}\right) + \sin^2\phi\left(X_{101}^{100} - X_{110}^{100}\right) = 0,\\
&2\cos^2\phi X_{110}^{100} + \sin\phi\cos\phi\left(X_{110}^{110} + X_{110}^{101} + X_{111}^{100} - X_{100}^{100}\right) + \sin^2\phi\left(X_{110}^{111} - X_{101}^{100}\right) = 0,\\
&2\cos^2\phi X_{110}^{111} + \sin\phi\cos\phi\left(X_{111}^{111} - X_{100}^{111} - X_{110}^{101} - X_{110}^{110}\right) + \sin^2\phi\left(X_{110}^{100} - X_{101}^{111}\right) = 0,\\
&2\cos^2\phi X_{111}^{101} + \sin\phi\cos\phi\left(X_{111}^{111} - X_{111}^{100} - X_{110}^{101} - X_{101}^{101}\right) + \sin^2\phi\left(X_{100}^{101} - X_{111}^{110}\right) = 0,\\
&2\cos^2\phi X_{111}^{110} + \sin\phi\cos\phi\left(X_{111}^{111} - X_{111}^{100} - X_{101}^{110} - X_{110}^{110}\right) + \sin^2\phi\left(X_{100}^{110} - X_{111}^{101}\right) = 0.
\end{aligned}
$$

The remaining equations are trivial, i.e $X_{0kl}^{0ij} = X_{0kl}^{0ij}$ or identical with the previous equations. From the form of these equations, we can see that the number of solutions for $\phi \neq \frac{\pi}{2}$ cannot be higher than the number of solutions for $\phi = \frac{\pi}{2}$ as the equations have the same form for both of these cases and, furthermore, for the case $\phi = \frac{\pi}{2}$ some of the attractor equations connect fewer matrix elements than for the case $\phi \neq \frac{\pi}{2}$.

First, we can notice that the equations (A.1) are exactly the same as the equations for eigenvectors. They have two solutions $|\varphi_N^+\rangle$ and $|\varphi_N^-\rangle$. Together with an obvious eigenvector $|\mathbf{0}_N\rangle$ they form the basis of mutual eigenvectors of all possible operators $\hat{U}_{a,bc}^{(\phi)}$.

For $\phi = \frac{\pi}{2}$ each of the previous equations creates a connection between elements of set A_j for some j. By identifying those equations, we can immediately solve the attractor equations and get the following result:

For the set A_1 the corresponding attractor is obviously given by $|\mathbf{0}_N\rangle\langle\mathbf{0}_N|$. Attractors corresponding to sets A_2 and A_3 are given by $|\mathbf{0}_N\rangle\left(\langle\varphi_N^+| + \langle\varphi_N^-|\right)$ and $|\mathbf{0}_N\rangle\left(\langle\varphi_N^+| - \langle\varphi_N^-|\right)$. Similarly attractors corresponding to sets A_4 and A_5 are given by $\left(|\varphi_N^+\rangle + |\varphi_N^-\rangle\right)\langle\mathbf{0}_N|$ and $\left(|\varphi_N^+\rangle - |\varphi_N^-\rangle\right)\langle\mathbf{0}_N|$. Attractors corresponding to the sets A_6 and A_7 are given by $\hat{I}_{N,E}$ and $\hat{I}_{N,O}$. Attractors corresponding to the remaining sets $A_8, \ldots, A_{11}$ are given by $\left(|\varphi_N^+\rangle \pm |\varphi_N^-\rangle\right)\left(\langle\varphi_N^+| \pm \langle\varphi_N^-|\right)$. By orthonormalizing these solution we arrive to the

eigenspace corresponding to value $\lambda = 1$ stated in the theorem 4.3.1.

For the case $\phi \neq \frac{\pi}{2}$ we already have 10 solution of attractor equations for the eigenvalue $\lambda = 1$. We know that if $|x\rangle$ and $|y\rangle$ are mutual eigenvectors to all of the operators $\hat{U}^{(\phi)}_{a,bc}$, then all the operators of the form $|x\rangle\langle y|$ belong to the attractor space. In our case we have three mutual eigenvectors which are given by $|\mathbf{0}_N\rangle, |\varphi^+_N\rangle$ and $|\varphi^-_N\rangle$ thus we got 9 solutions of attractor equations. Furthermore, the operator $\hat{I}_N$ always belongs to the attractor space.

First, we will discuss the possible form of the last attractor which can possibly exist. From the general properties of attractors we know that it must be a symmetric or an antisymmetric operator $\hat{Y}$, since if $\hat{Y}$ is an attractor corresponding to the eigenvalue $\lambda = 1$, then $\hat{Y}^\dagger$ is an attractor corresponding to the same eigenvalue. From the symmetry of attractor equations for $\lambda = 1$ we can immediately discard the possibility of an antisymmetric solution. Furthermore, the last attractor cannot have the form $|x\rangle\langle x|$, since then there would exist attractors of the form $|x\rangle\langle\varphi^+_N|$. Since we consider the maximal interaction F_1-graph, all solutions must be symmetric with respect to all possible permutations of qubits.

Based on the previous considerations we know the following: $X^{\mathbf{i}}_{\mathbf{j}} = X^{\mathbf{j}}_{\mathbf{i}}$, $X^{\mathbf{i}}_{\mathbf{j}} = X^{\tau(\mathbf{i})}_{\tau(\mathbf{j})}$. We can thus make a further simplification in the notation. In the following we will denote the matrix elements by $X^{\tau(\mathbf{i})+\tau(\mathbf{j})} \equiv X^{\mathbf{i}+\mathbf{j}}$. Furthermore, we will separate diagonal and off-diagonal matrix elements. The diagonal elements will be denoted by $X^{2\mathbf{k}}_{\text{diag}}$.

Now we must show that those 10 solutions form all the solutions of attractor equations for the case $\phi \neq \frac{\pi}{2}$. As attractor equations form a standard system of linear equations, the number of solutions is given by the number of variables and the number of equations. We will thus compare the number of equations and the number of variables for $\phi = \frac{\pi}{2}$ and $\phi \neq \frac{\pi}{2}$.

In the simplified notation attractor equations for the case $\phi = \frac{\pi}{2}$ have the form

$$X^{\mathbf{k}} = X^{\mathbf{k}+2}, k \in \{1, \ldots, 2N-3\},$$
$$X^{\mathbf{2k}}_{\text{diag}} = X^{\mathbf{2k}+4}_{\text{diag}}, k \in \{1, \ldots, N-2\}.$$

We can thus see that this is the system of $3N-5$ equations for $3N-1$ variables and thus there are four solutions. Those solutions are given by the operators $\hat{I}_{N,E}$, $\hat{I}_{N,O}$, $\left(|\varphi_N^+\rangle + |\varphi_N^-\rangle\right)\left(\langle\varphi_N^+| + \langle\varphi_N^-|\right)$ and $\left(|\varphi_N^+\rangle - |\varphi_N^-\rangle\right)\left(\langle\varphi_N^+| - \langle\varphi_N^-|\right)$.

For the case $\phi \neq \frac{\pi}{2}$ simplified attractor equations have the form

$$\sin\phi\left(X^{\mathbf{k}} - X^{\mathbf{k}+2}\right) = 2\cos\phi X^{\mathbf{k}+1}, k \in \{1, \ldots, 2N-3\},$$
$$\sin\phi\left(X^{\mathbf{2k}}_{\text{diag}} - X^{\mathbf{2k}+2}_{\text{diag}}\right) = 2\cos\phi X^{\mathbf{2k}+1}, k \in \{1, \ldots, N-1\}.$$

This is a system of $3N-4$ linear equations for $3N-1$ variables. There are three solutions which are given by $\hat{I}_N, |\varphi_N^+\rangle\langle\varphi_N^+|$ and $|\varphi_N^-\rangle\langle\varphi_N^-|$. Therefore, we have justified the form of the eigenspaces corresponding to eigenvalue $\lambda = 1$ for both cases $\phi = \frac{\pi}{2}$ and $\phi \neq \frac{\pi}{2}$ since the case $\phi \neq \frac{\pi}{2}$ has one solution less.

Let us turn our attention to the possible eigenvalue $\lambda = -1$. Since we are considering the maximal interaction F-graph, every qubit plays the role of the control qubit in some transformation $\hat{U}^{(\phi)}_{a,bc}$. Thus for $N > 3$ and for any set A_j we can always find the matrix element $X^{0\mathbf{i}}_{0\mathbf{j}}$ which yields the attractor equation $X^{0\mathbf{i}}_{0\mathbf{j}} = -X^{0\mathbf{i}}_{0\mathbf{j}}$ and because of the symmetry of the attractor equations with respect to all possible permutations of qubits, for $N > 3$ there is no non-trivial attractor corresponding to the eigenvalue $\lambda - 1$.

The previous argument fails for the case $N = 3$ and $\phi = \frac{\pi}{2}$, because for the set A_8 there is no such pair of indices $\mathbf{i} \neq \mathbf{j} \in I_{N-1}$. The resulting attractor equations for this case are

$$X^{110}_{101} = -X^{101}_{110}, \qquad X^{110}_{101} = -X^{011}_{101}, \qquad X^{110}_{011} = -X^{101}_{011},$$
$$X^{101}_{110} = -X^{011}_{110}, \qquad X^{110}_{011} = -X^{011}_{110}, \qquad X^{101}_{011} = -X^{011}_{101}.$$

The normalized solution of these equations is given by $\hat{X}_{12}$.

For the case $N = 3$ and $\phi \neq \frac{\pi}{2}$ the previous argument can be used again as the matrix elements which belong to the set A_8 are now connected with the matrix elements from other sets and thus for this case no non-trivial solution of attractor equations corresponding to the eigenvalue $\lambda = -1$ exists.

This concludes the derivation of the form of the base attractor space of control unitary three-qubit interactions with one control qubit.

A.2 The base attractor space of the controlled unitary three-qubit interactions with two control qubits

The derivation of the form of the base attractor space of controlled random unitary three-qubit interactions with two control qubits follows exactly the same path as the previous case, though it is simpler as the attractor equations have the same form for all $\phi \in (0, \pi)$.

Similarly to the previous case, we can rewrite the attractor equations for $\phi = \frac{\pi}{2}$ as

$$\sum_{\mathbf{i},\mathbf{j} \in I_N} X_{\mathbf{j}}^{\mathbf{i}} \left|\mathbf{i}\right\rangle \left\langle\mathbf{j}\right| = \lambda \sum_{\mathbf{i},\mathbf{j} \in I_N} X_{\mathbf{j}}^{\mathbf{i'}} \left|\mathbf{i}\right\rangle \left\langle\mathbf{j}\right| .$$

The matrix equations can be thus reduced to the equations for matrix elements which have the form

$$X_{\mathbf{j}}^{\mathbf{i}} = \lambda X_{\mathbf{j'}}^{\mathbf{i'}},$$

where for $\hat{U}_{kl,m}^{(\phi)}$ we have $\mathbf{i} = (i_1, \ldots, i_m, \ldots, i_N)$, $\mathbf{i'} = (i_1, \ldots, i_m \oplus i_k \cdot i_l, \ldots, i_N)$ and similarly for $\mathbf{j}$ and similar relation holds for $\mathbf{j}$.

If we divide the set of all pairs of indices $(\mathbf{i}, \mathbf{j}) \in I_N \times I_N$ as in the previous case, the resulting subsets will have the following form:

$$
\begin{aligned}
A_1 &= \{[\mathbf{0},\mathbf{0}]\}, & A_{1+i} &= \{[\mathbf{0},\mathbf{1}_i]\},\\
A_{N+1+i} &= \{[\mathbf{1}_i,\mathbf{0}]\}, & A_{(i+1)N+j+1} &= \{[\mathbf{1}_i,\mathbf{1}_j]\},\\
A_{(N+1)^2+1} &= \{[\mathbf{0},\mathbf{i}]|\tau(\mathbf{i})>1\}, & A_{(N+1)^2+i+1} &= \{[\mathbf{1}_i,\mathbf{j}]|\tau(\mathbf{j})>1\},\\
A_{(N+1)^2+N+2} &= \{[\mathbf{i},\mathbf{0}]|\tau(\mathbf{i})>1\}, & A_{(N+1)^2+N+2+i} &= \{[\mathbf{i},\mathbf{1}_j]|\tau(\mathbf{i})>1\},\\
A_{(N+2)^2} &= \{[\mathbf{i},\mathbf{i}]|\tau(\mathbf{i})>1\}, & A_{(N+2)^2+1} &= \{[\mathbf{i},\mathbf{j}]|\mathbf{i}\neq\mathbf{j},\tau(\mathbf{i})>1,\tau(\mathbf{j})>1\},
\end{aligned}
$$

where $\mathbf{1}_i = (0,\ldots,\underbrace{1}_{i},\ldots,0)$ and $i,j\in\hat{N}$.

In the next step we consider the maximal interaction F_2-graph of a quantum network consisting of $N\geq 3$ qubits. As previously we pick randomly three qubits which are interacting with each other through the operator $\hat{U}^{(\phi)}_{ab,c}$. The attractor equations for the case $\lambda=1$ which emerge from this interaction are the following ones

$$
\begin{aligned}
\cos\frac{\phi}{2}X^{111}_{0ij} &= \sin\frac{\phi}{2}X^{110}_{0ij},\\
\cos\frac{\phi}{2}X^{0ij}_{111} &= \sin\frac{\phi}{2}X^{0ij}_{110},\\
\cos\frac{\phi}{2}X^{111}_{i0j} &= \sin\frac{\phi}{2}X^{110}_{i0j},\\
\cos\frac{\phi}{2}X^{i0j}_{111} &= \sin\frac{\phi}{2}X^{i0j}_{110},\\
X^{110}_{111} &= X^{111}_{110},\\
2\cos\frac{\phi}{2}X^{110}_{111} &= \sin\frac{\phi}{2}\left(X^{110}_{110}-X^{111}_{111}\right).
\end{aligned}
$$

The remaining equations have the trivial form $X^{\mathbf{i}}_{\mathbf{j}} = X^{\mathbf{i}}_{\mathbf{j}}$. From the form of the equations, we can see that number of the solutions of these equations is same for all possible values of the parameter $\phi\in(0,\pi)$ as the coefficients $\sin\frac{\phi}{2}$ and $\cos\frac{\phi}{2}$ never vanish. By solving the eigenvector equations, we find that there exist $N+2$ eigenvectors corresponding to the eigenvalue $\lambda=1$ which are mutual eigenvectors of all operators $\hat{U}^{(\phi)}_{ab,c}$. These eigenvectors are $|\mathbf{0}_N\rangle$, $|\varphi^+_N\rangle$ and $|\mathbf{1}_i\rangle$, $i\in\hat{N}$. With the help of these eigenvectors we can immediately construct the $(N+2)^2$ independent solutions of attractor equations. Each of these solutions corresponds for $\phi=\frac{\pi}{2}$ to one of the subset of indices A_j. The only subset which is left without solution is the set $A_{(N+2)^2}$. By putting $X^{\mathbf{i}}_{\mathbf{j}}=0$ for $[\mathbf{i},\mathbf{j}]\notin A_{(N+2)^2}$ we arrive to the set of equations corresponding to this subset. Their

solution is the operator

$$\hat{Y} = \sum_{\tau(z)>1, z\in I_N} |z\rangle \langle z| .$$

By orthonormalizing all the solutions we arrive to the base attractor space which is stated in theorem 4.4.1.

Appendix B

Base graphs

In this appendix we focus on the derivation of the form of base graphs of the considered three-qubit interactions. Although we will derive the form of the base attractor space for different cases, all of these derivations share similar steps. First, we need to find a suitable condition for an interaction graph to be a base graph. This condition is provided by the so-called index graph, which is an undirected colored graph corresponding to a given interaction graph. Index graph is going to be defined in the next section. Then we show, that a maximal interaction graph fulfils this condition. In the last step of the proof, we are concerned about the number of edges which can be removed from the maximal interaction graph without loosing the validity of this condition.

B.1 Index graph

The index graph $g = (I_N, e_G, \varphi)$ is an undirected colored graph which is associated with an interaction graph (or an interaction F-graph) $G = (V, E)$. Each vertex of the index graph corresponds to an element of the computational basis, i.e. vertex $\mathbf{i}$ corresponds to the vector $|\mathbf{i}\rangle$. The index graph g is equipped with a map $C : e_G \to \{e \in E\}$. Thus the edges of the interaction graph G denote the color of edges of the index graph g and to simplify the terminology, they will be referred to as colors in the context of the index graph. If the elements of the computational basis $|\mathbf{i}\rangle$ and $|\mathbf{j}\rangle$ with $\mathbf{i} \neq \mathbf{j}$ are connected through the operator $\hat{U}_e^{(\phi)}$, then the set e_G contains the edge $\mathbf{ij}$ with $C(\mathbf{ij}) = e$. Since each pair $\mathbf{i}$ and $\mathbf{j}$ can be connected by more than one operator, the index graph can

have multiple edges. To simplify the notation we will drop the map φ from the definition of the index graph g as if the vertices **i** and **j** are connected by more than one edge, these edges have different colors and thus the concerned edge can be always uniquely specified. Furthermore it is clear that if **i** is connected with **j** then **j** is connected with **i**, thus the index graph is undirected.

Since for the controlled unitary interactions the element of the computational basis $|\mathbf{0}\rangle$ is not connected with any of the other elements of computational basis, the vertex **0** of the index graph always forms a single element component. For the control unitary three-qubit interactions with two control qubits there are even more of vertices with this property. Since such vertices are not our concern, we will slightly modify the definition of the index graph so it corresponds to the particular case.

With the help of the index graph we can form a necessary and sufficient condition for an interaction graph to be a base graph. For an eigenvalue $\lambda = 1$, we can easily see that the corresponding eigenspace stays minimal as long as the number of the subsets of pairs of indices $\{A_l |\ l \in \hat{m}\}$ stays as low as possible. This condition can be for all cases which are discussed in the following sections expressed in terms of two-connectedness. For the case of the eigenvalue $\lambda = -1$ the condition is easily found from the proof of the form of the corresponding eigenspace.

B.2 Base graphs of the controlled unitary three-qubit interactions with one control qubit

For the case of controlled unitary three-qubit interactions with one control qubit we exclude the vertex **0** from the definition of the index graph g corresponding to the interaction F_1-graph G.

There are two different cases for the controlled unitary three-qubit interactions with one control qubit, $\phi = \frac{\pi}{2}$ and $\phi \neq \frac{\pi}{2}$. The differences are so significant that we will treat them separately. First we will discuss the simpler case $\phi \neq \frac{\pi}{2}$.

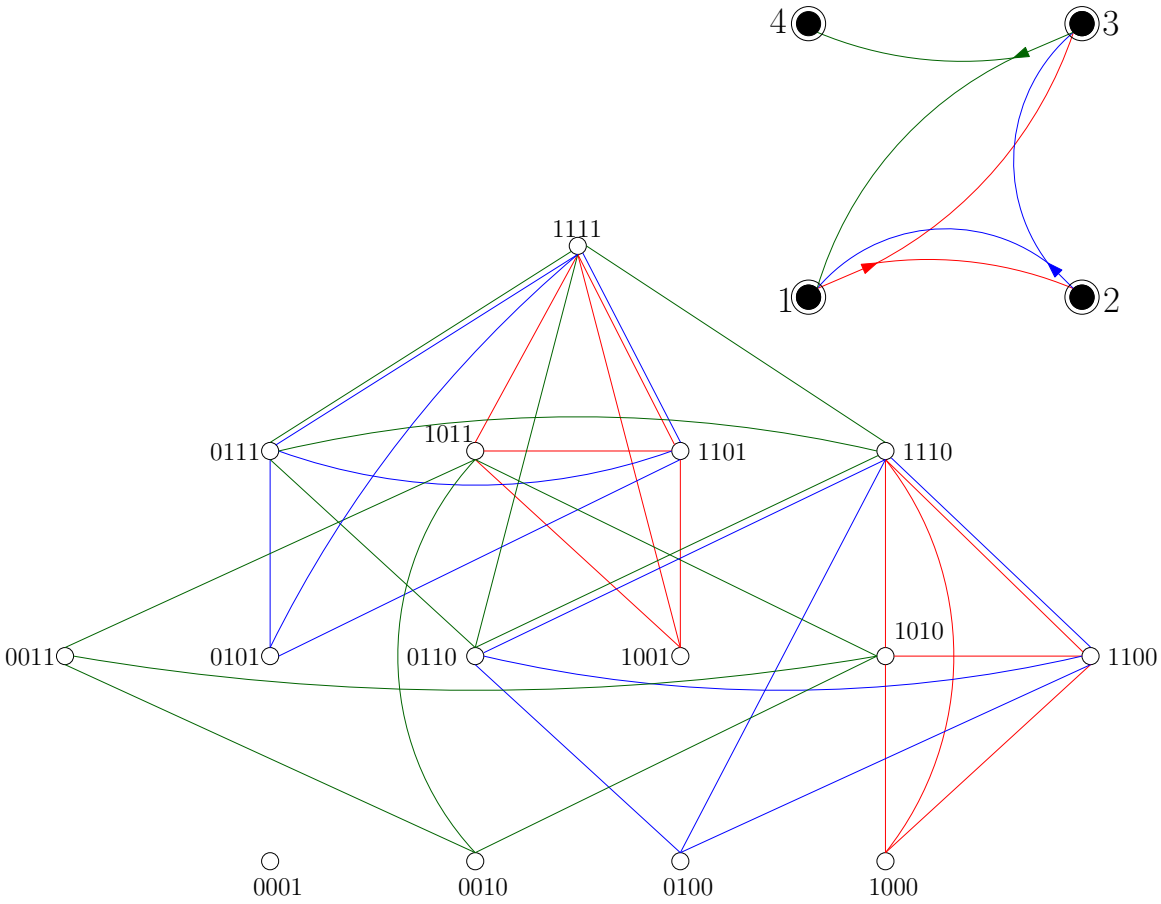


Figure B.1: An example of an interaction F_1-graph and the corresponding index graph for $\phi \neq \frac{\pi}{2}$. The vertex **0** is not included as it would only create a trivial component in the index graph. As we can see even for low number of vertices and edges the corresponding index graph can be rather complicated.

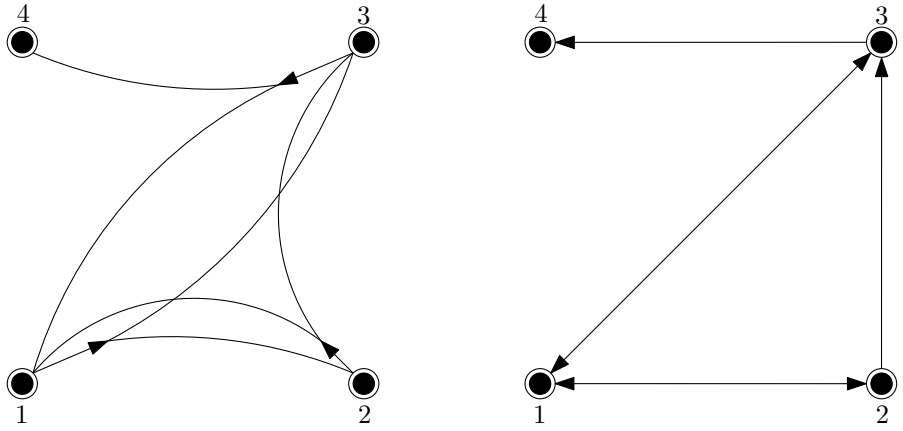


Figure B.2: An example of an interaction F_1-graph $G^{(3,1)}$ and corresponding two-qubit interaction graph $G^{(2)}$.

For $\phi \neq \frac{\pi}{2}$ the set of all pairs of indices $(\mathbf{i}, \mathbf{j})$ can be divided into four subsets which are invariant under all possible transformations $\hat{U}^{(\phi)}_{j,kl}$. These subsets are given by

$$A_1 = \{[\mathbf{0}, \mathbf{0}]\}, \qquad A_2 = \{[\mathbf{0}, \mathbf{i}] | \tau(\mathbf{i}) > 0\}, \qquad A_3 = \{[\mathbf{0}, \mathbf{i}] | \tau(\mathbf{i}) > 0\},$$
$$A_4 = \{[\mathbf{i}, \mathbf{j}] | \tau(\mathbf{i}) > 0,\ \tau(\mathbf{j}) > 0\}.$$

We can easily see that the subset A_1 remains invariant as it contains only one element. The subsets A_2 and A_3 remain invariant, if the corresponding index graph g is connected. The subset A_4 remains invariant, if the corresponding index graph g is two-connected. As two-connectedness implies connectedness, we arrive to the statement that the interaction F-graph of given three-qubit RUO $\Phi^{(3,1)}$ with one control qubit and $\phi \neq \frac{\pi}{2}$ is the base graph if and only if the corresponding index graph is two-connected.

For the proof of the two-connectedness of strongly connected interaction F_1-graphs we can use the proof of the form of base graphs of controlled unitary two-qubit interactions. Because three-qubit transformations $\hat{U}^{(\phi)}_{j,kl}$ have the property $\hat{U}^{(\phi)}_{j,kl} = \hat{U}^{(\phi)}_{jk}\hat{U}^{(\phi)}_{jl}$ we can construct for a given three-qubit RUO $\Phi^{(3,1)}$ a corresponding two-qubit RUO $\Phi^{(2)}$ by replacing transformations $\hat{U}^{(\phi)}_{j,kl}$ in the convex decomposition of $\Phi^{(3,1)}$ by two transformations $\hat{U}^{(\phi)}_{j,k}$ and $\hat{U}^{(\phi)}_{j,l}$ with $p_{jk} = p_{jl} = \frac{1}{2}p_{j,kl}$. An example of such a construction of the corresponding RUO $\Phi^{(2)}$ is given on the figure B.2.

Now we can easily show that if index graph $g^{(2)}$ which is associated with an interaction graph $G^{(2)}$ of a two-qubit interaction $\Phi^{(2)}$ is two-connected, then the index graph $g^{(3,1)}$ which is associated with an interaction F_1-graph $G^{(3,1)}$ of a three-qubit interaction $\Phi^{(3,1)}$ must be also two-connected. This follows from the fact that if we do not consider coloring $C^{(\cdot)}$ of graphs $g^{(3,1)}$ and $g^{(2)}$, then $g^{(2)} \subset g^{(3,1)}$. This follows from the fact that if the vertices $\mathbf{i}$ and $\mathbf{j}$ are connected in the index graph $g^{(2)}$ via operator $\hat{U}_{kl}^{(\phi)}$ then they are connected in the index graph $g^{(3,1)}$ via operator $\hat{U}_{k,lm}^{(\phi)}$ for any m. Since $g^{(2)}$ is two-connected, each pair of vertices $[\mathbf{i}_1, \mathbf{i}_2]$ is connected to any pair of vertices $[\mathbf{j}_1, \mathbf{j}_2]$ via paths $P_{\mathbf{i}_1,\mathbf{i}_2} = (\mathbf{i}_1\mathbf{u}_2, \mathbf{u}_2\mathbf{u}_3 \ldots, \mathbf{u}_k\mathbf{i}_2)$ and $P_{\mathbf{j}_1,\mathbf{j}_2} = (\mathbf{j}_1\mathbf{v}_2, \mathbf{v}_2\mathbf{v}_3 \ldots, \mathbf{v}_k\mathbf{j}_2)$ such that for every $l \in \hat{k}$ the equation $C^{(2)}(\mathbf{u}_l, \mathbf{u}_{l+1}) = C^{(2)}(\mathbf{v}_l, \mathbf{v}_{l+1})$ with $\mathbf{u}_1 = \mathbf{i}_1$, $\mathbf{u}_{k+1} = \mathbf{i}_2$, $\mathbf{v}_1 = \mathbf{j}_1$ and $\mathbf{v}_{k+1} = \mathbf{j}_2$. We can simply check that the paths $P_{\mathbf{i}_1,\mathbf{i}_2}$ and $P_{\mathbf{j}_1,\mathbf{j}_2}$ are connecting the pairs $[\mathbf{i}_1, \mathbf{i}_2]$ and $[\mathbf{j}_1, \mathbf{j}_2]$ in the index graph $g^{(3,1)}$ (due to the property $g^{(2)} \subset g^{(3,1)}$) and, furthermore, the equation $C^{(3,1)}(\mathbf{u}_l, \mathbf{u}_{l+1}) = C^{(3,1)}(\mathbf{v}_l, \mathbf{v}_{l+1})$ holds for every $l \in \hat{k}$. As a consequence, $g^{(3,1)}$ is two-connected and $G^{(3,1)}$ is thus a base F_1-graph. From the construction of $G^{(2)}$ it is obvious that $G^{(2)}$ is strongly connected if and only if $G^{(3,1)}$ is strongly connected. As a result all strongly connected interaction F_1-graphs $G^{(3,1)}$ of RUO $\Phi^{(3,1)}$ have the minimal eigenspace corresponding to the eigenvalue $\lambda = 1$. On the contrary if $G^{(3,1)}$ is not strongly connected then the corresponding index graph $g^{(3,1)}$ has more than one component and thus it cannot be two-connected. The conclusion is that the eigenspace of $\Phi^{(3,1)}$ corresponding to the eigenvalue $\lambda = 1$ is minimal if and only if $G^{(3,1)}$ is strongly connected interaction F_1-graph.

The case $\phi = \frac{\pi}{2}$ is more complicated. Although the relation $\hat{U}_{12}^{(\frac{\pi}{2})}\hat{U}_{13}^{(\frac{\pi}{2})} = \hat{U}_{1,23}^{(\frac{\pi}{2})}$ still holds the index graph of the controlled unitary two-qubit interaction made in the same way as in the case $\phi \neq \frac{\pi}{2}$ is not a subgraph of the corresponding index graph of the controlled unitary three-qubit interaction with one control qubit. Nevertheless, we begin our proof for the case $\lambda = 1$ also by dividing the set of pairs of indices $[\mathbf{i}, \mathbf{j}]$ into invariant subsets. These subsets are the subsets A_i, $i \in \hat{11}$ which are given by (A.1). As an analog to the previous case these subsets remain invariant as long as the following properties of the index graph hold: The subgraph of the index graph $\mathfrak{g}_1$ which is formed by the vertices $\mathbf{i}$ with $\tau(\mathbf{i}) = 2k,\ k \in N$ must be two-connected. The same applies also to the subgraph of the index graph $\mathfrak{g}_2$ which is formed by the vertices $\mathbf{i}$ with $\tau(\mathbf{i}) = 2k - 1,\ k \in N$.

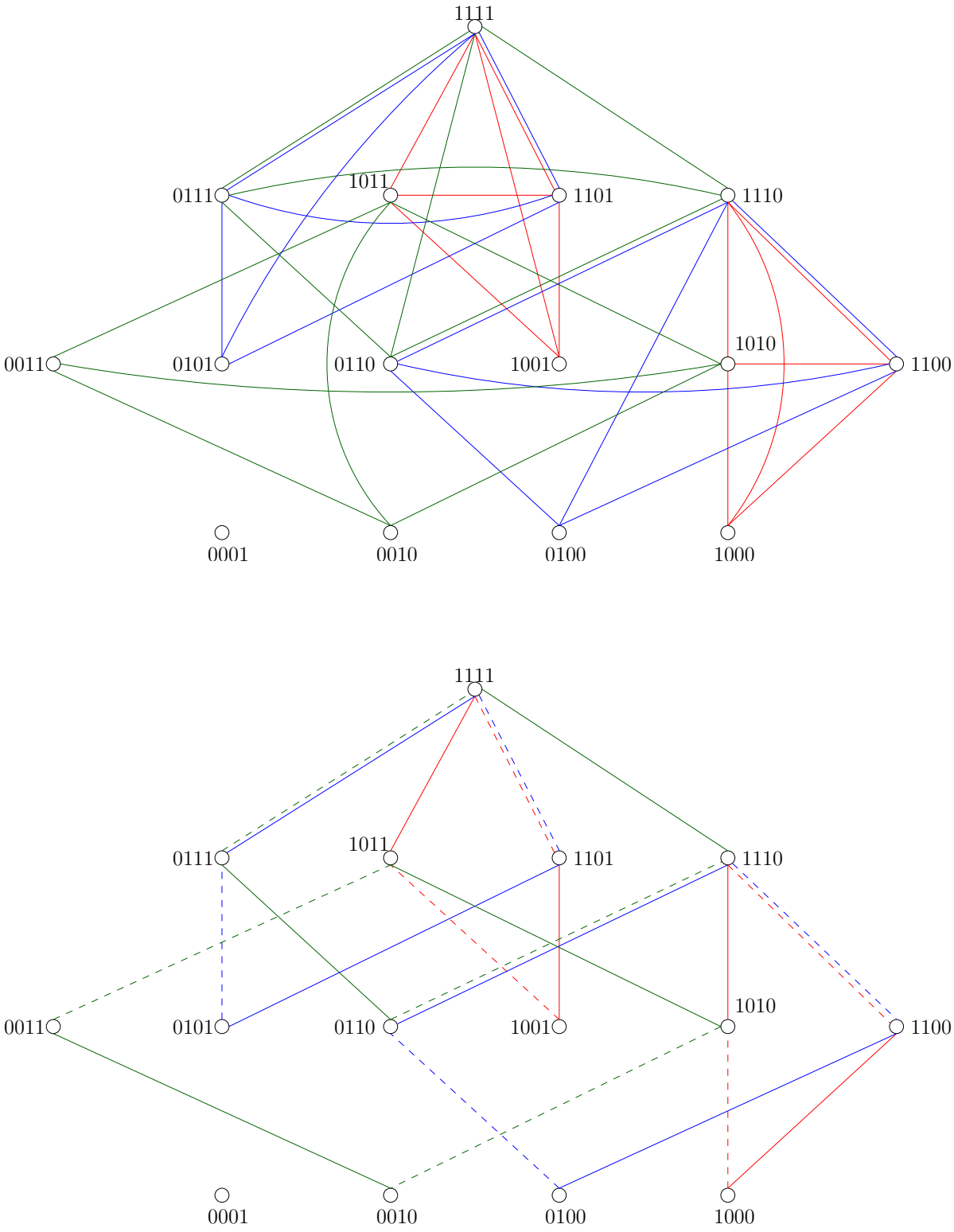


Figure B.3: The index graph $g^{(3,1)}$ associated with the interaction F-graph $G^{(3,1)}$ (figure B.2) and the corresponding index graph $g^{(2)}$.

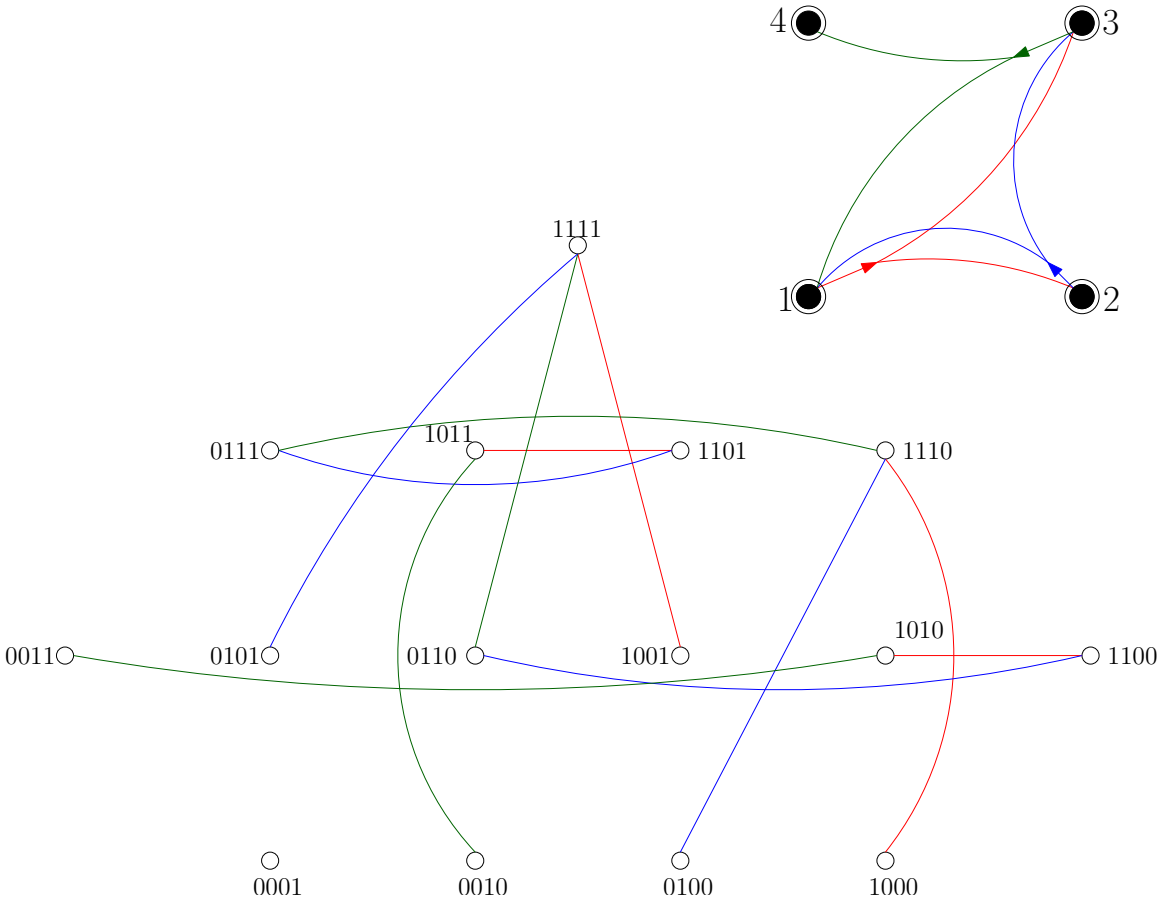


Figure B.4: The index graph associated with the interaction F-graph on the figure B.2 for the case $\phi = \frac{\pi}{2}$.

Furthermore, these components must be jointly connected in the sense that for any pair of vertices $[\mathbf{i}_1, \mathbf{i}_2] \in \mathfrak{g}_1$ and any pair of vertices $[\mathbf{j}_1, \mathbf{j}_2] \in \mathfrak{g}_2$ must be simultaneously connected by paths $P_{\mathbf{i}_1,\mathbf{i}_2} = (\mathbf{i}_1\mathbf{u}_2, \mathbf{u}_2\mathbf{u}_3 \ldots, \mathbf{u}_k\mathbf{i}_2)$ and $P_{\mathbf{j}_1,\mathbf{j}_2} = (\mathbf{j}_1\mathbf{v}_2, \mathbf{v}_2\mathbf{v}_3 \ldots, \mathbf{v}_k\mathbf{j}_2)$ such that for every $l \in \hat{k}$ the equation $C(\mathbf{u}_l\mathbf{u}_{l+1}) = C(\mathbf{v}_l\mathbf{v}_{l+1})$ with $\mathbf{u}_1 = \mathbf{i}_1$, $\mathbf{u}_{k+1} = \mathbf{i}_2$, $\mathbf{v}_1 = \mathbf{j}_1$ and $\mathbf{v}_{k+1} = \mathbf{j}_2$. From the form of the given transformations we can easily see that satisfying of one of these condition is sufficient for satisfying of the other conditions.

As a first step towards the proof of the form of base graphs we prove that the subgraph $\mathfrak{g}_2$ of the index graph associated with the maximal interaction F_1-graph is two-connected. To prove this we create a special type of an interaction F_1-graph $G^{(N)}$ which we call star F_1-graph on $N \geq 3$ vertices and we prove the two-connectedness of the subgraph $\mathfrak{g}_2^{(N)}$ of the index graph associated with this type of graph.

For $N = 3$, the star F_1-graph is the hypergraph $G^{(3)} = (V^{(3)}, E^{(3)})$ with $V^{(3)} = \{1, 2, 3\}$ and $E^{(3)} = \{(1; 2, 3), (2; 3, 1), (3; 1, 2)\}$. Suppose we have the star F_1-graph on $N \geq 3$ vertices $G^{(N)} = (V^{(N)}, E^{(N)})$. The star F_1-graph on $N + 1$ vertices is then defined as

$$G^{(N+1)} = (V^{(N+1)}, E^{(N+1)}), \quad V^{(N+1)} = V^{(N)} \cup \{N + 1\},$$
$$E^{(N+1)} = E^{(N)} \cup \{(1; 2, N + 1), (2; 1, N + 1), (N + 1; 1, 2)\}.$$

This induction step is for $N = 3$ visualized on figure B.5. The two-connectedness of the component $\mathfrak{g}_2^{(N)}$ of the index graph associated with the star F_1-graph on N vertices can be proved by induction on the number of vertices of this graph. For $N = 3$ the component $\mathfrak{g}_2^{(3)}$ is apparently two-connected. Let us suppose that the component $\mathfrak{g}_2^{(N)}$ is two-connected. Thus the component $\mathfrak{g}_1^{(N)}$ is also two-connected. The component $\mathfrak{g}_2^{(N+1)}$ can be created in two steps. In the first step we take components $\mathfrak{g}_1^{(N)}$ and $\mathfrak{g}_2^{(N)}$ and we relabel their vertices. The vertex $\mathbf{i} \in \mathfrak{g}_1^{(N)}$ is labeled as $\mathbf{i}1$ and the vertex $\mathbf{i} \in \mathfrak{g}_2^{(N)}$ is labeled as $\mathbf{i}0$. We mark the newly emerged graphs as $\mathfrak{g}_1^{\prime(N)}$ and $\mathfrak{g}_2^{\prime(N)}$. Next we create the graph $\mathfrak{g}^{(N+1)} = \mathfrak{g}_1^{\prime(N)} \cup \mathfrak{g}_1^{\prime(N)} \cup \{\mathbf{0}1\}$. This graph has three components: the component $\mathfrak{h}_1$ is consisted of vertices $\{\mathbf{i}0 \in I_{N+1} | \tau(\mathbf{i}) > 0\}$, the component $\mathfrak{h}_2$ is consisted of vertices $\{\mathbf{i}1 \in I_{N+1} | \tau(\mathbf{i}) > 0\}$ and the last component is consisted of the single vertex $\{\mathbf{0}1\}$. From the construction, components $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two-connected. We finish the construction of the component $\mathfrak{g}_2^{(N+1)}$ by addition of edges to the graph $\mathfrak{g}^{(N+1)}$ which

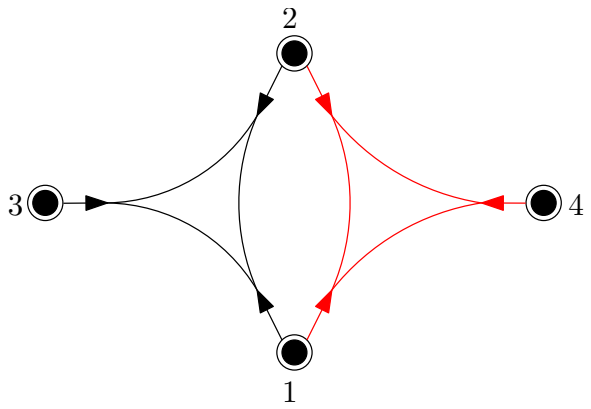


Figure B.5: The construction of star F_1-graph on 4 vertices. The edges which are not present in the star F_1-graph on 3 vertices are red.

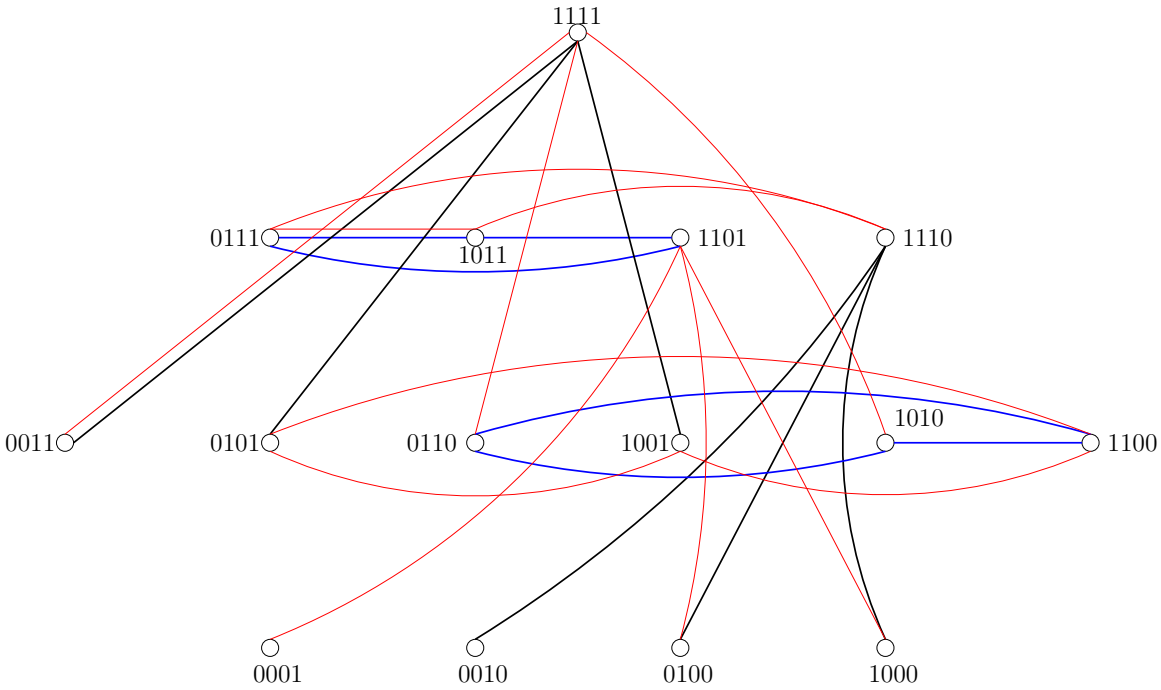


Figure B.6: The index graph associated with the star F_1-graph on 4 vertices constructed from the index graph associated with the star F_1-graph on 3 vertices. This graph has 2 components $\mathfrak{G}_1^{(4)}$ and $\mathfrak{G}_2^{(4)}$. Black edges correspond to the subgraphs of these components which are isomorphic to the component $\mathfrak{G}_2^{(3)}$, blue edges correspond to the subgraphs of these components which are isomorphic to the component $\mathfrak{G}_1^{(3)}$ of the index graph associated with the star F_1-graph on 3 vertices.

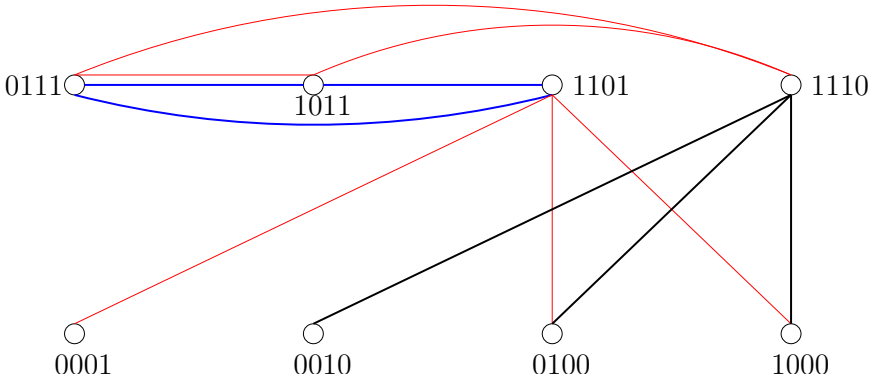


Figure B.7: The component $\mathfrak{G}_2^{(4)}$ of the index graph associated with the star F_1-graph on 4 vertices.

correspond to the F-arcs from the set $\{(1;2,N+1),(2;1,N+1),(N+1;1,2)\}$ of the interaction F_1-graph. These edges are connecting the trivial component $\{\mathbf{01}\}$ with the component $\mathfrak{h}_2$ as well as they are connecting the component $\mathfrak{h}_1$ with the component $\mathfrak{h}_2$. The important fact is that the added edges also connect vertices within the component $\mathfrak{h}_2$. From the construction it is straightforward to prove that the component $\mathfrak{g}_2^{(N+1)}$ is two-connected. The only non-trivial case is, if we want to connect the pairs of vertices $(\mathbf{i}_1,\mathbf{i}_2)$ and $(\mathbf{j}_1,\mathbf{j}_2)$ with $\mathbf{i}_k \in \mathfrak{h}_1$ and $\mathbf{j}_k \in \mathfrak{h}_2$. In this case the two-connectedness is the result of the newly added edges in the component $\mathfrak{h}_2$. As these edges have no counterpart in the component $\mathfrak{h}_1$ we can freely move between vertices in the component $\mathfrak{h}_2$ without moving within component $\mathfrak{h}_1$.

Since the component $\mathfrak{g}_2^{(N)}$ of the index graph corresponding to the star F_1-graph on N vertices is a subgraph of the component $\mathfrak{g}_2$ of the index graph corresponding to the maximal interaction F_1-graph on N vertices, the component $\mathfrak{g}_2$ of the index graph corresponding to the maximal interaction F_1-graph on N vertices must be also two-connected as the addition of edges cannot disturb the property of two-connectedness.

In the last step of the proof we show that we do not violate the property of two-connectedness of the index graph by removing edges from the interaction F_1-graph as long as the corresponding interaction F_1-graph stays strongly connected. We prove this by induction on the length d of the path connecting the vertices i_1 and i_{d+1}

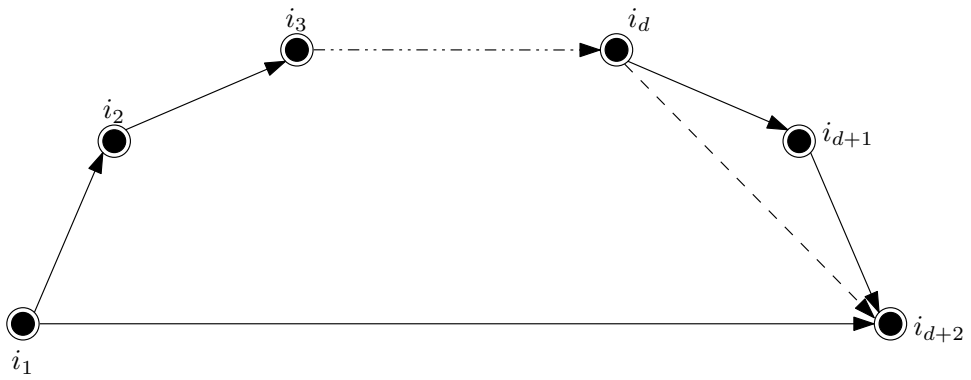


Figure B.8: The removal of the F-arc containing the directed edge (i_1, i_{d+2}) from the interaction F_1-graph.

in the interaction F_1-graph. The considered transformations $\hat{U}_{j,kl}^{(\frac{\pi}{2})}$ have the property $\hat{U}_{k,lm}^{(\frac{\pi}{2})}.\hat{U}_{j,kl}^{(\frac{\pi}{2})}.\hat{U}_{k,lm}^{(\frac{\pi}{2})}.\hat{U}_{j,kl}^{(\frac{\pi}{2})} = \hat{U}_{j,km}^{(\frac{\pi}{2})}$. Suppose that the vertices i_1 and i_3 are connected by the path $P = (i_1i_2, i_2i_3)$ of length $d = 2$. Let us mark $(i_1; i_2, j) = e$, $(i_2; i_3, k) = f$. Suppose the edges i_1 and i_3 are also connected with the F-arc $(i_1; i_3, l) = g$. Because of the previously mentioned property, applying the sequence of colors $fefe$ in the index graph has the same result as applying the color g. Thus all the edges with color g can be removed from the index graph without loosing the property of two-connectedness. In the corresponding interaction F_1-graph we remove the F-arc $(i_1; i_3, l)$. The resulting interaction F_1-graph is a base graph and it is still strongly connected. Let us suppose that for every path of length $d \geq 2$ the F-arc connecting the beginning and the end of the path can be removed from the interaction graph as long as it is not violating desired connectivity of other vertices. Let us take a path of length $d + 1$ of the form $P = (i_1i_2, i_2i_3, \ldots, i_di_{d+1}, i_{d+1}i_{d+2})$. Suppose that the vertices i_1 and i_{d+2} are connected with the F-arc $(i_1; i_{d+2}, m)$ We will temporarily add the edge i_di_{d+2} to the interaction F_1-graph. By this addition, the vertices i_1 and i_{d+2} are connected by path P' of length d. We can thus remove the F-arc $(i_1; i_{d+2}, m)$ from the interaction F_1-graph. This completes the proof by induction.

Concerning the case $\lambda = -1$ as it is easily seen from the proof of the form of the corresponding eigenspace, this eigenspace stays minimal as long as all qubits i play the role of control qubit of some transformation $\hat{U}_{i,jk}^{(\phi)}$ for any $\phi \in (0, \pi)$. Thus for the strongly connected interaction F_1-graphs the eigenspace corresponding to the eigenvalue $\lambda = -1$ is the minimal eigenspace.

To summarize, the base graphs of controlled unitary three-qubit interactions with one control qubit base graphs for any parameter $\phi \in (0, \pi)$ are exactly all strongly connected interaction F_1-graphs.

B.3 Base graphs of the controlled unitary three-qubit interactions with two control qubits

For the case of controlled unitary three-qubit interactions with two control qubits we exclude the vertices $\mathbf{z}$ with $\tau(\mathbf{z}) \leq 1$ from the definition of the index graph g corresponding to the interaction F_2-graph G. Although the proof of the form of base graphs for this case is analogous to the previous case, there are differences which need to be pointed out. They are the result of the different structure of the visual representation - the interaction F_2-graph.

Because of the form of transformations $\hat{U}_{j,kl}^{(\phi)}$ and the corresponding attractor space it is clear that there are no technical differences between the cases $\phi = \frac{\pi}{2}$ and $\phi \neq \frac{\phi}{2}$. Thus the proof of the form of the base attractor space for $\phi = \frac{\pi}{2}$ is also proof for general $\phi \in (0, \pi)$. Similar to the proof from the previous section, we start our consideration for

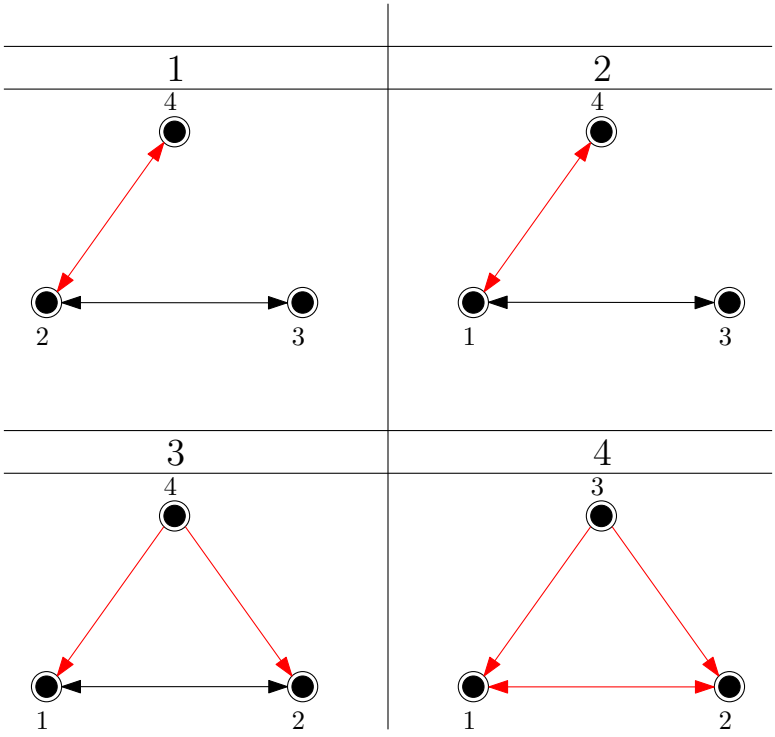


Figure B.9: The construction of graph of symmetries corresponding to the star F_2-graph on 4 vertices. The edges which are not present in the graph of symmetries corresponding to the star F_2-graph on 3 vertices are red.

$\lambda = 1$ by dividing the set of all pairs of indices $[\mathbf{i}, \mathbf{j}]$ into subsets which are invariant under transformations $\hat{U}_{jk,l}^{\left(\frac{\pi}{2}\right)}$. This division was made in the appendix A.2. It is consisting of $(N+2)^2 + 1$ subsets A_i. Analogous to the previous case, we can easily see that the number of invariant subsets does not increase if and only if the index graph g of given RUO $\Phi^{(3,2)}$ is two-connected.

The proof of two-connectedness of the index graph associated with the maximal interaction F_2-graph can be done in two steps. First, we construct a certain interaction F_2-graph $G^{(N)}$ on $N \geq 3$ vertices, its associated index graph $g^{(N)}$ and we prove that $g^{(N)}$ is two-connected. This interaction F_2-graph can be viewed as an analogy of the oriented star graph. For sake of simplicity we will call this interaction F_2-graph as star F_2-graph. We construct the star F_2-graph by constructing the corresponding graph of symmetries. This construction is done by induction.

For $N = 3$, the graph of symmetries corresponding to the star F_2-graph is the set of three graphs $G_i^{(3)} = \{V_i^{(3)}, E_i^{(3)}\}$, $i \in \hat{3}$ and $G_i^{(3)}$ being maximal graph for all i. Suppose we got the graph of symmetries corresponding to the star F_2-graph on N vertices. The graph of symmetries corresponding to the star F_2-graph on $N+1$ vertices is constructed by defining following

$$
\begin{aligned}
&G_i^{(N+1)} = (V_i^{(N+1)}, E_i^{N+1}), \quad V_i^{(N+1)} = V_I^{(N)} \cup \{N+1\}, i \in \hat{N},\\
&E_i^{(N+1)} = E_i^{(N)} \cup \{(i, N+1), (N+1), i\}, i \in \{1, 2\},\\
&E_i^{(N+1)} = E_i^{(N)} \cup \{(N+1, 1), (N+1, 2)\}, i \in \{3, \ldots, N\},\\
&G_{N+1}^{(N+1)} = \{V_{N+1}^{(N+1)}, E_{N+1}^{(N+1)}\}, \quad V_{N+1}^{(N+1)} = \hat{N},\\
&E_{N+1}^{(N+1)} = \{(i,1) | i \in \{2, \ldots, N\}\} \cup \{(j,2) | j \in \{1, 3, 4, \ldots, N\}\}.
\end{aligned}
$$

This procedure is for $N = 3$ visualized on the figure B.9. Next we prove that the two-connectedness of the index graph associated with the star F_2-graph for any number of vertices $N \geq 3$. This is done also by induction. The two-connectedness of the index graph $g^{(3)}$ corresponding to the star F_2-graph on 3 vertices is trivial. Let us suppose that the index graph corresponding to the star F_2-graph on N vertices is two-

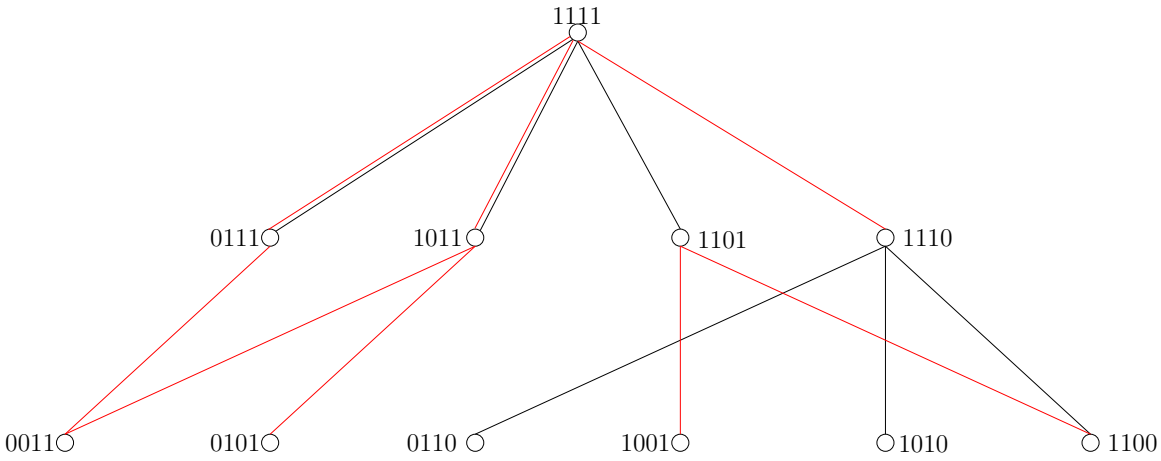


Figure B.10: The construction of the index graph corresponding to the star F_2-graph on 4 vertices from the index graph corresponding to the star F_2-graph on 3 vertices without the real marking of color of the edges. The black edges correspond to the subgraph of this index graph, which is isomorphic to the index graph corresponding to the star F_2-graph on 3 vertices.

connected. From this index graph we create the index graph corresponding to the star F_2-graph on $N+1$ vertices in two steps. First, we add $2^N - 1$ new vertices to the index graph, which correspond to the new qubit which was added to the system. The newly-emerged index graph has $N+2$ components. The first component $\mathfrak{g}_1$ is formed by vertices $\{(\mathbf{i},0)|\mathbf{i} \in I_N, \tau(\mathbf{i} > 1)\}$. The second component $\mathfrak{g}_2$ is formed by vertices $\{(\mathbf{i},1)|\mathbf{i} \in I_N, \tau(\mathbf{i} > 1)\}$. These two components are isomorphic to the index graph corresponding to the star F_2-graph on N vertices and thus both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two-connected. The remaining components are formed by single vertices $\{\mathbf{1}_i 1\}$ for $i \in \hat{N}$. In the second step, we create the index graph corresponding to the star F_2-graph on $N+1$ vertices by adding edges to this index graph. The whole procedure is visualized on the figure B.10 for the case $N = 3$. The added edges connect the components $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as well as they connect the trivial components with component $\mathfrak{g}_2$. What is important is that they also connect vertices within the component $\mathfrak{g}_2$. From the construction it is straightforward to prove that the index graph corresponding to the star F_2-graph on $N+1$ vertices is two-connected. The only non-trivial case is if we want to connect the pairs of vertices $[\mathbf{i}_1, \mathbf{i}_2]$ and $[\mathbf{j}_1, \mathbf{j}_2]$ with $\mathbf{i}_k \in \mathfrak{g}_1$ and $\mathbf{j}_k \in \mathfrak{g}_2$. In this case the two-connectedness is the result of the newly added edges in the component $\mathfrak{g}_2$. As these edges have no counterpart

in the component $\mathfrak{g}_1$ we can freely move between vertices in the component $\mathfrak{g}_2$ without moving within component $\mathfrak{g}_1$.

Since the index graph corresponding to the star F_2-graph on N vertices is a subgraph of the index graph corresponding to the maximal F_2-graph on N vertices, the index graph corresponding to the maximal F_2-graph on N vertices must be also two-connected as the addition of edges cannot disturb the property of two-connectedness.

In the last step of the proof we show that we do not violate the condition of two-connectedness of the index graph by removing edges from the maximal interaction F_2-graph as long as the condition that any vertex $ij \in V$ is connected to a vertex kl_k for every possible $k \neq i, j$ and for any $l_k \in \{1, \ldots, N\}$ holds. We prove this by induction on the length of the path connecting the vertices ij and kl_k. The considered transformations $\hat{U}^{(\frac{\pi}{2})}_{jk,l}$ have the property $\hat{U}^{(\frac{\pi}{2})}_{kl,m}.\hat{U}^{(\frac{\pi}{2})}_{jk,l}.\hat{U}^{(\frac{\pi}{2})}_{kl,m}.\hat{U}^{(\frac{\pi}{2})}_{jk,l} = \hat{U}^{(\frac{\pi}{2})}_{jk,m}$. Suppose we are given a path of length $d = 2$: $P_{i_1 i_2, i_3 i_4} = ((i_1 i_2, i_2 i_3), (i_2 i_3, i_3 i_4))$. Let us note $e = (i_1, i_2; i_3), f = (i_2, i_3; i_4)$ and $g = (i_1, i_2; i_4)$. Because of the previously mentioned property, applying the sequence of edges colors $fefe$ in the index graph has the same result as applying the edge with the color g. Thus all the edges with color g can be removed from the index graph without loosing the property of two-connectedness. In the corresponding interaction F_2-graph we remove the F-arc $(\imath_1, i_2; i_4)$. The resulting interaction F_2-graph is a base graph and there exists a path connecting vertices $i_1 i_2$

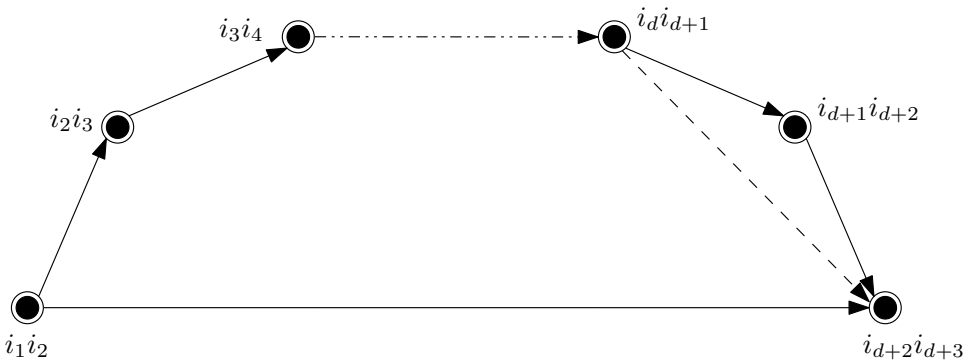


Figure B.11: The removal of the F-arc containing the directed edge $(i_1 i_2, i_{d+2}, i_{d+3})$ from the interaction F-graph.

with $i_4 l_4$ with $l_4 \in \hat{N}$. Let us suppose that for every path of length $d \geq 2$ the F-arc connecting the beginning and the end of the path can be removed from the interaction graph as long as it is not violating desired connectivity of other vertices. Let us take a path of length $d+1$. Without loss of generality let us assume that this path has the form $P = ((i_1 i_2, i_2 i_3), \ldots, (i_d i_{d+1}, i_{d+1} i_{d+2}), (i_{d+1} i_{d+2}, i_{d+2} i_{d+3}))$. We will temporarily add the edge $(i_d i_{d+1}, i_{d+2} i_{d+3})$ to the interaction F-graph. By this addition, the vertices $\{i_1, i_2\}$ and $\{i_{d+2}, i_{d+3}\}$ are connected by path P' of length d. We can thus remove the F-arc containing the vertices $i_1 i_2$ and $i_{d+2} i_{d+3}$ as long as it does not violate the connectivity of other vertices. The same holds for any path of length $d+1$. This completes the proof by induction.

The condition that $\{i, j\} \in V$ is connected to a vertex $\{k, l_k\}$ for every possible $k \neq i, j$ and for any $l_k \in \hat{N}$ holds for any vertex $\{i, j\} \in V$ is a needed condition. We can easily check that if this condition is violated, the corresponding index graph has more than one components and thus it is not two-connected. Thus we have proved that this condition is sufficient and needed for the minimality of eigenspace of RUO Φ corresponding to the eigenvalue $\lambda = 1$.

As can be easily see from the proof of the form of the base attractor space which was made in the appendix A.1, as long as all pairs of qubits play role of the control qubits in some transformation $\hat{U}^{(\phi)}_{jk,l}$, there are no non-trivial solutions of attractor equations for the eigenvalue $\lambda = -1$. Thus all interaction F-graph which satisfy the condition have trivial eigenspace corresponding to the eigenvalue $\lambda = -1$ and thus these graphs are the all base graphs corresponding to controlled unitary interactions with two control qubits.

To summarize, we have proved that the sufficient and needed condition for an interaction F_2-graph to be a base graph is that any vertex $ij \in V$ is connected to a vertex kl_k for every possible $k \neq i, j$ and for any $l_k \in \hat{N}$.

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