Czech Technical University in Prague
Faculty of Nuclear Science and Physical Engineering

DIPLOMA THESIS

Shadowing in continuous dynamical systems

Lucie Strmisková

Supervisor: doc. Ing. Pavel Soldán, Dr.

Acknowledgements

I would like to thank my supervisor, doc. Ing. Pavel Soldán, Dr., for great support and patience and for providing access to all the important references.
I am also grateful to my father for his care and encouraging words during my studies.

Název práce:
Stínování ve spojitých dynamických systémech

Autor: Lucie Strmisková

Obor: Matematické inženýrství

Zaměření: Matematická fyzika

Druh práce: Diplomová práce

Vedoucí práce: doc. Ing. Pavel Soldán, Dr., Katedra chemické fyziky a optiky, Matematicko-fyzikální fakulta, Karlova univerzita Praha

Abstrakt: Načrtneme problémy, které způsobuje nepřesnost numerických metod při řešení chaotických differenciálních rovnic, a zadefinujeme pojem stínovaní. Poté vysvětlíme, proč je hyperbolicita systému klíčovou vlastností pro stínování trajektorií. Představíme zajímavé dynamické systémy s chaotickým chováním. Uvedeme větu o stínování vhodnou pro numerické počítání. Poté za pomoci této věty ukážeme existenci stínu pro numerická řešení vybraných dynamických systémů. Nakonec ukážeme příklad nestínovatelné trajektorie.

Klíčová slova: diferenciální rovnice, chaos, stínování, hyperbolické systémy, nestínovatelnost

Title:
Shadowing theorem in continuous dynamical systems

Author: Lucie Strmisková

Abstract: We sketch the problem of numerical solutions of chaotic differential equations and shadowing. Then we explain why the hyperbolicity of the system is crucial property for shadowing. We introduce some interesting dynamical systems that exhibit chaotic behaviour. Then we present shadowing theorem convenient for numerical computation. We use this theorem to prove the existence of shadows of the numerical solutions of presented dynamical systems. Finally we show an example of the unshadowable trajectory.

Key words: differential equations, chaos, shadowing, hyperbolic systems, unshadowability

Contents

Introduction **3**

1 Shadowing theorem for differential equations **5**
- 1.1 Differential equations 5
- 1.2 Shadowing in discrete dynamical systems 6
  - 1.2.1 Hyperbolicity for difference equations 6
  - 1.2.2 Shadowing for difference equations 10
- 1.3 Poincaré map 16
- 1.4 Linear differential equations 19
- 1.5 Hyperbolic stationary point 20
- 1.6 Hyperbolic set 22
- 1.7 Shadowing theorem for hyperbolic sets 26

2 Numerical shadowing **28**
- 2.1 Lorenz equations 28
- 2.2 Chaos generator 31
- 2.3 Finite time shadowing theorem 33
- 2.4 Finite precision computations 38
- 2.5 Implementation of the theorem 39
  - 2.5.1 Lorenz equations 41
  - 2.5.2 Chaos generator 42
- 2.6 Map method 43

3 Unshadowability **46**
- 3.1 Glitches 46
- 3.2 Shadowing test 47

Conclusion and future work **49**

A Lyapunov functions **50**

B Lyapunov exponents **52**

List of used symbols

The following list covers some of the mathematical symbols and conventions used in the work without prior definition.

Symbol	**Explanation**
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
x	the real number
$\mathbf{x}$	the vector
I	the identity matrix
$A^\dagger$	the hermitian conjugate of a matrix A
A^T	the transpose of a matrix A
$U = U^\circ$	the open set
$U = \overline{U}$	the closed set
∂U	the bound of a set U
$U^\perp$	the orthogonal complement to a set U
$f \in C^r$	the function continuously differentiable to the order r
f^n	the composition of n functions f
$\|x\|$	the absolute value of a number x
$\\|\mathbf{x}\\|$	the norm of $\mathbf{x}$
$\langle \mathbf{x}, \mathbf{y} \rangle$	the scalar product of $\mathbf{x}$ and $\mathbf{y}$
$T_{\mathbf{x}}W$	the tangent space of a manifold W at a point $\mathbf{x}$

Introduction

One of the significant characteristics of a chaotic dynamical system is a sensitive dependence on initial conditions - two solutions that start nearby will diverge from each other with the future evolution of the system.

Chaotic dynamical systems are described by nonlinear differential equations. Most of them are not solvable analytically and we must rely on numerical simulations. We require that the behaviour of a solution generated by the computer is the same as the behaviour of a true solution. However we should realize that all computers work with a finite precision. The rounding error made at any step of the computation causes a numerical trajectory (pseudoorbit) to differ from the true one and this difference will be amplified exponentially due to the chaotic nature of the system.

So it is natural to ask if making numerical simulations for chaotic systems is purposeful and to what these computer generated orbits actually correspond.

Famous shadowing theorem says that for hyperbolic dynamical systems the true trajectory will really diverge from the computer generated one (for chaotic systems) but there always exists a shadow - true trajectory with slightly different initial condition which stays arbitrarily close to the computed orbit for arbitrarily long time.

In the previous work [26] we dealt with shadowing in discrete dynamical systems. We would like to extend the knowledge of shadowing and use it to study continuous dynamical systems.

In the first chapter, we sum up our knowledge about shadowing in discrete dynamical systems. We explain in detail what the hyperbolicity is and why it is so important for shadowing. Then we review the history of shadowing and present three shadowing theorems. Two of them can be applied also for non-hyperbolic sets (with the bigger or smaller success).

We can look at the numerical solutions of the differential equations as the discrete pseudoorbits of the used numerical method, so it might seem that we can directly apply the shadowing methods constructed for maps to these numerical solutions. But there is a fundamental difference between a discrete solution to an ordinary differential equation and a discrete map. The errors have only "space like" character for discrete maps, whereas the numerical solutions can have errors both in space and in time. A true orbit and a pseudoorbit can have the same trajectory, but different time scale. Therefore it is necessary to include time rescaling in the shadowing definition for the continuous cases.

We spent a plenty of time by the study of hyperbolic systems because we wanted to explain the essence of the shadowing. For the same reason, there are many examples in this diploma thesis.

In the second chapter, we introduce two interesting dynamical systems that exhibit chaotic behaviour. Then we present two useful numerical methods for searching for the shadows. One method is a generalization of the map method. The other method comprises the lack of hyperbolicity in the direction of the vector field and therefore it gives better results. Both methods are purely algebraical and the existence of the shadow depends on invertibility of a certain linear operator.

The shadowing methods with their advantages and disadvantages are discussed in full detail. Finally, we use one method to verify that the numerical solutions of the presented chaotic differential equations are shadowable.

In the last chapter, we examine some systems with unshadowable trajectories and try to understand the consequences of unshadowability for numerical computation.

Chapter 1

Shadowing theorem for differential equations

1.1 Differential equations

We are interested in the solutions of the autonomous ordinary differential equations

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)), \tag{1.1}$$

where $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^T$ is an n-dimensional vector and $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector-valued function. For each $\mathbf{x}_0 \in U$ there is a unique solution $\mathbf{x}(t)$ of (1.1) with $\mathbf{x}(0) = \mathbf{x}_0$ defined on a maximal open interval $J(\mathbf{x}_0) \subset \mathbb{R}$ [15].

The set $\Omega = \{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} | t \in J(\mathbf{x})\}$ is open and we can define a map $\phi(\mathbf{x}, t) : \Omega \to \mathbb{R}^n$ such that $\phi(\mathbf{x}, t)$ is the solution of the differential equation (1.1) at time t with initial condition $\mathbf{x}$. Hence

$$\frac{d}{dt}\phi(\mathbf{x}, t) = f(\phi(\mathbf{x}, t))$$

for all t such that the solution through $\mathbf{x}$ exists and $\phi(\mathbf{x}, 0) = \mathbf{x}$. We call ϕ a *flow* of the equation (1.1) and write $\phi(t, \mathbf{x}) = \phi^t(\mathbf{x})$.

Following properties of the flow are simple consequences of its definition.

Lemma 1. [15] Properties of the flow:

1. $\phi^{s+t}(\mathbf{x}) = \phi^s(\phi^t(\mathbf{x}))$.

2. If f is C^r, then $\phi : \Omega \to \mathbb{R}^n$ is a C^r map.

3. If f is C^r, then ϕ^t is C^r diffeomorphism between $U^t = \{\mathbf{x} \in U | t \in J(\mathbf{x})\}$ and U^{-t}.

Although the unique solution exists for all $\mathbf{x}_0$, generally it cannot be written analytically and we search the solutions using numerical simulations. We are interested in validity of these numerical solutions, i.e., if for every numerical solution exists true solution (a shadow), which is close to it for sufficiently long time.

The numerical methods lead to difference equations $\mathbf{x}_{n+1} = g(\mathbf{x}_n)$. Also using the Poincaré sections, the qualitative study of certain differential equations can be reduced to the study of the associated difference equations. Moreover the shadowing problem for the difference equations is much more easier to analyze. Therefore we are first concerned with the existence of shadows for the numerically generated solutions of the difference equations.

1.2 Shadowing in discrete dynamical systems

We present the shadowing problem for diffeomorphisms in this section. By a *diffeomorphism* $f : U \to V$ we mean a one-to-one map such that both f and $f^{-1} : V \to U$ are differentiable.

1.2.1 Hyperbolicity for difference equations

Consider

$$\mathbf{x}_{n+1} = g(\mathbf{x}_n), \tag{1.2}$$

where $g : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1 diffeomorphism.

As we have said in the introduction shadowing and hyperbolicity of the system go hand in hand. We prefer the approach that is presented in [20], where a general hyperbolic set is deduced from a hyperbolic fixed point, rather than state quite difficult definition without any explanation.

Definition 1. A point $\mathbf{x}_0$ is said to be *hyperbolic fixed point* of the map g if $g(\mathbf{x}_0) = \mathbf{x}_0$ and all the eigenvalues of $Dg(\mathbf{x}_0)$ lie off the unit circle. The sum of generalized eigenvectors corresponding to the eigenvalues inside (outside) the unit circle is called the stable (unstable) subspace and is denoted as E^s (E^u).

These subspaces are invariant under $Dg(\mathbf{x}_0)$ and there exist constants [16] $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0, 1)$ such that for all $k \geq 0$

$$\|(Dg(\mathbf{x}_0))^k \xi\| \leq K_1 {\lambda_1}^k \|\xi\| \quad \text{for} \quad \xi \in E^s, \tag{1.3}$$

$$\|(Dg(\mathbf{x}_0))^{-k} \xi\| \leq K_2 {\lambda_2}^k \|\xi\| \quad \text{for} \quad \xi \in E^u. \tag{1.4}$$

If $g = A \in \mathbb{R}^{n,n}$, the stable subspace E^s is the set of points attracted by the fixed point $\mathbf{x}_0$. We can find such a set also for nonlinear map g, but it is not the linear subspace any more.

Definition 2. Let $\mathbf{x}_0$ be a hyperbolic fixed point of the C^1 diffeomorphism $g : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$. The set

$$W^s(\mathbf{x}_0) = \{\mathbf{x} \in U : g^k(\mathbf{x}) \to \mathbf{x}_0 \quad \text{as} \quad k \to \infty\}$$

is called the *stable manifold* of $\mathbf{x}_0$ and the set

$$W^u(\mathbf{x}_0) = \{\mathbf{x} \in U : g^k(\mathbf{x}) \to \mathbf{x}_0 \quad \text{as} \quad k \to -\infty\}$$

is called the *unstable manifold* of $\mathbf{x}_0$.

The stable manifold need not to be generally a submanifold of $\mathbb{R}^n$, therefore we introduce the term local stable manifold (stable manifold with an extra condition) which already fulfils the manifold definition.

Definition 3. Let $\mathbf{x}_0$ be a hyperbolic fixed point of the C^1 diffeomorphism $g : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$. For given $\varepsilon > 0$, the set

$$W^{s,\varepsilon}(\mathbf{x}_0) = \{\mathbf{x} \in U : g^k(\mathbf{x}) \to \mathbf{x}_0 \quad \text{as} \quad k \to \infty, \quad \|g^k(\mathbf{x}) - \mathbf{x}_0\| < \varepsilon \quad \forall k \geq 0\}$$

is called the *local stable manifold* of $\mathbf{x}_0$.

The following theorem shows that the behaviour of points in the neighbourhood of hyperbolic fixed point $\mathbf{x}_0$ corresponds, at least locally, to the behaviour generated by the linearization $Dg(\mathbf{x}_0)$.

Theorem 1 (Stable manifold theorem)**.** [20] Let $g : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^r(r \geq 1)$ diffeomorphism with hyperbolic fixed point $\mathbf{x}_0$ and associated stable subspace E^s. Then for ε sufficiently small, $W^{s,\varepsilon}(\mathbf{x}_0)$ is a C^r submanifold of $\mathbb{R}^n$ containing $\mathbf{x}_0$ and moreover $T_{\mathbf{x}_0} W^{s,\varepsilon}(\mathbf{x}_0) = E^s$.

It is obvious that previous theorem also proves that $W^{u,\epsilon}(\mathbf{x}_0)$ is a submanifold of $\mathbb{R}^n$. If we change the direction of iterations the unstable submanifold changes to a stable one and vice versa. The precise value of ε depends on g and it can be found in the proof of the stable manifold theorem.

It also follows from the proof that for $\mathbf{x} \in U$ the following implication holds

$$\|g^k(\mathbf{x}) - \mathbf{x}_0\| \leq \delta \quad \text{for} \quad k \geq 0 \quad \Rightarrow \quad g^k(\mathbf{x}) \to \mathbf{x}_0 \quad \text{as} \quad k \to \infty. \tag{1.5}$$

Therefore if a hyperbolic fixed point is Lyapunov stable, then it is automatically also asymptotic stable. (The Lyapunov and asymptotic stability are defined in the Appendix A.) It is a very significant property and for non-hyperbolic systems it need not be true.

Example 1. Consider $A = \begin{pmatrix} -1 & 0 \\ 0 & \lambda \end{pmatrix}$ with $|\lambda| \neq 1$. The origin is a non-hyperbolic fixed point. If we take point $\mathbf{x} = \binom{\varepsilon}{0}$ from ε neighbourhood of the origin ($\varepsilon > 0$), we see that it will stay in the ε neighbourhood for all its future evolution but it will never approach the origin.

Now we extend the hyperbolicity definition from a fixed point to a general set. First we look at a periodic point because it helps us to understand why the hyperbolic set is defined in such a complicated way.

Definition 4. A point $\mathbf{x}_0$ is called a *periodic point* of the map g with period $m \geq 1$ if $g^m(\mathbf{x}_0) = \mathbf{x}_0$ and m is a minimal integer with this property. The periodic point $\mathbf{x}_0$ is said to be hyperbolic if $\mathbf{x}_0$ is hyperbolic as a fixed point of g^m.

Example 2. Let us look at the so called Hénon map[1] $g(x, y) = (1 - ax^2 + y, -bx)$ with a special choice of parameters $a = 2$, $b = 1$. There are two periodic points with period 3: $\mathbf{x}_0 = \left(0, -\frac{1}{2}\right)$ and $\mathbf{y}_0 = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Let us evaluate the Jacobian matrix of g^3 in these points in order to see if they are hyperbolic or not.

$$Dg^3(\mathbf{x}_0) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad Dg^3(\mathbf{x}_1) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad Dg^3(\mathbf{x}_2) = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}.$$

All these matrices have the same eigenvalues $2 \pm \sqrt{3}$ therefore $\mathbf{x}_0$ is a hyperbolic periodic point.

The matrices

$$Dg^3(\mathbf{y}_0) = \begin{pmatrix} -20 & -9 \\ 9 & 4 \end{pmatrix}, \quad Dg^3(\mathbf{y}_1) = \begin{pmatrix} -14 & 3 \\ 9 & -2 \end{pmatrix}, \quad Dg^3(\mathbf{y}_2) = \begin{pmatrix} -14 & -9 \\ -3 & -2 \end{pmatrix}$$

have the eigenvalues $-8 \pm 3\sqrt{7}$ so the point $\mathbf{y}_0$ is also a hyperbolic periodic point.

We will see that the conservation of hyperbolicity, and even of the eigenvalues along the hyperbolic orbit is a general property.

Consider the orbit $S = \{\mathbf{x}_0, g(\mathbf{x}_0), \ldots, g^{m-1}(\mathbf{x}_0)\}$. If the starting point $\mathbf{x}_0$ is a hyperbolic periodic point with period m, then there exists splitting $\mathbb{R}^n = E^s \oplus E^u$, where E^s, E^u are stable and unstable subspaces of $\mathbf{x}_0$ as a fixed point of g^m. These subspaces are invariant under $Dg^m(\mathbf{x}_0)$, and there exist constants $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0, 1)$ such that for all $k \geq 0$

$$\|(Dg^m(\mathbf{x}_0))^k \xi\| \leq K_1 {\lambda_1}^k \|\xi\| \quad \text{for} \quad \xi \in E^s, \tag{1.6}$$

$$\|(Dg^m(\mathbf{x}_0))^{-k} \xi\| \leq K_2 {\lambda_2}^k \|\xi\| \quad \text{for} \quad \xi \in E^u. \tag{1.7}$$

Let us realize that if $\mathbf{x}_0$ is a periodic point, then also the other points of S are periodic with the same period. We would like to know if they are also hyperbolic.

If $E^s = \{\vec{y}_1, \ldots, \vec{y}_m\}$, $m \leq n$, is the subspace that contains vectors from the tangent space at the point $\mathbf{x}_0$ and $\mathbf{x}_k = g^k(\mathbf{x}_0)$, then $Dg^k(\mathbf{x}_0)E^s$ is the subspace with the same dimension consisting of the tangent vectors at point $g^k(\mathbf{x}_0)$.

[1] Hénon map was introduced by French astronomer Michel Hénon in 1976 as a simplified model of the Poincaré section of the Lorenz model. It is one of the most famous dynamical systems that exhibit chaotic behaviour and the map is not uniformly hyperbolic.

Let us notice that

$$Dg^m(x_0) = \prod_{i=1}^{m} Dg(g^{m-i}(x_0))$$

and

$$Dg^m(g^k(\mathbf{x}_0)) = \prod_{i=1}^{m} Dg(g^{m+k-i}(\mathbf{x}_0)) =$$
$$= (Dg(g^{k-1}(\mathbf{x}_0))\dots Dg(\mathbf{x}_0))Dg^m(\mathbf{x}_0)(Dg(g^{k-1}(\mathbf{x}_0))\dots Dg(\mathbf{x}_0))^{-1}.$$

Therefore $Dg^m(\mathbf{x}_0)$ and $Dg^m(g^k(\mathbf{x}_0))$, $k \in \hat{m}$, are really similar matrices and apparently $E^s(g^k(\mathbf{x}_0)) = Dg^k(\mathbf{x}_0)E^s$, $E^u(g^k(\mathbf{x}_0)) = Dg^k(\mathbf{x}_0)E^u$.

Thus for $\mathbf{x} = g^k(\mathbf{x}_0)$, $k \in \widehat{m-1}$, there is a splitting $\mathbb{R}^n = E^s(\mathbf{x}) \oplus E^u(\mathbf{x})$, where

$$E^s(\mathbf{x}) = Dg^k(\mathbf{x}_0)E^s, \quad E^u(\mathbf{x}) = Dg^k(\mathbf{x}_0)E^u.$$

Let us look at invariance properties of these subspaces

$$Dg(\mathbf{x})(E^s(\mathbf{x})) = Dg(g^k(\mathbf{x}_0))(Dg^k(\mathbf{x}_0))(E^s) = Dg^{k+1}(\mathbf{x}_0) = E^s(g(\mathbf{x})).$$

If we realize that

$$(Dg^m(\mathbf{x}_0))^k = \Big(\prod_{i=1}^{m} Dg(g^{m-i}(\mathbf{x}_0))\Big)^k, \quad Dg^{mk}(\mathbf{x}_0) = \prod_{i=1}^{mk} Dg(g^{mk-i}(\mathbf{x}_0))$$

and $g^m(\mathbf{x}_0) = \mathbf{x}_0$, we see that we can replace the inequalities (1.6), (1.7) by much more useful ones

$$\|Dg^{mk}(\mathbf{x}_0)\xi\| \le K_1{\lambda_1}^k\|\xi\| \quad \text{for} \quad \xi \in E^s, \tag{1.8}$$

$$\|Dg^{-mk}(\mathbf{x}_0)\xi\| \le K_2{\lambda_2}^k\|\xi\| \quad \text{for} \quad \xi \in E^u. \tag{1.9}$$

By this, we have derived that when $k \ge 0$ is a multiple of m, then

$$\|Dg^k(\mathbf{x}_0)\xi\| \le K_1\tilde{\lambda}_1^k\|\xi\|, \quad \xi \in E^s, \quad \tilde{\lambda}_1 = {\lambda_1}^{\frac{1}{m}}.$$

If k is not a multiple of m, it can be written in the form $k = ma + b$, $a \in \mathbb{N}$, $b \in \widehat{m-1}$.

$$\|Dg^k(\mathbf{x}_0)\xi\| \le \|Dg^{ma+b}(\mathbf{x}_0)\xi\| \le \|Dg^b(g^{ma}(\mathbf{x}_0))\|\|Dg^{ma}(\mathbf{x}_0)\xi\| \le \tilde{K}_1\tilde{\lambda}_1^{ma}\|\xi\| \le$$
$$\le L_1\tilde{\lambda}_1^k\|\xi\|.$$

The previous inequality also holds for the other points of S ($\mathbf{x} = g^l(\mathbf{x}_0), \tilde{\xi} \in E^s(x)$).

$$\|Dg^k(\mathbf{x})\tilde{\xi}\| = \|Dg^k(g^l(\mathbf{x}_0))Dg^l(\mathbf{x}_0)\xi\| \le L_1\tilde{\lambda}_1^{l+k}\|\xi\| \le \tilde{L}_1\tilde{\lambda}_1^k\|\tilde{\xi}\|.$$

We could imagine the general invariant set S of g as an orbit with period $m = \infty$. Therefore we hope that the definition of a hyperbolic set is not surprising after the previous discussion about hyperbolic orbits.

Definition 5. A compact set $S \subset U$ is called a *hyperbolic set* of the map g if

1. S is invariant, i.e., $g(S) = S$
2. there is a continuous splitting $\mathbb{R}^n = E^s(\mathbf{x}) \oplus E^u(\mathbf{x})$, $\mathbf{x} \in S$

such that the subspaces $E^s(\mathbf{x})$ and $E^u(\mathbf{x})$ have constant dimensions $\forall \mathbf{x} \in S$, moreover these subspaces have the invariance properties

$$Dg(\mathbf{x})(E^s(\mathbf{x})) = E^s(g(\mathbf{x})), \quad Dg(\mathbf{x})(E^u(\mathbf{x})) = E^u(g(\mathbf{x}))$$

and there are constants $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0, 1)$ such that for $k \geq 0$ and for all $\mathbf{x} \in S$

$$\|Dg^k(\mathbf{x})\xi\| \leq K_1 {\lambda_1}^k \|\xi\|, \quad \xi \in E^s(\mathbf{x}), \tag{1.10}$$

$$\|Dg^{-k}(\mathbf{x})\xi\| \leq K_2 {\lambda_2}^k \|\xi\|, \quad \xi \in E^u(\mathbf{x}). \tag{1.11}$$

K_1, K_2 are called constants and λ_1, λ_2 exponents for the hyperbolic set S.

The continuous splitting means that if $P(\mathbf{x})$ is a projection of $\mathbb{R}^n$ onto $E^s(\mathbf{x})$ along $E^u(\mathbf{x})$, then $P(\mathbf{x})$ is a continuous function.The continuity of P need not to be involved in the hyperbolicity definition because it follows from the other assumptions.

In the shadowing literature, there is often the hyperbolic set S defined for the diffeomorphism $g : M \to M$, where M is a smooth submanifold of $\mathbb{R}^n$. In this case, the condition of the splitting $\mathbb{R}^n = E^s(\mathbf{x}) \oplus E^u(\mathbf{x})$, $\mathbf{x} \in S$, is replaced by the condition $T_{\mathbf{x}}M = E^s(\mathbf{x}) \oplus E^u(\mathbf{x})$, $\mathbf{x} \in S$, where $T_{\mathbf{x}}M$ is the tangent space to M at point $\mathbf{x}$. We use the fact that for any linear space V we can identify the tangent space $T_{\mathbf{x}}V$ with the original space V.

1.2.2 Shadowing for difference equations

In this subsection, we explain exactly what shadowing is and why the hyperbolicity is so important for it.

Mostly we are unable to solve the difference equations $\mathbf{x}_i = g(\mathbf{x}_{i-1})$ explicitly, we search the solutions numerically and hope that such gained solutions, δ pseudoorbits, are close to the true ones.

Definition 6. A sequence $\{\mathbf{y}_i\}_{i=0}^{\infty}$ of points in $U \subset \mathbb{R}^n$ is a δ pseudoorbit of g if $\|\mathbf{y}_{i+1} - g(\mathbf{y}_i)\| < \delta$ for all i.

The following definition shows what exactly we imagine under the term "the computed orbit is close to the true one."

Definition 7. The true orbit $\{\mathbf{x}_i\}_{i=0}^{\infty}$ ($\mathbf{x}_{i+1} = g(\mathbf{x}_i)$) ε shadows the δ pseudoorbit $\{\mathbf{y}_i\}_{i=0}^{\infty}$ if

$$\|\mathbf{x}_i - \mathbf{y}_i\| < \varepsilon \qquad \forall i.$$

Our key question if the numerical generated trajectory corresponds to some true trajectory can be reformulated: "For given δ pseudoorbit, is there a true trajectory which ε shadows it? And under what conditions?"

The following theorem gives us the answer.

Theorem 2 (Shadowing theorem)**.** [2] Let S be a hyperbolic set for a diffeomorphism g. For each $\varepsilon > 0$ there is a $\delta > 0$ such that every δ pseudoorbit in S is ε shadowed by a unique true orbit lying in S.

This theorem was first presented by Anosov [2]. Bowen [3] proved that it is sufficient to assume that certain neighborhood of δ pseudoorbit is hyperbolic for shadowing.

To understand why the hyperbolicity is the key property for shadowing, let us look at two simple examples.

Example 3. Consider the contracting mapping $g : \mathbb{R}^n \to \mathbb{R}^n$,

$$\|g^k(\mathbf{x}) - g^k(\mathbf{y})\| \leq K^k \|\mathbf{x} - \mathbf{y}\|, \quad K \in (0, 1).$$

It is easy to show that any pseudoorbit $\{\mathbf{x}_0, \mathbf{x}_1 \ldots, \mathbf{x}_N\}$ is shadowed by the true orbit beginning at its own initial condition.

$$\|g(\mathbf{x}_0) - \mathbf{x}_1\| < \delta,$$

$$\|g^2(\mathbf{x}_0) - \mathbf{x}_2\| \leq \|g^2(\mathbf{x}_0) - g(\mathbf{x}_1)\| + \|g(\mathbf{x}_1) - \mathbf{x}_2\| \leq (K + 1)\delta,$$

$$\vdots$$

$$\|g^k(\mathbf{x}_0) - \mathbf{x}_n\| \leq (K^{k-1} + K^{k-2} + \ldots + 1)\delta.$$

Therefore the true orbit $\{\mathbf{x}_0, g(\mathbf{x}_0), \ldots, g^N(\mathbf{x}_0)\}$ shadows the pseudoorbit $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_N\}$ within $\frac{\delta}{1-K}$.

Example 4. Consider the expanding mapping: $g : \mathbb{R}^n \to \mathbb{R}^n$,

$$\|g^k(\mathbf{x}) - g^k(\mathbf{y})\| \geq C^k \|\mathbf{x} - \mathbf{y}\|, \quad C > 1.$$

In contrast to the previous, the expanding mapping is sensitive to the change of initial condition. Despite of this sensitivity, all δ pseudoorbits $\{\mathbf{x}_0, \mathbf{x}_1 \ldots, \mathbf{x}_N\}$ can be shadowed. The inverse map g^{-1} is contracting with $K = \frac{1}{C}$ so the true orbit $\{g^{-N}(\mathbf{x}_N), g^{-N+1}(\mathbf{x}_N), \ldots, \mathbf{x}_N\}$ shadows the pseudoorbit within $\frac{\delta}{1-\frac{1}{C}}$.

Previous examples seem to be trivial but we should realize that a general hyperbolic dynamical system is a combination of these two cases. At each point we can find some expanding and some contracting directions. The idea of shadowing hyperbolic sets is based on this simple observation – we can shadow contracting and expanding maps.

Unfortunately, the theorem 2 is not particulary convenient for practical computation. The first problem is that δ that is produced can be smaller than machine epsilon of currently existing computers. The second and more important problem is that the hyperbolicity assumption is too restrictive. The hyperbolicity is very difficult to verify. Moreover, the most of currently studied dynamical systems are not uniformly hyperbolic.

Although the results of Anosov and Bowen are not directly applicable in practice, they helped us to understand the background of shadowing and showed the direction of research when attempting to shadow non-hyperbolic systems.

Hammel *et al.* [9], [11] first proved the existence of shadows of nontrivial lengths for twodimensional non-hyperbolic systems. Their method consists of two parts: refinement and containment.

Refinement is a numerical procedure that for a given δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ produces a nearby pseudoorbit $\{\tilde{\mathbf{y}}_k\}_{k=0}^N$ with less noise.

Let $\mathbf{p}_k$ represent the magnitude of a noise at each step

$$\mathbf{p}_{k+1} = \mathbf{y}_{k+1} - g(\mathbf{y}_k). \tag{1.12}$$

The refined orbit $\{\tilde{\mathbf{y}}_k\}_{k=0}^N$ is constructed from the original pseudoorbit as

$$\tilde{\mathbf{y}}_k = \mathbf{y}_k + \mathbf{c}_k,$$

where $\mathbf{c}_k$ is the correction term. It satisfies the relation

$$\mathbf{c}_{k+1} = g(\tilde{\mathbf{y}}_k) - g(\mathbf{y}_k) - \mathbf{p}_{k+1}.$$

Expanding $g(\tilde{\mathbf{y}}_k)$ about $\mathbf{y}_k$ in Taylor series we get

$$\mathbf{c}_{k+1} = Dg(\mathbf{y}_k)\mathbf{c}_k - \mathbf{p}_{k+1}. \tag{1.13}$$

There is a splitting of $\mathbb{R}^2$ as $\mathbb{R}^2 = \operatorname{span}(\mathbf{s}_k) \oplus \operatorname{span}(\mathbf{u}_k)$ for all k, so we can write $\mathbf{c}_k$ as $\mathbf{c}_k = \alpha_k \mathbf{u}_k + \beta_k \mathbf{s}_k$ and $\mathbf{p}_k$ as $\mathbf{p}_k = \gamma_k \mathbf{u}_k + \delta_k \mathbf{s}_k$.

Given $\{\mathbf{y}_k\}_{k=0}^N$ the coefficients γ_k and δ_k can be computed directly from equation (1.12). We rewrite the equation (1.13) to get the coefficients α_k and β_k

$$\alpha_{k+1}\mathbf{u}_{k+1} + \beta_{k+1}\mathbf{s}_{k+1} = Dg(\mathbf{y}_k)(\alpha_k \mathbf{u}_k + \beta_k \mathbf{s}_k) - (\gamma_{k+1}\mathbf{u}_{k+1} + \delta_{k+1}\mathbf{s}_{k+1}).$$

The unit vectors $\mathbf{u}_k$ and $\mathbf{s}_k$ have the following "invariance" property

$$\mathbf{u}_{k+1} = \frac{Dg(\mathbf{y}_k)\mathbf{u}_k}{\|Dg(\mathbf{y}_k)\mathbf{u}_k\|}, \quad \mathbf{s}_{k+1} = \frac{Dg(\mathbf{y}_k)\mathbf{s}_k}{\|Dg(\mathbf{y}_k)\mathbf{s}_k\|}.$$

Then

$$\begin{aligned}\alpha_{k+1} &= \alpha_k \|Dg(\mathbf{y}_k)\mathbf{u}_k\| - \gamma_{k+1},\\ \beta_{k+1} &= \beta_k \|Dg(\mathbf{y}_k)\mathbf{s}_k\| - \delta_{k+1}.\end{aligned}$$

To achieve numerical stability, we must solve the recursion relation backwards along the unstable direction

$$\alpha_k = \frac{\alpha_{k+1} + \gamma_{k+1}}{\|Dg(\mathbf{y}_k)\mathbf{u}_k\|}, \quad \alpha_N = 0 \tag{1.14}$$

and forwards along the stable direction

$$\beta_{k+1} = \beta_k \|Dg(\mathbf{y}_k)\mathbf{s}_k\| - \delta_{k+1}, \quad \beta_0 = 0. \tag{1.15}$$

These recursion relations with such chosen initial points ensure that the refined orbit is less noisy than the original one.

The next step is containment. We construct a sequence of small parallelograms $\{M_k\}_{k=0}^N$ in this procedure. The points of refined orbit $\{\tilde{\mathbf{y}}_k\}_{k=0}^N$ serve as the centers of these parallelograms. We require that the image $g(M_k)$ lies across M_{k+1}. Each parallelogram M_k has expanding sides parallel to the unstable unit vector $\mathbf{u}_k$ at point $\tilde{\mathbf{y}}_k$ and contracting sides parallel to stable unit vector $\mathbf{s}_k$. The image of expanding sides of M_k must intersect two contracting sides of M_{k+1} but cannot intersect the expanding ones as shown in the figure 1.1.

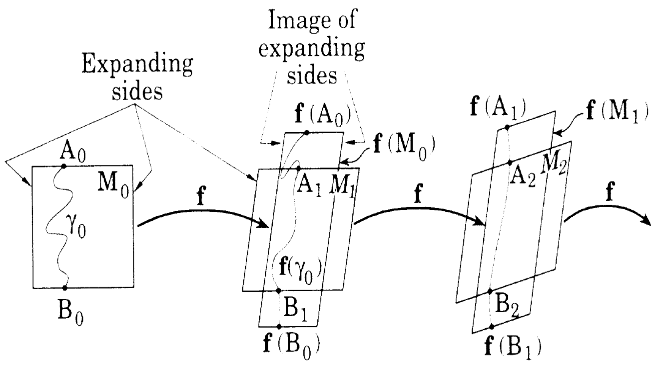


Figure 1.1: Containment of a true trajectory [9].

There is a true orbit $\{\mathbf{x}_k\}_{k=0}^N$ such that $\mathbf{x}_k \in M_k$ for all k. Let Γ_0 be a curve in M_0 connecting one contracting side of M_0 with the other. Then $g(\Gamma_0)$ contains the curve Γ_1 connecting one contracting side of M_1 with the other. We can map the curve Γ_0 by g^k and restrict it to the M_k to gain the sequence of curves $\{\Gamma_k\}_{k=0}^N$. Then we choose arbitrary point $\mathbf{x}_N$ from the curve Γ_N and construct backwards the true orbit $\{\mathbf{x}_k\}_{k=0}^N$, where $\mathbf{x}_k = g^{-1}(\mathbf{x}_{k+1})$. This true orbit is close to the δ pseudoorbit.

Containment cannot continue forever for non-hyperbolic systems. This method fails when the angle between the stable and unstable subspace is nearly zero, then the parallelogram is "squeezed" to the line. The points, at which this situation occurs, are called *glitches.*

Although it worked in some cases, their method was not rigorous in the sense that there is no guarantee that the refinement process is convergent for all initial points. The rigorous verification of shadowing based on the refinement was done by Sauer and Yorke in [23]. For a given δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ they found the stable subspace $S_k = \text{span}(\mathbf{s}_k)$ and unstable subspace $U_k = \text{span}(\mathbf{u}_k)$ at each point $\mathbf{y}_k$ and defined the positive number r_k to be an upper bound for the expansion rate of $Dg(\mathbf{y}_k)$ along S_k and positive number t_k to be an upper bound for the expansion rate of $Dg^{-1}(\mathbf{y}_k)$ along U_k. Then they defined recursively the sequence of numbers $C_k : C_k = \csc\theta_k + r_{k-1}C_{k-1}$, $C_0 = 0$, where θ_k is the angle between subspaces S_k and U_k. Similarly they defined the sequence $D_k : D_k = \csc\theta_k + t_k D_{k+1}$, $D_N = 0$. If C_k and D_k are sufficiently bounded, then the δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ can be shadowed. Precise formulation is given below.

Theorem 3. [23] Assume $\delta < \frac{1}{20n^2}$ and let B a bound on the first and second partial derivatives of g and g^{-1}. If

$$\max(C_k, D_k) \le \frac{1}{n^{5/2}B^2\sqrt{\delta}}$$

for all $k = 0, \ldots, N$, then there exists a true orbit $\{\mathbf{x}_k\}_{k=0}^N$ of g such that $\|\mathbf{x}_k - \mathbf{y}_k\| < \sqrt{\delta}$ for $k = 0, \ldots, N$.

The proof is constructive. The δ pseudoorbit $\mathbf{y} = \{\mathbf{y}_k\}_{k=0}^N$ is taken as the initial orbit and then a sequence of refined δ_l pseudoorbits $\mathbf{y}^l = \{\mathbf{y}_k^l\}_{k=0}^N$ with decreasing noise δ_l is defined. It can be shown that the sequence $\{\mathbf{y}^l\}$ has a limit under the assumptions of the theorem 3 and this limit is the true orbit which is sufficiently close to the original δ pseudoorbit.

If g is hyperbolic, then r_k and t_k are less than one for all k and the angle between the stable and unstable subspaces at each point is bounded away from zero, so the constants C_k and D_k are bounded for all k. Therefore the original shadowing theorem follows from the theorem 3.

Hayes [12] also made the refinement and containment method rigorous. He generalized it to maps of arbitrary dimensions and finally to differential equations.

The previous shadowing methods were geometrical, i.e., the splitting of the tangent space at each point to stable and unstable subspaces was used to construct true orbit forwards along the contracting directions and backwards along the expanding ones. Palmer´s approach [20] was purely algebraical. He considered the δ pseudoorbits $\mathbf{y} = \{\mathbf{y}_k\}_{k=0}^\infty$ as the elements of $l^\infty(\mathbb{R}^n)$ and tried to find true orbits $\mathbf{x} = \{\mathbf{x}_k\}_{k=0}^\infty$ such that $\|\mathbf{x} - \mathbf{y}\|_\infty < \varepsilon$.

He defined the map $G : l^\infty(\mathbb{R}^n) \to l^\infty(\mathbb{R}^n)$ such that

$$[G(\mathbf{x})]_k = \mathbf{x}_{k+1} - g(\mathbf{x}_k).$$

The true orbit $\mathbf{x} = \{\mathbf{x}_k\}_{k=0}^\infty$ is the root of the equation $G(\mathbf{x}) = 0$, while the δ pseudoorbit $\mathbf{y} = \{\mathbf{y}_k\}_{k=0}^\infty$ satisfies the condition $\|G(\mathbf{y})\|_\infty < \delta$.

The equations $G(\mathbf{x}) = 0$, $\|\mathbf{x} - \mathbf{y}\|_\infty < \varepsilon$ have a unique solution if the derivative $L = DG(\mathbf{y})$ is invertible and the norm of the inverse $\|L^{-1}\|$ is suitably bounded.

The linear operator $L = DG(\mathbf{y}) : l^\infty(\mathbb{R}^n) \to l^\infty(\mathbb{R}^n)$ is defined for $\mathbf{u} = \{\mathbf{u}_k\}_{k=0}^\infty$ as follows

$$(L\mathbf{u})_k = \mathbf{u}_{k+1} - Dg(\mathbf{y}_k)\mathbf{u}_k.$$

L is invertible and its inverse is bounded if the difference equation

$$\mathbf{u}_{k+1} = Dg(\mathbf{y}_k)\mathbf{u}_k \tag{1.16}$$

has so called exponential dichotomy on $\langle 0, \infty)$.

The following two definitions explain what is exactly meant by the exponential dichotomy property.

Definition 8. The *transition matrix* $\phi(k, m)$ for the difference equation

$$\mathbf{u}_{k+1} = A_k \mathbf{u}_k, \quad \text{where} \quad A_k \in \mathbb{R}^{n,n}, \quad \det A_k \neq 0 \quad \forall k \in J \subset \mathbb{Z}$$

is defined by:

$$\phi(k, m) = \begin{cases} A_{k-1} \dots A_m & \text{for } k > m \\ I & \text{for } k = m \\ (A_{k-1} \dots A_m)^{-1} & \text{for } k < m. \end{cases}$$

Definition 9. The difference equation $\mathbf{u}_{k+1} = A_k \mathbf{u}_k$ has an *exponential dichotomy* on J if there are projections P_k and constants $K_1, K_2 > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$ such that for $k, m \in J$ the invariance conditions

$$\phi(k, m) P_m = P_k \phi(k, m)$$

are satisfied and the inequalities

$$\|\phi(k, m) P_m\| \leq K_1 {\lambda_1}^{k-m} \quad k \geq m$$

$$\|\phi(k, m)(I - P_m)\| \leq K_2 {\lambda_2}^{m-k} \quad k \leq m$$

hold. K_1, K_2 are called *constants* and λ_1, λ_2 *exponents* associated with the dichotomy.

The following theorem connects hyperbolicity of the system with exponential dichotomy.

Theorem 4. A compact invariant set S for the diffeomorphism $g : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ is hyperbolic iff for all $\mathbf{x} \in S$ the difference equation

$$\mathbf{u}_{k+1} = Dg(g^k(\mathbf{x}_0))\mathbf{u}_k \tag{1.17}$$

has an exponential dichotomy on $\langle 0, \infty)$ with constants, exponents and rank of projections independent on $\mathbf{x}$.

The equation (1.17) has the exponential dichotomy, but the matrices $Dg(\mathbf{x})$ are evaluated at the points of the true orbit, while we need the exponential dichotomy for the equation (1.16) for purposes of shadowing. Fortunately, the exponential dichotomy property is robust under perturbation as the following lemma shows.

Lemma 2. [20] Let S be a compact hyperbolic set with exponents λ_1, λ_2 for the diffeomorphism $g : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$, where U is convex. Suppose that α_1, α_2 are numbers satisfying

$$\lambda_1 < \alpha_1 < 1, \quad \lambda_2 < \alpha_2 < 1.$$

Then if δ is sufficiently small (depending on g, S, α_1 and α_2), the difference equation (1.16) has an exponential dichotomy on $\langle 0, \infty)$ with exponents λ_1, λ_2, with the rank of the projection equal to dim E^s and with constants depending only on g, S, α_1 and α_2.

We must be satisfied with shadows $\mathbf{x} = \{\mathbf{x}_k\}_{k=0}^N$ of finite lengths for δ pseudoorbits $\mathbf{y} = \{\mathbf{y}_k\}_{k=0}^N$ lying in non-hyperbolic sets. The operator L is now finite-dimensional matrix of the form

$$L = \begin{pmatrix} -Dg(y_0) & \mathbb{1} & 0 & \dots & 0 \\ 0 & -Dg(\mathbf{y}_1) & \mathbb{1} & 0 & \\ & & \ddots & & \\ & & & -Dg(\mathbf{y}_N) & \mathbb{1} \end{pmatrix}.$$

It is natural to hope that if $\|L^{-1}\|$ will be bounded, the existence of shadows is guaranteed also in this case. More precisely:

Theorem 5. [20] Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 map, $\{\mathbf{y}_k\}_{k=0}^N$ be its δ pseudoorbit and $L : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ be the linear operator defined for $\mathbf{u} = \{\mathbf{u}_k\}_{k=0}^N$ by:

$$(L\mathbf{u})_k = \mathbf{u}_{k+1} - Dg(\mathbf{y}_k)\mathbf{u}_k, \quad k = 0, \dots, N-1.$$

Suppose $\epsilon = 2\|L^{-1}\|\delta$, where L^{-1} is the right inverse of L, and

$$M = \sup\{\|D^2 g(\mathbf{x})\|, \quad \mathbf{x} \in \mathbb{R}^n, \quad \|\mathbf{x} - \mathbf{y}_k\| \le \epsilon \quad \text{for} \quad \text{some} \quad k = 0, \dots, N-1\}.$$

Then if $2M\|L^{-1}\|^2\delta \le 1$, the δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ is ϵ shadowed by a true orbit $\{\mathbf{x}_k\}_{k=0}^N$ of g.

In the previous work [26], we used this theorem to prove the existence of a $\epsilon \le 1.68 \times 10^{-12}$ shadow length $N = 10000$ for the pseudoorbit of the Hénon map $f(x, y) = (1 - 1.4x^2 + y, 0.3x)$ starting at the origin (figure 1.3).

1.3 Poincaré map

We begin this section with an example.

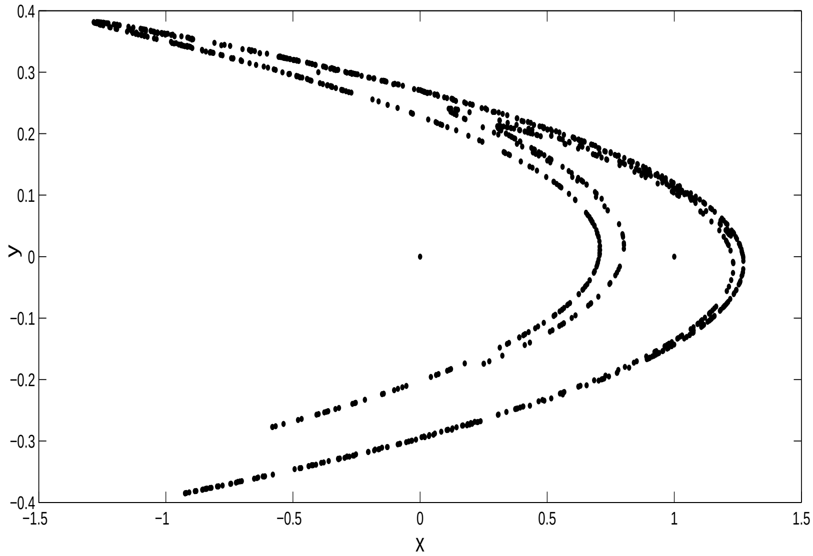


Figure 1.2: Hénon attractor.

Example 5. Consider the system of differential equations

$$\begin{aligned} \dot{x} &= x - y - (x^2 + y^2)x, \\ \dot{y} &= x + y - (x^2 + y^2)y, \end{aligned} \tag{1.18}$$

that has the following form in polar coordinates ($r \in \langle 0, \infty)$, $\varphi \in \langle 0, 2\pi)$)

$$\dot{r} = r - r^3, \quad \dot{\varphi} = 1.$$

The differential equations can be easily integrated.

$$\varphi(t) = \varphi_0 + t.$$

$$\frac{1}{r^3}\dot{r} = \frac{1}{r^2} - 1, \quad \frac{dr^{-2}}{dt} + 2r^{-2} - 2 = 0, \quad r^2(t) = \frac{{r_0}^2}{{r_0}^2 + (1 - {r_0}^2)e^{-2t}}.$$

We immediately see that one of the solutions of the equations (1.18) is periodic orbit Γ ($r = 1$) and this orbit attracts all neighboring points. So the orbit is stable.

Now we choose the half line $\Sigma = \{(r, \varphi) | \varphi = 0\}$. The orbit Γ crosses Σ at point $r = 1$. We consider the solutions of (1.18) starting at Σ near $r = 1$ and look at the times $\tau(r)$ and coordinates $P(r)$ these solutions take to hit Σ again. It is obvious that $\tau(r) = 2\pi$ for all r and

$$P(r) = \Big(\frac{1}{1 + (\frac{1}{r^2} - 1)e^{-4\pi}}\Big).$$

Therefore we can regard the orbit Γ as the fixed point of the map P. Because $DP(1) = e^{-4\pi} < 1$, this fixed point is stable.

The time $\tau(r)$ is called *first return time*, the half line Σ is called *Poincaré section* and the map P the *Poincaré map*. The Poincaré map gives us the correspondence between the differential equations and diffeomorphisms.

We can construct the Poincaré map for all differential equations

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)), \quad \mathbf{x} \in \mathbb{R}^n \tag{1.19}$$

with periodic orbit Γ. We choose the point $\mathbf{x}_0 \in \Gamma$, then we find the hyperplane Σ ($\mathbf{x}_0 \in \Sigma$) transversal to the flow of the system and let us evolve the points $\mathbf{x}$ starting at Σ. Then we look at the points $P(\mathbf{x})$. The transversality of the section means that periodic orbits starting on the section do not flow parallel to it. In this way we can reduce the problem of analyzing the flow near the periodic orbit to the study of the discrete map with a state space that is one dimension smaller than the original system. The Poincaré map preserves many of important properties of the original system, so it is often used to for analyzing the original system. Unfortunately there is no universal way how to construct the Poincaré map, because we must practically solve the original system.

Let $\phi^t(\mathbf{x}_0)$ be a periodic solution of the system (1.19) with period T. Consider the Poincaré section

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^n | \langle \mathbf{x} - \mathbf{x}_0, f(\mathbf{x}_0) \rangle = 0\}.$$

The following theorem ensures that Poincaré map is well defined on some neighborhood of $\mathbf{x}_0$.

Theorem 6. [20] Let $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^r $(r \geq 1)$ vectorfield, let ϕ be the flow associated with (1.19) and let $\phi^t(\mathbf{x}_0)$ be a periodic solution of (1.19) with period T. Then there exists $\Delta > 0$ and C^r function $\tau : B(\mathbf{x}_0, \Delta) \cap \Sigma \to \mathbb{R}$ such that

1. $\phi^{\tau(\mathbf{x})}(\mathbf{x}) \in \Sigma$,
2. $\tau(\mathbf{x}_0) = T$ and $|\tau(\mathbf{x}) - T| \leq \frac{4\|\phi^T(\mathbf{x}) - \mathbf{x}_0\|}{\|f(\mathbf{x}_0)\|}$,
3. $\tau'(\mathbf{x})\mathbf{h} = -\frac{\langle D\phi^{\tau(\mathbf{x})}(\mathbf{x})\mathbf{h}, f(\mathbf{x}_0)\rangle}{\langle f(\phi^{\tau(\mathbf{x})}(\mathbf{x})), f(\mathbf{x}_0)\rangle}$ if $\mathbf{h}$ is orthogonal to $f(\mathbf{x}_0)$ and
4. $(\exists \alpha > 0)(\forall \mathbf{x} \in \mathbb{R}^n)(\mathbf{x}, \phi^t(\mathbf{x}) \in B(\mathbf{x}_0, \Delta) \cap \Sigma, -\alpha \leq t \leq T + \alpha)(t = 0 \quad \text{or} \quad t = \tau(\mathbf{x}))$.

The proof of the theorem is based on the fact that we can find the root of the equation

$$g(\tau(\mathbf{x}), \mathbf{x}) = \langle \phi^{\tau(\mathbf{x})}(\mathbf{x}) - \mathbf{x}_0, f(\mathbf{x}_0) \rangle = 0. \tag{1.20}$$

We can calculate $\tau'(\mathbf{x})$ differentiating the equation (1.20) with respect to $\mathbf{x}$. For $\mathbf{h}$ orthogonal to $f(\mathbf{x}_0)$, we get

$$\langle f(\phi^{\tau(\mathbf{x})}(\mathbf{x})), f(\mathbf{x}_0)\rangle \tau'(\mathbf{x})\mathbf{h} + \langle D\phi^{\tau(\mathbf{x})}(\mathbf{x})\mathbf{h}, f(\mathbf{x}_0)\rangle.$$

It can be shown that $\frac{\partial g}{\partial t}(\tau(\mathbf{x}), \mathbf{x}) = \langle f(\phi^{\tau(\mathbf{x})}(\mathbf{x})), f(\mathbf{x}_0)\rangle > 0$. Hence

$$\tau'(\mathbf{x})\mathbf{h} = -\frac{\langle D\phi^{\tau(\mathbf{x})}(\mathbf{x})\mathbf{h}, f(\mathbf{x}_0)\rangle}{\langle f(\phi^{\tau(\mathbf{x})}(\mathbf{x})), f(\mathbf{x}_0)\rangle}.$$

Therefore it follows that Poincaré map $P : B(\mathbf{x}_0, \Delta) \cap \Sigma \to \Sigma$ defined by

$$P(\mathbf{x}) = \phi^{\tau(\mathbf{x})}(\mathbf{x})$$

is a C^r map with $\mathbf{x}_0$ as the fixed point. For $\mathbf{h} \in (f(\mathbf{x}_0))^\perp$

$$DP(\mathbf{x}_0)\mathbf{h} = -\|f(\mathbf{x}_0)\|^{-2}\langle D\phi^T(\mathbf{x}_0)\mathbf{h}, f(\mathbf{x}_0)\rangle f(\mathbf{x}_0) + D\phi^T(\mathbf{x}_0)\mathbf{h}. \tag{1.21}$$

ϕ is the flow of the system (1.19), therefore $D\phi^T(\mathbf{x}_0)$ is invertible and moreover

$$D\phi^T(\mathbf{x}_0)f(\mathbf{x}_0) = f(\mathbf{x}_0), \quad f(\mathbf{x}_0) \neq 0. \tag{1.22}$$

The expression (1.21) is the orthogonal projection of $D\phi^T(\mathbf{x}_0)\mathbf{h}$ to $(f(\mathbf{x}_0))^\perp$, so $DP(\mathbf{x}_0)$ is invertible map of $(f(\mathbf{x}_0))^\perp$ to itself. Therefore it follows from the inverse function theorem that $P : B(\mathbf{x}_0, \Delta) \cap \Sigma \to P(B(\mathbf{x}_0, \Delta) \cap \Sigma)$ is a C^r diffeomorphism for sufficiently small Δ.

1.4 Linear differential equations

We first remind some basic, but important facts about the solutions of the system of the first order linear differential equations with constant coefficients

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n,n}. \tag{1.23}$$

The flow for system (1.23) is $\phi^t(\mathbf{x}) = e^{tA}\mathbf{x}$, where e^A is defined by the power series $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. This sum is absolutely convergent for all A. So there is only one problem - how to calculate with exponentials of matrices.

We are interested only in the case when the matrix is diagonalizable. To calculate e^A is very simple matter if A is a diagonal matrix $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then $e^{tA} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

Suppose now that the matrix A has n real eigenvalues $\lambda_1, \ldots, \lambda_n$ with associated eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let P be the matrix with the eigenvectors of A as columns. Obviously P is regular. Then

$$AP = [\lambda_1\mathbf{e}_1, \ldots, \lambda_n\mathbf{e}_n] = P\text{diag}(\lambda_1, \ldots, \lambda_n) \quad \Rightarrow A = P\text{diag}(\lambda_1, \ldots, \lambda_n)P^{-1}.$$

So $\phi^t(\mathbf{x}) = e^{tA}\mathbf{x} = P\text{diag}(\mathbf{e}^{\lambda_1 t}, \ldots, \mathbf{e}^{\lambda_n t})P^{-1}\mathbf{x}$.

Now suppose that $A \in \mathbb{R}^{2,2}$ with a pair of conjugate eigenvalues $\rho \pm i\omega$. Then there is a complex eigenvector $\mathbf{z}$ such that $A\mathbf{z} = (\rho + i\omega)\mathbf{z}$. Let $P = [\text{Im}\mathbf{z}, \text{Re}\mathbf{z}]$.

$$AP = [\rho\text{Im}\mathbf{z} + \omega\text{Re}\mathbf{z}, \rho\text{Re}\mathbf{z} - \omega\text{Im}\mathbf{z}] = P\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

We can decompose matrix Λ as $\Lambda = \Lambda_1 + \Lambda_2$. Then $e^{\Lambda} = e^{\Lambda_1} e^{\Lambda_2}$.

$$e^{\Lambda_1 t} = \begin{pmatrix} e^{\rho t} & 0 \\ 0 & e^{\rho t} \end{pmatrix}, \quad e^{\Lambda_2 t} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}.$$

The generalization of the previous computation to higher dimensions in straightforward and it is sum up in the following theorem.

Theorem 7. Let $A \in \mathbb{R}^{n,n}$ has k distinct real eigenvalues $\lambda_1, \dots, \lambda_k$ and $m = \frac{1}{2}(n-k)$ distinct pairs of complex eigenvalues $\rho_1 \pm i\omega_1, \dots, \rho_m \pm i\omega_m$. Then there exists regular matrix P such that $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_k, B_1, \dots, B_m)$, where $B_i = \begin{pmatrix} \rho_i & -\omega_i \\ \omega_i & \rho_i \end{pmatrix}$.

Furthermore, $e^{tA} = Pe^{t\Lambda}P^{-1}$ and $e^{t\Lambda} = \text{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_k t}, e^{B_1 t}, \dots, e^{B_m t})$, where $e^{tB_i} = e^{\rho_i t} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$.

There is an eigenspace associated with each eigenvalue. When the eigenvalue λ is multiple, then its eigenspace is defined as $E_\lambda = \text{span}\{\mathbf{x} \in \mathbb{R}^n | (A - \lambda I)^k \mathbf{x} = 0 \quad \text{for} \quad \text{some} \quad k \in \mathbb{N}\}$.

There are three possibilities for an eigenvalue of A. It has negative, positive or zero real part. First let us assume that all eigenvalues of A have negative real parts. From theorem 7 it is obvious that all points in $\mathbb{R}^n$ moving as dictates (1.23) converge exponentially to the origin. If real parts of all eigenvalues of A are positive, all points diverge to infinity. These properties motivate the following definition.

Definition 10. The set $E^s(A) = \bigoplus_{\lambda, \text{Re}\lambda < 0} E_\lambda$ is called the *stable subspace* of A and the set $E^u(A) = \bigoplus_{\lambda, \text{Re}\lambda > 0} E_\lambda$ is the *unstable subspace* of A.

1.5 Hyperbolic stationary point

We would like to deduce the definition of general hyperbolic set for differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}(t)), \tag{1.24}$$

where $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^r $(r \geq 1)$ vectorfield, analogously as in the discrete case. The simplest hyperbolic set is hyperbolic stationary point.

Definition 11. A point $\mathbf{x}_0$ is said to be *hyperbolic stationary point* of the equation (1.24) if $\phi^t(\mathbf{x}_0) = \mathbf{x}_0$ for all t and all the eigenvalues of $Df(\mathbf{x}_0)$ has no zero or purely imaginary eigenvalues. $\mathbf{x}_0$ is a *sink* if all eigenvalues of $Df(\mathbf{x}_0)$ have negative real parts and a *source* if all the eigenvalues of $Df(\mathbf{x}_0)$ have positive real parts. Otherwise $\mathbf{x}_0$ is a *saddle*.

For linear systems, we have defined the stable, resp. unstable subspace as the set of points that converge to, resp. diverge away from the origin (the only fixed point of (1.23)). We can find these sets also for nonlinear differential equations.

Definition 12. Let $\mathbf{x}_0$ be a hyperbolic stationary point of the equation (1.24). The set

$$W^{s,\varepsilon}(\mathbf{x}_0) = \{\mathbf{x} \in U : \phi^t(\mathbf{x}) \to x_0 \quad \text{as} \quad t \to \infty, \quad \|\phi^t(\mathbf{x}) - \mathbf{x}_0\| \leq \varepsilon \quad \forall t \geq 0\}$$

is called the *local stable manifold* of $\mathbf{x}_0$ and the set

$$W^{u,\varepsilon}(\mathbf{x}_0) = \{\mathbf{x} \in U : \phi^t(\mathbf{x}) \to x_0 \quad \text{as} \quad t \to -\infty, \quad \|\phi^t(\mathbf{x}) - \mathbf{x}_0\| \leq \varepsilon \quad \forall t \leq 0\}$$

is called the *local unstable manifold* of x_0.

The correspondence between local stable manifold and stable subspace is given by the following theorem.

Theorem 8 (Stable manifold theorem)**.** [7] Suppose that the origin is a hyperbolic stationary point for $\dot{\mathbf{x}} = f(\mathbf{x})$ and E^s and E^u are the stable and unstable manifolds of the linear system $\dot{\mathbf{x}} = Df(0)\mathbf{x}$. Then there exist local stable and unstable manifolds $W^{s,\varepsilon}(0)$ and $W^{u,\varepsilon}(0)$ of the same dimension as E^s and E^u respectively. These manifolds are (respectively) tangential to E^s and E^u at the origin and as smooth as the original function f.

Example 6. Consider the equations

$$\begin{aligned} \dot{x} &= 3x + 2y^2 + xy \\ \dot{y} &= -y + 3y^2 + x^2y - 4x^3 \end{aligned} \tag{1.25}$$

with the origin as the stationary point.

The linearized system is $\dot{x} = 3x$, $\dot{y} = -y$. Therefore the origin is a saddle with invariant linear subspaces

$$E^s = \{(x,y) | x = 0\} \quad \text{and} \quad E^u = \{(x,y) | y = 0\}.$$

Local stable manifold is smooth and tangential to E^s, so it can be described by a smooth function $y = V(x)$ satisfying the condition $\frac{\partial V}{\partial x}(0) = 0$. We try to approximate the function V as the power series

$$V(x) = \sum_{i \geq 2} v_i x^i.$$

The linear term is omitted because it has to be tangential to E^s at the origin.

Now we know that

$$\dot{y} = -y + 3y^2 + x^2y - 4x^3 = -\sum_{i\geq 2} v_i x^i + 3\Big(\sum_{i\geq 2} v_i x^i\Big)\Big(\sum_{k\geq 2} v_k x^k\Big) + \sum_{i\geq 2} v_i x^{i+2} - 4x^3$$

and also

$$\dot{y} = \dot{x}\frac{\partial V}{\partial x} = \Big[3x + 2\Big(\sum_{i\geq 2} v_i x^i\Big)\Big(\sum_{k\geq 2} v_k x^k\Big) + \sum_{i\geq 2} v_i x^{i+1}\Big]\sum_{i\geq 2} k v_k x^{k-1}.$$

We approximate it to the cubic term, so equating the terms of order x^2 and x^3 we get

$$-v_2 = 6v_2 \quad \text{and} \quad -v_3 - 4 = 9v_3.$$

Therefore

$$W^{u,\varepsilon}(\mathbf{x}_0) = \{(x,y)|y = -\frac{2}{5}x^3\}.$$

Similarly, we can describe the local stable manifold by $x = S(y)$ such that $\frac{\partial S}{\partial y}(0) = 0$. It gives us that

$$W^{s,\varepsilon}(\mathbf{x}_0) = \{(x,y)|x = -\frac{2}{5}y^2 - \frac{1}{3}y^3\}.$$

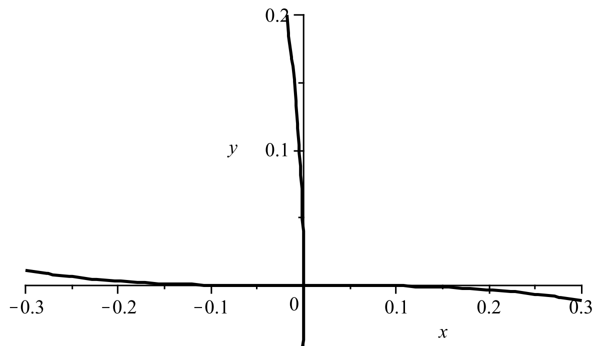


Figure 1.3: Stable and unstable manifolds of the equation (1.25).

1.6 Hyperbolic set

Definition 13. Let $\mathbf{u}(t)$ be a periodic solution of the equation (1.24) with period T. Then $\mathbf{u}(t)$ is called *hyperbolic* if all but one of the eigenvalues of $D\phi^T(\mathbf{u}(0))$ lie off the unit circle.

We decompose $\mathbb{R}^n$ as $\mathbb{R}^n = \mathrm{span}(f(\mathbf{x}_0)) \oplus f(\mathbf{x}_0)^\perp$ and now using the equations (1.22) and (1.21) we see that the matrix $D\phi^T(\mathbf{x}_0)$ has the following form

$$D\phi^T(\mathbf{x}_0) = \begin{pmatrix} 1 & \dots \\ 0 & \\ \vdots & DP(\mathbf{x}_0) \\ 0 & \end{pmatrix}.$$

Therefore $\mathbf{u}(t)$ is hyperbolic if and only if $x_0 = \mathbf{u}(0)$ is hyperbolic fixed point of the Poincaré map P.

We can find the points attracted by, resp. repelled from the hyperbolic periodic solution.

Definition 14. Let $\mathbf{u}(t)$ be a hyperbolic periodic solution of the equation (1.24). The set

$$W^s(\mathbf{u}) = \{\mathbf{x} \in U : \varrho(\phi^t(\mathbf{x}), \mathbf{u}) = \min_{0 \le s \le T} \|\phi^t(\mathbf{x}) - \mathbf{u}(s)\| \to 0 \quad \text{as} \quad t \to \infty\}$$

is called the *stable manifold* of $\mathbf{u}(t)$ and the set

$$W^u(\mathbf{u}) = \{\mathbf{x} \in U : \varrho(\phi^t(\mathbf{x}), \mathbf{u}) = \min_{0 \le s \le T} \|\phi^t(\mathbf{x}) - \mathbf{u}(s)\| \to 0 \quad \text{as} \quad t \to -\infty\}$$

is called the *unstable manifold* of $\mathbf{u}(t)$.

The next proposition connects the sets $W^s(x_0)$, $W^u(x_0)$ with the sets $W^s(\mathbf{u})$, $W^u(\mathbf{u})$.

Proposition 1. Let $\mathbf{u}(t)$ be a hyperbolic periodic solution of the equation (1.24), $\mathbf{x}_0 = \mathbf{u}(0)$. Then the stable and unstable manifolds are given by

$$W^s(\mathbf{u}) = \bigcup_{t<0} \phi^t(W^s(x_0))$$

and

$$W^u(\mathbf{u}) = \bigcup_{t>0} \phi^t(W^u(x_0)).$$

Assume that the equation (1.24) has a periodic solution $\mathbf{u}(t)$, $\mathbf{x}_0 = \mathbf{u}(0)$. Then $\mathbb{R}^n = \mathrm{span}(f(\mathbf{x}_0)) \oplus f(\mathbf{x}_0)^\perp$ and we construct Poincaré map in the same way as in the section 1.3, i.e., $P(\mathbf{x}) = \phi^{\tau(\mathbf{x})}(\mathbf{x})$.

From Riesz theorem, we know that there exists such $\mathbf{z} \in f(\mathbf{x}_0)^\perp$ that for every $\mathbf{w} \in f(\mathbf{x}_0)^\perp \quad \tau'(\mathbf{x}_0)\mathbf{w} = -\mathbf{z}^\dagger \mathbf{w}$.

From equation (1.21), it follows that

$$D\phi^T(\mathbf{x}_0)\mathbf{w} = (\mathbf{z}^\dagger \mathbf{w}) f(\mathbf{x}_0) + DP(\mathbf{x}_0)\mathbf{w}, \quad \mathbf{w} \in f(\mathbf{x}_0)^\perp \tag{1.26}$$

and

$$DP(\mathbf{x}_0) f(\mathbf{x}_0) = f(\mathbf{x}_0). \tag{1.27}$$

The point $\mathbf{x}_0$ is a hyperbolic fixed point of P, therefore 1 is not an eigenvalue of $DP(\mathbf{x}_0)$ and consequently 1 is not an eigenvalue of $DP(\mathbf{x}_0)^+$. We may define the vector $\tilde{\mathbf{z}} = -(I - (DP(\mathbf{x}_0))^\dagger)^{-1}\mathbf{z}$. Denote as $\tilde{E}^s(\mathbf{x}_0)$, resp. $\tilde{E}^u(\mathbf{x}_0)$ the stable, resp. unstable subspace of $\mathbf{x}_0$ considered as the fixed point of P. We can define the subspaces $E^s(\mathbf{x}_0)$ and $E^u(\mathbf{x}_0)$ as follows

$$E^s(\mathbf{x}_0) = \{(\tilde{\mathbf{z}}^\dagger \mathbf{w})f(\mathbf{x}_0) + \mathbf{w}, \quad \mathbf{w} \in \tilde{E}^s(\mathbf{x}_0)\}, \tag{1.28}$$

$$E^u(\mathbf{x}_0) = \{(\tilde{\mathbf{z}}^\dagger \mathbf{w})f(\mathbf{x}_0) + \mathbf{w}, \quad \mathbf{w} \in \tilde{E}^u(\mathbf{x}_0)\}. \tag{1.29}$$

$$\begin{aligned} D\phi^T(\mathbf{x}_0)((\tilde{\mathbf{z}}^\dagger \mathbf{w})f(\mathbf{x}_0) + \mathbf{w}) &= DP(\mathbf{x}_0)\mathbf{w} + (\tilde{\mathbf{z}}^\dagger + \mathbf{z}^\dagger)\mathbf{w}f(\mathbf{x}_0) = \\ &= DP(\mathbf{x}_0)\mathbf{w} + \tilde{\mathbf{z}}^\dagger DP(\mathbf{x}_0)\mathbf{w}f(\mathbf{x}_0). \end{aligned} \tag{1.30}$$

Because $\tilde{E}^s(\mathbf{x}_0)$ and $\tilde{E}^u(\mathbf{x}_0)$ are invariant subspaces for $DP(\mathbf{x}_0)$, the subspaces $E^s(\mathbf{x}_0)$, $E^u(\mathbf{x}_0)$ are invariant under $D\phi^T(\mathbf{x}_0)$ and moreover they have the same dimensions as $\tilde{E}^s(\mathbf{x}_0)$ and $\tilde{E}^s(\mathbf{x}_0)$. So we can decompose $\mathbb{R}^n$ as $\mathbb{R}^n = \mathrm{span}(f(\mathbf{x}_0)) \oplus E^s(\mathbf{x}_0) \oplus E^u(\mathbf{x}_0)$.

Using the mathematical induction on the equation (1.30), we get

$$[D\phi^T(\mathbf{x}_0)]^k((\tilde{\mathbf{z}}^\dagger \mathbf{w})f(\mathbf{x}_0) + \mathbf{w}) = DP(\mathbf{x}_0)^k\mathbf{w} + (\tilde{\mathbf{z}}^\dagger DP(\mathbf{x}_0)^k\mathbf{w})f(\mathbf{x}_0).$$

We know that there exist constants $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0,1)$ such that for $k \geq 0$

$$\|(DP(\mathbf{x}_0))^k\xi\| \leq K_1{\lambda_1}^k\|\xi\|, \quad \xi \in \tilde{E}^s(\mathbf{x}_0)$$

$$\|(DP(\mathbf{x}_0))^{-k}\xi\| \leq K_2{\lambda_2}^k\|\xi\|, \quad \xi \in \tilde{E}^u(\mathbf{x}_0).$$

If we realize that

$$(D\phi^{kT}(\mathbf{x}_0)) = D(\phi^T \circ \ldots \circ \phi^T)(\mathbf{x}_0) = \prod_{i=1}^{k} D\phi^T(\phi^{(k-i)T}(\mathbf{x}_0))$$

and $\phi^T(\mathbf{x}_0) = \mathbf{x}_0$, we see that $D\phi^{kT}(\mathbf{x}_0) = (D\phi^T(\mathbf{x}_0))^k$.

Hence for $k \geq 0$ and $\xi = (\tilde{\mathbf{z}}^\dagger \mathbf{w})f(\mathbf{x}_0) + \mathbf{w} \in E^s(\mathbf{x}_0)$

$$\|D\phi^{kT}(\mathbf{x}_0)\xi\| \leq \|(\tilde{\mathbf{z}}^\dagger[DP(\mathbf{x}_0)]^k\mathbf{w})f(\mathbf{x}_0)\| + \|[DP(\mathbf{x}_0)]^k\mathbf{w}\| \leq K_3{\lambda_1}^k\|\xi\|. \tag{1.31}$$

For $t \geq 0$ there exists nonnegative integer k such that $kT \leq t \leq (k+1)T$.

$$\|D\phi^t(\mathbf{x}_0)\xi\| = \|D\phi^{t-kT}(\mathbf{x}_0)D\phi^{kT}(\mathbf{x}_0)\xi\| \leq \|D\phi^{t-kT}(\mathbf{x}_0)D\phi^{kT}(\mathbf{x}_0)\xi\|\|D\phi^{kT}(\mathbf{x}_0)\xi\|.$$

We use Gronwall´s inequality [15] to bound the norm of $\|D\phi^{t-kT}(\mathbf{x}_0)\|$.

$$\begin{aligned} \|D\phi^{t-kT}(\mathbf{x}_0)\| &\leq \|\int_0^{t-kT} \frac{d}{ds}(D\phi^s(\mathbf{x}_0))ds + D\phi^0(\mathbf{x}_0)\| \\ &\leq \|\mathbf{x}_0\| + \int_0^T \|\frac{d}{ds}D\phi^s(\mathbf{x}_0)\|ds \leq \|\mathbf{x}_0\|e^{MT}, \end{aligned}$$

where $M = \sup_{0\leq s\leq T} Df(\phi^s(\mathbf{x}_0))$.

Therefore for $t \geq 0$ and $\xi \in E^s(\mathbf{x}_0)$,

$$\|D\phi^t(\mathbf{x}_0)\xi\| \leq e^{MT} K_3 {\lambda_1}^k \|\mathbf{x}_0\| \|\xi\| \leq K_4 e^{-\alpha_1 t} \|\xi\|, \tag{1.32}$$

where $\alpha_1 = -\frac{\ln \lambda_1}{T} > 0$.

Finally we can define the subspaces $E^s(\mathbf{u}(t))$ and $E^u(\mathbf{u}(t))$ for each point of the periodic orbit

$$E^s(\mathbf{u}(t)) = D\phi^t(\mathbf{x}_0)E^s(\mathbf{x}_0), \quad E^u(\mathbf{u}(t)) = D\phi^t(\mathbf{x}_0)E^u(\mathbf{x}_0).$$

Obviously they have the invariance property

$$D\phi^t(\mathbf{u}(\tau))E^s(\mathbf{u}(\tau)) = E^s(\phi^t(\mathbf{u}(\tau))), \quad D\phi^t(\mathbf{u}(\tau))E^u(\mathbf{u}(\tau)) = E^u(\phi^t(\mathbf{u}(\tau)))$$

and $\mathbb{R}^n$ can be decomposed as

$$\mathbb{R}^n = \text{span}(f(\mathbf{u}(t))) \oplus E^s(\mathbf{u}(t)) \oplus E^u(\mathbf{u}(t)) \quad \forall t.$$

Let us take $\xi \in E^s(\mathbf{u}(\tau))$ and $t \geq 0$. Then

$$\begin{aligned}\|D\phi^t(\mathbf{u}(\tau))\xi\| &= \|D\phi^t(\mathbf{u}(\tau))D\phi^\tau(\mathbf{x}_0)\tilde{\xi}\| = \|D\phi^{t+\tau}(\mathbf{x}_0)\tilde{\xi}\| \leq K_4 e^{-\alpha_1(t+\tau)}\|\tilde{\xi}\| \\ &\leq C_1 e^{-\alpha_1 t}\|\xi\|.\end{aligned}$$

Similarly, for $\xi \in E^u(\mathbf{u}(\tau))$ and $t \geq 0$

$$\|D\phi^{-t}(\mathbf{u}(\tau))\xi\| \leq C_2 e^{-\alpha_2 t}\|\xi\|.$$

We hope that the definition of a hyperbolic set follows from the previous computation.

Definition 15. A compact set $S \subset U$ is called *hyperbolic set* for the equation (1.24) if

1. $f(x) \neq 0$ for all $x \in S$
2. S is invariant, i.e., $\phi^t(S) = S$ for all t
3. there is a continuous splitting $\mathbb{R}^n = E^0(\mathbf{x}) \oplus E^s(\mathbf{x}) \oplus E^u(\mathbf{x})$, $\mathbf{x} \in S$

such that $E^0(\mathbf{x}) = \text{span}(f(x_0))$ and the subspaces $E^s(\mathbf{x})$ and $E^u(\mathbf{x})$ have constant dimensions $\forall \mathbf{x} \in S$, moreover these subspaces have the invariance properties

$$D\phi^t(\mathbf{x})(E^s(\mathbf{x})) = E^s(\phi^t(\mathbf{x})), \quad D\phi^t(\mathbf{x})(E^u(\mathbf{x})) = E^u(\phi^t(\mathbf{x}))$$

and there are positive constants $K_1, K_2, \lambda_1, \lambda_2$ such that for $t \geq 0$ and for all $\mathbf{x} \in S$

$$\|D\phi^t(\mathbf{x})\xi\| \leq K_1 e^{-\lambda_1 t}\|\xi\| \quad \xi \in E^s(\mathbf{x}), \tag{1.33}$$

$$\|D\phi^{-t}(\mathbf{x})\xi\| \leq K_2 e^{-\lambda_2 t}\|\xi\| \quad \xi \in E^u(\mathbf{x}). \tag{1.34}$$

Similarly as in the discrete case, the continuity of the splitting need not to be involved in the hyperbolicity definition because it follows from the other assumptions.

1.7 Shadowing theorem for hyperbolic sets

In this section, we present shadowing definitions and shadowing theorem for hyperbolic sets of differential equations of the form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)), \tag{1.35}$$

where U is convex open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^1 vector field.

The shadowing for differential equations is much more complicated than in the discrete case. There are problems even with the definition of a shadow. Should it be a continuous function of time or a sequence of points? Farther problem is caused by the lack of hyperbolicity in the direction of the vector field as the simple example shows.

Example 7. Consider the differential equation $\ddot{x} = 0$ with initial condition $x(0) = 0$. The solution of this equation is straight-line motion $x = v_0 t$, where v_0 is a constant velocity. Let us assume that numerically generated velocity has the following form

$$v = \begin{cases} v_0 & \text{for } t < 0 \\ v_0 + \delta & \text{for } t \geq 0. \end{cases}$$

It is obvious that every exact solution with $\tilde{v}_0$ close to v_0 will linearly diverge from it, so for this δ pseudoorbit there is no true trajectory that shadows it for all t. Although if we allow linear rescaling of time, this δ pseudoorbit will be shadowable.

When we compute the orbit of a discrete map, the errors are only in space, while when we compute the solution of differential equations, the numerically generated trajectory can have errors also in time. The errors in the length of each timestep can accumulate and although the numerically generated trajectory will follow the trajectory of the exact solution, these two solutions can have different time scale. For this reason, the rescaling of time is necessary for shadowing in continuous dynamical systems.

Since the subject of our interest are numerical solutions of differential equations, we will be dealt with discrete pseudoorbits and our definition of shadow allows to rescale time.

Definition 16. For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=0}^{\infty}$ in U is called a *discrete δ pseudoorbit* for equation (1.35) if there is a bounded sequence of positive times $\{h_k\}_{k=0}^{\infty}$ such that

$$\|\mathbf{y}_{k+1} - \phi^{h_k}(\mathbf{y}_k)\| < \delta \quad \text{for} \quad \forall k \geq 0.$$

Definition 17. A discrete δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^{\infty}$ with associated times $\{h_k\}_{k=0}^{\infty}$ is said to be ε shadowed by a true trajectory of the equation (1.35) if there are sequences $\{\mathbf{x}_k\}_{k=0}^{\infty}$ and $\{t_k\}_{k=0}^{\infty}$ such that $\mathbf{x}_{k+1} = \phi^{t_k}(\mathbf{x}_k)$ and

$$\|x_k - y_k\| < \varepsilon, \quad |t_k - h_k| < \varepsilon \quad \text{for} \quad \forall k \geq 0.$$

Similarly as for diffeomorphisms, the δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^{\infty}$ lying in hyperbolic set can be shadowed along all its length.

Theorem 9 (Shadowing theorem)**.** [20] Let S be a compact hyperbolic set for equation (1.35). For a given $\varepsilon > 0$, there is a δ such that any δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^{\infty}$ of equation (1.35) lying in S can be ε shadowed by a unique true orbit $\{\mathbf{x}_k\}_{k=0}^{\infty}$. Moreover

$$\langle f(\mathbf{y}_k), \mathbf{x}_k - \mathbf{y}_k \rangle = 0 \quad \forall k.$$

The shadowing theorem for non-hyperbolic systems will be discussed in the next chapter.

Chapter 2

Numerical shadowing

First we introduce two interesting dynamical systems that exhibit chaotic behaviour. Then we state two shadowing theorems convenient for the practical usage. All their assumptions can be verified by the computer, and mainly they can be applied also for non-hyperbolic dynamical systems. Finally we use one of the theorems to show that the numerical solutions of the presented chaotic differential equations are shadowable.

2.1 Lorenz equations

In 1963 Edward Lorenz published the article called Deterministic nonperiodic flow [18]. He studied a fluid cell with the temperature difference ΔT between the bottom and upper edge maintained at a constant value. For large ΔT, there occurs a convection which is modeled by two partial differential equations. The variables in these equations are expanded in Fourier series which leads to an infinite set of ordinary differential equations. All but three modes are omitted and thus the following set of three ordinary differential equations is obtained.

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= xy - \beta z. \end{aligned} \tag{2.1}$$

The variable x is proportional to the rate of convective motion. Variables y and z correspond to the horizontal and vertical temperature variation, respectively. The constants σ, ρ and β are positive.

Because $\mathrm{div} f(x, y, z) = -\sigma - 1 - b < 0$, the dynamical system governed by the Lorenz equations is dissipative and from the Liouville theorem we know that every volume V is squeezed into the volume $Ve^{-(\sigma+1+b)t}$ during the time t. Moreover we will show that all trajectories tend towards some bounded ellipsoid E.

These observations suggest that there is a bounded set of zero volume in E to which all trajectories are attracted. The nature of this set depends on the parameter ρ as we will see.

Consider the Lyapunov function $V = \rho x^2 + \sigma y^2 + \sigma(z-2\rho)^2$ with $\dot{V} = -2\sigma(\rho x^2 + y^2 + \beta z^2 - 2\beta\rho z)$. The theory of Lyapunov functions is explained in the Appendix A. There is certainly a bounded region D with $\dot{V} \leq 0$. Let c be the maximum value of V inside D and consider the bounded ellipsoid $E = \{(x,y,z)|\rho x^2 + \sigma y^2 + \sigma(z-2\rho)^2 \leq K\}$ in which $V \leq c + \varepsilon$, $\varepsilon > 0$.

If $\mathbf{x}$ does not lie in E, then $\dot{V} \leq -\delta(\varepsilon)$ for some small positive $\delta(\varepsilon)$. V decreases along the trajectory beginning at point $\mathbf{x}$ and thus the trajectory must eventually enter the ellipsoid E. Of course, both ε and δ can be zero. In this case it would take an infinite time to enter E and our argument does not sound very convincing. But the theorem 15 ensures that this entering really occurs.

The ellipsoid E is invariant as the following theorem shows ($g(\mathbf{x}) = V(\mathbf{x}) - K$).

Theorem 10. [7] Suppose $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and there is a continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ such that the set $D = \{\mathbf{x} \in \mathbb{R}^n | g(\mathbf{x}) < 0\}$ is a simply connected bounded domain with smooth boundary ∂D. If

$$\langle \nabla f, g \rangle < 0 \quad \text{on} \quad \partial D$$

then for all $\mathbf{x} \in D$, $\phi^t(\mathbf{x}) \in D$ for all $t \geq 0$.

We now determine the size of the ellipsoid E. Obviously, the Lyapunov function V reaches its maximum on the boundary ∂D, where $\rho x^2 + y^2 + \beta z^2 - 2\beta\rho z = 0$. Using the Lagrange function with Lagrange multiplier λ, we find that the maximum must satisfy the following equations

$$x = \lambda x, \quad \sigma y = \lambda y, \quad \sigma(z-2\rho) = \beta\rho(z-\rho).$$

There are three possible solutions of the previous equations. We are interested in classical parameter values $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$, where the maximum occurs for

$$x_{\max} = 0, \quad \lambda = \sigma, \quad z_{\max} = \frac{\rho(\beta-2)}{\beta-1}, \quad y^2_{\max} = \frac{\beta^2\rho^2(\beta-2)}{(\beta-1)^2}.$$

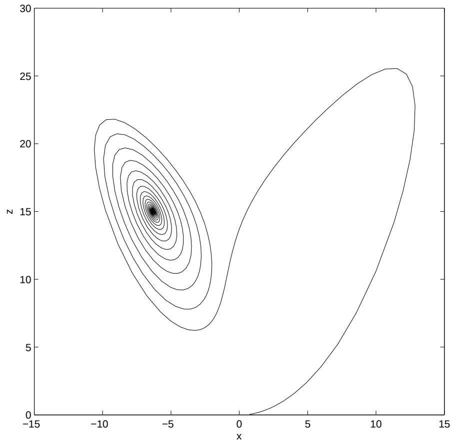
Therefore $\beta \geq 2$ and $V(x_{\max}, y_{\max}, z_{\max}) = \frac{\sigma\rho^2\beta^2}{\beta-1}$.

Thus we have proved that all trajectories tend towards an attracting set of a zero volume contained in the invariant ellipsoid

$$E = \left\{(x,y,z)|\rho x^2 + \sigma y^2 + \sigma(z-2\rho)^2 \leq \frac{\sigma\rho^2\beta^2}{\beta-1}\right\}.$$

There are only four possibilities, the attracting set can be a stationary point (figure 2.1), a periodic orbit (figure 2.1) or quasi-periodic orbit and the so called strange attractor (figure 2.2). The initial point is $\mathbf{x}_0 = (1;0;0)$ in these simulations.

We look at the stationary points of the Lorenz equations. We fix the values of σ and β as $\sigma = 10$ and $\beta = \frac{8}{3}$, the parameter ρ is allowed to vary. We immediately see that the origin is a stationary point for all values of ρ. It is stable for $\rho \in (0,1)$. When



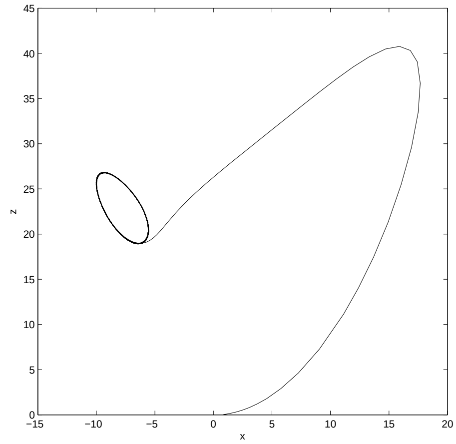


Figure 2.1: Left: stationary point for $\rho = 16$; right: periodic orbit for $\rho = 24$.

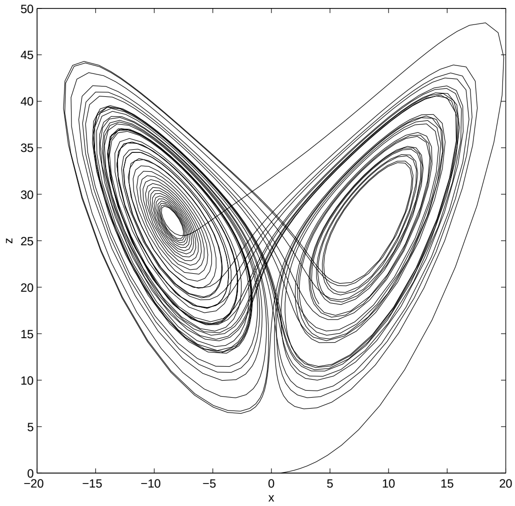


Figure 2.2: Lorenz attractor for $\rho = 28$.

parameter ρ increases above 1, the origin loses its stability and two other stationary points $\mathbf{x}_1 = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$, $\mathbf{x}_2 = (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)$ appear. These points are stable for $\rho \in (1, \frac{470}{19})$ because the Jacobian matrix at $\mathbf{x}_1$, $\mathbf{x}_2$ have three eigenvalues with negative real parts. For $\rho = \frac{470}{19}$ the Hopf bifurcation occurs, i.e., the complex conjugate pair of eigenvalues cross the y-axis. Numerical simulations prove that the Hopf bifurcation is subcritical – the periodic orbit is unstable and the points are attracted by the strange attractor.

The chaotic solution (figure 2.2), calculated when $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$, was described in Lorenz´s paper. This solution has some special features which are common to all chaotic solutions. The trajectory is non-periodic, it wraps around the attractor, first on one side, then on the other and this wrapping continues forever.

The general form of an attractor does not depend on the initial conditions and the used numerical method. Any method gives us the same picture, the difference is in the exact sequence of loops which the trajectory passes through. However this sequence of loops is extremely sensitive to the change of initial conditions or the integrating routine. This sensitivity makes impossible any long time prediction of the chaotic trajectory.

2.2 Chaos generator

The next interesting example of a dynamical system with chaotic behaviour is a nonlinear circuit designed at the Technische Universität in Kaiserslautern [17]. Its simplified scheme is shown in the figure 2.3. This circuit demonstrates the classical route to chaos. The behaviour of the system is very simple at the beginning, then the so called period doubling occurs and the sequence of period doublings tends to the chaos.

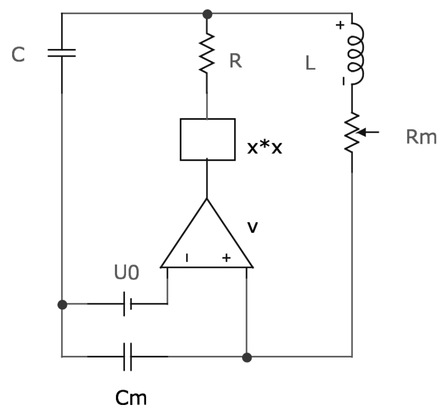


Figure 2.3: Chaos generator.

The circuit consists of two capacitors C_m, C, resistors R_m, R, an inductor L, an amplifier and a squaring module. The variable resistor R_m serves as a control parameter.

We apply the second Kirchhoff law to the loop RCC_m to gain the equation

$$\frac{Q}{C} + R(\dot{Q} + \dot{Q}_m) - v^2(U - U_0)^2 = 0, \tag{2.2}$$

where Q and Q_m are the charges at C, C_m and U is the voltage at the capacitor C_m.

Similarly for the loop C_mCLR_m, we obtain

$$\frac{Q}{C} - L\ddot{Q}_m - R_m\dot{Q}_m - \frac{Q_m}{C_m} = 0. \tag{2.3}$$

Using a long series of substitutions (the details can be found in [17]), the equations (2.2), (2.3) can be rewritten as a system of three differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= \mu x(1-x) - y - \beta z. \end{aligned} \tag{2.4}$$

The constants β, μ depend on the values of circuit components in a complicated way. The system (2.4) has two stationary points $x_1 = 0$ and $x_2 = 1$. They correspond to $U_1 = \frac{1}{2v^2}(1 + 2v^2U_0 + \sqrt{1 + 4v^2U_0})$ and $U_2 = \frac{1}{2v^2}(1 + 2v^2U_0 - \sqrt{1 + 4v^2U_0})$. The stationary point x_1 is unstable. The stationary point $x_2 = 1$ is stable for $\beta > \mu$. The behaviour in the neighbourhood of this stationary point will be studied in a more detail using the numerical simulations.

The values of circuit components used in the numerical simulations are the following

$$v = 1.2\,V^{-1/2} \quad R = 3300\,\Omega \quad C = C_m = 47 \times 10^{-9}\,F \quad L = 0.1\,H \quad U_0 = 4\,V.$$

The stability condition $\beta > \mu$ tends to quadratic equation for R_m with only one positive root. Hence for large values of R_m the voltage U remains at $U_2 = 2.6447V$. When R_m decreases its critical value $R_{m\text{crit}} = 770.6113\,\Omega$, a Hopf bifurcation occurs, i.e. the limit cycle appears. For even smaller values of R_m, the system becomes chaotic.

The system is damped down for $R_m = 1000\ \Omega$ and all points from a certain neighbourhood of the stationary point x_2 tend to it. This situation is well illustrated in the figure 2.4 for the initial condition $(1.78; 0; 0)$.

The time evolution of the point $(1.78; 0; 0)$ is quite different when $R_m = 500\ \Omega$ as figure 2.5 shows. The stationary point x_2 repels the nearby points and they are attracted by a limit cycle. This behaviour is not much surprising from a physical point of view. The resistor is not able to damp the signal down anymore so the signal can begin to oscillate. The variables $U, \dot{U}$ (scaled x, y) oscillate with the same frequency so the phase trajectory is an ellipse.

For even lower values of the resistance, the period doubling occurs, i.e., the signal oscillates with several different amplitudes. We can see a period-2 behaviour for $R_m = 150\ \Omega$ and period-4 behaviour for $R_m = 130\ \Omega$ in the figure 2.6.

For even lower resistance, the behaviour of the circuit becomes chaotic (figure 2.7).

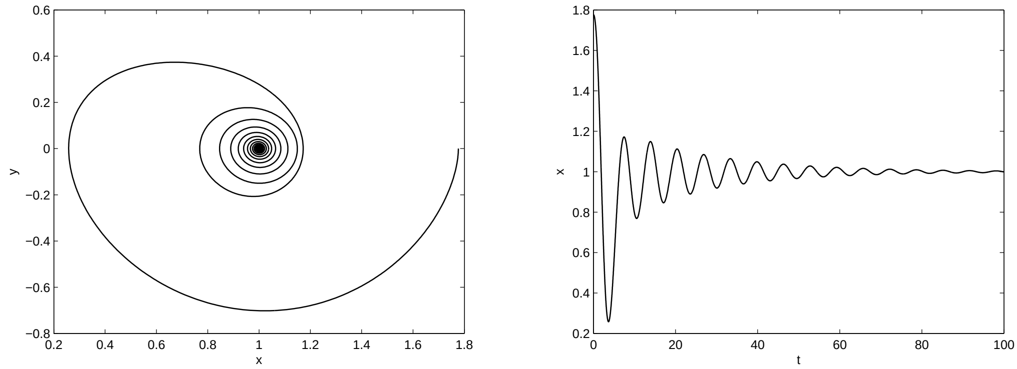


Figure 2.4: Stable stationary point. Left: phase picture of scaled U and $\dot{U}$; right: time evolution of the scaled voltage.

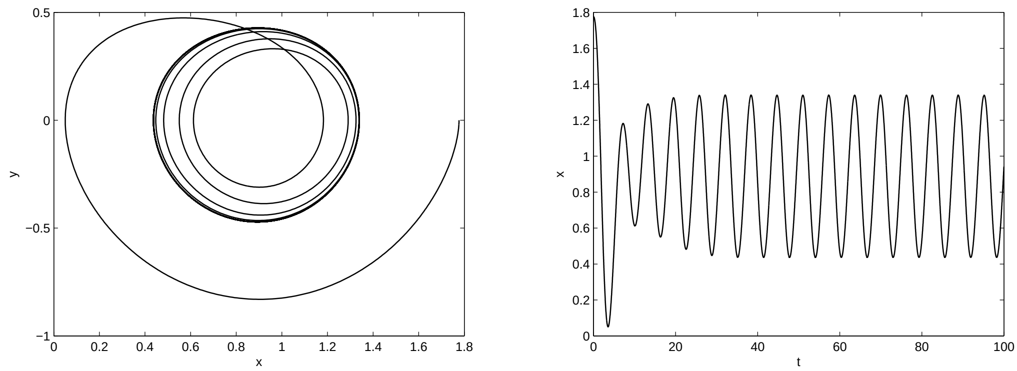


Figure 2.5: Limit cycle. Left: phase picture of scaled U and $\dot{U}$; right: time evolution of the scaled voltage.

2.3 Finite time shadowing theorem

In this section, we demonstrate that the pseudoorbits of the autonomous system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is} \quad C^2 \quad \text{vector field}, \tag{2.5}$$

are shadowed for long time by true solutions.

We introduce some notation before we state the shadowing theorem. We use the Euclidean norm for vectors and associated operator norm for matrices.

Let $\{\mathbf{y}_k\}_{k=0}^{N}$ be a δ pseudoorbit of (2.5) with associated times $\{h_k\}_{k=0}^{N-1}$. We first construct three sequences of matrices.

We find such a sequence $\{Y_k\}_{k=0}^{N-1}$ ($Y_k \in \mathbb{R}^{n,n}$) that

$$\|Y_k - D\phi^{h_k}(\mathbf{y}_k)\| \leq \delta \quad \text{for} \quad k = 0, \ldots, N-1.$$

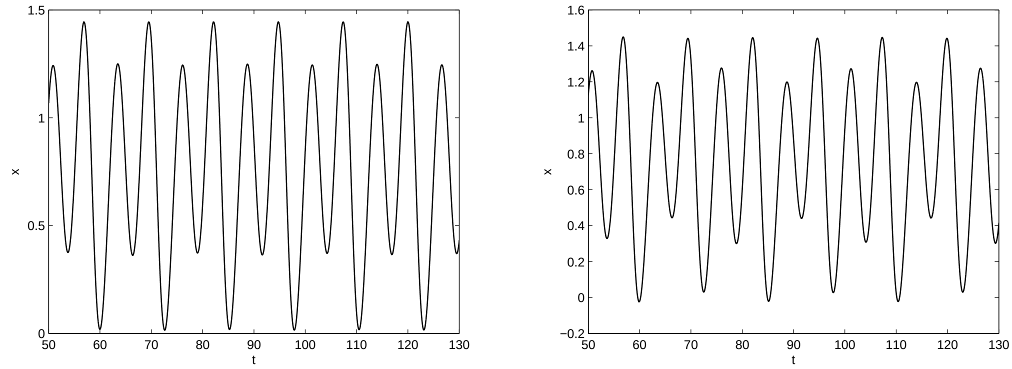


Figure 2.6: Period doubling. Left: period-2; right: period-4.

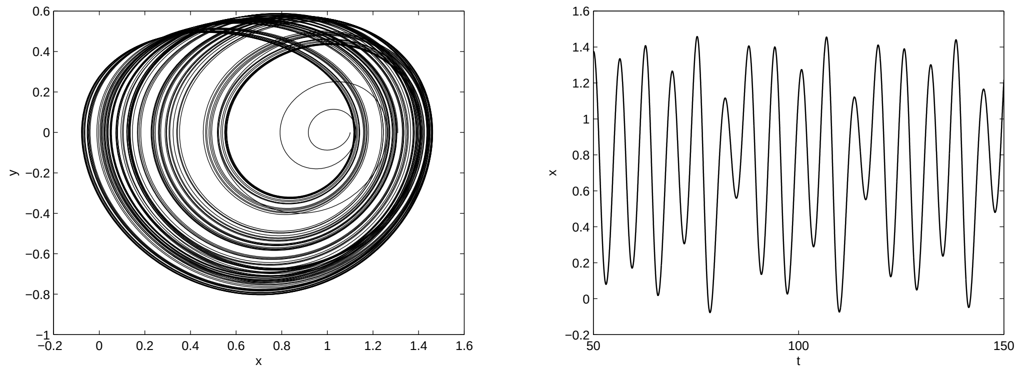


Figure 2.7: Chaotic behaviour - $R_m = 100\,\Omega$. Left: phase picture of scaled U and $\dot{U}$; right: time evolution of the scaled voltage.

Then we define a sequence $\{S_k\}_{k=0}^{N}$ ($S_k \in \mathbb{R}^{n,n-1}$) of matrices such that the columns of S_k form the orthonormal basis of $f(y_k)^{\perp}$. Therefore S_k have to satisfy the following conditions

$$\|S_k^T f(\mathbf{y}_k)\| \leq \delta_1 \quad \|S_k^T S_k - I\| \leq \delta_1$$

for some small positive δ_1. The value of δ_1 depends on the machine epsilon. This dependence will be discussed in the next section.

Next we compute a sequence $\{A_k\}_{k=0}^{N-1}$ ($A_k \in \mathbb{R}^{n-1,n-1}$), where A_k is Y_k restricted to $f(\mathbf{y}_k)^{\perp}$ and projected to $f(\mathbf{y}_{k+1})^{\perp}$, i.e.

$$\|A_k - S_{k+1}^T Y_k S_k\| \leq \delta_1.$$

Finally we introduce a linear operator $L : (\mathbb{R}^{n-1})^{N+1} \to (\mathbb{R}^{n-1})^N$ defined for $\mathbf{u} = \{\mathbf{u}_k\}_{k=0}^{N}$ by

$$(\mathbf{Lu})_k = \mathbf{u}_{k+1} - A_k \mathbf{u}_k \quad \text{for} \quad k = 0, \ldots, N-1.$$

Now we define various constants on the convex set $U \subset \mathbb{R}^n$ containing the δ pseudoorbit in its interior.

$$M_0 = \sup_{\mathbf{x}\in U} \|f(\mathbf{x})\|, \quad M_1 = \sup_{\mathbf{x}\in U} \|Df(\mathbf{x})\|, \quad M_2 = \sup_{\mathbf{x}\in U} \|D^2 f(\mathbf{x})\|$$

$$\Delta = \inf_{0\le k\le N} \|f(\mathbf{y}_k)\|, \quad \overline{M}_0 = \sup_{0\le k\le N} \|f(\mathbf{y}_k)\|, \quad \overline{M}_1 = \sup_{0\le k\le N} \|Df(\mathbf{y}_k)\|$$

$$\Theta = \sup_{0\le k\le N-1} \|Y_k\|, \quad h_{\min} = \inf_{0\le k\le N-1} h_k, \quad h_{\max} = \sup_{0\le k\le N-1} h_k.$$

We choose such positive $\varepsilon_0 \le h_{\min}$ that for all $k = 0, \ldots, N-1$ if $\|x - \phi^t(y_k)\| \le \varepsilon_0$ then the solution $\phi^t(x)$ exists and remains in U for $0 \le t \le h_k + \varepsilon_0$.

Theorem 11 (Finite time shadowing theorem). [5] Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudoorbit of (2.5) with associated times $\{h_k\}_{k=0}^{N-1}$, let

$$C = \max\{\Delta^{-1}(\theta\|L^{-1}\|(1+\delta_1)+1), \|L^{-1}\|\sqrt{1+\delta_1}\},$$

$$\bar{\delta} = \frac{C((M_1 + \sqrt{1+\delta_1})\delta + 3\delta_1(\sqrt{1+\delta_1} + \Delta^{-1}))}{1 - \delta_1(1+\Delta^{-2})}$$

and let

$$\begin{aligned}\overline{M} =& (\overline{M}_0 + M_1\nu\delta)(\overline{M}_1 + M_2\nu\delta) + 2(\overline{M}_1 + M_2\nu\delta)\sqrt{1+\delta_1}e^{M_1(h_{\max}+\varepsilon_0)} + \\ & M_2(h_{max} + \varepsilon_0)(1+\delta_1)e^{2M_1(h_{\max}+\varepsilon_0)},\end{aligned}$$

where

$$\nu = 2C(e^{M_1(h_{\max}+\varepsilon_0)}\sqrt{1+\delta_1} + M_0)(1-\bar{\delta})^{-1} + 1.$$

If δ, δ_1 and ε_0 satisfy the inequalities

1. $\delta_1(1+\Delta^{-2}) < 1$
2. $\bar{\delta} < 1$
3. $2C(1-\bar{\delta})^{-1}\sqrt{1+\delta_1}\delta < \varepsilon_0$
4. $2\overline{M}C^2(1-\bar{\delta})^{-2}\delta \le 1$

then the pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ is ε shadowed by a true orbit $\{\mathbf{x}_k\}_{k=0}^N$ with the shadowing distance $\varepsilon \le 2C(1-\bar{\delta})^{-1}\sqrt{1+\delta_1}\delta$.

We now sketch the proof of the previous theorem in order to know why we compute the sequences of the matrices $\{Y_k\}_{k=0}^{N-1}$, $\{S_k\}_{k=0}^N$ and $\{A_k\}_{k=0}^{N-1}$.

For a given δ pseudoorbit $(\{\mathbf{y}_k\}_{k=0}^N, \{h_k\}_{k=0}^{N-1})$, we would like to find its shadow $(\{\mathbf{x}_k\}_{k=0}^N, \{t_k\}_{k=0}^{N-1})$. To do this, we construct a hyperplane H_k through $\mathbf{y}_k$, that is approximately normal to $f(\mathbf{y}_k)$ and then we show that $\mathbf{x}_k$ is contained in H_k (figure 2.8).

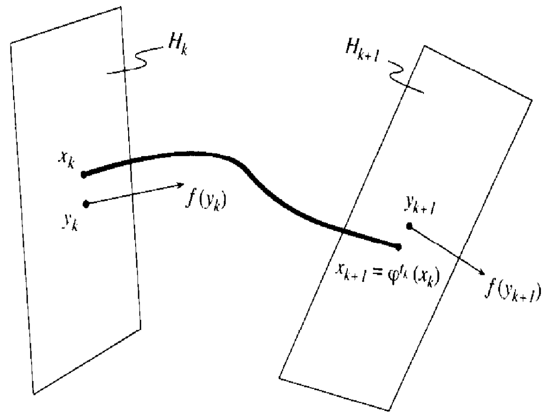


Figure 2.8: Hyperplanes [5].

We first identify H_k with $\mathbb{R}^{n-1}$ via the map $\mathbf{z} \mapsto \mathbf{y}_k + S_k\mathbf{z}$, $\mathbf{z} \in \mathbb{R}^{n-1}$. Then the problem of finding a shadow transforms to the problem of finding a sequence of times $\{t_k\}_{k=0}^{N-1}$ and a sequence of points $\{\mathbf{z}_k\}_{k=0}^{N}$ in $\mathbb{R}^{n-1}$ such that

$$\mathbf{y}_{k+1} + S_{k+1}\mathbf{z}_{k+1} = \phi^{t_k}(\mathbf{y}_k + S_k\mathbf{z}_k), \quad k = 0, \dots, N-1.$$

We introduce two Banach spaces: $X = (\mathbb{R}^{n-1})^{N+1} \times \mathbb{R}^N$ with norm

$$\|(\{\mathbf{w}_k\}_{k=0}^{N}, \{s_k\}_{k=0}^{N-1})\| = \max\{\max_{0\le k\le N} \|\mathbf{w}_k\|, \max_{0\le k\le N-1} |s_k|\}$$

and $Y = (\mathbb{R}^n)^N$ with norm

$$\|\{\mathbf{g}_k\}_{k=0}^{N-1}\| = \max_{0\le k\le N-1} \|\mathbf{g}_k\|.$$

Let $O = O^\circ \subset X$ be the set containing the points $\mathbf{v} = (\{\mathbf{w}_k\}_{k=0}^{N}, \{s_k\}_{k=0}^{N-1})$ with $|s_k - h_k| < \varepsilon_0$ and $\|\mathbf{w}_k\| < \frac{\varepsilon_0}{\sqrt{1+\delta_1}}$. We define a function $G : O \to Y$ by

$$[G(\mathbf{v})]_k = \mathbf{y}_{k+1} + S_{k+1}\mathbf{w}_{k+1} - \phi^{s_k}(\mathbf{y}_k + S_k\mathbf{w}_k), \quad k = 0, \dots, N-1.$$

The theorem 11 will be proved if we find a root $\tilde{\mathbf{v}} = (\{\mathbf{z}_k\}_{k=0}^{N}, \{t_k\}_{k=0}^{N-1})$ of the equation

$$G(\tilde{\mathbf{v}}) = 0 \tag{2.6}$$

in the closed ball of radius ε about $\mathbf{v}_0 = (\{\vec{0}\}_{k=0}^{N}, \{h_k\}_{k=0}^{N-1})$. The equation is solvable under the conditions of the following lemma.

Lemma 3. Let X, Y be finite-dimensional vector spaces and let $G : O = O^\circ \subset X \to Y$ be C^2 function satisfying the following properties.

1. The derivative $DG(\mathbf{v}_0)$ at $\mathbf{v}_0 \in O$ has a right inverse $(DG(\mathbf{v}_0))^{-1}$

2. $\overline{B}(\mathbf{v}_0, \varepsilon) \subset O$, where $\varepsilon = 2\|(DG(\mathbf{v}_0))^{-1}\|\|DG(\mathbf{v}_0)\|$.

3. $2M\|(DG(\mathbf{v}_0))^{-1}\|^2\|G(\mathbf{v}_0)\| \leq 1$, where $M = \sup\{\|D^2G(\mathbf{v})\| : \|\mathbf{v} - \mathbf{v}_0\| \leq \varepsilon\}$.

Then there is a solution $\tilde{\mathbf{v}}$ of the equation

$$G(\tilde{\mathbf{v}}) = 0,$$

satisfying $\|\tilde{\mathbf{v}} - \mathbf{v}_0\| \leq \varepsilon$.

The lemma follows from the Brouwer fixed point theorem applied to the map $H(\mathbf{v}) = \mathbf{v}_0 - (DG(\mathbf{v}_0))^{-1}(G(\mathbf{v}) - DG(\mathbf{v}_0)(\mathbf{v} - \mathbf{v}_0))$.

The most difficult part in the verification of the lemma 3 assumptions is the construction of the right inverse of $DG(\mathbf{v}_0)$ and the estimation of the bound of its norm.

Let $\mathbf{v} = (\{\mathbf{w}_k\}_{k=0}^{N}, \{s_k\}_{k=0}^{N-1}) \in X$. Then the derivative of G at $\mathbf{v}_0$ acts as

$$[DG(\mathbf{v}_0)\mathbf{v}]_k = -s_k f(\phi^{h_k}(\mathbf{y}_k)) + S_{k+1}\mathbf{w}_{k+1} - D\phi^{h_k}(\mathbf{y}_k)S_k\mathbf{w}_k.$$

We approximate $DG(\mathbf{v}_0)$ by the linear operator $T : X \to Y$ defined by

$$[T\mathbf{v}]_k = -s_k f(\mathbf{y}_{k+1}) + S_{k+1}\mathbf{w}_{k+1} - D\phi^{h_k}(\mathbf{y}_k)S_k\mathbf{w}_k.$$

It is a standard result of functional analysis that because

$$\begin{aligned}\|T^{-1}DG(\mathbf{v}_0) - I\| &\leq \|T^{-1}\|\|DG(\mathbf{v}_0) - T\| \leq \|T^{-1}\| \sup_{0\leq k\leq N-1} \|f(\phi^{h_k}(y_k)) - f(y_{k+1})\| \\ &\leq M_1\|T^{-1}\| \sup_{0\leq k\leq N-1} \|\phi^{h_k}(y_k) - y_{k+1}\| \leq CM_1\delta < 1,\end{aligned}$$

the operator $T^{-1}DG(\mathbf{v}_0)$ is invertible and

$$(DG(\mathbf{v}_0))^{-1} = (I - T^{-1}(T - DG(\mathbf{v}_0)))^{-1}T^{-1}. \tag{2.7}$$

In order to find a right inverse of T we must solve the equations

$$(T\mathbf{u})_k = g_k$$

for a given $\mathbf{g} = \{\mathbf{g}_k\}_{k=0}^{N-1}$, i.e.,

$$-s_k f(\mathbf{y}_{k+1}) + S_{k+1}\mathbf{w}_{k+1} - D\phi^{h_k}(\mathbf{y}_k)S_k\mathbf{w}_k = \mathbf{g}_k. \tag{2.8}$$

We multiply the equation (2.8) first by $f(\mathbf{y}_{k+1})^T$ to gain

$$-s_k\|f(\mathbf{y}_{k+1})\|^2 - f(\mathbf{y}_{k+1})^T D\phi^{h_k}(\mathbf{y}_{k+1})S_k\mathbf{w}_k = f(\mathbf{y}_{k+1})^T\mathbf{g}_k$$

and then by S_{k+1}^T to obtain

$$\mathbf{w}_{k+1} - A_k\mathbf{w}_k = S_{k+1}^T\mathbf{g}_k.$$

Let us denote as $\mathbf{g}$ the sequence $\mathbf{g} = \{S_{k+1}^T \mathbf{g}_k\}_{k=0}^{N-1}$, then

$$\mathbf{w}_k = (L^{-1}\mathbf{g})_k$$

and

$$s_k = -\frac{f(\mathbf{y}_{k+1})^T}{\|f(\mathbf{y}_{k+1})\|}(D\phi^{h_k}(\mathbf{y}_{k+1})S_k(L^{-1}\mathbf{g})_k + \mathbf{g}_k).$$

It follows that

$$\|T^{-1}\| = \max(\|L^{-1}\|, \Delta^{-1}(\Theta\|L^{-1}\| + 1)).$$

Of course, the situation is not so simple in practice, because we do not know the matrices $D\phi^{h_k}(\mathbf{y}_k)$, we just know their approximations Y_k. Also the matrices S_k are only floating point approximations of orthogonal matrices. This is the reason, why δ_1 appears in the assumptions of the theorem 11.

As a consequence of the relation (2.7)

$$\|(DG(\mathbf{v}_0))^{-1}\| \leq \|T^{-1}\|(1 - \|T^{-1}\|M_1\delta)^{-1}.$$

The verification of the other lemma 3 assumptions is quite straightforward.

We should mention some practical remarks to the theorem 11. The vector valued function f and its derivatives Df, D^2f must be bounded over the entire convex set U containing the δ pseudoorbit, therefore this theorem is not applicable to the dynamical systems which may contain poles in U. Moreover computing the bound on the second derivative of f can be very expensive.

The theorem 3 is directly applicable to differential equations, the map g is the flow ϕ of the system in that case. It gives quite satisfactory shadowing times and shadowing distances, but computing the bounds on the first and second derivatives of ϕ is very difficult. Therefore the need of the determination of the bounds of Df and D^2f (however difficult to estimate) is still large improvement. Also the theorem 3 doubles the numerical accuracy, i.e., we need $\delta = 10^{-20}$ to ensure the existence of the shadow with a shadowing distance $\varepsilon = 10^{-10}$. As we will see, the numerical accuracy required by the theorem 11 is not so high.

2.4 Finite precision computations

All numerical computations are affected by roundoff errors – a computer does not work with all real numbers, but only with their subset. This subset is denoted as $\mathbb{F}$ and its elements are called the floating point numbers. The term is derived from the fact that there is no fixed number of digits before and after the decimal point, i.e., the decimal point can float.

A floating point number is the number of the form

$$f = \pm .d_1 d_2 \dots d_t \times \beta^e, \quad d_1 \neq 0, \quad 0 \leq d_i < \beta.$$

It is characterized by 4 integers the base β, the precision t and the exponent range $[L, U]$. The typical values for (β, t, L, U) are $(2, 16, -64, 64)$.

In order to model the effect of rounding errors for a numerical algorithm, we define the operator $fl : G \subset \mathbb{R} \to \mathbb{F}$, where G is the interval determined by the values L and U, as the operator which rounds the number to the nearest floating point number. It can be shown that

$$fl(x) = x(1 + \varepsilon), \quad |\varepsilon| \leq \frac{1}{2}\beta^{1-t}. \tag{2.9}$$

We assume that the equation 2.9 holds also when for $x = x_1 \diamond x_2$, where $\diamond$ denotes one of the four basic arithmetic operations between x_1 and x_2.

Example 8. We try to estimate the error of the extended product $x_1x_2 \dots x_n$. Let us denote $p_n = fl(x_1x_2 \dots x_n)$. Then p_i, $i \in \hat{n}$, are given by the reccurence

$$\begin{aligned} p_1 &= x_1 \\ p_k &= fl(p_{k-1}x_k) = p_{k-1}x_k(1 + \varepsilon_k), \end{aligned}$$

where $|\varepsilon_k| \leq 2^{-t}$ for binary computations.

Therefore

$$p_n = x_1x_2 \dots x_n(1 + \varepsilon_2)(1 + \varepsilon_3) \dots (1 + \varepsilon_n),$$

which implies that

$$fl(x_1x_2 \dots x_n) = x_1x_2 \dots x_nK,$$

where

$$(1 - 2^{-t})^{n-1} \leq K \leq (1 + 2^{-t})^{n-1}.$$

Similarly, we can find error bounds for all numerical algorithms. For more detail see [28], [29].

2.5 Implementation of the theorem

We use the ode45 MATLAB solver for all our numerical integrations of differential equations. This function implements Runge-Kutta method with a variable time step.

To determine M_0, M_1 and M_2, we must choose some bounded convex set U, in which the δ pseudoorbit is contained. It can happen that the pseudoorbit will leave the set U during the computation and we are therefore forced to enlarge U and determine constants M_0, M_1 and M_2 again. However, if we choose U, which is forward invariant under the flow ϕ^t of the system (2.5), we can compute these constants once and for all.

We must determine such an $\varepsilon_0 \leq h_{\min}$ that the solution $\phi^t(\mathbf{x})$ is defined and lies in U for all $t \in (0, h_k + \varepsilon_0)$. For forward invariant set U this is true if ε_0 satisfies the condition

$$\varepsilon_0 < \text{dist}(\mathbf{y}_k, \partial U). \tag{2.10}$$

The flow $\phi^t(x)$ is defined by the equation

$$\frac{d}{dt}\phi^t(\mathbf{x}) = f(\phi^t(\mathbf{x})).$$

We assume that the function f is C^2 and the flow is as smooth as f, therefore the matrix $D\phi^t(x)$ must satisfy the equation

$$\frac{d}{dt}D\phi^t(x) = Df(\phi^t(x))D\phi^t(x).$$

So if we are looking for the matrix Y_k, we must solve the enlarged set of differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \dot{Y} = Df(\mathbf{x})Y \tag{2.11}$$

with initial conditions

$$\mathbf{x}(0) = \mathbf{y}_k, \quad Y(0) = I.$$

After this computation, we can determine the values of the constants Δ, $\overline{M}_0$, $\overline{M}_1$ and Θ.

The sequence of $n\times(n-1)$ matrices $\{S_k\}_{k=0}^{N-1}$ is computed recursively. We choose S_0 in such a way that the matrix

$$\left(\frac{f(\mathbf{y}_0)}{\|\mathbf{y}_0\|}\Big| S_0\right)$$

is orthogonal.

In the next step, we use the fact that every regular matrix $A \in \mathbb{R}^{n,n}$ can be decomposed as a product $A = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix. This decomposition is unique if R has positive diagonal elements.

In order to obtain S_{k+1} and A_k we compute the QR factorization of the matrix

$$\left(\frac{f(\mathbf{y}_{k+1})}{\|f(\mathbf{y}_{k+1})\|}\Big| Y_kS_k\right) = \left(\frac{f(\mathbf{y}_{k+1})}{\|f(\mathbf{y}_{k+1})\|}\Big| S_{k+1}\right) R_k.$$

The matrix A_k is $(n-1)\times(n-1)$ submatrix placed in the right low corner of the matrix R_k.

The last problem is to estimate the norm $\|L^{-1}\|$. To find a right inverse of L, we have to solve the equation

$$\mathbf{u}_{k+1} = A_k\mathbf{u}_k + \mathbf{g}_k, \tag{2.12}$$

i.e., to find $u = \{\mathbf{u}_k\}_{k=0}^N$ for given $g = \{\mathbf{g}_k\}_{k=0}^N$.

We will deal only with three-dimensional dynamical systems, therefore the matrices A_k are two-dimensional and we may write

$$\mathbf{u}_k = \begin{pmatrix} v_k \\ w_k \end{pmatrix}, \quad A_k = \begin{pmatrix} a_k & b_k \\ 0 & c_k \end{pmatrix}.$$

The equation (2.12) is in the components

$$v_{k+1} = a_k v_k + b_k w_k + g_k^{(1)}, \tag{2.13}$$
$$w_{k+1} = c_k w_k + g_k^{(2)}. \tag{2.14}$$

We construct L^{-1} by solving (2.14) forwards starting with $w_0 = 0$ and then solving (2.13) backwards starting with $v_N = 0$. The reasons for such a choice are explained in full detail in [20].

Now we define new variables $\tilde{v}_k$, $\tilde{w}_k$, $k = 0, \dots, N$, by the recursions

$$\tilde{w}_0 = 0, \quad \tilde{w}_{k+1} = |c_k|\tilde{w}_k + 1, \quad k = 0, \dots, N-1$$

and

$$\tilde{v}_N = 0, \quad \tilde{v}_k = |a_k^{-1}|(\tilde{v}_{k+1} + |b_k|\tilde{w}_k), \quad k = N-1, \dots, 0.$$

We see that $|w_k| \le \tilde{w}_k$ and $|v_k| \le \tilde{v}_k$ for $g = \{g_k\}_{k=0}^N$ such that $\|g\| \le 1$. Therefore

$$\|L^{-1}\| = \max_{0 \le k \le N} \sqrt{v_k^2 + w_k^2} \le \max_{0 \le k \le N} \sqrt{\tilde{v}_k^2 + \tilde{w}_k^2}. \tag{2.15}$$

2.5.1 Lorenz equations

Lorenz chose the values of the parameters $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$, so we will solve the equations

$$\begin{aligned} \dot{x} &= -10x + 10y, \\ \dot{y} &= 28x - y - xz, \\ \dot{z} &= xy - \frac{8}{3}z. \end{aligned} \tag{2.16}$$

We will compute only the δ pseudoorbits starting in $U = \{(x, y, z) | 28x^2 + \frac{8}{3}y^2 + \frac{8}{3}(z - 56)^2 \le 33450\}$, so we can now determine the constants M_0, M_1 and M_2.

$$M_0 \le 5519, \quad M_1 \le 161, \quad M_2 = 2.$$

We computed many pseudoorbits starting in U and the values of the constants presented in the theorem 11 were very similar. We choose the δ pseudoorbit pictured in the figure 2.2 as a representative sample and write the values of these constants only for it.

We chose $\varepsilon = 0.0001$. Because our solution lies in the set $\langle -20; 20\rangle \times \langle -30; 30\rangle \times \langle 0; 50\rangle$, the condition $\varepsilon_0 < \operatorname{dist}(\mathbf{y}_k, \partial U)$ is certainly satisfied.

The values of the constants presented in the theorem 11 computed along the δ pseudoorbit are

$$\overline{M}_0 \le 438, \quad \overline{M}_1 \le 41, \quad \Delta \ge 6,$$
$$\theta \le 1.43, \quad h_{\min} = h_{\max} = 0.01.$$

The most time-consuming part of the computation was the estimation that the norm of L^{-1} satisfy the inequality $\|L^{-1}\| \leq 5326$. MATLAB computes with $\delta = 2.22 \times 10^{-16}$. Considerations stated in [28] allow us to estimate that $\delta_1 \leq 5 \times 10^{-14}$. The constant $\delta \leq 3.5 \times 10^{-13}$. It gives us $C \leq 5327$, $\overline{\delta} = 3.1 \times 10^{-7}$, $\nu \leq 5.9 \times 10^7$ and $\overline{M} \leq 1.84 \times 10^4$.

The assumptions

1. $\delta_1(1 + \Delta^{-2}) < 5.2 \times 10^{-14} < 1$
2. $\bar{\delta} \leq 3.1 \times 10^{-7} < 1$
3. $2C(1 - \bar{\delta})^{-1}\sqrt{1 + \delta_1}\delta \leq 3.8 \times 10^{-9} < 0.0001$
4. $2\overline{M}C^2(1 - \bar{\delta})^{-2}\delta \leq 0.37 \leq 1$

are satisfied. Therefore our chaotic pseudoorbit is shadowed by a true orbit with the shadowing distance $\varepsilon \leq 3.8 \times 10^{-9}$.

2.5.2 Chaos generator

The values of circuit components used for the numerical simulations are the following

$$v = 1.2\,V^{-1/2}, \quad R = 3300\,\Omega, \quad R_m = 100\,\Omega, \quad C = C_m = 47 \times 10^{-9}\,F,$$

$$L = 0.1\,H, \quad U_0 = 4\,V,$$

so we will solve the set of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= 0.7491401416x(1 - x) - y - 0.3583243186z. \end{aligned} \tag{2.17}$$

We choose the set $U = \langle -0.2; 2\rangle \times \langle -1; 0.8\rangle \times \langle -0.8; 0.8\rangle$, which contains the chaotic trajectory (figure 2.7). Then

$$M_0 \leq 2.83, \quad M_1 \leq 2.86 \quad \text{and} \quad M_2 = \mu \leq 1.5.$$

We computed the pseudo trajectory starting at point $(1; 0; 0)$ of the length $t = 10^6 s$. The first one hundred seconds of its time evolution are pictured in the figure 2.7. The values of the constants $\overline{M}_0$ and $\overline{M}_1$ computed along the δ pseudoorbit are

$$\overline{M}_0 \leq 2.83, \quad \overline{M}_1 \leq 2.86.$$

The values of these constants are smaller than in the previous case, so it could seem that the chaotic pseudo trajectory is shadowable. However there is still one constant left – Δ, and unfortunately its zero value does not allow to use the theorem 11 for the proof of the existence of the shadow for our chaotic trajectory. The constant

Δ is equal to zero, because the set U contains unstable stationary point and the pseudo trajectory gets arbitrarily close to it during the time evolution.

We chose the simple set of differential equations (2.17) to show the limitations of the theorem 11. The verification of the assumptions of this theorem is not difficult in practice, it gives us very long shadowing times and the shadowing distance is more than sufficient, but it is not applicable for the sets containing the stationary points. Let us notice that in the definition of hyperbolic set we have also assumed that the hyperbolic set is without stationary points. The shadowing theorem for the sets with stationary points was proved by Pilyugin [21].

We should note that the impossibility to use the theorem 11 does not mean that the trajectory is unshadowable. The chaotic circuit described above was set up in our school laboratory and the picture in the oscilloscope screen really corresponds to the picture gained by the numerical simulation.

2.6 Map method

We have spent plenty of time looking for shadows for the δ pseudoorbits $\{\mathbf{y}_k\}_{k=0}^N$ of diffeomorphisms. We can imagine the δ pseudoorbit $(\{\mathbf{y}_k\}_{k=0}^N, \{h_k\}_{k=0}^{N-1})$ of the differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$ as the δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ of the sequence of diffeomorphisms $\{\phi^{h_k}\}_{k=0}^{N-1}$,

$$\|\mathbf{y}_{k+1} - \phi^{h_k}(\mathbf{y}_k)\| < \delta, \quad k = 0, \ldots, N-1,$$

where ϕ^{h_k} is the ODE solver with the time step h_k. Thus it is natural to try to apply the map method (theorem 5) to the δ pseudoorbit $\{\mathbf{y}_k\}_{k=0}^N$ of the sequence of diffeomorphisms $\{\phi^{h_k}\}_{k=0}^{N-1}$. The theorem 5 can be used directly when the time step h_k is fixed for all steps of the computation. Mostly the times h_k are varied so we must generalize the theorem 5 in order to be applicable to a sequence of maps.

The generalization to the sequence of diffeomorphisms $\{\phi^{h_k}\}_{k=0}^{N-1}$ is straightforward. We define the linear operator $L : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ $\mathbf{u} = \{\mathbf{u}_k\}_{k=0}^N$ by the relation

$$(L\mathbf{u})_k = \mathbf{u}_{k+1} - D\phi^{h_k}(\mathbf{y}_k)\mathbf{u}_k, \quad k = 0, \ldots, N-1$$

and the constant M will be

$$M = \sup\{\|D^2\phi^{h_k}(\mathbf{x})\|, \quad \|\mathbf{x} - \mathbf{y}_k\| \leq \epsilon, \quad k = 0, \ldots, N\}.$$

To use the theorem 5 we must determine the norm $\|L^{-1}\|$. We know only the approximations Y_k of the matrices $D\phi^{h_k}(\mathbf{y}_k)$

$$\|Y_k - D\phi^{h_k}(\mathbf{y}_k)\| < \delta. \tag{2.18}$$

These matrices define the linear operator $\tilde{L} : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ for $\mathbf{u} = \{\mathbf{u}_k\}_{k=0}^N$ by

$$(\tilde{L}\mathbf{u})_k = \mathbf{u}_{k+1} - Y_k\mathbf{u}_k, \quad k = 0, \ldots, N-1.$$

Because $\|L - \tilde{L}\| < \delta$, the following inequality holds

$$\|L^{-1}\| < (1 - \delta\|\tilde{L}^{-1}\|)^{-1}\|\tilde{L}^{-1}\|.$$

To find the right inverse of $\tilde{L}$ means to solve the equation $\tilde{L}\mathbf{u} = \mathbf{g}$ for given $\mathbf{g} = \{\mathbf{g}_k\}_{k=0}^{N-1}$, i.e., the difference equation

$$\mathbf{u}_{k+1} = Y_k\mathbf{u}_k + \mathbf{g}_k \quad k = 0, \ldots, N-1. \tag{2.19}$$

Let us take the orthogonal matrix $Q_0 = I$ and compute the following QR factorizations

$$Y_k Q_k = Q_{k+1}R_k, \quad k = 0, \ldots, N-1.$$

Now we can introduce new variables $\tilde{\mathbf{u}}_k : \mathbf{u}_k = Q_k\tilde{\mathbf{u}}_k$. The equation (2.19) transforms into

$$\tilde{\mathbf{u}}_{k+1} = R_k\tilde{\mathbf{u}}_k + Q_{k+1}^T\mathbf{g}_k, \quad k = 0, \ldots, N-1.$$

We define the linear operator $T : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ by:

$$(T\mathbf{u})_k = \mathbf{u}_{k+1} - R_k\mathbf{u}_k \quad \text{for} \quad \mathbf{u} = \{\mathbf{u}_k\}_{k=0}^N.$$

$$(\tilde{L}^{-1}\mathbf{g})_k = Q_k\tilde{\mathbf{u}}_k = Q_kT^{-1}Q_{k+1}^T\mathbf{g}_k. \tag{2.20}$$

So the inverse T^{-1} of T defines the inverse $\tilde{L}^{-1}$.

Unfortunately, the situation is not so simple, because the matrices R_k and Q_k are only the floating point approximations of the orthogonal and upper triangular matrices, respectively, i.e.

$$\|Q_k^TQ_k - I\| < \delta_1, \quad \|R_k - Q_{k+1}^TY_kQ_k\| < \delta_1.$$

For this reason we define another linear operator $V : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ by

$$(V\mathbf{u})_k = Q_{k+1}^TQ_{k+1}\mathbf{u}_{k+1} - Q_{k+1}^TY_kQ_k\mathbf{u}_k.$$

Therefore $\|\tilde{L}^{-1}\| < (1 + \delta_1)\|V^{-1}\|$. Because $\|V - T\| < 2\delta_1$, the norm $\|V^{-1}\|$ satisfies the inequality

$$\|V^{-1}\| < (1 - 2\delta_1\|T^{-1}\|)^{-1}\|T^{-1}\|.$$

The summary of these considerations leads us to the following theorem.

Theorem 12. [5] Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudoorbit of (2.5) with associated times $\{h_k\}_{k=0}^{N-1}$, let

$$C = (1 - (\delta + \delta\delta_1 + 2\delta_1)\|T^{-1}\|)^{-1}(1 + \delta_1)\|T^{-1}\|$$

and

$$\overline{M} = M_2h_{\max}e^{2M_1h_{\max}}.$$

If the following inequalities are satisfied

1. $\delta_1 < 1$,
2. $(\delta + \delta\delta_1 + 2\delta_1)\|T^{-1}\| < 1$,
3. $2C\delta < \varepsilon_0$,
4. $2\overline{M}C^2\delta \leq 1$,

then there is a true orbit $\{\mathbf{x}_k\}_{k=0}^N$ such that $\mathbf{x}_{k+1} = \phi^{h_k}(\mathbf{x}_k)$ for $k = 0, \dots, N-1$ and $\{x_k\}_{k=0}^N$ ε shadows $\{\mathbf{y}_k\}_{k=0}^N$ within $\varepsilon \leq 2C\delta$.

We should note that the previous theorem provides only the lower bounds for the shadowing time and the shadowing distance because in contrast to the theorem 11 the time is not allowed to fluctuate and as we have seen (example 7) the time rescaling is necessary for the shadowing in the continuous case. Coomes *et al.* have found shadows for pseudoorbits of the Lorenz equations lasting only $10\,s$, while shadowing time, when they used theorem 11, was $10^5\,s$.

Chapter 3

Unshadowability

3.1 Glitches

We begin this section with the simple example.

Example 9. Consider one-dimensional map $f(x) = 1-2x^2$. The interval $I = \langle -1, 1\rangle$ is a compact invariant set of f. It is not a hyperbolic set of f, because $Df(-\frac{1}{4}) = 1$, $Df(\frac{1}{4}) = -1$.

Let us compute a δ pseudoorbit beginning at point $x_0 = 0$. If get the inaccurate value $x_1 = 1+\epsilon$, $0 < \epsilon < \delta$ in the following step and then our computation continues without any error

$$x_2 = f(x_1) = -1 - 4\epsilon - 2\epsilon^2 < -1, \quad \dots \quad .$$

The δ pseudoorbit diverges to $-\infty$. But I is an invariant set of f so all true orbits beginning in I stay in this interval for all future evolution and therefore there is no shadow for our δ pseudoorbit.

The point $x = -1$ is a hyperbolic fixed point of f but $E^u = \mathbb{R}$ and therefore it repels all nearby orbits.

Previous example illustrates that even for very simple dynamical systems there exists points in the phase space at which all true trajectories diverge from the computer generated one. These points are called *glitches*.

One type of glitches has the same cause as the glitch in example 9. There is a fixed point $\mathbf{x}$, which has both stable and unstable manifolds nontrivial. The rounding error pushes the trajectory across the stable manifold, while true trajectories move to the stable manifold. These manifolds separate exponentially during the time evolution so no shadow can exist (figure 3.1).

The other type occurs when the system has a Lyapunov exponent which fluctuates about zero. The theory of Lyapunov exponents is outlined in the Appendix B.

The Lyapunov exponent of a trajectory fluctuates about zero if for any positive T the time-T Lyapunov exponent is arbitrarily long positive and arbitrarily long negative along the trajectory.

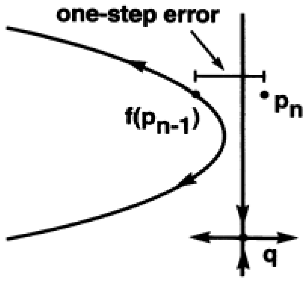


Figure 3.1: Glitch [6].

The way, in which a fluctuating Lyapunov exponent causes the unshadowability of trajectories, was theoretically described by Abraham and Smale [1].

3.2 Shadowing test

There is no other way how to find out whether the pseudoorbit is shadowable or not than to make the computer assisted proof. However as we have seen the shadowing methods are very expensive both for time and memory. Dawson *et al.* [6] introduced a practical algorithm for a diagnostic of the shadowability.

They introduced so called *continuously shadowable pseudo trajectory.* It is such a pseudo trajectory that can be continuously deformed to a true trajectory. The errors at each point must decrease monotically to zero during the deformation. Although it seems that the continuous shadowability is stronger property than shadowability, it can be shown that the pseudo trajectories, whose shadowability is ensured by the Anosov-Bowen theorems and also by the theorem 3, are also continuously shadowable.

Dawson *et al.* defined the so called *brittleness* as a constant of proportionality between the distance between the initial point and the point, in which the δ pseudo trajectory is deformed into a true one, and the δ. Brittleness of a pseudo trajectory is a measure of its shadowability. It is necessary that brittleness multiplied by the error δ is smaller than the size of the attractor for shadowing.

For example, we computed pseudoorbits with noise 10^{-13} in the neighbourhood of the attractor of the length of the order 10. So if the brittleness will be greater than 10^{14}, then we cannot expect the existence of a shadow.

It can be shown (and it is not surprising) that even pseudoorbits of infinite lengths in hyperbolic systems have finite brittleness. Although the value of the brittleness is finite, it can be very large, which requires very small δ for shadowing. It can happen that the δ will be smaller than the machine epsilon of currently existing computers. This is the same problem which arises from the original Anosov-Bowen shadowing theorems. For non-hyperbolic systems, the brittleness increases with the length of the pseudoorbit.

To determine the distance at which the pseudoorbit is deformed to a true one (and thus to determine the brittleness) we need to know the true trajectory. This knowledge is usually not available, so we cannot determine the exact value of the brittleness in the most cases. But we can gain a first-order approximation using the Jacobian matrices evaluated at points of the pseudoorbit.

Let δ_i be the vector of errors at step i ($\delta_i \leq \delta$) and $\mathbf{c}_i$ be the correction term to the orbit $\{\mathbf{x}_i\}_{i=0}^N$. Then

$$f(\mathbf{x}_i + \mathbf{c}_i) = \mathbf{x}_{i+1} + \mathbf{c}_{i+1}. \tag{3.1}$$

Let us assume that the correction term can be decomposed as $\mathbf{c}_i = \mathbf{s}_i + \mathbf{u}_i$, where $\mathbf{s}_i$, resp. $\mathbf{u}_i$ are stable, resp. unstable directions at the point $\mathbf{x}_i$, i.e., the system is nearly hyperbolic. It follows

$$\delta_i + Df(\mathbf{x}_i)\mathbf{c}_i = \mathbf{x}_{i+1} + \mathbf{c}_{i+1}$$

from the equation (3.1).

To achieve numerical stability, we must solve the recursion relation backwards along the unstable direction

$$\mathbf{u}_i = U((Df(\mathbf{x}_i))^{-1}\mathbf{u}_{i+1} - \delta_i), \quad \alpha_N = 0 \tag{3.2}$$

and forwards along the stable direction

$$\mathbf{s}_{i+1} = S(Df(\mathbf{x}_i)\mathbf{s}_i + \delta_i), \quad \beta_0 = 0. \tag{3.3}$$

S is the projection onto the stable subspace, while U to the unstable one.

Then the brittleness is the maximum of the ratios of the norm of correction term $\mathbf{c}_i$ and to the norm of δ_i.

Dawson *et al.* computed the brittleness for the double rotor map [10]. In contrast to the system studied by Abraham and Smale, the double rotor is a real physical system with a chaotic behaviour whose one Lyapunov exponent fluctuates about zero. The test brittleness is greater than 10^{40} for this system, when we want to find a shadow consisted of $N = 10^4$ points. For $N = 10^5$, the brittleness is even 10^{100}. This huge number suggest that this system is unshadowable.

Conclusion and future work

Many fundamental physical laws are described by the differential equations. In general, it is not possible to find a closed form solutions and we are obliged to rely on numerical simulations. All computers work with finite precision, numerical errors at each step can accumulate, so there is a question to what the pseudoorbits actually correspond. The shadow is a true trajectory, which is sufficiently close to the computer generated one. We have tried to find the conditions under which are the pseudoorbits shadowable.

First we have introduced the idea of shadowing in the discrete dynamical systems. The shadowing is the property of hyperbolic sets, therefore we have dealt with hyperbolic systems in order to understand why the hyperbolicity is the key property. We have presented three shadowing theorems. We have tried to confront their advantages and disadvantages from the practical usage point of view.

Then we have defined a hyperbolic set for continuous dynamical systems. The fundamental difference between the discrete and continuous case is that the latter has the error both in space and time, therefore the shadowing definition in the continuous case has to allow the time rescaling.

We have generalized the shadowing method developed for maps to the differential equations. The time is not allowed to fluctuate in this method, therefore it gives us only lower bounds for shadowing times. We have presented the shadowing theorem which allows time rescaling, and we have used it to prove the existence of a shadow for a chaotic solution of the Lorenz equations. Finally we have shown an example of an unshadowable trajectory.

The aim of this work was to understand the idea of shadowing in the continuous case and acquaint ourselves with different shadowing methods. These are essential for the study of the unshadowability, which will be subject of our further research.

Appendix A

Lyapunov functions

There are three types of the solutions of the differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$ passing though the point $\mathbf{x}_0$

- stationary solutions $\mathbf{x}_0 = \phi^t(\mathbf{x}_0)\ \forall t$, where $f(\mathbf{x}_0) = 0$,
- periodic solutions $\phi^t(\mathbf{x}_0)$ with period T for which there exists such $T > 0$ that $\phi^t(\mathbf{x}_0) = \phi^{t+T}(\mathbf{x}_0)$ and $\phi^{t_1}(\mathbf{x}_0) \neq \phi^{t_2}(\mathbf{x}_0)$ for $|t_1 - t_2| < T$,
- the solutions, where $\phi^{t_1}(\mathbf{x}_0) = \phi^{t_2}(\mathbf{x}_0)$ for $t_1 \neq t_2$.

The simplest case is the stationary point $\mathbf{x}_0$.

Definition 18. A point $\mathbf{x}_0$ is *Lyapunov stable* iff

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall \mathbf{y}_0 \in \mathbb{R}^n)(|\mathbf{x}_0 - \mathbf{y}_0| < \delta \Rightarrow |\phi^t(\mathbf{x}_0) - \phi^t(\mathbf{y}_0)| < \epsilon \quad \forall t \geq 0).$$

It means that all points from a certain neighbourhood of $\mathbf{x}_0$ remain in its neighbourhood during the time evolution. Sometimes $\mathbf{x}_0$ has even stronger property – it attracts all nearby points.

Definition 19. A point $\mathbf{x}_0$ is *quasi-asymptotically stable* iff

$$(\exists \delta > 0)(\forall \mathbf{y}_0 \in \mathbb{R}^n)(|\mathbf{x}_0 - \mathbf{y}_0| < \delta \Rightarrow |\phi^t(\mathbf{x}_0) - \phi^t(\mathbf{y}_0)| \to 0 \quad \text{as} \quad t \to \infty).$$

There is no connection between Lyapunov and quasi-asymptotic stability. There are some trajectories that are quasi-asymptotically stable, but that are not Lyapunov stable and vice versa [25]. Therefore the asymptotic stability is defined.

Definition 20. A point $\mathbf{x}_0$ is *asymptotically stable* if it is both Lyapunov and quasi-asymptotically stable.

In order to determine if the stationary point is stable, or not without solving the original differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$ we introduce so called Lyapunov function.

Definition 21. Suppose that the origin $\mathbf{x}_0 = 0$ is a stationary point for the differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$. Let U be an open neighbourhood of $\mathbf{x}_0$ and $V : \overline{U} \to \mathbb{R}$ be a continously differentiable function. V is called a *Lyapunov function* on U if

1. $V(\mathbf{x}_0) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \overline{U} \setminus \{\mathbf{x}_0\}$
2. $\dot{V}(\mathbf{x}_0) \leq 0$ for all $\mathbf{x} \in U$.

The following two theorems connects the Lyapunov function and the stability of the origin.

Theorem 13 (Lyapunov´s first stability theorem)**.** Suppose that a Lyapunov function can be defined on a neighbourhood of the origin, $\mathbf{x}_0 = 0$, which is a stationary point of the differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$. Then the origin is Lyapunov stable.

Theorem 14 (Lyapunov´s second stability theorem)**.** Suppose $\mathbf{x}_0 = 0$ is a stationary point for $\dot{\mathbf{x}} = f(\mathbf{x})$ and let V be a Lyapunov function on a neighbourhood U of $\mathbf{x}_0 = 0$. If $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in U \setminus \{\mathbf{x}_0\}$, then $\mathbf{x}_0$ is asymptotically stable.

Example 10. Consider the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x. \end{aligned} \tag{A.1}$$

The origin is a stationary point. Let us try a standard guess $V(x,y) = x^2 + y^2$ with

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 2xy - 2xy = 0.$$

Therefore V is really a Lyapunov function and the origin is Lyapunov stable, but it is not asymptotically. The equations (A.1) are simply solvable, the solutions are concentric circles about the origin which confirms its Lyapunov stability.

The theory of Lyapunov functions is very elegant, but unfortunatelly there is no general rule how to find them and sometimes the problem of finding Lyapunov function can be very difficult.

The theorem 14 ensures that all trajectories from a certain neighborhood of the origin tend to it. For many dynamical systems there is no stable stationary point, but all trajectories are attracted by some bounded set in the phase space. We can define Lyapunov function V also for the nontrivial set and use it in order to prove that this set is attracting.

Theorem 15. Let $U \subset \mathbb{R}^n$ be a simply connected, compact domain and $V : \mathbb{R}^n \to \mathbb{R}$ a continuously differentiable function. Suppose that for each $k > 0$, $V_k = \{\mathbf{x} \in \mathbb{R}^n | V(\mathbf{x}) < k\}$ is a simply connected, bounded domain with $V_k \subset V_l$ if $k < l$. If there exists $\kappa > 0$ such that $U \subset V_\kappa$ and $\delta > 0$ such that $\dot{V}(\mathbf{x}) \leq -\delta < 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus U$ then for all $\mathbf{x}$ there exists $t(\mathbf{x}) \geq 0$ such that $\phi^t(\mathbf{x}) \in V_\kappa$ for all $t > t(\mathbf{x})$.

Appendix B

Lyapunov exponents

Consider the dynamical system governed by the differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \tag{B.1}$$

where μ is a set of parameters. The behaviour of the system can depend significantly on the values of the parameters as we have seen in the section 2.5.2. We would like to have some measure of the degree of chaos in order to see how the behaviour changes with the change of the parameters. We introduce Lyapunov exponents as a measure of chaos.

We consider one-dimensional phase space for the simplicity. Let us look at two trajectories $\phi^t(x_1)$, $\phi^t(x_2)$ starting close to each other. Their distance $s(t) = \phi^t(x_1) - \phi^t(x_2)$ evolves under the equation

$$\frac{ds}{dt} = \frac{d}{dt}\phi^t(x_1) - \frac{d}{dt}\phi^t(x_2) = f(x_1, \mu) - f(x_2, \mu).$$

We assumed that x_1 is close to x_2, therefore we can expand $f(x_1)$ by a Taylor series

$$f(x_1) = f(x_2) + \frac{df}{dx}(x_2)(x_1 - x_2).$$

Thus

$$\frac{ds}{dt} = \frac{df}{dx}(x_2)(x_1 - x_2). \tag{B.2}$$

We assume that the convergence or the divergence of close trajectories is exponential

$$s(t) = s_0 e^{\lambda t} \quad \Rightarrow \quad \dot{s} = \lambda s. \tag{B.3}$$

From equations (B.2) and (B.3), it follows that

$$\lambda = \frac{df}{dx}(x_2).$$

The λ is called *Lyapunov exponent* for the dynamical system $\dot{x} = f(x, \mu)$. We see that close trajectories will diverge from each other when $\lambda > 0$.

In the phase space with a dimension n, we associate Lyapunov exponent with the rate of divergence, resp. convergence of close trajectories for each of directions in phase space. The Lyapunov exponents will be the eigenvalues of the Jacobian matrix $\{\lambda_1, \ldots, \lambda_n\}$.

In practice, Lyapunov exponents $\{\lambda_1, \ldots, \lambda_n\}$ varies with $\mathbf{x}$ (except the simplest case, when f is a constant matrix A). Therefore λ_i, $i \in \hat{n}$, are defined as a time average over the trajectory.

System which has at least one positive average Lyapunov exponent is called chaotic.

Bibliography

[1] Abraham, Ralph H.; Smale, Stephen: *Nongenerity of Ω-stability*, Proceedings of Symposia in Pure Mathematics 14, 5-8, 1970

[2] Anosov, Dmitrij V.: *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proceedings of the Steklov Institute of Mathematics 90, 1967

[3] Bowen, Rufus: *Ω-limit sets for Axiom A diffeomorphisms*, Journal of Differential Equations 18, 333-339, 1975

[4] Coomes, Brian A.; Koçak, Hüsseyn and Palmer, Kenneth J.: *Shadowing orbits of ordinary differential equations*, Journal of Computational and Applied Mathematics 52, 35-43, 1994

[5] Coomes, Brian A.; Koçak, Hüsseyn and Palmer, Kenneth J.: *Rigorous computational shadowing of orbits of ordinary differential equations*, Numerische Mathematik 69, 401-421, 1995

[6] Dawson, Silvina; Grebori, Celso; Sauer, Tim; Yorke, James A.: *Obstructions to shadowing when a Lyapunov exponent fluctuates about zero*, Physical Review Letters, Volume 73, Number 14, 1927-1930, 1994

[7] Glendinning, Paul: *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations*, Cambridge University Press, Cambridge, 1996

[8] Golub, Gene H.; Van Loan, Charles F.: *Matrix computations*, The Johns Hopkins University Press, Maryland, 1996

[9] Grebogi, Celso; Hammel, Stephen M.; Yorke, James A. and Sauer, Tim: *Shadowing of physical trajectories in chaotic dynamics: containment and refinement*, Physical Review Letters, Volume 65, Number 13, 1527-1530, 1990

[10] Grebogi, Celso; Kostelich, Erich J.; Ott, Edward; Yorke, James A.: *Multidimensioned intertwined basin boundaries: Basin structure of the kicked double rotor*, Physica 25D, 347-360, 1987

[11] Hammel, Stephen M; Yorke, James A. and Grebogi, Celso: *Numerical orbits of chaotic processes represent true orbits*, Bulletin of the American Mathematical Society, Volume 19, Number 2, 465-469, 1988

[12] Hayes, Wayne: *Rigorous shadowing of numerical solutions of ordinary differential equations by containment*, Ph.D. thesis, Department of computer science, University of Toronto, 2001

[13] Hayes, Wayne and Jackson, Kenneth R.: *A survey of shadowing methods for numerical solutions of ordinary differential equations*, Applied Numerical Mathematics 53, 299-321, 2005

[14] Hilborn, Robert C.: *Chaos and nonlinear dynamics: An introduction for scientists and engineers*, Oxford Univerisity Press, Oxford, 2004

[15] Hirsch, Morris W. and Smale, Stephen: *Differential equations, dynamical systems and linear algebra*, Academic Press, New York, 1974

[16] Katok, Anatole and Hasselblatt, Boris: *Introduction to the modern theory of dynamical systems*, Cambridge University Press, New York, 1995

[17] Korsch, Hans J. and Jodl, Hans-Jörg *Chaos: A program collecting for the PC*, Springer Verlag, Berlin, 1994

[18] Lorenz, Edward N.: *Deterministic nonperiodic flow*, Journal of the Atmospheric Sciences 20, 130-141, 1963

[19] MathWorks, "MATLAB," http://www.mathworks.com/

[20] Palmer, Kenneth J.: *Shadowing in dynamical systems - theory and applications*, Kluwer Academic Publishers, Dordrecht, 2000

[21] Pilyugin, Sergei Yu.: *Shadowing in structurally stable flows*, Journal of Differential Equations 140, 238-265, 1997

[22] Sauer, Tim; Grebogi, Celso and Yorke, James A.: *How long do numerical chaotic solutions remain valid?*, Physical Review Letters, Volume 79, Number 1, 59-62, 1997

[23] Sauer, Tim and Yorke, James A.: *Rigorous verification of trajectories for the computer simulation of dynamical systems*, Nonlinearity 4, 961-979, 1991

[24] Sparrow, Colin: *The Lorenz equations: Bifurcations, chaos and strange attractor*, Springer, New York, 1982

[25] Strmisková, Lucie: *Chaotic oscillator*, Bachelor thesis, Department of Physics, Czech Technical University, 2007

[26] Strmisková, Lucie: *Shadowing in chaotic systems*, Research work, Department of Physics, Czech Technical University, 2009

[27] "Wikipedia, the free encyklopedia," http://www.wikipedia.org/

[28] Wilkinson, James H.: *Rounding errors in algebraic processes*, Prentice-hall, New Yersey, 1963

[29] Wilkinson, James H.: *The algebraic eigenvalue problem*, Oxford University Press, New York, 1965

[30] Yuan, Guo-Cheng; Yorke, James A.: *An open set of maps for which every point is absolutely nonshadowable*, Proceedings of the American Mathematical Society, Volume 128, Number 3, 909-918, 1999

Prohlášení

Prohlašuji, že jsem svou bakalářskou práci vypracovala samostatně a použila jsem pouze podklady uvedené v přiloženém seznamu.

Nemám žádný důvod proti užití tohoto školního díla ve smyslu §60 Zákona č. 121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

Declaration

I declare that I wrote my bachelor thesis independently and exclusively with the use of cited bibliography.

I agree with the usage of this thesis in a purport of the Act 121/2000 (Copyright Act).

Praha, May 5, 2010

Lucie Strmisková