Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Department of Physics

The discrete spectrum of non-Hermitian operators

RESEARCH THESIS

Author: Radek Novák
Supervisor: Mgr. David Krejčiřík, Ph.D.
Consultant: Ing. Petr Siegl, Ph.D.
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Abstract:

The weakly-coupled bound states of the Hamiltonian $-\Delta + \varepsilon V$ with a complex-valued potential V are investigated. We study the perturbation on the threshold of the essential spectrum using the Birman-Schwinger principle. Our main point of interest is a sufficient condition for existence and uniqueness of the bound state. The influence of the singularity of the Green function of the free Hamiltonian on the existence of bound state is discovered. We conclude with an asymptotic formula for the eigenvalue of the bound state.

Keywords: Birman-Schwinger principle, bound states, essential spectrum, non-Hermitian potential, relatively bounded potential

Název: **Diskrétní spektrum nehermitovských operátorů**

Autor: Bc. Radek Novák

Abstrakt:

V této práci jsou zkoumány slabě vázané stavy Hamiltoniánu $-\Delta+\varepsilon V$ s připuštěním možných komplexních hodnot potenciálu V. Pomocí Birman-Schwingerova principu se zabýváme poruchou prahu esenciálního spektra a soustředíme se především na postačující podmínku na existenci a jednoznačnost vázaného stavu. Ukazuje se, že za jeho existenci je zodpovědná singularita Greenovy funkce Hamiltoniánu volné částice. Na závěr uvádíme asymptotický vztah pro vlastní hodnotu příslušející vázanému stavu.

Klíčová slova: Birman-Schwingerův princip, vázané stavy, esenciální spektrum, nehermitovský potenciál, relativně omezený potenciál

Contents

Chapter 1

Introduction

The emergence of quantum physics was inspired by several paradoxes that classical physic could not explain. One of these were observations of discrete frequencies of radiation absorbed or emitted by an atom which was in a dispute with the classical prediction that any value should be admissible. The phenomenon of discrete energy levels appeared consequently in many other physical systems and was explained as late as in the framework of quantum mechanics by the existence of a discrete spectrum of the Hamiltonian describing the system.

Finding these eigenvalues from the discrete spectrum can be a challenging task. In this paper we are interesting in describing the discrete spectrum of a Hamiltonian corresponding to a free particle which is weakly perturbed by a potential. Mathematically speaking, we are going to work with Hamiltonian

$$H_\varepsilon = -\Delta + \varepsilon V,$$

where ε is a small and positive coupling parameter. We consider V generally a complex-valued potential, motivated by a recent interest in non-Hermitian operators caused by the emergence of phenomenological Hamiltonians in the optical models of nuclear scattering or the study of unstable lasers (see for instance [3] and references therein). We generalise known solution of this problem [18] for self-adjoint Hamiltonians to a more general non-Hermitian case.

In this paper we focus on the problem in one dimension only. There are already some known results in this area, let us mention for example the bound on the absolute value of the eigenvalue obtained in [5] for Schrödinger operators with potential εV, $V \in L^1(\mathbb{R})$,

$$|\lambda| \leq \varepsilon^2 \frac{\|V\|_1^2}{4}. \tag{1.1}$$

This thesis is organized as follows. In Chapter 2 we introduce the Birman-Schwinger principle, a technique for analysing partial differential equations, which we then in Chapter 3 apply on the problem of finding weakly-coupled bound states. We summarise our results in Chapter 4.

Chapter 2

Birman-Schwinger principle

In this chapter we introduce a useful technique for studying certain types of partial differential equations, particularly in the analysis of the point spectrum of differential operators. It was developed independently by M. Sh. Birman [1] and J. Schwinger [16] in the year 1961 for estimating the number of negative eigenvalues of a self-adjoint Schrödinger operator. Since its origin it was applied also in finding eigenvalue bounds in non-Hermitian case [4][5][6][7][11], studying behaviour of the resolvent [10] and finding weakly coupled bound states [18]. Generally it enables us to solve an eigenvalue problem for differential operators by solving an eigenvalue problem for integral operators. The most significant advantages of this procedure include the fact that integral equations are much more suitable for solving. Among the well established numerical methods for solving such equations let us mention e.g. the method of sequential approximations.

2.1 Definition of the Hamiltonian

First we introduce a free Hamiltonian

$$\begin{aligned} H_0 &:= -\Delta, \\ \mathrm{Dom}(H_0) &:= H^2(\mathbb{R}), \end{aligned} \tag{2.1}$$

acting in a Hilbert space $\mathscr{H} := L^2(\mathbb{R}, \mathrm{d}x)$. We consider potentials which can be regarded in some sense as small perturbations of H_0. Such potential enable us to properly introduce operator H_ε as a closed operator acting on $\mathscr{H}$. We introduce the notion of relative boundedness to ensure that the considered sum of a self-adjoint operator H_0 and a multiplication operator V is well defined.

Definition 2.1.1 ([12, Sec.X.2]). *Let T and V be closed operators in $\mathscr{H}$. We call V relatively bounded with respect to T (or T-bounded), if* $\mathrm{Dom}(T) \subset \mathrm{Dom}(V)$ *and there exist nonnegative constants a, b such that*

$$\|V\psi\| \leq a\|T\psi\| + b\|\psi\|$$

holds for all $\psi \in \mathrm{Dom}(T)$. *The infimum of constants a for which this relation holds is called the H_0-bound of T (or simply the relative bound).*

Investigation of the relative boundedness can be simplified by using the following criterion.

Theorem 2.1.2. *Let T be closed operator in $\mathscr{H}$ such that $\sigma(T) \neq \mathbb{C}$ and V an operator in $\mathscr{H}$. V is $T-$bounded if, and only if, $V(T-z)^{-1}$ is bounded for some $z \in \rho(T)$.*

Proof. $\Rightarrow$) Let $\psi \in \mathscr{H}$. It can be written as $\psi = (T-z)\phi$ for some $\phi \in \mathrm{Dom}(T)$ and $z \in \rho(T)$. Then $\phi = (T-z)^{-1}\psi \in \mathrm{Dom}(V)$ and we have

$$\begin{aligned} \|V(T-z)^{-1}\psi\| &\leq a\|T(T-z)^{-1}\psi\| + b\|(T-z)^{-1}\psi\| \\ &= a\|(T-z+z)(T-z)^{-1}\psi\| + b\|(T-z)^{-1}\psi\| \\ &\leq a\|\psi\| + (a|z|+b)\|(T-z)^{-1}\psi\| < +\infty. \end{aligned}$$

Dividing by $\|\psi\|$ and taking supremum over all $\psi \in \mathscr{H}$ we get the claim.
$\Leftarrow$) Again, taking $\psi \in \mathscr{H}$ written as $\psi = (T-z)\phi$ for some $\phi \in \mathrm{Dom}(T)$, then from the assumption there is such constant $C \in \mathbb{R}$ such that

$$\|V\phi\| = \|V(T-z)^{-1}\psi\| \leq C\|\psi\| = C\|(T-z)\phi\| \leq C\|T\phi\| + C|z|\|\phi\| < +\infty$$

for all $\phi \in \mathrm{Dom}(T)$. Then $\mathrm{Dom}(T) \subset \mathrm{Dom}(V)$ and our proof is concluded. □

The following theorem specifies conditions under which the sum of operators H_0 and multiplication operator V is well defined for H_0-bounded potentials V.

Theorem 2.1.3 ([9, Thm. IV-1.1]). *Let S and T be operators in $\mathscr{H}$ and let T be $S-$bounded with the relative bound smaller than 1. Then $S+T$ is closed if, and only if, S is closed.*

Now we can define the operator

$$\begin{aligned} H &= H_0 + V \\ \mathrm{Dom}(H) &:= \mathrm{Dom}(H_0) = H^2(\mathbb{R}) \end{aligned} \tag{2.2}$$

as closed operator for the H_0-bounded potential V with the H_0-bound smaller than 1. In the following text we are going to consider only potentials

$$V \in L^2(\mathbb{R}) + L^\infty(\mathbb{R}),$$

i.e. $V = V_1 + V_2$, where $V_1 \in L^2(\mathbb{R})$ and $V_2 \in L^\infty(\mathbb{R})$. The considered class of potentials is natural in $\mathbb{R}$ in the sense of the following theorem:

Theorem 2.1.4 ([17, Thm. II.3])**.** *Let $V(x)$ be function on $\mathbb{R}$ with $\limsup_{|x|\to+\infty} |V(x)| < +\infty$. If $\mathrm{Dom}(V) \supset \mathrm{Dom}(H_0)$, then $V \in L^2(\mathbb{R})+L^\infty(\mathbb{R})$. If moreover $\limsup_{|x|\to+\infty} |V(x)| = 0$, then the $L^\infty(\mathbb{R})$ part can be chosen with arbitrary small supremum norm $\|\cdot\|_\infty$.*

Proof. Let us directly construct a decomposition of potential V into $V_1 \in L^2(\mathbb{R})$ and $V_2 \in L^\infty(\mathbb{R})$. Let $\limsup_{|x|\to+\infty} |V(x)| = n$. We set $V_2(x) = V(x)$ if $|V(x)| < 2n$ and $V_2(x) = 0$ otherwise. V_1 is then just its complement: $V_1(x) = V(x) - V_2(x)$. Clearly, $V_2 \in L^\infty(\mathbb{R})$ and $V_1(x)$ vanishes for $|x|$ larger than some R. It is possible to find $\psi \in \mathrm{Dom}(H_0)$ such that it is equal to 1 in $(-R, R)$. Then we have

$$\|V_1\|^2 = \int_{-R}^{R} |V_1(x)|^2 \,\mathrm{d}x = \int_{-R}^{R} |V_1(x)\psi(x)|^2 \,\mathrm{d}x \leq \int_{-R}^{R} |V(x)\psi(x)|^2 s \,\mathrm{d}x < +\infty$$

from the assumption that ψ lies also in $\mathrm{Dom}(V)$. The second part of the theorem can be proven in a similar manner. □

Since the property of the potential that its $L^\infty(\mathbb{R})$ part can be chosen arbitrarily small is going to turn out as very useful and later on we will restrict ourselves to this class of potentials, we therefore introduce a special notation for them.

Definition 2.1.5. *We say that potential V is from $L^2(\mathbb{R})+L^\infty_\delta(\mathbb{R})$ if there are for every $\delta > 0$ such $V_1 \in L^2(\mathbb{R})$ and $V_2 \in L^\infty(\mathbb{R})$ that $V = V_1 + V_2$ and $\|V_2\|_\infty < \delta$.*

In the following we are going to show that V is H_0-bounded with the relative bound zero. Thus the assumptions of Theorem 2.1.3 will be fulfilled and the operator (2.2) will be closed as we desire.

Theorem 2.1.6. *$V \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ is H_0-bounded with the relative bound equal to 0.*

Proof. The proof can be done separately for V_1 and V_2 and the result then follows from the Minkowski inequality. For V_2 we have

$$\|V_2\psi\| \leq \|V_2\|_\infty \|\psi\|.$$

Thus for $\psi \in \mathrm{Dom}(H_0) \subset L^2(\mathbb{R})$ we have $\psi \in \mathrm{Dom}(V_2)$ and V_2 is not only H_0-bounded with relative bound 0, but even bounded. Taking V_1 we make use of the fact that $\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [0, +\infty)$ (see e.g. [2, Ex. 7.2.1] for proof) so every $\lambda < 0$ lies in $\rho(H_0)$ and that the resolvent function can be for such λ expressed as an integral operator

$$(R_\lambda \phi)(x) := \left((H_0 - \lambda)^{-1}\phi\right)(x) = \int_{\mathbb{R}} \mathcal{R}_\lambda(x,y)\phi(y)\,\mathrm{d}y,$$

where the integral kernel of R_λ takes form

$$\mathcal{R}_\lambda(x,y) := \frac{\mathrm{e}^{-\sqrt{-\lambda}|x-y|}}{2\sqrt{-\lambda}} \tag{2.3}$$

as can be easily verified by using Fourier transform. We select the branch of the square root so that $\mathrm{Re}\sqrt{-\lambda} > 0$, as long as λ is not along the non-negative real axis. For $1 \leq p, q, r \leq +\infty$ such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$ holds the Young inequality $\|f * g\|_q \leq \|f\|_p \|g\|_r$, where $f \in L^p(\mathbb{R}), g \in L^r(\mathbb{R})$ and $\|\cdot\|_p$ denotes the norm in the space $L^p(\mathbb{R})$ (see [12, Ex. IX.4.1] for proof). Applying this to our case we obtain

$$\|R_\lambda \phi\|_\infty \leq \|\phi\| \|\mathcal{R}'_\lambda\|,$$

where

$$\mathcal{R}'_\lambda(x) := \mathcal{R}_\lambda(x, 0), \tag{2.4}$$

i.e. $\|\mathcal{R}'_\lambda\|^2 = \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\sqrt{-\lambda}\,|x|}}{-4\lambda}\,\mathrm{d}x$. Now we can easily see that $\lim_{\lambda \to -\infty} \|\mathcal{R}'_\lambda\| = 0$, therefore for every $\delta > 0$ we can find $\lambda(\delta)$ sufficiently negative so that $\|\mathcal{R}'_\lambda\| \leq \frac{\delta}{\|V_1\|}$ holds. (We assume $V_1 \neq 0$ because for $V_1 = 0$ the claim is fulfilled trivially.) Since every $\psi \in \mathrm{Dom}(H_0)$ can be written as $\psi = R_\lambda \phi$ for some $\phi \in \mathscr{H}$ and $\lambda \in \rho(H_0)$, we conclude with the inequality

$$\begin{aligned} \|V_1\psi\| = \|V_1 R_\lambda \phi\| &\leq \|V_1\|_2 \, \|R_\lambda\phi\|_\infty \leq \|V_1\|_2 \, \|\phi\| \|\mathcal{R}'_\lambda\| \\ &\leq \delta\|\phi\| = \delta\|(H_0 - \lambda)\psi\| \leq \delta\|H_0\psi\| + \delta\lambda\|\psi\| < +\infty, \end{aligned}$$

which holds for every $\psi \in \mathrm{Dom}(H_0)$ and every $\delta > 0$. We emphasized by the bottom index 2 that we norm V as a function in $\mathscr{H}$, not as an operator on $\mathscr{H}$. (An operator norm of a multiplication operator V is $\|V\| = \|V\|_\infty$.) □

Remark 2.1.7. *The proof of Theorem 2.1.6 can be easily generalised for dimension two only by changing the integral kernel of the resolvent.*

2.2 Birman-Schwinger principle

In the following we present a proof of the generalised Birman-Schwinger principle for the case of a free particle Hamiltonian with a complex-valued potential.

Theorem 2.2.1 (Birman-Schwinger principle). *Let H be an operator defined in (2.2), where V is H_0-bounded with the relative bound smaller than 1, and let $\lambda \in \mathbb{C} \setminus [0, +\infty)$. Then*

$$\lambda \in \sigma_p(H) \qquad \Leftrightarrow \qquad -1 \in \sigma_p(K_\lambda), \tag{2.5}$$

where $K_\lambda := |V|^{1/2}(H_0 - \lambda)^{-1} V_{1/2}$ with $V_{1/2} := |V|^{1/2} \mathrm{e}^{\mathrm{i}\arg(V)}$.

Proof. $\Rightarrow$: If there is such $\psi \in L^2(\mathbb{R})$ that $(H_0 + V)\psi = \lambda\psi$ then by definition we have $\psi \in \mathrm{Dom}(H) = H^2(\mathbb{R})$. Function $\phi := |V|^{1/2}\psi$ is in $L^2(\mathbb{R})$, since

$$\begin{aligned}
\||V|^{1/2}\psi\| &= \||V|^{1/2}(H_0 + 1)^{-1}\xi\| \\
&\leq \||V|^{1/2}(H_0 + 1)^{-1}\|\|\xi\| \\
&= \|(H_0 + 1)^{-1}|V|(H_0 + 1)^{-1}\|^{1/2}\|\xi\| \\
&\leq \|(H_0 + 1)^{-1}\|\||V|(H_0 + 1)^{-1}\|\|\xi\| < +\infty,
\end{aligned}$$

where $\xi \in L^2(\mathbb{R})$ and we used the equality $\|T^*T\| = \|T\|^2$ which holds for every bounded operator T and the fact that $-1 \in \rho(H_0)$ and therefore the operator $(H_0 + 1)^{-1}$ exists and is bounded. Using the assumption we get

$$K_\lambda\phi = |V|^{1/2}(H_0 - \lambda)^{-1}V\psi = -|V|^{1/2}(H_0 - \lambda)^{-1}(H_0 - \lambda)\psi = -|V|^{1/2}\psi = -\phi.$$

$\Leftarrow$: Other way round, we assume that there is such $\phi \in L^2(\mathbb{R})$ that $K_\lambda\phi = -\phi$. Our ψ can be defined as $-(H_0 - \lambda)^{-1}V_{1/2}\phi$. Then $\psi \in \mathrm{Dom}(H)$ (since the resolvent $(H_0 - \lambda)^{-1}$ displays $L^2(\mathbb{R})$ to $\mathrm{Dom}(H)$ and $V_{1/2}\phi \in L^2(\mathbb{R})$ using the same arguments as in the proof of the opposite implication) and

$$\begin{aligned}
(H_0 - \lambda)\psi &= -(H_0 - \lambda)(H_0 - \lambda)^{-1}V_{1/2}\phi = -V_{1/2}\phi = V_{1/2}K_\lambda\phi \\
&= V_{1/2}|V|^{1/2}(H_0 - \lambda)^{-1}V_{1/2}\phi = -V\psi.
\end{aligned}$$

□

We note that Theorem 2.2.1 can be straightforwardly generalised to higher dimensions since our proof does not depend on the explicit form of the resolvent kernel. Further on, the resolvent $R_{H_0}(\lambda) = (H_0 - \lambda)^{-1}$ is an integral operator whose kernel $\mathcal{R}_z(x, y)$ is known in all dimensions for all $\lambda \in \mathbb{C} \setminus [0, +\infty)$ (i.e. $(R_\lambda\psi)(x) = \int_{\mathbb{R}^n} \mathcal{R}_\lambda(x, y)\psi(y)\,\mathrm{d}y$). Specifically in the dimension one the integral kernel of R_λ takes form (2.3) Therefore K_λ is for $\lambda < 0$ also an integral operator with the integral kernel $\mathcal{K}_\lambda(x, y) = |V|^{1/2}(x)\mathcal{R}_\lambda(x, y)V_{1/2}(y)$ and we acquired our pledged integral equation.

Chapter 3

Weakly coupled bound states in one dimension

We now apply the results of the preceding chapter on the perturbations of the threshold of the essential spectrum of the free particle Hamiltonian with a weakly coupled potential (see Figure 3.1). Our goal is to state sufficient conditions which guarantee the existence and uniqueness of a bound state for a Hamiltonian

$$\begin{aligned} H_\varepsilon &:= H_0 + \varepsilon V, \\ \mathrm{Dom}(H_\varepsilon) &:= H^2(\mathbb{R}), \end{aligned} \tag{3.1}$$

for ε small and positive where V is a relatively bounded potential with respect to H_0. (Further condition on V are going to be imposed later.) We do not have to demand the relative bound to be smaller than 1 since we can always achieve this by taking ε sufficiently small. The Birman-Schwinger principle (Theorem 2.2.1) says that $\lambda \in \mathbb{C} \setminus [0, +\infty)$ is an eigenvalue of H_ε if, and only if, -1 is an eigenvalue of an integral operator εK_λ. The integral kernel of K_λ takes in this case form

$$\mathcal{K}_\lambda(x, y) = |V(x)|^{1/2} \, \frac{\mathrm{e}^{-\sqrt{-\lambda}|x-y|}}{2\sqrt{-\lambda}} \, V_{1/2}(y). \tag{3.2}$$

3.1 Preliminary results

The appearance of weakly-coupled eigenvalue is caused by the singularity of $\mathcal{K}_\lambda$ as $\lambda \to 0$. To exploit it, we use the decomposition of K_λ into two integral operators, L_λ and M_λ,

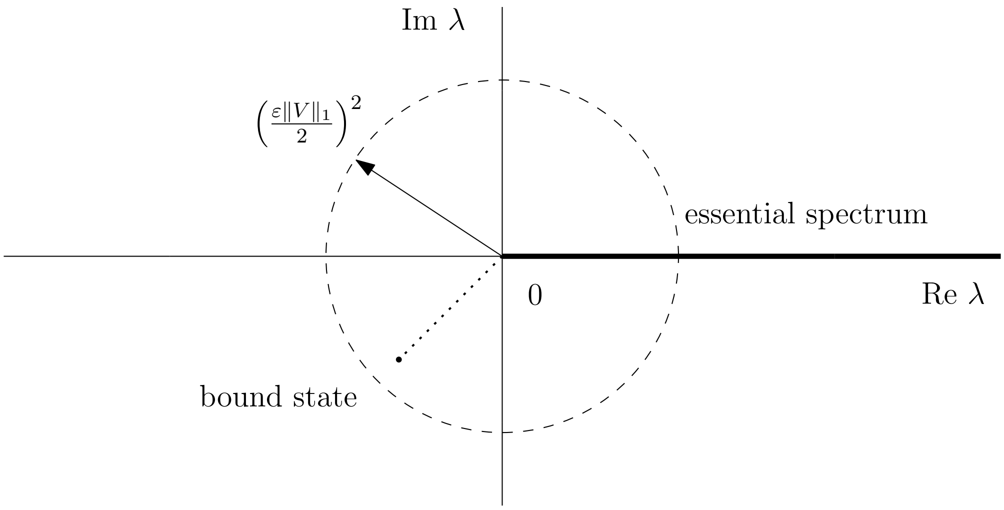

Figure 3.1: Possible spectrum of H_ε with the essential spectrum located on the positive real axis and a unique bound state. The circle denotes the bound (1.1) on the size of possible eigenvalues.

separating the singularity in L_λ [18]. Their integral kernels are

$$\begin{aligned}\mathcal{L}_\lambda(x,y) &:= |V(x)|^{1/2}\,\frac{1}{2\sqrt{-\lambda}}\,V_{1/2}(y),\\ \mathcal{M}_\lambda(x,y) &:= |V(x)|^{1/2}\,\frac{\mathrm{e}^{-\sqrt{-\lambda}|x-y|}-1}{2\sqrt{-\lambda}}\,V_{1/2}(y)\end{aligned} \tag{3.3}$$

respectively. Before proving the main results of this chapter, we state the following lemma about the operator M_λ.

Lemma 3.1.1. *Let $V \in L^1(\mathbb{R},(1+x^2)\mathrm{d}x) \cap \big(L^2(\mathbb{R},\mathrm{d}x) + L^\infty_\delta(\mathbb{R},\mathrm{d}x)\big)$. Then M_λ converges for $\lambda \to 0-$ in a Hilbert-Schmidt norm to the integral operator M_0 with the kernel*

$$\mathcal{M}_0(x,y) := -|V(x)|^{1/2}\,\frac{|x-y|}{2}\,V_{1/2}(y), \tag{3.4}$$

i.e. $\lim_{\lambda\to 0-}\|M_\lambda - M_0\|_{HS} = 0$, *where* $\|\cdot\|_{HS}$ *denotes the Hilbert-Schmidt norm on* $L^2(\mathbb{R})$.

Proof. We can check that M_0 is a Hilbert-Schmidt operator using simple estimates since

$$\begin{aligned} \|M_0\|_{HS}^2 &= \int_{\mathbb{R}^2} |\mathcal{M}_0(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{1}{2}\int_{\mathbb{R}^2} |V(x)|(x^2+y^2)|V(y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}} |V(x)|x^2\,\mathrm{d}x \int_{\mathbb{R}} |V(y)|\,\mathrm{d}y \\ &\le \frac{1}{2}\left(\int_{\mathbb{R}} |V(x)|(1+x^2)\,\mathrm{d}x\right)^2 < +\infty. \end{aligned} \tag{3.5}$$

M_λ converges to M_0 pointwise as $\lambda \to 0-$ and

$$|M_\lambda(x,y)| \le |M_0(x,y)| \tag{3.6}$$

for all $x, y \in \mathbb{R}$ and for all $\lambda \in \mathbb{C}\setminus[0,+\infty)$. Using the dominated converge theorem yields desired result. To verify the inequality (3.6) it is sufficient to see that

$$\left|\frac{\mathrm{e}^{a+\mathrm{i}b}-1}{-(a+\mathrm{i}b)}\right|^2 \le 1$$

holds for all $a, b \in \mathbb{R}, a < 0$. After an explicit calculation of the absolute value on left-hand side of the inequality and a simple algebraic manipulation, we reformulate our problem to verification that

$$1+\mathrm{e}^{2a}-2\mathrm{e}^a\cos b - a^2 - b^2 \le 0$$

holds. We employ the estimate $\cos b \ge 1 - b^2/2$ which holds for all $b \in \mathbb{R}$ to get

$$\begin{aligned} 1+\mathrm{e}^{2a}-2\mathrm{e}^a\cos b - a^2 - b^2 &\le 1+\mathrm{e}^{2a}-2\mathrm{e}^a(1-\frac{b^2}{2}) - a^2 - b^2 \\ &\le 1+\mathrm{e}^{2a}-2\mathrm{e}^a 1 + b^2 - a^2 - b^2 \\ &= 1+\mathrm{e}^{2a}-2\mathrm{e}^a - a^2. \end{aligned}$$

Using calculus of functions of one variable it is now easy to check that $F(a) := 1+\mathrm{e}^{2a} - 2\mathrm{e}^a - a^2 \le 0$. □

Corollary 3.1.2. *M_λ converges in the operator norm to M_0 for $\lambda \to 0-$.*

Proof. The corollary follows immediately from the inequality $\|T\| \le \|T\|_{HS}$ valid for all Hilbert-Schmidt operators T [13, Thm. VI.22]. □

In the end of this section we introduce a theorem for finding a fixed point of a function which will come handy in Section 3.3.

Theorem 3.1.3 (Banach contraction theorem, [8, Thm. I.1.1])**.** *Let (X, d) be a complete metric space anf $F : X \to X$ be contractive, i.e. $d(F(x), F(y)) \leq C\, d(x, y)$ for all $x, y \in X$ and some constant $C < 1$. Then F has a unique fixed point u, i.e. $F(u) = u$, and $\lim_{n\to+\infty} F^n(x) = u$ for each $x \in X$*

3.2 Stability of the essential spectrum

Since we perturb the threshold of the essential spectrum, another condition on the potential arises. The essential spectrum of H_ε could spread to the area where the bound state should occur. Certain class of potentials in fact leaves $\sigma_{\text{ess}}(H_0)$ invariant. It requires some decline of the potential in the infinity which is in the case of $L^2(\mathbb{R})+L^\infty_\delta(\mathbb{R})$ potential fulfilled. In the following we discuss this condition further. Since the definition of the essential spectrum for non-Hermitian operators differs in literature, let us specify this notion.

Definition 3.2.1 ([2])**.** *Let T be a closed operator on $\mathscr{H}$. We say that $\lambda \in \mathbb{C}$ belongs to the essential spectrum of T (denoted $\sigma_{ess}(T)$) if there exists a sequence $(\psi_n)_{n=1}^{+\infty}$, $\|\psi_n\| = 1$ for all n, such that it does not contain any convergent subsequence and $\lim_{n\to+\infty}(T - \lambda)\psi_n = 0$.*

Let us also remind a known notion of a compact operator.

Definition 3.2.2 ([13])**.** *A bounded operator T is compact if for any bounded sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathscr{H}$, the sequence $(T\psi_n)_{n\in\mathbb{N}}$ contains a Cauchy subsequence.*

Again, we would like to view V as a small perturbation of H_0 in some sense, we therefore introduce relative compactness.

Definition 3.2.3 ([14, Def. XIII.4.1])**.** *Let T be a self-adjoint operator. An operator V with $\operatorname{Dom}(T) \subset \operatorname{Dom}(V)$ is called relatively compact with respect to T if $V(T - \mathrm{i})^{-1}$ is compact.*

We should note that if $V(T - \mathrm{i})^{-1}$ is compact, then $V(T - z)^{-1}$ is compact for all $z \in \rho(T)$ thanks to the first resolvent formula. Therefore it is sufficient to check compactness only for an operator $V(T - z)^{-1}$ with an arbitrary $z \in \rho(T)$.

Theorem 3.2.1. *$V \in L^2(\mathbb{R}) + L^\infty_\delta(\mathbb{R})$ is relatively compact with respect to H_0.*

Proof. Let us check that the operator $V(H_0-\lambda)^{-1}$ is compact for some $\lambda \in \rho(H_0) = \mathbb{C}\setminus[0,+\infty)$. We restrict ourselves to only such $\lambda\in\mathbb{R}$ that $\lambda<0$. Let $(\psi_n)_{n\in\mathbb{N}} \subset \mathscr{H}$ be an arbitrary bounded sequence and $\varepsilon>0$. We now want to check that we can find such strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and some n_0 that for every $k,l>n_0$ is $\|V(H_0-\lambda)^{-1}\psi_{n_k} - V(H_0-\lambda)^{-1}\psi_{n_l}\| < \varepsilon$. (In other words we are looking for a Cauchy subsequence in sequence $V(H_0-\lambda)^{-1}\psi_n$.) We can set $V = V_1+V_2$, where $V_1\in L^2(\mathbb{R})$ and $V_2\in L^\infty(\mathbb{R})$ with $\|V_2\|_\infty<\delta$. Then $V_1(H_0-\lambda)^{-1}$ is Hilbert-Schmidt. Indeed,

$$\begin{aligned}
\|V_1(H_0-\lambda)^{-1}\|_{HS}^2 &= \int_{\mathbb{R}^2} \left| V_1(x)\frac{\mathrm{e}^{-\sqrt{-\lambda}|x-y|}}{2\sqrt{-\lambda}}\right|^2 \mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \left(|V_1|^2 * |\mathcal{R}'_\lambda|^2\right)(x)\,\mathrm{d}x \\
&\le \||V_1|^2 * |\mathcal{R}'_\lambda|^2\|_1 \\
&\le \||V_1|^2\|_1 \||\mathcal{R}'_\lambda|^2\|_1 \\
&= \|V_1\|_2 \|\mathcal{R}'_\lambda\|_2 < +\infty,
\end{aligned}$$

where $\mathcal{R}'_\lambda$ is a function defined in (2.4) and we used Young inequality described in the proof of Theorem 2.1.6. As a consequence, operator $V_1(H_0-\lambda)^{-1}$ is compact so there is $n_0\in\mathbb{N}$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ such that $\|V_1(H_0-\lambda)^{-1}\psi_{n_k} - V_1(H_0-\lambda)^{-1}\psi_{n_l}\| < \delta$ for $k,l>n_0$. $V_2(H_0-\lambda)^{-1}$ is for a fixed δ bounded with an arbitrarily small bound since we still have freedom in the choice of λ:

$$\|V_2(H_0-\lambda)^{-1}\| \le \|V_2\|\|(H_0-\lambda)^{-1}\| < \delta\frac{1}{|\lambda|}.$$

In this estimated we used known formula for the norm of the resolvent of a self-adjoint operator: $\|(H_0-\lambda)^{-1}\| = 1/\mathrm{dist}(\lambda,\sigma(H_0)) = 1/|\lambda|$. For $k,l>n_0$ we have:

$$\|V(H_0-\lambda)^{-1}\psi_{n_k} - V(H_0-\lambda)^{-1}\psi_{n_l}\| \le \delta + K\frac{\delta}{|\lambda|},$$

which we wanted to show. □

The following theorem gives us the desired result about the essential spectrum of H_ε.

Theorem 3.2.4 (Weyl theorem, [14, Cor. XIII.4.2])**.** *Let T be a self-adjoint operator and let V be a relatively compact perturbation of S. Then:*

- *operator $T+V$ defined with* $\mathrm{Dom}(T+V) = \mathrm{Dom}(T)$ *is a closed operator*
- *if V is symmetric, then $T+V$ is self-adjoint*

- $\sigma_{ess}(T+V) = \sigma_{ess}(T)$

This theorem also gives us immediately that the operator $H_0 + V$ is closed, which we derived using other method in Section 2.1.

3.3 Existence and uniqueness of the weakly coupled bound state

Equipped with the results of the previous section we may state our first theorem regarding the existence of the bound state.

Theorem 3.3.1. *Let* $V \in L^1(\mathbb{R}, (1+x^2)\mathrm{d}x) \cap \left(L^2(\mathbb{R}, \mathrm{d}x) + L^\infty_\delta(\mathbb{R}, \mathrm{d}x)\right)$, $\lambda \in \mathbb{C} \setminus [0, +\infty)$ *and* $\varepsilon > 0$ *so small that the inequality* $\varepsilon \int_{\mathbb{R}} |V(x)|(1+x^2)\,\mathrm{d}x < \sqrt{2}$ *holds. Then*

$$\lambda \in \sigma_p(H_\varepsilon) \qquad \Leftrightarrow \qquad \sqrt{-\lambda} = -\frac{\varepsilon}{2}\left(V_{1/2}, (I+\varepsilon M_\lambda)^{-1}|V|^{1/2}\right). \tag{3.7}$$

Proof. Our goal is to find condition to ensure that the operator εK_λ has eigenvalue -1. Under the assumptions and the estimate (3.5) we have $\|\varepsilon M_\lambda\| \leq \varepsilon \|M_0\|_{HS} < 1$ so the operator $(I+\varepsilon M_\lambda)^{-1}$ exists and is bounded. We may write

$$\begin{aligned}(I+\varepsilon K_\lambda)^{-1} &= \left((I+\varepsilon M_\lambda)(I+(I+\varepsilon M_\lambda)^{-1}\varepsilon L_\lambda)\right)^{-1}\\ &= \left(I+(I+\varepsilon M_\lambda)^{-1}\varepsilon L_\lambda\right)^{-1}(I+\varepsilon M_\lambda)^{-1}.\end{aligned}$$

and therefore look only if the operator $P^\varepsilon_\lambda := (I+\varepsilon M_\lambda)^{-1}\varepsilon L_\lambda$ has eigenvalue -1. Since L_λ is a rank-one operator by definition, we can write

$$P^\varepsilon_\lambda = \phi(\psi, \cdot),$$

with

$$\psi := \varepsilon\frac{1}{2\sqrt{-\lambda}}V_{1/2}, \qquad \phi := (I+\varepsilon M_\lambda)^{-1}|V|^{1/2}.$$

The operator P^ε_λ can have only one eigenvalue, namely (ψ, ϕ). Putting it equal to -1 we get the condition

$$-1 = \frac{\varepsilon}{2\sqrt{-\lambda}}\left(V_{1/2}, (I+\varepsilon M_\lambda)^{-1}|V|^{1/2}\right),$$

which we wanted to show. □

The preceding theorem reduced our problem to solving an algebraic equation. We can further work with this result and get an asymptotic expansion for the bound state and a sufficient condition for its existence and uniqueness, as it is summarized in the following theorem.

Theorem 3.3.2. *Let* $V \in L^1(\mathbb{R}, (1+x^2)\mathrm{d}x) \cap \big(L^2(\mathbb{R}, \mathrm{d}x) + L^\infty_\delta(\mathbb{R}, \mathrm{d}x)\big)$ *and* $\varepsilon > 0$ *so small that the inequality* $\varepsilon \int_{\mathbb{R}} |V(x)|(1+x^2)\,\mathrm{d}x < 1$ *holds. Then* H_ε *possesses the unique eigenvalue* $\lambda = \lambda(\varepsilon) \in \mathbb{C} \setminus [0, +\infty)$ *if* $\int_{\mathbb{R}} \operatorname{Re} V(x)\,\mathrm{d}x < 0$*. The asymptotic expansion*

$$\sqrt{-\lambda(\varepsilon)} = -\frac{\varepsilon}{2} \int_{\mathbb{R}} V(x)\,\mathrm{d}x - \frac{\varepsilon^2}{4} \int_{\mathbb{R}^2} V(x)|x-y|V(y)\,\mathrm{d}x\,\mathrm{d}y + \mathcal{O}(\varepsilon^3) \tag{3.8}$$

holds as $\varepsilon \to 0$*.*

Proof. We use the following identity to obtain the asymptotic expansion of the implicit equation (3.7):

$$(I + \varepsilon M_\lambda)^{-1} = I - \varepsilon M_\lambda (I + \varepsilon M_\lambda)^{-1} = I - \varepsilon M_\lambda + \varepsilon^2 M_\lambda^2 (I + \varepsilon M_\lambda)^{-1}$$

as $\varepsilon \to 0+$. This expansion should be fulfilled by every solution of (3.7). The use of the formula (3.7) is justified when $\int_{\mathbb{R}} \operatorname{Re} V(x)\,\mathrm{d}x < 0$ since then $\lambda \notin [0, +\infty)$. For the sake of the simplicity of the formulae we introduce notation $k := \sqrt{-\lambda}$.

Although it might appear that we have obtained our solution, we have yet no evidence that it actually exists. To ensure its existence we use the Banach fixed point theorem 3.1.3. As our metric space X we set a disc $B(k_0, r)$ with the centre in the point $k_0 := \int_{\mathbb{R}} V(x)\,\mathrm{d}x$ and sufficiently small radius r so that the whole disc lies in the half-plane $\operatorname{Re} k > 0$.

We first prepare few estimates before we continue with the proof of existence. Since $\int_{\mathbb{R}} V(x)\,\mathrm{d}x \neq 0$, we can immediately see from the asymptotic expansion that for every k, solution of (3.7), holds

$$\frac{1}{|k|} \leq \frac{C_1}{\varepsilon}, \tag{3.9}$$

where C_1 is a positive constant. This inequality also holds for k_0.

The operator valued function M_λ is analytic as a function of λ in the region $\operatorname{Re} k > 0$, therefore we can apply Cauchy integral formula with a curve $\gamma = k_0 + r\mathrm{e}^{\mathrm{i}\varphi}$ (constant $r > 0$ so small that the whole curve lies in this region) to get

$$\begin{aligned} \left\| \frac{\partial M_\lambda}{\partial k} \right\| &= \int_{\mathbb{R}^2} \left(\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathcal{M}_\lambda(x,y)}{(k-k')^2}\,\mathrm{d}k' \right) \mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{\|M_0\|}{2\pi} \left| \oint_\gamma \frac{1}{(k-k')^2}\,\mathrm{d}k' \right| \\ &= \frac{\|M_0\|}{2\pi} \left| \int_0^{2\pi} \frac{1}{r^2 \mathrm{e}^{2\mathrm{i}\varphi}} \mathrm{i}r\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{d}\varphi \right| \\ &= \frac{\|M_0\|}{2\pi r} \left| \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\varphi}\,\mathrm{d}\varphi \right| \leq \frac{\|M_0\|}{r} = \frac{C_2}{|k|} \end{aligned} \tag{3.10}$$

by setting $r = \operatorname{Re} k/2$.
We also prepare prescription for differentiating $(1+\varepsilon M_\lambda)^{-1}$ using the second resolvent formula:

$$\begin{aligned}\frac{\partial}{\partial k}(1+\varepsilon M_\lambda)^{-1} &= \lim_{k'\to k}\frac{(1+\varepsilon M_\lambda)^{-1}-(1+\varepsilon M_{\lambda'})^{-1}}{k-k'}\\ &= \lim_{k'\to k}\varepsilon\frac{(1+\varepsilon M_\lambda)^{-1}(\varepsilon M_\lambda-\varepsilon M_{\lambda'})(1+\varepsilon M_{\lambda'})^{-1}}{k-k'}\\ &= \varepsilon(1+\varepsilon M_\lambda)^{-1}\frac{\partial M_\lambda}{\partial k}(1+\varepsilon M_\lambda)^{-1}.\end{aligned} \tag{3.11}$$

Now back to the proof of existence - we define a function

$$G(\lambda,\varepsilon) := -\frac{\varepsilon}{2}\left(V_{1/2},(I+\varepsilon M_\lambda)^{-1}|V|^{1/2}\right) \tag{3.12}$$

and estimate its derivative for Re $k > 0$ using the hypothesis, estimates (3.9),(3.10) and formula (3.11):

$$\begin{aligned}\left|\frac{\partial G(\lambda,\varepsilon)}{\partial k}\right| &= \left|\frac{\varepsilon^2}{2}\left(V_{1/2},\varepsilon(1+\varepsilon M_\lambda)^{-1}\frac{\partial M_\lambda}{\partial k}(1+\varepsilon M_\lambda)^{-1}|V|^{1/2}\right)\right|\\ &\le \frac{\varepsilon^2}{2}\|V_{1/2}\|^2\|(1+\varepsilon M_\lambda)^{-1}\|^2\left\|\frac{\partial M_\lambda}{\partial k}\right\|\\ &\le \frac{\varepsilon^2}{2}\|V_{1/2}\|^2\|(1+\varepsilon M_\lambda)^{-1}\|^2\frac{C_1C_2}{\varepsilon} = K\,\varepsilon.\end{aligned} \tag{3.13}$$

Our goal is to show that $G(\lambda,\varepsilon)$ is contractive in the disc $B(k_0,r)$. We take arbitrary $k_1,k_2 \in B(k_0,r)$. Taking r sufficiently small we can expand these in Taylor series in the neighbourhood of the point k_0

$$G(k_j,\varepsilon) = G(k_0,\varepsilon) + (k_j-k_0)\frac{\partial G(\lambda,\varepsilon)}{\partial k}(k_0) + \mathcal{O}((k_j-k_0)^2),$$

where $j = 1,2$. Using this series we can sum up

$$\begin{aligned}|G(k_1,\varepsilon)-G(k_2,\varepsilon)| &= |G(k_1,\varepsilon)-G(k_0,\varepsilon)+G(k_0,\varepsilon)-G(k_2,\varepsilon)|\\ &= \left|(k_1-k_2)\frac{\partial G(\lambda,\varepsilon)}{\partial k}(k_0)+\mathcal{O}((k_1-k_0)^2)+\mathcal{O}((k_2-k_0)^2)\right|\\ &\le \left|\frac{\partial G(\lambda,\varepsilon)}{\partial k}(k_0)+\mathcal{O}(k_1-k_2)\right||k_1-k_2|.\end{aligned}$$

Setting ε and r sufficiently small, we can make the coefficient by $|k_1-k_2|$ strictly smaller than one and therefore conclude that the equation (3.7) has a unique solution in $B(k_0,\varepsilon)$. To show that it is unique in the half-plane $\operatorname{Re} k > 0$ we assume that there are two solutions to (3.7): k_1 and k_2. We connect them by a straight line and realise that for all

k on this line holds estimate (3.9). Writing the equation (3.7) as $\alpha = G(\alpha, \varepsilon)$ and using 3.11 we get a contradiction in the following equation:

$$|k_2 - k_1| = \left| \int_{k_1}^{k_2} \frac{\partial G(\lambda, \varepsilon)}{\partial k} \, \mathrm{d}k \right| \leq K\varepsilon |k_2 - k_1| < |k_2 - k_1| \tag{3.14}$$

for ε sufficiently small.

□

We note that there is an alternative possibility how to carry out the proof using Rouché's theorem [15, Thm. 10.43]. We notice that the bound state indeed arises from the threshold of the essential spectrum and not from any other point of the essential spectrum. We emphasize that the condition $\int_{\mathbb{R}} \operatorname{Re} V(x) \, \mathrm{d}x < 0$ relates only to the case when the ε is small enough. It is possible for a bound state to appear even when this condition is violated, however it needs to happen for large ε. In such case Theorem 3.3.2 gives us no information regarding its existence.

Chapter 4

Conclusions

In this thesis we investigated the influence of the weakly coupled complex-valued potential on the spectrum of a free particle Hamiltonian, especially on the emergence of the bound state from the threshold of the essential spectrum. Our results guarantee presence and uniqueness of this bound state in the case when the coupling is weak enough. Our result is consistent with the bound on the magnitude of the eigenvalue obtained in [7].
We reformulate the problem using the Birman-Schwinger principle which gives us an interesting insight into the problem. We can immediately see that there needs to be a singularity in the resolvent function of H_0 for $\lambda \to 0$, otherwise the bound state could not appear. Indeed, if operator K_λ with integral kernel defined in (3.2) were bounded, norm of εK_λ could be arbitrary small with ε going to 0 and therefore it would be impossible for it to posses -1 as an eigenvalue. (We recall that the norm of an operator needs to be larger than or equal to absolute value of any of its eigenvalues.)
We come to the conclusion that the singularity is responsible for the existence of the bound state. Nevertheless the resolvent function has the singularity only in dimensions one and two. Therefore it could be expected that the bound state should also appear in two dimensional case while in more dimensions it could not be ensured for ε small enough, just as it happens in the selfadjoint case. Generalisation of our results to dimension two seems as a natural topic of a further research.

Chapter 5

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