Czech Technical University in Prague
Faculty of Nuclear Science and Physical Engineering

BACHELOR´S THESIS

Lucie Strmisková

Supervisor: Dr. Ing. Pavel Soldán

July 21, 2007

Acknowledgements

I would like to thank Dr. Ing. Pavel Soldán for his support, valueable consultations and careful corrections.

Název práce:
Chaotický oscilátor

Autor: Lucie Strmisková

Obor: Matematické inženýrství

Zaměření: Matematická fyzika

Druh práce: Bakalářská práce

Vedoucí práce: Dr. Ing. Pavel Soldán, Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Abstrakt: Uvedeme definice stability a shrneme důležité vlastnosti řešení převážně lineárních diferenciálních rovnic. Seznámíme se s jejich využitím pro určení stability řešení rovnic nelineárních. Popíšeme nejběžnější bifurkace, které nastávají pro jednoparametrický systém. Získané znalosti uplatníme při rozboru chování dvou jednoduchých elektrických obvodů.

Klíčová slova: stacionární body, orbity, stabilita, Hurwitzovo kritérium, bifurkace

Title:
Chaotic oscillator

Author: Lucie Strmisková

Abstract: We present the stability definitions and we summarize the important solution properties of mainly linear differential equations. We use gained knowledge for determining the stability of nonlinear differential equations. We describe the most common bifurcations in systems with one parameter. We apply the previous by investigating the behaviour of two simple electrical circuits.

Key words: stationary points, periodic orbits, Hurwitz´s criterion, stability, bifurcations

Contents

Introduction

Chaos is a term used for an erratic, almost random, behaviour of time-dependent dynamical systems, which we otherwise consider to be simple and expectable.

Roots of the chaos theory date back to the beginning of the twentieth century, when Henri Poincaré, while working on the three-body problem, discovered that there can exist orbits that are aperiodic and that are not still increasing nor approaching to a fixed point.

In the sixties, when it became evident, that the linear theory was not able to explain most of the observed phenomena, the nonlinear theory progressed more rapidly. Stimulating factor had been, of course, the birth of efficient computers.

In 1960, the American meteorologist Edward Lorenz was concerned by the problem of weather predictions. He set up twelve differential equations to model a climate. He noted the results gained by numerical simulation, and one year later, he tried to repeat these calculations. To his large surprise, the results were totally different. He noticed that in the first calculation he inserted the number 0,506127 but then he used rounded off value 0,506. Thus he discovered that nonlinear equations are very sensitive to initial conditions. This sensitivity is popularly called the butterfly effect (Lorenz summarized his results in a lecture with a name: ”Does a flap of butterfly wings in Brazil set off a tornado in Texas?”)

Presently, the chaos theory is a very popular research topic and it is not limited only to meteorology or physics. In biology for example, it is used for modeling the population growth or brain behaviour by an epileptic fit. Chaotic behaviour also occurs in some chemical reactions. Some mathematicians try to explain the move of shares at exchange using the chaos theory. In this bachelor´s thesis, we will be deal with chaotic circuits.

Chapter 1

Differential equations and stability of their solutions

1.1 Differential equations

We summarize some important properties of differential equations in this section. Consider a system of the first order differential equations in its standard form:

$$\dot{x}_1 = f_1(x_1, \ldots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \ldots, x_n),$$

where f_i are real functions defined in some domain $G \subseteq \mathbb{R}^n$, x_i are real variables and the dot denotes the differentiation with respect to time. The system is often written in a vector form:

$$\dot{\vec{x}} = f(\vec{x}), \qquad \vec{x} \in \mathbb{R}^n. \tag{1.1}$$

Such equations, where time does not appear explicitly on the right hand side of the equations, are called *autonomous.* On the other hand, the term *non-autonomous* denotes the equations where time does appear explicitly. In this bachelor's thesis, we will focus mainly on the autonomous systems.

Definition 1. Vector function $\vec{x}(t)$ is a *solution of the equation* (1.1) in an open interval I iff:
1. the differentiation $\dot{\vec{x}}$ exists and it is continuous in I
2. $\vec{x}(t) \in G \quad \forall t \in I$
3. $\dot{\vec{x}}(t) = f(\vec{x}(t)) \quad \forall t \in I$.

Now we would like to know when the differential equation is soluble. The following theorem gives us the answer.

Theorem 1 (Local existence and uniqueness of the solution). Suppose $\dot{\vec{x}} = f(\vec{x})$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Then there exist maximal $t_1 > 0$ and $t_2 > 0$ such that a solution $\vec{x}(t)$ with initial condition $\vec{x}(t_0) = \vec{x}_0$ exists and is unique for all $t \in (t_0 - t_1, t_0 + t_2)$.

The proof of this theorem can be found in almost all textbooks on differential equations (*e.g.* [2]) a we will not give it here.

We can imagine the solution of differential equation (1.1) with initial condition $\vec{x}(0) = \vec{x}_0$ as a point of an n-dimensional space called *the phase space*. The value of $\vec{x}(t)$ represents the state of the dynamical system described by the differential equation at given time t, so the phase space is a set of all possible states of the system in this sense. The vector $\vec{x}(t)$ traces out a curve in $\mathbb{R}^n$. This curve is often called an *integral curve, orbit* or *trajectory* through x_0. It will be signed as $x(x_0, t)$ in the following pages whether x is one-dimensional or not.

Several significant trajectories exist in the phase space. For us, stationary points and periodic orbits are the most important.

Definition 2. A point x_0 is called a *stationary* (*fixed* or *equlibrium*) *point* iff it does not change during the time evolution, i.e.:

$$\frac{d}{dt}x(x_0, t) = 0 \quad \Leftrightarrow \quad x(x_0, t) = x_0 \quad \forall t \geq 0.$$

Definition 3. A point x_0 is *periodic with period* T $(T > 0)$ iff

$$x(x_0, t + T) = x(x_0, t) \quad \forall t \in \mathbb{R} \quad \text{and} \quad x(x_0, t + s) \neq x(x_0, t) \quad \forall s \in (0, T).$$

Closed curve $\Gamma = \{y \in \mathbb{R}^n | y = x(x_0, t), 0 \leq t \leq T\}$ is called a *periodic orbit.*

1.2 Stability

The problem of the solution stability of differential equations has been mentioned in the introduction. In this paragraph, we would like to analyze this problem in a more detail. We chose the most commonly used definitions of stability from about 60 different definitions and we will concentrate on them.

Definition 4. A point x_0 is *Liapounov stable* iff

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall y_0 \in \mathbb{R}^n)(|x_0 - y_0| < \delta \Rightarrow |x(x_0, t) - y(y_0, t)| < \epsilon \quad \forall t \geq 0).$$

It means that the distance of two trajectories, which start nearby, does not break certain bound during the time evolution. By the sign $|.|$ is meant the norm in $\mathbb{R}^n$. All norms are equivalent in $\mathbb{R}^n$ so we will choose the most convenient norm for concrete examples.

Definition 5. A point x_0 is *quasi-asymptotically stable* iff

$$(\exists\delta > 0)(\forall y_0 \in \mathbb{R}^n)(|x_0 - y_0| < \delta \Rightarrow |x(x_0,t) - y(y_0,t)| \to 0 \quad \text{as} \quad t \to \infty).$$

It means that all nearby trajectories will approach the trajectory $x(x_0, t)$ through x_0. However, we should realize that this definition only says what happens when time tends to infinity so the orbits do not have to tend to each other at finite time.

The following two simple examples illustrate the fact that the Liapounov stability does not results from quasi-asymptotical stability and vice versa.

Example 1. Consider the system

$$\dot{x}_1 = -x_2 \qquad \dot{x}_2 = x_1$$

with solution

$$x_1 = r_0.\cos(t+\varphi) \qquad x_2 = r_0.\sin(t+\varphi).$$

Take two nearby points:

$$x = r_0.(\cos(t+\varphi), \sin(t+\varphi)) \quad y = (r_0 + \delta).(\cos(t+\varphi), \sin(t+\varphi))$$

$$|x - y| = |\delta.(\cos(t+\varphi), \sin(t+\varphi))| = \delta$$

The solutions are concentric circles about the origin. The distance between two nearby points remains constant so all points are Liapounov stable but none are quasi-asymptotically stable. ♠

Example 2. Take the non-autonomous system

$$\dot{x}_1 = \frac{x_1}{t} - t^2 x_1 {x_2}^2 \qquad \dot{x}_2 = -\frac{x_2}{t}$$

with solution

$$x_1(t) = x_{01}\frac{t}{t_0}e^{-(x_{02}t_0)^2(t-t_0)} \qquad x_2(t) = x_{02}\frac{t_0}{t} \quad \forall t \geq t_0 > 0.$$

We show that the stationary point $x = 0$ is quasi-asymptotically stable although it is not Liapounov stable. The definitions used for the stability of non-autonomous systems slightly differ from the ones for autonomous systems but the difference is not important for this purpose.

It is obvious that $\lim_{t\to\infty}|x(t)| = 0 \; \forall t_0, x_0$ so the condition for quasi-asymptotical stability is fulfilled.

Take $\epsilon = \frac{1}{e}, \quad t_0 = 1, \quad x_{01} = \delta^2, \quad x_{02} = \delta, \quad \forall\delta > 0.$

$$x_1(t) = \delta^2 t e^{-\delta^2(t-1)}, \quad x_2(t) = \frac{\delta}{t} \quad t \geq 1$$

In special time $t_1 = 1 + \frac{1}{\delta^2}$, the solution looks:

$$x_1(t_1) = \delta^2 \left(1 + \frac{1}{\delta^2}\right) e^{-1} = \frac{\delta^2 + 1}{e} > \frac{1}{e} = \epsilon$$

$$x_2(t) = \frac{\delta^3}{\delta^2 + 1}$$

$$|x(t_1)| = |x_1(t_1)| + |x_2(t_2)| > \epsilon$$

♠

Definition 6. A point x_0 is called *asymptotically stable* iff it is both Liapounov stable and quasi-asymptotically stable.

These three definitions are particularly useful in the case when x_0 is a stationary point as the figure 1.1 shows.

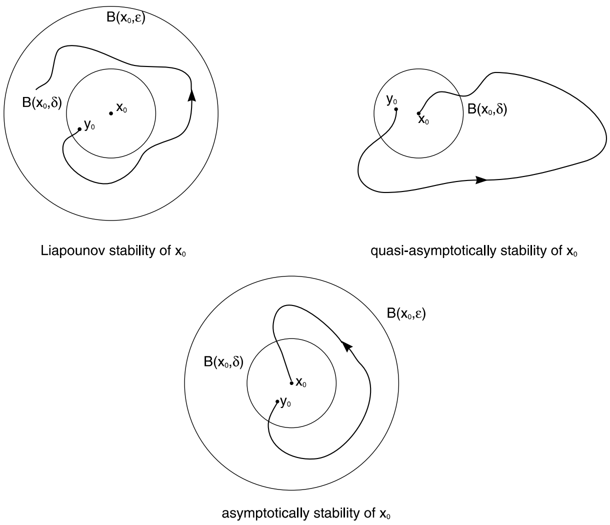

Figure 1.1: Stability of stationary points

The definition of the asymptotically stable stationary point x_0 says that there exists such a neighbourhood of x_0 that all points from this neighbourhood tend to it. This neighbourhood is often called *the domain of asymptotic stability of* x_0, *i.e.*:

$$D_{x_0} = \{y_0 \in \mathbb{R}^n | \quad |y(y_0, t) - x_0| \to 0 \quad \text{as} \quad t \to \infty\}.$$

In special case if $D_{x_0} = \mathbb{R}^n$ the point x_0 is called *globally asymptotically stable point.* The following example indicates the existence of globally asymptotically stable point.

Example 3. Take the system $\dot{x}_1 = -x_2 \quad \dot{x}_2 = 2x_1 - 3x_2$ with solution

$$x(x_0, t) = ((2x_{01} - x_{02}).e^{-t} + (x_{02} - x_{01}).e^{-2t}, (2x_{01} - x_{02}).e^{-t} + 2(x_{02} - x_{01}).e^{-2t}).$$

It is obvious that the stationary point $y_0 = 0$ is globally asymptotically stable because all trajectories with any initial condition x_0 approach it. ♠

Example 4. Consider the system $\dot{r} = 0 \quad \dot{\theta} = 1 + r$ with solution

$$r(t) = r_0 \quad \theta(t) = (1 + r_0)t + \theta_0.$$

This example is very similar to example 1. The solutions also lie on concentric circles around the origin and we expect that all points are Liapounov stable too. However the origin is the only point that fulfills the previous definition.

Take two nearby points $(r_0, 0)$ and $(r_0 + \delta, 0)$ and find phase lag

$$\Delta\theta = (1 + r_0)t - (1 + r_0 + \delta)t = -\delta t.$$

It is apparent that the distance of such chosen points is grater then $2r_0$ in some special times although the orbits as a whole remain nearby. We see that the definition used for points cannot be the most suitable for periodic orbits. For this reason we try to find another stability definition more convenient for closed orbits. ♠

Let $\Gamma = \{y \in \mathbb{R}^n | y = x(x_0, t), 0 \leq t \leq T\}$. We define the neighbourhood $N(\Gamma, \epsilon)$ as follows:

$$N(\Gamma, \epsilon) = \{x \in \mathbb{R}^n | \exists y \in \Gamma : \quad |x - y| < \epsilon\}.$$

Definition 7. A periodic orbit Γ is *orbital stable* iff

$$(\forall \epsilon > 0)(\exists \delta > 0)(x_0 \in N(\Gamma, \epsilon) \Rightarrow x(x_0, t) \in N(\Gamma, \epsilon) \quad \forall t \geq 0).$$

So the system from example 4 is orbital stable but it is not Liapounov stable. On the other hand, Liapounov stable orbits are always orbital stable. The orbital stability is illustrated in figure 1.2.

1.3 Linear stability

Until now, we have discussed the stability of solutions of differential equations using only the respective definitions. Sometimes it could be very difficult, and moreover we do know that closed form solutions are not always possible to find. For this reason we would like to know some useful criteria for assignment of stability. Very simple criterion exists for the system of the first order linear differential equations with constant coefficients

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n,n}. \tag{1.2}$$

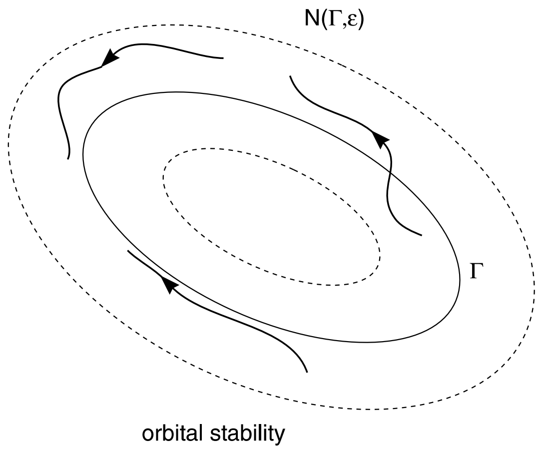

Figure 1.2: Orbital stability

We summarize the most important properties of the system (1.2) briefly at first. More detailed information could be found in [2].

1. The matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with respective eigenvectors $h^{(1)}, \dots, h^{(n)}$. Then the vector functions $h^{(1)}e^{\lambda_1 t}, \dots, h^{(n)}e^{\lambda_n t}$ form the fundamental system of solutions of (1.2).

2. The matrix A has m distinct eigenvalues $\lambda_1, \dots, \lambda_m$, $m < n$. Denote the multiplicity of λ_i as a root of characteristic multinominal of the matrix A as l_i and the number of linearly independent eigenvectors respective to λ_i as p_i. We know that $p_i \leq l_i$ so $\sum_{i=1}^n p_i \leq \sum_{i=1}^n l_i = n$. Therefore we do not suffice with eigenvectors to form the fundamental system.

We transfer the matrix A to its Jordan normal form $Q = H^{-1}AH, \det H \neq 0$ to solve the system $\dot{x} = Ax$. Let us take $x = Hy$. Then $\dot{x} = Ax \Leftrightarrow \dot{y} = Qy$.

Denote the identity matrix of the order k as I_k, the null matrix as O and suppose that the matrix of the order k, P_k, has the following form:

$$P_k = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

So the Jordan normal form of the matrix A can be written as a block diagonal matrix:

$$Q = \begin{pmatrix} \lambda_1 I_{k_1} + P_{k_1} & O & O & \dots & O \\ O & \lambda_2 I_{k_2} + P_{k_2} & O & \dots & O \\ \vdots & & & & \\ O & O & O & \dots & \lambda_r I_{k_r} + P_{k_r} \end{pmatrix}.$$

The number of diagonal blocks r is determined by the condition $r = \sum_{i=1}^{m} p_i \geq m$. For this reason the eigenvalues $\lambda_1, \ldots, \lambda_r$ do not have to be distinct. The size of blocks will be determined later.

We may imagine the system $\dot{y} = Qy$ as r independent subsystems and solve them separately. The fundamental matrix of each subsystem is then:

$$U_{\lambda_i k_i}(t) = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{k_i-1}}{(k_i-1)!} \\ 0 & 1 & t & \ldots & \frac{t^{k_i-2}}{(k_i-2)!} \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} e^{\lambda_i t}. \tag{1.3}$$

Hence the fundamental matrix $U(t)$ of system $\dot{y} = Qy$ is a block diagonal matrix with blocks (1.3).

$$U(t) = \begin{pmatrix} U_{\lambda_1 k_1}(t) & O & \ldots & O \\ O & U_{\lambda_2 k_2}(t) & \ldots & O \\ \vdots & & & \\ O & \ldots & O & U_{\lambda_r k_r}(t) \end{pmatrix} \tag{1.4}$$

It is obvious that the matrix $HU(t)$ is a fundamental matrix of the system $\dot{x} = Ax$.

We have derived the form of $U(t)$, but we still do not know which matrix H transfers A to its Jordan normal form Q. The following conditions for column vectors of the matrix H result from $Q = H^{-1}AH \Leftrightarrow HQ = AH$.

$$\begin{aligned} (A - \lambda_j I)h^{(k_1+\ldots+k_{j-1}+1)} &= 0 \\ (A - \lambda_j I)h^{(k_1+\ldots+k_{j-1}+2)} &= h^{(k_1+\ldots+k_{j-1}+1)} \\ &\vdots \\ (A - \lambda_j I)h^{(k_1+\ldots+k_{j-1}+k_j)} &= h^{(k_1+\ldots+k_{j-1}+k_j-1)} \end{aligned} \tag{1.5}$$

The vectors $h^{(k_1+\ldots+k_{j-1}+1)}, \ldots, h^{(k_1+\ldots+k_{j-1}+k_j)}$, that satisfy the conditions (1.5), are called the *chain respective to eigenvalue* λ_j ($j \in \hat{r}$). The number k_j is called the *length of the chain*.

If we denote the column vectors of $HU(t)$ as $v^{(i)}(t)$ we can express them explicitly in terms of vectors $h^{(i)}$.

$$\begin{aligned} v^{k_1+\ldots+k_{j-1}+l}(t) = h^{k_1+\ldots+k_{j-1}+1} \frac{t^{l-1}}{(l-1)!} e^{\lambda_j t} + \\ + h^{k_1+\ldots+k_{j-1}+2} \frac{t^{l-2}}{(l-2)!} e^{\lambda_j t} + \ldots + h^{k_1+\ldots+k_j} e^{\lambda_j t} \end{aligned} \tag{1.6}$$

Example 5. We try to find the fundamental matrix of the system

$$\begin{aligned} \dot{x}_1 &= 13x_1 - 28x_2 + 3x_3 \\ \dot{x}_2 &= 4x_1 - 8x_2 + x_3 \\ \dot{x}_3 &= -x_1 + 4x_2 + x_3. \end{aligned}$$

The matrix has only one eigenvalue $\lambda_1 = 2$ with multiplicity $l_1 = 3$ and eigenvector $(2,1,2)^T$. So $r = 3$ and $k_1 = 3$. We take $h^{(1)} = (2,1,2)^T$ and we find the other vectors $h^{(2)}, h^{(3)}$ from equations:

$$(A - 2I)h^{(2)} = h^{(1)}, \quad (A - 2I)h^{(3)} = h^{(2)}$$

We obtain $h^{(2)} = (5,2,1)^T, h^{(3)} = (3,1,0)^T$. We gain the fundamental matrix of the system $\dot{x} = Ax$ using the formula (1.6)

$$\begin{pmatrix} 2 & 2t+5 & t^2+5t+3 \\ 1 & t+2 & \frac{1}{2}t^2+2t+1 \\ 2 & 2t+1 & t^2+t \end{pmatrix} e^{2t}$$

♠

Lemma 1. Suppose that $\rho = \max\{\mathrm{Re}\lambda_i, i \in \hat{r}\}$ and $i_1, \ldots, i_p$ are all such indexes that $\rho = \mathrm{Re}\lambda_{i_s}$, $s \in \hat{p}$. Define $\mu = \max\{k_i, i = i_1, \ldots, i_p\}$. If $U(t)$ is a fundamental matrix (1.4) then there exists constant $K_1 > 0$ such that

$$|U(t)| \le K_1(1 + t^\mu)e^{\rho t} \quad \forall t \ge 0.$$

Proof. $U(t) = (u_{ij}(t)) \quad i,j \in \hat{n}$
The existence of such $C > 0$, that $|u_{ij}(t)| \le C(1 + t^\mu)e^{\rho t}$, results from the form of blocks (1.3). All matrix norms are equivalent, *i.e.*

$$\xi_1 \sum_{i=1}^{n} \sum_{j=1}^{n} |u_{ij}(t)| \le |U(t)| \le \xi_2 \sum_{i=1}^{n} \sum_{j=1}^{n} |u_{ij}(t)|, \quad \xi_1, \xi_1 \in \mathbb{R}$$

Therefore $|U(t)| \le n^2 \xi_2 C(1 + t^\mu)e^{\rho t}$. □

Lemma 2. Suppose that $V(t) = HU(t)H^{-1}$, where $HU(t)$ is the fundamental matrix of the system $\dot{x} = Ax$. Then there exists such constant $K_2 > 0$ that

$$|V(t)| \le K_2(1 + t^\mu)e^{\rho t} \quad \forall t \ge 0.$$

Hence

$$|x(t)| \le |x_0| K_2(1 + t^\mu)e^{\rho t} \quad \forall t \ge 0$$

for all solutions of $\dot{x} = Ax$.

Proof. The multiplication of the matrix $HU(t)$ by H^{-1} from the right creates a matrix with column vectors, which are linear combinations of the original column vectors, so the matrix $HU(t)H^{-1}$ is also the fundamental matrix of $\dot{x} = Ax$.

$$|V(t)| \le |U(t)||H||H^{-1}| \quad \Rightarrow \quad |V(t)| \le |H||H^{-1}|K_1(1 + t^\mu)e^{\rho t}$$

$V(t)x_0$ is the solution of $\dot{x} = Ax$. $V(0) = I$ because of $U(0) = I$.

$$V(0)x_0 = x(0) \quad \rightarrow \quad V(t)x_0 = x(t), \forall t \in \mathbb{R}.$$

$$|x(t)| \le |x_0| K_2(1 + t^\mu)e^{\rho t}$$

□

Theorem 2. 1.The stationary point $x = 0$ of the system $\dot{x} = Ax$ is Liapounov stable iff all eigenvalues of the matrix A have non-positive real parts, and moreover the lengths of chains respective to eigenvalues with zero real part are always one.
2.The stationary point $x = 0$ of $\dot{x} = Ax$ is asymptotically stable iff all eigenvalues of the matrix A have negative real parts.

Proof. 1. First we prove that the stationary point $x = 0$ of the system $\dot{x} = Ax$ is Liapounov stable iff the solution $x(x_0, t)$ is bounded for all initial values $x_0 \in \mathbb{R}^n$ and for all $t \geq 0$.

Suppose that all solutions $x(x_0, t)$ are bounded for all $t \geq 0$. Therefore the fundamental matrix $V(t)$ must be also bounded, *i.e.* $\exists K > 0 \; |V(t)| \leq K$.

Choose $\epsilon > 0$ and such initial condition x_0 that $|x_0| < \delta$.

$$|x(x_0, t)| = |V(t)x_0| \leq |V(t)||x_0| < K\delta.$$

For a special choice of $\delta = \frac{\epsilon}{K}$, the condition for Liapounov stability of the stationary point $x = 0$ is fulfilled.

Assume that the stationary point $x = 0$ is Liapounov stable, *i.e.*

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad |x_0| < \delta \quad \Rightarrow \quad |x(x_0, t)| < \epsilon \quad \forall t \geq 0.$$

Suppose that there exists such $x_0 \in \mathbb{R}^n$ that the solution $x(x_0, t)$ is not bounded for $t \geq 0$.

Let us take $y_0 = \frac{\delta}{2|x_0|} x_0$. (It is correct because x_0 cannot be 0.)

$$x(y_0, t) = V(t)y_0 = \frac{\delta}{2|x_0|} V(t)x_0 = \frac{\delta}{2|x_0|} x(x_0, t) \quad \forall t \geq 0$$

Therefore the solution $x(y_0, t)$ is also unbounded although $|y_0| < \delta$ and its norm must be less than ϵ from presumptions.

For this reason, it is enough to show that all solutions of $\dot{x} = Ax$ are bounded instead of they are Liapounov stable.

Suppose that $\mathrm{Re}\lambda_i \leq 0 \; \forall i \in \hat{r}$ and if $\mathrm{Re}\lambda_j = 0$ then $k_j = 1$ for all such j. We have verified that $|x(t)| \leq |x_0|K(1 + t)$ (lemma 2). So the solution could become unbounded as $t \to \infty$. However, all column vectors $v^{(i)}(t)$ of the fundamental matrix $HU(t)$ satisfy that $\lim_{t\to\infty} |v^{(i)}(t)| < \infty$. Hence all solutions (which are some linear combination of the column vectors $v^{(i)}(t)$) must be bounded.

Assume that the previous condition for the eigenvalues is not true. So there exists such $s \in \hat{r}$ that $\mathrm{Re}\lambda_s > 0$ or $\mathrm{Re}\lambda_s = 0$ but $k_s > 1$. It is evident that if $\mathrm{Re}\lambda s > 0$ then some solutions of $\dot{x} = Ax$ are unbounded. If $\mathrm{Re}\lambda_s = 0$ and $k_s > 1$ then the norm of the respective column vector $v^{(k_1 + k_{j-1} + 2)}$ tends to infinity as $t \to \infty$. Hence the statement 1 is proved.

2. Suppose that $\mathrm{Re}\lambda_i < 0 \quad \forall i \in \hat{r}$. Using the previous lemma, we get

$$|x(t)| \leq |x_0|K_2(1 + t^\mu)e^{\rho t}.$$

So $|x(t)| \to 0$ as $t \to \infty$ and the condition for asymptotic stability of $x = 0$ is fulfilled.

Suppose that $|x(t)| \to 0$ as $t \to \infty$. This condition is satisfied when $x(t) \to 0$. All eigenvalues must have negative real parts to fulfill $x(t) \to 0$ as can be seen from the form of fundamental matrix (1.4). □

Previous theorem is very useful but calculating the roots of characteristic polynomial analytically is very tedious and often impossible. Furthermore, the analytical form of roots of a polynomial with parameters is often so complicated that we are unable to gain any information from it. Fortunately for purposes of determination the stability, this problem can be obviated as the following two theorems show.

Theorem 3. Consider the polynomial

$$P_n(x) = a_0 + a_1 x + \ldots + a_n x^n, \quad n \geq 1, \quad a_0 > 0, \quad a_n \neq 0, \quad a_i \in \mathbb{R} \quad \forall i \in \hat{n}. \tag{1.7}$$

If all roots of this polynomial have negative real parts (such a polynomial is often called *Hurwitz's polynomial*), then all coefficients a_i, $i \in \{0, \ldots, n\}$ are positive.

Proof. Assume that $x_j = -\alpha_j \pm i\beta_j$ $(j \in \hat{p})$ are complex roots of polynomial (1.7) and $x_k = -\gamma_k$ $(k \in \hat{q})$ are the real ones. The polynomial is Hurwitz's, so $\alpha_j > 0$ $\forall j \in \hat{p}$, $\gamma_k > 0 \quad \forall k \in \hat{q}$. Denote the multiplicity of a root $x_k = -\gamma_k$ as m_k and multiplicity of $x_j = -\alpha_j + i\beta_j$ as n_j. The polynomial (1.7) have real coefficients so $x_j = -\alpha_j - i\beta_j$ has the same multiplicity. It is obvious that

$$\sum_{j=1}^{p} 2n_j \quad + \quad \sum_{k=1}^{q} m_k = n.$$

All polynomials can be written in the form:

$$P_n(x) = a_n \prod_{j=1}^{p} (x + \alpha_j - i\beta_j)^{n_j} (x + \alpha_j + i\beta_j)^{n_j} \prod_{k=1}^{q} (x + \gamma_k)^{m_k}$$

$$P_n(x) = a_n \prod_{j=1}^{p} (x^2 + 2\alpha_j x + {\beta_j}^2 + {\alpha_j}^2)^{n_j} \prod_{k=1}^{q} (x + \gamma_k)^{m_k}$$

If we compare the coefficients of terms with the same order of x we see that all coefficients a_i have the same sign. So all coefficients must be positive as a consequence of positivity of a_0. □

The roots of polynomial $P_2(x) = a_2 x^2 + a_1 x + a_0$ are either real or it is a complex conjugated pair. If $a_0, a_1, a_2 > 0$ then the complex conjugated pair has a negative real part. For the same reason both the real roots are negative as results from Viète´s formulas for roots of a quadratic equation. The roots of polynomial $P_3(x) = x^3 + x^2 + 4x + 30$ are $-3, 1+3i, 1-3i$. So, the previous condition is necessary but not sufficient for polynomials of the order grater than two. For this reason, we would like to know another criterion that will be both necessary and sufficient. Before writing this criterion,we introduce some important properties of polynomials.

Definition 8. Polynomial

$$F(x) = (1 + \alpha x)f(x) + f(-x), \quad \alpha > 0 \tag{1.8}$$

is called a *conjugated polynomial* to the polynomial $f(x)$.

Lemma 3. The polynomial (1.8) conjugated to the Hurwitz's polynomial $f(x)$ is also the Hurwitz's polynomial.

Proof. Consider the polynomials $\Phi_m(x) = (1+\alpha x)f(x)+mf(-x)$, where $0 \leq m \leq 1$. The polynomial $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ is Hurwitz's from the assumptions so $a_0, a_1, \ldots, a_n > 0$. Then

$$\Phi_m(x) = (1 + \alpha x)(a_0 + a_1 x + \ldots + a_n x^n) + m(a_0 - a_1 x + \ldots + a_n (-1)^n x^n) \tag{1.9}$$

Therefore, the coefficients a_i^m are linear functions of the parameter m and moreover the roots of this polynomial are closed in a sufficiently large circle ($\alpha a_n > 0 \Rightarrow |\Phi_m(x)| > 0 \quad |x| \geq R$), where R does not depend on m.

The polynomial $\Phi_0(x) = (1 + \alpha x)f(x)$ is the Hurwitz's polynomial because of positivity of α. We use the *absurdum* proof to show that Φ_m are Hurwitz's polynomials for all $m \in \langle 0, 1\rangle$. Assume that $\bar{m} \in \langle 0, 1\rangle$ is such a parameter that the polynomial $\Phi_{\bar{m}}$ is not the Hurwitz's polynomial. The roots of (1.9) are continuous bounded functions of the parameter m so there exists such $\tilde{m} \in \langle 0, 1\rangle$ that at least one root of the polynomial $\Phi_{\tilde{m}}$ leaves the left half-plane of the complex plane to realize that $\Phi_{\bar{m}}$ is not the Hurwitz's polynomial. Therefore $\Phi_{\tilde{m}}$ has an imaginary root $i\beta$.

$$\Phi_{\tilde{m}}(i\beta) = (1 + i\alpha\beta)f(i\beta) + \tilde{m}f(-i\beta) = 0$$

$$|1 + i\alpha\beta||f(i\beta)| = \tilde{m}|f(-i\beta)| \tag{1.10}$$

$f(\overline{x}) = \overline{f(x)}$ for all polynomials with real coefficients.

$$|f(-i\beta)| = |f(\overline{i\beta})| = |\overline{f(i\beta)}| = |f(i\beta)|$$

$|f(i\beta)| \neq 0$ because f is Hurwitz's polynomial so we can cancel it out from (1.10).

$$|1 + i\alpha\beta| = \tilde{m} \quad \Rightarrow \quad 1 + \alpha^2\beta^2 = \tilde{m}^2$$

Hence $\tilde{m} > 1$ ($\alpha > 0, \beta \neq 0$) and it is against the condition $\hat{m} \in \langle 0, 1\rangle$. So all polynomials $\Phi_m(x)$ are the Hurwitz's polynomials for all $m \in \langle 0, 1\rangle$ and the statement is proved (for special choice $m = 1$). □

Lemma 4. For all Hurwitz's polynomials $F(x)$ of the order $n+1$ there exists $\alpha > 0$ and Hurwitz's polynomial $f(x)$ of the order n such that

$$F(x) = (1 + \alpha x)f(x) + f(-x). \tag{1.11}$$

Proof.

$$F(-x) = (1 - \alpha x)f(-x) + f(x) \tag{1.12}$$

We compare the terms $f(-x)$ from equations (1.11) and (1.12) and we gain the polynomial $f(x)$ as a function of $F(x)$ this way.

$$\alpha^2 x^2 f(x) = (\alpha x - 1)F(x) + F(-x) \tag{1.13}$$

Suppose that $F(x) = A_0 + A_1 x + \ldots + A_{n+1}x^{n+1}, \quad A_0, \ldots, A_{n+1} > 0$. Therefore

$$\alpha^2 x^2 f(x) = A_0 \alpha x - 2A_1 x + A_1 \alpha x^2 + \ldots$$

Thus $f(x)$ is the polynomial of the order n and the condition (1.11) is fulfilled for special choice $\alpha = \frac{2A_1}{A_0}$. We prove that $f(x)$ is also the Hurwitz's polynomial using a similar trick as in the previous lemma.

Consider polynomials

$$\Phi_m(x) = (\alpha x - 1)F(x) + mF(-x), \quad m \in \langle 0, 1\rangle. \tag{1.14}$$

The roots of the polynomial (1.14) are continuous bounded functions of the parameter m. The polynomial $\Phi_0(x) = (\alpha x - 1)F(x)$ has $n+1$ roots in the left half of the complex plane and $x_{n+2} = \frac{1}{\alpha}$ is in the right one.

The roots of (1.14) are placed this way for all values of the parameter $m \in \langle 0, 1)$. The curve $x_i = x_i(m)$ must intersect the imaginary axis to cross to the other half-plane. However, it is impossible for $m \in \langle 0, 1)$. The prove is identical as in the previous lemma so we would not give it here again.

We know that the polynomials $\Phi_m(x)$ have n+1 roots with negative real part and one with positive real part for $m \in \langle 0.1)$. However, the polynomial $\Phi_1(x)$ $(m = 1)$ has two null roots. Suppose that $x_l(m) \to 0$ and $x_k(m) \to 0$ as $m \to 1-$. We use the relation between the polynomial $a_0 + \ldots + a_n x^n$ and its roots: $\sum_{j=1}^{n} \frac{1}{x_j} = -\frac{a_1}{a_0}$.

$$\sum_{j=1}^{n+2} \frac{1}{x_j(m)} = \frac{A_1}{A_0} \tag{1.15}$$

as results from the form of $\Phi_m(x)$. Therefore one of the roots $x_k(m)$, $x_l(m)$ must have positive real part. If not, the left part of the equation (1.15) would tend to $-\infty$ and the right part would remain positive and it is impossible.

Hence $\Phi_1(x) = (\alpha x - 1)F(x) + F(-x)$ has two null roots (the coefficient, which stands before the term x^2, is positive) and n roots with negative real part. $\alpha^2 x^2 f(x) = (\alpha x - 1)F(x) + F(-x)$ and therefore $f(x)$ is really the Hurwitz's polynomial. □

Definition 9. Consider the polynomial

$$P_n(x) = a_0 + a_1 x + \ldots + a_n x^n, \quad n \geq 1, \quad a_i > 0 \quad \forall i \in \{0, \ldots, n\}. \tag{1.16}$$

The matrix
$$\begin{pmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \dots & a_n \end{pmatrix},$$
where $a_j = 0$ for $j < 0$ and $j > n$, is called the *Hurwitz's matrix* of the polynomial $P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$.

Theorem 4 (Hurwitz's criterion). All roots of polynomial (1.16) have negative real parts iff all main minors of the Hurwitz's matrix are positive, *i.e.*

$$\begin{aligned} D_1 &= a_1 > 0 \\ D_2 &= \det \begin{pmatrix} a_1 & a_0 \\ a_3 & a_2 \end{pmatrix} > 0 \\ &\vdots \\ D_n &= a_n D_{n-1} > 0. \end{aligned}$$

Proof. 1. $\Rightarrow$ We prove it using the mathematical induction.

$$n = 1 \quad f(x) = a_0 + a_1 x \Rightarrow x = -\frac{a_0}{a_1} < 0 \Rightarrow \Delta_1 = a_1 > 0$$

Suppose that the theorem is true for all Hurwitz's polynomials of the order n and $F(x)$ is the Hurwitz's polynomial of the order $n+1$. $F(x)$ can be written as a conjugated polynomial to the Hurwitz's polynomial of the order n. $F(x) = (1 + 2cx)f(x) + f(-x)$, where $c > 0$ and $f(x) = a_0 + \ldots + a_n x^n$.

$$F(x) = (1 + 2cx)(a_0 + a_1 + \ldots + a_n x^n) + (a_0 - a_1 x + \ldots + (-1)^n a_n x^n)$$

$$F(x) = 2a_0 + 2\sum_{k=1}^{n}(ca_{k-1} + \frac{1 + (-1)^k}{2} a_k)x^k + 2a_n c x^{n+1}$$

So the main minors of the Hurwitz's matrix look like

$$D_{k+1} = 2^{k+1} \begin{vmatrix} ca_0 & a_0 & 0 & 0 & \dots & 0 \\ ca_2 & ca_1 + a_2 & ca_0 & 0 & \dots & 0 \\ \vdots & & & & & \\ c_{2k} & ca_{2k-1} + a_{2k} & ca_{2k-2} & \dots & & \end{vmatrix}$$

$$D_{k+1} = 2^{k+1} c^{k+1} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \dots & a_n \end{vmatrix} = \alpha^{k+1} a_0 \Delta_k$$

Δ_k is the main minor of the Hurwitz's matrix of the polynomial $f(x)$. We know that a_0, α and Δ_k are positive from premises so D_{k+1} is also positive.

2.$\Leftarrow$

$$n = 1 \quad f(x) = a_0 + a_1 x \quad a_0 > 0 \quad \Delta_1 = a_1 > 0 \Rightarrow x = -\frac{a_0}{a_1} < 0$$

Suppose that the statement is true for all Hurwitz's polynomials of the order n, $F(x) = A_0 + A_1 x + \ldots + A_{n+1}x^{n+1}$ and $A_0 > 0, D_1 = A_1 > 0, \ldots, D_{n+1} > 0$. We can imagine polynomial $F(x)$ as the conjugated one to $f(x) = a_0 + \ldots + a_n x^n \quad (a_0 > 0, a_n \neq 0)$. $D_{k+1} = \alpha^{k+1} a_0 \Delta_k > 0$ from the premises and proved part of the theorem.

$$\alpha > 0 \Rightarrow \Delta_k > 0 \quad k \in \hat{n}$$

Thus $f(x)$ is Hurwitz's polynomial and therefore $F(x)$ is also Hurwitz's one as a result of lemma 3. □

Example 6. Consider a linear system of differential equations with real parameters p, q.

$$\begin{aligned} \dot{x}_1 &= -x_1 + px_2 \\ \dot{x}_2 &= qx_1 - x_2 + px_3 \\ \dot{x}_3 &= qx_2 - x_3. \end{aligned} \tag{1.17}$$

We determine the values of p, q, for which the null solution of (1.17) is asymptotically stable.

First we find the eigenvalues of the respective matrix A, the characteristic polynomial of the matrix A is

$$\det(A - \lambda I) = -(\lambda + 1)(\lambda^2 + 2\lambda + 1 - 2pq) \quad \Rightarrow \quad \lambda_1 = -1 \quad \lambda_{2,3} = -1 \pm \sqrt{2pq}.$$

Therefore the null solution of the system (1.17) is asymptotically stable iff $pq < \frac{1}{2}$.

The theorem 3 is necessary and sufficient when looking for the roots with negative real parts of the equation $\lambda^2 + 2\lambda + 1 - 2pq$. So $1 - 2pq > 0 \Rightarrow pq < \frac{1}{2}$.

Finally, we use the Hurwitz's criterion to examine the asymptotic stability of the null solution of (1.17). $\det(A - \lambda I) = \lambda^3 + 3\lambda^2 + (3 - 2pq)\lambda + 1 - 2pq$. We do know that $\lambda_1 = -1$, but the application of the Hurwitz's criterion for polynomials of the order less than three loses the sense.

First we must satisfy the necessary condition to create the Hurwitz's matrix ($a_i > 0 \ i \in \{0, 1, 2, 3\}$). Hence $pq < \frac{1}{2}$. So the Hurwitz's matrix has the following form for such parameters p, q that $pq < \frac{1}{2}$:

$$\begin{pmatrix} 3 - 2pq & 1 - 2pq & 0 \\ 1 & 3 & 3 - 2pq \\ 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
D_1 &= 3 - 2pq > 0 \Rightarrow pq < \frac{3}{2}\\
D_2 &= 8 - 4pq > 0 \Rightarrow pq < 2\\
D_3 &= 1D_2 > 0 \Rightarrow D_2 > 0
\end{aligned}
$$

Therefore $pq < \frac{1}{2}$. This example illustrates the fact that we must not forget the condition for positivity of the respective polynomial coefficients. Without this condition, the Hurwitz's criterion need not to give us the right results. ♠

1.4 Linearization of nonlinear systems

We have spent plenty of time investigating the stability of solutions of the systems of linear differential equations, although the most of dynamical systems are described by the nonlinear ones. The question arises whether it was a waste of time and energy or not.

Theorem 5. Suppose that

$$\dot{x} = Ax + g(x), \quad g(0) = 0, \tag{1.18}$$

where $A \in \mathbb{R}^{n,n}$ and g is a vector function, which is continuous in some domain $H \in \mathbb{R}^n$ $(0 \in H)$ and moreover g satisfies the condition

$$\lim_{|x| \to 0} \frac{|g(x)|}{|x|} = 0. \tag{1.19}$$

Then:

1. If all eigenvalues of A have negative real parts the stationary point $x = 0$ of the system (1.18) is asymptotically Liapounov stable.
2. If there exists at least one eigenvalue of the matrix A with positive real part then the stationary point $x = 0$ is Liapounov unstable.

We need three simple but useful lemmas to prove this theorem.

Lemma 5. Suppose that functions $\phi(t)$ and $\psi(t)$ have derivatives in an interval $I = (a, a+b)$ $(b > 0)$ and $\phi(t) < \psi(t)$ for all $t \in (a, a+\epsilon)$, where $b > \epsilon > 0$. Then there occurs one of the following cases:

1.$\phi(t) < \psi(t)$ $\forall t \in I$

2.There exists $t_0 \in I$ such that $\phi(t) < \psi(t)$ $\forall t \in (a, t_0)$, $\phi(t_0) = \psi(t_0)$ and $\dot{\phi}(t_0) \geq \dot{\psi}(t_0)$.

Proof. The possibility, that case 1 occurs, is obvious. Suppose that 1 is not true. Then there exists minimal $t_0 > a$ such that $\phi(t_0) = \psi(t_0)$. Moreover for all $h > 0$, $\phi(t_0 - h) < \psi(t_0 - h)$. Hence

$$\frac{\phi(t_0) - \phi(t_0 - h)}{h} > \frac{\psi(t_0) - \psi(t_0 - h)}{h}.$$

We get the case 2 for $h \to 0$. □

Lemma 6. Suppose that all eigenvalues λ_i $(i \in \hat{n})$ of the matrix $A \in \mathbb{R}^{n,n}$ satisfy the condition $\mathrm{Re}\lambda_i < \alpha$. Then $|e^{At}| \le ce^{\alpha t}$ for all $t \ge 0$ and convenient constant c.

Proof. From the previous, we know that the differential equation $\dot{x} = Ax$ has n linearly independent solutions in the form: $x(t) = e^{\lambda t}p(t)$, where λ is the eigenvalue of the matrix A and $p(t) = (p_1(t), \dots, p_n(t))^T$ is a vector, which components are polynomials of the order $\le n$. Denote $\alpha - \mathrm{Re}\lambda$ as β. $|p_i(t)| \le c_i e^{\beta t}$ because of positivity of β. Hence

$$|e^{\lambda t}p_i(t)| \le e^{\beta + \mathrm{Re}\lambda t}c_i = c_i e^{\alpha t}$$

□

Lemma 7. Suppose that I is such an interval that $t_0 \in I$ and γ is a positive constant. Suppose that functions $\xi, \phi : I \to \mathbb{R}$ are continuous and nonnegative in I and moreover:

$$\xi(t) \le \gamma + \left| \int_{t_0}^{t} \rho(\tau)\xi(\tau)d\tau \right| \quad t \in I \tag{1.20}$$

Then

$$\xi(t) \le \gamma e^{|\int_{t_0}^{t} \rho(\tau)d\tau|} \quad t \in I.$$

Proof. We prove it for $t \ge t_0$, for $t < t_0$, the proof is analogous.

$$\frac{\rho(t)\xi(t)}{\gamma + \int_{t_0}^{t} \rho(\tau)\xi(\tau)d\tau} \le \rho(t) \quad t \ge t_0$$

as results from (1.20). We integrate both sides of the previous equation:

$$\ln(\gamma + \int_{t_0}^{t} \rho(\tau)\xi(\tau)d\tau) - \ln\gamma \le \int_{t_0}^{t} \rho(\tau)d\tau$$

$$\xi(t) \le \gamma + \int_{t_0}^{t} \rho(\tau)\xi(\tau)d\tau \le \gamma e^{\int_{t_0}^{t} \rho(\tau)d\tau}$$

□

Proof of the theorem 5. 1. From the lemma 6, we know that there exist $c > 1$, $\beta > 0$ such that $\mathrm{Re}\lambda_i < -\beta$ and $|e^{At}| \le ce^{-\beta t}$ for $t \ge 0$. The condition (1.19) implies:

$$\exists \delta > 0 \quad |x| < \delta \Rightarrow |g(x)| < \frac{\beta}{2c}|x|.$$

We want to show that if the norm x_0 is sufficiently small, then the norm of the solution of (1.18) with initial condition x_0 tends to zero.

All solutions of the equation (1.18) with initial condition $x(0) = x_0$ can be written in the form:

$$x(t) = e^{At}x_0 + \int_{0}^{t} e^{A(t-\tau)}g(x(\tau))d\tau.$$

Hence if $|x(t)| \leq \delta$

$$|x(t)| \leq |x_0|ce^{-\beta t} + \int_0^t \frac{\beta}{2} e^{-\beta(t-\tau)} |x(\tau)| d\tau.$$

Suppose that $|x_0| < \epsilon$ and $\phi(t) = |x(t)|e^{\beta t}$.

$$\phi(t) \leq c\epsilon + \frac{\beta}{2} \int_0^t \phi(\tau) d\tau$$

Using the previous lemma, we get $\phi(t) \leq c\epsilon e^{\beta t/2} \Leftrightarrow |x(t)| \leq c\epsilon e^{-\frac{\beta t}{2}}$.

2. We transfer the equation (1.18) in a more convenient form. Suppose that the matrix H transfers the matrix A to its Jordan normal form Q ($Q = H^{-1}AH$). Assume that $\alpha > 0$ and the matrix B is a diagonal matrix $B = \mathrm{diag}(\alpha, \alpha^2, \ldots, \alpha^n)$. It is easy to see that $B^{-1} = \mathrm{diag}(\alpha^{-1}, \ldots, \alpha^{-n})$.

By using the substitution $x(t) = HBy(t)$, we get the equation (1.18) in the form:

$$\dot{y} = B^{-1}H^{-1}(AHBy + g(HBy)), \quad \text{where} B^{-1}H^{-1}AHB = B^{-1}QB := C$$

$$\dot{y} = Cy + f(y) \quad \text{where} f(y) = B^{-1}H^{-1}g(HBy). \tag{1.21}$$

The matrix $C = B^{-1}QB \Leftrightarrow d_{ii} = \lambda_i \quad d_{i,i+1} = 0$ or α as results from the form of normal Jordan matrix Q.

We would like to know if the function $f(y)$ also satisfies the condition (1.19).

$$\lim_{|x| \to 0} \frac{|g(x)|}{|x|} = 0 \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(|x| < \delta \Rightarrow |g(x)| < \epsilon |x|).$$

$$|f(x)| = |B^{-1}H^{-1}g(HBx)| \leq |B^{-1}H^{-1}|\epsilon|HB||x| \quad \text{for} |x| < |HB|\delta$$

Hence $f(x)$ has similar properties as $g(x)$ for sufficiently small x.

Let us write the equation (1.21) in the components:

$$\dot{y}_i = \lambda_i y_i + [\alpha y_{i+1}] + f_i(y) \quad i \in \hat{n}. \tag{1.22}$$

From the form of the matrix C, or if you like Q, we know that the term in square brackets is nonzero iff i denotes a Q matrix row in the matrix cell of the order grater than 1 and moreover it is not the last such row.

Denote as $j(k)$ all such indexes, for which $\mathrm{Re}\lambda_j > 0$ ($\lambda_k \leq 0$) and define the functions $\phi(t) = \sum_{j=1}^n |y_j(t)|^2$ and $\psi(t) = \sum_{k=1}^n |y_k(t)|^2$, where $y(t)$ is the solution of (1.21). We choose such $\alpha > 0$ that $0 < 6\alpha < \mathrm{Re}\lambda_j \; \forall j$ and $\delta > 0$ so small that it satisfies: $|f(y)| < \alpha |y|$ for $|y| < \delta$.

Suppose that $y(t)$ is a solution with initial conditions:

$$|y(t_0)| < \delta \quad \psi(t_0) < \phi(t_0). \tag{1.23}$$

If $|y(t)| \leq \delta$ and $\psi(t) \leq \phi(t)$ we can write:

$$\dot{\phi}(t) = \sum_{j=1}^{n}(\dot{y}_j(t)\overline{y_j(t)} + \overline{\dot{y}_j(t)}y_j(t)) = 2\sum_{j=1}^{n}\mathrm{Re}(\dot{y}_j(t)\overline{y_j(t)})$$

Using the relation (1.22), we get:

$$\dot{\phi}(t) = 2\sum_{j=1}^{n}\mathrm{Re}(\lambda_j y_j(t)\overline{y_j(t)} + [\alpha y_{j+1}(t)\overline{y_j(t)}] + \overline{y_j(t)}f_j(y(t))) \tag{1.24}$$

Let us look at the particular components in a more detail.

$$\sum_{j=1}^{n}\mathrm{Re}(\lambda_j y_j(t)\overline{y_j(t)}) = \sum_{j=1}^{n}\mathrm{Re}\lambda_j|y_j(t)|^2 > 6\alpha\phi(t) \tag{1.25}$$

$$\sum_{j=1}^{n}\mathrm{Re}(y_{j+1}(t)\overline{y_j(t)}) \le \sum_{j=1}^{n}|y_j(t)y_{j+1}(t)| \le \sqrt{\sum_{j=1}^{n}|y_j(t)|^2.\sum_{j=1}^{n}|y_j(t)|^2} = \phi(t) \tag{1.26}$$

The last but one step results from the Schwartz inequality.

$$\sum_{j=1}^{n}\mathrm{Re}(\overline{y_j(t)}f_j(y(t))) \le \sqrt{\sum_{j=1}^{n}|y_j(t)|^2.\sum_{j=1}^{n}|f_j(y(t))|^2} \le \sqrt{\phi(t)}|f_j(y(t))| \le 2\alpha\phi(t) \tag{1.27}$$

The last inequality is a consequence of the following:

$$|f(y(t))| \le \alpha|y(t)| \le \alpha\sqrt{\phi(t)+\psi(t)} \le 2\alpha\sqrt{\phi(t)}$$

Hence

$$\frac{1}{2}\dot{\phi}(t) > 6\alpha\phi(t) - \alpha\phi(t) - 2\alpha\phi(t) = 3\alpha\phi(t). \tag{1.28}$$

The relation (1.24) is also true for $\psi(t)$ but we must not forget that $\mathrm{Re}\lambda_k \le 0$.

$$\sum_{j=1}^{n}\mathrm{Re}(\lambda_j y_j(t)\overline{y_j(t)}) \le 0$$

$$\sum_{j=1}^{n}\mathrm{Re}(y_{j+1}(t)\overline{y_j(t)}) \le \alpha\psi(t)$$

$$\sum_{j=1}^{n}\mathrm{Re}(\overline{y_j(t)}f_j((y(t)))) \le 2\alpha\phi(t)$$

Hence $\frac{1}{2}\dot{\psi}(t) \leq \alpha\psi(t) + 2\alpha\phi(t)$. We have assumed that $\psi(t) \leq \phi(t)$. We summarize the previous :

$$\dot{\phi}(t) > 6\alpha\phi(t)$$
$$\dot{\psi}(t) \leq 2\alpha\psi(t) + 4\alpha\phi(t) \leq 6\alpha\phi(t) < \dot{\phi}(t).$$

Moreover, $\psi(t_0) < \phi(t_0)$. As a consequence of lemma 5, we see that $\psi(t) < \phi(t)$. All solutions $y(t)$, which satisfy the initial condition (1.23) and moreover $|y(t)| \leq \delta$, fulfill also $\psi(t) < \phi(t)$ and $\dot{\phi}(t) > 6\alpha\phi(t)$. Hence $\phi(t) \geq \phi(t_0)e^{6\alpha t}$ and therefore for all such solutions, there exists such t_1 that $|y(t_1)| = \delta$. The stationary solution $y(t) = 0$ cannot be Liapounov stable. □

Example 7. Consider the system of differential equations

$$\begin{aligned} \dot{x} &= -x - 9y + 3x^2 - 24y^2 + 2x^5 \\ \dot{y} &= x - y + x^2 - 7xy. \end{aligned} \tag{1.29}$$

The linearization of the system has the following form:

$$\dot{x} = -x - 9y \qquad \dot{y} = x - y.$$

The eigenvectors are $-1 \pm 3i$ so the origin is asymptotically stable. The solution of the system (1.29) with initial condition near the origin really tend to it as can be seen in the figure 1.3. The solution was gained using the ode15s solver for stiff differential equations in the program MATLAB.

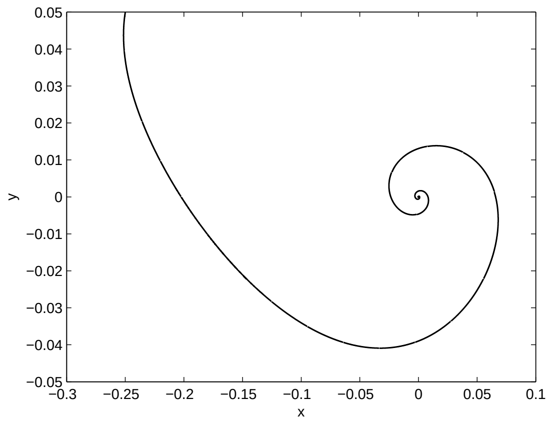

Figure 1.3: Solution of the system (1.29)

♠

Chapter 2

Bifurcations

The word bifurcation denotes a situation in which the solutions of a nonlinear system of differential equations alter their character with a change of a parameter on which the solutions depend. Bifurcation theory studies these changes (*e.g.* appearance and disappearance of the stationary points, dependence of their stability on the parameter etc.)

In this chapter, we use the stationary solutions of some simple differential equations to describe the most important types of bifurcations. We prefer the heuristic approach to the rigorous mathematical description, that can be found in *e.g.* [4] or [1].

2.1 Transcritical bifurcation

Let us take the first-order differential equation:

$$\frac{dx}{dt} = x(a - c - abx) \tag{2.1}$$

with positive constants b, c. We try to show how the stability of stationary points depends on the parameter a.

This equation has two stationary points:

$$x = 0 \quad \forall a \in \mathbb{R} \qquad\qquad x = \frac{a-c}{ab} \quad \forall a \in \mathbb{R} \setminus \{0\}$$

In order to investigate the stability of the null solution, we linearize the equation (2.1):

$$\frac{dx}{dt} = (a - c)x.$$

It is not too difficult to see that its solution is:

$$x(t) = x_0 e^{(a-c)t}, \quad \text{where} \quad x_0 = x(0).$$

Thus the null solution is stable for $a < c$ and unstable for $a > c$. The linearized system is not able to determine the stability of the null solution in case $a = c$. Fortunately, this equation is simply soluble.

$$\frac{dx}{dt} = -abx^2 \qquad -\frac{1}{x^2}\frac{dx}{dt} = ab \qquad \frac{dx^{-1}}{dt} = ab$$

$$x^{-1}(t) = abt + {x_0}^{-1} \qquad x(t) = \frac{x_0}{x_0 abt + 1}$$

We see that $x(t) \to \infty \quad$ as $\quad t \to -\frac{1}{abx_0}$, so the null solution is unstable for $a = c$.

To examine the stability of the stationary solution $x = \frac{a-c}{ab}$, we change the coordinates in order to arrange $x = \frac{a-c}{ab}$ to the origin. We may similarly as in the previous case show that the solution $x = \frac{a-c}{ab}$ is stable for $c < a$ and unstable for $c \geq a$.

We plot the stationary points versus the value of a parameter a in so called bifurcation diagram - figure 2.1 (the diagram is drawn for $c = b = 1$). It is conventional to draw the stable solutions as continuous curves and the unstable ones as dashed curves.

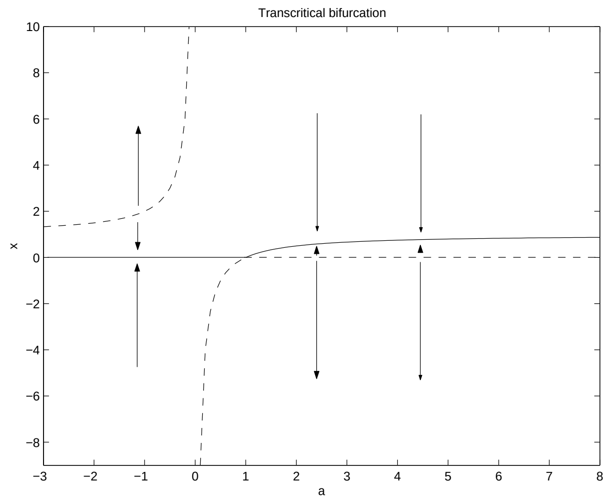

Figure 2.1: Transcritical bifurcation

The situation for $a = c$ is called a *transcritical bifurcation*. It occurs when one stable and one unstable fixed points cross each other. At the crossing point they exchange their stability property. So the stable stationary point becomes unstable and vice versa.

2.2 Saddle-node bifurcation

The number of stationary points of the differential equation

$$\frac{dx}{dt} = a - x^2 \tag{2.2}$$

depends on the value of the parameter a. There are two fixed points for $a > 0$, one fixed point for $a = 0$ and none for $a < 0$.

Take the stationary solution $x = A$, $A := \pm\sqrt{a}$ for $a > 0$. By changing the coordinates ($y = x - A$) we obtained the differential equation

$$\frac{dy}{dt} = -y^2 - 2Ay.$$

We can linearize it for the null solution:

$$\frac{dy}{dt} = -2Ay \qquad y = y_0 e^{-2At}$$

So the solution $x = A$ is stable for $A > 0$ and ustable for $A < 0$.

To determine the stability of the solution $x = 0$, we must solve the equation (2.2) explicitly. Similar equation was solved in previous section. So its solution is:

$$x = \frac{x_0}{1 + x_0 t}$$

and it is unstable.

Now we do know that the stationary points $x = \sqrt{a}$ are stable for $\forall a > 0$ and the stationary points $x = -\sqrt{a}$ are unstable for $\forall a \geq 0$. That is all that we need for plotting the bifurcation diagram - figure 2.2.

The situation at the origin is called a *saddle-node bifurcation* and occurs when a stable fixed point (a node) collides and annihilates with an unstable one (a saddle).

2.3 Pitchfork bifurcation

Consider the differential equation

$$\frac{dx}{dt} = ax - bx^3, \quad a, b \in \mathbb{R}. \tag{2.3}$$

The equilibrium points are $x = 0$ $\forall a$ and $x = \pm\sqrt{\frac{a}{b}}$ $\forall a, b$ such that $\frac{a}{b} > 0$. To determine the stability of the null solution, we take the linearized system: $\frac{dx}{dt} = ax$ with solution $x(t) = x_0 e^{at}$. It is obvious that the null solution is stable for $a < 0$ and unstable for $a > 0$. The linear criterion is not sufficient in case $a = 0$ and we have to solve the equation (2.3) explicitly.

$$\frac{dx}{dt} = -bx^3 \qquad -\frac{1}{x^3}\frac{dx}{dt} = b \qquad \frac{dx^{-2}}{dt} = 2b$$

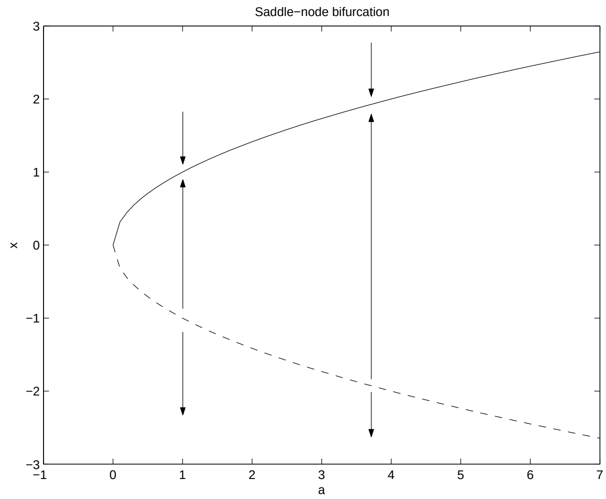

Figure 2.2: Saddle-node bifurcation

$$x^{-2}(t) = 2bt + {x_0}^{-2} \qquad x^2(t) = \frac{{x_0}^2}{1 + 2bt{x_0}^2} \quad x(t) = \sqrt{x^2(t)}\mathrm{sgn} x_0$$

So the solution $x = 0$ is stable for $b > 0$ and unstable for $b < 0$.

To investigate the stability of the solution $x = \pm\sqrt{\frac{a}{b}}$, we change the coordinates again and we obtain this linearized system for the null solution $\frac{dy}{dt} = -2ay$ with solution $y(t) = e^{-2at}$. We say that the stationary points $x = \pm\sqrt{\frac{a}{b}}$ are stable (unstable) for $a > 0$ ($a < 0$) on account of this solution.

We obtain two different bifurcation diagrams for $b > 0$ (figure 2.3) and $b < 0$ (figure 2.4).

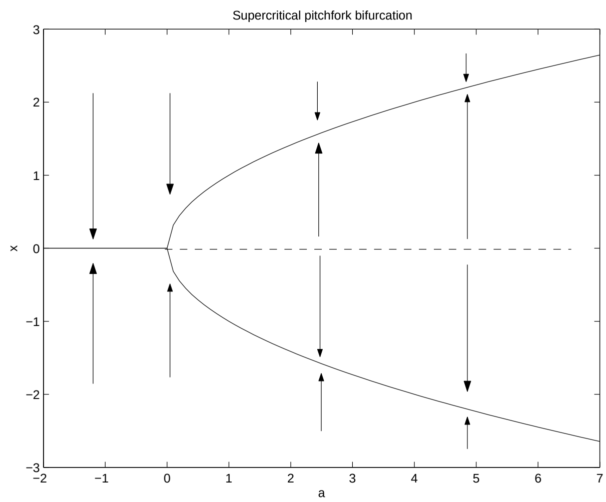

Figure 2.3: Supercritical pitchfork bifurcation

There is a unique stationary point $x = 0$ for $a \leq 0$ but three fixed points for $a > 0$. The bifurcated solutions $x = \pm\sqrt{\frac{a}{b}}$ are stable whenever they exist and they appear as the parameter a increases above its critical value. It is called *supercritical pitchfork bifurcation.* The name *pitchfork* is not much surprising considering the appearance of the bifurcation.

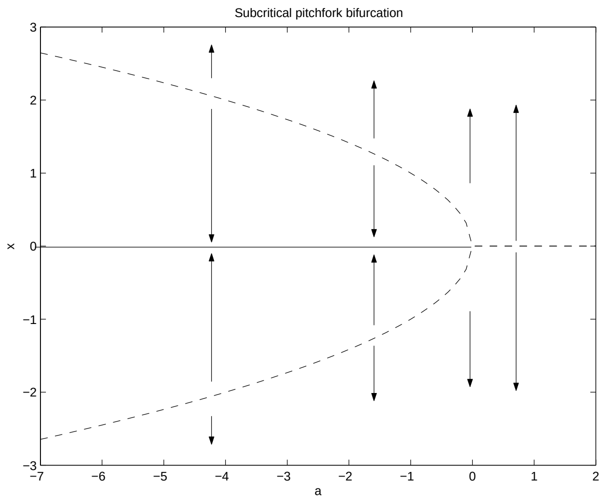

Figure 2.4: Subcritical pitchfork bifurcation

The situation for $b < 0$ is similar to the previous case, but the bifurcated solutions $x = \pm\sqrt{\frac{a}{b}}$ arise as a decreases underneath its critical value and they are unstable whenever they exist. This bifurcation is called a *subcritical pitchfork bifurcation.*

2.4 Hopf bifurcation

We have seen that the stationary point loses its stability when the eigenvalue crosses the imaginary axis (the control parameter must increase or decrease its critical value). We know that the eigenvalue can be a complex number. In this case, the conjugated pair of eigenvalues crosses the imaginary axis together. This phenomenon is called a *Hopf bifurcation.*

It is obvious that the phase space must be at least two-dimensional in order for Hopf bifurcation to occur.

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= ax - by - (x^2 + y^2)x \\ \dot{y} &= bx + ay - (x^2 + y^2)y, \end{aligned} \tag{2.4}$$

where a, b are real parameters. The origin is a stationary point and we try to investigate its stability. The linearization of the system (2.4) has the following form:

$$\dot{x} = ax - by \quad \dot{y} = bx + ay.$$

The eigenvalues are $a \pm ib$. Therefore the origin is stable for $a < 0$ and unstable for $a > 0$. We expect that some bifurcation occurs when $a = 0$ (similarly as in the previous examples). We use polar coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$) to solve the system (2.4) explicitly. We see that $x + iy = re^{i\varphi}$. Hence

$$\begin{aligned} \frac{d(re^{i\varphi})}{dt} &= \frac{dx}{dt} + i\frac{dy}{dt} \\ \frac{d(re^{i\varphi})}{dt} &= \Big(\frac{dr}{dt} + ir\frac{d\varphi}{dt}\Big)e^{i\varphi} \\ \frac{dx}{dt} + i\frac{dy}{dt} &= ax - by - (x^2 + y^2)x + ibx + iay - i(x^2 + y^2)y = (ar - r^3 + ibr)e^{i\varphi}. \end{aligned}$$

We compare the real and imaginary parts

$$\dot{r} = ar - r^3 \quad \dot{\varphi} = b. \tag{2.5}$$

We immediately see the solution $\varphi(t) = bt + \varphi_0$. The differential equation, which describes the radius evolution, is coincidentally the same as in the section on pitchfork bifurcations. Therefore, the origin is stable for $a \leq 0$. The condition $\dot{r} = 0$ is fulfilled for two points $r = 0$ and $r = \sqrt{a}$. We see, that for $a > 0$, the origin is unstable but a new stationary solution (a periodic orbit) appears and this orbit is stable. The stability of the orbit results from: $r^2(t) = \frac{ar_0^2}{r_0^2 + (a - r_0^2)e^{-2at}} \quad (a \neq 0)$. The appearance and disappearance of the cycle is called a Hopf bifurcation and it is quite common phenomenon.

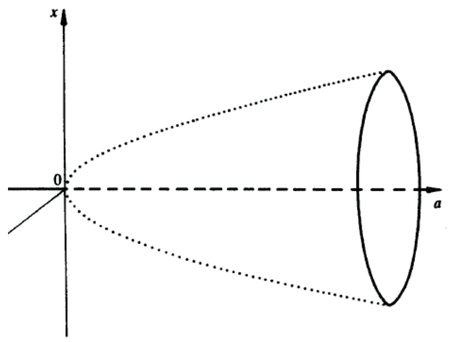

Figure 2.5: The bifurcation diagram - Hopf bifurcation

2.5 Classification of the stationary points

2.5.1 One-dimensional case

The phase space is just the x-axis in one-dimensional case and the time evolution of the point x_0 is determined by the equation $\dot{x} = f(x)$. Until now, we have met stable and unstable stationary points. We will extend our knowledge about the stationary points in this section.

Suppose that $x = a_0$ is a stationary point of the equation $\dot{x} = f(x)$. Take the point $x = a_0 + a$, where a is sufficiently small, and look at the possible behaviour of this point.

We know the Taylor expansion of the function $f(x)$ for $x = a_0 + a$.

$$f(x) = f(a_0) + af'(a_0) + \frac{a^2}{2} f''(a_0) + \dots$$

The first term on the right side is equal to zero by the definition of stationary point. The value of f' at a_0 is called the *eigenvalue of stationary point* a_0 (or often a *Liapounov exponent*) and it is denoted as $\lambda = \frac{df}{dx}(a_0)$.

Suppose that $\lambda < 0$. Then the point x decreases toward a_0 from the right and increases to a_0 from the left. Therefore a_0 attracts nearby trajectories. This type of stationary points is called a *node*.

Assume that $\lambda > 0$. Then conversely, the trajectories move away from a_0 on both sides. Such stationary point is called a *repellor*.

Finally, $\lambda = 0$. This case is more difficult than the previous ones because the stationary point a_0 can be both a node and a repellor, or the third possibility can occur when the stationary point will attract trajectories on one side and repel them on the other. Such stationary point is called a *saddle point*.

When $\lambda = 0$ the first nonzero term is f''. The change of the sign of the second derivative of f as x passes through a_0 is necessary in order to a_0 could be a repellor or a node. It is easy to see that the second derivative must be positive from the left and negative from the right for the node.

The last possibility, when the second derivative has the same sign on both sides of a_0, is a saddle point. There can occur two cases: the sign of the derivative is positive, such saddle point is called type I saddle point, and the sign is negative - type II saddle point. So the type I saddle point attract trajectories from the left and repel them from the right.

There can exist more than one stationary point for equation $\dot{x} = f(x)$. The smoothness of the function f sets bounds for the types of the stationary points that can be placed nearby themselves. Let us take two repellors. It is obvious that they cannot neighbour and moreover the stationary point between them must be a node. Conversely, two nodes need a repellor between them.

Type I saddle point cannot lie nearby the type II saddle. The stationary point between them is a node. Similarly, the type II saddle point and type I saddle point must have a repellor between them. On the other hand, the saddle points of the same type can have themselves as a neighbour.

Example 8. Consider the equation $\dot{x} = \cos x$. We see that there exists infinite number of fixed points $x_k = (2k+1)\frac{\pi}{2}$, where $k \in \mathbb{Z}$. The same way as the values of the Liapounov exponents -1 change for 1, the nodes change for the repellors. So we have a never-ending chain of the repellors and nodes (figure 2.6).

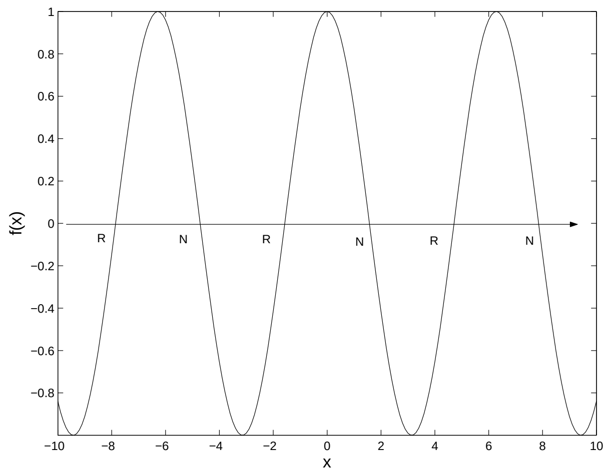

Figure 2.6: The stationary points of $\dot{x} = \cos x$

The solutions of the equation $\dot{x} = \cos x$ are illustrated on the following figure 2.7. We see that the solution with an initial condition near the repellor really moves away from it to the nearest node and then remains constant because the node is a stationary point.

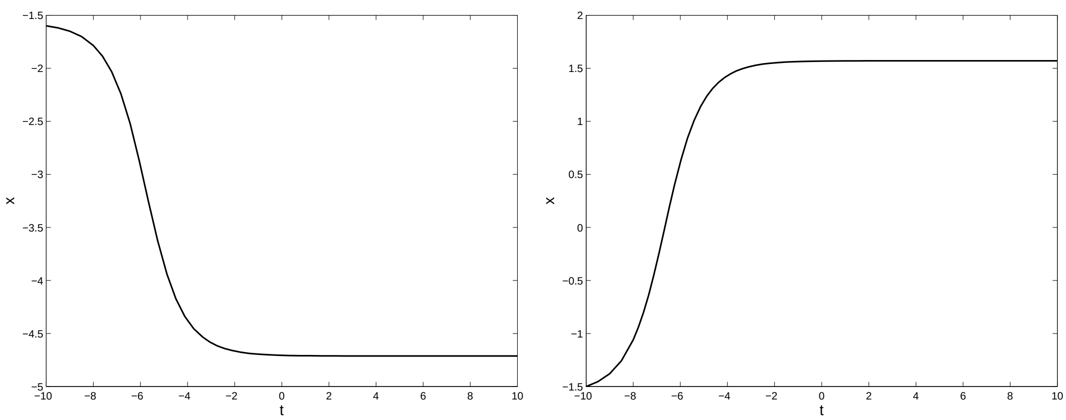

Figure 2.7: Solutions of the equation $\dot{x} = \cos x$

♠

2.5.2 Two-dimensional case

We would like to extend previous considerations about stationary points to two-dimensional phase space. Suppose that

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

are the differential equations describing the dynamical system with a stationary point $\vec{a}_0 = (a_{01}, a_{02})$. Just as in one dimension, we expect that the type of stationary point $\vec{a}_0$ depends on the partial derivatives: $\frac{\partial f_1}{\partial x_1}$, $\frac{\partial f_1}{\partial x_2}$, $\frac{\partial f_2}{\partial x_1}$ and $\frac{\partial f_2}{\partial x_2}$. The character of the dependence will be discussed in the following.

Let us again take the point $\vec{x} = (x_1, x_2)$, which is sufficiently close to the stationary point $\vec{a}_0$, and write the Taylor expansion of the functions $f_1(x), f_2(x)$.

$$f_1(\vec{x}) = (x_1 - a_{01})\frac{\partial f_1}{\partial x_1}(\vec{a}_0) + (x_2 - a_{02})\frac{\partial f_1}{\partial x_2}(\vec{a}_0) + \ldots \tag{2.6}$$

$$f_2(\vec{x}) = (x_1 - a_{01})\frac{\partial f_2}{\partial x_1}(\vec{a}_0) + (x_2 - a_{02})\frac{\partial f_2}{\partial x_2}(\vec{a}_0) + \ldots \tag{2.7}$$

We have omitted the terms $f_1(\vec{a}_0), f_2(\vec{a}_0)$ because they are zero by the definition of stationary point and ignored the derivatives of the order higher than the first. (The analysis, when $\frac{\partial f_i}{\partial x_j} = 0$ $i, j \in \hat{2}$, is analogous as in the one-dimensional phase space but we will be not do it here.)

We introduce new variables $y_1 = x_1 - a_{01}$, $y_2 = x_2 - a_{02}$ that represent the distance between the nearby point $\vec{x}$ and the stationary point $\vec{a}_0$. We have met a node, a repellor and a saddle point. We await that the distance tend to zero for a node and to infinity for a repellor. For a saddle point, we expect the different sings of the eigenvalues.

$$\dot{y}_1 = \frac{\partial f_1}{\partial x_1}(\vec{a}_0)y_1 + \frac{\partial f_1}{\partial x_2}(\vec{a}_0)y_2$$
$$\dot{y}_2 = \frac{\partial f_2}{\partial x_1}(\vec{a}_0)y_1 + \frac{\partial f_1}{\partial x_1}(\vec{a}_0)y_2. \tag{2.8}$$

It is a system of differential equations with constant coefficients. We have familiarized ourselves with properties of its solutions in the first chapter. First, we must create so-called *Jacobian matrix* J of the vector function f to find the eigenvalues of system (2.8).

$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$, where the partial derivatives are evaluated at the stationary point $\vec{a}_0$.

Using the terminology of the linear algebra, we obtain the eigenvalues in the form:

$$\lambda_{1,2} = \frac{\mathrm{Tr}J \pm \sqrt{(\mathrm{Tr}J)^2 - 4\mathrm{det}J}}{2}.$$

The solution of (2.8) is then:

$$y_1(t) = c_1 h_1^1 e^{\lambda_1 t} + c_2 h_1^2 e^{\lambda_2 t}$$
$$y_2(t) = c_1 h_2^1 e^{\lambda_1 t} + c_2 h_2^2 e^{\lambda_2 t},$$

where $c_i \in \mathbb{R}$ and $h^{(i)}$ is the eigenvector respective to eigenvalue λ_i, $i \in \hat{2}$.

First, we assume that the eigenvalues are real, *i.e.*

$$(\mathrm{Tr}J)^2 - 4\det J \geq 0 \quad \Leftrightarrow \quad \det J \leq \frac{1}{4}(\mathrm{Tr}J)^2$$

If $\det J < 0$ the eigenvalues have opposite signs and hence the stationary point is a saddle.

Both eigenvalues must be negative (positive) for $\vec{a}_0$ to be a node (a repellor). Therefore $\mathrm{Tr}J < 0$ ($\mathrm{Tr}J > 0$) and $\det J > 0$ in both cases.

Let us discuss the complex eigenvalues, *i.e.* $\det J > \frac{1}{4}(\mathrm{Tr}J)^2$. If we denote $\frac{\mathrm{Tr}}{2}J$ as a and $\frac{\sqrt{(TrJ)^2 - 4\det J}}{2}$ as b, we can write the solution in the form:

$$y_j(t) = \frac{e^{at}}{2}(c_1 h_j^1 e^{ibt} + c_2 h_j^2 e^{-ibt}), \quad j \in \hat{2}.$$

We see that $y_j(t)$ oscillates with increasing ($a > 0$) or decreasing ($a < 0$) amplitude. Therefore if $\mathrm{Tr}J < 0$ then the points in a neighbourhood of $\vec{a}_0$ tend to it on the spiral and $\vec{a}_0$ is called a *spiral node.* Analogously, if $\mathrm{Tr}J > 0$ the stationary point $\vec{a}_0$ is called a *spiral repellor.*

The results are summarized in the figure 2.8, the figure was taken up from [13].

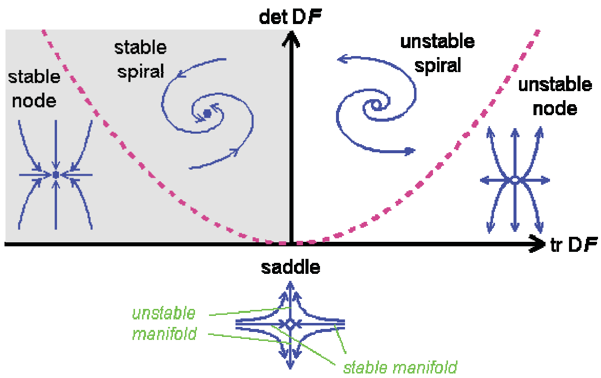

Figure 2.8: Stationary points in two dimensions

We have noticed that our classification of stationary points differs from the classification in [1]. The author of [1] made the following mistake in the solution of eigenvalues $\lambda_{1,2} = \frac{-\mathrm{Tr}J \pm \sqrt{(\mathrm{Tr}J)^2 - 4\det J}}{2}$ and therefore a node becomes a repellor and vice versa.

Example 9. We try to find the character of stationary points of the following system

$$\dot{x} = 3x - x^2 - 2xy \qquad \dot{y} = 2y - xy - y^2.$$

This system has four stationary points: $(0,0)$, $(0,2)$, $(1,1)$ and $(3,0)$. We make up the Jacobian matrix J and evaluate it in the stationary points to determine the type of these stationary points.

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$$

$$(0,0) \quad J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \mathrm{Tr}J = 5 \quad \mathrm{det}J = 6 \Rightarrow \text{a repellor}$$

$$(0,2) \quad J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \quad \mathrm{Tr}J = -3 \quad \mathrm{det}J = 2 \Rightarrow \text{a node}$$

$$(3,0) \quad J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \quad \mathrm{Tr}J = -4 \quad \mathrm{det}J = 3 \Rightarrow \text{a node}$$

$$(1,1) \quad J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \quad \mathrm{Tr}J = -2 \quad \mathrm{det}J = -1 \Rightarrow \text{a saddle}$$

♠

2.5.3 Three-dimensional case

We classify the stationary points in three-dimensional case quite quickly because it is practically the same as in the previous case. The dynamical system is described by a system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= f_3(x_1, x_2, x_3). \end{aligned}$$

The character of each stationary point x_0 is fully determined by the eigenvalues of the Jacobian matrix $J_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$. We do not try to find explicit form of each eigenvalue as in the previous case because it is tremendous and for purposes of investigating the stability also useless.

Definition 10. The number of eigenvalues of Jacobian matrix respective to stationary point x_0, whose real parts are positive, is called the index of the stationary point x_0.

This term is introduced for systems with three or more dimension and in geometric terms, it is the dimension of the unstable subset E^u.

In three dimensions, these stationary points can appear:

1. **A node** - All eigenvalues are real and negative. All nearby points are attracted to it without looping around the fixed point.

2.**A spiral node** - All eigenvalues have negative real parts and two of them have nonzero imaginary part. The nearby points spiral to it.

3.**A repellor**- All eigenvalues are real and positive. All nearby trajectories move away from it without looping around the fixed point.

4.**A spiral repellor** - All eigenvalues have positive real parts and two of them have nonzero imaginary part. All nearby trajectories spiral away from the fixed point.

5.**A saddle point - index 1** - All eigenvalues are real. One is positive and two negative. Trajectories approach it on a surface and move away from it along a curve.

6.**A spiral saddle point - index 1** - One positive eigenvalue and complex conjugated pair with negative real part. Nearby points spiral to it on a surface and diverge from it along a curve.

7.**A saddle point - index 2** - All eigenvalues are real. One is negative and two positive. Trajectories approach it on a curve and move away from it on a surface.

8.**A spiral saddle point - index 2** - One negative eigenvalue and complex conjugated pair with positive real part. Nearby points spiral away from it on a surface.

The figure 2.9 represents the stationary points in the three-dimensional phase space and their location in the complex plane (the figure was taken up from [5]).

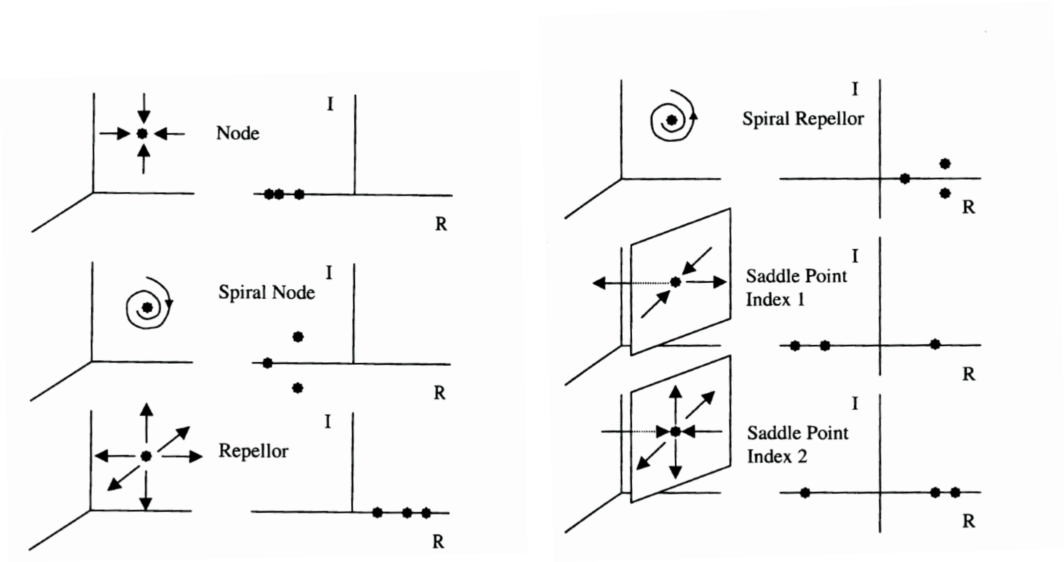

Figure 2.9: Stationary points in three dimensions

Chapter 3

Chaotic oscillators

In this chapter, we would like to show that the chaotic behaviour is not the subject limited only to theoretical models but it occurs in real physical systems and these systems are often very simple.

We have chosen two electrical circuits and we try to analyze their behaviour. First, we look at the circuits analytically, and then we simulate their behaviour using the ode15s MATLAB solver for stiff differential equations.

Before doing this, we briefly introduce the components of the circuit, their function, and how they affect to the circuit behaviour.

A resistor - The current passing through a resistor is directly proportional to the voltage across a resistor $V = IR$, where the proportionality constant R is called the resistance.

A capacitor - It is an electrical device that can store energy in the electrical field $V = \frac{Q}{C}$. The measure C of the amount of charge Q stored in a capacitor is called the capacitance.

An inductor - The important property of an inductor is that it produces an electrical potential difference across it: $V = L\frac{dI}{dt}$, where the proportionality constant L is called the inductance. It has to be emphasized that no chaotic circuit gets along without an inductor. Without it, the current and potential differences are so tightly joined that there is no possibility for chaotic behaviour.

A diode - A diode is an electrical device, which allows an electrical current to flow in one direction (this direction is called a forward-bias direction and it is indicated by the vertex of the triangle in diode´s circuit symbol), but blocks it in the other (reverse-bias direction). We can understand the basic function of a diode using a hydraulic analogy. We imagine a diode as a water pipe with a flap valve. The valve can deflect in one direction to allow water to flow but it closes when water tries to flow in the opposite direction.

The first important property of the pipe with a flap valve (a diode) is that the valve does not close immediately so a small amount of water (current) always passes through in the reverse-bias direction. The time necessary for closing is called the *reverse-recovery time* and it is usually a few microseconds.

The second property is that the reverse-recovery time depends on the flow volume (the size of current). If only a small amount of water flows, the valve is deflected just a little bit and it can close quite fast. If we increase the flow volume, the valve need more time for closing.

If we combine an inductor with a capacitor and if we put them in the circuit with a diode, while the frequency of circuit´s current oscillations ($f = \frac{2\pi}{\sqrt{LC}}$) is about the reverse-recovery time, the current is changing enough that the nonlinearity (changing the forward-bias to reverse-bias current) becomes important and chaos becomes possible.

3.1 Vilnius oscillator

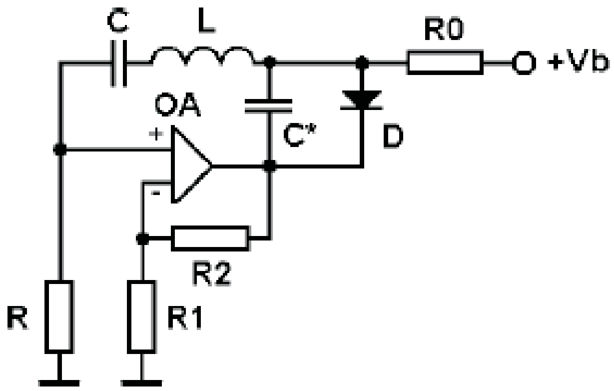

Figure 3.1: Vilnius oscillator

Figure 3.1 shows the circuit diagram of the chaotic oscillator (we will call it the Vilnius oscillator [8]). It consists of an operational amplifier (LM741), a RLC loop, an extra capacitor, three resistors (the variable resistor R_2 serves as a control parameter of this dynamical system) and finally, a diode (1N4148) as a nonlinear element.

There are three dynamical variables in the Vilnius oscillator: the voltages across the capacitors C, C^* - V_C, V_{C^*} and the current through the inductor L - I_L. We try to determine how the variables V_C, V_{C^*}, I_L depend on the other components of the circuit.

The current through the capacitor C must be equal to the current through the inductor L: $C\frac{dV_c}{dt} = I_L$. Applying the second Kirchhoff law to the loop CLC^*OA, we get: $L\frac{dI_L}{dt} = (k-1)RI_L - V_C - V_{C^*}$, where $k = \frac{R_2}{R_1} + 1$ is the gain of the amplifier. Finally, we use the first Kirchhoff law to the circuit node between capacitor C^* and inductor L : $C^*\frac{dC^*}{dt} = I_0 + I_L - I_D$, where $I_0 \approx \frac{V_b}{R_0}$ and I_D is the current through the diode. The current passing through well-behaved diodes satisfies the relation (Shockley diode equation):

$$I_D = I_S(e^{\frac{eV_D}{k_BT}} - 1), \tag{3.1}$$

where I_S is the saturation current (characteristic of a diode), e is the elementary charge, k_B is the Boltzmann constant, T is the absolute temperature and V_D is the voltage across the diode. With regard to the parallel connection between the diode and capacitor C^*, we obtain that $V_D = V_{C^*}$.

Thus, we have a system of three differential equations describing the circuit behaviour:

$$C\frac{dV_c}{dt} = I_L, \quad L\frac{dI_l}{dt} = (k-1)RI_L - V_C - V_{C^*}, \quad C^*\frac{dC^*}{dt} = I_0 + I_L - I_D. \tag{3.2}$$

We change the variables V_C, V_{C^*}, I_L (as well as the other characteristics of the circuit components) for dimensionless ones, which are more convenient for numerical simulations:

$$\begin{aligned} x &= \frac{V_C}{V_T} & y &= \frac{\rho I_L}{V_T} & z &= \frac{V_{C^*}}{V_T} & \theta &= \frac{t}{\tau} \\ V_T &= \frac{k_B T}{e} & \rho &= \sqrt{\frac{L}{C}} & \tau &= \sqrt{LC} & a &= (k-1)\frac{R}{\rho} \\ b &= \frac{\rho I_0}{V_T} & c &= \frac{\rho I_S}{V_T} & \epsilon &= \frac{C^*}{C}. \end{aligned} \tag{3.3}$$

The system (3.2) in new variables looks like:

$$\dot{x} = y, \quad \dot{y} = -x + ay - z, \quad \epsilon\dot{z} = b + y - c(e^z - 1), \tag{3.4}$$

where the dot denotes the differentiation with respect to θ.

In Semiconductor Physics Institute in Vilnius, the circuit was set up with the following parameters: $L = 100mH$, $C = 10nF$, $C^* = 15nF$, $V_b = 20V$, $R = 1k\Omega$, $R_1 = 10k\Omega$, $R_0 = 20k\Omega$. The resistance of the variable resistor R_2 ranges from 0 to $10k\Omega$. The room temperature is fixed at the value $T = 293,15K$. The saturation current of a diode is $I_S = 1.10^{-13}A$ (the value was derived from constant c used in an article [8]). We will use the same values of parameters for the numerical simulation.

Thus, $a \in \langle 0, 1\rangle$, $b = 39,57$, $c = 4.10^{-9}$ and $\epsilon = 0.15$.

The system (3.4) has the only stationary solution $(-\ln(1+\frac{b}{c}), 0, \ln(1+\frac{b}{c}))$. We determine how the nearby points behave and then we try to explain these results from a physical point of view.

We change the coordinates once more in order to arrange the stationary point into the origin:

$$u = x + \ln\Big(1 + \frac{b}{c}\Big) \quad v = y \quad w = z - \ln\Big(1 + \frac{b}{c}\Big). \tag{3.5}$$

The system (3.4) transforms to the new system:

$$\dot{u} = v, \quad \dot{v} = av - u - w, \quad \epsilon\dot{w} = b + v - c\Big(\frac{b+c}{c}e^w - 1\Big). \tag{3.6}$$

Unfortunately, we are unable to gain the eigenvalues of the linearized system

$$\dot{u} = v, \quad \dot{v} = av - u - w, \quad \dot{w} = \frac{v}{\epsilon} - \frac{b+c}{\epsilon} w$$

in an analytical form. The stationary point is a saddle point - index 2, *i.e.* the linearized system has two eigenvalues with positive real part and one with negative one, as results from the numerical evaluations, which were done in the program Maple.

The instability of this fixed point is not surprising because it corresponds to the situation when zero current passes through the inductor L although the current I_0 remains constant. This splitting of current (when the currents passing through C^* and D are much more bigger than the inductor current) is very unsymmetric and the circuit tries to return to the equilibrium.

The solution of the system (3.4) with initial condition $(-23, 0, 23)$ and $a = 0.4$ is illustrated in the figure 3.2.

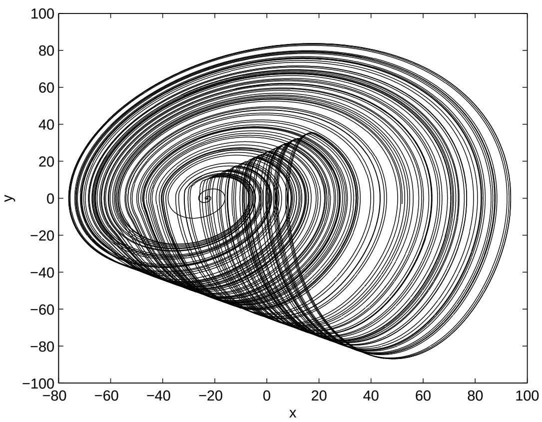

Figure 3.2: Vilnius oscillator

3.2 Chaos generator

The nonlinear circuit, shown in the figure 3.3, is a simple RLC circuit. It consists of two capacitors C_m, C, resistors R_m, R, an inductor L, an amplifier and a squaring module. The variable resistor R_m serves as a control parameter.

Applying the second Kirchhoff law to the loop RCC_m, we get:

$$\frac{Q}{C} + R(\dot{Q} + \dot{Q}_m) - v^2(U - U_0)^2 = 0, \tag{3.7}$$

where the dot denotes the differentiation with respect to time, Q and Q_m are the charges at C, C_m and U is the voltage at the capacitor C_m.

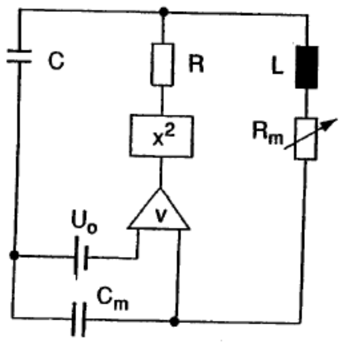

Figure 3.3: Chaos generator

For the loop C_mCLR_m, we obtain:

$$\frac{Q}{C} - L\ddot{Q}_m - R_m\dot{Q}_m - \frac{Q_m}{C_m} = 0. \tag{3.8}$$

Subtracting the equation (3.8) from (3.7) gives:

$$L\ddot{Q}_m + (R + R_m)\dot{Q}_m + \frac{Q_m}{C_m} + R\dot{Q} - v^2(U - U_0)^2 = 0. \tag{3.9}$$

We differentiate the equation (3.8) with respect to time and multiply it by RC :

$$R\dot{Q} - RLC\dddot{Q}_m - RCR_m\ddot{Q}_m - \frac{RC}{C_m}\dot{Q}_m = 0. \tag{3.10}$$

We must not forget that $U = \frac{Q_m}{C_m}$. Finally, we subtract (3.10) from (3.9) and we gain the third order differential equation this way:

$$\dddot{U} + a\ddot{U} + b\dot{U} + cU = cv^2(U - U_0)^2, \tag{3.11}$$

where

$$a = \Big(\frac{1}{RC} + \frac{R_m}{L}\Big) \qquad b = \frac{1}{LC}\Big(1 + \frac{R_m}{R} + \frac{C}{C_m}\Big) \qquad c = \frac{1}{LCRC_m}$$

We rescale the time by $t = b^{-1/2}\tau$:

$$\dddot{U} + \beta\ddot{U} + \dot{U} = F(U), \tag{3.12}$$

where $\beta = ab^{-1/2}$, the dot denotes the differentiation with respect to dimensionless time τ, and $F(U) = cb^{-3/2}(-U + v^2(U - U_0)^2)$. We introduce a new dimensionless variable x :

$$U = A - Bx$$

$$A = \frac{1}{2v^2}(1 + 2v^2U_0 + \sqrt{1 + 4v^2U_0}) \tag{3.13}$$

$$B = v^{-2}\sqrt{1 + 4v^2U_0}. \tag{3.14}$$

The equation (3.12) in new variables looks as follows:

$$\dddot{x} + \beta\ddot{x} + \dot{x} = f(x), \tag{3.15}$$

where the nonlinear function $f(x) = \mu x(1 - x)$ depends on a parameter $\mu = cb^{-3/2}\sqrt{1 + 4v^2U_0}$.

Using the standard substitution, we rewrite the third order differential equation as a system of three first order differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -\beta z - y + f(x). \end{aligned} \tag{3.16}$$

The system (3.16) has two stationary points $x_1 = 0$ and $x_2 = 1$. They correspond to $U_1 = \frac{1}{2v^2}(1 + 2v^2U_0 + \sqrt{1 + 4v^2U_0})$ and $U_2 = \frac{1}{2v^2}(1 + 2v^2U_0 - \sqrt{1 + 4v^2U_0})$. We use the Hurwitz´s criterion for determining their stability.

First, we create the Jacobian matrix of the system (3.16).

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu - 2\mu x & -1 & -\beta \end{pmatrix}$$

For $x_1 = 0$, we obtain this characteristic equation of the Jacobian matrix J:

$$\lambda^3 + \beta\lambda^2 + \lambda - \mu = 0. \tag{3.17}$$

We see that the equation (3.17) does not satisfy even the necessary condition for stability. Therefore it is an unstable stationary point.

For $x_2 = 1$, we obtain this characteristic equation of the Jacobian matrix J:

$$\lambda^3 + \beta\lambda^2 + \lambda + \mu = 0. \tag{3.18}$$

All coefficients are positive, so we can set up the Hurwitz´s matrix respective to the equation (3.18):

$$\begin{pmatrix} 1 & \mu & 0 \\ 1 & \beta & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$D_1 = 1 > 0 \qquad D_2 = \beta - \mu \qquad D_3 = D_2$$

The stationary point $x_2 = 1$ ($U_2 = \frac{1}{2v^2}(1 + 2v^2U_0 - \sqrt{1 + 4v^2U_0})$ is stable for $\beta > \mu$. The behaviour in the neighbourhood of this stationary point will be studied in a more detail using the numerical simulation.

For $\beta > \mu$, the stationary point is stable, for $\beta < \mu$, it becomes unstable. From the previous, it is obvious that a bifurcation occurs for $\beta = \mu$. β and μ are both functions of R_m so we expect that the bifurcation occurs when R_m increases or decreases its critical value.

The values of circuit components used for the numerical simulations are the following:

$$v = 1,2V^{-1/2} \quad R = 3300\Omega \quad C = C_m = 47.10^{-9}F \quad L = 0.1H \quad U_0 = 4V.$$

The stability condition $\beta > \mu$ tends to quadratic equation for R_m with only one positive root. Hence for large values of R_m the voltage U remains at $U_2 = 2,6447V$. When R_m decreases its critical value $R_{mcrit} = 770,6113\Omega$, a Hopf bifurcation occurs, *i.e.* the limit cycle appears. For even smaller values of R_m, the system becomes chaotic.

The fact that just Hopf bifurcation ocurs results from characteristic equation (3.18). For $\beta = \mu$, there is one real eigenvalue $\lambda = -\beta$ and a complex conjugated pair $\lambda_{1,2} = \pm i$. From (3.18), it is also obvious that none of bifurcation, when a single real eigenvalue crosses the imaginary axis, can occur because of positivity of μ.

For $R_m = 1000\Omega$, the system is damped down and all points from a neighbourhood of the stationary point x_2 tend to it. This situation is illustrated in the figure 3.4 for the initial condition $(1,78; 0; 0)$. The stationary point attraction is shown in the left figure and the signal damping in the right one.

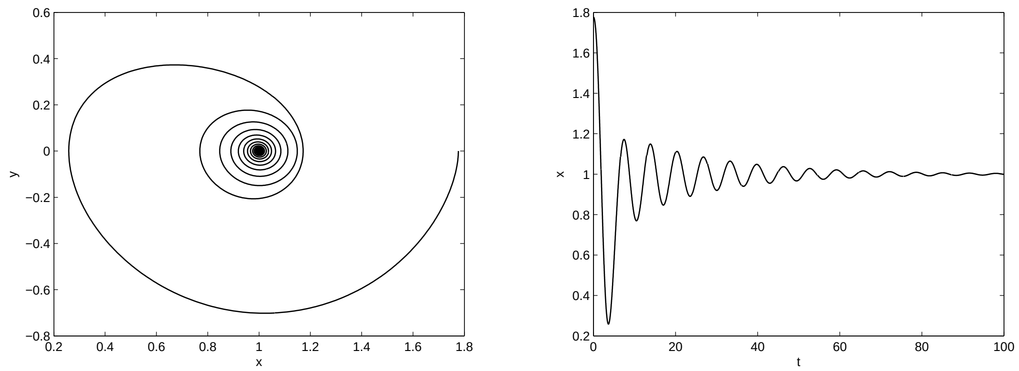

Figure 3.4: Stable stationary point

Time evolution of the point $(1,78; 0; 0)$ is quite different when $R_m = 500\Omega$ as figure 3.5 shows. The stationary point x_2 repels the nearby points and they are attracted by a limit cycle. This behaviour is not much surprising from a physical point of view. The resistor is not yet able to damp the signal down so the circuit can begin to oscillate. The variables $U, \dot{U}$ (scaled x, y) oscillate with the same frequency so the phase trajectory is an ellipse.

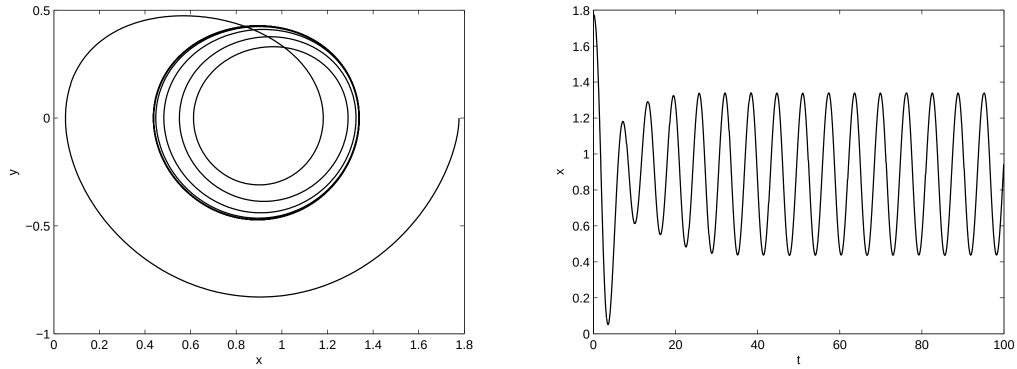

Figure 3.5: Limit cycle

For even lower values of the resistance, the period doubling occurs, *i.e.* the circuit oscillates with several different amplitudes. We can see a period-2 behaviour for $R_m = 150\Omega$ and period-4 behaviour for $R_m = 130\Omega$ in the figure 3.6.

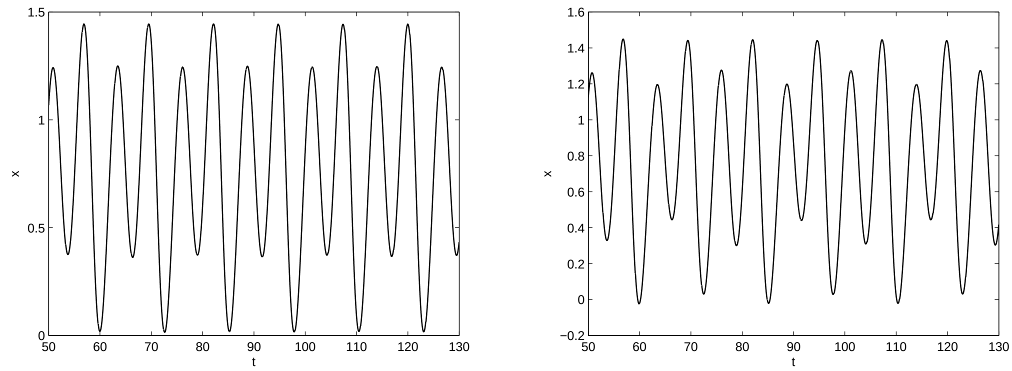

Figure 3.6: Period doubling

For even lower resistance, the behaviour of the circuit becomes chaotic (figure 3.7).

We would like to say that this circuit (with slightly different components) was set up in the practicum and it allows a comparison between numerical simulations and real behaviour of the circuit.

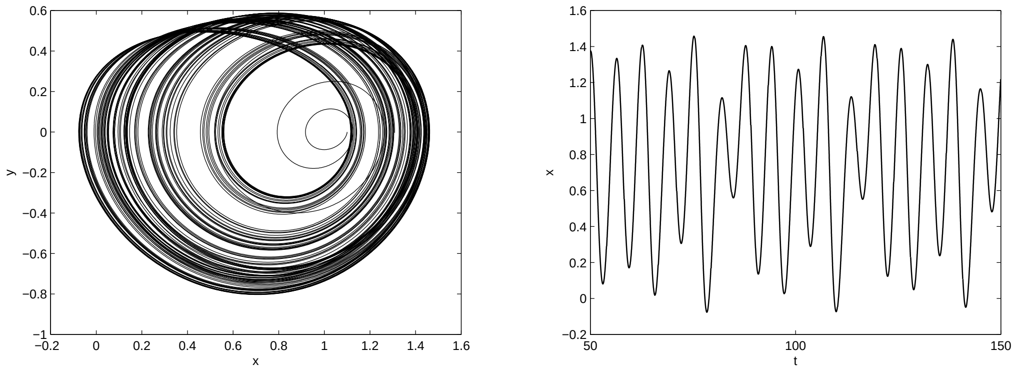

Figure 3.7: Chaotic behaviour - $R_m = 100\Omega$

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Prohlášení

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Praha, July 21, 2007

Lucie Strmisková