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Klasifikace tříd řešitelných Lieových algeber s vícerozměrnými centry

Classification of classes of solvable Lie algebras with higher dimensional centres

VÝZKUMNÝ ÚKOL

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Bc. Jindřich Prokop

Název práce: **Klasifikace tříd řešitelných Lieových algeber s vícerozměrnými centry**
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Abstrakt: Práce se zabývá hledáním řešitelných rozšíření zvolené posloupnosti nilpotentních Lieových algeber sudé dimenze. V první části je prozkoumán nejjednodušší, tj. šestirozměrný případ, ve druhé je pak rozebrán případ obecný. Cílem je jednak přispět k programu klasifikace řešitelných Lieových algeber vyšších dimenzí a jednak poodhalit možné vlastnosti řešitelných Lieových algeber obecně.
Klíčová slova: nilpotentní Lieovy algebry, řešitelné Lieovy algebry, řešitelná rozšíření, klasifikace řešitelných Lieových algeber

Title: **Classification of classes of solvable Lie algebras with higher dimensional centres**
Author: Bc. Jindřich Prokop
Abstract: The aim of this work is to find the solvable extensions of the given series of nilpotent Lie algebras of even dimension. The first part focuses on the simplest case, that is six-dimensional one. The second part generalizes the results of the former to arbitrary even dimension of the chosen algebra. One of the aims is the contribution towards the classification of solvable Lie algebras of higher dimensions, the other is to reveal at least a little some properties of solvable Lie algebras in general.
Key words: nilpotent Lie algebras, solvable Lie algebras, solvable extensions, classification of solvable Lie algebras

Contents

Introduction

Lie algebras play an important role in numerous fields of physics as well as of mathematics. They may emerge as a tangent spaces of Lie groups describing symmetries of some problems, also can they be found as abstract Lie algebras in others. In the first case it usually more efficient to work with corresponding Lie algebra rather than with given Lie group. When one encounters a Lie algebra identification of this algebra as isomorphic to some known type is one of the first concerns as it can help solving the problem the algebra is connected with. Hence, classifying Lie algebras is useful to many practical problems of mathematics and theoretical physics. According to Levi's theorem every Lie algebra $\mathfrak{g}$ is decomposable into the semidirect sum of its radical $\mathfrak{r}$ and a semisimple Lie algebra $\mathfrak{s}$ usually called Levi factor, that is

$$\mathfrak{g} = \mathfrak{s} \oplus\!\!\!\!\!\!\;\;\supset \mathfrak{r}, \qquad ([\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r}).$$

Via Levi's decomposition the problem of classifying all Lie algebras is simplified to classification of all semisimple and solvable ones. All complex semisimple Lie algebras were first classified by W. Killing in 1890. This was redone by E. Cartan in 1894 with extension to real semisimple algebras. Although this was done more than a century ago, there is no known way of classifying solvable or even nilpotent ones. Some results have been achieved in low dimension more or less using brute force. This approach stops being feasible in relatively low dimensions – the solvable algebras are classified up to dimension 6, the nilpotent ones up to dimensions 8. Thus a different approach emerged in this endeavour, namely construction of series of nilpotent algebras of arbitrarily large dimension and with similar structure, followed by finding of all of solvable extensions of every element in the series. One such series is constructed in below and all the solvable extensions are found as well.

The first chapter covers some preliminaries needed for the following ones. Also, the chosen series of nilpotent algebras is defined there. Second chapter investigates the simplest case of these algebras, that is the one with the lowest dimension to provide some insight for more complex cases. The solvable extensions are found there as well. Third chapter generalizes the results of the preceding one to the generic dimensions of the nilradical. With the exception of the most degenerate cases, the generalized Casimir invariants of the solvable extensions by one element are found in chapter 4.

Chapter 1

Introduction

1.1 Series of Nilradicals

Since the aim of this work is to find all solvable extensions of given nilpotent algebras, the first thing to do is to choose the nilpotent algebras, that is the nilradicals of the to be found extensions. The series of even-dimensional algebras $\mathfrak{r}_{2k} \equiv \mathrm{span}\{e_i\}_1^{2k}$ with commutation relations given as

$$[e_i, e_{2k}] = e_{i-1}, \qquad \forall i \in \{3, \dots, 2k-1\} \tag{1.1}$$

with $k \geq 3$ has been chosen.

1.2 Theoretical Background

Basic knowledge of the theory of Lie algebras is assumed. A good reference for that is [1]. This work uses the methods described and utilized in [5], [2], [3] and [4].

Let $\mathfrak{r}$ be a nilpotent Lie algebra with basis $(e_i)_1^n$ and the structure coefficients given by relations

$$[e_i, e_j] = {c_{ij}}^k e_k. \tag{1.2}$$

Note that Einstein summation convention is assumed here as well as in the remainder of the text. If we want $\mathfrak{s} \equiv \mathrm{span}(e_i, f_j)_{i=1,j=1}^{n,m}$ to be a solvable Lie algebra with nilradical $\mathfrak{r}$ the commutation relations defining $\mathfrak{s}$ must be of the form

$$\begin{aligned} [e_i, f_j] &= {(\mathbb{D}_j)_i}^k e_k, \qquad && \forall i \in \{1, \dots, n\},\ \forall j \in \{1, \dots, m\} \\ [f_i, f_j] &= {\gamma_{ij}}^k e_k, && \forall i, j \in \{1, \dots, m\} \end{aligned} \tag{1.3}$$

to ensure the condition $[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{r}$ holds and the Jacobi identities must be satisfied. This, however does not suffice as adding an element f_k to $\mathfrak{r}$ such that

$\mathfrak{n} \equiv \text{span}(e_i, f_k)$ satisfies the preceding conditions while remaining nilpotent even after addition of f_k yields a larger nilradical – the extension itself. The validity of Jacobi identities for (f_k, e_i, e_j) for every $i, j \in \{1, \dots, n\}$ is equivalent to $\mathbb{D}_k$ defined above being a derivation of $\mathfrak{r}$. Moreover, $\mathbb{D}_k$ is an outer derivation of $\mathfrak{r}$ if and only if $\mathfrak{n}$ is a solvable extension of $\mathfrak{r}$ with $\mathfrak{r}$ as its nilradical. This allows for seeking the solvable extensions of a given nilpotent algebra $\mathfrak{r}$ by one element in the following way. We first find the automorphisms and the inner and generic derivations of $\mathfrak{r}$. Then we find the classes of outer derivations w. r. t. addition of arbitrary inner derivation, multiplication by a number from the field over which we assume the algebra $\mathfrak{r}$ and conjugation by an automorphism of $\mathfrak{r}$. There is one to one correspondence between these classes and the non-isomorphic solvable extensions of $\mathfrak{r}$ by one element. Obviously, two derivations from the same class correspond to two isomorphic extensions. To construct extensions by more than one element we must observe the conditions mentioned above for all the added vectors $f_1, \dots, f_m$. This means looking for outer derivations that are compatible, i.e. the corresponding extending elements must not disrupt Jacobi identities, nor may they yield an extension that is either of lower dimension than $n + m$ or its nilradical is larger than $\mathfrak{r}$. The first condition translates to derivatives as

$$[\mathbb{D}_i, \mathbb{D}_j] \in \mathfrak{Inn}, \qquad \forall i, j \in \{1, \dots, m\}, \tag{1.4}$$

where $\mathfrak{Inn}$ denotes the algebra of inner derivations of $\mathfrak{r}$, the second translates as the requirement that all the derivation must be linearly nil-independent (only the trivial linear combination of the derivations yields a nilpotent matrix). This is neatly described in subsection 2.2.3 for the case of adding two elements, while the generalization to the addition of arbitrary number of elements is straightforward.

Let $\mathfrak{g}$ be an arbitrary Lie algebra. Then we define the derived series of $\mathfrak{g}$ as

$$\begin{aligned} D^0\mathfrak{g} &:= \mathfrak{g}, \\ D^k\mathfrak{g} &:= [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}], \qquad k \geq 1, \end{aligned} \tag{1.5}$$

the lower central series of $\mathfrak{g}$ as

$$\begin{aligned} C^0\mathfrak{g} &:= \mathfrak{g}, \\ C^k\mathfrak{g} &:= [C^{k-1}\mathfrak{g}, \mathfrak{g}], \qquad k \geq 1, \end{aligned} \tag{1.6}$$

and the upper central series of $\mathfrak{g}$ as

$$\begin{aligned} \mathfrak{z}^1 &:= \mathfrak{z} \equiv \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}\ [x, y] = 0\}, \\ \mathfrak{z}^k &:= \left\{x \in \mathfrak{g}\ \middle| \forall y \in \mathfrak{g}\ [x, y] \in \mathfrak{z}^{k-1}\right\}, \qquad k \geq 2. \end{aligned} \tag{1.7}$$

This notation is used throughout the text. Any element $\mathfrak{e}$ of all of these series is an invariant ideal of $\mathfrak{g}$, that is, it is an ideal of $\mathfrak{g}$ and its image under any automorphism from $\mathfrak{Aut}(\mathfrak{g})$ is equal to $\mathfrak{e}$ itself.

Chapter 2

Solvable Extensions of Six-dimensional Nilradical

To get some intuition for more general case we first find all solvable extensions of $\mathfrak{r} \equiv \mathfrak{r}_6 = \mathrm{span}(e_1, \dots, e_6)$ given by relations

$$[e_i, e_6] = e_{i-2}, \qquad \forall i \in \{3, 4, 5\}. \tag{2.1}$$

Towards this aim we will use methods described in previous chapter; firstly we must examine the structure of $\mathfrak{r}_6$ itself. This is done in the first section. The second one is devoted to finding all solvable extensions of $\mathfrak{r}_6$.

2.1 Properties of $\mathfrak{r}_6$

Basic properties of $\mathfrak{r}_6$ are found in this section. The inner derivations are listed; it is shown that the automorphisms and as a consequence the derivations are of upper triangular form in the canonical basis. The form of generic outer derivation and generic automorphisms are explicitly listed as they will be needed in the next section.

2.1.1 Inner Derivations

The inner derivations are easily found to be

$$
\begin{aligned}
&\mathrm{ad}_{e_1} = 0, && \mathrm{ad}_{e_2} = 0,\\
&\mathrm{ad}_{e_3} = \begin{pmatrix} 0&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, && \mathrm{ad}_{e_4} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},\\
&\mathrm{ad}_{e_5} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, && \mathrm{ad}_{e_6} = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}.
\end{aligned} \tag{2.2}
$$

2.1.2 Derived Series, Flag of Ideals

It is readily seen that

$$
\begin{gathered}
D^1\mathfrak{r}_6 = C^1\mathfrak{r}_6 = [\mathfrak{r}_6, \mathfrak{r}_6] = \mathrm{span}(e_1, e_2, e_3),\\
C^2\mathfrak{r}_6 = \mathrm{span}(e_1), \quad C^3\mathfrak{r}_6 = 0, \quad D^2\mathfrak{r}_6 = 0,
\end{gathered}
$$

and

$$\mathfrak{z} = \mathrm{span}(e_1, e_2), \quad \mathfrak{z}^2 = \mathrm{span}(e_1, e_2, e_3, e_4), \quad \mathfrak{z}^3 = \mathfrak{r}_6.$$

If we can prove that $\mathrm{span}(e_1, \dots, e_5)$ is an invariant ideal, we will know that all automorphisms of $\mathfrak{r}_6$ are represented by upper-triangular matrices. To show that it indeed is an invariant ideal, we consider the centralizer of $D^1\mathfrak{r}_6$:

$$
\begin{aligned}
Z_{\mathfrak{r}}(D^1\mathfrak{r}_6) &= \{x \in \mathfrak{r}_6 | \forall y \in D^1\mathfrak{r}_6\ [x, y] = 0\} = \{x \in \mathfrak{r}_6 | \forall k = 1, 2, 3 [x, e_k] = 0\}\\
&= \mathrm{span}(e_1, \dots, e_5);
\end{aligned}
$$

hence we have the flag of invariant ideals:

$$0 \subsetneq C^2\mathfrak{r}_6 \subsetneq \mathfrak{z} \subsetneq D^1\mathfrak{r}_6 \subsetneq \mathfrak{z}^2 \subsetneq Z_{\mathfrak{r}}(D^1\mathfrak{r}_6) \subsetneq \mathfrak{r}_6.$$

Therefore, any automorphism of $\mathfrak{r}_6$ is an upper-triangular matrix in basis $(e_i)_1^6$.

2.1.3 Automorphisms

We exploit the fact that every upper-triangular matrix admits a decomposition into the product of the diagonal and unitriangular matrix $\Phi = \Phi_{diag} \circ \Phi_{uni}$ and that automorphisms form a group to restrict ourselves only to diagonal and unitriangular automorphisms. Consider a diagonal automorphism Φ of $\mathfrak{r}_6$ and assume that

$$\Phi e_6 = \alpha e_6, \quad \Phi e_5 = \beta e_5, \quad \Phi e_4 = \gamma e_4.$$

Then by the commutation relations (2.1) we get

$$\begin{aligned} \Phi e_3 &= \Phi[e_5, e_6] = [\Phi e_5, \Phi e_6] = \alpha\beta[e_5, e_6] = \alpha\beta e_3, \\ \Phi e_2 &= \Phi[e_4, e_6] = [\Phi e_4, \Phi e_6] = \alpha\gamma[e_5, e_6] = \alpha\gamma e_3, \\ \Phi e_1 &= \Phi[e_3, e_6] = [\Phi e_3, \Phi e_6] = \alpha^2\beta[e_3, e_6] = \alpha^2\beta e_3, \end{aligned}$$

and the generic diagonal automorphism is of the form

$$\Phi_{diag} = \begin{pmatrix} \alpha^2\beta & 0 & 0 & 0 & 0 & 0 \\ & \alpha\gamma & 0 & 0 & 0 & 0 \\ & & \alpha\beta & 0 & 0 & 0 \\ & & & \gamma & 0 & 0 \\ & & & & \beta & 0 \\ & & & & & \alpha \end{pmatrix}. \tag{2.3}$$

Now assume that Φ is unitriangular and that

$$\Phi e_6 = \sum_{i=1}^{5} \alpha_i e_i + e_6, \quad \Phi e_5 = \sum_{i=1}^{4} \beta_i e_i + e_5, \Phi e_4 = \sum_{i=1}^{3} \gamma_i e_i + e_4.$$

Then again using relations (2.1) we obtain

$$\begin{aligned} \Phi e_3 = \Phi[e_5, e_6] &= \left[\sum_{i=1}^{4} \beta_i e_i + e_5, \sum_{i=1}^{5} \alpha_i e_i + e_6\right] = \sum_{i=1}^{2} \beta_{i+2} e_i + e_3 \\ \Phi e_2 = \Phi[e_4, e_6] &= \left[\sum_{i=1}^{3} \gamma_i e_i + e_4, \sum_{i=1}^{5} \alpha_i e_i + e_6\right] = \gamma_3 e_1 + e_2 \\ \Phi e_1 &= \Phi[e_3, e_6] = [\Phi e_3, \Phi e_6] = e_3. \end{aligned}$$

Hence the generic form of a unitriangular automorphism is

$$\Phi_{uni} = \begin{pmatrix} 1 & \gamma_3 & \beta_3 & \gamma_1 & \beta_1 & \alpha_1 \\ & 1 & \beta_4 & \gamma_2 & \beta_2 & \alpha_2 \\ & & 1 & \gamma_3 & \beta_3 & \alpha_3 \\ & & & 1 & \beta_4 & \alpha_4 \\ & & & & 1 & \alpha_5 \\ & & & & & 1 \end{pmatrix}. \tag{2.4}$$

2.1.4 Generic Outer Derivation

We obtain generic outer derivation by differentiating generic automorphism at the point $\mathbb{1}$. Evidently $\Phi = \Phi_{diag} \circ \Phi_{uni} = \mathbb{1} \iff \alpha = \beta = \gamma = 1$ and $\alpha_i = \beta_j = \gamma_k = 0$. Thus we vary these parameters around this value, where we denote the variation of the original parameter $(\alpha, \alpha_i, ...)$ with the corresponding latin letter, e. g. $\alpha = 1 + a$. From the resulting matrix we subtract $\mathbb{1}$ to obtain a generic derivation in the form

$$\begin{pmatrix} 2a+b & c_3 & b_3 & c_1 & b_1 & a_1 \\ & a+c & b_4 & c_2 & b_2 & a_2 \\ & & a+b & c_3 & b_3 & a_3 \\ & & & c & b_4 & a_4 \\ & & & & b & a_5 \\ & & & & & a \end{pmatrix}. \tag{2.5}$$

We may eliminate parameters a_1, a_2, a_3 by subtracting multiples of inner derivations $\mathrm{ad}_{e_3}, \mathrm{ad}_{e_4}, \mathrm{ad}_{e_5}$ given in (2.2) and without loss of generality we may eliminate b_3 by subtracting a multiple of ad_{e_6}; this changes $\gamma_2 \to \gamma_2 - \beta_3$ so we rename this parameter accordingly to obtain the generic form of the outer derivation of $\mathfrak{r}_6$:

$$D := \begin{pmatrix} 2a+b & c_3 & 0 & c_1 & b_1 & 0 \\ & a+c & b_4 & c_2 & b_2 & 0 \\ & & a+b & c_3 & 0 & 0 \\ & & & c & b_4 & a_4 \\ & & & & b & a_5 \\ & & & & & a \end{pmatrix}. \tag{2.6}$$

2.2 Extensions

We are interested in equivalence classes of outer derivations w. r. t. conjugation by arbitrary automorphisms both of the diagonal form (2.3) and of the unitriangular one (2.4), multiplication by an arbitrary constant and addition of linear combinations of inner derivations. These classes correspond to the extensions by one vector. We firstly resolve under which conditions it is possible to simplify the derivations, then we find all nonequivalent classes. The extensions by two vectors, i. e. the eight-dimensional extensions are found in the third section, and in the last section we show the unique nine-dimensional extension. Henceforth, the equalities are understood mod inner derivations, where appropriate.

2.2.1 Conditions for Complete Diagonalizability

Provided that a, b, c are nonequal, we can always diagonalize the lower 3×3 submatrix in the following manner. By setting $\beta_4 := \frac{b_4}{b-c}$, $\alpha_5 := \frac{a_5}{a-b}$, $\alpha_4 := \frac{\frac{a_5 b_4}{a-b}+a_4}{a-c}$ and omitting (i. e. setting them to 0) the other parameters in (2.4) we obtain the automorphism

$$\mathbb{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & \frac{b_4}{b-c} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & \frac{b_4}{b-c} & \frac{\frac{a_5 b_4}{a-b}+a_4}{a-c} \\ & & & & 1 & \frac{a_5}{a-b} \\ & & & & & 1 \end{pmatrix}. \tag{2.7}$$

Then $\mathbb{A}^{-1} D \mathbb{A}$ yields

$$\begin{pmatrix} 2a+b & c_3 & 0 & c_1 & \frac{c_1 b_4 + b_1 b - b_1 c}{b-c} & 0 \\ & a+c & 0 & \frac{c_2 b - c_2 c - b_4 c_3}{b-c} & \frac{c_2 b_4 b - c_2 b_4 c + b_2 b^2 - 2 b_2 b c + b_2 c^2 - b_4^2 c_3}{(b-c)^2} & 0 \\ & & a+b & c_3 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}. \tag{2.8}$$

Here we adopt the convention of denoting the obtained components with the same symbol as before, but with tilde. E.g. (2.8) becomes

$$\begin{pmatrix} 2a+b & c_3 & 0 & c_1 & \tilde{b}_1 & 0 \\ & a+c & 0 & \tilde{c}_2 & \tilde{b}_2 & 0 \\ & & a+b & c_3 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}.$$

If $c \neq a+b$ we are able to transform out c_3 by setting $\gamma_3 := \frac{c_3}{c-a-b}$ in (2.4). The resulting automorphism is

$$\mathbb{B} := \begin{pmatrix} 1 & \frac{c_3}{c-a-b} & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & \frac{c_3}{c-a-b} & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \tag{2.9}$$

and the derivation is transformed to

$$\mathbb{B}^{-1}\mathbb{A}^{-1}D\mathbb{A}\mathbb{B} = \begin{pmatrix} 2a+b & 0 & 0 & \tilde{c}_1 & \tilde{b}_1 & 0 \\ & a+c & 0 & \tilde{c}_2 & \tilde{b}_2 & 0 \\ & & a+b & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}.$$

In order to completely diagonalize D, $a \neq 0$, $b \neq a+c$, $c \neq 2a+b$ must be satisfied. These conditions allow us to use the automorphism

$$\mathbb{C} := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{2a+b-c} & \frac{-\tilde{b}_1}{2a} & 0 \\ & 1 & 0 & \frac{-\tilde{c}_2}{a} & \frac{\tilde{b}_2}{a+c-b} & 0 \\ & & 1 & 0 & & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}, \tag{2.10}$$

to obtain

$$\mathbb{C}^{-1}\mathbb{B}^{-1}\mathbb{A}^{-1}D\mathbb{A}\mathbb{B}\mathbb{C} = \begin{pmatrix} 2a+b & 0 & 0 & 0 & 0 & 0 \\ & a+c & 0 & 0 & 0 & 0 \\ & & a+b & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}.$$

2.2.2 Outer Derivation Classes

As was shown in the preceding, the possibility of simplifying, or rather transforming one derivation to another via permitted operations depends on the values of parameters a, b, c, a_i, b_j, c_k. Thus, the problem is divided into several cases in the following way. The first level of division corresponds to the lower 3×3 submatrix, the second one determines whether c_3 vanishes or not, and the possibility of elimination of the remaining parameters is determined by the third level. This copies the general approach, which is to simplify the lower 3×3 submatrix as much as possible via an automorphism denoted by $\mathbb{A}_i$; the next step is to eliminate c_3, if possible, via $\mathbb{B}$. Via the third unitriangular automorphism $\mathbb{C}_j$ we maximally simplify block containing b_1, b_2, c_1, c_2. The last step is scaling the remaining parameters to $(\pm)1$ if possible.

1. $a \neq b \neq c \neq a$
With this condition satisfied we can diagonalize the lower 3×3 submatrix by the automorphism $\mathbb{A}$ described in the preceding subsection.

1.1. $c \neq a + b$
This condition allows for the elimination of c_3 using the automorphism $\mathbb{B}$ given above.

1.1.1. $a \neq 0$, $c \neq a + b$, $b \neq a + c$, $c \neq 2a + b$
With all the conditions satisfied we can completely diagonalize D. Since we already have $a \neq 0$, we can divide by a to obtain

$$D_1(\tilde{b}, \tilde{c}) = \begin{pmatrix} 2+\tilde{b} & 0 & 0 & 0 & 0 & 0 \\ & 1+\tilde{c} & 0 & 0 & 0 & 0 \\ & & 1+\tilde{b} & 0 & 0 & 0 \\ & & & \tilde{c} & 0 & 0 \\ & & & & \tilde{b} & 0 \\ & & & & & 1 \end{pmatrix}.$$

Note, that this class includes all diagonal derivations with $a \neq 0$. Hence, these classes will be ommited in the following discussion.
1.1.2. $a = 0$ $\quad (\implies b \neq a + c,\ c \neq 2a + b)$
Automorphisms $\mathbb{A}$ and $\mathbb{B}$ remain the same as in previous case, eliminating c_3, b_4, a_4, a_5. With $a = 0$ we set $\gamma_2 := 0, \beta_1 := 0$, remaining parameters as in (2.10). This choice leads to

$$\mathbb{C}_2 := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{b-c} & 0 & 0 \\ & 1 & 0 & 0 & \frac{\tilde{b}_2}{c-b} & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

$$\mathbb{C}_2^{-1}\mathbb{B}^{-1}\mathbb{A}^{-1}D\mathbb{A}\mathbb{B}\mathbb{C}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \tilde{b}_1 & 0 \\ & \tilde{c} & 0 & \tilde{c}_2 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \tilde{c} & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

If $\tilde{c}_2 \neq 0$ we are able to transform it to one by setting $\alpha = \tilde{c}_2$ in (2.3). This yields

$$D_2(\tilde{b}_1, \tilde{c}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \tilde{b}_1 & 0 \\ & \tilde{c} & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \tilde{c} & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

If $\tilde{c}_2 = 0$, we arrive to

$$D_3^{(\sigma)}(\tilde{c}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \sigma & 0 \\ & \tilde{c} & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \tilde{c} & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}, \quad \sigma \in \{0, \pm 1\},$$

where $\pm$ sign only refers to the case $\mathbb{F} = \mathbb{R}$.

1.1.3. $b = a + c \quad (\Longrightarrow\ c \neq 2a + b, a \neq 0)$

As in the previous case we only have to redefine $\mathbb{C}$:

$$\mathbb{C}_3 := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{3a} & \frac{-\tilde{b}_1}{2a} & 0 \\ & 1 & 0 & \frac{-\tilde{c}_2}{a} & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

we divide D by a and with the help of diagonal automorphism we arrive to

$$D_4(\tilde{c}) = \begin{pmatrix} 3+\tilde{c} & 0 & 0 & 0 & 0 & 0 \\ & 1+\tilde{c} & 0 & 0 & 1 & 0 \\ & & 2+\tilde{c} & 0 & 0 & 0 \\ & & & \tilde{c} & 0 & 0 \\ & & & & 1+\tilde{c} & 0 \\ & & & & & 1 \end{pmatrix}.$$

1.1.4. $c = 2a + b \quad (\Longrightarrow\ b \neq a + c, a \neq 0)$

$$\mathbb{C}_4 := \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{-\tilde{b}_1}{2a} & 0 \\ & 1 & 0 & \frac{-\tilde{c}_2}{a} & \frac{\tilde{b}_2}{3a} & 0 \\ & & 1 & 0 & & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

we divide D by a and with the help of diagonal automorphism we arrive to

$$D_5(\tilde{b}) = \begin{pmatrix} 2+\tilde{b} & 0 & 0 & 1 & 0 & 0 \\ & 3+\tilde{b} & 0 & 0 & 0 & 0 \\ & & 1+\tilde{b} & 0 & 0 & 0 \\ & & & 2+\tilde{b} & 0 & 0 \\ & & & & \tilde{b} & 0 \\ & & & & & 1 \end{pmatrix}.$$

1.2. $c = a + b \quad (\implies a \neq 0, b \neq a + c,\ c \neq 2a + b)$
In this case, it is impossible to transform out c_3. Nevertheless, using $\mathbb{A}$ and $\mathbb{C}$ we obtain

$$D_6(\tilde{b}) = \begin{pmatrix} 2+\tilde{b} & 1 & 0 & 0 & 0 & 0 \\ & 2+\tilde{b} & 0 & 0 & 0 & 0 \\ & & 1+\tilde{b} & 1 & 0 & 0 \\ & & & 1+\tilde{b} & 0 & 0 \\ & & & & \tilde{b} & 0 \\ & & & & & 1 \end{pmatrix}.$$

2. $a = b \neq c$
We eliminate a_4 and b_4 using the automorphism

$$\mathbb{A}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & \frac{b_4}{a-c} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & \frac{b_4}{a-c} & \frac{a_4 - \frac{a_5 b_4}{a-c}}{a-c} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

2.1. $c \neq 2a$

2.1.1. $a \neq 0, c \neq 0, c \neq 3a$

$$\mathbb{C}^{-1}\mathbb{B}^{-1}\mathbb{A}_2^{-1} D \mathbb{A}_2 \mathbb{B}\mathbb{C} = \begin{pmatrix} 3a & 0 & 0 & 0 & 0 & 0 \\ & a+c & 0 & 0 & 0 & 0 \\ & & 2a & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & a & a_5 \\ & & & & & a \end{pmatrix}$$

Dividing by a and using diagonal automorphism we get

$$D_7(\tilde{c}) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ & 1+\tilde{c} & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & \tilde{c} & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

2.1.2. $c \neq 0 \quad (\implies a \neq 0, c \neq 3a)$

$$D_8 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

2.1.3. $c = 3a \quad (\implies a \neq 0 \neq c)$

$$D_9 = \begin{pmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ & 4 & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 3 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

2.1.4. $a = 0 \quad (\implies c \neq 3a, c \neq 0)$
In case of $\tilde{c}_2 \neq 0$, we obtain two non-equivalent one-parameter sets:

$$D_{10}(\tilde{b}_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{b}_1 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \; D_{11}(\tilde{b}_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{b}_1 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

In the second case, we are able to scale $\tilde{b}_1$ to 1, resp. ± 1 for field $\mathbb{C}$, resp. $\mathbb{R}$:

$$D_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & \pm 1 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \; D_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & \pm 1 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

if it is not already zero. $\tilde{b}_1$ being zero leads to non-equivalent cases:

$$D_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \ D_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

2.2. $c = 2a \quad (\implies a \neq 0 \neq c, c \neq 3a)$

$$D_{16} = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & 0 \\ & & 2 & 1 & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

3. $a \neq b = c$

The automorphism

$$\mathbb{A}_3 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & \frac{\frac{a_5 b_4}{a-b} + a_4}{a-b} \\ & & & & 1 & \frac{a_5}{a-b} \\ & & & & & 1 \end{pmatrix}$$

transforms the matrix of the derivation into the form

$$\begin{pmatrix} 2a+b & \tilde{c}_3 & 0 & \tilde{c}_1 & \tilde{b}_1 & 0 \\ & a+b & \tilde{b}_4 & \tilde{c}_2 & \tilde{b}_2 & 0 \\ & & a+b & \tilde{c}_3 & 0 & 0 \\ & & & b & \tilde{b}_4 & 0 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}.$$

3.1. $a \neq 0$

To transform out the remaining parameters we use the automorphism $\mathbb{B}$ as in (2.9) and $\mathbb{C}_3$, where $\mathbb{C}_3$ is obtained from (2.4) by setting $\gamma_1 = \frac{-\tilde{c}_1}{2a}$, $\gamma_2 = \frac{-\tilde{c}_2}{a}$, $\beta_1 = \frac{-2a\tilde{b}_1 - \tilde{c}_1\tilde{b}_4}{4a^2}$, $\beta_2 = \frac{-a\tilde{b}_2 - \tilde{c}_2\tilde{b}_4}{a^2}$. This along with the division by a

and conjugation by suitable diagonal automorphism, yields

$$D_{17}(\tilde{b}) = \begin{pmatrix} 2+\tilde{b} & 0 & 0 & 0 & 0 & 0 \\ & 1+\tilde{b} & 1 & 0 & 0 & 0 \\ & & 1+\tilde{b} & 0 & 0 & 0 \\ & & & \tilde{b} & 1 & 0 \\ & & & & \tilde{b} & 0 \\ & & & & & 1 \end{pmatrix}.$$

3.2. $a = 0$

3.2.1. $b_4 \neq 0$

We use the same $\mathbb{A}_3$ as in the previous case. Setting $\gamma_1 := \frac{\tilde{b}_1}{b_4}$, $\gamma_2 := \frac{\tilde{b}_2}{b_4}$ and $\gamma_3 := \frac{-\tilde{c}_2}{b_4}$ in (2.4) we define the automorphism

$$\mathbb{C}_{31} := \begin{pmatrix} 1 & 0 & 0 & \frac{\tilde{b}_1}{b_4} & 0 & 0 \\ & 1 & 0 & \frac{\tilde{b}_2}{b_4} & 0 & 0 \\ & & 1 & \frac{-\tilde{c}_2}{b_4} & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

which allows the elimination of $\tilde{c}_2$, $\tilde{b}_1$, $\tilde{b}_2$. In this case, it is impossible to eliminate $\tilde{c}_1$ and it is in general case impossible to set it to ± 1 along with c_3 and b_4. Thus, we obtain the following cases

$$D_{18}(\tilde{c}_1) = \begin{pmatrix} 1 & 1 & 0 & \tilde{c}_1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$D_{19} = \begin{pmatrix} 1 & 0 & 0 & \pm 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}, \quad D_{20} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

3.2.2. $b_4 = 0$

a) $c_3 \neq 0$. Then we set $\beta_2 := -\frac{b_1}{c_3}, \beta_3 := \frac{c_1}{c_3}, \beta_4 := \frac{c_2}{c_3}$ to get the cases

$$D_{21} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & \pm 1 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}, \quad D_{22} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

b) $c_3 = 0$. In the case of $c_2 \neq 0$ we pick the automorphism with $\beta_4 := -\frac{b_2}{c_2}, \gamma_3 := \frac{c_1}{c_2}$ eliminating c_1 and b_2 only to fall back to D_2 with $\tilde{c} = 1$. If on the other hand $c_2 = 0$ we are able to eliminate only $\tilde{b}_1$ in the case of $\tilde{c}_1 \neq 0$ or $\tilde{b}_2 \neq 0$, the condition being $\beta_4 \tilde{c}_1 - \gamma_3 \tilde{b}_2 = \tilde{b}_1$. Thus we come to the cases

$$D_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}, \quad D_{24} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

and since the case $\tilde{c}_1 = \tilde{b}_2 = 0$ is covered by D_2 with $\tilde{c} = 1$, the only remaining case

$$D_{25} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

4. $a = c \neq b$

Via the automorphism

$$\mathbb{A}_4 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & \frac{b_4}{b-a} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & \frac{b_4}{b-a} & 0 \\ & & & & 1 & \frac{a_5}{a-b} \\ & & & & & 1 \end{pmatrix},$$

we eliminate parameters b_4 and a_5 thus obtaining the derivation in the form

$$\begin{pmatrix} 2a+b & \tilde{c}_3 & 0 & \tilde{c}_1 & \tilde{b}_1 & 0 \\ & 2a & 0 & \tilde{c}_2 & \tilde{b}_2 & 0 \\ & & a+b & \tilde{c}_3 & 0 & 0 \\ & & & a & 0 & \tilde{a}_4 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}. \tag{2.11}$$

4.1. $b \neq 0$
Applying the automorphism

$$\mathbb{B}_4 := \begin{pmatrix} 1 & \frac{\tilde{c}_3}{b} & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & \frac{\tilde{c}_3}{b} & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$$

to the derivation (2.11) yields

$$\begin{pmatrix} 2a+b & 0 & 0 & \tilde{c}_1 & \tilde{b}_1 & 0 \\ & 2a & 0 & \tilde{c}_2 & \tilde{b}_2 & 0 \\ & & a+b & 0 & 0 & 0 \\ & & & a & 0 & \tilde{a}_4 \\ & & & & b & 0 \\ & & & & & a \end{pmatrix}.$$

4.1.1. $a \neq 0, 2a \neq b, b \neq -a$
We use the automorphism

$$\mathbb{C}_{41} := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{a+b} & \frac{\tilde{b}_1}{2a} & 0 \\ & 1 & 0 & \frac{-\tilde{c}_2}{a} & \frac{\tilde{b}_2}{b-2a} & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

to obtain

$$D_{26}(\tilde{b}) = \begin{pmatrix} 2+\tilde{b} & 0 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 & 0 \\ & & 1+\tilde{b} & 0 & 0 & 0 \\ & & & 1 & 0 & 1 \\ & & & & \tilde{b} & 0 \\ & & & & & 1 \end{pmatrix}.$$

4.1.2. $b = -a \quad (\implies a \neq 0, 2a \neq b)$
The implication holds, since we seek non-nilpotent derivations only. Instead of $\mathbb{C}_{41}$ we use

$$\mathbb{C}_{42} := \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\tilde{b}_1}{2a} & 0 \\ & 1 & 0 & \frac{-\tilde{c}_2}{a} & \frac{\tilde{b}_2}{-3a} & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

and we get the resulting derivation

$$D_{27} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ & 2 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 1 \\ & & & & -1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

4.1.3. $b = 2a \quad (\implies a \neq 0, b \neq -a)$
The implication holds, since we seek non-nilpotent derivations only. Instead of $\mathbb{C}_{41}$ we use

$$\mathbb{C}_{43} := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{a+b} & \frac{\tilde{b}_1}{2a} & 0 \\ & 1 & 0 & \frac{-\tilde{c}_2}{a} & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

and we get the resulting derivation

$$D_{28} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 1 & 0 \\ & & 3 & 0 & 0 & 0 \\ & & & 1 & 0 & 1 \\ & & & & 2 & 0 \\ & & & & & 1 \end{pmatrix}.$$

4.1.4. $a = 0 \quad (\implies b \neq -a, 2a)$
Instead of $\mathbb{C}_{41}$ we use

$$\mathbb{C}_{44} := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{b} & 0 & 0 \\ & 1 & 0 & 0 & \frac{\tilde{b}_2}{b} & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

and obtain the resulting derivations

$$D_{29}(\tilde{b}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \tilde{b}_1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$D_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 & \pm 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}, \; D_{31} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

4.2. $b = 2a$

We cannot transform out c_3, but c_1, c_2, b_1, b_2 are eliminated using the automorphism

$$\mathbb{C}_{45} := \begin{pmatrix} 1 & 0 & 0 & \frac{\tilde{c}_3\tilde{c}_2 - \tilde{c}_1 a}{a^2} & \frac{\tilde{c}_3\tilde{b}_2 - 2\tilde{b}_1 a}{4a^2} & 0 \\ & 1 & 0 & -\frac{\tilde{c}_2}{a} & -\frac{\tilde{b}_2}{2a} & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

We come to the class:

$$D_{32} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 1 \end{pmatrix}.$$

5. $a = c = b$
The case of $a = 0$ is of no interest as we seek non-nilpotent outer derivations. We firstly divide derivation by a thus obtaining:

$$\tilde{D} = \begin{pmatrix} 3 & \tilde{c}_3 & 0 & \tilde{c}_1 & \tilde{b}_1 & 0 \\ & 2 & \tilde{b}_4 & \tilde{c}_2 & \tilde{b}_2 & 0 \\ & & 2 & \tilde{c}_3 & 0 & 0 \\ & & & 1 & \tilde{b}_4 & \tilde{a}_4 \\ & & & & 1 & \tilde{a}_5 \\ & & & & & 1 \end{pmatrix}.$$

Regardless of values of other parameters it is always possible to eliminate $\tilde{c}_3$ using

$$\mathbb{B}_5 := \begin{pmatrix} 1 & -\tilde{c}_3 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & -\tilde{c}_3 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

as well as $\tilde{b}_1, \tilde{b}_2, \tilde{c}_1$ and $\tilde{c}_2$ using

$$\mathbb{C}_5 := \begin{pmatrix} 1 & 0 & 0 & \frac{-\tilde{c}_1}{2} & \frac{2\tilde{b}_1 + \tilde{b}_4\tilde{c}_1}{-4} & 0 \\ & 1 & 0 & -\tilde{c}_2 & -\tilde{b}_2 - \tilde{b}_4\tilde{c}_2 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

In case of $\tilde{b}_4$ or $\tilde{a}_5$ being nonzero, we can eliminate $\tilde{a}_4$ using automorphism (2.4) satisfying the condition $\tilde{b}_4\alpha_5 + \tilde{a}_4 - \beta_4\tilde{a}_5 = 0$. We thus obtain the class with the canonical representative

$$D_{33} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ & 2 & 1 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

The cases of $\tilde{b}_4 = 0$ or $\tilde{a}_5 = 0$ fall back to previous cases.

We have only listed the representatives of the classes of outer derivations. Commutation relations of the corresponding solvable extensions of $\mathfrak{r}_6$ along with the dimensions of the elements of their characteristic series are ommited as they are contained in the section 3.3 (for $k = 3$).

2.2.3 Solvable Eight-dimensional Extensions

All solvable algebras with the nilradical $\mathfrak{r}$ of dimension eight are found in this section. To accomplish this we must seek all non-equivalent compatible outer derivations of $\mathfrak{r} \equiv \mathfrak{r}_6$, that is:

i) The derivations must be linearly nil-independent. Otherwise we would either get seven-dimensional extension or the nilradical would be larger. In the case of upper-triangular matrices this translates as the matrices having linearly independent diagonals.

ii) Commutator of the derivations must lie in $\mathfrak{Inn} \equiv \mathfrak{Inn}(\mathfrak{r})$.

iii) We may apply automorphism from $\mathrm{Aut}(\mathfrak{r})$ simultaneously on both derivations to obtain an equivalent extension.

iv) Assuming that we have two compatible derivations D and d we can choose arbitrary $\tilde{D}, \tilde{d}$ from the plane $\mathrm{span}\{D, d\}$ as long as they keep generating this plane. Of course, extension obtained using D, d is the same as that obtained using $\tilde{D}, \tilde{d}$. It is readily seen that if D and d conform to i) and ii) then $\tilde{D}$ and $\tilde{d}$ conform to these rules as well.

v) We can add an arbitrary inner derivation to D or d.

vi) $[D, d]$ is an ad-preimage of the commutator $[f_1, f_2]$ of the extending vectors f_1 and f_2. In all the cases below requirement ii) leads to $[D, d] = 0$, which implies that in all cases $[f_1, f_2]$ must be in $Z(\mathfrak{r}) = \mathrm{span}\{e_1, e_2\}$. Assume that

$$[f_1, f_2] = \alpha_1 e_1 + \alpha_2 e_2, \qquad [f_1, e_1] = \beta_1 e_1, \qquad [f_1, e_2] = \gamma_1 e_1 + \gamma_2 e_2,$$
$$[f_2, e_1] = \delta_1 e_1, \qquad [f_2, e_2] = \varepsilon_1 e_1 + \varepsilon_2 e_2,$$

and let us set $\beta_2 = \delta_2 = 0$ for the integrity of the notation below. We can add an arbitrary vector from $Z(\mathfrak{r})$ to f_1 and f_2 without changing ad-action of f_1 or f_2 on $\mathfrak{r}$. Thus we try and make $[f_1, f_2]$ as simple as possible by changing $f_1 \to f_1 + A_1 e_1 + A_2 e_2$ and $f_2 \to f_2 + B_1 e_1 + B_2 e_2$. We find that

$$[f_1 + A_i e_i, f_2 + B_j e_j] = (\alpha_\iota - A_1 \delta_\iota - A_2 \varepsilon_\iota + B_1 \beta_\iota + B_2 \gamma_\iota) e_\iota.$$

Thus for given D, d all extensions with nonvanishing $[f_1, f_2]$ are equivalent to the otherwise same extension with $[f_1, f_2] = 0$ if for arbitrary α_1, α_2 we are able to find A_1, B_1, C_1, D_1 such that the equations

$$\alpha_\iota = A_1\delta_\iota + A_2\varepsilon_\iota - B_1\beta_\iota - B_2\gamma_\iota \qquad \iota = 1, 2$$

are satisfied. If $\beta_\iota, \gamma_\iota, \delta_\iota, \varepsilon_\iota = 0$ then ι. equation may not be satisfied and we must discriminate non-equivalent extensions with parameter α_ι. On the other hand if

$$\operatorname{rank}\begin{pmatrix}\beta_1 & \gamma_1 & \delta_1 & \varepsilon_1\\ 0 & \gamma_2 & 0 & \varepsilon_2\end{pmatrix} = 2$$

it is possible to satisfy both equations and thus we may set $[f_1, f_2] := 0$ without any loss of generality.

Throughout this section we denote one derivation D, its parameters with capital letters, the other is denoted d, its parameters with lower case letters:

$$D := \begin{pmatrix} 2A+B & C_3 & 0 & C_1 & B_1 & 0\\ & A+C & B_4 & C_2 & B_2 & 0\\ & & A+B & C_3 & 0 & 0\\ & & & C & B_4 & A_4\\ & & & & B & A_5\\ & & & & & A \end{pmatrix}, \tag{2.12}$$

$$d := \begin{pmatrix} 2a+b & c_3 & 0 & c_1 & b_1 & 0\\ & a+c & b_4 & c_2 & b_2 & 0\\ & & a+b & c_3 & 0 & 0\\ & & & c & b_4 & a_4\\ & & & & b & a_5\\ & & & & & a \end{pmatrix}. \tag{2.13}$$

The lower 3×3 submatrices of D, d are denoted S, s respectively.

Case 1

$A \neq 0$ or $a \neq 0$. Via iv) we know that we can use

$$S := \begin{pmatrix} C & B_4 & A_4\\ & B & A_5\\ & & 2 \end{pmatrix}, \quad s := \begin{pmatrix} c & b_4 & a_4\\ & b & a_5\\ & & 0 \end{pmatrix}.$$

a) $b \neq 0$.

$$S := \begin{pmatrix} C & B_4 & A_4 \\ & 1 & A_5 \\ & & 2 \end{pmatrix}, \quad s := \begin{pmatrix} c & b_4 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix}.$$

Given that $C \neq 1, 2$ it is possible to eliminate B_4, A_4, A_5. Taking commutator we find out that b_4, a_4, a_5 must be zeroes as well to allow for the commutator to be in the algebra of inner derivations. If we further assume that $C \neq 3$, the first derivation can be completely diagonalized.

$$D := \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ & 2+C & 0 & 0 & 0 & 0 \\ & & 3 & 0 & 0 & 0 \\ & & & C & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}, \quad d := \begin{pmatrix} 1 & c_3 & 0 & c_1 & b_1 & 0 \\ & c & 0 & c_2 & b_2 & 0 \\ & & 1 & c_3 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

We demand that $[D, d] \in \mathfrak{Inn}$. This requires $c_3, c_2, b_1 \stackrel{!}{=} 0$. If $C \neq 5$ we also have $c_1 \stackrel{!}{=} 0$ and if $C \neq -1$ then $b_2 \stackrel{!}{=} 0$.

i) $C \neq \pm 1, 2, 3, 5$

$$D(C) := \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ & 2+C & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & C & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 2 \end{pmatrix}, \quad d(c) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & c & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

According to vi) it is possible to set $[f_1, f_2] := 0$ if $C \neq -2$ or $c \neq 0$. In that case we obtain the extension:

$\mathfrak{s}_{6+2,1}(C, c)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$4e_1$	$(2+C)e_2$	$2e_3$	Ce_4	0	$2e_6$	0
f_2	e_1	ce_2	e_3	ce_4	e_5	0	0

$$C \neq 0 \vee c \neq 0 \quad \Longrightarrow \quad CS = [8, 6] \quad DS = [8, 6, 3, 0] \quad US = [0]$$

$$C = 0 \wedge c = 0 \quad \Longrightarrow \quad CS = [8, 5] \quad DS = [8, 5, 2, 0] \quad US = [0]$$

In the case $C = -2,\ c = 0$ we get $[f_1, f_2] = \alpha_2 e_2$. The action of f_1 and f_2 on $\mathfrak{r}$ is diagonal; hence, assuming that $\alpha_2 \neq 0$, it will be the same in the basis $(e_1, \tilde{e_2}, e_3, e_3, \tilde{e_4}, e_5, e_6)$ with $\tilde{e_2} = \alpha_2 e_2$ and $\tilde{e_4} = \alpha_2 e_4$ and the action of e_6 will remain the same as well. But then $[f_1, f_2] = \tilde{e}_2$ and omitting the tildes we get the extension:

$\mathfrak{s}_{6+2,1'}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$4e_1$	0	$2e_3$	$-2e_4$	0	$2e_6$	0
f_2	e_1	0	e_3	0	e_5	0	e_2

$$CS = [8,6] \quad DS = [8,6,3,0] \quad US = [1]$$

ii) $C = -1$
Every nondiagonal parameter but B_2 is eliminated via the automorphism $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}_3$ given in the preceding. Nevertheless, in the case of $c \neq 1$ it is possible to eliminate b_2 in d using the automorphism that leaves D unchanged. The condition $[D,d] \in \mathfrak{Inn}$ yields $a_5, a_4, b_4, B_2, b_1, c_3, c_2, c_1 \stackrel{!}{=} 0$. Thus we get the special case of the extension above, namely $\mathfrak{s}_{6+2,1}(-1,c)$. Hence, we assume that $c = 1$. In that case we are unable to eliminate b_2 as well and the commutator of the derivations is the inner derivation regardless of the value of parameters B_2 and b_2. The resulting derivations are of the form

$$D(B_2) := \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & B_2 & 0 \\ & & 3 & 0 & 0 & 0 \\ & & & -1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}, \quad d(c) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & c & 0 & 0 & b_2 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

where one of the remaining parameters can be scaled to 1. Thus we obtain a one-parametric family of extensions (in the case of $B_2 \neq 0$)

$\mathfrak{s}_{6+2,2}(b_2)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$5e_1$	e_2	$3e_3$	$-e_4$	$e_2 + e_5$	$2e_6$	0
f_2	e_1	e_2	e_3	e_4	$b_2e_2 + e_5$	0	0

$$CS = [8,6] \qquad DS = [8,6,3,0] \qquad US = [0]$$

and in the case $B_2 = 0$ one more extension

$\mathfrak{s}_{6+2,2'}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$5e_1$	e_2	$3e_3$	$-e_4$	e_5	$2e_6$	0
f_2	e_1	e_2	e_3	e_4	$e_2 + e_5$	0	0

$$CS = [8,6] \qquad DS = [8,6,3,0] \qquad US = [0]$$

iii) $C = 5$

This is analogous to ii). The matrices are of the form

$$D := \begin{pmatrix} 5 & 0 & 0 & C_1 & 0 & 0 \\ & 7 & 0 & 0 & 0 & 0 \\ & & 3 & 0 & 0 & 0 \\ & & & 5 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}, \quad d(c) := \begin{pmatrix} 1 & 0 & 0 & c_1 & 0 & 0 \\ & c & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

while only the case $c_1 = 1$ is of interest.

$\mathfrak{s}_{6+2,3}(c_1)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$5e_1$	$7e_2$	$3e_3$	$e_1 + 5e_4$	e_5	$2e_6$	0
e_2	e_1	e_2	e_3	$c_1 e_1 + e_4$	e_5	0	0

$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$

$\mathfrak{s}_{6+2,3'}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$5e_1$	$7e_2$	$3e_3$	$5e_4$	e_5	$2e_6$	0
e_2	e_1	ce_2	e_3	$e_1 + e_4$	e_5	0	0

$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$

iv) $C = 1$

If $c \neq 1$ then we can use automorphisms to get the derivations in the form

$$S := \begin{pmatrix} 1 & B_4 & 0 \\ & 1 & 0 \\ & & 2 \end{pmatrix}, \quad s := \begin{pmatrix} c & 0 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix},$$

and taking commutator we immediately see that $B_4, a_4, a_5 \stackrel{!}{=} 0$. We can completely diagonalize D corresponding to S and from the condition $[D, d] \in \mathfrak{Inn}$ we get $c_1, c_2, c_3, b_1, b_2 \stackrel{!}{=} 0$. Thus we obtain derivations

$$D := \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & 0 \\ & & 3 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}, \quad d(c) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & c & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

corresponding to the special case of i) $\mathfrak{s}_{6+2,1}(1, c)$. If $c = 1$ we pick linear combinations

$$S := \begin{pmatrix} 0 & B_4 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}, \quad s := \begin{pmatrix} 1 & b_4 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix},$$

where $a_4, a_5 \overset{!}{=} 0$ to allow for $[D, d]$ to be in $\mathfrak{Inn}$. Each parameter but B_4 may be eliminated from D using suitable automorphism and again $c_1, c_2, c_3, b_1, b_2 \overset{!}{=} 0$ if the commutator of D and d is an inner derivation of $\mathfrak{r}$. Thus we obtain derivations

$$D(B_4) := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ & 1 & B_4 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & B_4 & 0 \\ & & & & 0 & 0 \\ & & & & & 1 \end{pmatrix}, \quad d(b_4) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & b_4 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & b_4 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

If $B_4 \neq 0$ we rescale it to 1 using diagonal automorphism and we subtract $b_4 D$ from d. This leads to the extension

$\mathfrak{s}_{6+2,4}(b_4)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$2e_1$	e_2	$e_2 + e_3$	0	e_4	e_6	0
f_2	$(1-2b_4)e_1$	$(1-b_4)e_2$	$(1-b_4)e_3$	e_4	e_5	$-b_4 e_6$	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

If $B_4 = 0$ we can rescale b_4 instead to obtain

$\mathfrak{s}_{6+2,5}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$5e_1$	$3e_2$	$3e_3$	e_4	e_5	$2e_6$	0
e_2	e_1	e_2	$e_2 + e_3$	e_4	$e_4 + e_5$	0	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

If both B_4 and b_4 vanish we get the special case of $\mathfrak{s}_{6+2,1}(C, c)$ with $C = c = 1$.

v) $C = 2$
If $c \neq 0$ we can use automorphisms to obtain lower submatrices in the form

$$S := \begin{pmatrix} 2 & 0 & A_4 \\ & 1 & 0 \\ & & 2 \end{pmatrix}, \quad s := \begin{pmatrix} c & b_4 & 0 \\ & 1 & a_5 \\ & & 0 \end{pmatrix},$$

where $b_4, A_4, a_5 \stackrel{!}{=} 0$ since $[S, s] \stackrel{!}{=} 0$. D with lower submatrix S may be diagonalized and if we demand that $[D, d] \in \mathfrak{Inn}$ we must set b_1, b_2, c_1, c_2, c_3 in d to zeroes. This does not lead to new extension of $\mathfrak{r}$, since it is a special case of $\mathfrak{s}_{6+2,1}(C, c)$ with $C = 2$.
If $c = 0$ we cannot eliminate a_4 in s (d), but we can eliminate every parameter in D but A_4. As well, all the parameters in d but a_4 must be equal to zero due to the condition $[d, D] \in \mathfrak{Inn}$. Hence we get the derivations

$$D(A_4) := \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ & 4 & 0 & 0 & 0 & 0 \\ & & 3 & 0 & 0 & 0 \\ & & & 2 & 0 & A_4 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}, \; d(a_4) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & a_4 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

Again rescaling the parameters via diagonal automorphisms and subtracting suitable multiple of the first derivation from the second one to diagonalize the second one in case $A_4 \neq 0$ leads to the extensions:

$\mathfrak{s}_{6+2,6}(a_4)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$5e_1$	$4e_2$	$3e_3$	$2e_4$	e_5	$e_4{+}2e_6$	0
f_2	$(1{-}5a_4)e_1$	$-4a_4e_2$	$(1{-}3a_4)e_3$	$-2a_4e_4$	$(1{-}a_4)e_5$	$-2a_4e_6$	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

$\mathfrak{s}_{6+2,7}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$5e_1$	$4e_2$	$3e_3$	$2e_4$	e_5	$2e_6$	0
f_2	e_1	0	e_3	0	e_5	e_4	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

The case $A_4 = a_4 = 0$ corresponds to $\mathfrak{s}_{6+2,1}(C = 2, c = 0)$.

vi) $C = 3$
S can be diagonalized using automorphisms, s is easily shown to be diagonal as well. We are unable to eliminate C_3 in D, on the other hand we can eliminate c_3 in d provided that $c \neq 1$. Taking commutator we see that D, d must be diagonal, that is we came to the special case of $\mathfrak{s}_{6+2,1}(C, c)$ with $C = 3$.
If $c = 1$ it is not possible to eliminate c_3 as well. Nevertheless, other parameters than C_3 are easily transformed out from D and by taking

commutator $[D, d]$ we readily see that $c_1, c_2, b_1, b_2 \overset{!}{=} 0$ in d. The derivations take up the form

$$D(C_3) := \begin{pmatrix} 5 & C_3 & 0 & 0 & 0 & 0 \\ & 5 & 0 & 0 & 0 & 0 \\ & & 3 & C_3 & 0 & 0 \\ & & & 3 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}, \ d(c_3) := \begin{pmatrix} 1 & c_3 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & c_3 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

and we obtain two new extensions

$\mathfrak{s}_{6+2,8}(c_3)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$5e_1$	e_1+5e_2	$3e_3$	e_3+3e_4	e_5	$2e_6$	0
f_2	$(1-5c_3)e_1$	$(1-5c_3)e_2$	$(1-3c_3)e_3$	$(1-3c_3)e_4$	$(1-c_3)e_5$	$-2c_3e_6$	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

$\mathfrak{s}_{6+2,9}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$5e_1$	$5e_2$	$3e_3$	$3e_4$	e_5	$2e_6$	0
f_2	e_1	$e_1 + e_2$	e_3	$e_3 + e_4$	e_5	0	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

and $\mathfrak{s}_{6+2,1}(3, 1)$.

b) $b = 0$
It is evident that $c \neq 0$ since d must not be nilpotent, hence we can change basis to the linear combinations such that the lower submatrices take up the form

$$S := \begin{pmatrix} 0 & B_4 & A_4 \\ & B & A_5 \\ & & 2 \end{pmatrix}, \ s := \begin{pmatrix} 1 & b_4 & a_4 \\ & 0 & a_5 \\ & & 0 \end{pmatrix}.$$

i) $B \neq 2$
We eliminate A_5 in D and b_4, a_4 in d using suitable automorphism. Taking commutator we obtain the condition $a_5, B_4, A_4 \overset{!}{=} 0$. Thus we have the derivations in the form

$$D = \begin{pmatrix} 4+B & C_3 & 0 & C_1 & B_1 & 0 \\ & 2 & 0 & C_2 & B_2 & 0 \\ & & 2+B & C_3 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & B & 0 \\ & & & & & 2 \end{pmatrix},$$

$$d = \begin{pmatrix} 0 & c_3 & 0 & c_1 & b_1 & 0 \\ & 1 & 0 & c_2 & b_2 & 0 \\ & & 0 & c_3 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

Again we transform out B_1, b_2, c_1, C_2, c_3 via suitable automorphisms and we take the commutator to find that $b_1, B_2, C_1, c_2, C_3 \stackrel{!}{=} 0$ as well. Hence the derivations must look like

$$D(B) = \begin{pmatrix} 4+B & 0 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 & 0 \\ & & 2+B & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & B & 0 \\ & & & & & 2 \end{pmatrix},$$

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

According to vi) we are able to set $[f_1, f_2] := 0$ provided that $B \neq -4$, while we are unable to do so in case $B = -4$. Thus we obtain two one-parametric sets of non-isomorphic extensions:

$\mathfrak{s}_{6+2,10}(B)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$(4+B)e_1$	$2e_2$	$(2+B)e_3$	0	Be_5	$2e_6$	0
f_2	0	e_2	0	e_4	0	0	0

$$B \neq 0 \implies CS = [8,6] \qquad DS = [8,6,3,0] \qquad US = [0]$$

$$B = 0 \implies CS = [8,5] \qquad DS = [8,5,2,0] \qquad US = [0]$$

By the same argument as with $\mathfrak{s}_{6+2,1'}$ we set $[f_1, f_2] := e_1$.

$\mathfrak{s}_{6+2,10'}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	0	$2e_2$	$-2e_3$	0	$-4e_5$	$2e_6$	0
e_2	0	e_2	0	e_4	0	0	e_1

$$CS = [8,6] \qquad DS = [8,6,3,0] \qquad US = [1]$$

ii) $B = 2$

In this case the parameters A_5, a_5 remain in the derivations. Nevertheless all the other parameters may be either eliminated using automorphisms or shown to be null to allow for $[D, d]$ to be the inner derivation. Hence we get the matrices

$$D(A_5) = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 & 0 \\ & & 4 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 2 & A_5 \\ & & & & & 2 \end{pmatrix}, \quad d(a_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 0 & a_5 \\ & & & & & 0 \end{pmatrix},$$

and the corresponding extensions

$\mathfrak{s}_{6+2,11}(a_5)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$6e_1$	$2e_2$	$4e_3$	0	$2e_5$	$e_5{+}2e_6$	0
f_2	$-6a_5e_1$	$(1{-}2a_5)e_2$	$-4a_5e_3$	e_4	$-a_5e_5$	$-2a_5e_6$	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

$\mathfrak{s}_{6+2,12}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$6e_1$	$2e_2$	$4e_3$	0	$2e_5$	$2e_6$	0
f_2	0	e_2	0	e_4	0	e_5	0

$$CS = [8, 6] \qquad DS = [8, 6, 3, 0] \qquad US = [0]$$

with a special case of $\mathfrak{s}_{6+2,10}(B)$ with $B = 2$.

Case 2

Assume that the component in lower right corner of matrices is zero in both outer derivations. Then there exists linear combinations of these two derivations such that their lower 3×3 submatrices are

$$S := \begin{pmatrix} 1 & B_4 & A_4 \\ & 2 & A_5 \\ & & 0 \end{pmatrix}, \quad s := \begin{pmatrix} 0 & b_4 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix},$$

since we demand that they are linearly nil-independent. According to the subsection 2.2.1 we can use automorphism to obtain the matrices in the form

$$S := \begin{pmatrix} 1 & 0 & 0 \\ & 2 & 0 \\ & & 0 \end{pmatrix}, \quad s := \begin{pmatrix} 0 & b_4 & a_4 \\ & 1 & a_5 \\ & & 0 \end{pmatrix}.$$

Commuting S and s we find $b_4, a_4, a_5 \overset{!}{=} 0$. From (2.6) we see that whole matrices must look like

$$D := \begin{pmatrix} 2 & C_3 & 0 & C_1 & B_1 & 0 \\ & 1 & 0 & C_2 & B_2 & 0 \\ & & 2 & C_3 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 0 \end{pmatrix}, \; d := \begin{pmatrix} 1 & c_3 & 0 & c_1 & b_1 & 0 \\ & 0 & 0 & c_2 & b_2 & 0 \\ & & 1 & c_3 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

where C_3, C_1 and B_2 can be eliminated by using suitable automorphism. Taking the commutator of these two and comparing it with inner derivations, we see that $c_1, c_3, b_2 \overset{!}{=} 0$. Finally we can use diagonal automorphism to rescale some of the remaining parameters. If $C_2 \neq 0$ we rescale it to one via diagonal automorphism and we subtract $c_2 D$ from d to get

$$D(B_1) := \begin{pmatrix} 2 & 0 & 0 & 0 & B_1 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$d(c_2, b_1) := \begin{pmatrix} 1-2c_2 & 0 & 0 & 0 & b_1 & 0 \\ & -c_2 & 0 & 0 & 0 & 0 \\ & & 1-2c_2 & 0 & 0 & 0 \\ & & & -c_2 & 0 & 0 \\ & & & & 1-2c_2 & 0 \\ & & & & & 0 \end{pmatrix}.$$

These derivations correspond to the extensions

$\mathfrak{s}_{6+2,13}(B_1,b_1,c_2)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
f_1	$2e_1$	e_2	$2e_3$	e_2+e_4	$B_1e_1+2e_5$	0	0
f_2	$(1-2c_2)e_1$	$-c_2e_2$	$(1-2c_2)e_3$	$-c_2e_4$	$b_1e_1+(1-2c_2)e_5$	0	0

$$CS = [8,5] \qquad DS = [8,5,0] \qquad US = [0]$$

In case of $C_2 = 0$ and $c_2 \neq 0$ we rescale c_2 to obtain two-parametric set of compatible derivations ($B_1, b_1 \in \mathbb{F}$):

$$D(B_1) := \begin{pmatrix} 2 & 0 & 0 & 0 & B_1 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 0 \end{pmatrix}, \quad d(b_1) := \begin{pmatrix} 1 & 0 & 0 & 0 & b_1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 0 \end{pmatrix}$$

and the set of algebras:

$\mathfrak{s}_{6+2,14}(B_1, b_1)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$2e_1$	e_2	$2e_3$	e_4	$B_1e_1 + 2e_5$	0	0
e_2	e_1	0	e_3	e_2	$b_1e_1 + e_5$	0	0

$$CS = [8, 5] \qquad DS = [8, 5, 0] \qquad US = [0]$$

If both C_2 and c_2 vanish, there is a possibility of rescaling either B_1 or b_1 to $(\pm)1$ if they are not already null. Hence, we obtain one-parametric set:

$$D := \begin{pmatrix} 2 & 0 & 0 & 0 & \pm 1 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 0 \end{pmatrix}, \quad d(b_1) := \begin{pmatrix} 1 & 0 & 0 & 0 & b_1 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}.$$

We subtract suitable multiple of D from d to diagonalize d and redefine parameter b_1 to get the extensions in the form:

$\mathfrak{s}_{6+2,15}(b_1)$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$2e_1$	e_2	$2e_3$	e_4	$\pm e_1 + 2e_5$	0	0
e_2	$(1 + 2b_1)e_1$	b_1e_2	$(1 + 2b_1)e_3$	b_1e_4	e_5	0	0

$$CS = [8, 5] \qquad DS = [8, 5, 0] \qquad US = [0]$$

and finally two (in case $\mathbb{F} = \mathbb{R}$ three) cases

$$D := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 0 \end{pmatrix}, \quad d(b_1) := \begin{pmatrix} 1 & 0 & 0 & 0 & \pm 1, 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

corresponding to the algebras

$\mathfrak{s}_{6+2,16}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$2e_1$	e_2	$2e_3$	e_4	$2e_5$	0	0
e_2	e_1	0	e_3	0	$\pm e_1 + e_5$	0	0

$CS = [8, 5] \qquad DS = [8, 5, 0] \qquad US = [0]$

$\mathfrak{s}_{6+2,17}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1
e_1	$2e_1$	e_2	$2e_3$	e_4	$2e_5$	0	0
e_2	e_1	0	e_3	0	e_5	0	0

$CS = [8, 5] \qquad DS = [8, 5, 0] \qquad US = [0]$

2.2.4 Solvable Nine-dimensional Extension

The outer derivations D_1, D, d must be linearly nil-independent. It is convenient to change to linear combinations such that $diag(D_1) = (4, 3, 2, 1, 0, 2)$, $diag(D) = (0, 1, 0, 1, 0, 0)$ and $diag(d) = (1, 0, 1, 0, 1, 0)$. Then we can diagonalize D_1 and since the commutators must be in $\mathfrak{Inn}$ we find that D and d are diagonal as well. Thus we only have one possibility for the action of f_1, f_2 and f_3 on $\mathfrak{r}$. We desire to simplify commutators of these vectors as much as possible.
Assume that

$$[f_1, f_2] = \alpha_\iota e_\iota, \quad [f_1, f_3] = \beta_\iota e_\iota, \quad [f_2, f_3] = \gamma_\iota e_\iota.$$

The action of f_1, f_2, f_3 on $Z(\mathfrak{r})$ is given by relations

$$\begin{aligned} &[f_1, e_1] = 4e_1, \qquad &[f_2, e_1] = 0, \qquad &[f_3, e_1] = e_1, \\ &[f_1, e_2] = 3e_2, \qquad &[f_2, e_2] = e_2, \qquad &[f_3, e_2] = 0. \end{aligned}$$

If we define $\tilde{f}_1 := f_1 + A_\iota e_\iota$, $\tilde{f}_2 := f_1 + B_\iota e_\iota$ and $\tilde{f}_3 := f_1 + C_\iota e_\iota$ we get

$$\begin{aligned} [\tilde{f}_1, \tilde{f}_2] &= (\alpha_1 + 4B_1)e_1 + (\alpha_2 - A - 2 + 3B_2)e_2, \\ [\tilde{f}_1, \tilde{f}_3] &= (\beta_1 + 4C_1 - A_1)e_1 + (\beta_2 + 3C_2)e_2, \\ [\tilde{f}_2, \tilde{f}_3] &= (\gamma_1 + B_1)e_1 + (\gamma_2 + 3C_2)e_2. \end{aligned}$$

Setting $B_1 := -\frac{\alpha_1}{4}$, $C_2 := -\frac{\beta_2}{3}$, $A_2 := \alpha_2$, $A_1 := \beta_1$ and $C_1 = B_2 := 0$, we get $[\tilde{f}_1, \tilde{f}_2] = 0$, $[\tilde{f}_1, \tilde{f}_3] = 0$. Furthermore, by Jacobi identity

$$0 \overset{!}{=} [[f_1, f_2], f_3] + [[f_3, f_1], f_2] + [[f_2, f_3], f_1],$$

we have that $[f_2, f_3] \overset{!}{=} 0$ as well. Therefore, there is only one nine-dimensional solvable extension of $\mathfrak{r}$:

$\mathfrak{s}_{6+3}$	e_1	e_2	e_3	e_4	e_5	e_6	f_1	f_2	f_3
e_1		0	0	0	0	0	$4e_1$	0	e_1
e_2			0	0	0	0	$3e_2$	e_2	0
e_3				0	0	e_1	$2e_3$	0	e_3
e_4					0	e_2	e_4	e_4	0
e_5						e_3	0	0	e_5
e_6							$2e_6$	0	0
f_1								0	0
f_2									0

$$CS = [9, 6] \qquad DS = [9, 6, 3, 0] \qquad US = [0]$$

Chapter 3

Extensions of Even-dimensional Nilradical

We wish to generalize the results of previous chapter to $\mathfrak{r}_{2k} \equiv \operatorname{span}(e_1, \ldots, e_{2k})$ with the sole non-trivial ad-action $[e_{2k}, e_i] = e_{i-2}$ for every $i \in \{3, \ldots, e_{2k-1}\}$. We will advance in the very same manner as with the six-dimensional case.

3.1 Properties of $\mathfrak{r}_{2k}$

First, we investigate properties of the nilpotent algebra $\mathfrak{r}_{2k}$, that is we find flag of invariant ideals with codimensions equal to 1, its inner and outer derivations and the algebra of automorphism.

3.1.1 Inner Derivations

Since we will operate modulo inner derivations of $\mathfrak{r}_{2k}$, it is useful to list them explicitly. The first two elements constitute the centre of $\mathfrak{r}_{2k}$

$$\mathrm{ad}_{e_1} = \mathrm{ad}_{e_2} = 0,$$

the ad-action of e_i for $i = 3, \ldots, 2k-1$ is given by

$$(\mathrm{ad}_{e_i})_{j,2k} = -\delta_{i-2,j}, \quad (\mathrm{ad}_{e_i})_{j,l} = 0, \qquad \forall j \in \{1, \ldots, 2k\}, \forall l < 2k, \tag{3.1}$$

and the action of the last element is of the form

$$\mathrm{ad}_{e_{2k}} = \begin{pmatrix} 0 & 0 & 1 & & & & & \\ & 0 & 0 & 1 & & & & \\ & & 0 & 0 & \ddots & & & \\ & & & 0 & \ddots & 1 & & \\ & & & & \ddots & 0 & 1 & \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}, \tag{3.2}$$

that is

$$(\mathrm{ad}_{e_{2k}})_{i,j} = \begin{cases} \delta_{i-2,j} & \forall i \in \{3, \dots, 2k-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

3.1.2 Derived Series, Flag of Ideals

The lower central series is easily found to be

$$\begin{aligned} &C^j \mathfrak{r}_{2k} = \mathrm{span}(e_1, \dots, e_{2(k-j)-1}), \qquad \forall j \in \{1, \dots, k-1\}, \\ &C^k \mathfrak{r}_{2k} = 0, \end{aligned}$$

similarly the upper central series is effortlessly computed as

$$\mathfrak{z}_j = \mathrm{span}(e_1, \dots, e_{2j}), \qquad \forall j \in \{1, \dots, k\}.$$

It is evident that the derived series is

$$\begin{aligned} &D^1 \mathfrak{r}_{2k} = \mathrm{span}(e_1, \dots, e_{2k-3}), \\ &D^2 \mathfrak{r}_{2k} = 0. \end{aligned}$$

Furthermore

$$Z_{\mathfrak{r}_{2k}}(C^1 \mathfrak{r}_{2k}) = \{x \in \mathfrak{r}_{2k} \mid [x, y] = 0 \quad \forall y \in \mathfrak{r}^1_{2k}\} = \mathrm{span}(e_1, \dots, e_{2k-1}).$$

Thus

$$C^{k-1}\mathfrak{r}_{2k} \subsetneq \mathfrak{z}_1 \subsetneq C^{k-2}\mathfrak{r}_{2k} \subsetneq \mathfrak{z}_2 \subsetneq \dots \subsetneq C^1\mathfrak{r}_{2k} \subsetneq \mathfrak{z}_{k-1} \subsetneq Z_{\mathfrak{r}_{2k}}(C^1\mathfrak{r}_{2k}) \subsetneq \mathfrak{r}_{2k} \tag{3.4}$$

is the flag of characteristic ideals in $\mathfrak{r}_{2k}$, where the ideals are equal to

$$\mathrm{span}(e_1) \subsetneq \mathrm{span}(e_1, e_2) \subsetneq \dots \subsetneq \mathrm{span}(e_1, \dots e_{2k-1}) \subsetneq \mathfrak{r}_{2k},$$

respectively. Consequentially all the automorphisms are upper-triangular matrices in the basis $(e_1, \dots, e_{2k})$. Dimensions of the series are listed for reference:

$$\begin{aligned} \mathrm{DS} &= [2k, 2k-3, 0], \\ \mathrm{CS} &= [2k, 2k-3, 2k-5, \dots, 3, 1, 0], \\ \mathrm{US} &= [2, 4, \dots, 2k-4, 2k-2, 2k]. \end{aligned} \tag{3.5}$$

3.1.3 Automorphisms, Generic Outer Derivation

By the same argument as in subsection 2.1.3 it is sufficient to find general diagonal and unitriangular automorphisms separately.

Generic Diagonal Automorphism

Let Φ be an automorphism of r_{2k} and $\Phi(e_{2k}) = \alpha e_{2k}$, $\Phi(e_{2k-1}) = \beta e_{2k-1}$, $\Phi(e_{2k-2}) = \gamma e_{2k-2}$. Then

$$\Phi(e_{2k-3}) = \Phi([e_{2k}, e_{2k-1}]) = [\Phi(e_{2k}), \Phi(e_{2k-1})] = [\alpha e_{2k}, \beta e_{2k-1}] = \alpha\beta e_{2k-3},$$

and

$$\Phi(e_{2k-4}) = \Phi([e_{2k}, e_{2k-2}]) = [\Phi(e_{2k}), \Phi(e_{2k-2})] = [\alpha e_{2k}, \gamma e_{2k-2}] = \alpha\gamma e_{2k-3}.$$

Since

$$e_{2(k-j)-1} = \mathrm{ad}_{e_{2k}}^{j} e_{2k-1}, \qquad e_{2(k-j)} = \mathrm{ad}_{e_{2k}}^{j-1} e_{2k-2}, \qquad \forall j \in \{1, \dots, k-1\}$$

we have for every $j \in \{1, \dots, k-1\}$

$$\begin{aligned} \Phi(e_{2(k-j)-1}) &= \alpha^j \beta e_{2(k-j)-1}, \\ \Phi(e_{2(k-j)}) &= \alpha^{j-1} \gamma e_{2(k-j)}. \end{aligned} \tag{3.6}$$

Hence the generic diagonal automorphism is of the form

$$\Phi_{diag} = \begin{pmatrix} \alpha^{k-1}\beta & & & & & & & \\ & \alpha^{k-2}\gamma & & & & & & \\ & & \alpha^{k-2}\beta & & & & & \\ & & & \ddots & & & & \\ & & & & \alpha^2\beta & & & \\ & & & & & \gamma & & \\ & & & & & & \beta & \\ & & & & & & & \alpha \end{pmatrix}. \tag{3.7}$$

Generic Unitriangular Automorphism

Let us assume that Φ is a unitriangular automorphism and that

$$\begin{aligned}
\Phi(e_{2k}) &= \sum_{j=1}^{2k-1} \alpha_j e_j + e_{2k}, \\
\Phi(e_{2k-1}) &= \sum_{j=1}^{2k-2} \beta_j e_j + e_{2k-1}, \\
\Phi(e_{2k-2}) &= \sum_{j=1}^{2k-3} \gamma_j e_j + e_{2k-2}.
\end{aligned}$$

Then by the same argument as in the previous subsection we have

$$\begin{aligned}
\Phi(e_{2(k-i)-1}) &= \sum_{j=2i+1}^{2k-2} \beta_j e_{j-2i} + e_{2(k-i)-1}, \qquad \forall i \in \{1, \dots, k-2\}, \\
\Phi(e_{2(k-i)}) &= \sum_{j=1}^{2(k-i)-1} \gamma_j e_{j-2(i-1)} + e_{2(k-i)}, \qquad \forall i \in \{2, \dots, k-1\}.
\end{aligned} \tag{3.8}$$

These relations lead to generic unitriangular automorphism in the form

$$\Phi_{uni} = \begin{pmatrix}
1 & \gamma_{2k-3} & \beta_{2k-3} & \gamma_{2k-5} & \cdots & \gamma_3 & \beta_3 & \gamma_1 & \beta_1 & \alpha_1 \\
 & 1 & \beta_{2k-2} & \gamma_{2k-4} & \cdots & \gamma_4 & \beta_4 & \gamma_2 & \beta_2 & \alpha_2 \\
 & & 1 & \gamma_{2k-3} & \cdots & \gamma_5 & \beta_5 & \gamma_3 & \beta_3 & \alpha_3 \\
 & & & 1 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & \ddots & \gamma_{2k-3} & \beta_{2k-3} & \gamma_{2k-5} & \beta_{2k-5} & \alpha_{2k-3} \\
 & & & & & 1 & \beta_{2k-2} & \gamma_{2k-4} & \beta_{2k-4} & \alpha_{2k-4} \\
 & & & & & & 1 & \gamma_{2k-3} & \beta_{2k-3} & \alpha_{2k-3} \\
 & & & & & & & 1 & \beta_{2k-2} & \alpha_{2k-2} \\
 & & & & & & & & 1 & \alpha_{2k-1} \\
 & & & & & & & & & 1
\end{pmatrix}. \tag{3.9}$$

Outer Derivations

General automorphism is given by composition of diagonal automorphism (3.7) with unitriangular automorphism (3.9) and is thus described by $6k-3$ parameters $\alpha, \beta, \gamma, \alpha_1, \dots, \alpha_{2k-1}, \beta_1, \dots, \beta_{2k-2}, \gamma_1, \dots, \gamma_{2k-3}$. Obviously, it is equal to $\mathbb{1}$ for the choice $\alpha, \beta, \gamma := 1$, $\alpha_i, \beta_j, \gamma_\ell = 0$. Assume that $\alpha = 1 + a$, $\beta = 1 + b$, $\gamma = 1 + c$, $\alpha_i = a_i$, $\beta_j = b_j$, $\gamma_\ell = c_\ell$ where all the parameters on the right-hand side of the equations are small. Then we find the general form of the derivation as an infinitesimal automorphism minus $\mathbb{1}$, i.e.

$$
\begin{pmatrix}
(k-1)a+b & c_{2k-3} & b_{2k-3} & c_{2k-5} & \cdots & c_3 & b_3 & c_1 & b_1 & a_1 \\
 & (k-2)a+c & b_{2k-2} & c_{2k-4} & \cdots & c_4 & b_4 & c_2 & b_2 & a_2 \\
 & & (k-2)a+b & c_{2k-3} & \cdots & c_5 & b_5 & c_3 & b_3 & a_3 \\
 & & & (k-3)a+c & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & \ddots & c_{2k-3} & b_{2k-3} & c_{2k-5} & b_{2k-5} & a_{2k-5} \\
 & & & & & a+c & b_{2k-2} & c_{2k-4} & b_{2k-4} & a_{2k-4} \\
 & & & & & & a+b & c_{2k-3} & b_{2k-3} & a_{2k-3} \\
 & & & & & & & c & b_{2k-2} & a_{2k-2} \\
 & & & & & & & & b & a_{2k-1} \\
 & & & & & & & & & a
\end{pmatrix}. \quad (3.10)
$$

Since we are interested in outer derivation classes w. r. t. conjugation by arbitrary automorphism, multiplication by a number from the field and addition of linear combinations of inner derivations, by subtracting suitable multiple of derivation (3.1) we can eliminate $a_1, \dots, a_{2k-3}$ and similarly via derivation (3.2) we may eliminate b_{2k-3}. Hence the chosen form of a representative of the given outer derivation class is

$$
\begin{pmatrix}
(k-1)a+b & c_{2k-3} & 0 & c_{2k-5} & \cdots & c_3 & b_3 & c_1 & b_1 & 0 \\
 & (k-2)a+c & b_{2k-2} & c_{2k-4} & \cdots & c_4 & b_4 & c_2 & b_2 & 0 \\
 & & (k-2)a+b & c_{2k-3} & \cdots & c_5 & b_5 & c_3 & b_3 & 0 \\
 & & & (k-3)a+c & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & \ddots & c_{2k-3} & 0 & c_{2k-5} & b_{2k-5} & 0 \\
 & & & & & a+c & b_{2k-2} & c_{2k-4} & b_{2k-4} & 0 \\
 & & & & & & a+b & c_{2k-3} & 0 & 0 \\
 & & & & & & & c & b_{2k-2} & a_{2k-2} \\
 & & & & & & & & b & a_{2k-1} \\
 & & & & & & & & & a
\end{pmatrix}. \quad (3.11)
$$

Notice that both the automorphism and the outer derivation is determined by its last three columns. Furthermore if $\mathbb{A} \in \mathfrak{Aut}$ and $D \in \mathfrak{Der}$ then $\mathbb{A} D \mathbb{A}^{-1} \in \mathfrak{Der}$ and it is of course always possible to subtract inner derivations so that we get the derivation in the form (3.11). Hence the resulting derivation $\mathbb{A} D \mathbb{A}^{-1}$ is determined by the last three columns as well. Thus, for the sake of simplicity, only these columns will be written henceforth.

3.2 Conditions for Complete Diagonalization

The lower 4×4 matrix is an exact copy of that in the six-dimensional case. Thus the conditions and the used automorphisms are the same as well. The lower 3×3 matrix is diagonalized via automorphism (3.9) with $\alpha_{2k-1} := \frac{a_{2k-1}}{a-b}$, $\beta_{2k-2} := \frac{b_{2k-2}}{b-c}$, $\alpha_{2k-2} := \frac{\frac{a_{2k-1} b_{2k-2}}{a-b} + a_{2k-2}}{a-c}$ given that $a \neq b \neq c \neq a$. Whenever $c \neq a+b$ we may eliminate c_{2k-3} via $\Phi|_{\gamma_{2k-3} = \frac{c_{2k-3}}{c-a-b}}(0, \dots, 0)$. Note that this automorphism eliminates c_{2k-3} regardless of the outcome of the first step and it does not change the lower 3×3 submatrix. The matrix is then in the form

$$\begin{pmatrix}
(k-1)a+b & 0 & 0 & c_{2k-5} & b_{2k-5} & \cdots & c_1 & b_1 & 0 \\
 & (k-2)a+c & 0 & c_{2k-4} & b_{2k-4} & \cdots & c_2 & b_2 & 0 \\
 & & (k-2)a+b & 0 & 0 & \cdots & c_3 & b_3 & 0 \\
 & & & (k-3)a+c & 0 & \ddots & \vdots & \vdots & \vdots \\
 & & & & (k-3)a+b & \ddots & c_{2k-4} & b_{2k-4} & 0 \\
 & & & & & \ddots & 0 & 0 & 0 \\
 & & & & & & c & 0 & 0 \\
 & & & & & & & b & 0 \\
 & & & & & & & & a
\end{pmatrix}$$

and it can be rewritten using

$$\mathbb{N}_\ell := \begin{pmatrix} 0 & 0 & \\ \ell a + c & 0 & \\ & \ell a + b & \end{pmatrix}, \quad \mathbb{M}_\ell := \begin{pmatrix} c_{2\ell-1} & b_{2\ell-1} \\ c_{2\ell} & b_{2\ell} \end{pmatrix} \tag{3.12}$$

as

$$\begin{pmatrix}
d_1 & \mathbb{N}_{k-2} & \mathbb{M}_{k-2} & \mathbb{M}_{k-3} & \cdots & \mathbb{M}_2 & \mathbb{M}_1 & \begin{matrix} 0 \\ 0 \end{matrix} \\
 & & \mathbb{N}_{k-3} & \mathbb{M}_{k-2} & & \mathbb{M}_3 & \mathbb{M}_2 & \begin{matrix} 0 \\ 0 \end{matrix} \\
 & & & \mathbb{N}_{k-4} & \ddots & \vdots & \vdots & \vdots \\
 & & & & \ddots & \mathbb{M}_{k-2} & \mathbb{M}_{k-3} & \begin{matrix} 0 \\ 0 \end{matrix} \\
 & & & & & \mathbb{N}_1 & \mathbb{M}_{k-2} & \begin{matrix} 0 \\ 0 \end{matrix} \\
 & & & & & & \mathbb{N}_0 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\
 & & & & & & & a
\end{pmatrix}.$$

Given that the first two steps were successful, we may eliminate block $\mathbb{M}_\ell$ using the automorphism

$$\mathbb{C}_\ell := \Phi\big|_{\gamma_{2\ell-1} = -\frac{c_{2\ell-1}}{(k-\ell)a+b-c}, \gamma_{2\ell} = -\frac{c_{2\ell}}{(k-\ell-1)a}, \beta_{2\ell-1} = -\frac{b_{2\ell-1}}{(k-\ell)a}, \beta_{2\ell} = -\frac{b_{2\ell}}{(k-\ell-1)a+c-b}} (0, \dots, 0)$$

provided that $a \neq 0$, $(k-\ell-1)a+c-b \neq 0$ and $(k-\ell)a+b-c \neq 0$. Applying $\mathbb{C}_\ell$ modifies $\mathbb{M}_1, \dots, \mathbb{M}_\ell$ while it leaves the rest of the derivation unchanged; hence we apply $\mathbb{C}_\ell$ from the last one ($\mathbb{C}_{k-2}$) to $\mathbb{C}_1$.

It is now evident that the generic outer derivation may be completely diagonalized if and only if the conditions $0 \neq a \neq b \neq c \neq a$, $\ell a + b \neq c$ for $\ell = 1, \dots, k-1$, and $\ell a + c \neq b$ for $\ell = 1, \dots, k-2$ are satisfied.

If one or more of these conditions does not hold we are usually unable to eliminate some non-diagonal element. In that case we try to scale it to $(\pm)1$ via diagonal automorphism; hence, it is useful to list the action of the diagonal automorphism (3.7) on the derivation (3.11). The action on the lower 4×4 block is left unchanged from the six-dimensional case:

$$\begin{pmatrix} a+b & \frac{\gamma}{\alpha\beta} c_{2k-3} & 0 & 0 \\ & c & \frac{\beta}{\gamma} b_{2k-2} & \frac{\alpha}{\gamma} a_{2k-2} \\ & & b & \frac{\alpha}{\beta} a_{2k-1} \\ & & & a \end{pmatrix}, \tag{3.13}$$

the action on the rest of the derivation is given by

$$\begin{aligned} c_{2\ell-1} &\longrightarrow \frac{\gamma c_{2\ell-1}}{\beta \alpha^{k-\ell-1}}, & b_{2\ell-1} &\longrightarrow \frac{b_{2\ell-1}}{\alpha^{k-\ell}}, \\ c_{2\ell} &\longrightarrow \frac{c_{2\ell}}{\alpha^{k-\ell-1}}, & b_{2\ell} &\longrightarrow \frac{\beta b_{2\ell}}{\gamma \alpha^{k-\ell-1}}. \end{aligned} \tag{3.14}$$

3.3 Classes of Outer Derivations, Extensions by One Element

We proceed similarly as in the six-dimensional case, that is at the first level we divide the cases leading to different 3×3 lower submatrices, at the second level we discriminate whether it is possible to eliminate c_{2k-3} and the third one corresponds to the rest of the parameters of the derivation. Classes that would be a subcase of some previously obtained family of classes are omitted.

1. $a \neq b \neq c \neq a$
By the results of the preceding section we can eliminate parameters a_{2k-1}, a_{2k-2} and b_{2k-2}.

1.1. $c \neq a+b$
This condition allows for the elimination of c_{2k-3}.

1.1.1. $a \neq 0$, $ja+b \neq c$ for $j \in \{2, \dots, k-1\}$ and $\ell a + c \neq b$ for every $\ell \in \{1, \dots, k-2\}$
The derivation is diagonalized via the automorphisms described in the preceding section. Since $a \neq 0$ we can further multiply by a^{-1} to get

$$D_1(b,c) = \begin{pmatrix} \ddots & & & \\ & c & & \\ & & b & \\ & & & 1 \end{pmatrix}$$

and the corresponding extension $\mathfrak{s}_{2k+1,1}(b,c) = \operatorname{span}(e_1, \dots, e_{2k}, f_1)$ with

$$\begin{aligned} [f_1, e_{2\ell-1}] &= (k-\ell+b)e_{2\ell-1}, & \forall \ell = 1, \dots, k, \\ [f_1, e_{2\ell}] &= (k-\ell-1+c)e_{2\ell}, & \forall \ell = 1, \dots, k-1, \\ [f_1, e_{2k}] &= e_{2k}, \end{aligned}$$

and dimensions of elements of lower and derived series

$$b,c \neq 0 \implies CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0],$$

$$b\cdot c = 0, b+c \neq 0 \implies CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0],$$

$$b,c = 0 \implies CS = [2k+1, 2k-2], \quad DS = [2k+1, 2k-2, 2k-5, 0]$$

and upper central series

$$b \neq 1-k, c \neq 2-k \implies US = [0],$$

$$b = 1-k, c \neq 2-k \vee b \neq 1-k, c = 2-k \implies US = [1],$$

$$b = 1-k, c = 2-k \implies US = [2].$$

1.1.2. $ia+b=c$ for some $i\in\{2,\dots,k-1\} \implies a\neq 0$ and $ja+c\neq b$ for $j=1,\dots,k-1$

We first justify the implication above. Since $ia+b=c$ and $b\neq c$ it is evident that $a\neq 0$. Further, assume that $ja+c=b$ for some j. Then $ia+b=ia+ja+c=c$, that is $i+j\overset{!}{=}0$ since $a\neq 0$, thus $j=-i$. Therefore, the inequality $ja+c\neq b$ holds for every positive j.

Let us denote $\ell:=k-i$; then by the same argument as in the preceding section we can eliminate $\mathbb{M}_j$ for every $j\neq\ell$. The components $c_{2\ell}$, $b_{2\ell-1}$, $b_{2\ell}$ of the remaining block are eliminated by (3.9) with $\gamma_{2\ell}=-\frac{c_{2\ell}}{(k-\ell-1)a}$, $\beta_{2\ell-1}=-\frac{b_{2\ell-1}}{(k-\ell)a}$, $\beta_{2\ell}=-\frac{b_{2\ell}}{(2k-2\ell-1)a}$ and all the other parameters set to zero. It is not possible to eliminate $c_{2\ell-1}$ but we are able to scale to 1 via the diagonal automorphism (3.7). After dividing by a we obtain $k-2$ different one-parametric sets of classes

$$D_2^{(\ell)}(b)=\begin{pmatrix} & & \vdots & & \\ & & 1 & & \\ & \ddots & \vdots & & \\ & & i+b & & \\ & & & b & \\ & & & & 1 \end{pmatrix},\quad i\in\{2,\dots,k-1\}$$

and the corresponding extensions $\mathfrak{s}^{(i)}_{2k+1,2}(b)$ given by the relations

$$\begin{aligned}
[f_1,e_{2j-1}]&=(k-j+b)e_{2j-1}, && \forall j=1,\dots,k,\\
[f_1,e_{2j}]&=(k-1-j+i+b)e_{2j}, && \forall j=1,\dots,i-1,\\
[f_1,e_{2j}]&=e_{2(j-i)+1}+(k-1-j+i+b)e_{2j}, && \forall j=i,\dots,k-1,\\
[f_1,e_{2k}]&=e_{2k}.
\end{aligned}$$

Characteristic series depend on the value of b:

$$b\neq 0, b\neq \ell-k \implies CS=[2k+1,2k],\quad DS=[2k+1,2k,2k-3,0],$$

$$b=0\vee b=\ell-k \implies CS=[2k+1,2k-1],\quad DS=[2k+1,2k-1,2k-4,0],$$

while the centre is always empty regardless of the value of b. Note that only one of the latter conditions may be satisfied for given b and each one leads to different characteristic series; their dimensions are equal nonetheless.

1.1.3. $ia+c=b$ for some $i\in\{1,\dots,k-2\} \implies a\neq 0$ and $\ell a+b\neq c$ for every $\ell\in\{2,\dots,k-1\}$

We again eliminate $\mathbb{M}_i$ for every $i\neq j:=k-i-1$ and via (3.9) with

$\gamma_{2j-1} := -\frac{c_{2j-1}}{(2k-2j-1)a}$, $\gamma_{2j} := -\frac{c_{2j}}{(k-j-1)}$ and $\beta_{2j-1} := -\frac{b_{2j-1}}{(k-j)a}$ we eliminate $c_{2j-1}, c_{2j}, b_{2j-1}$. Further, we scale β_{2j} to one and obtain

$$D_3^{(i)}(c) = \begin{pmatrix} & & \vdots & \\ & & 1 & \\ \ddots & & \vdots & \\ & c & & \\ & & i+c & \\ & & & 1 \end{pmatrix} \quad i \in \{1, \dots, k-2\}$$

and the corresponding extensions $\mathfrak{s}_{2k+1,3}^{(i)}(c)$ with action of f_1 given by

$$\begin{aligned} [f_1, e_{2\ell-1}] &= (k-\ell+i+c)e_{2(k-\ell)-1}, && \forall \ell = 1, \dots, i+1, \\ [f_1, e_{2\ell-1}] &= e_{2(\ell-i-1)} + (k-\ell+i+c)e_{2(k-\ell)-1}, && \forall \ell = i+2, \dots, k, \\ [f_1, e_{2\ell}] &= (k-\ell-1+c)e_{2\ell}, && \forall \ell = 1, \dots, k-1, \\ [f_1, e_{2k}] &= e_{2k}. \end{aligned}$$

The characteristic series depend on c similarly as in the previous case:

$$c \neq 0 \ \wedge \ c \neq -i \implies CS = [2k+1, 2k], \ DS = [2k+1, 2k, 2k-3, 0],$$

$$c = 0 \ \vee \ c = -i \implies CS = [2k+1, 2k-1], \ DS = [2k+1, 2k-1, 2k-4, 0],$$

the centre is again empty regardless of the value of c.

1.1.4. $a = 0 \implies ja + b \neq c$ and $\ell a + c \neq b$ since $b \neq c$

Using modified $\mathbb{C}_\ell$ with $\beta_{2\ell} = \frac{b_{2\ell}}{b-c}$ and $\gamma_{2\ell-1} = \frac{c_{2\ell-1}}{b-c}$ we eliminate $b_{2\ell}$ and $c_{2\ell-1}$. Changing basis from $(e_i)_1^{2k}$ to $(e_1, e_3, \dots, e_{2k-1}, e_2, e_4, \dots, e_{2k})$ we obtain the derivation matrix in the Jordan form; thus it is not possible to further simplify it. We obtain the derivation in the form

$$\begin{pmatrix} b & 0 & \cdots & & \cdots & 0 & b_1 & 0 \\ & c & \cdots & & \cdots & c_2 & 0 & 0 \\ & & \ddots & & & \vdots & \vdots & \vdots \\ & & & & & 0 & b_{2k-5} & 0 \\ & & & & & c_{2k-4} & 0 & 0 \\ & & & & \ddots & 0 & 0 & 0 \\ & & & & & c & 0 & 0 \\ & & & & & & b & 0 \\ & & & & & & & 0 \end{pmatrix}.$$

The action of the diagonal automorphism is given by relations (3.14). Thus we set the last (the one with the greatest index) element with odd power of α to 1 and in the case of all elements with the odd power of α being zero, we set the last remaining element to $(\pm)1$ and obtain corresponding families of extensions with number of parameters varying from k to 2. All of these extensions have the same characteristic series:

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 0], \quad US = [0].$$

1.2. $c = a + b$
It is not needed to assume anything else as this condition implies $a \neq 0$, $\ell a + b \neq c$ for every $\ell \neq 1$ and $ja + c \neq b$ for every $j \neq -1$. It is not possible to eliminate c_{2k-3} and we cannot use $\mathbb{C}_i$ from the previous subsection. However, it is possible to eliminate all $\mathbb{M}_i$ via $\mathbb{C}_i^{(2)}$ given by

$$\gamma_{2i} := -\frac{c_2 i}{(k-i-1)a}, \quad \gamma_{2i-1} := \frac{(l-k+1)ac_{2i-1} + c_{2k-3}c_{2i}}{(k-i-1)^2 a^2},$$
$$\beta_{2i} := -\frac{b_2 i}{(k-i)a}, \quad \beta_{2i-1} := \frac{(l-k)ab_{2i-1} + b_{2k-3}c_{2i}}{(k-i)^2 a^2}.$$

The resulting derivation classes are

$$D_5(b) = \begin{pmatrix} \ddots & & & & \\ & 1+b & 1 & & \\ & & 1+b & & \\ & & & b & \\ & & & & 1 \end{pmatrix},$$

the corresponding extension $\mathfrak{s}_{2k+1,5}(b)$ is given by the relations

$$\begin{aligned} [f_1, e_{2\ell-1}] &= (k-\ell+b)e_{2\ell}, && \forall \ell = 1, \dots, k, \\ [f_1, e_{2\ell}] &= e_{2\ell-1} + (k-\ell+b)e_{2\ell}, && \forall \ell = 1, \dots, k-1, \\ [f_1, e_{2k}] &= e_{2k}. \end{aligned}$$

The centre is again empty regardless of the parameter b while

$$b \neq 0, -1 \implies CS = [2k+1, 2k], \ DS = [2k+1, 2k, 2k-3, 0],$$
$$b = 0, -1 \implies CS = [2k+1, 2k-1], \ DS = [2k+1, 2k-1, 2k-4, 0].$$

2. $a = b \neq c$
Parameters a_{2k-2} and b_{2k-2} are eliminated using unitriangular automorphism (3.9) with $\beta_{2k-2} = \frac{b_{2k-2}}{a-c}$ and $\alpha_{2k-2} = \frac{a_{2k-2} - \frac{a_{2k-1}b_{2k-2}}{a-c}}{a-c}$, while the parameter a_{2k-1} cannot be eliminated.

2.1. $c \neq 2a$

Similarly as in the case 1.1, we eliminate c_{2k-3} by the automorphism with $\gamma_{2k-3} := \frac{c_{2k-3}}{c-2a}$.

2.1.1. $a \neq 0$, $ja \neq c$ for $j \in \{3, \dots, k\}$ and $\ell a + c \neq 0$ for every $\ell \in \{0, \dots, k-3\}$.

The rest of the parameters is eliminated via automorphisms $\mathbb{C}_\ell$ from the previous section. Using diagonal automorphism, we set a_{2k-1} to 1 and obtain the derivation in the form

$$D_6(c) = \begin{pmatrix} k & 0 & \cdots & 0 & 0 & 0 \\ & k-2+c & \ddots & \vdots & \vdots & \vdots \\ & & \ddots & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}.$$

The corresponding extension $\mathfrak{s}_{2k+1,6}(c)$ is given by the commutation relations

$$\begin{aligned} [e_{2\ell-1}, f_1] &= (k-\ell-1)e_{2\ell-1}, & \forall \ell \in \{1, \dots, k\}, \\ [e_{2\ell}, f_1] &= (k-\ell-1+c)e_{2\ell}, & \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2k}, f_1] &= e_{2k-1} + e_{2k}. \end{aligned}$$

Its characteristic series depend on the value of c:

$$\begin{gathered} c \neq 0 \implies CS = [2k+1, 2k],\ DS = [2k+1, 2k, 2k-3, 0], \\ c = 0 \implies CS = [2k+1, 2k-1],\ DS = [2k+1, 2k-1, 2k-4, 0], \\ c \neq k-2 \implies US = [0], \qquad c = k-2 \implies US = [1]. \end{gathered}$$

2.1.2. $ia = c$ for some $i \in \{3, \dots, k\} \implies a \neq 0$ and $ja + c \neq 0$ for $j = 0, \dots, k-3$.

In this case blocks $\mathbb{M}_j$ for every $j \neq k-i+1$ are easily eliminated using the same automorphisms $\mathbb{C}_j$ as in the previous case. We modify $\mathbb{C}_{k-i+1}$ by setting $\gamma_{2(k-i)+1} := 0$ and with the modified automorphism we eliminate every parameter in $\mathbb{M}_{k-i+1}$ but $c_{2(k-i)+1}$. According to the action (3.13, 3.14) of diagonal automorphism it is possible to simultaneously scale a_{2k-1} and $c_{2(k-i)+1}$ to 1. The resulting derivation is of the form

$$D_7^{(i)} = \begin{pmatrix} & \vdots & & \\ & 1 & & \\ \ddots & \vdots & & \\ & i & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \quad i \in \{3, \dots, k\},$$

the corresponding extension $\mathfrak{s}_{2k+1,7}^{(i)}$ is given by relations

$$\begin{aligned}
[e_{2\ell-1}, f_1] &= (k-\ell-1)e_{2\ell-1}, && \forall \ell \in \{1,\dots,k\},\\
[e_{2\ell}, f_1] &= e_{2\ell}, && \forall \ell \in \{1,\dots,i-2\},\\
[e_{2\ell}, f_1] &= e_{2(\ell-i)+3} + (k-\ell)e_{2\ell}, && \forall \ell \in \{i-1,\dots,k-1\},\\
[e_{2k}, f_1] &= e_{2k-1} + e_{2k}.
\end{aligned}$$

Characteristic series are the same regardless of the value of i:

$$CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0], \quad US = [0].$$

2.1.3. $ia + c = 0$ for some $i \in \{0,\dots,k-3\}$.
Similarly as in the previous case we eliminate all $\mathbb{M}_\ell$ for every $\ell \neq k-2-i$ and we eliminate every element in $\mathbb{M}_{k-2-i}$ but $b_{2(k-2-i)}$ by accordingly modified $\mathbb{C}_{k-2-i}$. It is again possible to scale remaining non-diagonal parameters to one simultaneously and we thus obtain classes of derivations with the canonical representatives in the form

$$D_8^{(i)} = \begin{pmatrix} & & \vdots & \\ & & 1 & \\ \ddots & & \vdots & \\ & -i & 0 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \quad i \in \{0,\dots,k-3\}.$$

The commutation relations of the extensions $\mathfrak{s}_{2k+1,8}^{(i)}$ are

$$\begin{aligned}
[e_{2\ell-1}, f_1] &= (k-\ell-1)e_{2\ell-1}, && \forall \ell \in \{1,\dots,i+2\},\\
[e_{2\ell-1}, f_1] &= e_{2(\ell-2-i)} + (k-\ell-1)e_{2\ell-1}, && \forall \ell \in \{i+3,\dots,k\},\\
[e_{2\ell}, f_1] &= (k-\ell-1-i)e_{2\ell}, && \forall \ell \in \{1,\dots,k-1\},\\
[e_{2k}, f_1] &= e_{2k-1} + e_{2k}.
\end{aligned}$$

The centre is empty regardless of the value of i, while the dimensions of elements of lower and derived series depend on i like:

$$i \neq 0 \implies CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0], \quad US = [0],$$

$$i = 0 \implies CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0], \quad US = [0].$$

2.1.4. $a = 0$
As in the case 1.1.4 it is not possible to eliminate c_{2i} and b_{2i-1} for every $i \in \{1,\dots,k-2\}$. Thus we come to the matrix of the derivation in the form

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & b_1 & 0 \\ & c & \cdots & \cdots & c_2 & 0 & 0 \\ & & \ddots & & \vdots & \vdots & \vdots \\ & & & & 0 & b_{2k-5} & 0 \\ & & & & c_{2k-4} & 0 & 0 \\ & & & \ddots & 0 & 0 & 0 \\ & & & & c & 0 & 0 \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{pmatrix},$$

where it is also possible to scale one of the parameters to $(\pm)1$. Depending on the number of remaining parameters b_i and c_j that are null we get one of the families of extensions whose number of parameters ranges from $2k-5$ to 0. All of these extensions share the dimensions of elements of their characteristic series:

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 0], \quad US = [1].$$

2.2. $c = 2a$

Again the case is very similar to the case 1.2. It is not possible to eliminate parameters a_{2k-1} and c_{2k-3}, they may be scaled to 1 nonetheless. The resulting derivation is of the form

$$D_9 = \begin{pmatrix} \ddots & \ddots & & & \\ & 2 & 1 & & \\ & & 2 & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix},$$

the corresponding extension $\mathfrak{s}_{2k+1,9}$ is given by the commutation relations

$$\begin{aligned} [e_{2\ell-1}, f_1] &= (k+1-\ell)e_{2\ell-1}, && \forall \ell \in \{1, \dots, k\}, \\ [e_{2\ell}, f_1] &= e_{2\ell-1} + (k-\ell+1)e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2k}, f_1] &= e_{2k-1} + e_{2k}, \end{aligned}$$

and the characteristic series are

$$CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0], \quad US = [0].$$

3. $a \neq b = c$

The parameter b_{2k-2} may not be eliminated from the lower 3×3 block if nonvanishing, while the parameters a_{2k-2} and a_{2k-1} are easily set to zero by a suitable automorphism.

3.1. $a \neq 0$
Again, the parameter c_{2k-3} is eliminated analogously to the six-dimensional case. To transform out the rest of the parameters we use the automorphisms $\mathbb{C}_\ell^{(3)}$ given as a special case of unitriangular automorphism (3.9) with

$$\gamma_{2\ell-1} := -\frac{c_{2\ell-1}}{(k-\ell)a}, \qquad \beta_{2\ell-1} := -\frac{(k-\ell)b_{2\ell-1}a + c_{2\ell-1}b_{2k-2}}{(k-\ell)^2a^2},$$
$$\gamma_{2\ell} := -\frac{c_{2\ell}}{(k-\ell-1)a}, \qquad \beta_{2\ell} := -\frac{(k-\ell-1)b_{2\ell}a + c_{2\ell}b_{2k-2}}{(k-\ell-1)^2a^2}.$$

The resulting derivation is of the form

$$D_{10}(b) = \begin{pmatrix} k-1+b & 0 & \cdots & 0 & 0 & 0 \\ & k-2+b & & \vdots & \vdots & \vdots \\ & & \ddots & 0 & 0 & 0 \\ & & & b & 1 & 0 \\ & & & & b & 0 \\ & & & & & 1 \end{pmatrix},$$

the resulting commutation relations of $\mathfrak{s}_{2k+1,10}(b)$ are

$$\begin{aligned} [e_{2\ell-1}, f_1] &= e_{2\ell-2} + (k+b-\ell)e_{2\ell-1}, && \forall \ell \in \{1, \dots, k\}, \\ [e_{2\ell}, f_1] &= (k-1-\ell+b)e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2k}, f_1] &= e_{2k}. \end{aligned}$$

Its characteristic series depend on b like

$$b \neq 0 \implies CS = [2k+1, 2k], \ DS = [2k+1, 2k, 2k-3, 0],$$

$$b = 0 \implies CS = [2k+1, 2k-1, 2k-2], \ DS = [2k+1, 2k-1, 2k-4, 0],$$
$$b \neq 1-k, 2-k \implies US = [0],$$
$$b = 1-k, 2-k \implies US = [1].$$

3.2. $a = 0$
In this case it is not possible to eliminate c_{2k-3}.

3.2.1. $b_{2k-2} \neq 0$
It is possible to eliminate parameters $b_1, \dots, b_{2k-4}$ using automorphisms $\mathbb{C}_\ell^{(3)}$ given by

$$\gamma_{2\ell-1} := \frac{b_{2\ell-1}}{b_{2k-2}}, \qquad \gamma_{2\ell} := \frac{b_{2\ell}}{b_{2k-2}},$$

where ℓ goes from 1 to $k-2$. Similarly as with $\mathbb{C}_\ell$, we have to apply $\mathbb{C}_\ell^{(3)}$ from the one with the highest index to the one with lowest index. Moreover, it is possible to transform out c_{2k-4} via the automorphism with $\gamma_{2k-3} := -\frac{c_{2k-4}}{b_{2k-2}}$. It is not possible to eliminate any of the remaining parameters and scaling b_{2k-2} along with c_{2k-3} to 1 disallows to scale any of the remaining parameters; thus we arrive to the family of classes:

$$D_{11}(c_{2k-5},\ldots,c_1) = \begin{pmatrix} 1 & 1 & \cdots & & \cdots & c_1 & 0 & 0 \\ & 1 & \ddots & & & \vdots & \vdots & \vdots \\ & & \ddots & & & c_{2k-5} & 0 & 0 \\ & & & & \ddots & 0 & 0 & 0 \\ & & & & \ddots & 1 & 0 & 0 \\ & & & & & 1 & 1 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 0 \end{pmatrix},$$

the corresponding extensions $\mathfrak{s}_{11}(c_{2k-5},\ldots,c_1)$ are given by commutation relations

$$\begin{aligned} [e_{2\ell-1}, f_1] &= e_{2\ell-2} + e_{2\ell-1}, && \forall \ell \in \{1,\ldots,k\}, \\ [e_2, f_1] &= e_1 + e_2, \\ [e_{2\ell}, f_1] &= \sum_{r=1}^{2\ell-3} c_{r+2(k-\ell+1)} e_r + e_{2\ell-1} + e_{2\ell}, && \forall \ell \in \{2,\ldots,k-1\}, \\ [e_{2k}, f_1] &= 0. \end{aligned}$$

If $c_{2k-3} = 0$ there is no need to scale it and we are left with free parameters to rescale one of the remaining nonvanishing parameters. Leaving the family of extensions overparametrized (different values of a parameter may lead to isomorphic algebras) we can write the commutation relations in the form

$$\begin{aligned} [e_{2\ell-1}, f_1] &= e_{2\ell-2} + e_{2\ell-1}, && \forall \ell \in \{1,\ldots,k\}, \\ [e_{2\ell}, f_1] &= e_{2\ell}, && \forall \ell \in \{1,2\}, \\ [e_{2\ell}, f_1] &= \sum_{r=1}^{2\ell-3} c_{r+2(k-\ell+1)} e_r + e_{2\ell}, && \forall \ell \in \{3,\ldots,k-1\}, \\ [e_{2k}, f_1] &= 0. \end{aligned}$$

Characteristic series of all of these algebras are readily seen to be

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 0], \quad US = [0].$$

3.2.2. $b_{2k-2} = 0$ and $c_{2k-3} \neq 0$
Using automorphisms $\mathbb{C}_\ell^{(32)}$ given by

$$\beta_{2\ell} := \frac{b_{2\ell-1}}{b_{2k-2}}, \qquad \gamma_{2\ell} := \frac{c_{2\ell-1}}{b_{2k-2}},$$

we eliminate $b_{2\ell-1}$ and $c_{2\ell-1}$ for every $\ell \in \{1, \dots, k-2\}$. Furthermore, it is possible to eliminate c_{2k-4} via the automorphism with $\beta_{2k-2} = \frac{c_{2k-4}}{c_{2k-3}}$. Thus, we have transformed the derivation into the form

$$D_{12}(b_i, c_j) = \begin{pmatrix} 1 & 1 & \cdots & & \cdots & 0 & 0 & 0 \\ & 1 & \cdots & & \cdots & c_2 & b_2 & 0 \\ & & \ddots & & & \vdots & \vdots & \vdots \\ & & & & & c_{2k-6} & b_{2k-6} & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & b_{2k-4} & 0 \\ & & & & \ddots & 1 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 0 \end{pmatrix}.$$

It is of course possible to scale one of the remaining parameters b_i to $(\pm)1$ provided that it is not null. The corresponding extensions $\mathfrak{s}_{2k+1,12}(b_i, c_j)$ are given by relations

$$\begin{aligned} [e_{2\ell-1}, f_1] &= \sum_{r=1}^{\ell-2} b_{2r} e_{2(r-k+\ell)} + e_{2\ell-1}, && \forall \ell \in \{1, \dots, k\}, \\ [e_{2\ell}, f_1] &= \sum_{r=1}^{\ell-2} c_{2r} e_{2(r-k+\ell+1)} + e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2k}, f_1] &= 0, \end{aligned}$$

their characteristic series are

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 0], \quad US = [0].$$

3.2.3. $b_{2k-2} = 0$ and $c_{2k-3} = 0$
The matrix of the derivation is of the form

$$\begin{pmatrix}
1 & 0 & \cdots & & \cdots & c_1 & b_1 & 0 \\
 & 1 & \cdots & & \cdots & c_2 & b_2 & 0 \\
 & & \ddots & & & \vdots & \vdots & \vdots \\
 & & & & & c_{2k-6} & b_{2k-6} & 0 \\
 & & & & & c_{2k-5} & b_{2k-5} & 0 \\
 & & & & & c_{2k-4} & b_{2k-4} & 0 \\
 & & & & \ddots & 0 & 0 & 0 \\
 & & & & & 1 & 0 & 0 \\
 & & & & & & 1 & 0 \\
 & & & & & & & 0
\end{pmatrix}. \tag{3.15}$$

The possibility of its simplification depend on its parameters, e.g. if $b_{2k-4} \neq 0$ we are able to eliminate $b_{2k-5}, \dots, b_1$. In case $b_{2k-4} = 0$ the problem becomes rather complicated. Nevertheless, any solvable $2k+1$-dimensional algebra with nilradical $\mathfrak{r}_{2k}$ and the corresponding part of the action of the element $f \notin \mathfrak{r}_{2k}$ given by the lower 4×4 submatrix can be transformed into the form, where the action of f is given by the matrix (3.15), while some of parameters might be redundant. Characteristic series of these extensions are the same regardless of values of the parameters:

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 0], \quad US = [0].$$

4. $a = c \neq b$
a_{2k-2} remains in the lower 3×3 block.

4.1. $b \neq 0$
c_{2k-3} is transformed out by the automorphism analogous to the one in the six-dimensional case 4.1.

4.1.1. $a \neq 0,\ b \neq \ell a,\ \forall \ell \in \{2, \dots, k-1\} \cup \{2-k, \dots, -1\}$
The rest of the parameters is eliminated via $\mathbb{C}_\ell$ defined in the previous section with $c = a$. The resulting derivation is of the form

$$D_{13}(b) = \begin{pmatrix}
k-1+b & \cdots & \cdots & 0 & 0 & 0 \\
 & \ddots & & \vdots & \vdots & \vdots \\
 & & \ddots & 0 & 0 & 0 \\
 & & & 1 & 0 & 1 \\
 & & & & b & 0 \\
 & & & & & 1
\end{pmatrix},$$

the corresponding extension $\mathfrak{s}_{2k+1,13}(b)$ is given by commutation relations

$$\begin{aligned}
[e_{2\ell-1}, f_1] &= (k+b-\ell)e_{2\ell-1}, && \forall \ell \in \{1,\dots,k\},\\
[e_{2\ell}, f_1] &= (k-\ell)e_{2\ell}, && \forall \ell \in \{1,\dots,k-1\},\\
[e_{2k}, f_1] &= e_{2k-2} + e_{2k},
\end{aligned}$$

its characteristic series depend on the parameter b:

$$b \neq 0 \implies CS = [2k+1, 2k],\ DS = [2k+1, 2k, 2k-3, 0],$$

$$b = 0 \implies CS = [2k+1, 2k-1, 2k-2],\ DS = [2k+1, 2k-1, 2k-4, 0],$$

$$b \neq 1-k \implies US = [0], \qquad b = 1-k \implies US = [1].$$

4.1.2. $b = ia$ for some $i \in \{2,\dots,k-1\}$

Using the special form of automorphisms $\mathbb{C}_\ell$ from the preceding case with $b = ia$ we eliminate all $\mathbb{M}_\ell$ for every $\ell \neq k-i$. To eliminate every parameter in $\mathbb{M}_{k-i}$ but $b_{2(k-i)}$ we modify $\mathbb{C}_{k-i}$ by setting $\beta_{2(k-i)} := 0$. It is possible to simultaneously scale remaining non-diagonal parameters a_{2k-2} and $b_{2(k-i)}$ to 1 using diagonal automorphism. Thus we obtain the classes given by the matrices of the form

$$D_{14}^{(i)} = \begin{pmatrix}
k-1+i & 0 & \cdots & \cdots & 0 & 0 & 0\\
 & k-1 & \ddots & & \vdots & \vdots & \vdots\\
 & & \ddots & & 0 & 1 & 0\\
 & & & \ddots & \vdots & \vdots & \vdots\\
 & & & \ddots & 0 & 0 & 0\\
 & & & & 1 & 0 & 1\\
 & & & & & i & 0\\
 & & & & & & 1
\end{pmatrix}, \quad i \in \{2,\dots,k-1\},$$

the corresponding extensions $\mathfrak{s}_{2k+1,14}^{(i)}$ are given by the commutation relations

$$\begin{aligned}
[e_{2\ell-1}, f_1] &= (k+i-\ell)e_{2\ell-1}, && \forall \ell \in \{1,\dots,k-i-1\},\\
[e_{2\ell-1}, f_1] &= e_{2(k-i-\ell+1)} + (k+i-\ell)e_{2\ell-1}, && \forall \ell \in \{k-i,\dots,k\},\\
[e_{2\ell}, f_1] &= (k-\ell)e_{2\ell}, && \forall \ell \in \{1,\dots,k-1\},\\
[e_{2k}, f_1] &= e_{2k-2} + e_{2k}.
\end{aligned}$$

Characteristic series of these extension are the same for every i:

$$CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0], \quad US = [0].$$

4.1.3. $b = ia$ for some $i \in \{2-k, \dots, -1\}$
Similarly to the preceding case we eliminate every $\mathbb{M}_\ell$ for $\ell \neq k+i-1$ using the same automorphisms. Changing $\gamma_{2(k+i-1)-1}$ to 0 in $\mathbb{C}_{k+i-1}$ we obtain the automorphism that allows to transform out the remaining parameters in $\mathbb{M}_{k+i-1}$ except for $c_{2(k+i-1)-1}$. Again, we scale the remaining non-diagonal parameters to 1 using diagonal automorphism and obtain the derivation classes given by representatives

$$D_{15}^{(i)} = \begin{pmatrix} k-1+i & 0 & \cdots & \cdots & 0 & 0 & 0 \\ & k-1 & \ddots & & \vdots & \vdots & \vdots \\ & & \ddots & & 1 & 0 & 0 \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & \ddots & 0 & 0 & 0 \\ & & & & 1 & 0 & 1 \\ & & & & & i & 0 \\ & & & & & & 1 \end{pmatrix}, \quad i \in \{2-k, \dots, -1\},$$

the corresponding extensions $\mathfrak{s}_{2k+1,15}^{(i)}$ are given by the commutation relations

$$\begin{aligned} [e_{2\ell-1}, f_1] &= (k+i-\ell)e_{2\ell-1}, && \forall \ell \in \{1, \dots, k\}, \\ [e_{2\ell}, f_1] &= (k-\ell)e_{2\ell}, && \forall \ell \in \{1, \dots, -i\}, \\ [e_{2\ell}, f_1] &= e_{2(\ell+i)-1} + (k-\ell)e_{2\ell}, && \forall \ell \in \{1-i, \dots, k-1\}, \\ [e_{2k}, f_1] &= e_{2k-2} + e_{2k}. \end{aligned}$$

Characteristic series of these extensions are the same for every i:

$$CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0], \quad US = [0].$$

4.1.4. $a = 0$
This is completely analogous to the cases 1.1.4 and 2.1.4. The resulting derivations are of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & b_1 & 0 \\ & 0 & \cdots & \cdots & c_2 & 0 & 0 \\ & & \ddots & & \vdots & \vdots & \vdots \\ & & & & 0 & b_{2k-5} & 0 \\ & & & & c_{2k-4} & 0 & 0 \\ & & & \ddots & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 \\ & & & & & 1 & 0 \\ & & & & & & 0 \end{pmatrix}$$

where additional scaling may be done in the same manner as in the above mentioned cases. Again the characteristic series are the same for every member of this family:

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 0], \quad US = [1].$$

4.2. $b = 0$
The resulting derivation is of the form

$$D_{16} = \begin{pmatrix} k-1 & \cdots & & \cdots & 0 & 0 & 0 \\ & \ddots & & & \vdots & \vdots & \vdots \\ & & & & 0 & 0 & 0 \\ & & & \ddots & 1 & 0 & 0 \\ & & & & 1 & 0 & 1 \\ & & & & & 0 & 0 \\ & & & & & & 1 \end{pmatrix},$$

the corresponding extension $\mathfrak{s}_{2k+1,16}$ is given by the commutation relations

$$\begin{aligned} [e_{2\ell-1}, f_1] &= (k-\ell)e_{2\ell-1}, & \forall \ell \in \{1, \dots, k\}, \\ [e_{2\ell}, f_1] &= e_{2\ell-1} + (k-\ell)e_{2\ell}, & \forall \ell \in \{1, \dots, k-1\}, \\ [e_{2k}, f_1] &= e_{2k-2} + e_{2k}. \end{aligned}$$

Characteristic series of these extensions are the same for every i:

$$CS = [2k+1, 2k-1], \quad DS = [2k+1, 2k-1, 2k-4, 0], \quad US = [0].$$

5. $a = b = c$
As in the six-dimensional case we do not take the case of $a = 0$ into account and the operations done in the lower 4×4 block are the same as well. To transform out the non-diagonal parameters outside the lower block we use the automorphisms $\mathbb{C}_\ell^{(3)}$. Hence we obtain the class of derivation given by the representative:

$$D_{17} = \begin{pmatrix} k & \cdots & \cdots & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \ddots & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix},$$

that is, we obtain the extension $\mathfrak{s}_{2k+1,17}$ with relations:

$$
\begin{aligned}
[e_1, f_1] &= ke_1, && \\
[e_{2\ell-1}, f_1] &= e_{2\ell-2} + (k-\ell+1)e_{2\ell-1}, && \forall \ell \in \{1, \dots, k\}, \\
[e_{2\ell}, f_1] &= (k-\ell)e_{2\ell}, && \forall \ell \in \{1, \dots, k-1\}, \\
[e_{2k}, f_1] &= e_{2k-1} + e_{2k}. &&
\end{aligned}
$$

Its characteristic series are easily found to be:

$$CS = [2k+1, 2k], \quad DS = [2k+1, 2k, 2k-3, 0], \quad US = [0].$$

3.4 Extensions by Two Elements

All the solvable $2k+2$-dimensional extensions of the nilpotent algebra $\mathfrak{r}_{2k}$ are found in this section. The approach is very similar to that in subsection 2.2.3, the points i) through vi) from the beginning of the aforementioned subsection are referenced throughout this section and the notation D, d and S, s has the analogous meaning. Apparently, the behaviour in the lower 4×4 submatrices is exactly the same regardless of k. Thus the previous results obtained for the six-dimensional ($k = 3$) case are used here without further notice and the cases are divided analogously. For the sake of simplicity the action of f_1 and f_2 on $\mathfrak{r}_{2k}$ is written in a rather compact form; the elements e_i with $i \leq 0$ are to be understood as zeroes.

Case 1 $A \neq 0 \vee a \neq 0$

Without loss of generality we assume that $A \neq 0$.

a) $b \neq 0$

By the same argument as in the six-dimensional case the lower 3×3 blocks can be transformed in the form

$$S(C) = \begin{pmatrix} C & 0 & 0 \\ & 1 & 0 \\ & & 2 \end{pmatrix}, \quad s(c) = \begin{pmatrix} c & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix},$$

and this form is assumed while imposing the conditions below.

i) $C \notin \{1, 2, 3\} \cup \{5, 7, \dots, 2k-1\} \cup \{-1, -3, \dots, 5-2k\}$
This leads to both derivation being diagonal, thus fully determined by lower

3×3 blocks:

$$S(C) = \begin{pmatrix} C & 0 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}, \quad s(c) = \begin{pmatrix} c & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}.$$

Note that we have subtracted d from D and the tilde above C was omitted. Analogously to the six-dimensional case if either $c \neq 0$ or $C \neq 4 - 2k$ the commutator $[f_1, f_2]$ can be set to 0. In that case we obtain the extensions $\mathfrak{s}_{2k+2,1}(C, c)$:

$\mathfrak{s}_{2k+2,1}(C,c)$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-1-\ell)+C]e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell-1}$	$ce_{2\ell}$	0	0

$$C \neq 0 \vee c \neq 0 \implies CS = [2k+2, 2k] \quad DS = [2k+2, 2k, 2k-3, 0]$$
$$C = c = 0 \implies CS = [2k+2, 2k-1] \quad DS = [2k+2, 2k-1, 2k-4, 0]$$
$$US = [0]$$

If on the other hand $c = 0$ and $C = 4-2k$ we must assume that $[f_1, f_2] = \alpha_2 e_2$, where we change the basis so that $\alpha_2 = 1$ in the same manner as in the the case of $k = 3$. The resulting extension is

$\mathfrak{s}_{2k+2,1'}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$[2(k-\ell)+1]e_{2\ell-1}$	$(2-2\ell)e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell}$	0	0	e_2

$$S = [2k+2, 2k] \quad DS = [2k+2, 2k, 2k-3, 0] \quad US = [1]$$

ii) $C = i$ for some $i \in \{5, 7, \dots, 2k-1\}$
If $c \neq 1$ we are able to eliminate c_{2k-i} in d and every non-diagonal parameter but C_{2k-i} in D via suitable automorphism. The remaining parameters must be set to 0 due to the condition $[D, d] \in \mathfrak{Inn}$. Therefore, we obtain the special case of the preceding extension $\mathfrak{s}_{2k+2,1}(i, 1)$. Hence, we assume that $c = 1$. In that case, we are unable to eliminate C_{2k-i} as well as c_{2k-i} via the automorphisms, while the condition $[D, d] \in \mathfrak{Inn}$ is satisfied regardless of the values of these parameters. Using suitable diagonal automorphism (3.7) we scale C_{2k-i} to 1 provided that it is not null. Thus the derivations take up

the form

$$D = \begin{pmatrix} \cdots & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & 1 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \ddots & 0 & 0 & 0 \\ & i & 0 & 0 \\ & & 1 & 0 \\ & & & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} \cdots & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \cdots & c_{2k-i} & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \ddots & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 0 \end{pmatrix}.$$

Subtracting $c_{2k-i}D$ from d we come to $k-2$ one-parameters families of extensions:

$\mathfrak{s}^{(i)}_{2k+2,2}(\tau)$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$[2(k-\ell)+1]e_{2\ell-1}$	$e_{2(\ell+1)-i}+(2-2\ell)e_{2\ell}$	$2e_{2k}$	0
f_2	$\{1-\tau[2(k-\ell)+1]\}e_{2\ell-1}$	$[1-\tau(2-2\ell)]e_{2\ell}$	$-2\tau e_{2k}$	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

If C_{2k-i} is 0 we scale c_{2k-i} instead to obtain the extensions

$\mathfrak{s}^{(i)}_{2k+2,2'}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$[2(k-\ell)+1]e_{2\ell-1}$	$(2-2\ell)e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell}$	$e_{2(\ell+1)-i}+e_{2\ell}$	0	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

iii) $C=i$ for some $i\in\{-1,-3,\dots,5-2k\}$
As in the preceding case, if $c\neq 1$ we obtain the special case of the preceding extension $\mathfrak{s}_{2k+2,1}(i,1)$. Thus, let us assume that $c=1$. In this case, we are unable to eliminate B_{2k-3+i} as well as b_{2k-3+i} via the automorphisms. It is possible to scale one of these parameters to 1 if they do not equal 0. Hence we obtain $k-2$ one-parametric families

$\mathfrak{s}^{(i)}_{2k+2,3}(\tau)$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$e_{2\ell-3+i}+2(k-\ell)e_{2\ell-1}$	$[2(k-1-\ell)+i]e_{2\ell}$	$2e_{2k}$	0
f_2	$[1-2\tau(k-\ell)]e_{2\ell-1}$	$\{1-\tau[2(k-1-\ell)+i]\}e_{2\ell}$	$-2\tau e_{2k}$	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

and $k-2$ extensions

$\mathfrak{s}^{(i)}_{2k+2,3'}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-1-\ell)+i]e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell-3+i}+e_{2\ell-1}$	$e_{2\ell}$	0	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

iv) $C=3$
Similarly as in the preceding cases we assume that $c=1$; otherwise we would find a special case of $\mathfrak{s}_{2k+2,1}$. Parameters C_{2k-3} and c_{2k-3} do not vanish, hence we obtain one-parametric family

$\mathfrak{s}_{2k+2,4}(\tau)$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$e_{2\ell-1}+[2(k-\ell)+1]e_{2\ell}$	$2e_{2k}$	0
f_2	$[1-2\tau(k-\ell)]e_{2\ell-1}$	$\{1-\tau[2(k-\ell)+1]\}e_{2\ell}$	$-2\tau e_{2k}$	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

and one extension

$\mathfrak{s}_{2k+2,4'}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-\ell)+1]e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell-1}$	$e_{2\ell-1}+e_{2\ell}$	0	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

v) $C=2$
Again, we assume that $c=0$ to obtain a new extension. Parameters A_{2k-2} and a_{2k-2} may be of arbitrary value. Using diagonal automorphism we obtain one-parametric family

$\mathfrak{s}_{2k+2,5}(\tau)$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-\ell)]e_{2\ell}$	$e_{2k-2}+2e_{2k}$	0
f_2	$[1-2\tau(k-\ell)]e_{2\ell-1}$	$-\tau[2(k-\ell)]e_{2\ell}$	$-2\tau e_{2k}$	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

and one extension

$\mathfrak{s}_{2k+2,5}$	$e_{2\ell-1},\ \ell\in\{1,\dots,k\}$	$e_{2\ell},\ \ell\in\{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-\ell)]e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell-1}$	0	e_{2k-2}	0

$$CS=[2k+2,2k] \qquad DS=[2k+2,2k,2k-3,0] \qquad US=[0]$$

vi) $C = 1$

Let us assume that $c = 1$ to obtain a new extension. Parameters B_{2k-2} and b_{2k-2} may be of arbitrary value. Using diagonal automorphism we obtain a one-parametric family

$\mathfrak{s}_{2k+2,6}(\tau)$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$e_{2\ell-2} + 2(k-\ell)e_{2\ell-1}$	$[2(k-1-\ell)+1]e_{2\ell}$	$2e_{2k}$	0
f_2	$[1-2\tau(k-\ell)]e_{2\ell-1}$	$\{1-\tau[2(k-1-\ell)+1]\}e_{2\ell}$	$-2\tau e_{2k}$	0

$$CS = [2k+2, 2k] \qquad DS = [2k+2, 2k, 2k-3, 0] \qquad US = [0]$$

and one extension

$\mathfrak{s}_{2k+2,6'}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-1-\ell)+1]e_{2\ell}$	$2e_{2k}$	0
f_2	$e_{2\ell-2} + e_{2\ell-1}$	$e_{2\ell}$	0	0

$$CS = [2k+2, 2k] \qquad DS = [2k+2, 2k, 2k-3, 0] \qquad US = [0]$$

b) b = 0

The transforms carried out in the lower 4×4 blocks are evidently the same as in the six-dimensional case. Also, it is always possible to eliminate non-diagonal parameters outside these blocks. The only difference in the generic even-dimensional case is that we must devise a more general form of the condition leading to the the possibility of setting $[f_1, f_2] := 0$, namely $B \neq -4$ becomes $B \neq 2(1-k)$

i) $B \neq 2$

If $B \neq 2(1-k)$ we are able to set $[f_1, f_2]$ to zero. Hence we obtain the extensions

$\mathfrak{s}_{2k+2,7}(B)$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$[2(k-\ell)+B]e_{2\ell-1}$	$2(k-1-\ell)e_{2\ell}$	$2e_{2k}$	0
f_2	0	$e_{2\ell}$	0	0

$$CS = [2k+2, 2k] \qquad DS = [2k+2, 2k, 2k-3, 0] \qquad US = [0]$$

On the other hand, if $B = 2(1-k)$ then we set $[f_1, f_2] := e_1$ and get the extension

$\mathfrak{s}_{2k+2,7}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$2(1-\ell)e_{2\ell-1}$	$2(k-1-\ell)e_{2\ell}$	$2e_{2k}$	0
f_2	0	$e_{2\ell}$	0	e_1

$$CS = [2k+2, 2k] \qquad DS = [2k+2, 2k, 2k-3, 0] \qquad US = [1]$$

ii) $B = 2$
In complete analogy to the six-dimensional case and preceding results we obtain

$\mathfrak{s}_{2k+2,8}(\tau)$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$[2(k-\ell)+B]e_{2\ell-1}$	$2(k-1-\ell)e_{2\ell}$	$e_{2k-1}+2e_{2k}$	0
f_2	$-\tau[2(k-\ell)+B]e_{2\ell-1}$	$[1-2\tau(k-1-\ell)]e_{2\ell}$	$-2\tau e_{2k}$	0

$$CS = [2k+2, 2k] \qquad DS = [2k+2, 2k, 2k-3, 0] \qquad US = [0]$$

On the other hand, if $B = 2(k-1)$ then we set $[f_1, f_2] := e_1$ and get the extension

$\mathfrak{s}_{2k+2,8'}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1
f_1	$[2(k-\ell)+B]e_{2\ell-1}$	$2(k-1-\ell)e_{2\ell}$	$2e_{2k}$	0
f_2	0	$e_{2\ell}$	0	e_1

$$CS = [2k+2, 2k] \qquad DS = [2k+2, 2k, 2k-3, 0] \qquad US = [0]$$

Case 2 $A = 0 \wedge a = 0$

The situation in the lower 4×4 corner is the same as in the six-dimensional case. By the case **1.1.4** of the preceding section we know that we are unable to eliminate $C_2, C_4, \dots C_{2k-4}$ and $B_1, B_3, \dots B_{2k-5}$ in D as well as corresponding parameters in d via the automorphisms. Evidently, $[D, d] \in \mathfrak{Inn}$ regardless of the values of these parameters, hence we have the derivations in the form

$$D = \begin{pmatrix} \cdots & 0 & B_1 & 0 \\ \cdots & C_2 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \ddots & 0 & B_{2k-5} & 0 \\ \ddots & C_{2k-4} & 0 & 0 \\ \ddots & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 2 & 0 \\ & & & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \cdots & 0 & b_1 & 0 \\ \cdots & c_2 & 0 & 0 \\ & \vdots & \vdots & \vdots \\ \ddots & 0 & b_{2k-5} & 0 \\ \ddots & c_{2k-4} & 0 & 0 \\ \ddots & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ & & & 0 \end{pmatrix}.$$

The diagonal automorphism (3.7) allows to scale one of the remaining nonzero parameters to $(\pm)1$. As any number of these parameters may be zero, we obtain $2k-4$ families of non-isomorphic $2k+2$-dimensional solvable extension of $\mathfrak{r}_{2k}$ with number of parameters ranging from 0 to $2k-5$. All of these extensions have the same characteristic series

$$CS = [2k+2, 2k-1] \qquad DS = [2k+2, 2k-1, 0] \qquad US = [0]$$

3.5 Extension by Three Elements

By the very same arguments as in the case $k = 3$ we get exactly one $2k+3$-dimensional solvable extension of $\mathfrak{r}_{2k}$ given by

$\mathfrak{s}_{2k+3}$	$e_{2\ell-1},\ \ell \in \{1,\dots,k\}$	$e_{2\ell},\ \ell \in \{1,\dots,k-1\}$	e_{2k}	f_1	f_2
f_1	$2(k-\ell)e_{2\ell-1}$	$[2(k-\ell)-1]e_{2\ell}$	$2e_{2k}$	0	0
f_2	0	$e_{2\ell}$	0	0	0
f_3	$e_{2\ell-1}$	0	0	0	0

$$CS = [2k+3, 2k] \qquad DS = [2k+3, 2k, 2k-3, 0] \qquad US = [0]$$

Chapter 4

Generalized Casimir Invariants

The invariants of the coadjoint representation of the nilpotent algebra $\mathfrak{r}_{2k}$ and of its solvable extensions, that is their generalized Casimir invariants are found in this chapter. Assume that c^i_{jk} are the structure constants of a given Lie algebra $\mathfrak{g}$ w. r. t. a chosen basis $(x_i)_1^n$. Then the basis of the coadjoint representation of $\mathfrak{g}$ is given by the partial differential operators of the form

$$\hat{X}_k = x_i c^i_{jk} \frac{\partial}{\partial x_j}. \tag{4.1}$$

A generalized Casimir invariant of $\mathfrak{g}$ is then a smooth function $I = I(x_i)$ acting on $\mathfrak{g}^*$ such that

$$\hat{X}_k I \equiv 0, \qquad \forall k. \tag{4.2}$$

More on this subject can be found in the third chapter of [6].

4.1 Method of Characteristics

Finding generalized Casimir invariants of $\mathfrak{g}$ involves solving the system of linear PDEs (4.2). Method of characteristics allows to solve 1st order linear homogeneous PDE by transforming the problem of one PDE into a system of ODEs. Let $\vec{x} = (x_i)_1^n$ and let us have a linear PDE

$$\hat{X} I(\vec{x}) = f_j(\vec{x}) \frac{\partial}{\partial x_j} I(\vec{x}) = 0, \tag{4.3}$$

then solving the following system

$$\begin{aligned} \dot{\alpha}_j(t) &= f_j(\alpha_1(t), \dots, \alpha_n(t)) \\ \alpha_j(0) &= x_j \end{aligned} \qquad \forall j \in \{1, \dots, n\} \tag{4.4}$$

of ODEs called characteristic yields n functions $\alpha_i(t)$. These are n components of the flow of the vector field $\hat{X}$. By setting any suitable, that is one that does not depend on t trivially, α_i to an arbitrary fixed value we obtain the corresponding value of t. Substituting t back into the remaining solutions α_j, $j \neq i$ then provides $n-1$ functionally independent solutions of equation (4.3).

To solve a system of linear PDEs (4.2) we first pick one of these equations and find its solutions $\vec{\alpha} \equiv (\alpha_1, \dots, \alpha_{n-1})$ using method of characteristics. The next step is to pick another equation, say $\hat{X}_\ell$ and find the solutions of the equation

$$\hat{X}_\ell I(\vec{\alpha}) = (\hat{X}_\ell \alpha_i) \frac{\partial}{\partial \alpha_i} I(\vec{\alpha}) \equiv 0. \tag{4.5}$$

If we are able to find $\hat{X}_\ell \alpha_i$ as a function $h_i(\vec{\alpha})$ we can rewrite the equation in the form

$$\hat{X}_\ell I(\vec{\alpha}) = h_i(\vec{\alpha}) \frac{\partial}{\partial \alpha_i} I(\vec{\alpha}), \tag{4.6}$$

which is the same form as that of equation (4.3). Hence, we can solve it by the method of characteristics thus obtaining $n-2$ solutions β_i satisfying both chosen equations. To solve larger systems we simply repeat this process.

We have assumed that the system of PDEs consists of independent equations. Thus assuming that we have ℓ operators $\hat{X}_i$ acting on n-dimensional space we obtain $n-\ell$ solutions of the system. Evidently, the equation are not necessarily independent and the number of functionally independent solutions may be larger than $n-\ell$. The number of independent equations in the system (4.1) is equal to the rank of the following matrix

$$C = \left((x_i c^i_{jk})_{jk} \right). \tag{4.7}$$

Note that C is antisymmetric, thus its rank is even. The number of invariants to be found is $n - \mathrm{rank} C$.

4.2 Invariants of $\mathfrak{r}_{2k}$

We are looking for $2k-2$ invariants, they are easily found via the method of characteristics. The partial differential operators (4.1) are in the basis (e_i) of the form

$$\hat{E}_1 = \hat{E}_2 = 0,$$

$$\hat{E}_\ell = e_{\ell-2} \frac{\partial}{\partial e_{2k}}, \qquad \forall \ell \in \{1, \dots, 2k-1\},$$

$$\hat{E}_{2k} = \sum_{\ell=3}^{2k-1} e_{\ell-2} \frac{\partial}{\partial e_\ell}.$$

From the form of $\hat{E}_\ell$, it is evident that any invariant cannot depend on e_{2k}, thus only the last equation must be solved, that is we seek $I = I(e_1, \dots, e_{2k-1})$ such that

$$\hat{E}_{2k} I \equiv 0.$$

We first obtain the system of equations

$$\begin{aligned} &\dot{\alpha}_\iota(t) = 0, && \alpha_\iota(0) = e_\iota, && \iota \in \{1,2\}, \\ &\dot{\alpha}_\ell(t) = -\alpha_{\ell-2}(t), && \alpha_\ell(0) = e_\ell, && \ell \in \{3, \dots, 2k-1\}. \end{aligned}$$

The solution is readily found as

$$\begin{aligned} \alpha_\iota(t) &= e_\iota, && \iota \in \{1,2\}, \\ \alpha_{2\ell-1}(t) &= e_{2\ell-1} + \sum_{j=1}^{\ell-1} \frac{(-t)^j}{j!} e_{2(\ell-j)-1}, && \ell \in \{2, \dots, k\}, \\ \alpha_{2\ell}(t) &= e_{2\ell} + \sum_{j=1}^{\ell-1} \frac{(-t)^j}{j!} e_{2(\ell-j)}, && \ell \in \{2, \dots, k-1\}. \end{aligned}$$

Setting $\alpha_3 := 0$ we get $t = \frac{e_3}{e_1}$. Substituting this value back to α_i yields the invariants of $\mathfrak{r}_{2k}$ in the form

$$\begin{aligned} \alpha_\iota(\vec{e}) &= e_\iota, && \iota \in \{1,2\}, \\ \alpha_{2\ell-1}(\vec{e}) &= e_{2\ell-1} + \sum_{j=1}^{\ell-1} \left(-\frac{e_3}{e_1}\right)^j \frac{1}{j!} e_{2(\ell-j)-1}, && \ell \in \{3, \dots, k\}, \\ \alpha_{2\ell}(\vec{e}) &= e_{2\ell} + \sum_{j=1}^{\ell-1} \left(-\frac{e_3}{e_1}\right)^j \frac{1}{j!} e_{2(\ell-j)}, && \ell \in \{2, \dots, k-1\}. \end{aligned}$$

Multiplying invariants α_ℓ by $\alpha_1^{\ell-2}$ and $\alpha_1^{\ell-1}$ respectively, we obtain the polynomial functional basis of the generalized Casimir invariants of $\mathfrak{r}_{2k}$:

$$\begin{aligned} \alpha_\iota(\vec{e}) &= e_\iota, && \iota \in \{1,2\}, \\ \alpha_{2\ell-1}(\vec{e}) &= e_1^{\ell-2} e_{2\ell-1} + \sum_{j=1}^{\ell-1} \frac{(-e_3)^j e_1^{\ell-2-j}}{j!} e_{2(\ell-j)-1}, && \ell \in \{3, \dots, k\}, \\ \alpha_{2\ell}(\vec{e}) &= e_1^{\ell-1} e_{2\ell} + \sum_{j=1}^{\ell-1} \frac{(-e_3)^j e_1^{\ell-1-j}}{j!} e_{2(\ell-j)}, && \ell \in \{2, \dots, k-1\}. \end{aligned} \tag{4.8}$$

4.3 Invariants of Extensions by One Element

$\mathfrak{s}_{2k+1,1}(b, c)$

There are $2k-3$ functionally independent invariants to be found. The partial differential operators (4.1) are of the form

$$\hat{E}_1 = (k-1+b)e_1 \frac{\partial}{\partial f_1}, \quad \hat{E}_2 = (k-2+c)e_1 \frac{\partial}{\partial f_1},$$

$$\hat{E}_{2\ell-1} = (k-1-\ell+b)e_{2\ell-1} \frac{\partial}{\partial f_1} + e_{2\ell-3} \frac{\partial}{\partial e_{2k}}, \qquad \forall \ell \in \{2, \dots, k\},$$

$$\hat{E}_{2\ell} = (k-2-\ell+c)e_{2\ell} \frac{\partial}{\partial f_1} + e_{2\ell-2} \frac{\partial}{\partial e_{2k}}, \qquad \forall \ell \in \{2, \dots, k-1\},$$

$$\hat{E}_{2k} = e_{2k} \frac{\partial}{\partial f_1} + \sum_{\ell=3}^{2k-1} e_{\ell-2} \frac{\partial}{\partial e_\ell},$$

$$\hat{F}_1 = -\sum_{\ell=1}^{k} (k-\ell+b)e_{2\ell-1} \frac{\partial}{\partial e_{2\ell-1}} - \sum_{\ell=1}^{k-1} (k-1-\ell+c)e_{2\ell} \frac{\partial}{\partial e_{2\ell}} - e_{2k} \frac{\partial}{\partial e_{2k}}.$$

It is readily seen that the invariants must not depend on e_{2k} and on f_1. Thus the operators above effectively take up the form

$$\hat{E}_\ell = 0, \qquad \forall \ell \in \{1, \dots, 2k-1\},$$

$$\hat{E}_{2k} = \sum_{\ell=3}^{2k-1} e_{\ell-2} \frac{\partial}{\partial e_\ell},$$

$$\hat{F}_1 = -\sum_{\ell=1}^{k} (k-\ell+b)e_{2\ell-1} \frac{\partial}{\partial e_{2\ell-1}} - \sum_{\ell=1}^{k-1} (k-1-\ell+c)e_{2\ell} \frac{\partial}{\partial e_{2\ell}}.$$

The invariants (4.8) of $\mathfrak{r}_{2k}$ are the solutions of the equation $\hat{E}_{2k} I = 0$. The action of $\hat{F}_1$ on these solutions is given by equations

$$\begin{aligned} \hat{F}_1 \alpha_1 &= -(k-1+b)\alpha_1 \\ \hat{F}_1 \alpha_{2\ell} &= -[\ell(k-2) + (\ell-1)b + c]\alpha_{2\ell}, & \ell \in \{1, \dots, k-1\}, \\ \hat{F}_1 \alpha_{2\ell-1} &= -(\ell-1)(k-2+b)\alpha_{2\ell-1}, & \ell \in \{3, \dots, k\}. \end{aligned}$$

To find the solutions of the equation $\hat{F}_1 I = 0$ we construct the corresponding system of ODEs which looks as follows

$$\begin{aligned} \dot{\beta}_1(t) &= -(k-1+b)\beta_1(t), & \beta_1(0) &= \alpha_1, \\ \dot{\beta}_{2\ell}(t) &= -[\ell(k-2) + (\ell-1)b + c]\beta_{2\ell}(t), & \beta_{2\ell}(0) &= \alpha_{2\ell}, \ \ell \in \{1, \dots, k-1\}, \\ \dot{\beta}_{2\ell-1}(t) &= -(\ell-1)(k-2+b)\beta_{2\ell-1}(t), & \beta_{2\ell-1}(0) &= \alpha_{2\ell-1}, \ \ell \in \{3, \dots, k\}. \end{aligned}$$

their solutions are easily found to be

$$\begin{aligned}
\beta_1(t) &= \alpha_1 \mathrm{e}^{-(k-1+b)t}, \\
\beta_{2\ell}(t) &= \alpha_{2\ell} \mathrm{e}^{-[\ell(k-2)+(\ell-1)b+c]t}, && \ell \in \{1, \dots, k-1\}, \\
\beta_{2\ell-1}(t) &= \alpha_{2\ell-1} \mathrm{e}^{-(\ell-1)(k-2+b)t}, && \ell \in \{3, \dots, k\}.
\end{aligned}$$

In case of $b \neq 1-k$ we set $\beta_1(t) := 1$ thus obtaining $t = \frac{\ln \alpha_1}{k-1+b}$. Substituting this value to the remaining solutions β_i yields the invariants of $\mathfrak{s}_{2k+1,1}(b,c)$ in the form

$$\begin{aligned}
\beta_{2\ell} &= \alpha_{2\ell} \alpha_1^{-\frac{\ell(k-2)+(\ell-1)b+c}{k-1+b}}, && \ell \in \{1, \dots, k-1\}, \\
\beta_{2\ell-1} &= \alpha_{2\ell-1} \alpha_1^{-\frac{(\ell-1)(k-2+b)}{k-1+b}}, && \ell \in \{3, \dots, k\},
\end{aligned} \tag{4.9}$$

which after changing to suitable powers look like

$$\begin{aligned}
\beta_{2\ell} &= \alpha_{2\ell}^{k-1+b} \alpha_1^{-\ell(k-2)-(\ell-1)b-c}, && \ell \in \{1, \dots, k-1\}, \\
\beta_{2\ell-1} &= \alpha_{2\ell-1}^{k-1+b} \alpha_1^{-(\ell-1)(k-2+b)}, && \ell \in \{3, \dots, k\}.
\end{aligned}$$

If $b = 1-k$ we fix $\beta_5(t) := 1$ and get $t = \frac{\ln \alpha_5}{-2}$. The resulting invariants are obtained as

$$\begin{aligned}
\beta_1 &= \alpha_1, \\
\beta_{2\ell} &= \alpha_{2\ell} \alpha_5^{\frac{\ell-1-b+c}{2}}, && \ell \in \{1, \dots, k-1\}, \\
\beta_{2\ell-1} &= \alpha_{2\ell-1} \alpha_5^{1-\ell}, && \ell \in \{4, \dots, k\}.
\end{aligned}$$

Again, by using suitable powers we obtain a simpler functional basis of invariants in the form

$$\begin{aligned}
\beta_1 &= \alpha_1, \\
\beta_{2\ell} &= \alpha_{2\ell}^2 \alpha_5^{\ell-1-b+c}, && \ell \in \{1, \dots, k-1\}, \\
\beta_{2\ell-1} &= \alpha_{2\ell-1} \alpha_5^{1-\ell}, && \ell \in \{4, \dots, k\}.
\end{aligned}$$

$\mathfrak{s}^{(i)}_{2k+1,2}(b)$

Let us denote the simplified form of the partial differential operator corresponding to the added element by $\hat{F}_1^{(2)}$. The most useful form of this operator for our purpose is the following

$$\hat{F}_1^{(2)} = \hat{F}_1 - \sum_{j=i}^{k-1} e_{2(j-i)+1} \frac{\partial}{\partial e_{2j}}.$$

According to the relations (4.8) the invariants $\alpha_{2\ell-1}$ do not contain any element e_r with even r. Thus the action of $\hat{F}_1^{(2)}$ on these invariants is the same as the action of $\hat{F}_1$. However, its action on some of the basis elements with even indices changes to

$$\hat{F}_1^{(2)}\alpha_{2\ell} = -[\ell(k-2+b)+i]\alpha_{2\ell} - \alpha_1^i\alpha_{2(\ell-i)+1}, \quad \ell \in \{i, \dots, k-1\}.$$

Therefore, the corresponding ODEs change to

$$\dot{\beta}_{2\ell}(t) = -[\ell(k-2+b)+i]\beta_{2\ell}(t) - \beta_1^i(t)\beta_{2(\ell-i)+1}(t), \quad \ell \in \{i, \dots, k-1\}, \tag{4.10}$$

while the remaining ODEs remain unchanged. Hence, the solutions other than $\beta_{2\ell}$, $\ell \in \{i, \dots, k-1\}$ remain unchanged as well. Substituting them to (4.10) yields

$$\dot{\beta}_{2\ell}(t) = -[\ell(k-2+b)+i]\beta_{2\ell}(t) - \alpha_1^i\alpha_{2(\ell-i)+1}\mathrm{e}^{-[\ell(k+2-b)+i]t}, \quad \ell \in \{i, \dots, k-1\}.$$

Their solutions look like

$$\beta_{2\ell}(t) = (\alpha_{2\ell} - \alpha_1^i\alpha_{2(\ell-i)+1}t)\mathrm{e}^{-[\ell(k-2+b)+i]t}, \quad \ell \in \{i, \dots, k-1\},$$

the final step again depends on the value of b. If $b \neq 1-k$ we get the changed solutions in the form

$$\beta_{2\ell} = \left(\alpha_{2\ell} - \alpha_1^i\alpha_{2(\ell-i)+1}\frac{\ln\alpha_1}{k-1+b}\right)\alpha_1^{-\frac{[\ell(k-2+b)+i]}{k-1+b}}, \quad \ell \in \{i, \dots, k-1\},$$

while if $b = 1-k$, we obtain

$$\beta_{2\ell} = \left(\alpha_{2\ell} + \alpha_1^i\alpha_{2(\ell-i)+1}\frac{\ln\alpha_5}{2}\right)\alpha_5^{\frac{i-\ell}{2}}, \quad \ell \in \{i, \dots, k-1\}.$$

Thus, it is not possible to obtain the polynomial functional basis of generalized Casimir invariants of $\mathfrak{s}_{2k+1,2}(b)$.

Henceforth only the invariants differing from those of $\mathfrak{s}_{2k+1,1}(b,c)$ will be listed.

$\mathfrak{s}_{2k+1,3}(c)$

$c \neq 1-k-i:$

$$\beta_{2(i+2)-1} = \frac{\alpha_1^i \alpha_2}{k-2+i+c} \alpha_1^{-\frac{i(k+i+c)+2(k-2+c)}{k-1+i+c}} + \\ + \left(\alpha_{2(i+2)-1} - \frac{\alpha_1^i \alpha_2}{k-2+i+c}\right) \alpha_1^{-\frac{(i+1)(k-2+i+c)}{k-1+i+c}},$$

$$\beta_{2\ell-1} = \left(\alpha_{2\ell} - \frac{\alpha_1^i \alpha_{2(\ell-i-1)} \ln \alpha_1}{k-1+i+c}\right) \alpha_1^{-\frac{(\ell-1)(k-2+i+c)}{k-1+i+c}}, \qquad \ell \in \{i+3, \dots, k\},$$

$c = 1-k-i:$

$$\beta_{2(i+2)-1} = \frac{\alpha_1^i \alpha_2}{k-2+i+c} \alpha_5^{-\frac{2+i}{2}} + \\ + \left(\alpha_{2(i+2)-1} - \frac{\alpha_1^i \alpha_2}{k-2+i+c}\right) \alpha_5^{-\frac{i+1}{2}},$$

$$\beta_{2\ell-1} = \left(\alpha_{2\ell} + \frac{\alpha_1^i \alpha_{2(\ell-i-1)} \ln \alpha_5}{2}\right) \alpha_5^{\frac{(1-\ell)}{2}}, \qquad \ell \in \{i+3, \dots, k\}.$$

$\mathfrak{s}_{2k+1,5}(b)$

$b \neq 1-k:$

$$\beta_{2\ell} = \left(\alpha_{2\ell} - \alpha_1^i \alpha_{\ell-1} \frac{\ln \alpha_1}{k-1+b}\right) \alpha_1^{-\frac{\ell(k-2+b)+1}{k-1+b}}, \qquad \ell \in \{1, \dots, k-1\},$$

$b = 1-k:$

$$\beta_{2\ell} = \left(\alpha_{2\ell} + \alpha_1^i \alpha_{\ell-1} \frac{\ln \alpha_5}{2}\right) \alpha_5^{\frac{1-\ell}{2}}, \qquad \ell \in \{1, \dots, k-1\}.$$

$\mathfrak{s}_{2k+1,6}(c)$

This extension has the same invariants as $\mathfrak{s}_{2k+1,1}(1,c)$.

$\mathfrak{s}_{2k+1,7}^{(i)}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,2}^{(i-1)}(1)$.

$\mathfrak{s}_{2k+1,8}^{(i)}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,3}^{(i+1)}(1)$.

$\mathfrak{s}_{2k+1,9}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,5}(1)$.

$\mathfrak{s}_{2k+1,10}(b)$

$b \neq 1-k:$

$$\beta_{2\ell} = \left[\sum_{j=1}^{\ell} \left(\frac{\ln \alpha_1}{k-1+b}\right)^{\ell-j} \frac{1}{(\ell-j)!}\alpha_{2j}\alpha_2^{\ell-j}\right] \alpha_1^{-\frac{\ell(k-2+b)}{k-1+b}}, \quad \ell \in \{1,\dots,k-1\}$$

$$\beta_{2\ell-1} = \left[\sum_{j=1}^{\ell-1} \left(\frac{\ln \alpha_1}{k-1+b}\right)^{\ell-j} \frac{(\ell-1-j)}{(\ell-j)!}\alpha_2^{\ell-1-j}\alpha_{2j}+ \right.$$
$$\left. + \sum_{j=0}^{\ell-3} \left(\frac{\ln \alpha_1}{k-1+b}\right)^{j} \frac{1}{j!}\alpha_2^{j}\alpha_{2(\ell-j} - 1 + \frac{t^3}{6}\alpha_2^2\alpha_{\ell-3}\right] \alpha_1^{-\frac{(\ell-1)(k-2+b)}{k-1+b}},$$

$b = 1-k:$

$$\beta_{2\ell} = \left[\sum_{j=1}^{\ell} \left(-\frac{\ln \alpha_5}{2}\right)^{\ell-j} \frac{1}{(\ell-j)!}\alpha_{2j}\alpha_2^{\ell-j}\right] \alpha_5^{-\frac{\ell}{2}}, \quad \ell \in \{1,\dots,k-1\}$$

$$\beta_{2\ell-1} = \left[\sum_{j=1}^{\ell-1} \left(-\frac{\ln \alpha_5}{2}\right)^{\ell-j} \frac{(\ell-1-j)}{(\ell-j)!}\alpha_2^{\ell-1-j}\alpha_{2j}+ \right.$$
$$\left. + \sum_{j=0}^{\ell-3} \left(-\frac{\ln \alpha_5}{2}\right)^{j} \frac{1}{j!}\alpha_2^{j}\alpha_{2(\ell-j} - 1 + \frac{t^3}{6}\alpha_2^2\alpha_{\ell-3}\right] \alpha_5^{\frac{1-\ell}{2}}.$$

$\mathfrak{s}_{2k+1,11}$ and $\mathfrak{s}_{2k+1,12}$

It is rather difficult to find the exact form of the Casimir invariants of these extensions. On the other hand, we can establish that they can be found in the form

$$\beta_2 = \alpha_2,$$
$$\beta_\ell = \alpha_\ell + \mathfrak{P}(\alpha_1,\dots,\alpha_{\ell-1},\ln \alpha_1), \qquad \ell \in \{4,\dots,k-1\},$$

where $\mathfrak{P}$ is a polynomial.

$\mathfrak{s}_{2k+1,13}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,1}(b,1)$.

$\mathfrak{s}^{(i)}_{2k+1,14}$

This extension has the same invariants as $\mathfrak{s}^{(i-1)}_{2k+1,3}(1)$.

$\mathfrak{s}^{(i)}_{2k+1,15}$

This extension has the same invariants as $\mathfrak{s}^{(i+1)}_{2k+1,2}(1)$.

$\mathfrak{s}_{2k+1,16}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,5}(1)$.

$\mathfrak{s}_{2k+1,17}$

This extension has the same invariants as $\mathfrak{s}_{2k+1,10}(1)$.

4.4 Invariants of $\mathfrak{s}_{2k+3}$

To find the generalized Casimir invariants of the extension $\mathfrak{s}_{2k+3}$ given in section 3.5 we use the results obtained above, namely the already found invariants (4.8) of $\mathfrak{r}_{2k}$ and a special case of invariants (4.9) of family $\mathfrak{s}_{2k+1,1}(b,c)$ with $b=0$ and $c=1$. Notice that the invariants of $\mathfrak{s}_{2k+1,1}(0,1)$ are used in the form (4.9) and not in the final form given below the mentioned one. These invariants are the functional basis of the solutions of system

$$\hat{E}_{2k}I(\vec{e},f_1,f_2,f_3)=0,\qquad \hat{F}_1I(\vec{e},f_1,f_2,f_3)=0,$$

where $\hat{E}_{2k}$ and $\hat{F}_1$ are the same operators as in the case $\mathfrak{s}_{2k+1,1}(0,1)$. The sought invariants are the solutions of the system

$$\begin{aligned} &\hat{E}_{2k}I(\vec{e},f_1,f_2,f_3)=0, &&\hat{F}_1I(\vec{e},f_1,f_2,f_3)=0,\\ &\hat{F}_2I(\vec{e},f_1,f_2,f_3)=0, &&\hat{F}_3I(\vec{e},f_1,f_2,f_3)=0, \end{aligned}$$

where

$$\hat{F}_2=-\sum_{\ell=1}^{k}e_{2\ell-1}\frac{\partial}{\partial e_{2\ell-1}},\qquad \hat{F}_3=-\sum_{\ell=1}^{k-1}e_{2\ell}\frac{\partial}{\partial e_{2\ell}}.$$

By straightforward computation we obtain

$$\begin{aligned} &\hat{F}_3\beta_{2\ell}=-(k-1)\beta_{2\ell}, &&\forall\ell\in\{1,\dots,k-1\},\\ &\hat{F}_3\beta_{2\ell-1}=0, &&\forall\ell\in\{3,\dots,k\}. \end{aligned}$$

Solving the corresponding characteristic system we obtain

$$\gamma_{2\ell}(t) = \beta_{2\ell}\mathrm{e}^{-t}, \qquad \forall \ell \in \{1, \dots, k-1\},$$
$$\gamma_{2\ell-1}(t) = \beta_{2\ell-1}, \qquad \forall \ell \in \{3, \dots, k\},$$

fixing $\gamma_2(t) := 1$ we get the value of t as $t = \ln \beta_2$, which leads to

$$\gamma_{2\ell} = \frac{\beta_{2\ell}}{\beta_2}, \qquad \forall \ell \in \{2, \dots, k-1\},$$
$$\gamma_{2\ell-1} = \beta_{2\ell-1}, \qquad \forall \ell \in \{3, \dots, k\}.$$

Another straightforward calculation leads to the equations

$$\hat{F}_2\gamma_{2\ell} = -\frac{\ell-1}{k-1}\gamma_{2\ell}, \qquad \forall \ell \in \{2, \dots, k-1\},$$
$$\hat{F}_2\gamma_{2\ell-1} = -\frac{\ell-1}{k-1}\gamma_{2\ell-1}, \qquad \forall \ell \in \{3, \dots, k\}.$$

Again solving the corresponding characteristic system we obtain

$$\delta_{2\ell}(t) = \delta_{2\ell}\mathrm{e}^{-\frac{\ell-1}{k-1}t}, \qquad \forall \ell \in \{2, \dots, k-1\},$$
$$\delta_{2\ell-1}(t) = \delta_{2\ell-1}\mathrm{e}^{-\frac{\ell-1}{k-1}t}, \qquad \forall \ell \in \{3, \dots, k\}.$$

By setting $\delta_4(t) := 1$ we get $t = \ln \gamma_4^{-\frac{1}{k-1}}$. Substituting this value back into the remaining equations yields

$$\delta_{2\ell-1} = \gamma_{2\ell-1}\gamma_4^{\frac{\ell-1}{(k-1)^2}}, \qquad \forall \ell \in \{3, \dots, k\},$$
$$\delta_{2\ell} = \gamma_{2\ell}\gamma_4^{\frac{\ell-1}{(k-1)^2}}, \qquad \forall \ell \in \{3, \dots, k-1\}.$$

Using the invariants of $\mathfrak{s}_{2k+1,1}(0,1)$ (4.9) we write down $2k-5$ functionally independent invariants of the extension $\mathfrak{s}_{2k+3}$ in the form

$$\delta_{2\ell-1} = \beta_{2\ell-1}\left(\frac{\beta_4}{\beta_2}\right)^{\frac{\ell-1}{(k-1)^2}}, \qquad \forall \ell \in \{3, \dots, k\},$$
$$\delta_{2\ell} = \frac{\beta_{2\ell}}{\beta_2}\left(\frac{\beta_4}{\beta_2}\right)^{\frac{\ell-1}{(k-1)^2}}, \qquad \forall \ell \in \{3, \dots, k-1\}.$$

Furthermore, we can write them in terms of invariants (4.8) of $\mathfrak{r}_{2k}$ as

$$\delta_{2\ell-1} = \alpha_{2\ell-1}\alpha_1^{-\frac{(\ell-1)(k-2+b)}{k-1}}\left(\frac{\alpha_4}{\alpha_2}\alpha_1^{-\frac{2k-3}{k-1}}\right)^{\frac{\ell-1}{(k-1)^2}}, \qquad \forall \ell \in \{3, \dots, k\},$$
$$\delta_{2\ell} = \frac{\alpha_{2\ell}\alpha_1^{-\frac{\ell(k-2)+1}{k-1}}}{\alpha_2}\left(\frac{\alpha_4}{\alpha_2}\alpha_1^{-\frac{2k-3}{k-1}}\right)^{\frac{\ell-1}{(k-1)^2}}, \qquad \forall \ell \in \{3, \dots, k-1\}.$$

Conclusion

We have constructed the series of nilpotent Lie algebras $\mathfrak{r}_{2k}$ of arbitrary even dimension $d = 2k \geq 6$. Solvable extensions of thus obtained nilpotent algebras with $\mathfrak{r}_{2k}$ as its nilradical have been classified. Regardless of the dimension exactly one class of isomorphic $2k+3$-dimensional extensions of $\mathfrak{r}_{2k}$ exists. Of course many more of the lower dimensional extensions were found; there are $2k + 12$ families of isomorphic classes of solvable extensions by two elements with number of parameters ranging from 0 to $2k - 5$ and even more families of extensions by one element.

One of the questions raised was whether it is possible for a solvable extension $\mathfrak{s} \equiv (e_i, f_j)$ of the nilpotent algebra $\mathfrak{NR}(\mathfrak{s}) \equiv (e_i)$ to have added elements (f_j) such that there is at least one intrinsically nonvanishing ${\gamma_{ij}}^k$, where $[f_i, f_j] = {\gamma_{ij}}^k e_k \notin \mathfrak{z}$. Unfortunately, the extensions of the chosen nilradicals do not have this property and thus this question remains unanswered.

The generalized Casimir invariants of the $2k-1$-dimensional extensions were found with the exception of the most degenerate cases, in which the calculations turned out to be not feasible. The invariants of the extension $\mathfrak{s}_{2k+3}$ were found as well, while the invariants of solvable extensions of $\mathfrak{r}_{2k}$ by two vectors were omitted, as they are out of the scope of this work.

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