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Geometrically induced properties od the ground state of point-interaction Hamiltonians

Geometricky podmíněné vlastnosti základního stavu hamiltoniánů s bodovými interakcemi

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Prohlášení

Prohlašuji, že jsem svou bakalářskou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW, atd.) uvedené v přiloženén seznamu.

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Abstract

In this work we discuss the ground state of the point interaction Hamiltonian in one dimension. We introduce a mathematically rigorous definition of this Hamiltonian as the self-adjoined operator on the appropriate Sobolev space with the boundary condition at the point interactions sites. We study the spectral problem for the point interaction Hamiltonian and the ground state of the operator with help of the Krein's formula. We demonstrate the relation between the distance of the point interactions sites and the ground state on the line, the halfline and the star graph. We prove that the increase in the distance between the point interactions results in increase of the energy of the ground state.

Key words

The point interaction operator, point interactions in one dimension, the ground state of the point interaction Hamiltonian, Krein's formula

Abstrakt

V této práci se zabýváme základním stavem operátoru bodových interakcí. Zavedeme matematicky rigorozní definici tohoto operátoru, jakožto samosdruženého operátoru na odpovídajícím Sobolevově prostoru s vhodnými okrajovými podmínkami v místech bodových interakcí. Zabýváme se spektrálním problémem operátoru bodových interakcí a jeho základním stavem za pomoci Kreinovy formule. Dokážeme vztah mezi vzdáleností jednotlivých bodových interackcí a energií základního stavu na přímce, polopřímce a hvězdicovém grafu. Ukážeme že zvětšení vzdálenosti mezi bodovými interakcemi vede k zvýšení energie základního stavu.

Klíčová slova

Operátor bodových interakcí, bodové interakce v jedné dimenzi, základní stav operátoru bodových interakcí, Kreinova formule

Contents

Chapter 1

Introduction

In this work we will discuss the ground state of Hamiltonian operator, which can be formally written in the form:

$$\hat{H} = -\Delta + \sum_{y \in Y} \lambda_y \delta_y(\cdot) \tag{1.1}$$

where Δ denotes self-adjoint Laplacian with the domain $H^{2,2}(\mathbb{R})$. The quantum mechanical particle with Hamiltonian of this form moves in the field of contact potentials placed at the the points of a discrete set Y with the interaction strengths λ_y. Such operators are interesting, because the corresponding spectral and scattering problem can be solved exactly and used as an approximation to more realistic models of such systems. These Hamiltonians are useful in atomic and nuclear physics (as approximations of short range interactions), in solid state physics (for instance, as a description of nonrelativistic electron which moves freely in crystal lattice with fixed atoms, usually called Kronig-Penney model), or in electromagnetism (propagation in dielectric media), etc. A survey of such solvable models can be found in the monograph [1].

In this work we are going to discuss properties of the ground state of the operator (1.1) in one-dimensional systems, that is, for a particle living on line or a more complicated configuration space composed of line segments. The ground state of the system as the state with the lowest energy is of natural physical importance. While in an isolated system the particle remains in a given eigenstate, in reality any physical system interacts with the rest of the universe which can be regarded as a heat bath, often in form of an electromagnetic field. This leads to an energy dissipation which brings the system eventually into the ground state as its stable configuration. We will be particularly interested here in the case with attractive point interactions, i.e. with the coupling constants $\lambda_y < 0$. In this situation the existence of the spectral gap is guaranteed, i.e. the ground state is an isolated eigenvalue, whenever Y is a finite and non-empty set. We will introduce a mathematically rigorous definition of the formal expression (1.1) as a self-adjoint operator defined on the appropriate Sobolev space with the help of suitable boundary conditions at the point interactions sites. To find the ground state we have to solve the equation

$$\hat{H}\psi = -\Delta\psi + \sum_{y\in Y} \lambda_y \delta_y(\cdot)\psi = E\psi \tag{1.2}$$

This spectral problem can be simplified to an algebraic one using the so-called Krein's formula which expresses the resolvent of the operator, or rather its difference from that of the Laplacian as a specific finite-rank operator. Our main result is that a change of the set Y which increases distances between the point interaction sites results in an increase of the ground-state energy.

After solving the problem on the line we will present some generalizations. In particular, we will consider point interactions on a halfline with various boundary conditions at the endpoint. We will show that a similar result is valid regarding to changes of distances between the point interactions. With respect to the distances from the halfline end the situation is more complicated. The behavior is similar as long as the boundary condition is Neumann or Robin with a negative parameter, $g'(0+) = \alpha_0 g(0)$ with $\alpha_0 < 0$ while for its positive value or Dirichlet condition it needs not to be true. We also generalize this result to point interactions on a star graph under additional assumptions.

The result may look natural but it is general and new to our knowledge. Moreover, it opens interesting questions if we impose constraints on the minimum distance between the point interactions, as a possible simple model of the crystallization process. Further generalizations may concern a version of our result in presence of a regular potential and, in particular, its analogue for point interactions in dimension two and three. These questions, however, go beyond the scope of the present thesis and will be a subject of a future work.

Chapter 2

Point interactions in one dimension

In this chapter we summarize basic properties of point interactions in one dimension. We begin with properties of a single point interaction on the line. Afterwards we introduce finite number of point interactions on the line.

2.1 The one-center point interaction

We have more than one way how to introduce the quantum Hamiltonian which describes δ-interaction in one dimension. One of the easiest ways is to employ self-adjoint extension of a suitable densely defined symmetric operator. First we take closed and nonnegative operator

$$\dot{H}_y = -\frac{d^2}{dx^2} \tag{2.1}$$

with the domain $\mathrm{D}(\dot{H}_y) = \{g \in H^{2,2}(\mathbb{R}) | g(y) = 0\}$. Its adjoint acts as

$$\dot{H}_y^* = -\frac{d^2}{dx^2} \tag{2.2}$$

with the domain $\mathrm{D}(\dot{H}_y^*) = H^{2,2}(\mathbb{R} \setminus \{y\}) \cap H^{2,1}(\mathbb{R})$, where $H^{m,n}(\mathbb{R})$ are corresponding the Sobolev spaces. Solution of the equation

$$\dot{H}_y^* \psi(k) = k^2 \psi(k) \tag{2.3}$$

is given by

$$\psi(k,x) = e^{ik|x-y|} \tag{2.4}$$

where $\psi(k) \in \mathrm{D}(\dot{H}_y^*)$, $k^2 \in \mathbb{C} - \mathbb{R}$ and $\Im k > 0$. From this we can infer that $\dot{H}_y$ has deficiency indices (1,1).

According to [4, Section X.1] and [1, Appendix A] we know that all self-adjoint extensions can be then parameterized by $\theta \in [0, 2\pi)$ in the following way.

Theorem 2.1. *Let $\dot{H}$ be densely defined, closed, symmetric operator in Hilbert space $\mathcal{H}$ with deficiency indices (1,1). Let $\psi(z) \in \mathrm{D}(\dot{H}^*)$, $\Im z > 0$ fullfill* (2.3). *Then we may parameterize all self-adjoint extensions H_θ of $\dot{H}$ with $\theta \in [0, 2\pi)$ as follows: the domain of H_θ is*

$$\mathrm{D}(H_\theta) = \{g + c\psi_+ + ce^{i\theta}\psi_- | g(z) \in \mathrm{D}(\dot{H}), c \in \mathbb{C}\} \tag{2.5}$$

and the operator acts as

$$H_\theta(g + c\psi_+ + ce^{i\theta}\psi_-) = \dot{H}_y g + ic\psi_+ + ice^{i\theta}\psi_- \tag{2.6}$$

where $\psi_\pm = \psi(\pm i, \cdot)$, $\|\psi_-\| = \|\psi_+\|$.

When we apply this theorem to extensions of our operator $\dot{H}_y$ we see that the self-adjoint extensions $H_{\theta,y}$ satisfy the condition

$$\begin{aligned} \lim_{\epsilon \downarrow 0} [(g + c\psi_+ + ce^{i\theta}\psi_-)'(y+\epsilon) - (g + c\psi_+ + ce^{i\theta}\psi_-)'(y-\epsilon)] = \\ -c(1 + e^{i\theta}) = \alpha[g(y) + c\psi_+(y) + ce^{i\theta}\psi_-(y)] \end{aligned} \tag{2.7}$$

where $\alpha = \frac{-2\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}$ and the functions in the domain are continuous at the point y. We can thus parameterize the self-adjoint extensions using boundary conditions.

Theorem 2.2. *Let the operator $-\Delta_{\alpha,y}$ be defined as*

$$\begin{aligned} -\Delta_{\alpha,y} = -\frac{d^2}{dx^2}, \\ \mathrm{D}(-\Delta_{\alpha,y}) = \{g \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R}\backslash\{y\}) | \, g'(y+) - g'(y-) = \alpha g(y)\}, \\ -\infty < \alpha \leq \infty \end{aligned} \tag{2.8}$$

The family $\{-\Delta_{\alpha,y} | -\infty < \alpha \leq \infty\}$ *coincides with all self-adjoint extensions of* $\dot{H}_y$. *The case* $\alpha = 0$ *gives us the free Hamiltonian, or kinetic energy operator in* $L^2(\mathbb{R})$,

$$-\Delta_{\alpha,y} = -\frac{d^2}{dx^2}, \quad \mathrm{D}(-\Delta_{\alpha,y}) = H^{2,2}(\mathbb{R}) \tag{2.9}$$

The case $\alpha = \infty$ *corresponds to separated halflines with Dirichlet boundary condition at* y,

$$\begin{aligned} -\Delta_{\infty,y} = (-\Delta_{D-}) \oplus (-\Delta_{D-}), \\ \mathrm{D}(-\Delta_{\alpha,y}) =& \{g \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus \{y\}) | g(y) = 0\} \\ =& H_0^{2,2}((-\infty, y)) \cup H_0^{2,2}((y, \infty)) \end{aligned} \tag{2.10}$$

where $-\Delta_{D\pm}$ *is Dirichlet Laplacian (for the definition see [3, Section XIII.15]) on* (y, ∞) *and* $(-\infty, y)$, *respectively.*

Proof can be found in [1, Section I.3.1]

The operator $-\Delta_{\alpha,y}$ describes one dimensional quantum particle with δ-point interaction of strength α at $y \in \mathbb{R}$. According to [1, Theorem 3.2.3] we can obtain the self-adjoint operator $-\Delta_{\alpha,y}$ as limit of the operator

$$H_{\epsilon,y} = -\frac{d^2}{dx^2} + \epsilon^{-1} V(\frac{\cdot - y}{\epsilon}) \tag{2.11}$$

where $\epsilon > 0$, $y \in \mathbb{R}$, and $V(x) \in L^1(\mathbb{R})$. For the $\epsilon \searrow 0$ the operator $H_{\epsilon,y}$ converges to the operator $-\Delta_{\alpha,y}$ in the norm resolvent sense that means

$$\lim_{\epsilon \searrow 0} \| (H_{\epsilon,y} - k^2)^{-1} - (-\Delta_{\alpha,y} - k^2)^{-1} \| = 0.$$

Also we know that $\alpha = \int_{\mathbb{R}} dx\, V(x)$.This kind of approximation scheme automatically yields finite strength of the point interaction, $|\alpha| < \infty$.

The proof of this result is not easy but one can illustrate its essence using a formal argument. When we integrate Schrödinger equation corresponding to $H = -\frac{d^2}{dx^2} + \alpha\delta(x - y)$ from $x = y - \epsilon$ to $x = y + \epsilon$ for $\epsilon > 0$, we get

$$\psi'(y + \epsilon) - \psi'(y - \epsilon) + \alpha\psi(y) = E \int_{y-\epsilon}^{y+\epsilon} \psi(x) dx \tag{2.12}$$

which is the boundary condition in (2.8). If we replace $\delta(x-y)$ by $\frac{1}{\epsilon}V\left(\frac{x-y}{\epsilon}\right)$ we get the same result with $\alpha = \int_{\mathbb{R}} dx V(x)$.

Next we will express the resolvent of $-\Delta_{\alpha,y}$ by means of Krein's formula.

Theorem 2.3. *The resolvent of $-\Delta_{\alpha,y}$ is given by*

$$\begin{aligned}(-\Delta_{\alpha,y}-k^2)^{-1} &= G_k - 2\alpha k(i\alpha+2k)^{-1}(\overline{G_k(\cdot-y)},\cdot)G_k(\cdot-y), \\ k^2 &\in \rho(-\Delta_{\alpha,y}), \quad \Im k>0, \quad -\infty<\alpha\le\infty, \quad y\in\mathbb{R}\end{aligned} \tag{2.13}$$

where

$$G_k(x,x') = (i/2k)e^{ik|x-x'|},\ \Im k>0 \tag{2.14}$$

is the integral kernel of $(-\Delta_{\alpha,y}-k^2)^{-1}$ in $L^2(\mathbb{R})$ which means that the integral kernel of $(-\Delta_{\alpha,y}-k^2)^{-1}$ is

$$\begin{aligned}(-\Delta_{\alpha,y}-k^2)^{-1}(x,x') &= (i/2k)e^{ik|x-x'|} + \alpha(2k)^{-1}(i\alpha+2k)^{-1}e^{ik[|x-y|+|y-x'|]}, \\ & k^2\in\rho(-\Delta_{\alpha,y}), \quad \Im k>0, \quad x,x'\in\mathbb{R}.\end{aligned} \tag{2.15}$$

Proof can be found in [1, Section I.3.1]

Next we will say a few things about spectral properties of $-\Delta_{\alpha,y}$

Theorem 2.4. *Let $-\infty<\alpha\le\infty$, $y\in\mathbb{R}$ then we have*

$$\sigma_{ess}(-\Delta_{\alpha,y}) = \sigma_{ac}(-\Delta_{\alpha,y}) = [0,\infty), \quad \sigma_{sc}(-\Delta_{\alpha,y}) = \varnothing \tag{2.16}$$

If $-\infty<\alpha<0$, then $-\Delta_{\alpha,y}$ has one simple, negative eigenvalue, namely

$$\sigma_p(-\Delta_{\alpha,y}) = \left\{\frac{-\alpha^2}{4}\right\}, \quad -\infty<\alpha<0, \tag{2.17}$$

with an eigenfunction which can be chosen strictly positive:

$$\psi(x) = (\alpha/2)^{1/2}e^{\alpha|x-y|/2} \tag{2.18}$$

If $0\le\alpha\le\infty$, the operator $-\Delta_{\alpha,y}$ has no eigenvalues, i.e.

$$\sigma_p(-\Delta_{\alpha,y}) = \varnothing, \quad 0\le\alpha\le\infty. \tag{2.19}$$

Proof can be found in [1, Section I.3.1]

2.2 Finitely many point interactions

In this section, we will generalize the case of one point-interaction to the case of finitely many point-interactions. First we define the minimal operator $\dot{H}_Y$ as

$$\begin{aligned} \dot{H}_Y &= -\frac{d^2}{dx^2}, \\ \mathrm{D}(\dot{H}_Y) &= \{g \in H^{2,2}(\mathbb{R}) |\, g(y_j) = 0, y_j \in Y, j = 1, \ldots N\}, \\ Y &= \{y_1, \ldots, y_N\}, N \in \mathbb{N} \end{aligned} \tag{2.20}$$

$\dot{H}_Y$ is a closed and nonnegative operator. The adjoint operator $\dot{H}_Y^*$ to $\dot{H}_Y$ is given by

$$\begin{aligned} \dot{H}_Y^* &= -\frac{d^2}{dx^2}, \\ \mathrm{D}(\dot{H}_Y^*) &= H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus \{Y\}) \end{aligned} \tag{2.21}$$

The solution of $\dot{H}_Y^* \psi(k) = k^2 \psi(k)$, for $\psi(k) \in \mathrm{D}(\dot{H}_Y^*)$, $k^2 \in \mathbb{C} - \mathbb{R}$, $\Im k > 0$, is

$$\psi_j(k, x) = e^{ik|x-y_j|},\ \Im k > 0,\ y_j \in Y,\ j = 1, \ldots, N \tag{2.22}$$

thus $\dot{H}_Y$ has deficiency indices (N, N). Consequently, all the self-adjoint extensions of $\dot{H}_Y$ form an N^2-parameter family of self-adjoint operators. We restrict ourselves to the case of *local* boundary conditions coupling the boundary values at each point y_j, $j = 1, \ldots, N$ separately. Similarly as for one point interactions we introduce self-adjoint extension of $\dot{H}_Y$ as

$$\begin{aligned} -\Delta_{\alpha,Y} &= -\frac{d^2}{dx^2}, \\ \mathrm{D}(-\Delta_{\alpha,Y}) &= \{g \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus Y)| \\ g'(y_j+) - g'(y_j-) &= \alpha_j g(y_j), j = 1, \ldots, N\} \end{aligned} \tag{2.23}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $-\infty < \alpha_j \leq \infty$, $j = 1, \ldots, N$. The operator $-\Delta_{\alpha,Y}$ is self-adjoint([1, Section II.2.1]). The case $\alpha = 0$ leads to the kinetic energy operator $-\Delta$ on $H^{2,2}(\mathbb{R})$. The case $\alpha_{j_0} = \infty$ leads to the Dirichlet boundary condition at y_{j_0}, that is, $g(y_{j_0}+) = g(y_{j_0}-) = 0$. According to [1, Section II.2.2] the operator $-\Delta_{\alpha,Y}$ can be approximated by the operators

$$H_{\epsilon,Y} = -\frac{d^2}{dx^2} + \epsilon^{-1} \sum_{j=1}^{N} V_j\left(\frac{\cdot - y_j}{\epsilon}\right). \tag{2.24}$$

It can be proven that the operators $H_{\epsilon,Y}$ converge to the operator $-\Delta_{\alpha,Y}$ in the norm resolvent sense and $\alpha_j = \int_{\mathbb{R}} dx\, \epsilon^{-1} V_j(\frac{\cdot - y_j}{\epsilon})$ for all $j = 1, \ldots, N$. From this we can see that $-\Delta_{\alpha,Y}$ describes N δ-interactions at points $y_j \in Y$ of strength α_j, where $j = 1, \ldots, N$.

Now we will point out some properties of $-\Delta_{\alpha,Y}$. First we will state Krein's formula in this case:

Theorem 2.5. *Let $\alpha_j \neq 0$, $j = 1, \ldots, N$. Then the resolvent of $-\Delta_{\alpha,Y}$ is given by*

$$(-\Delta_{\alpha,Y} - k^2)^{-1} = G_k + \sum_{j,j'=1}^{N} [\Gamma_{\alpha,Y}(k)]^{-1}_{jj'} (\overline{G_k(\cdot - y_{j'})}, \cdot) G_k(\cdot - y_j), \tag{2.25}$$
$$k^2 \in \rho(-\Delta_{\alpha,Y}),\ \Im k > 0,\ -\infty < \alpha_j \leq \infty,\ y_j \in Y,\ j = 1, \ldots, N,$$

where

$$[\Gamma_{\alpha,Y}(k)]_{jj'} = -[\alpha_j^{-1} \delta_{jj'} + G_k(y_j - y_{j'})]_{j,j'=1}^{N} \tag{2.26}$$

where G_k is the free resolvent kernel given by (2.14) *,i.e.*

$$G_k(y_j - y_{j'}) = \frac{i}{2k} e^{ik|y_j - y_{j'}|}. \tag{2.27}$$

Proof can be found in [1, Section II.2.1]

Theorem 2.6. *Let $\alpha_j \neq 0$, $y_j \in Y, j \in \{1, 2, ..., N\}$. Assume that there is at most one index $j = j_0$ for which $\alpha_{j_0} = \infty$. Then $-\Delta_{\alpha,Y}$ has at most N eigenvalues which are all negative and simple. If $\alpha_j = \infty$ for at least two different values $j \in \{1, 2, ..., N\}$, then $-\Delta_{\alpha,Y}$ has at most N negative eigenvalues (counting multiplicity) and infinitely many eigenvalues embedded in $[0, \infty)$ accumulating at ∞. In particular, $k^2 \in \sigma_p(-\Delta_{\alpha,Y}) \cap (-\infty, 0)$ if $\det[\Gamma_{\alpha,Y}(k)] = 0$, $\Im k > 0$, and the multiplicity of eigenvalue $k^2 < 0$ equals the multiplicity of the eigenvalue zero of the matrix $\Gamma_{\alpha,Y}(k)$. Moreover, if $E_0 = k_0^2 < 0$ is an eigenvalue of $-\Delta_{\alpha,Y}$, the corresponding eigenfunctions are of the form*

$$\psi_0 = \sum_{j=1}^{N} c_j G_{k_0}(x - y_j), \quad \Im k_0 > 0, \tag{2.28}$$

where $(c_1, c_2, ..., c_N)$ are eigenvectors of the matrix $\Gamma_{\alpha,Y}(k_0)$ corresponding to the eigenvalue zero. If $\Delta_{\alpha,Y}$ has a ground state (the lowest isolated eigenvalue) it is nondegenerate and the corresponding eigenfunction can be chosen *to be strictly positive, i.e. the associated eigenvector* $(c_1, c_2, ..., c_N)$ *fulfills* $c_j > 0,\ j \in \{1, 2, ..., N\}$.

Proof can be found in [1, Section II.2.1]

Chapter 3

Energy of the ground state

3.1 Finite number of point interactions on the line

In this section we shall discuss how the ground state of $-\Delta_{\alpha,Y}$ is affected by a change of distance between point interactions. We prove that an increase in distance between point interactions causes an increase of the energy of the ground state. We can consider a change of the distance between two neighboring interactions which will, of course, result into a modification of distances between the other point interactions as well. As a preliminary, let us recall a useful tool for comparing operators.

Lemma 3.1 (About Neumann bracketing). *Let* Ω_1,Ω_2 *be disjoined subsets such that* $\overline{\Omega_1 \cup \Omega_2}^{int} = \Omega$, *and* $\Omega \backslash \Omega_1 \cup \Omega_2$ *has Lebesgue measure zero. Then* $0 \leq \Delta_N^{\Omega_1 \cup \Omega_2} \leq \Delta_N^{\Omega}$.

Proof can be found in [3, Section XIII.15]

There are different methods to address our problem. First we will prove a slightly weaker claim under an additional assumption, namely that the ground-state eigenfunction derivative changes sign between the two point interactions in question. A stronger claim will follow in the next theorem.

Theorem 3.1. *Let* $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ *be the point interaction Hamiltonians defined above where* $Y_i = \{y_{i,1}, \dots, y_{i,N}\}$ *and* $y_{i,1} \leq y_{i,2} \leq \dots \leq y_{i,n}$. *Suppose that* $\operatorname{card} Y_1 = \operatorname{card} Y_2$, $\alpha_k < 0$ *for all* k. *Suppose that there is an* i *such that*

$y_{2,j} = y_{1,j}$ for $j = 1, \dots, i$ and $y_{2,j} = y_{1,j} + \eta$ for $j = i+1, \dots, N$. Suppose further that for the ground state of the first operator we have $\psi'(y_i+) < 0$ and $\psi'(y_{i+1}-) > 0$. If $\eta \geq 0$ then the ground states of the two operators $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ satisfy $\min \sigma_p(-\Delta_{\alpha,Y_1}) \leq \min \sigma_p(-\Delta_{\alpha,Y_2})$

Proof. We consider the operator $-\Delta_{\alpha,Y}$ discussed above. From Theorem 2.6 we know that for this operator the eigenfunction of the ground state can be chosen as strictly positive. That means $\forall x \in \mathbb{R} \setminus Y \Rightarrow \psi(x) > 0$. We also have

$$-\Delta_{\alpha,Y}\psi = -E_0\psi$$

where $E_0 > 0$. This means

$$\frac{\partial^2\psi}{\partial x^2} = E_0\psi$$

$\forall x \in \mathbb{R}\setminus Y$. Because $\psi(x) > 0$ and $E_0 > 0$, we can conclude that this eigenfunction is convex for all $x \in \mathbb{R} \setminus \{Y\}$. By assumption we also have $\psi'(y_i-) > 0$ and $\psi'(y_{i+1}+) < 0$. Thanks to that we can find $x \in (y_i, y_{i+1})$ such that $\psi'(x) = 0$ Now we add Neumann condition at x to $-\Delta_{\alpha,Y}$. We denote the operator with the Neumann condition as $-\Delta^{(1)}_{\alpha,Y}$. The domain of $-\Delta^{(1)}_{\alpha,Y}$ is

$$\begin{aligned} \mathrm{D}(-\Delta^{(1)}_{\alpha,Y}) = \{g \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus Y)| \\ g'(y_j+) - g'(y_j-) = \alpha_j g(y_j),\ g'(x) = 0\}. \end{aligned}$$

Properties of the ground state for $-\Delta_{\alpha,Y}$ and $-\Delta^{(1)}_{\alpha,Y}$ are the same, because we chose x in such a way the ψ fulfills Neumann condition for the wave function of the ground state. Next we write $-\Delta^{(1)}_{\alpha,Y}$ as direct sum of two self-adjoint operators

$$-\Delta^{(1)}_{\alpha,Y} = -\Delta_{\beta,Y_1,x} \oplus -\Delta_{\gamma,Y_2,x} \tag{3.1}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\beta = (\alpha_1, \alpha_2, ..., \alpha_i)$, $\gamma = (\alpha_{i+1}, \alpha_{i+2}, ..., \alpha_n)$, $Y = (y_1, y_2, ..., y_n)$, $Y_1 = (y_1, y_2, ..., y_i)$, $Y_2 = (y_{i+1}, y_{i+2}, ..., y_n)$, $1 \leq i \leq n$, and

$$\begin{aligned} \mathrm{D}(-\Delta_{\beta,Y_1,x}) &= \{g \in H^{2,1}((-\infty, x)) \cap H^{2,2}((-\infty, x) \setminus Y_1)| \\ &\quad g'(y_j+) - g'(y_j-) = \alpha_j g(y_j), g'(x) = 0\} \\ \mathrm{D}(-\Delta_{\gamma,Y_2,x}) &= \{g \in H^{2,1}((x, \infty)) \cap H^{2,2}((x, \infty) \setminus Y_2)| \\ &\quad g'(y_j+) - g'(y_j-) = \alpha_j g(y_j), g'(x) = 0\} \end{aligned}$$

Now we define $-\Delta^{(2)}_{\alpha,Y}$

$$-\Delta^{(2)}_{\alpha,Y} = -\Delta_{\beta,Y_1,x} \oplus -\Delta^{(x,y)}_N \oplus -\Delta_{\gamma,Y_2,x} \tag{3.2}$$

where $-\Delta^{x,y}_N$ is Neumann Laplacian at the interval (x, y) (for the definition see [3, Section XIII.15]). We choose $x \leq y$ and denote $y - x = \eta$. The domain of the newly constructed operator is

$$\mathrm{D}(-\Delta^{(2)}_{\alpha,Y}) = \mathrm{D}(-\Delta_{\beta,Y_1,x}) \oplus \mathrm{D}(-\Delta^{(x,y)}_N) \oplus \mathrm{D}(-\Delta_{\gamma,Y_1,x}).$$

Neumann Laplacian is a positive operator, in particular, all its eigenvalues are positive. We are interested in the ground state of $-\Delta^{(2)}_{\alpha,Y}$. The discrete spectrum of $-\Delta^{(2)}_{\alpha,Y}$ is the union of discrete spectra of the orthogonal sum components,

$$\sigma_p(-\Delta^{(2)}_{\alpha,Y}) = \sigma_p(-\Delta_{\beta,Y_1,x}) \cup \sigma_p(-\Delta^{(x,y)}_N) \cup \sigma_p(-\Delta_{\gamma,Y_2,x})$$

The ground state of $-\Delta^{(2)}_{\alpha,Y}$ is negative which implies that the ground state is not affected by $\Delta^{(x,y)}_N$ because $\Delta^{(x,y)}_N \geq 0$. Next we define $-\Delta^{(3)}_{\alpha,Y}$ which is obtained from the operator $-\Delta^{(2)}_{\alpha,Y}$ by removing the Neumann conditions at the points x and y. It can be easily seen that $-\Delta^{(3)}_{\alpha,Y}$ is equal to $-\Delta_{\alpha,Y'}$ where $Y' = (y_1, y_2, \ldots, y_i, y_{i+1} + \eta, \ldots, y_n + \eta)$. According to Lemma 3.1 we have $-\Delta^{(2)}_{\alpha,Y} \leq -\Delta^{(3)}_{\alpha,Y}$. Also as we pointed out earlier we can write $-\Delta_{\alpha,Y} = -\Delta^{(1)}_{\alpha,Y}$ and $-\Delta^{(3)}_{\alpha,Y} = -\Delta_{\alpha,Y'}$. In combination with minmax principle [3, Section XIII.1] we arrived at the inequality:

$$\inf \sigma(-\Delta_{\alpha,Y}) = \inf \sigma(-\Delta^{(1)}_{\alpha,Y}) \leq \inf \sigma(-\Delta^{(2)}_{\alpha,Y}) \leq \inf \sigma(-\Delta^{(3)}_{\alpha,Y} \equiv -\Delta_{\alpha,Y'}) \tag{3.3}$$

From this we have $-\Delta_{\alpha,Y} \leq -\Delta_{\alpha,Y'}$. We note that

$$Y' = (y'_1, y'_2, \ldots, y'_n) = (y_1, y_2, \ldots, y_{i+1} + \eta, \ldots, y_n + \eta),$$

which implies that $y_j - y_{j+1} \leq y'_j - y'_{j+1}$ for all $j \in \{1, 2, \ldots, n-1\}$ which completes the proof. □

Remark 1. Now we would like to note a few things concerning the assumptions about the first derivative of the wave function. According to Theorem 2.6 we can write eigenfunction of the ground state as

$$\psi_0(x) = \sum_{i=1}^{N} c_i G_{k_0}(x - y_i) = \sum_{i=1}^{N} c_i \frac{i}{2k} e^{ik|x-y_i|},$$

where $c_k > 0$ for all k and $\Im k_0 > 0$. The vector $C = (c_1, \ldots, c_N)$ according to Theorem 2.6 fulfills $\Gamma_{\alpha,Y}(k_0)C = 0$. From this we have

$$c_k \left(\frac{1}{\alpha_k} + \frac{i}{2k} \right) + \sum_{j=1, j\neq k}^{N} c_j \frac{i}{2k} e^{ik|y_k - y_j|} = 0. \tag{3.4}$$

After algebraic manipulations we get

$$\begin{aligned} c_k &= -\left(\frac{1}{\alpha_k} + \frac{i}{2k} \right)^{-1} \sum_{j=1, j\neq k}^{N} c_j \frac{i}{2k} e^{ik|y_k - y_j|} \\ &= -\left(\frac{i\alpha_k}{2k + i\alpha_k} \right) \sum_{j=1, j\neq k}^{N} c_j e^{ik|y_k - y_j|}, \end{aligned}$$

We can choose the vector $C = (c_1, \ldots, c_N)$ to fulfill $\sum_{i=1}^{N} c_i = 1$. We are interested in the values of $\psi'(y_k-)$ and $\psi'(y_{k+1}+)$. A simple calculation will show that for $x \in \mathbb{R} \setminus Y$ we have

$$\psi_0'(x) = \sum_{j=1}^{N} c_j \frac{i}{2k} ik \operatorname{sgn}(x - y_j) e^{ik|x - y_j|}, \tag{3.5}$$

We substitute $x - y_k = -\epsilon$ with $\epsilon > 0$ into (3.5) obtaining

$$\psi_0'(y_k - \epsilon) = \sum_{j=1}^{N} c_j \frac{i}{2k} ik \operatorname{sgn}(y_k - \epsilon - y_j) e^{ik|y_k - \epsilon - y_j|}$$

$$= \sum_{j=1}^{k-1} c_j \frac{i}{2k} ik e^{ik|y_k - \epsilon - y_j|} - c_j \frac{i}{2k} ik e^{ik\epsilon} - \sum_{j=k+1}^{N} c_j \frac{i}{2k} ik e^{ik|y_k - \epsilon - y_j|}$$

$$= -\sum_{j=1}^{k-1} c_j \frac{1}{2} e^{ik|y_k - \epsilon - y_j|} + c_k \frac{1}{2} e^{ik\epsilon} + \sum_{j=k+1}^{N} c_j \frac{1}{2} e^{ik|y_k - \epsilon - y_j|}$$

From this equation we can see that our premise about $\psi'(y_k-)$ will be fulfilled when $c_k e^{ike} \geq |\sum_{j=1, j\neq k}^{N} c_j \frac{1}{2} e^{ik|y_k - \epsilon - y_j|}|$. This is true when we place point interactions far enough from each other. Similar conclusion can be made about $\psi'(y_{k-1}+)$ in the same way as for $\psi'(y_k-)$ so we omit the details.

The previous result is special in several respects. First of all, it requires the derivative sign assumption. The relation between Y_1 and Y_2 is also particular: the latter is obtained from the former y splitting it and shifting one group of points. This is rather a problem of an elegant formulation, of course, since one can compose more complicated changes from elementary ones. What is more important, finally, is that the inequality we have obtained is not sharp. All these deficiencies can be removed if we use another method based on the secular equation derived from Krein's formula according to Theorem 2.6.

Theorem 3.2. *Let* $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ *be the point interaction Hamiltonians defined above. Suppose that* $\operatorname{card} Y_1 = \operatorname{card} Y_2$, $\alpha_k < 0$ *for all* k *and that* $y_{1,i} - y_{1,j} \leq y_{2,i} - y_{2,j}$ *holds for all* i, j *and* $y_{1,i} - y_{1,j} < y_{2,i} - y_{2,j}$ *for at least one pair of* i, j *then the ground states of the operators* $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ *satisfy* $\min \sigma_p(-\Delta_{\alpha,Y_1}) < \min \sigma_p(-\Delta_{\alpha,Y_2})$

Proof. We are interested in behavior of the ground state of $-\Delta_{\alpha,Y}$. In view of the secular equation $\det \Gamma_{\alpha,Y}(\kappa) = 0$, we have to investigate the lowest eigenvalue λ_0 of $\Gamma_{\alpha,Y}(\kappa)$. The latter is given by

$$\lambda_0(\alpha, Y; \kappa) = \min_{|C|=1} (C, \Gamma_{\alpha,Y}(\kappa) C) \tag{3.6}$$

where $C \in \mathbb{C}^n$ with $|C| = 1$. The energy of the ground state $-\kappa^2$ corresponds to the value of κ which fulfills $\lambda_0(\alpha, Y; \kappa) = 0$. From Theorem 2.6 (after introducing substitution $-\kappa = ik < 0$) we know that

$$\Gamma_{\alpha,Y}(\kappa)_{ij} = -\frac{\delta_{ij}}{\alpha_i} - \frac{1}{2\kappa} e^{-\kappa L_{ij}}, \tag{3.7}$$

where $L_{ij} = |y_i - y_j|$. From this we can calculate $(C, \Gamma_{\alpha,Y}(\kappa) C)$ as

$$(C, \Gamma_{\alpha,Y}(\kappa) C) = \sum_{i=1}^{N} |c_i|^2 \left(-\frac{1}{\alpha_i} - \frac{1}{2\kappa} \right) - 2 \sum_{i=1}^{N} \sum_{j=1}^{i-1} Re \left(\overline{c_i} c_j \frac{e^{-\kappa L_{ij}}}{2\kappa} \right), \tag{3.8}$$

We will notice that the semigroup $\left\{ e^{-t\Gamma_{\alpha,Y(\kappa)}} \right\}$ is positivity improving according to [3, Section XIII.12] and [3, Problem XIII.97] and therefore C for which the minimum is achieved can be chosen strictly positive, i.e. $c_i > 0$ for all $i = 1, \ldots, n$. Put together we have

$$\lambda_0(\alpha, Y; \kappa) = \min_{|C|=1, C>0} (C, \Gamma_{\alpha,Y}(\kappa) C). \tag{3.9}$$

Now we take two configurations of point interactions, namely (α, Y), $(\alpha, \tilde{Y})$ which fulfill $L_{ij} \leq \tilde{L_{ij}}$ for all (i,j) and $L_{ij} < \tilde{L_{ij}}$ for at least one pair of (i,j). For any fixed $C > 0$ we have

$$(C, \Gamma_{\alpha,Y}(\kappa)C) < (C, \Gamma_{\alpha,\tilde{Y}}(\kappa)C) \tag{3.10}$$

which can be seen from explicit form of $(C, \Gamma_{\alpha,Y}(\kappa)C)$ given above and thus we see that an increase in distance will make the exponential smaller. Because this is valid for every fixed C we have the second term smaller, from which we have

$$\lambda_0(\alpha, Y; \kappa) < \lambda_0(\alpha, \tilde{Y}; \kappa). \tag{3.11}$$

Sharp inequality is a consequence of the fact that we know that C for which the minimum is achieved does exist. This inequality implies the statement of the theorem. □

3.2 Finite number of point interactions on the halfline

In this section we will discuss the ground state of N point interactions placed on the halfline. Properties of the ground state have similar properties as on the line with small differences based on the condition which we put at the endpoint of the halfline. Without loss of generality we can renumerate point interactions to fullfill $y_i < y_{i+1}$ $\forall i \in \{1, 2, \ldots, n-1\}$.

First we will discuss the situation with Neumann condition at the endpoint. We define self-adjoined operator $-\Delta_{\alpha,Y}$ introduced in the first chapter with the domain of

$$\begin{aligned} \mathrm{D}(-\Delta_{\alpha,Y}) = \{g \in H^{2,1}(\mathbb{R}_0^+) \cap H^{2,2}(\mathbb{R}_0^+ \setminus Y) | \\ g'(y_j+) - g'(y_j-) = \alpha_j g(y_j), g'(0) = 0\} \end{aligned} \tag{3.12}$$

where $Y = \{y_1, y_2, \ldots, y_n\}$ $y_i > 0$ $\forall i \in \{1, 2, \ldots, n\}$

Theorem 3.3. *Let $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ be operators defined above. Suppose either:*
a) that there is an i such that $y_{2,j} = y_{1,j}$ for $j = 1, \ldots, i$ and $y_{2,j} = y_{1,j} + \eta$ for $j = i+1, \ldots, N$ and further that for the ground state of the first operator we have $\psi'(y_i+) < 0$ and $\psi'(y_{i+1}-) > 0$ or
b) that for $y_{2,j} = y_{1,j} + \eta$ for all $j \in \mathbb{N}$.

If either a) or b) is fulfilled for $\eta \geq 0$ *then the ground states of the two operators* $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ *is* $\min \sigma_p(-\Delta_{\alpha,Y_1}) \leq \min \sigma_p(-\Delta_{\alpha,Y_2})$

Proof. The proof is quite similar as proof for the case on the line. The only difference is when we are enlarging distance between the endpoint and the first point interaction. We split the proof to two parts:
a) Changing the distance between the endpoint and the first interaction
b) Changing the distance between two point interactions

a) First we have the operator $-\Delta_{\alpha,Y}$ defined above. Because we already have Neumann boundary condition fulfilled at the endpoint of the halfline, we can construct new operator as

$$-\Delta^{(1)}_{\alpha,Y} = -\Delta_{\alpha,Y'} \oplus -\Delta^{(0,\eta)}_{N} \tag{3.13}$$

where $-\Delta^{(0,\eta)}_{N}$ is Neumann Laplacian with the domain $\mathrm{D}(-\Delta^{(0,\eta)}_{N}) = \{g \in H^{2,2}((0,\eta)| g'(0) = 0, g'(\eta) = 0\}$ and the operator $-\Delta_{\alpha,Y'}$ with the domain

$$\begin{aligned} \mathrm{D}(-\Delta_{\alpha,Y'}) = \{g \in H^{2,1}((\eta,\infty)) \cap H^{2,2}((\eta,\infty) \setminus Y')| \\ g'(y'_j+) - g'(y'_j-) = \alpha_j g(y'_j), g'(\eta) = 0\}, \end{aligned} \tag{3.14}$$

where $Y' = Y + \eta$. We know that the eigenvalues of $-\Delta^{(1)}_{\alpha,Y}$ are calculated as union of eigenvalues of operators which our operator is made from. That is

$$\sigma_p(-\Delta^{(1)}_{\alpha,Y}) = \sigma_p(-\Delta^{(0,\eta)}_{N}) \cup \sigma_p(-\Delta_{\alpha,Y'})$$

Because Neumann Laplacian have nonnegative eigenvalues and because we are interested in negative eigenvalues, negative point spectrum remains the same that is why we can write $\min \sigma_p(-\Delta_{\alpha,Y}) = \min \sigma_p(-\Delta^{(1)}_{\alpha,Y})$. Next we define $-\Delta^{(2)}_{\alpha,Y}$ as $-\Delta^{(1)}_{\alpha,Y}$ without Neumann condition in η. We can easily see that $-\Delta^{(2)}_{\alpha,Y} = -\Delta_{\alpha,Y+\eta}$. Put all together we have for the ground state

$$\min \sigma_p(-\Delta_{\alpha,Y}) = \min \sigma_p(-\Delta^{(1)}_{\alpha,Y}) \leq \min \sigma_p(-\Delta^{(2)}_{\alpha,Y}) = \min \sigma_p(-\Delta_{\alpha,Y+\eta})$$

From there we have $-\Delta_{\alpha,Y} \leq -\Delta_{\alpha,Y'}$. We note $Y' = (y'_1, y'_2, \ldots, y'_n) = Y + \eta$. The relation $\eta \leq 0$ implies that $y_1 - 0 \leq y'_1 - 0$ which completes the proof.
b) Same as for the problem on the line, first we note that the function for the ground state can be chosen strictly positive, i.e. $\forall x \Rightarrow \psi(x) > 0$. Also we have

$$-\Delta_{\alpha,Y}\psi = -E_0\psi \tag{3.15}$$

where $E_0 > 0$. This means

$$\frac{d^2}{dx^2}\psi = E_0\psi \tag{3.16}$$

for all $x \in \mathbb{R}\setminus Y$. From this we can conclude that ψ is convex for $\forall x \in \mathbb{R}\setminus Y$. By assumptions we have $\psi(y_i+) > 0$ and $\psi(y_{i+1}-) < 0$. Put together we have that we can find $x \in (y_i, y_{i+1})$ such that $\psi'(x) = 0$. Now we add the Neumann condition at x to $-\Delta_{\alpha,Y}$. We denote the operator with Neumann condition as $-\Delta^{(1)}_{\alpha,Y}$. The domain of $-\Delta^{(1)}_{\alpha,Y}$ is

$$\begin{aligned} \mathrm{D}(-\Delta^{(1)}_{\alpha,Y}) = \{g \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R}\setminus Y)| \\ g'(y_j+) - g'(y_j-) = \alpha_j g(y_j),\, g'(0) = 0,\, g'(x) = 0\}. \end{aligned}$$

Properties of the ground state for $-\Delta_{\alpha,Y}$ and $-\Delta^{(1)}_{\alpha,Y}$ are the same, because we chose x in such a way the ψ fulfill Neumann condition for the wave function of the ground state. Next we write $-\Delta^{(1)}_{\alpha,Y}$ as direct sum of two self-adjoint operators

$$-\Delta^{(1)}_{\alpha,Y} = -\Delta_{\beta,Y_1,x} \oplus -\Delta_{\gamma,Y_2,x} \tag{3.17}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\beta = (\alpha_1, \alpha_2, ..., \alpha_i)$, $\gamma = (\alpha_{i+1}, \alpha_{i+2}, ..., \alpha_n)$, $Y = (y_1, y_2, ..., y_n)$, $Y_1 = (y_1, y_2, ..., y_i)$, $Y_2 = (y_{i+1}, y_{i+2}, ..., y_n)$, $1 \le i \le n$, and

$$\begin{aligned} \mathrm{D}(-\Delta_{\beta,Y_1,x}) = \{g \in H^{2,1}((0,x)) \cap H^{2,2}((0,x)\setminus Y_1)| \\ g'(y_j+) - g'(y_j-) = \alpha_j g(y_j),\, g'(0) = 0,\, g'(x) = 0\} \\ \mathrm{D}(-\Delta_{\gamma,Y_2,x}) = \{g \in H^{2,1}((x,\infty)) \cap H^{2,2}((x,\infty)\setminus Y_2)| \\ g'(y_j+) - g'(y_j-) = \alpha_j g(y_j),\, g'(x) = 0\} \end{aligned}$$

Now we define $-\Delta^{(2)}_{\alpha,Y}$

$$-\Delta^{(2)}_{\alpha,Y} = -\Delta_{\beta,Y_1,x} \oplus -\Delta^{(x,y)}_N \oplus -\Delta_{\gamma,Y_2,x} \tag{3.18}$$

where $-\Delta^{x,y}_N$ is Neumann Laplacian at the interval (x,y). We choose $x \le y$ and denote $y - x = \eta$. The domain of the newly constructed operator is

$$\mathrm{D}(-\Delta^{(2)}_{\alpha,Y}) = \mathrm{D}(-\Delta_{\beta,Y_1,x}) \oplus \mathrm{D}(-\Delta^{(x,y)}_N) \oplus \mathrm{D}(-\Delta_{\gamma,Y_1,x}).$$

Neumann Laplacian is a positive operator, in particular, all its eigenvalues are positive. We are interested in the ground state of $-\Delta^{(2)}_{\alpha,Y}$. The discrete spectrum of $-\Delta^{(2)}_{\alpha,Y}$ is the union of discrete spectra of the orthogonal sum components,

$$\sigma_p(-\Delta^{(2)}_{\alpha,Y}) = \sigma_p(-\Delta_{\beta,Y_1,x}) \cup \sigma_p(-\Delta^{(x,y)}_N) \cup \sigma_p(-\Delta_{\gamma,Y_2,x})$$

The ground state of $-\Delta^{(2)}_{\alpha,Y}$ is negative which implies that the ground state is not affected by $\Delta^{(x,y)}_N$ because $\Delta^{(x,y)}_N \geq 0$. Next we define $-\Delta^{(3)}_{\alpha,Y}$ which is obtained from the operator $-\Delta^{(2)}_{\alpha,Y}$ by removing the Neumann conditions at the points x and y. It can be easily seen that $-\Delta^{(3)}_{\alpha,Y}$ is equal to $-\Delta_{\alpha,Y'}$ where $Y' = (y_1, y_2, \ldots, y_i, y_{i+1} + \eta, \ldots, y_n + \eta)$. According to Lemma 3.1 we have $-\Delta^{(2)}_{\alpha,Y} \leq -\Delta^{(3)}_{\alpha,Y}$. Also as we pointed out earlier we can write $-\Delta_{\alpha,Y} = -\Delta^{(1)}_{\alpha,Y}$ and $-\Delta^{(3)}_{\alpha,Y} = -\Delta_{\alpha,Y'}$. In combination with minmax principle [3, Section XIII.1] we arrived at the inequality:

$$\inf \sigma(-\Delta_{\alpha,Y}) = \inf \sigma(-\Delta^{(1)}_{\alpha,Y}) \leq \inf \sigma(-\Delta^{(2)}_{\alpha,Y}) \leq \inf \sigma(-\Delta^{(3)}_{\alpha,Y} \equiv -\Delta_{\alpha,Y'}) \tag{3.19}$$

From this we have $-\Delta_{\alpha,Y} \leq -\Delta_{\alpha,Y'}$. We note that

$$Y' = (y'_1, y'_2, \ldots, y'_n) = (y_1, y_2, \ldots, y_{i+1} + \eta, \ldots, y_n + \eta),$$

which implies that $y_j - y_{j+1} \leq y'_j - y'_{j+1}$ for all $j \in \{1, 2, \ldots, n-1\}$ which completes the proof. □

Next we will discuss the case when at the endpoint of the halfline is the Robin condition $g'(0+) = \alpha_0 g(0)$ where $\alpha_0 < 0$. We will show that for this similar properties are met. This means that we will introduce the operator $-\Delta_{\alpha_0,\alpha,Y}$ with the domain $\mathrm{D}(-\Delta_{\alpha_0,\alpha,Y}) = \{g \in H^{2,1}(\mathbb{R}^+_0) \cap H^{2,2}(\mathbb{R}^+_0 \setminus Y) | g'(y_j+) - g'(y_j-) = \alpha_j g(y_j),\, g'(0+) = \alpha_0 g(0)\}$.

Theorem 3.4. *Let* $-\Delta_{\alpha_0,\alpha,Y_1}$, $-\Delta_{\alpha_0,\alpha,Y_2}$ *be operators defined above. Suppose that* $\operatorname{card} Y_1 = \operatorname{card} Y_2$, $\alpha_k < 0$ *for all* $k = 0, 1, \ldots, n$, $y_{1,i} - y_{1,j} \leq y_{2,i} - y_{2,j}$, $y_{1,1} \leq y_{2,1}$ *holds for all* i, j *and* $y_{1,i} - y_{1,j} < y_{2,i} - y_{2,j}$, *for at least one pair of* i, j *then the ground states of the operators* $-\Delta_{\alpha_0,\alpha,Y_1}$, $-\Delta_{\alpha_0,\alpha,Y_2}$ *is* $\min \sigma_p(-\Delta_{\alpha_0,\alpha,Y_1}) < \min \sigma_p(\alpha_0, -\Delta_{\alpha,Y_2})$

Proof. First we restate the problem to the operator on the line. We have the operator $-\Delta_{\alpha_0,\alpha,Y}$ defined above. We define operator $-\Delta_{\alpha',Y'}$ where

$\alpha' = \{\alpha_n, \ldots, \alpha_1, 2\alpha_0, \alpha_1, \alpha_n\}$ and $Y = \{-y_n, \ldots, -y_1, 0, y_1, y_n\}$ with the domain

$$\mathrm{D}(-\Delta_{\alpha',Y'}) = \{g \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus Y')| \\ g'(y_j+) - g'(y_j-) = \alpha'_j g(y_j)\}.$$

Operator $-\Delta_{\alpha',Y'}$ commutes with the parity operator which implies that when $g(x)$ solves

$$-\Delta_{\alpha',Y'} g = -Eg \tag{3.20}$$

then $g(x) = g(-x)$. We prove that when $g(x)$ solves (3.20), $E > 0$ then $f(x) = \frac{g(x)+g(-x)}{2} = g(x)$ for $x \in \mathbb{R}_0^+$ solves $-\Delta_{\alpha_0,\alpha,Y} f = -Ef$. We can see that every condition of the type $f'(y_j+) - f'(y_j-) = \alpha_j f(y_j)$ is fulfilled, because $g(x)$ fulfills $g'(y_j+) - g'(y_j-) = \alpha'_j g(y_j)$. The boundary condition $f'(0+) = \alpha_0 f(0)$ is fullfiled because $g'(0+) - g'(0-) = 2\alpha_0 g(0)$ and $g'(x) = -g'(-x)$. The operator $-\Delta_{\alpha',Y'}$ fulfills conditions of Theorem 3.2 that means that increase in distance between α'_i, α'_{i+1} and α'_{n+i+1}, α'_{n+i+2} results in elevation of the energy of the ground state of $-\Delta_{\alpha',Y'}$ which is equivalent to increase in distance between α_i, α_{i+1} for the operator $-\Delta_{\alpha_0,\alpha,Y}$ and the same elevation in its ground state energy. □

Remark 2. As with the problem on the line we could formulate weaker version of the previous theorem with condition placed on the first derivative around the point of interactions, which could be proven in the same way as on the line. The advantage of this approach is that the theorem will be valid for any condition at the endpoint of the halfline. On the other hand we get only unsharp inequality and the first point interaction cannot be moved.

Theorem 3.5. *Let $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ be operators with the domain $\mathrm{D}(-\Delta_{\alpha,Y_i}) = \{g \in H^{2,1}(\mathbb{R}_0^+) \cap H^{2,2}(\mathbb{R}_0^+ - Y_i)| g'(y_{i,j}+) - g'(y_{i,j}-) = \alpha_j g(y_{i,j})\}$. Suppose that there is an i such that $y_{2,j} = y_{1,j}$ for $j = 1, \ldots, i$ and $y_{2,j} = y_{1,j} + \eta$ for $j = i+1, \ldots, N$. Suppose further that for the ground state of the first operator we have $\psi'(y_i+) < 0$ and $\psi'(y_{i+1}-) > 0$. If $\eta \geq 0$ then the ground states of the two operators $-\Delta_{\alpha,Y_1}$, $-\Delta_{\alpha,Y_2}$ we have $\min \sigma_p(-\Delta_{\alpha,Y_1}) \leq \min \sigma_p(-\Delta_{\alpha,Y_2})$.*

Proof. Proof is analogous to the one of finitely many interactions on the line so we omit the details. □

3.3 Finite number of point interactions on a star graph

In this section we will discuss properties of the ground state on the star graph. We have a star graph from N halflines connected by the endpoint of the halflines. We focus our attention to the condition at the endpoint of a star graph.We impose two conditions. The first is

$$\psi_i(0) = \psi_j(0) = \psi(0), \tag{3.21}$$

and the second is

$$\sum_{i=1}^{N} \psi_i'(0) = \alpha_0 \psi(0) \tag{3.22}$$

where ψ_i is wave function on the i-th halfline and $\alpha_0 < 0$. We denote the operator

$$-\Delta_{\alpha_0,\alpha,Y^{(i)}} = -\frac{d^2}{dx^2} \tag{3.23}$$

where $\alpha = \alpha_{i,j}$, $Y^{(i)} = y_{j,k}^{(i)}$ $i \in \{1,\ldots,n\}, j \in \{1,\ldots,k_n\}$ with the domain

$$\mathrm{D}(-\Delta_{\alpha_0,\alpha,Y^{(j)}}) = \{g(x) \in \bigcup_{i\in N}^{\oplus} \mathrm{D}(-\Delta_{\alpha_i,Y_i^{(j)}})\} \tag{3.24}$$

where $-\Delta_{\alpha_i,Y_i^{(j)}}$ is the operator on the i-th halfline. Without loss of generality we can presume $y_{1,j} < y_{2,j} < \ldots < y_{n,j}$ for all j. Similarly as the problem on the halfline there can be proven that for $\alpha_{i,j}, \alpha_{k,l} < 0$ for all i, j, k, l, where first index is number of the point interaction on the halfline and the second is the number of the halfline, decreasing distance between 2 point interactions on the one halfline will lead to decrease of the energy of the ground state. This can be easily proven in the same manner as the problem for the line. That is

Theorem 3.6. *Let $-\Delta_{\alpha_0,\alpha,Y^{(i)}}$ be the operator defined above. Suppose that* $\operatorname{card} Y^{(1)} = \operatorname{card} Y^{(2)}$*, $\alpha_0 < 0$, $\alpha_{k,l} < 0$ for all k, l. Suppose*

$$\forall j\, \exists k_j\, \forall l \in \{1,\ldots k_j\}\; y_{l,j}^{(1)} = y_{l,j}^{(2)} \tag{3.25}$$

and

$$\forall j\, \forall i \in \{k_{j+1}, k_n\}\, \exists \eta_j \geq 0,\; y_{i,j}^{((1))} = y_{i,j}^{(2)} + \eta_j. \tag{3.26}$$

Suppose further that for the ground state of the first operator we have $\psi'_j(y_{k_j,j}+) < 0$ *and* $\psi'_j(y_{k_{j+1},j}-) > 0$. *Then the ground states of the operators* $-\Delta_{\alpha_0,\alpha,Y^{(1)}}$, $-\Delta_{\alpha_0,\alpha,Y^{(2)}}$ *satisfy*

$$\min \sigma_p(-\Delta_{\alpha_0,\alpha,Y^{(1)}}) \leq \min \sigma_p(-\Delta_{\alpha_0,\alpha,Y^{(2)}}) \tag{3.27}$$

Proof. We create the operator $-\Delta_{\alpha_0,\alpha,Y^{(i)}}$ as

$$-\Delta_{\alpha_0,\alpha,Y^{(i)}} = -\Delta_{\alpha_1,Y_1^{(i)}} \bigoplus -\Delta_{\alpha_2,Y_2^{(i)}} \bigoplus \ldots \bigoplus -\Delta_{\alpha_N,Y_N^{(i)}} \tag{3.28}$$

where $-\Delta_{\alpha_i,Y_k^{(i)}}$ is self-adjoint operator describing point interactions on the k-th halfline. We know that $\sigma_p(-\Delta_{\alpha,Y^{(i)}}) = \bigcup_{k\in N} \sigma_p(-\Delta_{\alpha_k,Y_k^{(i)}})$. From this we have $\min \sigma_p(-\Delta_{\alpha,Y^{(i)}}) = \min_k \sigma_p(-\Delta_{\alpha_k,Y_k^{(i)}})$. Now we use Theorem 3.5 on $-\Delta_{\alpha_0,\alpha_k,Y_k^{(i)}}$ which completes the proof. □

Chapter 4

Examples

4.1 Solution of two point interactions on the line

We are interested in the ground state of the operator $-\Delta_{\alpha,Y}$ where $\alpha = \{\alpha_1, \alpha_2\}$ and $Y = \{y_1, y_2\}$. According to Theorem 2.6 we know that every eigenvalue k^2 of the point spectrum of the operator $-\Delta_{\alpha,Y}$, is a solution of the secular equation $\det \Gamma_{\alpha,Y}(k) = 0$, where $\{\Gamma_{\alpha,Y}(k)\}_{i,j} = -[\frac{\delta_{i,j}}{\alpha_i} + \frac{e^{ik|y_i - y_j|}}{2ik}]$. Without loss of generality we can choose $y_1 \leq y_2$. We introduce substitution $-\kappa = ik < 0, y_1 - y_2 = L$. We have $\Im k > 0$, because we are interested in the ground state which has negative eigenvalue. The secular equation is explicitly

$$\begin{vmatrix} \frac{1}{\alpha_1} - \frac{1}{2\kappa} & \frac{1}{2\kappa}e^{-\kappa L} \\ \frac{1}{2\kappa}e^{-\kappa L} & \frac{1}{\alpha_2} - \frac{1}{2\kappa} \end{vmatrix} = (\frac{1}{\alpha_1} - \frac{1}{2\kappa})(\frac{1}{\alpha_2} - \frac{1}{2\kappa}) - \frac{e^{-2\kappa L}}{4\kappa^2} = 0, \qquad (4.1)$$

which is trancendent equation for κ. However, we can say a few things about the ground state introducing the substitution $\kappa L = L'$ where $L' > 0$. According to Theorem 2.6 and because we know that $-\infty < \alpha_i$ we can infer that every eigenvalue of the operator $-\Delta_{\alpha,Y}$ fulfills the condition $-\infty < -\kappa^2$. This implies that $\infty > |\kappa|$ from which we can state that $L \searrow 0 \Rightarrow \kappa L \searrow 0$. Solutions for κ with respect of L' are

$$\kappa_1 = \frac{1}{4}(\alpha_1 + \alpha_2 - e^{-2L'}\sqrt{4\alpha_1\alpha_2 e^{2L'} + \alpha_1^2 e^{4L'} - 2\alpha_1\alpha_2 e^{4L'} + \alpha_2^2 e^{4L'}})$$
$$\kappa_2 = \frac{1}{4}(\alpha_1 + \alpha_2 + e^{-2L'}\sqrt{4\alpha_1\alpha_2 e^{2L'} + \alpha_1^2 e^{4L'} - 2\alpha_1\alpha_2 e^{4L'} + \alpha_2^2 e^{4L'}})$$

Note that those are still equations for unknown κ. When we perform the limit for $L' \searrow 0$ we get:

a) $E_0 = -(\frac{\alpha_1+\alpha_2}{2})^2$ for $\alpha_i < 0$ when $L \longmapsto 0+$
b) $E_0 = -(\frac{\alpha_1+\alpha_2}{2})^2$ for $(\alpha_1 < 0 < \alpha_2) \wedge (|\alpha_1| > |\alpha_2|)$ when $L \longmapsto 0+$
c) $E_0 = 0$ for $(\alpha_1 < 0 < \alpha_2) \wedge (|\alpha_1| < |\alpha_2|)$ when $L \longmapsto 0+$

This will also give us a starting point for the numeric solution of the equation (4.1).

There are a few plots to illustrate properties of the ground state. First we will illustrate the claim of Theorem 3.2. On figures 4.1 and 4.2 we can see that increasing distance between two point interactions with negative strength results in increase of the ground state energy. If some $\alpha_j > 0$ the behavior could be different. We can see that on figures 4.3 and 4.4. Also on the figure 4.4 we can see that when total sum of the interaction strength is greater than zero, it results in disappearing of the eigenvalue when the point interactions are close to each other. It is also worth noticing that when the point interactions are far away from each other the ground state energy goes to the value $E_0 \approx -\left(\frac{\alpha}{2}\right)^2$ where α is the minimum of the negative point interaction strength, i.e. $\alpha = \min\{\alpha_j\}$ where $\alpha_j < 0$ for all j.

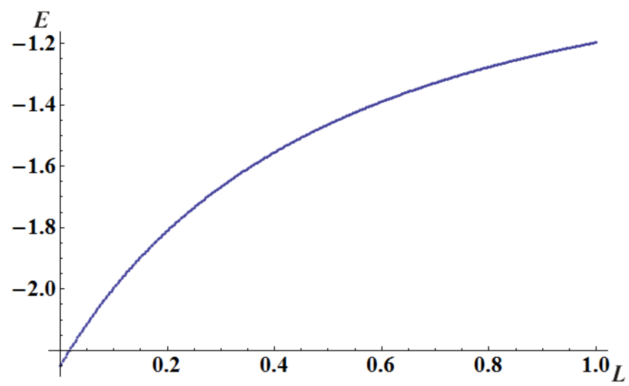

Figure 4.1: Energy of the ground state for $\alpha_1 = -1, \alpha_2 = -2$ as a function of L

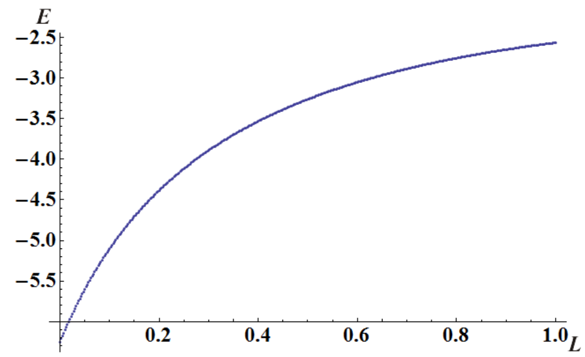

Figure 4.2: Energy of the ground state for $\alpha_1 = -3, \alpha_2 = -2$ as a function of L

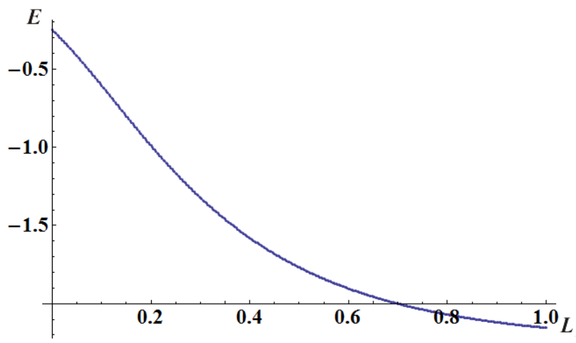

Figure 4.3: Energy of the ground state for $\alpha_1 = -3, \alpha_2 = 2$ as a function of L

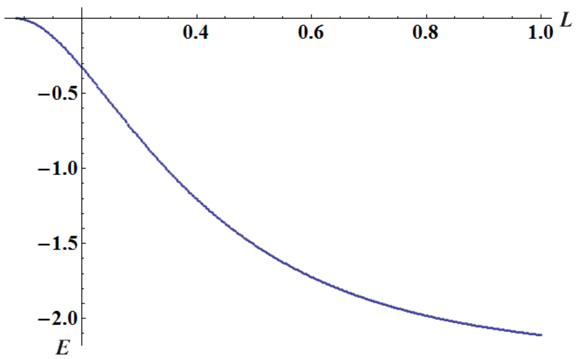

Figure 4.4: Energy of the ground state for $\alpha_1 = -3, \alpha_2 = 4$ as a function of L

4.2 Solution of three point interactions on the line

We are interested in the ground state of the operator $-\Delta_{\alpha,Y}$ where $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$ and $Y = (y_1, y_2, y_3)$. According to Theorem 2.6 we know that every eigenvalue k^2 of the point spectrum of the operator $-\Delta_{\alpha,Y}$ is the solution of the secular equation

$$\det \Gamma_{\alpha,Y}(k) = 0, \tag{4.2}$$

where $\{\Gamma_{\alpha,Y}(k)\}_{i,j} = -\left[\frac{\delta_{i,j}}{\alpha_i} + \frac{e^{ik|y_i-y_j|}}{2ik}\right]$. Without loss of generality we can choose $y_1 \leq y_2 \leq y_3$. The Solution of the equation (4.2) is similar to the problem with two point interactions. We introduce substitution $-\kappa = ik < 0, y_i - y_{i+1} = L_i$. We know that $L_i \in \mathbb{R}_0^+$. As in the two dimensional case we come to the transcendent equation. As in the two point interactions case we introduce substitution $\kappa L_i = -L_i'$ where $L_i' > 0$. According to the Theorem 2.6 and because we know that $-\infty < \alpha_i$ we can infer that every eigenvalue of the operator $-\Delta_{\alpha,Y}$ fulfills $-\infty < -\kappa^2$. This implies that $\infty > |\kappa|$ from which we can state that $L_i \searrow 0 \Rightarrow \kappa L_i \searrow 0$. Solution for κ with respect of L_1', L_2' can be solved exactly however we omit the details of solution because it can be easily done with help of Cardano's method.

As for the two point interactions problem there are a few things we can say from the form of this limit solution for the ground state. These are:

a) $E_0 = (\frac{1}{6}(\alpha_1 + \alpha_2 + \alpha_3 - (-(\alpha_1 + \alpha_2 + \alpha_3)^3)^{\frac{1}{3}} + \frac{(-(\alpha_1+\alpha_2+\alpha_3)^3)^{\frac{2}{3}}}{\alpha_1+\alpha_2+\alpha_3}))^2$ for $\alpha_1 + \alpha_2 + \alpha_3 \leq 0$ when $L_1 \longmapsto 0+$, $L_2 \longmapsto 0+$
b) $E_0 = 0$ for $\alpha_1 + \alpha_2 + \alpha_3 \geq 0$ when $L_1 \longmapsto 0+$, $L_2 \longmapsto 0+$

There are a few plots to illustrate properties of the ground state.

First we will illustrate the claim of Theorem 3.2. On figures 4.5, 4.6 and 4.7 it can be easily seen that increase of distance between point interactions with negative strength results in increase of the energy of the ground state. On practically every figure for the three point interactions it can be seen the same thing as for the case of two point interactions which is that when we place the point interactions far from each other the ground state energy goes to the ground state energy of the point interaction with lowest negative strength, i.e. $E_0 \approx -\left(\frac{\alpha}{2}\right)^2$ where α is $\alpha = \min\{\alpha_j\}$ where $\alpha_j < 0$ for all

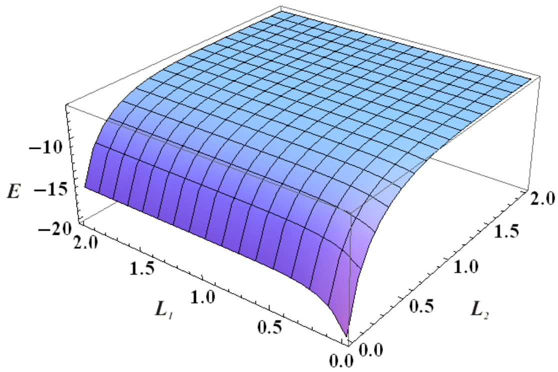

Figure 4.5: Energy of the ground state for $\alpha_1 = -5, \alpha_2 = -3, \alpha_3 = -1$ as a function of L_1 and L_2

j. If some $\alpha_j > 0$ the behavior could be quite different from the claim of Theorem 3.2. This can be seen on figures from 4.8. to 4.13. We can see on the figure 4.8 that when we decrease the distance between the negative point interactions it results in decrease of the ground state energy even if the third point interaction is positive. It is worth mentioning that when the first or last point interaction on the line is positive, i.e. $\alpha_1 > 0$ or $\alpha_N > 0$ respectively, its moving away from the other interaction results in decrease of the ground state energy (see figures 4.10,4.11 and 4.13). It is worth noticing but not at all surprising fact that when we have middle point interaction with strength equal zero the three interactions problem will degenerate to the two interactions problem (see figure 4.12).

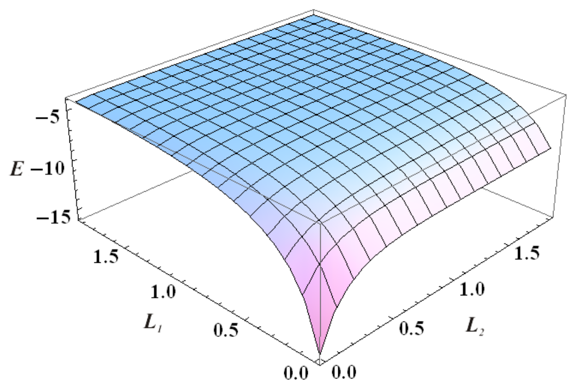

Figure 4.6: Energy of the ground state for $\alpha_1 = -2, \alpha_2 = -2, \alpha_3 = -4$ as a function of L_1 and L_2

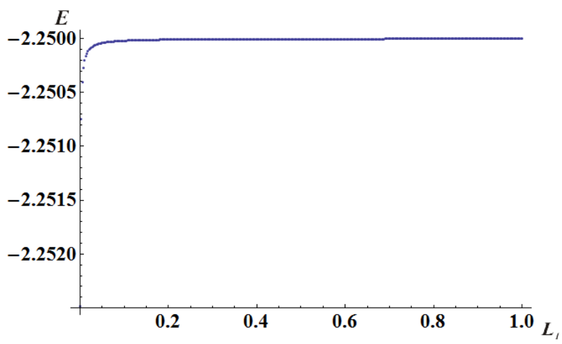

Figure 4.7: Energy of the ground state for $\alpha_1 = -1, \alpha_2 = -2, \alpha_3 = -3$ as a function of L_1 and fixed $L_2 = 5$

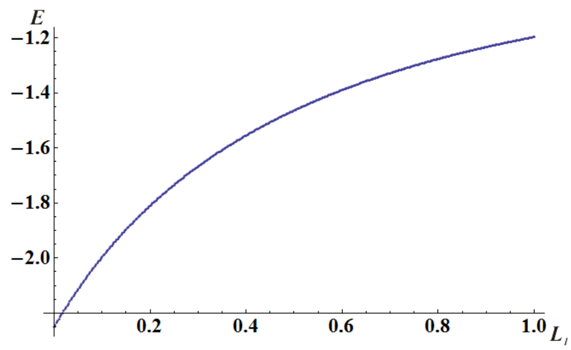

Figure 4.8: Energy of the ground state for $\alpha_1 = -1, \alpha_2 = -2, \alpha_3 = 3$ as a function of L_1 and fixed $L_2 = 5$

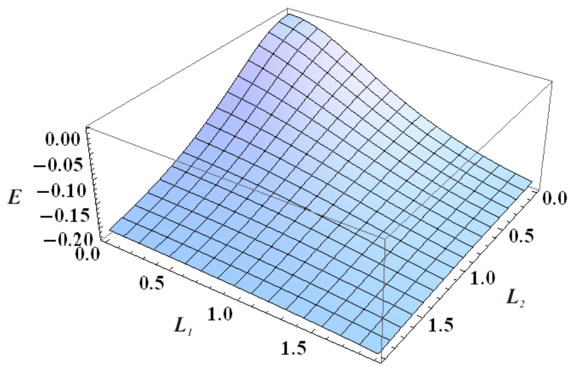

Figure 4.9: Energy of the ground state for $\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = -1$ as a function of L_1 and L_2

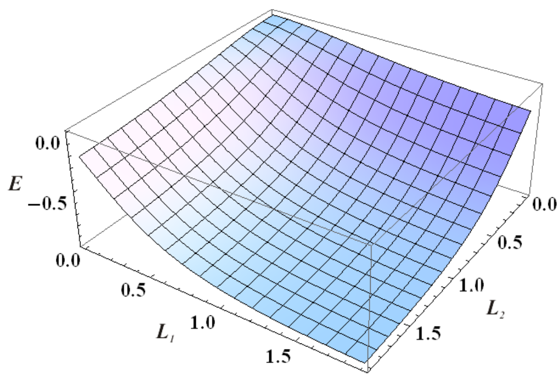

Figure 4.10: Energy of the ground state for $\alpha_1 = 1, \alpha_2 = -2, \alpha_3 = 1$ as a function of L_1 and L_2

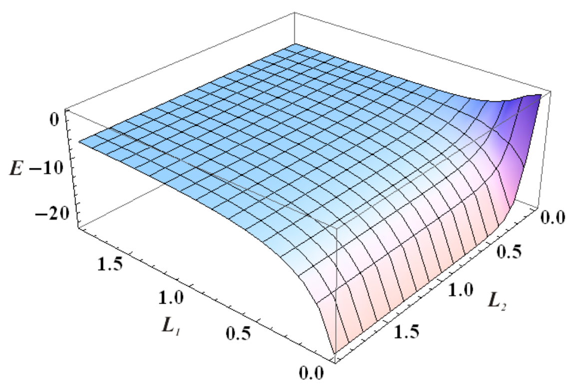

Figure 4.11: Energy of the ground state for $\alpha_1 = -5, \alpha_2 = -5, \alpha_3 = 10$ as a function of L_1 and L_2

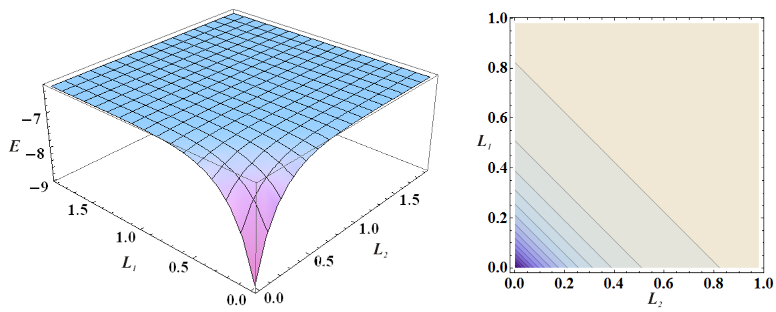

Figure 4.12: Energy of the ground state for $\alpha_1 = -5, \alpha_2 = 10^{-12}, \alpha_3 = -1$ as a function of L_1 and L_2

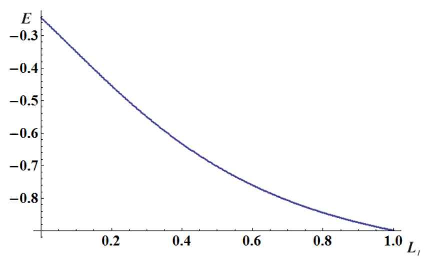

Figure 4.13: Energy of the ground state for $\alpha_1 = 1, \alpha_2 = -2, \alpha_3 = -3$ as a function of L_1 and fixed $L_2 = 5$

4.3 Conclusion

In this work we have shown a relation between the distance of attractive point interactions, i.e. point interactions with negative interaction strength $\alpha_j < 0$, and the energy of the ground state for some one dimensional objects, namely the line, the halfline and also for star graphs under some additional assumptions.

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