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RESEARCH WORK

The effective Hamiltonian in curved quantum waveguides as a consequence of strong resolvent convergence

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Abstract

The Dirichlet Laplacian in a curved two-dimensional strip built along a plane curve is investigated in the limit when the uniform cross-section of the strip diminishes. We show that the Laplacian converges in a strong resolvent sense to the well known one-dimensional Schroedinger operator whose potential is expressed solely in terms of the curvature of the reference curve. In comparison with previous results we allow curves whose curvature is not differentiable.

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Notation index

$\mathcal{B}(\mathcal{H})$	space of bounded operators on Hilbert space $\mathcal{H}$
$\mathbb{C}$	complex numbers
C^2	space of functions having continuous second derivative
C^0	space of continuous functions
$C^0(\bar{\Omega})$	space of uniformly continuous functions on Ω
C^∞	space of smooth funtions
$C_0^\infty(\bar{\Omega})$	subspace of $C^\infty(\Omega)$ defined in section 3.2
$C_c^\infty(\Omega)$	space of smooth functions with compact support in Ω
L^2	space of square-integrable functions
L^∞	space of bounded functions
$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$W^{1,2}$	Sobolev space introduced by Definition 3
$W_0^{1,2}$	Sobolev space introduced in section 3.2
κ	curvature of a curve defined in section 2.1
$\sigma(T)$	spectrum of an operator T
(a,b)	opened interval
∇	gradient operator
Δ_D^Ω	Laplacian on Ω with Dirichlet boundary conditions on $\partial\Omega$
$\xrightarrow{s}$	strong convergence
$\xrightarrow{w}$	weak convergence
$\oplus$	direct sum
$\otimes$	tensor product

1 Introduction

1.1 Motivation

The problem of constrained quantum systems is studied for a long time in different settings. There are works as [2], [10], [11] or [12] that cover many physical problems since they study the general case where the quantum system is constrained to a submanifold of a Riemannian manifold by some localizing potential. In these works they show that the effective Hamiltonian for the quantum system constrained to the submanifold depends not only on the inner properties of the submanifold but also on the extrinsic curvature of the constrained submanifold, on the curvature of the configuration space or the shape of the constraining potential. These are the quantum effects (the uncertainty principle which forbids us to say that the particle is localized on the submanifold plays here the crucial role) that do not occur in classical constrained systems. One can apply these results to problems such as rigid molecules or evolution of molecular systems along reaction paths, and also the problem of quantum waveguides. Quantum waveguides model the systems where the electron moves freely in microstructures of tubular shapes, the quantum wavequides are the subject of this work.

The problem of quantum waveguides, where the constraining potential is replaced by the Dirichlet boundary conditions, is studied in many papers separately. As the first example of such work let us mention the paper [5], where the existence of curvature-induced bound states in thin planar strips was discovered. The later paper [4] summarizes and extends this result, here the Laplacian on curved tubes of a constant cross section in two and three dimensions is investigated. Using the variational estimate, they show that there exists at least one isolated eigenvalue below the bottom of the essential spectrum, provided the tube is non-straight and its curvature vanishes asymptotically. By the means of the perturbation theory some results for the spectrum of thin tubes are derived. In this paper the assumption on the curvature κ of the axis curve of the waveguide is

$$\kappa \in C^2.$$

In paper [1] the Laplacian in the thin tube of $\mathbb{R}^3$ is considered, additionally the torsion effects are investigated. The effective potential q for the motion on the curve is found, it depends on the curvature and the torsion of the curve. It is proved that the eigenvalues and eigenvectors of the initial Hamiltonian converge to the eigenvalues and eigenvectors for the 1D problem with the potential q as the cross section of the tube goes to zero. These results are derived without any assumptions on the smoothness of the curvature, only the boundedness of the curvature is required:

$$\kappa \in L^\infty$$

However, only the waveguides of bounded length are considered and the methods used for proofs in this paper are rather different from the methods used in [4], the Γ-convergence theorem is used.

In conclusion our motivation is to generalize the results of papers [4] or [1]. We want to drop the assumption on the differentiability of the curvature needed in [4], we assume that κ is uniformly continuous,

$$\kappa \in C^0.$$

On the other hand we want to use the basic instruments of functional analysis while proving our statements. We were inspired by the methods used in [6], [8] or [9].

1.2 Main results

In this work we restrict to the planar waveguides, thus the Hamiltonian of the free particle on the thin strip along a plane curve Γ is investigated. We are looking for the approximation of this Hamiltonian in the limit of thin strip where only the movement on the curve is considered, we are looking for so called effective Hamiltonian. The convergence of the resolvent is investigated, we conclude as a main result that the initial Hamiltonian converges to the effective Hamiltonian with respect to the strong resolvent convergence in somewhat generalized way. Since we don't prove the strong resolvent convergence in the ordinary form, stating the consequences of our results is not straightforward.

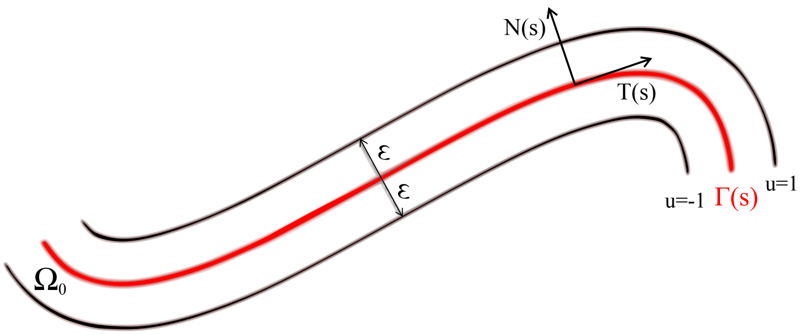
However, for the bounded waveguides the similar situation was considered in [1] and the convergence of eigenvalues and eigenfunctions was proved. In case of the infinite waveguide we don't state any subtle consequence of our result, but we believe that some consequences for the convergence of eigenvalues and eigenfunctions could be found using the results of the paper [13].

The effective Hamiltonian has the expected form, it is dependent on the scalar curvature of Γ and for the non-straight strip this Hamiltonian allows the bound states. However, in this work we assume that the curvature κ of the curve Γ is uniformly continuous and that the strip is non intersecting for thin enough strip, no other assumption are needed. In conclusion we see the sense of our work in proving the known results under the weaker conditions using some interesting mathematical tricks.

We summarize our results in Theorem 9 and the whole Section 4 is dedicated to its proof. The main point of this proof is introducing a function f_ε that is close to the curvature κ in the limit of thin strip but is differentiable. Also a Hilbert space decomposition plays the crucial role in the proof, we were inspired by [8] where this trick was used as well. In Section 2 we introduce the geometry of the strip and show the transformation of the Hamiltonian that was used already in [4]. In Section 3 we cite the mathematical results that we use in the proofs. Finally, the objectives of our future work are set in Section 5.

2 Preliminaries

2.1 Strip in plane



Let Γ be a unit-speed plane curve, *i.e.* the (image of the) C^2-smooth embedding $\Gamma\colon\ I \to \mathbb{R}^2$: $s \longmapsto (\Gamma^1(s), \Gamma^2(s))$ satisfying $|\dot{\Gamma}(s)| = 1$ for all $s \in I$. I is an interval in $\mathbb{R}$, we allow both finite and infinite intervals. The function $N := (-\dot{\Gamma}^2, \dot{\Gamma}^1)$ defines a unit normal vector field ($\dot{\Gamma} = \frac{\partial}{\partial s}\Gamma(s)$and the couple $(T, N) = (\dot{\Gamma}, N)$ gives a distinguished Frenet frame. We introduce a scalar function $\kappa(s)$ called the curvature, that satisfies the equation

$$\begin{pmatrix} \dot{T} \\ \dot{N} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

In the whole text we will assume that the curvature is bounded, i.e.

$$C_\kappa := \sup_{s\in I} |\kappa(s)| \leq \infty. \tag{1}$$

Since Γ is assumed to be C^2-smooth, we know that the curvature $\kappa(s)$ is continuous, however, for our calculation we have to add the assumption that $\kappa(s)$ is *uniformly* continuous.

Let $\Omega := I \times (-1, 1)$ be a straight strip in the plane. We define a curved strip $\Omega_0 := \mathcal{L}(\Omega)$ using the mapping

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2 : \{(s, u) \longmapsto \Gamma(s) + \varepsilon u N(s)\} \tag{2}$$

If we denote the coordinates (s, u) by $(1, 2)$, we can write the metric tensor of Ω_0 in the form

$$G_{ij} = (\partial_i \mathcal{L}) \cdot (\partial_j \mathcal{L}) = \begin{pmatrix} (1 - \varepsilon u\kappa(s))^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

We denote $|G|$ the determinant of the matrix G_{ij}:

$$|G| = \varepsilon^2 (1 - \varepsilon u\kappa(s))^2$$

To ensure that $|G| \neq 0$ we state the condition $\varepsilon C_\kappa < 1$. However, this is only the necessary condition for injectivity of the mapping $\mathcal{L}$, we have to require the injectivity of $\mathcal{L}$ as an extra condition.

Let us summarize the assumptions on the curve Γ that we make in this work.

Assumption 1 *The properties of the curve Γ are such that*

(i) the curvature κ is uniformly continuous and bounded: $C_\kappa := \sup_{s\in I} |\kappa(s)| < \infty$,

(ii) the mapping $\mathcal{L}$ introduced by (2) is injective for small enough ε.

2.2 The Hamiltonian

If we describe our curved strip Ω_0 with the cartesian coordinates (we denote them x), we have (for a free particle) a simple Hamiltonian:

$$\tilde{H} = -\Delta_D^{\Omega_0}$$

where D assigns that we have the Dirichlet boundary conditions on $\partial\Omega_0$. However, the Hilbert space $L^2(\Omega_0, \mathrm{d}x)$ and the domain of $\tilde{H}$ are rather complicated. For this reason we will work with new coordinates (s, u) introduced by the diffeomorphism $\mathcal{L}$ (see (2)). In this case the Hamiltonian is more complicated, but the Hilbert space becomes $L^2(\Omega, |G|^{1/2}\mathrm{d}s\mathrm{d}u)$ and the domain is much simpler.

Now we will in two steps come to a new Hamiltonian that will be more suitable for observing the constrained strip. In this section we will show the procedure for case $\kappa(s) \in C^2$ that was considered e.g. in [4], the result bellow can be found in paper [4] as well. However, in section 4 of this work we will change this procedure in order to allow $\kappa(s) \in C^0$, this generalization of the older results is the main scope of this work.

- Step one: change of coordinates. For the Laplacian in new coordinates we get

  $$\tilde{H}_\varepsilon = -|G|^{-1/2}\partial_i |G|^{1/2} G^{ij} \partial_j,$$

  where G^{ij} denotes the inverse matrix to G_{ij}.

  Using the new coordinates we can specify the domain of the operator $\tilde{H}_\varepsilon$. At first, the functions must fulfil the Dirichlet conditions on the boundary of the strip. Thus, for $I = \mathbb{R}$ we assume $\psi(s \to \infty, u) = \psi(s \to \infty, u) = 0$, for $I = (a, b)$ $\psi(a, u) = \psi(b, u) = 0$. Dirichlet conditions in u state: $\psi(s, -1) = \psi(s, 1) = 0$. These conditions would be written with difficulty for $\psi(x)$. In conclusion, we will state that $\psi \in \mathrm{Dom}\,(\tilde{H}_\varepsilon)$ if $\psi \in C^\infty$, if it fulfils the Dirichlet boundary conditions and if $\tilde{H}_\varepsilon \psi \in L^2(\Omega, |G|^{1/2}\mathrm{d}s\mathrm{d}u)$.

- Step two: Unitary transformation.

  In the second step we will "straighten" the strip: we will come to Hilbert space $L^2(\Omega, \mathrm{d}s\mathrm{d}u)$ so that the measure is independent of ε and u. The aim of this unitary transformation is also to get the Hamiltonian where the variables s and u are separated up to terms $\mathcal{O}(\varepsilon)$. This will enable us later to consider the behavior of the particle restricted to the very narrow neighborhood of Γ, where we want to observe just the 1D motion independent of u.

  We define a unitary transformation

  $$\begin{array}{rrcl} U: & \psi & \longmapsto & U\psi = |G|^{1/4}\psi \\ & \tilde{H}_\varepsilon & \longmapsto & U\tilde{H}_\varepsilon U^{-1} = |G|^{1/4}\tilde{H}_\varepsilon |G|^{-1/4} =: H_\varepsilon^{C^2} \end{array}$$

  The natation $H_\varepsilon^{C^2}$ was chosen because this unitary transformation has a good sense just for $\kappa(s) \in C^2$, in this work where we assume just $\kappa(s) \in C^0$ we will have to use a modified transformation. However, for the operator $H_\varepsilon^{C^2}$ we get

  $$H_\varepsilon^{C^2} = -\frac{1}{\varepsilon^2}\partial_u^2 - \partial_s(1 - \varepsilon u\kappa)^{-2}\partial_s + V_\varepsilon$$

  where

  $$V_\varepsilon = -\frac{\kappa^2}{4(1-\varepsilon u\kappa)^2} - \frac{\varepsilon u\ddot{\kappa}}{2(1-\varepsilon u\kappa)^3} - \frac{5\varepsilon^2 u^2\dot{\kappa}^2}{4(1-\varepsilon u\kappa)^4}$$

(the dot assigns the derivative with respect to s). As in the case of $\tilde{H}_\varepsilon$, this operator acts on the domain containing functions $\psi \in C^\infty$, fulfilling the Dirichlet boundary conditions and the condition $H_\varepsilon^{C^2}\psi \in L^2(\Omega, \mathrm{d}s\mathrm{d}u)$. The formulae for $H_\varepsilon^{C^2}$ can be easily derived by commuting the derivatives provided that $\kappa(s) \in C^2$, i.e. $\Gamma^1, \Gamma^2 \in C^4$. In the limit $\varepsilon \to 0$ (the thin strip), we don't get only the expected Laplacian part of Hamiltonian, but also some effective potential dependent on the geometrical properties of the curve. This causes the change of spectra and the existence of bound states. These results were derived in [4] and as we already said, in our work, we will generalize this result for $\kappa(s) \in C^0$.

3 Few results of the spectral theory

3.1 Quadratic forms

For the reasons mentioned bellow, we will sometimes work with the quadratic form associated with a non-negative self-adjoint operator instead of this operator.

Definition 1 *Let H be a non-negative self-adjoint operator. For $\phi, \psi \in \mathrm{Dom}\,(H^{1/2}) =: \mathcal{D}$ we define sesquilinear form $Q' : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$:*

$$Q'(\phi, \psi) := \langle H^{1/2}\phi, H^{1/2}\psi \rangle$$

and the quadratic form $Q : \mathcal{D} \to [0, +\infty)$ associated with Q':

$$Q[\psi] := Q'(\psi, \psi).$$

In the text, we will usually assign the Q and Q' by the same letter, which of these two is intended will be clear from the number of arguments and the shape of the parenthesis.

In the following lemma, we will introduce the term of closed quadratic form (for proof see Theorem 4.4.2 in [3]).

Lemma 2 *The following conditions are equivalent:*

(i) Q is the form arising form a non-negative self-adjoint operator H.

(ii) The domain $\mathcal{D}$ of Q is complete for the norm defined by

$$\|f\|_Q := (Q(f) + \|f\|^2)^{1/2}.$$

We say that the quadratic form fulfilling the conditions above is closed.

A form Q_2 is said to be an extension of Q_1 if it has a larger domain but coincides with Q_1 on the domain of Q_1. A form Q is said to be closable of it has a closed extension, the smallest closed extension is called its closure $\bar{Q}$.

3.2 Dirichlet boundary conditions and Sobolev spaces

In this text, we will work with differential operators with specified boundary conditions. This will restrict the domain of the operator and the domain of the associated quadratic form as well. In this section, we will in few steps introduce the Sobolev space $W_0^{1,2}(\Omega)$ and show, that it is suitable for the situation considered in this work. The details can be found in section 6.1 of [3], we give just an overview of the problem and definitions needed in the following text.

In general the operators in the form

$$Hf := -b(x)^{-1} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial f}{\partial x_j} \right). \tag{3}$$

acting on $L^2(\Omega, b(x)\mathrm{d}^N x)$ are studied. We assume, that $a(x) := \{a_{i,j}(x)\}$ is a real symmetric matrix depending upon the variable $x \in \Omega$ and also that the matrices $a(x)$ are uniformly positive and bounded in the sense that there exists a constant $c \geq 1$ such that

$$c^{-1}1 \leq a(x) \leq c1 \tag{4}$$

in the sense of matrices, for all $x \in \Omega$. In addition, we suppose that $b(x)$ is a positive (thus real) function on Ω satisfying

$$c^{-1} \leq b(x) \leq c \tag{5}$$

for all $x \in \Omega$. We introduce a space $C_0^\infty(\bar{\Omega})$ as the space of smooth functions on Ω all of whose partial derivatives can be extended continuously to $\bar{\Omega}$ and which fulfil the Dirichlet boundary conditions $\psi(x) = 0$ for $(x) \in \partial\Omega$. If we take $\mathrm{Dom}\,(H) = C_0^\infty(\bar{\Omega})$ and if we require that the functions $a_{i,j}(x)$ and $b(x)$ are in addition smooth enough, then the operator H is symmetric and non-negative. Moreover, under these conditions it can be shown that the associated quadratic form

$$Q(f,g) := \int_\Omega \sum_{i,j=1}^{N} a_{i,j}(x) \frac{\partial f}{\partial x_i} \frac{\partial \bar{g}}{\partial x_j} \mathrm{d}^N x \tag{6}$$

is closable on the domain $C_0^\infty(\bar{\Omega})$.

In this text we work with the operators in similar form as (3), however, the coefficients $b(x)$ and $a_{i,j}(x)$ do not fulfil the smoothness conditions, in fact, the operator H is not well defined any more. Thus, we have to understand the derivatives in the weak sense, i.e. we will work with the associated quadratic form only. The initial domain of such quadratic form will be $C_c^\infty(\Omega)$, the space of smooth functions with compact supports contained in Ω. In the terms where the classical derivatives would not be well defined, we will work with the functions as with distributions, where the distribution is defined to be a linear functional ϕ: $C_c^\infty(\Omega) \to \mathbb{C}$. If g is a function on Ω which is integrable when restricted to every compact subset of Ω, then g determines a distribution ϕ_g by means of the formula

$$\phi_g(f) := \int_\Omega f(x)g(x)\mathrm{d}^N x$$

If α is any multi-index, the weak derivative $D^\alpha \phi$ of the distribution ϕ is defined by

$$(D^\alpha \phi)(f) := (-1)^{|\alpha|} \phi(D^\alpha f).$$

If h is a smooth function on Ω, then we define the product $h\phi$ to be the distribution $(h\phi)(f) := \phi(hf)$. Now we can define the Sobolev space $W^{1,2}(\Omega)$ (the definition of Sobolev spaces is more general, but in this text this special case is sufficient).

Definition 3 *Let Ω be a region in $\mathbb{R}^N$ and $f \in L^2(\Omega)$. We say that f lies in the Sobolev space $W^{1,2}(\mathbb{R}^N)$, if the weak partial derivatives $\partial_i f := \frac{\partial f}{\partial x_i}$ lie in $L^2(\Omega)$. For these functions we define the Sobolev norm*

$$\|f\|_{1,2}^2 := \int_\Omega \left(|f|^2 + |\nabla f|^2 \right) \mathrm{d}^N x.$$

Finally, we define the subspace $W_0^{1,2}(\Omega)$ of $W^{1,2}(\Omega)$ to be the closure of the subspace $C_c^\infty(\Omega)$ for the norm $\|\cdot\|_{1,2}$. In [3], it is shown that the domain of the closure of the quadratic form (6) is precisely $W_0^{1,2}(\Omega)$. According to the lemma 2, we know that the closure $\bar{Q}$ is associated with a non-negative self-adjoint operator.

Even for the relaxed smoothness conditions on the coefficients $b(x)$ and $a_{i,j}(x)$ (more precisely it is enough that these coefficients depend on x measurably) we get the same result, i.e. the form Q defined by (6) is closed on the domain $W_0^{1,2}(\Omega)$ in the Hilbert space $L^2(\Omega, b(x)\mathrm{d}^N x)$ and there exists a non-negative self-adjoint operator H_D on $L^2(\Omega, b(x)\mathrm{d}^N x)$ associated to the form, in such a way that

$$\langle H_D^{1/2} f, H_D^{1/2} g\rangle = Q(f,g) \tag{7}$$

for all $f, g \in \mathrm{Dom}\,(H_D^{1/2}) = W_0^{1,2}(\Omega)$ (see theorem 6.1.4 in [3] for more details).

3.3 The Projection Theorem

Since we use the Hilbert space decomposition in the section 4.4 we recall here the projection theorem. Notice that for the subset M of the Hilbert space $\mathcal{H}$ we assign $M^\perp$ the set of all vectors in $\mathcal{H}$ that are orthogonal to all vectors of M.

Theorem 4 *Let $\mathcal{G}$ be a closed subspace in the Hilbert space $\mathcal{H}$. Then for all $x \in \mathcal{H}$ there exist unique vectors $y \in \mathcal{G}$ and $z \in \mathcal{G}^\perp$ such that $x = y + z$.*

3.4 The strong resolvent convergence and its consequences for the convergence of the spectrum

Let us start with the definition of the strong operator convergence.

Definition 5 *Let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on the Hilbert space $\mathcal{H}$. We say that the sequence of operators $\{B_n\}_{n=1}^\infty \subset \mathcal{B}(\mathcal{H})$ converges to the operator $B \in \mathcal{B}(\mathcal{H})$ in the sense of strong operator convergence if for all $x \in \mathcal{H}$, $B_n x \to Bx$.*

In [13], it is defined

Definition 6 *We say that the sequence of operators T_n ($n \in \mathbb{N}$) converges to T with respect to the strong resolvent convergence if $(z - T_n)^{-1}$ converges to $(z - T)^{-1}$ in the sense of strong operator convergence for every $z \in \mathbb{C} \setminus \mathbb{R}$.*

However, it is possible to show that the sufficient condition for the strong resolvent convergence is the existence of $k \in \mathbb{C} \setminus (\sigma(T_n) \cup \sigma(T))$ for all $n \in \mathbb{N}$ such that $(T_n + k)^{-1}$ converges to $(T + k)^{-1}$ in the sense of strong operator convergence (see Corollary 1.4 in chapter VII. of [7]).

The consequences for the convergence of spectrum can be found in the article [13], we cite the abstract of this article as a theorem (if we cited the main theorem of this article, too many definitions and further explanations would have to be given).

Theorem 7 *Let T_n ($n \in \mathbb{N}$), T and S be self-adjoint operators such that T_n converges to T with respect to the strong resolvent convergence. If S is bounded from below, $\sigma_{ess}(S) \cap (-\infty, 0) = \emptyset$, and $T_n \geq S$ for all $n \in \mathbb{N}$, then the negative eigenvalues of T_n converge to the negative eigenvalues of T. The corresponding eigenfunctions converge in norm.*

4 The proof of the strong resolvent convergence

4.1 The function $f_\varepsilon(s)$

To proof the strong resolvent convergence of the Hamiltonian on the thin strip in the case $\kappa(s) \in C^0$, we introduce a function f_ε that is close to κ in the limit $\varepsilon \to 0$, but unlike the function κ, f_ε is differentiable in I. This new function is defined as

$$f_\varepsilon(s) := \frac{\int_s^{s+\varepsilon} \kappa(\xi)\mathrm{d}\xi}{\varepsilon}$$

for all $s \in I$.

We assume, that the function $\kappa(s)$ is uniformly continuous in I, i.e.

$$(\forall\tilde{\varepsilon} > 0)(\exists\delta > 0)(\forall\xi_1, \xi_2 \in I, |\xi_2 - \xi_1| < \delta \Rightarrow |\kappa(\xi_2) - \kappa(\xi_1)| < \tilde{\varepsilon}).$$

We define

$$\sigma_\varepsilon := \sup_{s\in I} |\kappa(s+\varepsilon) - \kappa(s)| \tag{8}$$

and according to the definition of the uniform continuity we know that

$$(\forall\tilde{\varepsilon} > 0)(\exists\delta > 0)(\varepsilon < \delta \Rightarrow \sigma_\varepsilon \leq \tilde{\varepsilon}). \tag{9}$$

For the difference of the functions f_ε and κ we get

$$|\kappa(s) - f_\varepsilon(s)| \leq \sup_{\xi\in(s,s+\varepsilon)} |\kappa(s+\xi) - \kappa(s)| = |\kappa(s+\xi_s) - \kappa(s)| \leq \sigma_{\xi_s}, \tag{10}$$

here $s+\xi_s$ is some point in the interval $(s, s+\varepsilon]$ where the supremum is reached - this point exists due to the continuity of κ. Since $\xi_s \leq \varepsilon$ for all $s \in I$ we know that according to (9) we can achieve $\sigma_{\xi_s} \leq \tilde{\varepsilon}$ for any $\tilde{\varepsilon}$ by taking $\varepsilon < \delta$ and this relation will hold for all $s \in I$. Hence we know that the value $|\kappa(s) - f_\varepsilon(s)|$ goes to zero uniformly for all $s \in I$. Thus we can also formally introduce a function $\tilde{\sigma_\varepsilon}$ which vanishes for small ε and for which the relation

$$\sup_{s\in I} |\kappa(s) - f_\varepsilon(s)| \leq \tilde{\sigma_\varepsilon} \tag{11}$$

holds. This estimate and the fact that the function σ_ε as well as $\tilde{\sigma_\varepsilon}$ goes to zero will play the important role in the paragraphs bellow.

We have to note that if the interval I is finite resp. semi-infinite, i.e. $I = (a, b)$ resp. $I = (-\infty, b)$, we can define $\kappa(\xi) = \lim_{s\to b} \kappa(s)$ for $\xi \in [b, \infty)$ (the limit exists due to the uniform continuity of κ) and the function f_ε can be defined in the proceeding way on the whole I.

The crucial property of the function f_ε is its differentiability:[1]

$$\partial_s f_\varepsilon(s) = \frac{\kappa(s+\varepsilon) - \kappa(s)}{\varepsilon}.$$

When we use the function f_ε instead of κ in the step two, we can avoid the derivative of κ in the final formula for H_ε providing we will work with the quadratic forms instead of the operators. The unitary transformation becomes now:

$$\begin{aligned} \tilde{U}: \quad \psi &\longmapsto \tilde{U}\psi = |\tilde{G}|^{1/4}\psi \\ \tilde{H}_\varepsilon &\longmapsto \tilde{U}\tilde{H}_\varepsilon\tilde{U}^{-1} = |\tilde{G}|^{1/4}\tilde{H}_\varepsilon|\tilde{G}|^{-1/4} =: H'_\varepsilon \end{aligned} \tag{12}$$

[1] Through the text we use the notation $\partial_s f_\varepsilon$ etc. even if $f_\varepsilon = f_\varepsilon(s)$ is the function of a single variable. We are aware of the fact that this notation is not fully correct, but we hope that it can help the reader to orientate himself in the expressions where the dependence of the functions on individual variables is not noted.

where

$$|\tilde{G}| = \varepsilon^2(1-\varepsilon u f_\varepsilon(s))^2.$$

We will introduce $g_\varepsilon(s,u) := |G|^{1/2}|\tilde{G}|^{-1/2} = \frac{1-\varepsilon u\kappa(s)}{1-\varepsilon u f_\varepsilon(s)}$ and the Hilbert space now reads $L^2(\Omega, g_\varepsilon \mathrm{d}s\mathrm{d}u) =: \mathcal{H}_\varepsilon$. The corresponding scalar product is assigned $\langle .,. \rangle_{\mathcal{H}_\varepsilon}$.

H'_ε is an example of operator mentioned in section 3.2 that is in fact not well defined. Namely, some of the derivatives do not have a good sense since κ need not to be differentiable. Thus, we will work with the associated quadratic form Q'_ε instead of this operator. As we described in the section 3.2, the domain where this quadratic form is closed is the Sobolev space $W_0^{1,2}(\Omega)$ in the Hilbert space $\mathcal{H}_\varepsilon$. In the section 3.2 we also mentioned that the operator associated with the closure of Q'_ε is self-adjoint. Thus speaking about the operator H'_ε we will from now on mean this self-adjoint operator analogous to the operator H_D from (7).

Now we will rewrite the quadratic form $Q'_\varepsilon[\psi]$ in more illuminating form. We will use the integration by parts with respect to the variable s resp. u, where the zero boundary conditions on ∂I resp. for $u = \pm 1$ will be important as in the following example

$$\int_I \int_{-1}^{1} \bar{f}.\partial_u g \mathrm{d}u\mathrm{d}s = \int_I \left[\bar{f}(s,u) g(s,u)\right]_{u=-1}^{u=1} \mathrm{d}s - \int_I \int_{-1}^{1} \partial_u \bar{f}.g\mathrm{d}u\mathrm{d}s = 0 - \int_I \int_{-1}^{1} \partial_u \bar{f}.g\mathrm{d}u\mathrm{d}s.$$

The integrating by parts has a good sense for $\psi \in C_c^\infty(\Omega)$ which is dense in $W_0^{1,2}(\Omega)$.

$$\begin{aligned}
Q'_\varepsilon[\psi] \;&:= \langle \psi, H'_\varepsilon \psi \rangle_{\mathcal{H}_\varepsilon} = \left\langle \psi, -|\tilde{G}|^{1/4}|G|^{-1/2}\partial_i |G|^{1/2} G^{ij}\partial_j |\tilde{G}|^{-1/4}\psi \right\rangle_{\mathcal{H}_\varepsilon} = \\
&= \left\langle \psi, \left[-\frac{\sqrt{1-\varepsilon u f_\varepsilon}}{(1-\varepsilon u\kappa)} \partial_s \frac{1}{1-\varepsilon u\kappa} \partial_s \frac{1}{\sqrt{1-\varepsilon u f_\varepsilon}} - \frac{\sqrt{1-\varepsilon u f_\varepsilon}}{\varepsilon^2 (1-\varepsilon u\kappa)} \partial_u (1-\varepsilon u\kappa)\partial_u \frac{1}{\sqrt{1-\varepsilon u f_\varepsilon}} \right] \psi \right\rangle_{\mathcal{H}_\varepsilon} = \\
&= \int_\Omega \bar{\psi} \left[-\frac{1}{\sqrt{(1-\varepsilon u f_\varepsilon)}} \partial_s \frac{1}{1-\varepsilon u\kappa} \partial_s \frac{1}{\sqrt{1-\varepsilon u f_\varepsilon}} - \frac{1}{\varepsilon^2 \sqrt{(1-\varepsilon u f_\varepsilon)}} \partial_u (1-\varepsilon u\kappa)\partial_u \frac{1}{\sqrt{1-\varepsilon u f_\varepsilon}} \right] \psi \mathrm{d}s\mathrm{d}u = \\
&= \int_\Omega \frac{1}{1-\varepsilon u\kappa} \partial_s \left(\frac{1}{\sqrt{(1-\varepsilon u f_\varepsilon)}} \bar{\psi} \right) \partial_s \left(\frac{1}{\sqrt{1-\varepsilon u f_\varepsilon}} \psi \right) \mathrm{d}s\mathrm{d}u + \\
&\quad + \int_\Omega (1-\varepsilon u\kappa) \partial_u \left(\frac{1}{\varepsilon^2 \sqrt{(1-\varepsilon u f_\varepsilon)}} \bar{\psi} \right) \partial_u \left(\frac{1}{\sqrt{1-\varepsilon u f_\varepsilon}} \psi \right) \mathrm{d}s\mathrm{d}u = \\
&= \int_\Omega \frac{1}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)} |\partial_s \psi|^2 + \frac{(1-\varepsilon u\kappa)}{\varepsilon^2 (1-\varepsilon u f_\varepsilon)} |\partial_u \psi|^2 + \\
&\quad + \left(\frac{\frac{1}{2}\kappa f_\varepsilon}{(1-\varepsilon u f_\varepsilon)^2} - \frac{\frac{3}{4} f_\varepsilon^2}{(1-\varepsilon u f_\varepsilon)^3 (1-\varepsilon u\kappa)^{-1}} \right) |\psi|^2 + \\
&\quad + \frac{\frac{1}{4} u^2 \left(\kappa(s+\varepsilon) - \kappa(s)\right)^2}{(1-\varepsilon u f_\varepsilon)^3 (1-\varepsilon u\kappa)} |\psi|^2 + \frac{\frac{1}{2} u \left(\kappa(s+\varepsilon) - \kappa(s)\right)}{(1-\varepsilon u f_\varepsilon)^2 (1-\varepsilon u\kappa)} \left(\bar{\psi}\partial_s \psi + \psi \partial_s \bar{\psi} \right) \mathrm{d}s\mathrm{d}u
\end{aligned}$$

4.2 The shifted operator

In the previous section we derived the formula for the quadratic form associated with our Hamiltonian acting in $L^2(\Omega, g_\varepsilon \mathrm{d}s\mathrm{d}u) =: \mathcal{H}_\varepsilon$. While proving the strong-resolvent convergence of this operator we will work with the shifted operator

$$H_\varepsilon := H'_\varepsilon - \frac{E_1}{\varepsilon^2} \tag{13}$$

where E_1 is the first eigenvalue of the transverse Dirichlet Laplacian $-\Delta_D^{(-1,1)}$ acting on $L^2((-1,1), \mathrm{d}u)$, $E_1 = \frac{\pi^2}{4}$. This is done because of the term in the Hamiltonian H'_ε that is for small ε close to

$-\frac{1}{\varepsilon^2}\Delta_D^{(-1,1)}$. It can be easily shown that the eigenvalues of H'_ε have the form $\lambda_n(H'_\varepsilon) = \frac{E_1}{\varepsilon^2} + \mathcal{O}(1)$. Hence, to get the interesting part of spectra, we have to subtract $\frac{E_1}{\varepsilon^2}$.

The quadratic form associated with H_ε reads

$$\begin{aligned} Q_\varepsilon[\psi] \;&=\; \int_\Omega \frac{|\partial_s\psi|^2}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u + \frac{1}{\varepsilon^2}\int_\Omega |\partial_u\psi|^2 \frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u - \frac{E_1}{\varepsilon^2}\int_\Omega |\psi|^2\frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u \\ &+ \int_\Omega \left(V_\varepsilon^1 + V_\varepsilon^2\right)|\psi|^2\mathrm{d}s\mathrm{d}u + \int_\Omega \frac{u\left(\kappa(s+\varepsilon)-\kappa(s)\right)}{(1-\varepsilon u f_\varepsilon)^2(1-\varepsilon u\kappa)}\mathrm{Re}\left(\bar\psi\partial_s\psi\right)\mathrm{d}s\mathrm{d}u \end{aligned} \tag{14}$$

where

$$V_\varepsilon^1(s,u) \;:=\; \frac{\frac{1}{2}\kappa f_\varepsilon}{(1-\varepsilon u f_\varepsilon)^2} - \frac{\frac{3}{4}f_\varepsilon^2}{(1-\varepsilon u f_\varepsilon)^3(1-\varepsilon u\kappa)^{-1}} \tag{15}$$

$$V_\varepsilon^2(s,u) \;:=\; \frac{\frac{1}{4}u^2\left(\kappa(s+\varepsilon)-\kappa(s)\right)^2}{(1-\varepsilon u f_\varepsilon)^3(1-\varepsilon u\kappa)}. \tag{16}$$

This holds for all $\psi(s,u) \in \mathrm{Dom}\,(Q_\varepsilon) = W_0^{1,2}(\Omega)$. To shorten the formulas, the dependance $\kappa(s)$, $f_\varepsilon(s)$ and $\psi(s,u)$ is not emphasized if not necessary, we will use this shortening through the whole text.

We will now show, that there exists a positive constant k independent of ε such that $Q_\varepsilon(\psi) > -k\|\psi\|^2_{\mathcal{H}_\varepsilon}$ which implies that the operator $H_\varepsilon + k$ is positive. We will use the following estimates on individual terms in (14).

By introducing a function $\Phi := \sqrt{g_\varepsilon}\psi$ we get

$$q[\psi] \;:=\; \frac{1}{\varepsilon^2}\int_\Omega |\partial_u\psi|^2\frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u - \frac{E_1}{\varepsilon^2}\int_\Omega |\psi|^2\frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u = \tag{17}$$

$$= \frac{1}{\varepsilon^2}\int_\Omega |\partial_u\Phi|^2\mathrm{d}s\mathrm{d}u - \frac{E_1}{\varepsilon^2}\int_\Omega|\Phi|^2\mathrm{d}s\mathrm{d}u + \tag{18}$$

$$+\frac{1}{\varepsilon^2}\int_\Omega\left(\frac{\frac{1}{2}\kappa\varepsilon}{(1-\varepsilon u\kappa)^2} - \frac{\frac{1}{2}f_\varepsilon\varepsilon}{(1-\varepsilon u f_\varepsilon)^2}\right)\left(\Phi\partial_u\bar\Phi + \bar\Phi\partial_u\Phi\right)\mathrm{d}s\mathrm{d}u \tag{19}$$

$$\begin{aligned} &+\int_\Omega\left(\frac{\frac{1}{4}\kappa^2}{(1-\varepsilon u\kappa)^2} - \frac{\frac{1}{2}\kappa f_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)} + \frac{\frac{1}{4}f_\varepsilon^2}{(1-\varepsilon u f_\varepsilon)^2}\right)|\Phi|^2\mathrm{d}s\mathrm{d}u \geq \\ \geq\;& \int_\Omega\left(\frac{\frac{3}{4}f_\varepsilon^2}{(1-\varepsilon u f_\varepsilon)^2} - \frac{\frac{1}{2}\kappa f_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)} - \frac{\frac{1}{4}\kappa^2}{(1-\varepsilon u\kappa)^2}\right)|\Phi|^2\mathrm{d}s\mathrm{d}u \geq \\ \geq\;& -\frac{3C_\kappa^2}{4(1-C_\kappa\varepsilon)^2}\int_\Omega|\Phi|^2\mathrm{d}s\mathrm{d}u \geq \\ \geq\;& -3C_\kappa^2\int_\Omega|\Phi|^2\mathrm{d}s\mathrm{d}u =: -C_q\|\psi\|^2_{\mathcal{H}_\varepsilon} \end{aligned}$$

where C_q is a constant independent of ε. We used the positivity of the term (18) which follows from the fact that E_1 is the lowest eigenvalue of $-\Delta_D^{(-1,1)}$. The term with $\Phi\partial_u\bar\Phi$ on line (19) was integrated by parts. In the last estimate we assume that ε is so small that $\varepsilon C_\kappa \leq \frac{1}{2}$.

Next we will estimate the absolute value of the last term in (14) using the inequalities $\left|\int_\Omega f\mathrm{d}s\mathrm{d}u\right| \leq \int_\Omega |f|\mathrm{d}s\mathrm{d}u$ and $\mathrm{Re}f \leq |f|$, the Schwarz inequality in the Hilbert space $L^2(\Omega, \mathrm{d}s\mathrm{d}u)$ and the Young's inequality $2ab \leq a^2 + b^2$:

$$\begin{aligned} &\left|\int_\Omega \frac{u\left(\kappa(s+\varepsilon)-\kappa(s)\right)}{(1-\varepsilon u f_\varepsilon)^2(1-\varepsilon u\kappa)}\mathrm{Re}\left(\bar\psi\partial_s\psi\right)\mathrm{d}s\mathrm{d}u\right| \leq \\ \leq\;& \int_\Omega \frac{|\partial_s\psi|}{(1-\varepsilon u\kappa)^{1/2}(1-\varepsilon u f_\varepsilon)^{1/2}}\cdot\frac{|u\left(\kappa(s+\varepsilon)-\kappa(s)\right)|\cdot|\psi|}{(1-\varepsilon u\kappa)^{1/2}(1-\varepsilon u f_\varepsilon)^{3/2}}\mathrm{d}s\mathrm{d}u \leq \\ \leq\;& \frac{1}{2}\int_\Omega\frac{|\partial_s\psi|^2}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u + \frac{1}{2}\int_\Omega\frac{u^2\left(\kappa(s+\varepsilon)-\kappa(s)\right)^2}{(1-\varepsilon u f_\varepsilon)^3(1-\varepsilon u\kappa)}|\psi|^2\mathrm{d}s\mathrm{d}u \end{aligned} \tag{20}$$

Consequently we get

$$Q_\varepsilon[\psi] \geq \frac{1}{2}\int_\Omega \frac{|\partial_s\psi|^2}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u - C_q\|\psi\|^2_{\mathcal{H}_\varepsilon} + \int_\Omega V^1_\varepsilon - V^2_\varepsilon|\psi|^2\mathrm{d}s\mathrm{d}u. \tag{21}$$

The potential can be estimated by constant independent of ε as follows.

$$\begin{aligned} &\int_\Omega \left(V^1_\varepsilon - V^2_\varepsilon\right)|\psi|^2\mathrm{d}s\mathrm{d}u = \\ = \quad & \int_\Omega \left(\frac{\frac{1}{2}\kappa f_\varepsilon}{(1-\varepsilon u f_\varepsilon)(1-\varepsilon u\kappa)} - \frac{\frac{3}{4}f^2_\varepsilon}{(1-\varepsilon u f_\varepsilon)^2} - \frac{\frac{1}{4}u^2\left(\kappa(s+\varepsilon)-\kappa(s)\right)^2}{(1-\varepsilon u f_\varepsilon)^2(1-\varepsilon u\kappa)^2}\right)|\psi|^2\frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u \geq \\ \geq \quad & \left(-\frac{5}{4}\frac{C^2_\kappa}{(1-C_\kappa\varepsilon)^2} - \frac{(2C_\kappa)^2}{4(1-C_\kappa\varepsilon)^4}\right)\|\psi\|^2_{\mathcal{H}_\varepsilon} \geq -21C^2_\kappa\|\psi\|^2_{\mathcal{H}_\varepsilon} =: -C_V\|\psi\|^2_{\mathcal{H}_\varepsilon} \end{aligned}$$

Finally, noticing that the first term in (21) is positive, we get the required inequality

$$Q_\varepsilon[\psi] \geq -(C_q + C_V)\|\psi\|^2_{\mathcal{H}_\varepsilon}. \tag{22}$$

In consequence, the operator $H_\varepsilon + k$ for $k > C_q + C_V$ is positive, which in particular yields the boundedness of the inverse operator

$$\|(H_\varepsilon + k)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)} \leq \frac{1}{k - C_q - C_V}. \tag{23}$$

Hence we have $k \in \mathbb{R}\setminus\sigma(H_\varepsilon)$ for all ε. If we find a limit operator $(H+k)^{-1}$ of the sequence $(H_\varepsilon + k)^{-1}$ in the sense of strong operator convergence, we will prove the convergence of H_ε to H with respect to the strong resolvent convergence.

4.3 The resolvent equation

To investigate the convergence of the operator $(H_\varepsilon + k)^{-1}$ in the sense of strong operator convergence, we set any $F \in \mathcal{H}_\varepsilon$

$$\psi_\varepsilon := (H_\varepsilon + k)^{-1}F. \tag{24}$$

This function satisfies the resolvent equation $(H_\varepsilon + k)\psi_\varepsilon = F$. Using the quadratic forms, this means

$$Q_\varepsilon(\vartheta, \psi_\varepsilon) + k\langle\vartheta,\psi_\varepsilon\rangle_{\mathcal{H}_\varepsilon} = \langle\vartheta, F\rangle_{\mathcal{H}_\varepsilon} \tag{25}$$

for all $\vartheta \in \mathrm{Dom}\,(Q_\varepsilon)$. Especially for $\vartheta = \psi_\varepsilon$ we get

$$Q_\varepsilon[\psi_\varepsilon] + k\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = \langle\psi_\varepsilon, F\rangle_{\mathcal{H}_\varepsilon} \leq \frac{1}{2}\left(\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + \|F\|^2_{\mathcal{H}_\varepsilon}\right). \tag{26}$$

The behavior of the function ψ_ε in the limit $\varepsilon \to 0$ will be studied. In the first step we will find the upper bound for $\|\psi_\varepsilon\|_{\mathcal{H}_\varepsilon}$. Here, the inequality (26) will be used and we will use also the following estimate on $\|F\|^2_{\mathcal{H}_\varepsilon}$.

$$\begin{aligned} \|F\|^2_{\mathcal{H}_\varepsilon} &= \int_\Omega |F(s,u)|^2\frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u \leq \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon}\int_\Omega |F(s,u)|^2\mathrm{d}s\mathrm{d}u \leq \\ &\leq 3\|F\|^2_{\mathcal{H}_0} =: C_F \end{aligned} \tag{27}$$

The last inequality holds for $C_\kappa\varepsilon \leq \frac{1}{2}$, however, notice that C_F is independent of ε. Knowing that

$$(k - C_q - C_V)\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq Q_\varepsilon(\psi_\varepsilon) + k\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq \frac{1}{2}\left(\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + \|F\|^2_{\mathcal{H}_\varepsilon}\right)$$

we get

$$\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \le \frac{C_F}{2k - 2C_q - 2C_V - 1} =: C_\psi \tag{28}$$

where we assume that k is large enough so that the denominator in (28) is positive. More precisely, in all the estimates made in this section, it will be enough to assume $k > C_q + C_V + \frac{1}{2}$.

We will combine the inequalities (21) and (26) as follows

$$\begin{aligned}
& \frac{1}{2}\left(\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + \|F\|^2_{\mathcal{H}_\varepsilon}\right) \ge Q_\varepsilon(\psi_\varepsilon) + k\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \ge \\
\ge\ & \frac{1}{2}\int_\Omega \frac{|\partial_s\psi_\varepsilon|^2}{(1-\varepsilon u\kappa)^2}\frac{(1-\varepsilon u\kappa)}{(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u + (k - C_q - C_V)\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}\mathrm{d}s\mathrm{d}u \ge \\
\ge\ & \frac{1}{8}\|\partial_s\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + (k - C_q - C_V)\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}.
\end{aligned} \tag{29}$$

In this way we get the estimate on the longitudinal derivative of ψ_ε:

$$\|\partial_s\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \le 8(\frac{1}{8}\|\partial_s\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + (k - C_q - C_V - \frac{1}{2})\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) \le 4\|F\|^2_{\mathcal{H}_\varepsilon} \le 4C_F. \tag{30}$$

Since the estimates on $\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}$ and $\|\partial_s\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}$ independent of ε play an important role in next paragraphs we will state these results as a lemma.

Lemma 8 *Let ε be so small that $\varepsilon C_\kappa \le \frac{1}{2}$ and let $k > C_q + C_V + \frac{1}{2}$. Then under the conditions on the curve Γ stated in the section 2.1 we get for the function $\psi_\varepsilon := (H_\varepsilon + k)^{-1}F$ the following estimates:*

(i) $\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \le C_\psi$,

(ii) $\|\partial_s\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \le 4C_F$.

The constants C_ψ resp. C_F were introduced in (28) resp. (27) and are independent of ε.

4.4 Hilbert space decomposition

As we already mentioned in section 4.2, the only eigenvalue of the Laplacian $-\frac{1}{\varepsilon^2}\Delta_D^{(-1,1)}$ that plays the role in the spectra of H_ε is $\frac{E_1}{\varepsilon^2}$. Roughly said, it is only the projection on the first eigenspace of $-\frac{1}{\varepsilon^2}\Delta_D^{(-1,1)}$ that survives acting of the operator $(H_\varepsilon + k)^{-1}$ for small ε. To show this, we will use the following Hilbert space decomposition.:

$$\mathcal{H}_\varepsilon = \mathcal{H}^1_\varepsilon \oplus {}^\perp\mathcal{H}^1_\varepsilon \tag{31}$$

where

$$\mathcal{H}^1_\varepsilon := \left\{\psi \in \mathcal{H}_\varepsilon \mid \exists\varphi \in L^2(I),\, \psi(s,u) = \varphi(s)\chi_1(u)\right\}. \tag{32}$$

χ_1 is the eigenfunction of $-\Delta_D^{(-1,1)}$ on $L^2((-1,1),\mathrm{d}u)$, normalized to 1:

$$\chi_1(u) = \cos\frac{\pi u}{2}. \tag{33}$$

When multiplying this function with some function $\psi(s,u)$ we mean by the symbol χ_1 the function that acts in s as the identity: $1(s) \otimes \chi_1(u)$.

From the projection theorem and the closedness of $\mathcal{H}^1_\varepsilon$ it arises, that

$$(\forall\psi \in \mathcal{H}_\varepsilon)\left(\exists!\psi_1,\, \psi_1^\perp\right)\left(\psi_1 \in \mathcal{H}^1_\varepsilon,\, \psi_1^\perp \in {}^\perp\mathcal{H}^1_\varepsilon;\, \psi = \psi_1 + \psi_1^\perp = P_1^\varepsilon\psi + (1 - P_1^\varepsilon)\psi\right).$$

Here P_1^ε is the orthogonal projection in $\mathcal{H}_\varepsilon$ onto the subspace $\mathcal{H}_\varepsilon^1$ given by

$$P_1^\varepsilon \psi = \frac{\langle \chi_1, \psi \rangle_\varepsilon}{\langle \chi_1, \chi_1 \rangle_\varepsilon} \chi_1$$

where we define$\langle\ \cdot, \cdot\rangle_\varepsilon$ as a function dependent on s and ε:

$$\langle \chi_1, \psi \rangle_\varepsilon := \int_{-1}^{1} \chi_1(u)\psi(s,u) \frac{1-\kappa\varepsilon u}{1-f_\varepsilon \varepsilon u} \mathrm{d}u.$$

However, for the following estimates the projection P_1^0 independent of ε will be more convenient. Let us note that P_1^0 and $\langle \cdot, \cdot \rangle_0$ is given by the previous definitions with $\varepsilon = 0$ and $\langle \chi_1, \chi_1 \rangle_0 = 1$.

Operator P_1^0 acts on functions $\psi \in \mathcal{H}_0$, where $\mathcal{H}_0 := L^2(\Omega, \mathrm{d}s\mathrm{d}u)$. However, using the inequality

$$\frac{1-C_\kappa\varepsilon}{1+C_\kappa\varepsilon} \|\psi\|_{\mathcal{H}_0}^2 \le \|\psi\|_{\mathcal{H}_\varepsilon}^2 \le \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon} \|\psi\|_{\mathcal{H}_0}^2$$

we get, that

$$\psi \in \mathcal{H}_\varepsilon \quad \Leftrightarrow \quad \psi \in \mathcal{H}_0, \tag{34}$$

hence we can identify $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ as vector spaces. Nevertheless, we are still aware of the difference in topology of the two Hilbert spaces.

Finally, we identify the Hilbert space of the integrable functions on the curve Γ (which can be easily identified with $L^2(I)$) with the subspace $\mathcal{H}_\varepsilon^1$ in the following way. We define the mapping $\pi : L^2(I) \to \mathcal{H}_\varepsilon^1$,

$$(\pi\varphi)(s,u) := \varphi(s)\chi_1(u). \tag{35}$$

We can easily check, that $\mathrm{Dom}\,(\pi) = L^2(I)$ and $\mathrm{Ran}\,(\pi) = \mathcal{H}_\varepsilon^1$. The mapping π is not isometric, however the following inequality holds

$$\frac{\|\varphi \cdot \chi_1\|_{\mathcal{H}_\varepsilon}^2}{\|\varphi\|_{L^2(I)}^2} = \frac{\int_I \int_{-1}^1 |\varphi(s)|^2 \chi_1(u)^2 \frac{1-\kappa\varepsilon u}{1-f_\varepsilon \varepsilon u} \mathrm{d}u\mathrm{d}s}{\int_I |\varphi(s)|^2 \mathrm{d}s} \le \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon} \cdot \frac{\int_I |\varphi(s)|^2 \mathrm{d}s \cdot \int_{-1}^1 \chi_1(u)^2 \mathrm{d}u}{\int_I |\varphi(s)|^2 \mathrm{d}s} = \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon}.$$

By a similar estimate we also get a lower bound. Summing up, we have

$$\frac{1-C_\kappa\varepsilon}{1+C_\kappa\varepsilon} \le \frac{\|\varphi \cdot \chi_1\|_{\mathcal{H}_\varepsilon}^2}{\|\varphi\|_{L^2(I)}^2} \le \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon}.$$

for all sufficiently small ε. It follows that π is injective, thus π^{-1} exists. Hence, we can identify the operators acting in $L^2(I)$ with the operators acting in $\mathcal{H}_\varepsilon^1 \subset \mathcal{H}_\varepsilon$ which is the crucial point in looking for the effective Hamiltonian on the curve.

After few more estimates we will be prepared to show the strong-resolvent convergence of the operator H_ε.

4.5 The function ψ_ε in the limit $\varepsilon \to 0$

Referring to (34), we will consider ψ_ε as a vector in $\mathcal{H}_0$ and decompose it as

$$\psi_\varepsilon(s,u) = P_1^0 \psi_\varepsilon(s,u) + (1-P_1^0)\psi_\varepsilon(s,u) = \varphi(s)\chi_1(u) + \phi(s,u) \tag{36}$$

where we assigned $\varphi\chi_1 = P_1^0\psi_\varepsilon$ and $\phi = (1-P_1^0)\psi_\varepsilon$ (the projection P_1^0 was introduced in section 4.4). We can easily show that $\varphi(s) := \langle \chi_1, \psi_\varepsilon \rangle_0 \in L^2(I)$ and we also know, that for almost every $s \in I$, the orthogonality relation

$$\langle \chi_1, \phi \rangle_0 = \int_{-1}^{1} \chi_1(u)\phi(s,u)\mathrm{d}u = 0 \tag{37}$$

holds.

In the following paragraphs we will find bounds concerning the functions φ and ϕ, to conclude with the behavior of ψ_ε as $\varepsilon \to 0$.

Using the inequality (28) we show that $\|\varphi\|_{L^2}$ can be bounded from above by a constant independent of ε. For this purpose, we will estimate the absolute value of the scalar product $\langle \varphi\chi_1, \phi\rangle_{\mathcal{H}_\varepsilon}$ using the relation (37).

$$\begin{aligned}
|\langle \varphi\chi_1, \phi\rangle_{\mathcal{H}_\varepsilon}| &= \left|\int_\Omega \bar{\varphi}\chi_1\phi\left(1+\frac{(f_\varepsilon-\kappa)\varepsilon u}{1-f_\varepsilon\varepsilon u}\right)\mathrm{d}s\mathrm{d}u\right| = \\
&= \left|\int_I \bar{\varphi}\left(\int_{-1}^{1}\chi_1\phi\mathrm{d}u\right)\mathrm{d}s + \int_\Omega \bar{\varphi}\chi_1\phi\frac{(f_\varepsilon-\kappa)\varepsilon u}{1-f_\varepsilon\varepsilon u}\mathrm{d}s\mathrm{d}u\right| = \\
&= \left|0+\int_\Omega \bar{\varphi}\chi_1\phi\frac{(f_\varepsilon-\kappa)\varepsilon u}{1-f_\varepsilon\varepsilon u}\mathrm{d}s\mathrm{d}u\right| \leq \\
&\leq \frac{\varepsilon\tilde{\sigma_\varepsilon}}{1-C_\kappa\varepsilon}\int_\Omega |\varphi\chi_1|\cdot|\phi|\mathrm{d}s\mathrm{d}u \leq \frac{\varepsilon\tilde{\sigma_\varepsilon}}{1-C_\kappa\varepsilon}\cdot\frac{1}{2}\left(\|\varphi\|^2_{L^2(I)}+\frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon}\|\phi\|^2_{\mathcal{H}_\varepsilon}\right) \leq \\
&\leq \varepsilon\tilde{\sigma_\varepsilon}\left(\|\varphi\|^2_{L^2(I)}+3\|\phi\|^2_{\mathcal{H}_\varepsilon}\right) \qquad (38)
\end{aligned}$$

The last inequality holds for $C_\kappa\varepsilon \leq \frac{1}{2}$ and we also used the estimate (11): $|f_\varepsilon(s)-\kappa(s)| \leq \tilde{\sigma_\varepsilon}$ for all $s \in I$, the function $\tilde{\sigma_\varepsilon}$ goes to zero for small ε. However, sometimes the estimate $|f_\varepsilon(s)-\kappa(s)| \leq 2C_\kappa$ will be enough, especially if an estimate independent of ε is needed (as e.g. in the following estimate). Using the inequality (28), we get

$$\begin{aligned}
C_\psi \geq \|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} &= \langle \varphi\chi_1+\phi, \varphi\chi_1+\phi\rangle_{\mathcal{H}_\varepsilon} \geq \|\varphi\chi_1\|^2_{\mathcal{H}_\varepsilon} - 2\,|\langle\varphi\chi_1,\phi\rangle_{\mathcal{H}_\varepsilon}| + \|\phi\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq \frac{1-C_\kappa\varepsilon}{1+C_\kappa\varepsilon}\|\varphi\|^2_{L^2} - 4\varepsilon C_\kappa\left(\|\varphi\|^2_{L^2(I)}+3\|\phi\|^2_{\mathcal{H}_\varepsilon}\right)+\|\phi\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq \left(\frac{1}{4}-4C_\kappa\varepsilon\right)\|\varphi\|^2_{L^2} + (1-12C_\kappa\varepsilon)\,\|\phi\|^2_{\mathcal{H}_\varepsilon}. \qquad (39)
\end{aligned}$$

We have again supposed, that $C_\kappa\varepsilon \leq \frac{1}{2}$. Assuming in addition that $C_\kappa\varepsilon \leq \frac{1}{32}$ we get

$$\|\varphi\|^2_{L^2} \leq \frac{4\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}}{1-16C_\kappa\varepsilon} \leq 8\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq 8C_\psi. \qquad (40)$$

To get an estimate for the function ϕ, we will study the expression $q[\psi_\varepsilon]$, where the quadratic form $q[\psi]$ is introduced in (17). Using the decomposition (36) we can write

$$q[\psi_\varepsilon] = q[\varphi\chi_1] + 2\mathrm{Re}\, q(\varphi\chi_1,\phi) + q[\phi]. \qquad (41)$$

Integrating by parts twice and using that the function χ_1 is real and that $-\partial_u^2\chi_1 = E_1\chi_1$, we get

$$\begin{aligned}
q[\varphi\chi_1] &= \frac{1}{\varepsilon^2}\int_\Omega |\varphi\partial_u\chi_1|^2\,\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u} - \frac{E_1}{\varepsilon^2}\int_\Omega |\varphi\chi_1|^2\,\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u = \\
&= -\frac{1}{\varepsilon^2}\int_\Omega |\varphi|^2\chi_1\cdot\partial_u^2\chi_1\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u - \frac{1}{\varepsilon^2}\int_\Omega |\varphi|^2\chi_1\cdot\partial_u\chi_1\frac{\varepsilon(f_\varepsilon-\kappa)}{(1-\varepsilon u f_\varepsilon)^2}\mathrm{d}s\mathrm{d}u - \\
&\quad -\frac{E_1}{\varepsilon^2}\int_\Omega |\varphi\chi_1|^2\,\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u = \\
&= \frac{1}{\varepsilon^2}\int_\Omega |\varphi|^2\partial_u\chi_1\cdot\chi_1\frac{\varepsilon(f_\varepsilon-\kappa)}{(1-\varepsilon u f_\varepsilon)^2}\mathrm{d}s\mathrm{d}u + \frac{1}{\varepsilon^2}\int_\Omega |\varphi|^2\chi_1^2\frac{2f_\varepsilon\varepsilon^2(f_\varepsilon-\kappa)}{(1-\varepsilon u f_\varepsilon)^3}\mathrm{d}s\mathrm{d}u
\end{aligned}$$

which yields

$$q[\varphi\chi_1] = \int_\Omega |\varphi|^2\chi_1^2\frac{f_\varepsilon(f_\varepsilon-\kappa)}{(1-\varepsilon u f_\varepsilon)^3}\mathrm{d}s\mathrm{d}u. \qquad (42)$$

Integrating by parts once, we similarly get

$$q(\varphi\chi_1,\phi) = -\frac{1}{\varepsilon}\int_\Omega \bar{\varphi}\partial_u\chi_1\phi\frac{f_\varepsilon-\kappa}{(1-\varepsilon u f_\varepsilon)^2}\mathrm{d}s\mathrm{d}u. \tag{43}$$

To examine the last term in (41) we remember that ϕ is orthogonal to χ_1 in $\mathcal{H}_0$, thus

$$\int_{-1}^{1}|\partial_u\phi|^2\mathrm{d}u \geq E_2\int_{-1}^{1}|\phi|^2\mathrm{d}u, \tag{44}$$

where $E_2=\pi^2$ is the second eigenvalue of the Dirichlet Laplacian $-\Delta_D^{(-1,1)}$. In the case of planar strip, we know exactly the spectrum of the Hamiltonian in the cross section, i.e. it is clear that $E_2>E_1$, however, generally (e.g. in the 3D case) the fact that E_2 is strictly greater than E_1 plays an important role. Consequently, we have

$$\begin{aligned}
q[\phi] &= \frac{1}{\varepsilon^2}\int_\Omega|\partial_u\phi|^2\,\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u} - \frac{E_1}{\varepsilon^2}\int_\Omega|\phi|^2\,\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u \geq \\
&\geq \frac{E_2}{2\varepsilon^2}\frac{1-C_\kappa\varepsilon}{1+C_\kappa\varepsilon}\int_{-1}^{1}|\partial_u\phi|^2\mathrm{d}u + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} - \frac{E_1}{\varepsilon^2}\int_\Omega|\phi|^2\,\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u \geq \\
&\geq \frac{1}{\varepsilon^2}\left(\frac{E_2}{2}\frac{(1-C_\kappa\varepsilon)^2}{(1+C_\kappa\varepsilon)^2}-E_1\right)\|\phi\|^2_{\mathcal{H}_\varepsilon} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq \frac{\pi^2}{8\varepsilon^2}\|\phi\|^2_{\mathcal{H}_\varepsilon} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon}
\end{aligned} \tag{45}$$

where the last inequality holds for $C_\kappa\varepsilon \leq 7-4\sqrt{2}$.

Using the equations (42), (43) and inequality (45), we finally get

$$\begin{aligned}
q[\psi_\varepsilon] &\geq \int_\Omega|\varphi|^2\chi_1^2\frac{f_\varepsilon(f_\varepsilon-\kappa)}{(1-\varepsilon u f_\varepsilon)^3}\mathrm{d}s\mathrm{d}u - \frac{1}{\varepsilon}\int_\Omega\bar{\varphi}\partial_u\chi_1\phi\frac{f_\varepsilon-\kappa}{(1-\varepsilon u f_\varepsilon)^2}\mathrm{d}s\mathrm{d}u + \frac{\pi^2}{8\varepsilon^2}\|\phi\|^2_{\mathcal{H}_\varepsilon} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq -\frac{\tilde{\sigma_\varepsilon}C_\kappa}{(1-C_\kappa\varepsilon)^3}\|\varphi\|^2_{L^2(I)} - \frac{\tilde{\sigma_\varepsilon}\pi^2(1+C_\kappa\varepsilon)}{8(1-C_\kappa\varepsilon)^3}\cdot\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}\|\varphi\|_{L^2(I)} + \frac{\pi^2}{8}\cdot\frac{\|\phi\|^2_{\mathcal{H}_\varepsilon}}{\varepsilon^2} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq -16C_\kappa^2|\varphi\|^2_{L^2(I)} - \frac{3\pi^2C_\kappa}{4}\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}\|\varphi\|_{L^2(I)} + \frac{\pi^2}{8\varepsilon^2}\|\phi\|^2_{\mathcal{H}_\varepsilon} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} = \\
&= \frac{\pi^2}{8}\left(\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}-3C_\kappa\|\varphi\|_{L^2(I)}\right)^2 - C_\kappa^2\left(\frac{9\pi^2}{8}+16\right)\|\varphi\|^2_{L^2(I)} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq \frac{\pi^2}{8}\left(\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}-3C_\kappa\|\varphi\|_{L^2(I)}\right)^2 - C_\kappa^2\left(\frac{9\pi^2}{8}+16\right)8\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon}.
\end{aligned}$$

Similarly as in (29) we can use the inequality (26) to get an upper bound on $q[\psi_\varepsilon]$. Knowing that

$$\begin{aligned}
&\frac{1}{2}\left(\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}+\|F\|^2_{\mathcal{H}_\varepsilon}\right) \geq Q_\varepsilon[\psi_\varepsilon] + k\|\psi_\varepsilon\|_{\mathcal{H}_\varepsilon} \geq q[\psi_\varepsilon] + (k-C_V)\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \geq \\
&\geq \frac{\pi^2}{8}\left(\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}-3C_\kappa\|\varphi\|_{L^2(I)}\right)^2 + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} + \left(k-C_V-8C_\kappa^2\left(\frac{9\pi^2}{8}+16\right)\right)\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}
\end{aligned}$$

we get for $k > C_V + 8C_\kappa^2\left(\frac{9\pi^2}{8}+16\right)+\frac{1}{2}$

$$\begin{aligned}
&\frac{\pi^2}{8}\left(\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}-3C_\kappa\|\varphi\|_{L^2(I)}\right)^2 + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \leq \\
\leq\ &\frac{\pi^2}{8}\left(\frac{\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}-3C_\kappa\|\varphi\|_{L^2(I)}\right)^2 + \frac{1}{2\varepsilon^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} + \left(k-C_V-8C_\kappa^2\left(\frac{9\pi^2}{8}+16\right)-\frac{1}{2}\right)\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq \\
\leq\ &\frac{1}{2}\|F\|^2_{\mathcal{H}_\varepsilon} \leq \frac{1}{2}C_F.
\end{aligned}$$

This inequality yields

$$\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \le C_F\varepsilon^2 \tag{46}$$

and also the estimate on $\|\phi\|_{\mathcal{H}_\varepsilon}$: recalling (44) we have

$$\begin{aligned}
\|\phi\|^2_{\mathcal{H}_\varepsilon} &= \int_\Omega |\phi|^2 \frac{1-\varepsilon u\kappa}{1-\varepsilon u f_\varepsilon}\mathrm{d}s\mathrm{d}u \le \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon}\int_I\left(\int_{-1}^1 |\phi|^2\mathrm{d}u\right)\mathrm{d}s \le \frac{1+C_\kappa\varepsilon}{1-C_\kappa\varepsilon}\int_I\left(\int_{-1}^1 \frac{1}{E_2}|\partial_u\phi|^2\mathrm{d}u\right)\mathrm{d}s \le \\
&\le \frac{(1+C_\kappa\varepsilon)^2}{E_2(1-C_\kappa\varepsilon)^2}\int_\Omega |\partial_u\phi|^2\frac{1-\varepsilon u\kappa}{1-\varepsilon u f_\varepsilon}\mathrm{d}s\mathrm{d}u = \frac{(1+C_\kappa\varepsilon)^2}{E_2(1-C_\kappa\varepsilon)^2}\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \le \\
&\le \frac{(1+C_\kappa\varepsilon)^2}{E_2(1-C_\kappa\varepsilon)^2}C_F\varepsilon^2 =: C_\phi\varepsilon^2.
\end{aligned}$$

Similarly as in (39) we can use the inequality (30) combined with the fact, that

$$\int_{-1}^1 \chi_1\cdot\partial_s\phi\mathrm{d}u = \partial_s\left(\int_{-1}^1 \chi_1\phi\mathrm{d}u\right) = 0$$

and get the estimate on $\|\partial_s\varphi\|_{L^2(I)}$ and $\|\partial_s\phi\|_{\mathcal{H}_\varepsilon}$. Using exactly the same estimates as in (39) we have

$$\begin{aligned}
\frac{1}{4}\|F\|^2_{\mathcal{H}_\varepsilon} &\ge \|\partial_s\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = \langle\partial_s\varphi\cdot\chi_1+\partial_s\phi,\partial_s\varphi\cdot\chi_1+\partial_s\phi\rangle_{\mathcal{H}_\varepsilon} \ge \\
&\ge \left(\frac{1}{4}-4C_\kappa\varepsilon\right)\|\partial_s\varphi\|^2_{L^2} + (1-12C_\kappa\varepsilon)\|\partial_s\phi\|^2_{\mathcal{H}_\varepsilon}
\end{aligned}$$

which yields for $C_\kappa\varepsilon \le \frac{1}{32}$ the inequalities

$$\|\partial_s\varphi\|^2_{L^2(I)} \le 2\|F\|^2_{\mathcal{H}_\varepsilon} \le 2C_F \tag{47}$$

$$\|\partial_s\phi\|^2_{\mathcal{H}_\varepsilon} \le \frac{1}{2}\|F\|^2_{\mathcal{H}_\varepsilon} \le \frac{1}{2}C_F. \tag{48}$$

In conclusion, we got following estimates

$$\|\phi\|^2_{\mathcal{H}_\varepsilon} \le C_\phi\varepsilon^2 \tag{49}$$

$$\|\partial_u\phi\|^2_{\mathcal{H}_\varepsilon} \le C_F\varepsilon^2 \tag{50}$$

$$\|\partial_s\phi\|^2_{\mathcal{H}_\varepsilon} \le \frac{1}{2}C_F \tag{51}$$

$$\|\varphi\|^2_{L^2} \le 8C_\psi \tag{52}$$

$$\|\partial_s\varphi\|^2_{L^2(I)} \le 2C_F. \tag{53}$$

Apart from the assumption 1 the assumptions made to get these relations were:

(i) The ε is so small that $\varepsilon C_\kappa \le \frac{1}{32}$,

(ii) The constant k that figures in the definition of the function ψ_ε (24) fulfils $k > \max\left\{C_q + C_V + \frac{1}{2}, C_V + 8C_\kappa^2\left(\frac{9\pi^2}{8}+16\right)+\frac{1}{2}\right\}$.

However, these assumption can always be achieved.

4.6 The resolvent equation in the limit $\varepsilon \to 0$

In this section we will examine the behavior of the resolvent equation (25) as $\varepsilon \to 0$. The following properties of functions φ and ϕ will play in our estimates crucial role.

Since φ is bounded in $W^{1,2}(I)$ in consequence of inequalities (52) and (53) we know that we can choose a subsequence that converges in the weak sense. We will choose one fixed subsequence (we won't change the notation, but keep in mind that φ now assigns this subsequence) and we assign the limit function of this particular subsequence as φ_0 (other subsequences may have different limits):

$$\varphi \xrightarrow[\varepsilon\to 0]{w} \varphi_0 \qquad \text{in} \quad W^{1,2}(I). \tag{54}$$

Here the "w" assigns the weak convergence. In the following text we find the meaning of the function φ_0 and we conclude that all the subsequences must have the same limit.

Similarly, in consequence of inequalities (49), (50) and (51), we have that ϕ is bounded in $W^{1,2}(\Omega)$. Hence we can choose a subsequence that converges in a weak sense: $\phi \xrightarrow[\varepsilon\to 0]{w} \phi_0$. In this case we can directly find the function ϕ_0. For all $\psi \in C_c^\infty(\Omega)$ we can write

$$\begin{aligned}\langle \phi, \psi\rangle_{W^{1,2}(\Omega)} &= \int_\Omega \left(\bar\phi\psi + \partial_u\bar\phi\partial_u\psi + \partial_s\bar\phi\partial_s\psi\right) \mathrm{d}s\mathrm{d}u = \\ &= \int_\Omega \left(\bar\phi\psi + \partial_u\bar\phi\partial_u\psi - \bar\phi\partial_s^2\psi\right) \mathrm{d}s\mathrm{d}u\end{aligned}$$

where we got the last equality by integrating by parts. However, we know that the functions ϕ and $\partial_u\phi$ converge strongly to zero in $\mathcal{H}_0$ according to (49) and (50) (recall that the norms $\|\cdot\|_{\mathcal{H}_\varepsilon}$ and $\|\cdot\|_{\mathcal{H}_0}$ are comparable), thus all the three terms on the second line vanish for all $\psi \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, we get

$$\phi \xrightarrow[\varepsilon\to 0]{w} 0 \qquad \text{in} \quad W_0^{1,2}(\Omega). \tag{55}$$

While investigating the resolvent equation we will consider a special case $\vartheta(s,u) = \eta(s)\chi_1(u)$ where $\eta \in W_0^{1,2}(I)$. Then the left hand side of the resolvent equation reads:

$$\begin{aligned}&Q_\varepsilon(\eta\chi_1, \psi_\varepsilon) + k\,\langle \eta\chi_1, \psi_\varepsilon\rangle_{\mathcal{H}_\varepsilon} = \\ = \ &\int_\Omega \frac{\partial_s\bar\eta\cdot\chi_1\partial_s\psi_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u + \int_\Omega \left(V_\varepsilon^1 + V_\varepsilon^2\right)\bar\eta\chi_1\psi_\varepsilon \mathrm{d}s\mathrm{d}u + k\cdot\int_\Omega \bar\eta\chi_1\psi_\varepsilon\frac{1-\varepsilon u\kappa}{1-\varepsilon u f_\varepsilon}\mathrm{d}s\mathrm{d}u + \\ + \ &q(\eta\chi_1,\psi_\varepsilon) + \int_\Omega \frac{u\left(\kappa(s+\varepsilon)-\kappa(s)\right)}{2(1-\varepsilon u f_\varepsilon)^2(1-\varepsilon u\kappa)}\left(\bar\eta\chi_1\partial_s\psi_\varepsilon + \partial_s\bar\eta\cdot\chi_1\psi_\varepsilon\right)\mathrm{d}s\mathrm{d}u.\end{aligned} \tag{56}$$

Let us now examine the individual terms in the limit $\varepsilon \to 0$. We will use similar tricks as in (20), i.e. the Schwarz inequality in $L^2(\Omega, \mathrm{d}s\mathrm{d}u)$ or Young's inequality. We will observe the first term in more detail, other terms will be estimated in the similar way. Recalling $\psi_\varepsilon(s,u) = \varphi(s)\chi_1(u) + \phi(s,u)$ we get

$$\begin{aligned}&\int_\Omega \frac{\partial_s\bar\eta\cdot\chi_1\partial_s\psi_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u = \\ = \ &\int_\Omega \partial_s\bar\eta\cdot\chi_1\left(\partial_s\varphi\cdot\chi_1 + \partial_s\phi\right)\mathrm{d}s\mathrm{d}u + \int_\Omega \partial_s\bar\eta\cdot\chi_1\partial_s\psi_\varepsilon\frac{\varepsilon u(\kappa+f_\varepsilon) - \varepsilon^2u^2\kappa f_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon u f_\varepsilon)}\mathrm{d}s\mathrm{d}u.\end{aligned}$$

The property (54) yields

$$\int_\Omega \partial_s\bar\eta\cdot\chi_1\partial_s\varphi\cdot\chi_1\mathrm{d}s\mathrm{d}u \quad \xrightarrow[\varepsilon\to 0]{} \quad \int_\Omega \partial_s\bar\eta\partial_s\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u.$$

Using (55) and $\eta \in W_0^{1,2}(I)$ (i.e. $\eta\chi_1 \in W_0^{1,2}(\Omega)$), we get

$$\left|\int_\Omega \partial_s\bar\eta\cdot\chi_1\partial_s\phi\mathrm{d}s\mathrm{d}u\right| \le \left|\langle \eta\chi_1, \phi\rangle_{W^{1,2}(\Omega)}\right| \xrightarrow[\varepsilon\to 0]{} \quad 0.$$

Finally we estimate

$$\left|\int_\Omega \partial_s\bar\eta\cdot\chi_1\partial_s\psi_\varepsilon\frac{\varepsilon u(\kappa+f_\varepsilon)-\varepsilon^2u^2\kappa f_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon uf_\varepsilon)}\mathrm{d}s\mathrm{d}u\right|\le\int_\Omega|\partial_s\bar\eta\cdot\chi_1|\cdot|\partial_s\psi_\varepsilon|\frac{|\varepsilon u(\kappa+f_\varepsilon)-\varepsilon^2u^2\kappa f_\varepsilon|}{(1-\varepsilon u\kappa)(1-\varepsilon uf_\varepsilon)}\mathrm{d}s\mathrm{d}u\le$$
$$\le\ \varepsilon\cdot 12C_\kappa\|\partial_s\eta\|_{L^2(I)}\|\partial_s\psi_\varepsilon\|_{\mathcal{H}_0}\le\varepsilon\cdot 24C_\kappa\|\partial_s\eta\|_{L^2(I)}\|\partial_s\psi_\varepsilon\|_{\mathcal{H}_\varepsilon}\le\varepsilon\cdot 96C_\kappa\|\partial_s\eta\|_{L^2(I)}C_F\ \underset{\varepsilon\to0}{\longrightarrow}\ 0.$$

Here we assumed $C_\kappa\varepsilon\le\frac12$ and we will use this assumption through this section without noting it. In conclusion

$$\int_\Omega\frac{\partial_s\bar\eta\cdot\chi_1\partial_s\psi_\varepsilon}{(1-\varepsilon u\kappa)(1-\varepsilon uf_\varepsilon)}\mathrm{d}s\mathrm{d}u\quad\underset{\varepsilon\to0}{\longrightarrow}\quad\int_\Omega\partial_s\bar\eta\partial_s\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u.$$

The behavior of the term with the potentials is studied in the same way. It is possible to rewrite the potential V_ε^1 introduced by (15) as

$$V_\varepsilon^1(s,u)=-\frac14\kappa(s)^2+h_\varepsilon^1(s,u)+h_\varepsilon^2(s,u)$$

where

$$|h_\varepsilon^1(s,u)|\le\sigma_\varepsilon C_\kappa$$

and

$$|h_\varepsilon^2(s,u)|\le16\varepsilon C_\kappa^3$$

for all $(s,u)\in\Omega$. The function σ_ε was introduced by (8) and goes to zero for $\varepsilon\to0$. For the potential V_ε^2 introduced by (16) we have

$$|V_\varepsilon^2(s,u)|\le4\sigma_\varepsilon^2$$

for all $(s,u)\in\Omega$, thus by similar estimates as above we conclude with

$$\int_\Omega\left(V_\varepsilon^1+V_\varepsilon^2\right)\bar\eta\chi_1\psi_\varepsilon\mathrm{d}s\mathrm{d}u\quad\underset{\varepsilon\to0}{\longrightarrow}\quad\int_\Omega-\frac14\kappa^2\bar\eta\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u.$$

Since

$$\left|\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}-1\right|\le2\varepsilon\tilde{\sigma_\varepsilon},$$

we get

$$k\int_\Omega\bar\eta\chi_1\psi_\varepsilon\frac{1-\varepsilon u\kappa}{1-\varepsilon uf_\varepsilon}\mathrm{d}s\mathrm{d}u\quad\underset{\varepsilon\to0}{\longrightarrow}\quad k\int_\Omega\bar\eta\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u.$$

If we use the relation

$$\frac{u\left(\kappa(s+\varepsilon)-\kappa(s)\right)}{2(1-\varepsilon uf_\varepsilon)^2(1-\varepsilon u\kappa)}\le4\sigma_\varepsilon$$

which holds again for all $(s,u)\in\Omega$, we conclude that the last term in (56) also vanishes in the limit $\varepsilon\to0$.

Finally, we will look at the term $q(\eta\chi_1,\psi_\varepsilon)$ in more detail. In the same way as we derived (42) and (43) we get

$$q(\eta\chi_1,\varphi\chi_1)=\frac{1}{\varepsilon^2}\int_\Omega\bar\eta\varphi\left|\partial_u\chi_1\right|^2\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}-\frac{E_1}{\varepsilon^2}\int_\Omega\bar\eta\varphi\left|\chi_1\right|^2\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u=\int_\Omega\bar\eta\varphi\chi_1^2\frac{f_\varepsilon(f_\varepsilon-\kappa)}{(1-\varepsilon uf_\varepsilon)^3}\mathrm{d}s\mathrm{d}u$$

and

$$q(\eta\chi_1,\phi)=\frac{1}{\varepsilon^2}\int_\Omega\bar\eta\partial_u\chi_1\partial_u\phi\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}-\frac{E_1}{\varepsilon^2}\int_\Omega\bar\eta\chi_1\phi\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u=-\frac1\varepsilon\int_\Omega\bar\eta\partial_u\chi_1\phi\frac{f_\varepsilon-\kappa}{(1-\varepsilon uf_\varepsilon)^2}\mathrm{d}s\mathrm{d}u.$$

While deriving these formulas we used that $-\partial_u^2\chi_1=E_1\chi_1$. Using again our usual estimates we get

$$\left|\int_\Omega\bar\eta\varphi\chi_1^2\frac{f_\varepsilon(f_\varepsilon-\kappa)}{(1-\varepsilon uf_\varepsilon)^3}\mathrm{d}s\mathrm{d}u\right|\le4C_\kappa\tilde{\sigma_\varepsilon}\|\eta\|_{L^2(I)}\|\varphi\|_{L^2(I)}$$

and

$$\left|\frac{1}{\varepsilon}\int_{\Omega}\bar{\eta}\partial_u\chi_1\phi\frac{f_\varepsilon-\kappa}{(1-\varepsilon u f_\varepsilon)^2}\mathrm{d}s\mathrm{d}u\right| \leq \int_{\Omega}|\eta|\cdot|\partial_u\chi_1|\cdot\frac{|\phi|}{\varepsilon}\frac{|f_\varepsilon-\kappa|}{(1-\varepsilon u f_\varepsilon)^2}\mathrm{d}s\mathrm{d}u \leq 2\pi\tilde{\sigma_\varepsilon}\|\eta\|_{L^2(I)}\frac{2\|\phi\|_{\mathcal{H}_\varepsilon}}{\varepsilon}.$$

Hence from the inequalities (49) and (52) we get that the term $q(\eta\chi_1,\psi_\varepsilon)$ also vanishes.

Summing up all the estimates above we get that the left-hand side of the resolvent equation in the limit $\varepsilon\to 0$ reads:

$$\int_{\Omega}\partial_s\bar{\eta}\partial_s\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u-\int_{\Omega}\frac{1}{4}\kappa^2\bar{\eta}\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u+k\cdot\int_{\Omega}\bar{\eta}\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u.$$

The right-hand side of the resolvent equation reads

$$\langle\eta\chi_1,F\rangle_{\mathcal{H}_\varepsilon}=\int_{\Omega}\bar{\eta}\chi_1 F\frac{1-\varepsilon\kappa u}{1-\varepsilon f_\varepsilon u}\mathrm{d}s\mathrm{d}u.$$

Again, since $\|\eta\|_{L^2(I)}$ and $\|F\|_{\mathcal{H}_0}$ are bounded, in the limit $\varepsilon\to 0$ we get

$$\langle\eta\chi_1,F\rangle_{\mathcal{H}_\varepsilon}\quad\xrightarrow[\varepsilon\to 0]{}\quad\int_{\Omega}\bar{\eta}\chi_1 F\mathrm{d}s\mathrm{d}u.$$

In conclusion the resolvent equation in the limit $\varepsilon\to 0$ states:

$$\int_{\Omega}\partial_s\bar{\eta}\partial_s\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u-\int_{\Omega}\frac{1}{4}\kappa^2\bar{\eta}\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u+k\int_{\Omega}\bar{\eta}\varphi_0\chi_1^2\mathrm{d}s\mathrm{d}u=\int_{\Omega}\bar{\eta}\chi_1 F\mathrm{d}s\mathrm{d}u.$$

Introducing $f(s):=\int_{-1}^{1}\chi_1 F\mathrm{d}u=\pi^{-1}P_1^0F$ and recalling that χ_1 is normalized to 1, we get the one dimensional equation

$$\langle\dot{\eta},\dot{\varphi_0}\rangle_{L^2(I)}-\langle\eta,\frac{\kappa^2}{4}\varphi_0\rangle_{L^2(I)}+k\,\langle\eta,\varphi_0\rangle_{L^2(I)}=\langle\eta,f\rangle_{L^2(I)}$$

(here we come back to the notation $\dot{\eta}=\frac{\mathrm{d}\eta}{\mathrm{d}s}$ etc.) This was derived for any $\eta\in W_0^{1,2}(I)$, that is, the function φ_0 satisfies the resolvent equation

$$(H^{eff}+k)\varphi_0=f$$

where we introduced the operator H^{eff} acting on the 1D Hilbert space $L^2(I)$ called effective Hamiltonian:

$$H^{eff}:=-\Delta_D^I-\frac{\kappa^2}{4}.$$

The operator $-\Delta_D^I$ is the Dirichlet Laplacian on the interval I. The resolvent equation yields

$$\varphi_0=\left(H^{eff}+k\right)^{-1}f.$$

We found the meaning of the function φ_0 and we can see that this result was made independently of the choice of the subsequence φ. Hence all the subsequences must have the same limit φ_0 and we can understand the relation (54) as the relation for the sequence φ in whole (not only for some subsequence).

4.7 The generalized strong resolvent convergence

In this section we prove that ψ_ε converges in some sense to $\varphi_0\chi_1$. Since $\psi_\varepsilon=(H_\varepsilon+k)^{-1}F$ and $\varphi_0\chi_1=\left(H^{eff}+k\right)^{-1}f\,\chi_1=\pi\left(H^{eff}+k\right)^{-1}\pi^{-1}P_1^0F$, this will yield the strong resolvent convergence of the operator H_ε, however, in somewhat generalized form.

The relation (54) yields

$$\varphi - \varphi_0 \xrightarrow[\varepsilon\to 0]{w} 0 \qquad \text{in} \quad W^{1,2}(I).$$

In consequence also for every bounded subinterval $I' \subseteq I$ we have

$$\varphi - \varphi_0 \xrightarrow[\varepsilon\to 0]{w} 0 \qquad \text{in} \quad W_0^{1,2}(I')$$

(we can consider the scalar product of $\varphi - \varphi_0$ with the functions with the compact support in I' to get the second relation). Let $\varrho \in W_0^{1,2}(I')$, then

$$\langle \varphi - \varphi_0, \varrho \rangle_{W_0^{1,2}(I')} = \left\langle (-\Delta_D^{I'} + 1)^{1/2}(\varphi - \varphi_0), (-\Delta_D^{I'} + 1)^{1/2} \varrho \right\rangle_{L^2(I')}.$$

Since this term goes to 0 for every $\varrho \in W_0^{1,2}(I')$, we get the weak convergence of the sequence $(-\Delta_D^{I'}+1)^{1/2}(\varphi-\varphi_0)$ in $L^2(I')$. If we realize that the operator $(-\Delta_D^{I'}+1)^{-1/2}$ is compact and if we recall that the compact operators convert the weakly convergent sequences to strongly convergent ones, we get

$$\varphi - \varphi_0 = (-\Delta_D^{I'} + 1)^{-1/2}(-\Delta_D^{I'} + 1)^{1/2}(\psi_\varepsilon - \varphi_0\chi_1) \xrightarrow[\varepsilon\to 0]{s} 0. \tag{57}$$

In addition the relation (49) yields

$$\|\phi\|_{\mathcal{H}_\varepsilon} = \|\psi_\varepsilon - \varphi\chi_1\|_{\mathcal{H}_\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0. \tag{58}$$

Using the relation $\|\cdot\|_{\mathcal{H}_\varepsilon} \leq 2\|\cdot\|_{\mathcal{H}_0}$ that hold for $\varepsilon C_\kappa \leq \frac{1}{2}$ and other estimates that was used before, we get

$$\begin{aligned}
\|\chi_{I'}(\psi_\varepsilon - \varphi_0\chi_1)\|_{\mathcal{H}_\varepsilon} &\leq \|\chi_{I'}(\psi_\varepsilon - \varphi\chi_1)\|_{\mathcal{H}_\varepsilon} + \|\chi_{I'}(\varphi\chi_1 - \varphi_0\chi_1)\|_{\mathcal{H}_\varepsilon} \leq \\
&\leq \|\psi_\varepsilon - \varphi\chi_1\|_{\mathcal{H}_\varepsilon} + 2\|\chi_{I'}(\varphi - \varphi_0)\chi_1\|_{\mathcal{H}_0} = \|\psi_\varepsilon - \varphi\chi_1\|_{\mathcal{H}_\varepsilon} + \|\varphi - \varphi_0\|_{L^2(I')}.
\end{aligned}$$

In the consequence of relations (57) and (58) we know that both terms on the last line go to 0, thus

$$\|\chi_{I'}(\psi_\varepsilon - \varphi_0\chi_1)\|_{\mathcal{H}_\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0.$$

This is the result we were looking for and we will rewrite it as a theorem.

Theorem 9 *Let H_ε be the Hamiltonian defined by (13) unitarily equivalent to the shifted Dirichlet Laplacian $-\Delta_D^{\Omega_0} - \frac{E_1}{\varepsilon^2}$ acting on the strip of width 2ε along the curve $\Gamma : I \to \mathbb{R}^2$ and let the assumption 1 hold. Let H^{eff} be the one dimensional effective Hamiltonian acting on I:*

$$H^{eff} = -\Delta_D^I - \frac{\kappa^2}{4}$$

where κ is the curvature of Γ. Then

$$\left\|\chi_{I'}\left[(H_\varepsilon + k)^{-1} - \pi(H^{eff} + k)^{-1}\pi^{-1}P_1^0\right] F\right\|_{\mathcal{H}_\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0$$

for all $F \in \mathcal{H}_\varepsilon$ and for every bounded $I' \subseteq I$. Here π and P_1^0 are the projections introduced in section 4.4.

5 Conclusion

5.1 The consequences of the Theorem 9

In the section 3.4 we cited the conclusions of the paper [13]. It was assumed that the sequence of operators converges in the sense of strong resolvent convergence and although it was not emphasized, the Hilbert space, that the operators T_n were acting on, was assumed to be n-independent.

Our sequence of operators H_ε does not fulfil these assumptions since according to Theorem 9, it converges only locally and also the Hilbert space $\mathcal{H}_\varepsilon$ is dependent on ε.

However, since all the spaces $\mathcal{H}_\varepsilon$ are topologically equivalent and the strong convergence in $\mathcal{H}_\varepsilon$ is equivalent to the convergence in the fixed space $\mathcal{H}_0$, the problem of ε-dependent Hilbert space is not really serious (this problem was handled e.g. in [1]). Hence in case $I = (a, b)$ we can take $I' = I$ and it can be shown that the strong resolvent convergence yields the convergence of all the eigenvalues and all the eigenvectors in the norm.

In case of $I = \mathbb{R}$ the situation is more complicated. We leave this as an open problem although we believe that the results of the paper [13] yield some consequences for the spectrum of H_ε even in this case.

5.2 The future tasks

Apart from the tasks noted in the previous section, our main tasks for the future are following. At first we would like to show the *norm* resolvent convergence instead of the strong one, since it yields the stronger consequences for the convergence of the spectrum. Equally important is the task of extending the results on the three dimensional waveguide. It would be also nice to include other effects that should influence the effective Hamiltonian such as the changing of the waveguide cross section.

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