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RESEARCH WORK

Shadowing in chaotic systems

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Contents

Introduction

One of the significant characteristics of a chaotic dynamical system is a sensitive dependence on initial conditions - two solutions which start nearby will diverge from each other with the future evolution of the system.

Chaotic dynamical systems are described by nonlinear differential equations. Most of them are not solvable analytically and we must rely on numerical simulations. We require that the behaviour of a solution generated by the computer is the same as the behaviour of a true solution. However we should realize that all computers work with a finite precision. The rounding error made at any step of the computation causes a numerical trajectory to differ from the true one and this difference will be amplified exponentially due to the chaotic nature of the system.

So it is natural to ask if making numerical simulations for chaotic systems is purposeful and to what these computer generated orbits actually correspond.

Famous shadowing theorem says that for hyperbolic dynamical systems the true trajectory will really diverge from the computer generated (for chaotic systems) but there always exists a shadow - true trajectory with slightly different initial condition which stays arbitrarily close to the computed orbit for arbitrarily long time.

In this work we explain what the hyperbolicity is and why it is so important for shadowing. Because most of the interesting dynamical systems are non-hyperbolic, we try to modify the shadowing theorem for them. Finally we verify with the assistance of a computer the existence of a shadow for one concrete system of difference equation.

Chapter 1

Shadowing theorem for diffeomorphisms

1.1 A little linear algebra

We deal with difference equations rather than differential. For the purposes of shadowing they are much more easier to study and the determination whether the numerical solution of differential equation is shadowed or not can be often reduced to the study of shadowing for some difference equation. The study of difference equations is also convenient for one simple reason - mostly we are unable to solve differential equations analytically, and the numerical methods lead to difference equations anyway.

Consider the difference equation $x_{n+1} = f(x_n)$, where $x_i \in \mathbb{R}^n$ and $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1diffeomorphism.

The simplest case is when $f = A \in \mathbb{R}^{n,n}$ is a constant mapping. Let us remind some basic definitions.

Definition 1. A complex number λ is the eigenvalue of A if the operator $A - \lambda I$ is not injective. The set $E_\lambda = \text{span}\{x \in \mathbb{R}^n | (A - \lambda I)^k x = 0 \quad \text{for} \quad \text{some} \quad k \in \mathbb{N}\}$ is called the generalized eigenspace corresponding to the eigenvalue λ. Spectral radius $r(A)$ of A is defined as the maximal absolute value of the eigenvalues of A.

A linear map $A \in \mathbb{R}^{n,n}$ is bounded and using the Jordan normal form of matrix A it can be easily shown that for every $\delta > 0$ there exists a norm in $\mathbb{R}^{n,n}$ such that $\|A\| \leq r(A) + \delta$. On the other hand, all norms are equivalent in $\mathbb{R}^{n,n}$. Therefore

$$\|A^k x\| \leq C_\delta (r(A) + \delta)^k \|x\| \quad \forall x \in \mathbb{R}^n. \tag{1.1}$$

First let us assume that all eigenvalues of A have absolute values less than one. From (1.1) it is obvious that all points in $\mathbb{R}^n$ converge exponentially to the origin. If absolute values of all eigenvalues of A are grater than one, all points diverge to infinity. These properties motivate the following definition.

Definition 2. The set $E^s(A) = \bigoplus_{\lambda,|\lambda|<1} E_\lambda$ is called the *stable subspace* of A and the set $E^u(A) = \bigoplus_{\lambda,|\lambda|>1} E_\lambda$ is the *unstable subspace* of A.

Definition 3. A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if $\mathbb{R}^n = E^s(A) \bigoplus E^u(A)$.

The sets $E^s(A)$, $E^u(A)$ can be trivial and obviously they are invariant under A.

So for hyperbolic map A, there exist constants $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0,1)$ such that $|\lambda| < \lambda_1 < 1$ for all eigenvalues of A with absolute value less than one, $1 < \frac{1}{\lambda_2} < |\lambda|$ for all eigenvalues with absolute value grater than one and the following inequalities hold

$$\|A^k x\| \le K_1 {\lambda_1}^k \|x\| \quad x \in E^s(A), \tag{1.2}$$

$$\|A^{-k} x\| \le K_2 {\lambda_2}^k \|x\| \quad x \in E^u(A). \tag{1.3}$$

Example 1. Consider $A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ with $\lambda \in (0,1)$. Then A is a hyperbolic linear map. Its stable subspace is x axis, while y axis is the unstable one.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \lambda x_0 \\ \frac{1}{\lambda} y_0 \end{pmatrix}$$

Therefore hyperbolas $xy = const.$ are invariant curves of the map A. This property is actually the reason for the name hyperbolic map. ♠

1.2 Hyperbolic fixed point

As we have said in the introduction shadowing and hyperbolicity of the system go hand in hand. We prefer the approach that is presented in [4], where a general hyperbolic set is deduced from a hyperbolic fixed point, rather than state quite difficult definition without any explanation.

Definition 4. A point x_0 is said to be *hyperbolic fixed point* of the function f if $f(x_0) = x_0$ and all the eigenvalues of $Df(x_0)$ lie off the unit circle. The sum of generalized eigenvectors corresponding to the eigenvalues inside (outside) the unit circle is called the stable (unstable) subspace and is denoted as E^s (E^u).

Example 2. Consider so called Hénon map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (1 - ax^2 + y, -bx), \quad a \in \mathbb{R}, \quad b \in \mathbb{R} \setminus 0.$$

We find the fixed points of the Hénon map a determine if they are hyperbolic or not.

The equations

$$1 - ax^2 + y = x, \quad -bx = y$$

have the solutions:

$$x = \frac{-(b+1) \pm \sqrt{(b+1)^2 + 4a}}{2a}, \quad y = -bx \quad \text{for} \quad a \neq 0$$

and

$$x = \frac{1}{b+1}, \quad y = \frac{-b}{b+1} \quad \text{for} \quad a = 0.$$

These fixed point are non-hyperbolic if the eigenvalues of the matrix

$$Df(x) = \begin{pmatrix} -2ax & 1 \\ -b & 0 \end{pmatrix}$$

have absolute value equal to one. After short computation we find out that it occurs for $b = 1$ when $a \in \langle -1, 3 \rangle$ and for all other values of b when $a = -\frac{(b+1)^2}{4}$ or $a = \frac{3(b+1)^2}{4}$.

We have noticed that the intervals of non-hyperbolicity computed by us differ from the intervals presented in [4]. The author made the following mistake:

$$x = \frac{-(b+1) \pm \sqrt{(b+1)^2 - 4a}}{2a}$$

and as a consequence of the wrong sign of the element $4a$ the interval $\langle -1, 3 \rangle$ changes to the interval $\langle -3, 1 \rangle$ in the first case and there is an exchange of signs of a in the second one. ♠

For the linear diffeomorphism, there are some contracting (stable subspace) and expanding directions (unstable subspace). We can find these directions also in the nonlinear case.

Definition 5. Let x_0 be a hyperbolic fixed point of the C^1 diffeomorphism $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$. The set

$$W^s(x_0) = \{x \in U : f^k(x) \to x_0 \quad \text{as} \quad k \to \infty\}$$

is called the *stable manifold* of x_0 and the set

$$W^u(x_0) = \{x \in U : f^k(x) \to x_0 \quad \text{as} \quad k \to -\infty\}$$

is called the *unstable manifold* of x_0.

The stable manifold may not generally be a submanifold of $\mathbb{R}^n$ therefore we introduce the term local stable manifold (stable manifold with an extra condition) which already fulfils the manifold definition.

Definition 6. Let x_0 be a hyperbolic fixed point of the C^1 diffeomorphism $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$. For given $\epsilon > 0$, the set

$$W^{s,\epsilon}(x_0) = \{x \in U : f^k(x) \to x_0 \quad \text{as} \quad k \to \infty, \quad \|f^k(x) - x_0\| < \epsilon \quad \forall k \geq 0\}$$

is called the *local stable manifold* of x_0.

Consider the difference equation $x_{n+1} = f(x_n)$ with the hyperbolic fixed point x_0, $f(x_0) = x_0$. Take the starting point y_0 in a neighbourhood of x_0 and let us look at the behaviour of the orbit $S = \{y_0, y_1, \ldots\}$. Introduce the new variable $z_n = y_n - x_0$, *i.e.*, the distance between the points of orbit S and x_0. We can make the Taylor expansion of function f for the points y_n in a neighbourhood of x_0 :

$$y_{n+1} = f(x_0) + Df(x_0)(y_n - x_0) + O(y^2)$$

$$z_{n+1} \doteq Df(x_0) z_n.$$

We see that whether a given point y_0 will tend or move away from a fixed point x_0 depends on the properties of the matrix $Df(x_0)$. Therefore it would be very nice if there existed a connection between the stable and unstable subspaces of $Df(x_0)$ and the stable and unstable manifolds of x_0, which are not linear subspaces.

It can be shown that $W^{s,\epsilon}(x_0)$ is really a submanifold of $\mathbb{R}^n$ and moreover the tangent space of $W^{s,\epsilon}(x_0)$ at the point x_0 is E^s. So the behaviour of points in the neighbourhood of hyperbolic fixed point x_0 corresponds, at least locally, to the behaviour generated by the linearization.

Theorem 1 (Stable manifold theorem)**.** Let $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^r (r \geq 1)$ diffeomorphism with hyperbolic fixed point x_0 and associated stable subspace E^s. Then for ϵ sufficiently small, $W^{s,\epsilon}(x_0)$ is a C^r submanifold of $\mathbb{R}^n$ containing x_0 and moreover $T_{x_0} W^{s,\epsilon}(x_0) = E^s$.

Sketch of the proof of the theorem 1. Let x_0 be a hyperbolic fixed point with associated subspaces E^s, E^u. P is a projection of $\mathbb{R}^n$ onto E^s along E^u ($M^s = \|P\|, \quad M^u = \|I - P\|$) and as a consequence of hyperbolicity the inequalities (1.2), (1.3) hold. Moreover let us assume that the positive number δ satisfies the inequalities

$$\sigma = [K_1 M^s (1 - \lambda_1)^{-1} + K_2 M^u \lambda_2 (1 - \lambda_2)^{-1}] \omega(\delta) < 1, \tag{1.4}$$

where

$$\omega(\delta) = \sup\{\|Df(x) - Df(x_0)\|, \quad \|x - x_0\| \leq \delta\}$$

and

$$K_1 M^s \omega(\delta) < (1 - \sigma)(1 - \lambda_1). \tag{1.5}$$

Then for $x \in U$

$$\|f^k(x) - x_0\| \leq \delta \quad \text{for} \quad k \geq 0 \quad \Rightarrow \quad f^k(x) \to x_0 \quad \text{as} \quad k \to \infty. \tag{1.6}$$

Therefore if we are looking for points of $W^{s,\delta}(x_0)$, it suffices to find the points that stay in δ neighbourhood of x_0 all their future evolution.

To prove the so called *saddle-point property* (1.6) we define a sequence

$$y_k = f^k(x) - x_0 = A y_{k-1} + g(y_{k-1}), \tag{1.7}$$

where

$$A = Df(x_0), \quad g(y) = f(x_0 + y) - f(x_0) - Df(x_0)y.$$

We can write y_k as

$$y_k = Py_k + (I - P)y_k$$

and use (1.7) to each part.

$$Py_k = P(Ay_{k-1} + g(y_{k-1}))$$
$$Py_k = A^k Py_0 + \sum_{m=0}^{k-1} A^{k-m-1} Pg(y_m) \quad k \geq 0$$

$$(I - P)y_k = (I - P)(Ay_{k-1} + g(y_{k-1}))$$
$$(I - P)y_k = A^{-1}(I - P)y_{k+1} - A^{-1}(I - P)g(y_k)$$
$$(I - P)y_k = A^{-(l-k)}(I - P)y_l - \sum_{m=k}^{l-1} A^{-(m-k+1)}(I - P)g(y_m)$$

The norm of $A^{-(l-k)}(I-P)y_l$ tends to zero as $l \to \infty$. Therefore y_k can be written as

$$y_k = A^k Py_0 + \sum_{m=0}^{k-1} A^{k-m-1} Pg(y_m) - \sum_{m=k}^{\infty} A^{-(l-k+1)}(I - P)g(y_m). \tag{1.8}$$

Using the discrete Gronwall lemma ([4], p.7) we get

$$\|y_k\| \leq K_1 M^s (1 - \sigma)^{-1}[\lambda_1 + K_1 M^s (1 - \sigma)^{-1}\omega(\delta)]^k \|y_0\|, \tag{1.9}$$

which proves the property (1.6).

In the next step we show that for given $\xi \in E^s$ with norm $\|\xi\| \leq \frac{\delta(1-\sigma)}{K_1}$ there is a unique point $x \in \mathbb{R}^n$ such that

$$P(x - x_0) = \xi \tag{1.10}$$

and

$$\|f^k(x) - x_0\| \leq \delta \quad k \geq 0. \tag{1.11}$$

Consider $l^\infty(\mathbb{R}^n)$ - the Banach space of bounded sequences in $\mathbb{R}^n$ $y = \{y_k\}_{k=0}^\infty$ with norm $\|y\| = \sup_k \|y_k\|$ and let us define an operator T on the closed ball $\bar{B}_\delta$ of radius δ.

$$Ty = \{(Ty)_k\}_{k=0}^\infty \quad (Ty)_k = A^k \xi + \sum_{m=0}^{k-1} A^{k-m-1} Pg(y_m) - \sum_{m=k}^{\infty} A^{-(l-k+1)}(I - P)g(y_m) \tag{1.12}$$

Take two points $y, \tilde{y} \in \bar{B}_\delta$, then

$$
\begin{aligned}
\|(Ty)_k - (T\tilde{y})_k\| \le & \sum_{m=0}^{k-1} \|A^{k-m-1} P[g(y_m) - g(\tilde{y}_m)]\| + \\
& + \sum_{m=k}^{\infty} \|A^{-(m-k+1)}(I-P)[g(y_m) - g(\tilde{y}_m)]\| \le \\
& \le \Big[\sum_{m=0}^{k-1} K_1 {\lambda_1}^{k-m+1} M^s + \sum_{m=k}^{\infty} K_2 {\lambda_2}^{m-k+1} M^u \Big] \omega(\delta) \|y - \tilde{y}\| \le \\
& \le \sigma \|y - \tilde{y}\|.
\end{aligned}
$$

For special choice of $\tilde{y}$ - $\tilde{y} = \{\vec{0}\}_{k=0}^{\infty}$ it follows that

$$\|Ty\| \le K_1 \|\xi\| + \sigma\delta \le \delta.$$

So T is continuous and $\bar{B}_\delta$ is invariant under T. Moreover T is a contracting mapping:

$$\|Ty - T\tilde{y}\| \le \sigma \|y - \tilde{y}\|, \quad \sigma < 1.$$

$\bar{B}_\delta$ is a complete metric space therefore there exists a unique fixed point $y = \{y_k\}_{k=0}^{\infty}$ such that y_k satisfies (1.8), $Py_0 = \xi$ and $\|y_k\| \le \delta$ for $k \ge 0$ as we wanted to show.

So for $\xi \in E^s$ we find a unique point y_0 which is projected onto it. y_0 is a function of ξ : $y_0 = \phi(\xi)$. From (1.12) we see that T is C^r because g is. From implicit function theorem applied to $f(x, \mu) - x$, it follows that if contracting mapping depends smoothly on parameter, then also does its fixed point. So ϕ is C^r function of ξ and therefore the set

$$M = \left\{ x_0 + \phi(\xi); \quad \xi \in E^s \quad \|\xi\| \le \frac{\delta(1-\sigma)}{K_1} \right\}$$

is a C^r submanifold of $\mathbb{R}^n$ containing x_0. From the previous it is obvious that $M \subset W^{s,\delta}(x_0)$ but these sets may not be same.

Suppose that positive number ϵ satisfies

$$\epsilon \le \delta, \quad \delta < \frac{\delta(1-\sigma)}{K_1 M^s}. \tag{1.13}$$

It is clear that $W^{s,\epsilon}(x_0)$ is a subset of M. Moreover it is an open set.

If we take point $x \in W^{s,\epsilon}(x_0)$ and the point $y \in M$, $\|x - y\| < \delta$ from its δ neighbourhood it follows that $y \in W^{s,\epsilon}(x_0)$. From (1.9) we see that $\|f^k - y\| < \epsilon$ for sufficiently large k $(k > N)$. For $k < N$ we choose δ so small that $\max_k \|f^k(y) - f^k(x)\| < \epsilon - \max_k \|f^k(x) - x_0\|$.

Therefore $W^{s,\epsilon}(x_0)$ is a C^r submanifold of $\mathbb{R}^n$. □

It is obvious that previous theorem proves also that $W^{u,\epsilon}$ is a submanifold of $\mathbb{R}^n$. If we change the direction of iterations the unstable submanifold changes to a stable one and vice versa.

The main reason why we have dealt with the proof of the stable manifold theorem is that we wanted to show and verify the saddle point property (1.6). It follows that if a hyperbolic fixed point is Lyapunov stable, then it is automatically also asymptotic stable. It is very significant property and for non-hyperbolic systems it does not have to be true.

Example 3. Consider $A = \begin{pmatrix} -1 & 0 \\ 0 & \lambda \end{pmatrix}$ with $|\lambda| \neq 1$. The origin is non-hyperbolic fixed point. If we take point $x = \binom{\epsilon}{0}$ from ϵ neighbourhood of the origin ($\epsilon > 0$), we see that it will stay in the ϵ neighbourhood for all its future evolution but it will never approach the origin. ♠

It is a very common situation that the solution of the difference equation describing some physical process depends on some parameters such as temperature, pressure, *etc.* We would like to know if the local stable and unstable manifolds of a hyperbolic fixed point change continuously with the continuous variation of the parameters.

Consider a $C^r (r \geq 1)$ function $f : U \times V \to \mathbb{R}^n$, where $U = U^\circ \subset \mathbb{R}^n$, $V = V^\circ \subset \mathbb{R}^m$ and $f_\mu(x) = f(x, \mu)$ is a diffeomorphism. Suppose that for $\mu_0 \in V$, f_{μ_0} has a hyperbolic fixed point x_0.

From implicit function theorem, it follows that for μ sufficiently close to μ_0, the diffeomorphism f_μ has a unique fixed point $x(\mu)$ near to x_0 and the function $x(\mu)$ is C^r in μ.

Analogously as in the proof of the local stable manifold theorem (using the operator T and fixed point theorem) it can be shown that the stable and unstable manifolds of $x(\mu)$ depend smoothly (C^r) on μ.

1.3 Hyperbolic set

In this section we extend the hyperbolicity definition from fixed point to a general set. First we look at a periodic point because it helps us to understand why the hyperbolic set is defined in such a complicated way.

Definition 7. A point x_0 is called a *periodic point* of the map f with period $m \geq 1$ if $f^m(x_0) = x_0$ and m is a minimal integer with this property. The periodic point x_0 is said to be hyperbolic if x_0 is hyperbolic as a fixed point of f^m.

Example 4. Let us return to the Hénon map with a special choice of parameters $f(x, y) = (1 - 2x^2 + y, -x)$, $a = 2$, $b = 1$. There are two periodic points with period 3 – $x_0 = \left(0, -\frac{1}{2}\right)$ and $y_0 = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

$$x_0 = \left(0, -\frac{1}{2}\right), \quad x_1 = \left(\frac{1}{2}, 0\right), \quad x_2 = \left(\frac{1}{2}, -\frac{1}{2}\right), \quad x_3 = \left(0, -\frac{1}{2}\right)$$

$$y_0 = \left(-\frac{1}{2}, \frac{1}{2}\right), \quad y_1 = \left(1, \frac{1}{2}\right), \quad y_2 = \left(-\frac{1}{2}, -1\right), \quad y_3 = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Let us evaluate the Jacobian matrix of f^3 in these points to know if they are hyperbolic or not.

$$Df^3(x_0) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad Df^3(f(x_0)) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad Df^3(f^2(x_0)) = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}$$

All these matrices have the same eigenvalues $2 \pm \sqrt{3}$ therefore x_0 is a hyperbolic periodic point.

The matrices

$$Df^3(y_0) = \begin{pmatrix} -20 & -9 \\ 9 & 4 \end{pmatrix} \quad Df^3(f(y_0)) = \begin{pmatrix} -14 & 3 \\ 9 & -2 \end{pmatrix} \quad Df^3(f^2(y_0)) = \begin{pmatrix} -14 & -9 \\ -3 & -2 \end{pmatrix}$$

have the eigenvalues $-8 \pm 3\sqrt{7}$ so the point y_0 is also a hyperbolic periodic point.

We will see that the conservation of hyperbolicity, resp. the eigenvalues along the hyperbolic orbit is a general property. ♠

Consider the orbit $S = \{x_0, f(x_0), \ldots, f^{m-1}(x_0)\}$. If the starting point x_0 is a hyperbolic periodic point with period m, then there exists splitting $\mathbb{R}^n = E^s \oplus E^u$, where E^s, E^u are stable and unstable subspaces of x_0 as a fixed point of f^m. These subspaces are invariant under $Df^m(x_0)$ and there exist constants $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0, 1)$ such that for $k \geq 0$

$$\|(Df^m(x_0))^k \xi\| \leq K_1 {\lambda_1}^k \|\xi\| \quad \xi \in E^s \tag{1.14}$$

$$\|(Df^m(x_0))^{-k} \xi\| \leq K_2 {\lambda_2}^k \|\xi\| \quad \xi \in E^u. \tag{1.15}$$

Let us realize that if x_0 is a periodic point, then also the other points of S are periodic with the same period. We would like to know if they are also hyperbolic.

If $E^s = \{\vec{y}_1, \ldots, \vec{y}_m\}$, $m \leq n$, is the subspace that contains vectors from the tangent space at the point x_0 and $x_k = f^k(x_0)$, then $Df^k(x_0)E^s$ is the subspace with the same dimension consisted of the tangent vectors at point $f^k(x_0)$.

Take $k = 1$.

$$Df^m(x_0) = \prod_{i=1}^{m} Df(f^{m-i}(x_0))$$

$$\begin{aligned} Df^m(f(x_0)) &= Df(f^m(x_0)) Df(f^{m-1}(x_0)) \ldots Df(f(x_0)) = \\ &= Df(x_0) Df^m(x_0) (Df(x_0))^{-1} \end{aligned}$$

For general $k \in \hat{m}$:

$$Df^m(f^k(x_0)) = Df(f^{k-1}(x_0)) \ldots Df(x_0) Df^m(x_0) (Df(f^{k-1}(x_0)) \ldots Df(x_0))^{-1}.$$

So $Df^m(x_0)$ and $Df^m(f^k(x_0))$, $k \in \hat{m}$, are really similar matrices and apparently $E^s(f^k(x_0)) = Df^k(x_0)E^s$, $E^u(f^k(x_0)) = Df^k(x_0)E^u$.

Therefore for $x = f^k(x_0)$, $k \in \widehat{m-1}$, there is a splitting $\mathbb{R}^n = E^s(x) \oplus E^u(x)$, where

$$E^s(x) = Df^k(x_0)(E^s), \quad E^u(x) = Df^k(x_0)(E^u).$$

Let us look at invariance properties of these subspaces

$$Df(x)(E^s(x)) = Df(f^k(x_0))(Df^k(x_0))(E^s) = Df^{k+1}(x_0) = E^s(f(x)).$$

If we realize that

$$(Df^m(x_0))^k = \Big(\prod_{i=1}^{m} Df(f^{m-i}(x_0))\Big)^k, \quad Df^{mk}(x_0) = \prod_{i=1}^{mk} Df(f^{mk-i}(x_0))$$

and $f^m(x_0) = x_0$, we see that we can replace the inequalities (1.14), (1.15) by much more useful ones:

$$\|Df^{mk}(x_0)\xi\| \le K_1 {\lambda_1}^k \|\xi\| \quad \xi \in E^s \tag{1.16}$$

$$\|Df^{-mk}(x_0)\xi\| \le K_2 {\lambda_2}^k \|\xi\| \quad \xi \in E^u. \tag{1.17}$$

We have derived that when $k \ge 0$ is a multiple of m, then

$$\|Df^k(x_0)\xi\| \le K_1 \tilde{\lambda}_1^k \|\xi\|, \quad \xi \in E^s, \quad \tilde{\lambda}_1 = \lambda^{\frac{1}{m}}.$$

If k is not a multiple of m, it can be written in the form $k = ma - b$, $a \in \mathbb{N}$, $b \in \widehat{m-1}$.

$$\begin{aligned}\|Df^k(x_0)\xi\| \le \|Df^{ma-b}(x_0)\xi\| \le \|Df^{-b}(f^{ma}(x_0))\|\|Df^{ma}(x_0)\xi\| \le \tilde{K}_1 \tilde{\lambda}_1^{ma}\|\xi\| \le \\ \le \tilde{K}_1 \tilde{\lambda}_1^k \|\xi\|\end{aligned}$$

The previous inequality holds also for the other points of S ($x = f^k(x_0), \tilde{\xi} \in E^s(x)$).

$$\|Df^l(x)\tilde{\xi}\| = \|Df^l(f^k(x_0))Df^k(x_0)\xi\| \le K_1 \tilde{\lambda}_1^{l+k}\|\xi\| \le \tilde{K}_1 \tilde{\lambda}_1^l \|\xi\|$$

We could imagine the general invariant set S of f as an orbit with period $m = \infty$. Therefore we hope that the definition of a hyperbolic set is not surprising after the previous discussion about hyperbolic orbits.

Definition 8. A compact set $S \subset U$ is called *hyperbolic set* of the map f if

1. S is invariant, *i.e.*, $f(S) = S$
2. there is a continuous splitting $\mathbb{R}^n = E^s(x) \oplus E^u(x)$, $x \in S$

such that the subspaces $E^s(x)$ and $E^u(x)$ have constant dimensions $\forall x \in S$, moreover these subspaces have the "invariance" properties

$$Df(x)(E^s(x)) = E^s(f(x)), \quad Df(x)(E^u(x)) = E^u(f(x))$$

and there are constants $K_1, K_2 > 0$, $\lambda_1, \lambda_2 \in (0,1)$ such that for $k \geq 0$ and for all $x \in S$

$$\|Df^k(x)\xi\| \leq K_1 {\lambda_1}^k \|\xi\| \quad \xi \in E^s(x) \tag{1.18}$$

$$\|Df^{-k}(x)\xi\| \leq K_2 {\lambda_2}^k \|\xi\| \quad \xi \in E^u(x). \tag{1.19}$$

K_1, K_2 are called constants and λ_1, λ_2 exponents for the hyperbolic set S.

The continuous splitting means that if P is a projection of $\mathbb{R}^n$ onto $E^s(x)$ along $E^u(x)$, then P is a continuous function. The continuity of P need not to be involved in the hyperbolicity definition because it follows from the other assumptions.

In the shadowing literature, there is often hyperbolic set S defined for the diffeomorphism $f : M \to M$, where M is a smooth submanifold of $\mathbb{R}^n$. In this case, the condition of the splitting $\mathbb{R}^n = E^s(x) \oplus E^u(x)$, $x \in S$, is replaced by the condition $T_xM = E^s(x) \oplus E^u(x)$, $x \in S$, where T_xM is the tangent space to M at point x. We use the fact that for any linear space V we can identify the tangent space T_xV with the original space V.

1.4 Exponential dichotomy

Definition 9. The *transition matrix* $\phi(k,m)$ for the difference equation

$$u_{k+1} = A_k u_k, \quad \text{where} \quad A_k \in \mathbb{R}^{n,n}, \quad \det A_k \neq 0 \quad \forall k \in J \subset \mathbb{Z}$$

is defined by:

$$\phi(k,m) = \begin{cases} A_{k-1} \dots A_m & \text{for } k > m \\ I & \text{for } k = m \\ (A_{k-1} \dots A_m)^{-1} & \text{for } k < m. \end{cases} \tag{1.20}$$

The transition matrices satisfy the cocycle rule $\phi(k,p)\phi(p,m) = \phi(k,m)$.

Definition 10. The difference equation $u_{k+1} = A_k u_k$ has an *exponential dichotomy* on J if there are projections P_k and constants $K_1, K_2 > 0$ and $\lambda_1, \lambda_2 \in (0,1)$ such that for $k, m \in J$ the invariance conditions

$$\phi(k,m)P_m = P_k \phi(k,m) \tag{1.21}$$

are satisfied and the inequalities

$$\|\phi(k,m)P_m\| \leq K_1 {\lambda_1}^{k-m} \quad k \geq m \tag{1.22}$$

$$\|\phi(k,m)(I - P_m)\| \leq K_2 {\lambda_2}^{k-m} \quad k \leq m \tag{1.23}$$

hold. K_1, K_2 are called *constants* and λ_1, λ_2 *exponents* associated with the dichotomy.

Consider $A_k = A \in \mathbb{R}^{n,n}$ $\forall k$ such that A is a hyperbolic map. Then $\phi(k,m) = A^{k-m}$ for $k > m$. From inequalities (1.2), (1.3), we see that the difference equation $u_{k+1} = Au_k$ has an exponential dichotomy on $(-\infty, \infty)$ with all projections equal to P - the projection onto E^s along E^u.

We have dealt with hyperbolic sets for a long time. For this reason and the previous notice about hyperbolic maps we hope that the fact, that there exists a connection between hyperbolicity and exponential dichotomy, is not much surprising.

Theorem 2. A compact invariant set S for the diffeomorphism $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$ is hyperbolic iff for all $x \in S$ the difference equation

$$u_{k+1} = Df(f^k(x_0))u_k$$

has an exponential dichotomy on $(-\infty, \infty)$ with constants, exponents and rank of projection independent on x.

We will show the first implication, because the knowledge of constants and exponents associated with the dichotomy will be later very useful.

Suppose that S is a hyperbolic set, *i.e.*, it satisfies the definition 8. For equation $u_{k+1} = Df(f^k(x))u_k$, the transition matrix $\phi(k,m)$, $k > m$ has the form $\phi(k,m) = Df(f^{k-1}(x))Df(f^{k-2}(x)) \dots Df(f^m(x)) = Df^{k-m}(f^m(x))$.

Let $P(x)$ be a projection of $\mathbb{R}^n$ onto $E^s(x)$ along $E^u(x)$. The invariance property $Df(x)(E^s(x)) = E^s(f(x))$ implies the condition

$$Df(x)P(x) = P(f(x))Df(x), \quad \text{resp.} \quad Df^k(x)P(x) = P(f^k(x))Df^k(x). \quad (1.24)$$

If we take projections $P_k = P(f^k(x))$, the invariance conditions (1.21) are automatically satisfied. The inequalities (1.22), (1.23) also hold because of the inequalities (1.18), (1.19). Let $\xi \in \mathbb{R}^n$, then

$$\|\phi(k,m)P_m\xi\| = \|Df^{k-m}(f^m(x))P(f^m(x))\xi\| \le K_1{\lambda_1}^{k-m}M^s\|\xi\|$$

Thus the difference equation $u_{k+1} = Df(f^k(x_0))u_k$ has an exponential dichotomy on $(-\infty, \infty)$ with constants K_1M^s, K_2M^u and exponents λ_1, λ_2.

1.5 Shadowing theorem for hyperbolic sets

In this section we explain exactly what shadowing is and why the hyperbolicity is so important for it.

Definition 11. A sequence $\{y_i\}_{i=0}^n$ of points in $U \subset \mathbb{R}^n$ is a δ pseudoorbit of f if $\|y_{i+1} - f(y_i)\| < \delta$ for $0 \le i \le n$.

Definition 12. The true orbit $\{x_i\}_{i=0}^n$ $(x_{i+1} = f(x_i))$ ϵ shadows the δ pseudoorbit $\{y_i\}_{i=0}^n$ if

$$\|x_i - y_i\| < \epsilon \quad \text{for} \quad 0 \le i \le n.$$

Our key question if the numerical generated trajectory corresponds to some true trajectory can be reformulated: "For given δ pseudoorbit, is there a true trajectory which ϵ shadows it?"

Theorem 3 (Shadowing theorem)**.** Let S be a compact hyperbolic set for a C^1 diffeomorphism $f : U = U^\circ \subset \mathbb{R}^n \to \mathbb{R}^n$. Then there exist positive constants δ_0, σ_0, M depending only on f and S such that if: $g : U \to \mathbb{R}^n$ is a C^1 diffeomorphism satisfying

$$\|f(x) - g(x)\| + \|Df(x) - Dg(x)\| \leq \sigma \quad \text{for} \quad x \in U \tag{1.25}$$

with $\sigma \leq \sigma_0$, any δ pseudoorbit of f in S with $\delta \leq \delta_0$ is ϵ shadowed by a unique true orbit of g with $\epsilon = M(\sigma + \delta)$.

Later we outline the proof of the shadowing theorem to understand why the hyperbolicity is the key property, but first let us look at two simple examples.

Example 5. Consider the contracting mapping $f : \mathbb{R}^n \to \mathbb{R}^n$,

$$\|f^n(x) - f^n(y)\| \leq K^n \|x - y\|, \quad K \in (0, 1).$$

It is easy to show that any pseudoorbit $\{x_0, x_1 \ldots, x_N\}$ is shadowed by the true orbit beginning at its own initial condition.

$$\|f(x_0) - x_1\| < \delta$$

$$\|f^2(x_0) - x_2\| \leq \|f^2(x_0) - f(x_1)\| + \|f(x_1) - x_2\| \leq (K + 1)\delta$$

$$\vdots$$

$$\|f^n(x_0) - x_n\| \leq (K^{n-1} + K^{n-2} + \ldots + 1)\delta$$

Therefore the true orbit $\{x_0, f(x_0), \ldots, f^N(x_0)\}$ shadows the pseudoorbit $\{x_0, x_1, \ldots, x_N\}$ within $\frac{\delta}{1-K}$. ♠

Example 6. Consider the expanding mapping: $f : \mathbb{R}^n \to \mathbb{R}^n$,

$$\|f^n(x) - f^n(y)\| \geq C^n \|x - y\|, \quad C > 1.$$

In contrast to the contracting mapping, the expanding mapping is sensitive to the change of initial condition. Despite of this sensitivity, all δ pseudo orbits $\{x_0, x_1 \ldots, x_N\}$ can be shadowed. The inverse map f^{-1} is contracting with $K = \frac{1}{C}$ so the true orbit $\{f^{-N}(x_N), f^{-N+1}(x_N), \ldots, x_N\}$ shadows the pseudoorbit within $\frac{\delta}{1-\frac{1}{C}}$. ♠

Previous examples seem to be trivial but we should realize that a general hyperbolic dynamical system is a combination of these two cases. At each point we can find some expanding and some contracting directions. The idea of shadowing hyperbolic sets is based on this simple observation – we can shadow contracting and expanding mapping.

Sketch of the proof of the shadowing theorem. We seek a shadow $x = \{x_k\}_{k=0}^{\infty}$ ($x_{k+1} = g(x_k)$) for a given δ pseudoorbit $y = \{y_k\}_{k=0}^{\infty}$ of f. Let $O = O^{\circ} \subset l^{\infty}(\mathbb{R}^n)$ be the subset, which is consisting of all those sequences $z = \{z_k\}_{k=0}^{\infty}$ that

$$\|z - y\| < \operatorname{dist}(S, \partial U),$$

where $\operatorname{dist}(S, \partial U) := \infty$ for $U = \mathbb{R}^n$.

For $z = \{z_k\}_{k=0}^{\infty} \in O$ we define C^1 mapping $G : O \to l^{\infty}(\mathbb{R}^n)$

$$[G(z)]_k = z_{k+1} - g(z_k).$$

The shadowing theorem will be proved if we find a unique solution x of the equation $G(x) = 0$ with property $\|x - y\| \leq \epsilon$.

We use the following lemma to do this.

Lemma 1. Let X, Y be Banach spaces and $f : O = O^{\circ} \subset X \to Y$ be a C^1 function. Suppose that Δ and M are some positive constants and $y \in O$ is an element, for which

$$\|G(y)\| \leq \Delta$$

and the derivative $L = DG(y)$ is invertible with norm

$$\|L^{-1}\| \leq \frac{M}{2}.$$

Then if the closed ball $\bar{B}(y, M\Delta)$ lies in O and the inequality

$$\|DG(y) - DG(x)\| \leq \frac{1}{M}$$

holds for $\|x - y\| \leq M\Delta$, there is a unique solution of the equation

$$G(x) = 0$$

satisfying the condition $\|x - y\| \leq M\Delta$.

Previous lemma follows from the contracting mapping principle applied to the operator $F : O \to X$ defined by $F(x) = y - L^{-1}[G(x) - DG(y)(x - y)]$.

Lemma 1 proves the shadowing theorem if we show that for δ and σ sufficiently small, the operator L is invertible and bounded.

$L = DG(y) : l^{\infty}(\mathbb{R}^n) \to l^{\infty}(\mathbb{R}^n)$ is linear operator such that $(Lz)_k = z_{k+1} - Dg(y_k)z_k$.

$$L = \begin{pmatrix} -Dg(y_0) & \mathbb{1} & 0 & \dots \\ 0 & -Dg(y_1) & \mathbb{1} & 0 \\ & & \ddots & \end{pmatrix}$$

The following lemma gives us the necessary condition for invertibility of the operator L.

Lemma 2. Let $L : l^\infty(\mathbb{R}^n) \to l^\infty(\mathbb{R}^n)$ be the operator defined by

$$(Lz)_k = z_{k+1} - A_k z_k,$$

where $\{A_k\}_{k=0}^\infty$ is a bounded sequence of $n \times n$ invertible matrices.

Then if the difference equation $z_{k+1} = A_k z_k$ has an exponential dichotomy on $\langle 0, \infty)$ with exponents λ_1, λ_2 and constants K_1, K_2, the operator L is invertible with norm

$$\|L^{-1}\| \leq K_1(1-\lambda_1)^{-1} + K_2\lambda_2(1-\lambda_2)^{-1}.$$

It can be shown, but we will not do it here, that the difference equation

$$z_{k+1} = Dg(y_k)z_k$$

has an exponential dichotomy on $\langle 0, \infty)$ so L is invertible and its norm is bounded by the condition

$$\|L^{-1}\| \leq L_1(1-\alpha_1)^{-1} + L_2\alpha_2(1-\alpha_2)^{-1}, \tag{1.26}$$

where $\alpha_i = \frac{3+\lambda_i}{4}, \quad i \in \hat{2}$ and L_1, L_2 are constants depending only on g and S.

Now we know that $L = DG(y)$ is invertible and $\|L^{-1}\| \leq \frac{M}{2}$, where $M = 2L_1(1-\alpha_1)^{-1} + 2L_2\alpha_2(1-\alpha_2)^{-1}$.

$$\|G(y)\| = \sup_k \|y_{k+1} - g(y_k)\| \leq \sup_k\{\|y_{k+1} - f(y_k)\| + \|f(y_k) - g(y_k)\|\} \leq \delta + \sigma$$

Let us take all $x = \{x_k\}_{k=0}^\infty$ that satisfy the condition $\|x - y\| \leq M(\delta + \sigma)$ and look at the norm of the operator $DG(x) - DG(y)$

$$\begin{aligned}\|DG(x) - DG(y)\| \leq \sup_k \|Dg(y_k) - Dg(x_k)\| &\leq \sup_k\{\|Dg(y_k) - Df(y_k)\| + \\ &+ \|Df(y_k) - Df(x_k)\| + \|Df(x_k) - Dg(x_k)\|\} \leq \\ &\leq \omega(M(\delta+\sigma)) + 2\sigma,\end{aligned}$$

where $\omega(\epsilon) = \sup\{\|Df(x) - Df(y)\|, y \in S, \quad \|x - y\| \leq \epsilon\}$.

Therefore the shadowing theorem is proved for $M(\delta + \sigma) < \operatorname{dist}(S, \partial U)$ and $M[\omega(M(\delta+\sigma)) + 2\sigma] \leq 1$. □

1.6 Unshadowability

From the previous, it might seem that we can find a shadow of a finite length for all pseudotrajectories. The following simple example illustrates that this expectation is too optimistic.

Example 7. Consider one-dimensional map $f(x) = 1-2x^2$. The interval $I = \langle -1, 1\rangle$ is a compact invariant set of f. It is not a hyperbolic set of f, because $Df(-\frac{1}{4}) = 1$, $Df(\frac{1}{4}) = -1$.

Let us compute a δ pseudoorbit beginning at point $x_0 = 0$. We get the inaccurate value $x_1 = 1+\epsilon$, $0 < \epsilon < \delta$ in the following step and then our computation continues without any error.

$$x_2 = f(x_1) = -1 - 4\epsilon - 2\epsilon^2 < -1, \quad \ldots$$

The δ pseudoorbit diverges to $-\infty$. But I is an invariant set of f so all true orbits beginning in I stay in this interval for all future evolution and therefore there is no shadow for our δ pseudoorbit.

The point $x = -1$ is a hyperbolic fixed point of f but $E^u = \mathbb{R}$ and therefore it repels all nearby orbits.

The critical point $x = -1$ is called a glitch. At such glitches, all true trajectories diverge from the pseudotrajectory. Although this example is trivial, the existence of glitches is quite common phenomenon in chaotic systems. This is one of the reasons why we are unable to forecast the weather for a long time. It is not the problem of a wrong model but our inability to compute accurate or even to gain correct initial conditions. When our pseudoorbit hits the glitch, the accuracy of our prediction will be destroyed and we are unable to do anything to prevent it. ♠

Chapter 2

Numerical shadowing

In this section, we try to prove the existence of the shadow for the chaotic δ pseudo orbit of the Hénon map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (1 - ax^2 + y, -bx)$ with special choice of parameters $a = 1.4, b = -0.3$.

Hénon map was introduced by French astronomer Michel Hénon in 1976 as a simplified model of the Poincaré section of the Lorenz model. It is one of the most famous dynamical systems that exhibit chaotic behaviour. The behaviour of the system depends essentially on parameters a, b.

Before proving the existence of the shadow, we look at the behaviour of the origin, which moves as dictates Hénon map for different values of parameters a, b. We hope that the reader forgives us this interruption, but we consider the behaviour fascinating.

We used the program MATLAB 7.6.0 for all the simulations.

Hénon map has a stable periodic orbit of 7 points for $b = -0.3$, $a = 1.25$ as we can see in the figure 2.1. This orbit attracts all near points including the origin.

If we change parameter a slightly, the points are attracted by the strange attractor. The Hénon attractor is illustrated for $a = 1.4$, $b = -0.3$ in the figure 2.2.

The trajectory of the origin is presented in the figure 2.3, resp. 2.4 for $a = 0.2$, $b = 0.995$, resp. $a = 0.2$, $b = 0.999$. There are two fixed points x_1 and x_2, both are hyperbolic and their stable and unstable manifolds are nontrivial. It explains the strange behaviour of the origin. The origin moves to the unstable manifold of x_1 and then it is repelled from x_1 to the unstable manifold of x_2 and this jumping between the unstable manifolds continues. If we take $b > 1$ for $a = 0.2$, the origin diverges to infinity.

The theorem 3 is not particulary convenient for verifying the shadowability of the pseudo orbit. For this reason we present another shadowing theorem.

Theorem 4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 map, $\{y_k\}_{k=0}^N$ be its δ pseudo orbit and $L : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ be the linear operator defined for $u = \{u_k\}_{k=0}^N$ by:

$$(Lu)_k = u_{k+1} - Df(y_k)u_k, \quad k = 0, \dots, N-1.$$

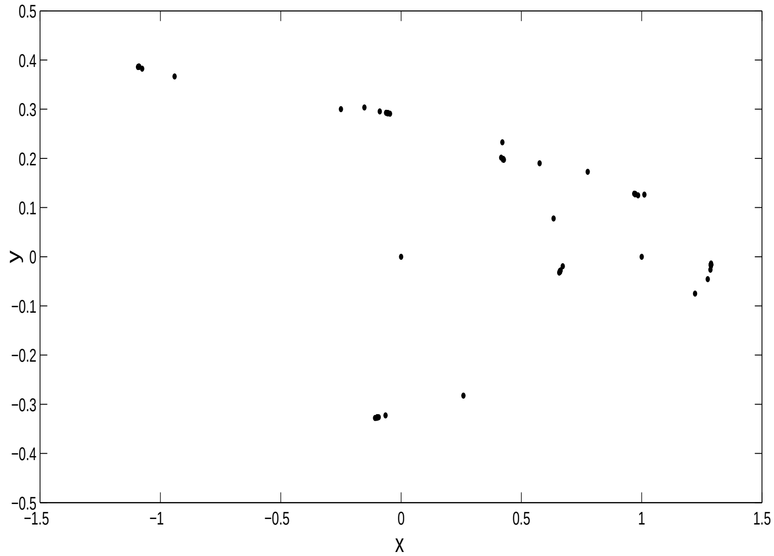

Figure 2.1: The stable orbit

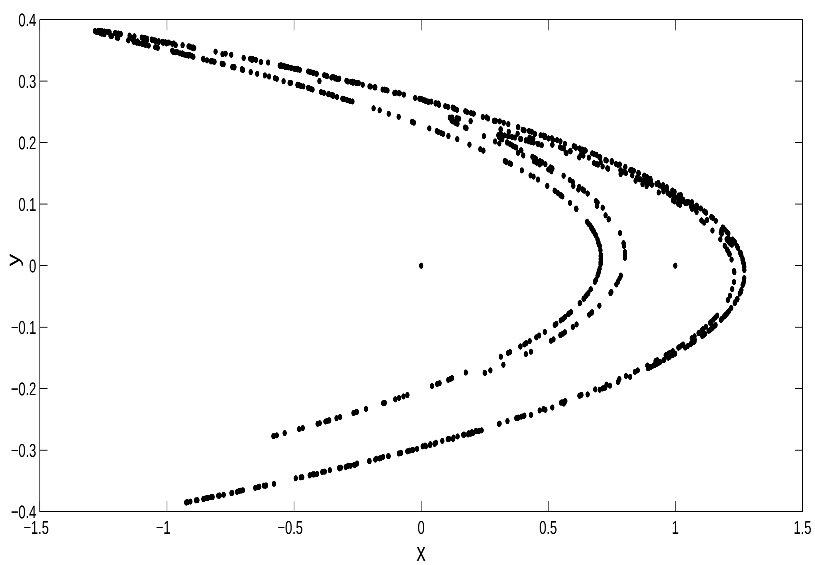

Figure 2.2: Hénon attractor

Suppose $\epsilon = 2\|L^{-1}\|\delta$, where L^{-1} is the right inverse of L, and

$$M = \sup\{\|D^2 f(x)\|, x \in \mathbb{R}^n, \quad \|x - y_k\| \leq \epsilon \quad \text{for} \quad \text{some} \quad k = 0, \ldots, N-1\}.$$

Then if $2M\|L^{-1}\|^2\delta \leq 1$, the δ pseudo orbit $\{y_k\}_{k=0}^{N}$ is ϵ shadowed by a true orbit

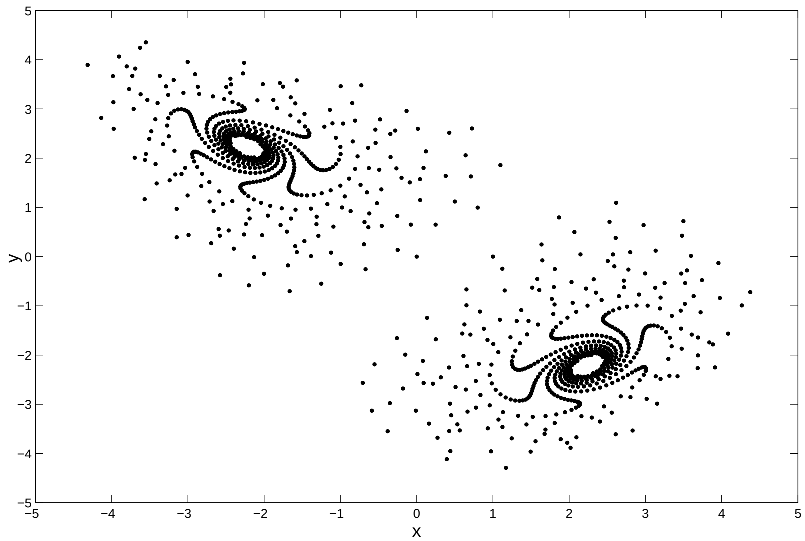

Figure 2.3: Two hyperbolic fixed points for b=0.995

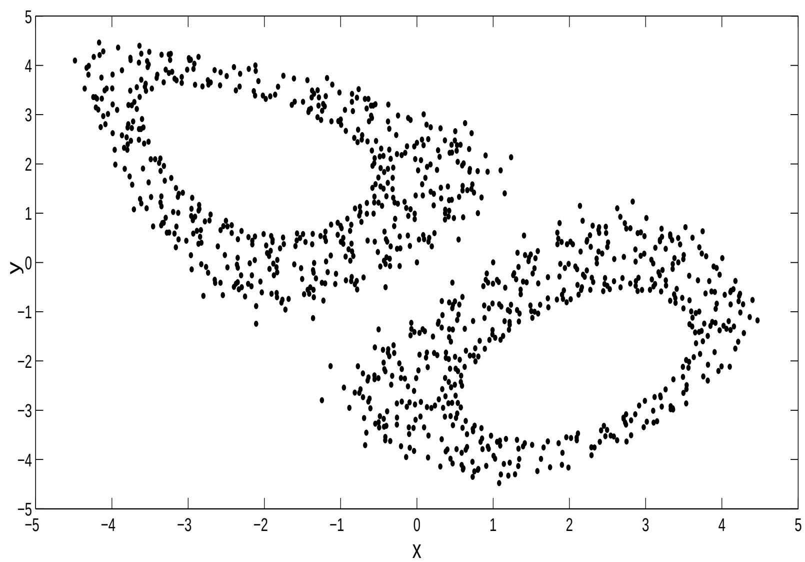

Figure 2.4: Two hyperbolic fixed points for b=0.999

$\{x_k\}_{k=0}^{N}$ of f.

The right inverse L^{-1} exists for hyperbolic maps and its norm is bounded by the condition (1.26), so the shadowability of hyperbolic maps follows from the previous theorem. Theorem 4 guarantees the shadowability also for the maps, which are not uniformly hyperbolic.

To find the right inverse of L means to solve the equation $Lu = g$ for given $g = \{g_k\}_{k=0}^{N-1}$, *i.e.* the difference equation

$$u_{k+1} = Df(y_k)u_k + g_k \quad k = 0, \ldots, N-1. \tag{2.1}$$

In the next step, we use the fact that every regular matrix $A \in \mathbb{R}^{n,n}$ can be decomposed as a product $A = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix. This decomposition is unique if R has positive diagonal elements.

Let us take the orthogonal matrix Q_0 and compute the following QR factorizations

$$Df(y_k)Q_k = Q_{k+1}R_k \quad k = 0, \ldots, N-1.$$

Now we can introduce new variables $\tilde{u}_k : u_k = Q_k\tilde{u}_k$. The equation (2.1) transforms into

$$\tilde{u}_{k+1} = R_k\tilde{u}_k + Q_{k+1}^T g_k, \quad k = 0, \ldots, N-1.$$

We define the linear operator $T : (\mathbb{R}^n)^{N+1} \to (\mathbb{R}^n)^N$ by:

$$(Tu)_k = u_{k+1} - R_k u_k \quad \text{for} \quad u = \{u_k\}_{k=0}^N.$$

$$(L^{-1}g)_k = Q_k\tilde{u}_k = Q_k T^{-1} Q_{k+1}^T g_k \tag{2.2}$$

So the inverse T^{-1} of T defines the inverse L^{-1} and moreover because the norm of orthogonal operator is equal to one, it follows

$$\|L^{-1}\| \leq \|T^{-1}\|. \tag{2.3}$$

The relation (2.3) is very useful, because T^{-1} is an upper triangular matrix.

To find a right inverse of T, we have to solve the equation

$$u_{k+1} = R_k u_k + g_k, \tag{2.4}$$

i.e., to find $u = \{u_k\}_{k=0}^N$ for given $g = \{g_k\}_{k=0}^N$.

The Hénon map is two-dimensional, so we may write

$$u_k = \begin{pmatrix} v_k \\ w_k \end{pmatrix} \quad R_k = \begin{pmatrix} a_k & b_k \\ 0 & c_k \end{pmatrix}.$$

The equation (2.4) is in the components:

$$v_{k+1} = a_k v_k + b_k w_k + g_k^{(1)} \tag{2.5}$$

$$w_{k+1} = c_k w_k + g_k^{(2)}. \tag{2.6}$$

We construct T^{-1} by solving (2.6) forwards starting with $w_0 = 0$ and then solving (2.5) backwards starting with $v_N = 0$. The reasons for such a choice are explained in full detail in [4].

Now we define new variables $\tilde{v}_k$, $\tilde{w}_k$, $k = 0, \ldots, N$, by the recursions

$$\tilde{w}_0 = 0 \quad \tilde{w}_{k+1} = |c_k|\tilde{w}_k + 1, \quad k = 0, \ldots, N-1$$

and

$$\tilde{v}_N = 0 \quad \tilde{v}_k = |a_k^{-1}|(\tilde{v}_{k+1} + |b_k|\tilde{w}_k), \quad k = N-1, \ldots, 0.$$

We see that $|w_k| \leq \tilde{w}_k$ and $|v_k| \leq \tilde{v}_k$ for $g = \{g_k\}_{k=0}^N$ such that $\|g\| \leq 1$. Therefore

$$\|T^{-1}\| = \max_{0 \leq k \leq N} \sqrt{v_k^2 + w_k^2} \leq \max_{0 \leq k \leq N} \sqrt{\tilde{v}_k^2 + \tilde{w}_k^2}. \tag{2.7}$$

We found the chaotic δ pseudo orbit $\{(x_k, y_k)\}_{k=0}^{10000}$ of the Hénon map $f(x, y) = (1 - 1.4x^2 + y, 0.3x)$ (figure 2.2) starting at the origin. Then the matrices $Df(y_k)$ and their QR decompositions were computed. Finally using the method explained above, we determined that

$$\|L^{-1}\| \leq \|T^{-1}\| \leq 3775.$$

$$M = \|D^2 f(x, y)\| \leq 2a = 2.8 \qquad \delta = 2.22 \times 10^{-16}$$

Therefore $2M\|L^{-1}\|^2\delta \leq 1.78 \times 10^{-8} < 1$. So the theorem 4 ensures that the δ pseudo orbit is ϵ shadowed by the true trajectory with $\epsilon \leq 1.68 \times 10^{-12}$.

Conclusion

First we introduced the hyperbolic fixed point with its properties. Then we explained the reasons for defining hyperbolic set in such a complicated way. We presented shadowing theorem and its modification more convenient for numerical computation. Finally we proved the existence of a shadow for a chaotic pseudo orbit of the Hénon map.

The aim of this work was to understand the idea of shadowing in discrete case, because it is necessary to present shadowing in continuous case, which will be the subject of our further research.

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