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DIPLOMA THESIS

Quasi-Hermitian Models

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Abstrakt: Zkoumáme vztah mezi $\mathcal{PT}$-symetrií a pseudohermitovostí. Nalezneme omezený pseudohermitovský operátor, který nemá antilineární symetrii a také operátor s antilineární symetrií, který není pseudohermitovský. Upozorníme na kriteria podobnosti samosdruženému operátoru a připomeneme metodu pro konstrukci metrického operátoru. Představíme tři typy modelů - řetězové modely, $\mathcal{PT}$-symetrické bodové interakce na přímce a supersymetrické bodové interakce na kružnici.
Klíčová slova: $\mathcal{PT}$-symmetrie, pseudohermitovost, kvazihermitovost, řetězové modely, bodové interakce, supersymetrie

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Abstract: We explore the relation between the $\mathcal{PT}$-symmetry and the pseudo-Hermiticity, we present a bounded pseudo-Hermitian operator without any antilinear symmetry and also an operator with antilinear symmetry which is not pseudo-Hermitian. We recall criteria for similarity to self-adjoint operators and the method for construction of metric operator. We present three types of $\mathcal{PT}$-symmetric and also quasi-Hermitian models - chain models, $\mathcal{PT}$-symmetric point interaction on a line and supersymmetric $\mathcal{PT}$-symmetric point interactions on a loop.
Key words: $\mathcal{PT}$-symmetry, pseudo-Hermiticity, quasi-Hermiticity, chain models, point interactions, supersymmetry

Prohlášení

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

Nemám závažný důvod proti užití tohoto školního díla ve smyslu §60 Zákona č.121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

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Contents

Introduction

Quasi-Hermitian models [37] play the essential role in a so called $\mathcal{PT}$-symmetric Quantum Mechanics. With regard of the fact that quasi-Hermitian operators are similar to the self-adjoint ones we may use them as observables. This can be done either by already mentioned similarity transformation or by the modification of the scalar product.

Whole $\mathcal{PT}$-symmetric Quantum Mechanics itself arised from the observation that spectrum of $\mathcal{PT}$-symmetric operators, despite they are non-Hermitian, can be real and positive. This fact was revealed by Caliceti et al [12] studied by Bessis and Zinn-Justin [44] and, finally, highlighted by Bender and Boettcher [5, 6]. Mostafazadeh [26] pointed out that $\mathcal{PT}$-symmetric operators are often also pseudo-Hermitian. This class of operators was introduced already in 1940's by Dirac [14] and Pauli [34].

From mathematical point of view there are still more open questions. The equivalence of the $\mathcal{PT}$-symmetry (or more general antilinear symmetry) and the pseudo-Hermiticity is not fully solved yet, nonetheless works [28, 38, 40] bring some results for special classes of operators. Discussion on the definition of quasi-Hermiticity is presented in [23], however we can find much older indeed mathematical work by Dieudonné [13]. Definition of quasi-Hermiticity presented there is more general then the special case usually considered within $\mathcal{PT}$-symmetric Quantum Mechanics. Since quasi-Hermitian operators in sense of Diedonné may have much more extensive spectral properties (e.g. non real spectrum) we do not adopt that definition in this work and we retain the "standard physical" one which, in fact, is equivalent to the similarity to some self-adjoint operator. Criteria for similarity to self-adjoint operators were found independently by several authors [41, 32, 25] and their importance and also applicability in the context of $\mathcal{PT}$-symmetry was stressed by Albeverio and

Kuzhel [3]. The construction of so called metric Θ is presented in [27], nevertheless only for very special class of operators and with very formal proof without necessary parts (convergence of the sums, domains of definition). Technically alternative point of view, however leading to the same result, provides [23]. Examples of metric operator (with careful verifications of all requirements) can be found in [22] and even for operator with non-compact resolvent [3].

Our aim is to review and add some missing parts to general theoretical results on $\mathcal{PT}$-symmetry, pseudo-Hermiticity and quasi-Hermiticity. Particularly we intend to investigate relation between $\mathcal{PT}$-symmetry and pseudo-Hermiticity and formulate more precisely the method of construction of metric Θ for pseudo-Hermitian operators with compact resolvent. The next part consists of concrete $\mathcal{PT}$-symmetric models - chain models, $\mathcal{PT}$-symmetric point interactions on a line and supersymmetric $\mathcal{PT}$-symmetric point interactions on a loop.

List of Symbols

$[\cdot,\cdot]$	the commutator
$\langle\cdot,\cdot\rangle$	the scalar product
$\mathscr{B}(\mathcal{H})$	bounded linear operators acting on $\mathcal{H}$
$\mathscr{C}(\mathcal{H})$	closed operators acting on $\mathcal{H}$
$\mathrm{Dom}(A)$	the domain of A
$\mathcal{H}$	the separable Hilbert space
$\mathrm{Ker}(A)$	the kernel of A
$\mathscr{L}(\mathcal{H})$	linear operators acting on $\mathcal{H}$
$\mathscr{L}(V_n)$	linear operators acting on V_n
$\overline{X}$	the closure of set X
$\mathrm{Ran}(A)$	the range of A
$\sigma(A)$	the spectrum of A
$\sigma_{d,ess}(A)$	the discrete and essential spectrum of A
$\sigma_{p,c,r}(A)$	the point,continuous and residual spectrum of A
$\varrho(A)$	the resolvent set of A
ϑ	Heaviside step function

w– lim, s– lim	the limit in weak and strong operator topology
$\{\cdot,\cdot\}$	the anticommutator
$^\perp$	the orthogonal complement
A^*	the adjoint of A
AC^2	absolute continuous functions with a.c. derivative and second derivative in L^2
C_0^∞	infinitely differentiable functions with compact support
$R_\lambda(A)$	the resolvent of A at λ
V_n	the linear vector space, dim $V_n = n$

Chapter 1

$\mathcal{PT}$-symmetry and pseudo-Hermiticity

1.1 Basic properties

The numerical study of Hamiltonians of the type $p^2 + m^2x^2 - (\mathrm{i}x)^N$, originally for $N = 3$, showed that these operators may have interesting spectral properties, i.e. spectrum is real, discrete, positive. $\mathcal{PT}$-symmetry was suggested to be fundamental property of Hamiltonians causing reality of spectra. Traditionally, the parity $\mathcal{P}$ is represented by the linear operator acting in $L_2(\mathbb{R})$ space

$$(\mathcal{P}\psi)(x) = \psi(-x), \quad \mathcal{P}^2 = I, \tag{1.1}$$

while the time-reversal symmetry $\mathcal{T}$ denotes a complex conjugation

$$(\mathcal{T}\psi)(x) = \overline{\psi}(x), \quad \mathcal{T}^2 = I, \tag{1.2}$$

and $\mathcal{PT}$-symmetry of Hamiltonian is understood as

$$\mathcal{PT}H\psi = H\mathcal{PT}\psi \quad \text{for all } \psi \in \mathrm{Dom}(H). \tag{1.3}$$

This relation implies a symmetry of the $\mathrm{Dom}(H)$ as well, i.e. $\psi \in \mathrm{Dom}(H)$ if and only if $\mathcal{PT}\psi \in \mathrm{Dom}(H)$. The immediate consequence of the $\mathcal{PT}$-symmetry for eigenvalues is that, if a complex number E is an eigenvalue of H then the complex conjugate $\overline{E}$ is the eigenvalue as well

$$H\psi_E = E\psi_E \quad \Rightarrow \quad \mathcal{PT}H\psi_E = H\mathcal{PT}\psi_E = \overline{E}\mathcal{PT}\psi_E. \tag{1.4}$$

Moreover, if eigenfunction ψ_E is also $\mathcal{PT}$-symmetric, i.e. $\mathcal{PT}\psi_E = \psi_E$, then eigenvalue E is real. This elementary proposition is often stressed and we speak about *unbroken $\mathcal{PT}$-symmetry* if all eigenfunctions are $\mathcal{PT}$-symmetric (and therefore all eigenvalues are real). It is obvious that the particular structure of $\mathcal{PT}$ operator is not necessary for validity of these conclusions, we can generalize it in a following way.

1.1.1 Antilinear symmetry

Definition 1.1. *Let $A \in \mathscr{L}(\mathcal{H})$. We say that A has an antilinear symmetry if there exists an antilinear bijective operator C and the relation*

$$AC\psi = CA\psi \tag{1.5}$$

holds for all $\psi \in \mathrm{Dom}(A)$.

Basic spectral properties of an operator having antilinear symmetry are given by following proposition

Proposition 1.1. *Let $A \in \mathscr{C}(\mathcal{H})$ have an antilinear symmetry C. Then $\lambda \in \mathbb{C}$ is in the spectrum of A if and only if $\overline{\lambda}$ is in the spectrum of A. Moreover, this equivalence is valid also for the disjoint parts of spectrum, i.e. $\lambda \in \sigma_{p,c,r}(A) \Longleftrightarrow \overline{\lambda} \in \sigma_{p,c,r}(A)$.*

Remark 1.2. *We use the definition of spectrum A.1 and its point, continuous and residual part presented in* [10].

Proof. Equation (1.5) and properties of C yield the relation between resolvents

$$(A - \overline{\lambda})^{-1} = C^{-1}(A - \lambda)^{-1}C. \tag{1.6}$$

Hence, $\lambda \in \varrho(A) \Longleftrightarrow \overline{\lambda} \in \varrho(A)$. Moreover, $\mathrm{Ker}(A - \overline{\lambda}) = C\ \mathrm{Ker}(A - \lambda)$ and $\overline{\mathrm{Ran}(A - \lambda)} = \overline{C\ \mathrm{Ran}(A - \overline{\lambda})}$, thus $\lambda \in \sigma_{p,c,r}(A) \Longleftrightarrow \overline{\lambda} \in \sigma_{p,c,r}(A)$ from the definition of spectrum A.1. □

Remark 1.3. *An $A \in \mathscr{L}(\mathcal{H})$ having an antilinear symmetry is not automatically closed.*

1.1.2 Pseudo-Hermiticity

Mostafazadeh drew attention to the class of pseudo-Hermitian [26] operators because the studied $\mathcal{PT}$-symmetric Hamiltonians possessed also this property.

Definition 1.2. *Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. A is called weakly pseudo-Hermitian, if there exists an operator η with properties*
(i) $\eta, \eta^{-1} \in \mathscr{B}(\mathcal{H})$,
(ii) $A = \eta^{-1} A^* \eta$.
If η is self-adjoint then A is called pseudo-Hermitian. If we want to specify particular η we say that A is η-(weakly)-pseudo-Hermitian.

Although pseudo-Hermiticity seems to be a stronger than weak pseudo-Hermiticity, we show that assuming the latter one only is sufficient for proving basic properties. Relation between the two properties is partly described by following theorem

Theorem 1.4 ([30])**.** *Let $A \in \mathscr{L}(\mathcal{H})$ be an η_w-weakly-pseudo-Hermitian operator and the spectrum $\sigma(\eta_w^{-1}\eta_w^*)$ of $\eta_w^{-1}\eta_w^*$ does not contain the unit circle S^1. Then A is pseudo-Hermitian.*

Corollary 1.5. *Let $A \in \mathscr{L}(V_n)$. Then A is weakly pseudo-Hermitian if and only if it is pseudo-Hermitian.*

Unlike operators having antilinear symmetry the (weakly)-pseudo-Hermitian ones are always closed.

Lemma 1.6. *Let $A \in \mathscr{L}(\mathcal{H})$ be a weakly pseudo-Hermitian operator. Then A is closed.*

Proof. We consider convergent sequence $\{x_n\} \subset \mathrm{Dom}(A)$, $x_n \to x$ for which $\{Ax_n\}$ is convergent. Since $\eta \in \mathscr{B}(\mathcal{H})$, sequence $\{y_n\}$, $y_n := \eta x_n$ is convergent, $y_n \to y = \eta x$. Since A^* is closed (theorem A.1), $A^* y_n \to A^* y$ and $y \in \mathrm{Dom}(A^*)$. Hence $x = \eta^{-1} y \in \mathrm{Dom}(A)$ and $Ax_n = \eta^{-1} A^* \eta\, x_n = \eta^{-1} A^* y_n \to \eta^{-1} A^* y = Ax$. □

Weak pseudo-Hermiticity implies also spectral properties.

Proposition 1.7. *Let $A \in \mathscr{L}(\mathcal{H})$ be a weakly-pseudo-Hermitian operator. Then point, continuous and residual spectrum $\sigma_{p,c,r}(A)$ of A and $\sigma_{p,c,r}(A^*)$ of A^* are equal.*

Proof. Relation (ii) of the definition 1.2 yields

$$(A-\lambda)^{-1} = \eta^{-1}(A^* - \lambda)^{-1}\eta. \tag{1.7}$$

Since η is bounded and bijective, the equality of $\sigma_{p,c,r}(A)$ and $\sigma_{p,c,r}(A^*)$ holds. □

1.2 Equivalence relations

Theorem A.2 shows that the equivalence $\lambda \in \sigma(A) \Leftrightarrow \overline{\lambda} \in \sigma(A)$ holds for every weakly-pseudo-Hermitian operator A. When we return back to the spectral properties of operators having antilinear symmetry and when we realize known examples of Hamiltonians which are both $\mathcal{PT}$-symmetric and pseudo-Hermitian, natural questions rise. Is every $\mathcal{PT}$-symmetric (or operator having an antilinear symmetry) weakly-pseudo-Hermitian? Or, possesses every weakly-pseudo-Hermitian operator an antilinear symmetry? We give some answers to these questions in the following.

In order to explain equivalence relations between $\mathcal{PT}$-symmetry and pseudo-Hermiticity we present another class of operators, namely J-self-adjoint ones.

Definition 1.3 ([16])**.** *Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. Let J be an antilinear isometric involution, i.e. $J^2 = I$ and $\langle Jx, Jy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. A is called J-symmetric if $A \subset JA^*J$. A is called J-self-adjoint if $A = JA^*J$.*

J-self adjoint operators were suggested by Borisov and Krejčiřík [11] to be "adequate for a rigorous formulation of $\mathcal{PT}$-symmetric problems". One of the reasons for this conclusion is a fact that residual spectrum of J-self adjoint operator is empty (in contrast to operators having antilinear symmetry and pseudo-Hermitian operators as we shall see later). It can be easily seen that every η-pseudo-Hermitian operator with antilinear symmetry C satisfying $\eta^2 = I$, $C^2 = I, [\eta, C] = 0$ and $\langle x, \eta C y\rangle = \langle y, x\rangle$ is ηC-self-adjoint.

Lemma 1.8 ([16])**.** *Let A be a J-self-adjoint operator. Then*
(i) $\dim(\mathrm{Ker}(A-\lambda)) = \dim(\mathrm{Ker}(A^* - \overline{\lambda}))$,
(ii) $\sigma_r(A) = \emptyset$.

1.2.1 Finite dimension

At first, we explore relations between operators having antilinear symmetry, weakly-pseudo-Hermitian, and J-self-adjoint operators in the finite dimensional spaces.

Lemma 1.9. *Every $A \in \mathscr{L}(V_n)$ is similar to the J-self-adjoint operator, i.e. there exists invertible $X \in \mathscr{L}(V_n)$ such that XAX^{-1} is J-self-adjoint.*

Proof. Every matrix A can be transformed to the Jordan form, i.e. $A = X^{-1}A_J X$, where A_J is composed of $k \times k$ Jordan blocks $J_k(\lambda)$

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \ddots \end{pmatrix}. \tag{1.8}$$

Every Jordan block $J_k(\lambda)$ is J-self-adjoint [18]. We provide J in the explicit form. $J = \mathcal{P}_k\mathcal{T}$, where $\mathcal{T}$ is a complex conjugation and $\mathcal{P}_k$ is a $k \times k$ parity, i.e.

$$\mathcal{P}_k = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}. \tag{1.9}$$

□

The above lemma is essential for the equivalence statement in finite dimensional spaces.

Proposition 1.10. *Let $A \in \mathscr{L}(V_n)$. Then A is pseudo-Hermitian if and only if it possesses an antilinear symmetry.*

Proof. Let A is pseudo-Hermitian, i.e. $A = \eta^{-1}A^*\eta$. Using the lemma above

$$XAX^{-1} = A_J = JA_J^*J = J(X^{-1})^*A^*X^*J = J(X^{-1})^*\eta A\eta^{-1}X^*J \tag{1.10}$$

we obtain $CA = AC$, where $C := \eta^{-1}X^*JX$ is the antilinear symmetry.

Let C be the antilinear symmetry of A. We find easily that

$$A = C^{-1}X^{-1}J(X^{-1})^*A^*X^*JXC, \tag{1.11}$$

hence A is η-weakly-pseudo-Hermitian, $\eta := X^*JXC$. By corollary 1.5, A is pseudo-Hermitian. □

1.2.2 Spectral operators of scalar type

Situation in Hilbert spaces with infinite dimension is of course more complicated. The problem is not solved even for bounded operators. Some results, however with not very correct proofs, provide articles of Mostafazadeh [26, 27, 28] and more correct works by Solombrino and Scolarici [40, 38]. All propositions of these articles require operators with discrete spectrum and with eigenvectors which form biorthonormal basis. Slight generalization is presented in [38] where operators with finite Jordan blocks in spectrum may appear. We provide extension of the propositions and we provide a new proof. We show that equivalence of weak pseudo-Hermiticity and antilinear symmetry is valid for spectral operators of scalar type. Proof for all spectral operators is not known yet (except finite Jordan blocks case [38]). See Appendix for the definitions and basic properties of spectral operators.

Important result for spectral operators of scalar type allows us to prove the desired equivalence very easily.

Theorem 1.11 ([15]). *Let $S_1, ..., S_k \in \mathscr{B}(\mathcal{H})$ be commuting scalar type operators in $\mathcal{H}$. Then there is a bounded self-adjoint operator X with a bounded everywhere defined inverse such that the operators XS_iX^{-1}, $i = 1, ..., k$ are all normal.*

Remark 1.12. *We say that $A \in \mathscr{L}(\mathcal{H})$ is similar to $B \in \mathscr{L}(\mathcal{H})$ if there is a $X \in \mathscr{B}(\mathcal{H})$ with a bounded everywhere defined inverse and $A = X^{-1}BX$. Thus, the particular case $k = 1$ in the theorem above states that every spectral operator of scalar type is similar to some normal operator.*

Proposition 1.13. *Let $S \in \mathscr{B}(\mathcal{H})$ be a spectral operator of scalar type. S is weakly pseudo-Hermitian if and only if it possesses an antilinear symmetry.*

Proof. S is similar to a normal operator N, $S = X^{-1}NX$. Every normal operator is J-self-adjoint [18] (proof is based on the spectral theorem for normal operators [36]). The rest of the proof is exactly the same procedure as for matrices.

If S is η-weakly-pseudo-Hermitian, then $C := \eta^{-1}X^*JX$ is antilinear symmetry of S. Conversely, if C is an antilinear symmetry of S then S is η-weakly-pseudo-Hermitian, where $\eta := X^*JXC$. □

1.2.3 Two examples

It may seem that the weak pseudo-Hermiticity and the antilinear symmetry are equivalent properties at least for bounded operators and only some technical proof is needed. However, we present examples of bounded operators having an antilinear symmetry which are not weakly pseudo-Hermitian and also bounded pseudo-Hermitian operator which cannot possess any antilinear symmetry.

Example 1.1 ([35])**.** Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis of $\mathcal{H} = l_2(\mathbb{N})$, i.e. $e_n(m) = \delta_{nm}$. Let T be an operator on $\mathcal{H}$ acting as

$$Te_n := e_{n-1}, \quad n \in \mathbb{N}, \quad e_0 := 0. \tag{1.12}$$

T is bounded and it possesses the antilinear symmetry $\mathcal{T}$-complex conjugation. Adjoint operator T^* acts as

$$T^*e_n := e_{n+1}, \quad n \in \mathbb{N}. \tag{1.13}$$

Every complex number λ with absolute value $|\lambda| < 1$ is in the point spectrum $\sigma_p(T)$ of T, corresponding eigenvector x_λ reads $x_\lambda = \sum_{n=1}^{\infty} \lambda^{n-1}e_n$. Spectrum $\sigma_p(T^*)$ of T^* is different. If we consider equation for eigenvalues

$$(T^* - \lambda)\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = 0, \tag{1.14}$$

we arrive at

$$\lambda\alpha_1 = 0, \tag{1.15}$$

$$\alpha_{n-1} - \lambda\alpha_n = 0, \quad n > 1. \tag{1.16}$$

Hence, the point spectrum of T^* is empty. In fact, the set $\{\lambda \in \mathbb{C} | |\lambda| < 1\} \subset \sigma_r(T^*)$ due to theorem A.3. Operator T is not weakly pseudo-Hermitian because the necessary condition is the equality of the spectrum of T and T^* as we have already shown (proposition 1.7).

Example 1.2. Let $\mathcal{H} = l^2(\mathbb{Z})$ and let $\{e_i\}_{-\infty}^{\infty}$ be the orthonormal basis, $e_n(m) = \delta_{nm}$. Operator T acts as

$$Te_i := \begin{cases} \lambda_0 e_i + e_{i+1}, & i \geq 1, \\ 0, & i = 0, \\ \overline{\lambda}_0 e_{-1}, & i = -1, \\ \overline{\lambda}_0 e_i + e_{i+1}, & i < -1, \end{cases} \tag{1.17}$$

$\lambda_0 \in \mathbb{C},\ \operatorname{Im} \lambda_0 > \frac{1}{2}$. We find T^* easily from the definition of the adjoint A.3,

$$\langle e_i, Te_j \rangle = \begin{cases} \lambda_0 \delta_{i,j} + \delta_{i,j+1}, & i,j \geq 1, \\ \overline{\lambda}_0 \delta_{i,j} + \delta_{i,j+1}, & i,j \leq -1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.18}$$

hence

$$T^* e_i = \begin{cases} \overline{\lambda}_0 e_i + e_{i-1} & i > 1, \\ \overline{\lambda}_0 e_1, & i = 1, \\ 0, & i = 0, \\ \lambda_0 e_i + e_{i-1}, & i \leq -1. \end{cases} \tag{1.19}$$

Let $\mathcal{P}$ be a parity, i.e.

$$\mathcal{P} e_i := e_{-i}. \tag{1.20}$$

We may show immediately from the definitions that T is $\mathcal{P}$-pseudo-Hermitian

$$T = \mathcal{P} T^* \mathcal{P}. \tag{1.21}$$

It is obvious that $\overline{\lambda}_0 \in \sigma_p(T) = \sigma_p(T^*)$. We show that $\lambda_0 \in \sigma_r(T) = \sigma_r(T^*)$. We express the equation

$$(T - \lambda_0) \sum_{i=-\infty}^{\infty} \alpha_i e_i = 0 \tag{1.22}$$

and determine coefficients α_i.

$$\begin{aligned} (T - \lambda_0) \sum_{i=-\infty}^{\infty} \alpha_i e_i &= \sum_{i=-\infty}^{-2} \alpha_i \left[(\overline{\lambda}_0 - \lambda_0) e_i + e_{i+1} \right] + \\ &\quad + \alpha_{-1} (\overline{\lambda}_0 - \lambda_0) e_{-1} - \alpha_0 \lambda_0 e_0 + \sum_{i=1}^{\infty} \alpha_i e_{i+1}, \end{aligned} \tag{1.23}$$

hence

$$\begin{aligned} \alpha_i &= 0 \text{ for } i \geq 0, \\ \alpha_i(\overline{\lambda}_0 - \lambda_0) + \alpha_{i-1} &= 0 \text{ for } i < 0. \end{aligned} \tag{1.24}$$

α_{-1} (and then all α_i) must be 0. If $\alpha_{-1} \neq 0$ then for $i < -1$, $\alpha_i = \alpha_{-1} (2\,\mathrm{i}\,\mathrm{Im}\,\lambda_0)^{-i-1}$. Since $\mathrm{Im}\,\lambda_0 > \frac{1}{2}$ from the definition of T,

$$\sum_{i=-\infty}^{\infty} |\alpha_i|^2 = +\infty. \tag{1.25}$$

Hence λ_0 is not an eigenvalue and $\overline{\mathrm{Ran}(T-\lambda_0)} \neq \mathcal{H}$ for

$$e_1 \notin \overline{\mathrm{span}(e_i)_{i\neq 1}} = \overline{\mathrm{Ran}(T-\lambda_0)}. \tag{1.26}$$

This proves that $\lambda_0 \in \sigma_r(T)$ and there holds also $\lambda_0 \in \sigma_r(T^*)$ by proposition 1.7.

Operator T cannot have any antilinear symmetry because the necessary condition is that $\lambda_0 \in \sigma_{p,c,r}(T) \Leftrightarrow \overline{\lambda_0} \in \sigma_{p,c,r}(T)$, by proposition 1.1.

Corollary 1.14. *Weak pseudo-Hermiticity and antilinear symmetry are not equivalent properties even for bounded operators on $\mathcal{H}$, see examples 1.1 and 1.2.*

Nonetheless, it is necessary to remark that operators from previous examples are not spectral. To justify this take into consideration

Theorem 1.15 ([15])**.** *If the space $\mathcal{H}$ is separable, then the point and residual spectra of a spectral operator are countable.*

Spaces $l^2(\mathbb{N})$ and $l^2(\mathbb{Z})$ are separable and we have already shown that the set $\{\lambda | |\lambda| < 1\}$ is included in the point spectrum of the operator from the example 1.1. Similarly, the set $\omega = \left\{\lambda \in \mathbb{C} | |\lambda - \overline{\lambda_0}| < 1\right\}$ is included in the point spectrum of the adjoint operator T^* from the example 1.2. Therefore, using theorem A.3, ω is included in the union of the point and residual spectrum of T.

For sake of completeness we discuss the relations of J-self-adjointness with antilinear symmetry and J-self-adjointness with weak pseudo-Hermiticity.

Example 1.1 presents the operator with antilinear symmetry with non-empty residual spectrum, thus it cannot be even similar to J-self-adjoint operator. Conversely, $A := (\mathrm{i})$ acting in $\mathbb{C}$ is J-self-adjoint, where J is the complex conjugation, however it cannot have any antilinear symmetry because $-\mathrm{i}$ is not in the spectrum of A.

Example 1.2 presents the pseudo-Hermitian operator with non-empty residual spectrum, which cannot be therefore similar to any J-self-adjoint operator. Again, operator $A := (\mathrm{i})$ is J-self-adjoint, however it is not weakly pseudo-Hermitian for $\mathrm{i} \notin \sigma(A^*)$.

Chapter 2

Quasi-Hermiticity

2.1 Basic properties

Quasi-Hermitian operators are very special class of pseudo-Hermitian operators. Their importance in physics was emphasized by F. G. Scholtz, H. B. Geyer and F. J. W. Hahne in [37].

Definition 2.1. *Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. A is called quasi-Hermitian, if there exists an operator Θ with properties*
(i) $\Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H})$,
(ii) Θ *is positive,*
(iii) $A = \Theta^{-1} A^* \Theta$.

In mathematics, the notion of quasi-Hermiticity appeared much earlier [13]. Later, mathematically oriented discussion on the definition of quasi-Hermiticity can be found in [23].

Remark 2.1. *A more general definition of quasi-Hermiticity is presented in [13], the inverse of Θ does not need to be bounded. However, we will not use this more general definition here, one of the reasons is that spectrum of such quasi-Hermitian operator may be non real, appropriate example is presented in [13].*

Remark 2.2 ([43])**.** *For bounded operators $A \in \mathscr{B}(\mathcal{H})$, the existence of such Θ that $0 \notin \overline{\{ \langle x, \Theta x \rangle \mid x \in \mathcal{H}, \ \|x\| = 1 \}}$ and condition (iii) of the definition (2.1) is satisfied implies quasi-Hermiticity of A.*

Reason why are quasi-Hermitian operators so important in the $\mathcal{PT}$-symmetric Quantum Mechanics is that they are self-adjoint in a Hilbert space $\mathcal{H}_\Theta$ with modified scalar product

$$\langle \cdot, \cdot \rangle_\Theta := \langle \cdot, \Theta \cdot \rangle . \tag{2.1}$$

The operator Θ is often called a 'metric' or 'metric operator' in physical literature. The properties of Θ, see the definition above, guarantee that $\langle \cdot, \cdot \rangle_\Theta$ fulfills all requirements for being scalar product and moreover

$$m \langle \psi, \psi \rangle \leq \langle \psi, \psi \rangle_\Theta \leq M \langle \psi, \psi \rangle \quad \forall \psi \in \mathcal{H}, \tag{2.2}$$

where $m = \inf\{\langle \psi, \Theta\psi \rangle \mid \psi \in \mathcal{H}, \|\psi\| = 1\}$ and $M = \|\Theta\|$. Since $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$ and Θ is positive, $m > 0$. It is possible to verify directly from the definition of the adjoint operator A.3 that quasi-Hermitian operator A is indeed self-adjoint in this scalar product.

2.1.1 Similarity to self-adjoint operator

Another point of view on quasi-Hermitian operators provide following proposition.

Proposition 2.3 ([3]). *Let $A \in \mathscr{L}(\mathcal{H})$ be a quasi-Hermitian operator with metric operator Θ. Then A is similar to the self-adjoint operator H,*

$$A = \varrho^{-1} H \varrho, \tag{2.3}$$

where $\varrho = \sqrt{\Theta}$.

The study of the problem of similarity to the self-adjoint operators may be found in the mathematical literature [32, 41, 25]. We recall integral-resolvent criterion

Theorem 2.4 ([32]). *Let $A \in \mathscr{L}(\mathcal{H})$. A is similar to a self-adjoint operator if and only if*

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A - \lambda I)^{-1}\psi\|^2 \mathrm{d}\xi \leq M\|\psi\|^2, \tag{2.4}$$

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A^* - \lambda I)^{-1}\psi\|^2 \mathrm{d}\xi \leq M\|\psi\|^2, \tag{2.5}$$

where $\lambda = \xi + \mathrm{i}\,\varepsilon, \psi \in \mathcal{H}$ and the integration is carried along an arbitrary straight line, parallel to the real axis, in the upper half plane.

This criterion is simplified when we restrict ourselves on pseudo-Hermitian operators.

Corollary 2.5 ([3])**.** *Let $A \in \mathscr{L}(\mathcal{H})$ be a pseudo-Hermitian operator. A is similar to a self-adjoint operator if and only if the spectrum of A is real and there exists a constant M such that*

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A-\lambda I)^{-1}\psi\|^2 \mathrm{d}\xi \leq M\|\psi\|^2, \tag{2.6}$$

where $\lambda = \xi + \mathrm{i}\,\varepsilon, \psi \in \mathcal{H}$ and the integration is carried along an arbitrary straight line, parallel to the real axis, in the upper half plane.

For purposes of $\mathcal{PT}$-symmetric Quantum Mechanics this version of the theorem may be very useful because many known and studied $\mathcal{PT}$-symmetric models are pseudo-Hermitian as well. As it is shown in [3], the latter criterion is extremely useful for the study of point interactions.

Another characterization of similarity of $\mathcal{P}$-pseudo-Hermitian operator acting in $L_2(\mathbb{R})$ to self-adjoint one is given using property of $\mathcal{C}$-symmetry [7, 9, 8, 3].

Definition 2.2 ([3])**.** *Let $A \in \mathscr{L}(L_2(\mathbb{R}))$ be a $\mathcal{P}$-pseudo-Hermitian. We say that A possesses the property of $\mathcal{C}$-symmetry if there exists bounded linear operator $\mathcal{C}$ in $L_2(\mathbb{R})$ such that the following conditions are satisfied*
(i) $A\mathcal{C} = \mathcal{C}A$,
(ii) $\mathcal{C}^2 = I$,
(iii) for each $f, g \in L_2(\mathbb{R})$ the sesquilinear form $(f,g)_{\mathcal{C}} := [\mathcal{C}f, g]_{\mathcal{P}}$,
where $[f,g]_{\mathcal{P}} := \langle \mathcal{P}f, g\rangle = \int_{-\infty}^{\infty} \overline{f(-x)}g(x)dx$, determines a scalar product in $L_2(\mathbb{R})$ that is equivalent to the initial one.

Proposition 2.6 ([3])**.** *Let $A \in \mathscr{L}(L_2(\mathbb{R}))$ be a $\mathcal{P}$-pseudo-Hermitian. Then the following statements are equivalent*
(i) A has the property of $\mathcal{C}$-symmetry,
(ii) A is similar to a self-adjoint operator.

It is obvious that $\mathcal{P}$-pseudo-Hermitian operator acting in $L_2(\mathbb{R})$ that has the property of $\mathcal{C}$-symmetry is quasi-Hermitian with the metric $\Theta = \mathcal{PC}$.

Authors of [3] emphasized also a necessary condition for similarity to self-adjoint operator, based on the theorem A.4.

Proposition 2.7. *Let $A \in \mathscr{L}(\mathcal{H})$ be similar to a self-adjoint operator. Then there exists real constant M such that*

$$\|R_{\lambda_0}(A)\| \leq \frac{M}{|\operatorname{Im} \lambda_0|} \tag{2.7}$$

for all $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.

Proof. It follows immediately from theorem A.4 and relation between resolvents of similar operators. □

With regard to the simplicity of this proposition it can be very useful for proving that an operator is not similar to self-adjoint one.

2.2 Metric operator

General resolvent criteria presented above do not provide any hint how to construct the similarity transformation ϱ or the metric operator Θ. Proposition 2.6 only modify the problem to the construction of $\mathcal{C}$-symmetry. Partial answer to this question can be found in [27, 31]. There is a simple criterion in finite dimensional spaces for operators to be quasi-Hermitian.

Proposition 2.8. *Let $A \in \mathscr{L}(V_n)$. Then A is quasi-Hermitian if and only if spectrum $\sigma(A)$ of A is real and A is diagonalizable. Operator Θ has form*

$$\Theta = \sum_{j=1}^{n} c_j \langle \phi_j, \cdot \rangle \phi_j, \tag{2.8}$$

where c_j are positive real constants and ϕ_j are eigenvectors of A^*.

Proof. If A is quasi-Hermitian matrix then it is similar to Hermitian one, hence it has real spectrum and it is diagonalizable.

If A has the real spectrum and it is diagonalizable, then

$$A = X^{-1} D X, \tag{2.9}$$

where D is a diagonal matrix composed of real eigenvalues. If we take adjoint of (2.9) we arrive at

$$X X^* A = A^* X^* X. \tag{2.10}$$

Whence $\Theta := X X^*$ is positive and it can be expressed in the form (2.8). □

Infinite dimensional case is of course more complicated, in fact there is no general result for all quasi-Hermitian operators yet. Some results for operators with pure discrete spectrum (see definition A.7) are known. We would like to present more rigorous approach to the construction of the metric Θ. In order to describe eigenvectors of quasi-Hermitian operator we recall a notion of Riesz basis and a basic criterion.

Definition 2.3 ([42]). *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Then the set $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is said to be a Riesz basis if there exists a bounded invertible operator U with bounded inverse and $x_n = Ue_n$ for all $n \in \mathbb{N}$.*

Theorem 2.9 ([42]). *Set $\{x_n\}_{n=1}^{\infty}$ is a Riesz basis if and only if there exist positive constants m, M such that*

$$m\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq M\|x\|^2 \tag{2.11}$$

for each $x \in \mathcal{H}$.

Proposition 2.10. *Let $A \in \mathscr{L}(\mathcal{H})$ be quasi-Hermitian operator with pure discrete spectrum and metric operator Θ. Let denote $\{\psi_n\}_{n=1}^{\infty}$, $\{\phi_n\}_{n=1}^{\infty}$ eigenvectors of A and A^*, respectively. It is possible to normalize them in a is such way that*

$$\phi_n = \Theta\psi_n, \quad \langle \psi_i, \phi_j \rangle = \delta_{ij} \quad \text{for all } i, j, n \in \mathbb{N}. \tag{2.12}$$

Both $\{\psi_n\}_{n=1}^{\infty}$ and $\{\phi_n\}_{n=1}^{\infty}$ are Riesz bases and

$$\Theta = \operatorname*{s-lim}_{N\to\infty} \sum_{j=1}^{N} \langle \phi_j, \cdot \rangle \, \phi_j. \tag{2.13}$$

Proof. We denote H self-adjoint operator which is similar to A and $\{e_n\}_{n=1}^{\infty}$ eigenvectors of H,

$$H = \varrho A \varrho^{-1}, \quad \varrho = \sqrt{\Theta}, \quad He_n = \lambda_n e_n, \ \|e_n\| = 1, \ \forall n \in N. \tag{2.14}$$

Since spectrum of H is pure discrete the resolvent of H is a normal compact operator, see theorem A.10. Hence normalized eigenvectors $\{e_n\}_{n=1}^{\infty}$ form orthonormal basis. Eigenvectors ψ_n, ϕ_n of A, A^* satisfy $\psi_n = \varrho^{-1}e_n$, $\phi_n = \varrho e_n$. Hence $\phi_n = \Theta\psi_n$

and $\langle \psi_i, \phi_j \rangle = \delta_{ij}$. Moreover, using functional calculus for self-adjoint operators, it follows from requirements for Θ (definition 2.1) that ϱ is bounded positive operator with bounded inverse and therefore $\{\psi_n\}_{n=1}^{\infty}$ and $\{\phi_n\}_{n=1}^{\infty}$ are Riesz bases.

We denote $\Theta_N := \sum_{j=1}^{N} \langle \phi_j, \cdot \rangle \phi_j$. Each $x \in \mathcal{H}$ can be uniquely expressed in the Riesz basis $\{\psi_n\}_{n=1}^{\infty}$

$$x = \sum_{n=1}^{\infty} \alpha_n \psi_n, \quad \alpha_n \in \mathbb{C}, \tag{2.15}$$

the action of Θ_N then reads

$$\Theta_N x = \sum_{j=1}^{N} \langle \phi_j, x \rangle \phi_j = \sum_{j=1}^{N} \sum_{n=1}^{\infty} \langle \phi_j, \alpha_n \psi_n \rangle \phi_j = \sum_{n=1}^{N} \alpha_n \phi_n. \tag{2.16}$$

Relation

$$\langle x, \Theta_N x \rangle = \sum_{n=1}^{N} |\alpha_n|^2 \tag{2.17}$$

shows that Θ_N is positive and by similar procedure, $\{\Theta_N\}_{N=1}^{\infty}$ is an increasing sequence of positive operators. Furthermore, $\Theta_N \leq \Theta$ for all $N \in \mathbb{N}$ due to

$$\langle (\Theta - \Theta_N) x, x \rangle = \sum_{n=N+1}^{\infty} |\alpha_n|^2. \tag{2.18}$$

Hence, $\{\Theta_N\}$ converges in a strong limit sense to some bounded positive operator (theorem A.11). In order to complete the proof, i.e. Θ is limit of $\{\Theta_N\}$, we show that

$$\Theta = \operatorname*{w-lim}_{N \to \infty} \Theta_N. \tag{2.19}$$

We take arbitrary $x, y \in \mathcal{H}$, then

$$|\langle (\Theta - \Theta_N) x, y \rangle| \leq \sum_{k=N+1}^{\infty} |\overline{\alpha_k} \beta_k| \leq \sqrt{\sum_{k=N+1}^{\infty} |\alpha_k|^2} \sqrt{\sum_{k=N+1}^{\infty} |\beta_k|^2}, \tag{2.20}$$

where

$$x = \sum_{n=1}^{\infty} \alpha_n \psi_n, \quad y = \sum_{n=1}^{\infty} \beta_n \psi_n. \tag{2.21}$$

Since $\varrho x = \sum_{n=1}^{\infty} \alpha_n e_n$ and $\varrho y = \sum_{n=1}^{\infty} \beta_n e_n$, the sums on the right side of (2.20) are finite and both converge to 0 for $N \to \infty$. □

We remark that if we insert positive constants c_j with property

$$\exists\, m, M > 0, \quad m \le c_j \le M \text{ for all } \; j \in \mathbb{N}, \tag{2.22}$$

into the sum (2.13) in the same way as in the finite dimensional case (2.8) we obtain another metric.

It is natural to ask then, if every pseudo-Hermitian operator (or/and with anti-linear symmetry) with real and pure discrete spectrum is quasi-Hermitian. Let us assume that A is a pseudo-Hermitian operator with compact resolvent. Then, by theorem A.7, the spectrum of A is pure discrete. Let us denote ψ_n, ϕ_n the eigenvectors of A, A^* respectively and λ_n the eigenvalues. We assume that the eigenvectors are normalized in a special way, $\|\phi_n\| = 1$ and $\langle \psi_n, \phi_n \rangle = 1$. Let us assume further that

$$\Theta = \operatorname*{s-lim}_{N \to \infty} \sum_{j=1}^{N} c_j \, \langle \phi_j, \cdot \rangle \, \phi_j \tag{2.23}$$

exists and it is a bounded operator for some set of positive numbers c_n which satisfies the condition (2.22).

Particurlary, it is not possible that $c_n \to 0$ (or some subsequence $c_{k_n} \to 0$), although this may seem to be useful for guaranteeing the convergence of (2.23). However, then $0 \in \sigma(\Theta)$ and therefore the inverse of Θ is not bounded. If we consider a sequence $\{\xi_n\} = \{\psi_n / \|\psi_n\|\}$ of unit vectors, then

$$\|\Theta \xi_n\| = \| \sum_{j=1}^{\infty} c_j \, \langle \phi_j, \xi_n \rangle \, \phi_j \| = \|c_n \, \langle \phi_n, \xi_n \rangle \, \phi_n\| \le c_n \to 0. \tag{2.24}$$

Without loss of generality we will assume that $c_n = 1$ for all $n \in \mathbb{N}$.

The crucial step is to show that eigenvectors ϕ_n form a Riesz basis. First of all it is necessary that span $\{\phi_1, \phi_2, ...\}$ is a dense set in $\mathcal{H}$, i.e. $\{\phi_n\}_{n=1}^{\infty}$ is a complete system in $\mathcal{H}$. We can show this for example by investigating Ker(Θ). If the point spectrum $\sigma_p(\Theta)$ does not contain 0, then each $x_0 \in \{\phi_1, \phi_2, ...\}^{\perp}$ is zero vector because it follows that $\Theta x_0 = 0$, by inserting x_0 into (2.23). Feasible analysis of the kernel of Θ using directly the sum expansion can be found e.g. in [22].

Further, the strong convergence of Θ implies the weak convergence and for all $x \in \mathcal{H}$

$$\Big\langle \sum_{n=1}^{\infty} \langle \phi_n, x \rangle \, \phi_n, x \Big\rangle = \sum_{n=1}^{\infty} | \, \langle \phi_n, x \rangle \, |^2 \le \|\Theta\| \|x\|^2. \tag{2.25}$$

Hence we obtained one of the inequalities (2.11). The second inequality exactly corresponds with existence of a bounded inverse of Θ, i.e. whether 0 is or is not in the continuous spectrum of Θ. This is probably the most technically difficult step in this procedure which cannot be skipped.

Nevertheless, it is satisfied automatically for η-pseudo-Hermitian operators with antilinear symmetry C, where η and C are involutive and commuting, i.e. $\eta^2 = I$, $C^2 = I$, $[\eta, C] = 0$. Then, it is obvious that A^* also has antilinear symmetry C and we may assume that eigenvectors ϕ_n are C invariant, it suffices to take $(\phi_n + C\phi_n)$. Furthermore, they can be normalized in such way that

$$\langle \phi_n, \eta\phi_m \rangle = 0 \quad \text{for } n \neq m \quad \text{and} \quad \langle \phi_n, \eta\phi_n \rangle = \pm 1 \quad \forall n \in \mathbb{N}. \tag{2.26}$$

Since η is typically indefinite we cannot exclude the '-1 case' in the latter. Properties of η, C and C invariance of ϕ_n yield

$$\langle \phi_n, \eta\phi_m \rangle = \langle \phi_n, C\eta\phi_m \rangle \quad \forall n, m \in \mathbb{N}. \tag{2.27}$$

Antilinear operator $C\eta$ is surely involutive and we can assume that the set $\{\phi_n\}_{n=1}^{\infty}$ is $C\eta$-orthonormal,

$$\langle \phi_n, C\eta\phi_m \rangle = \delta_{nm}, \tag{2.28}$$

it suffices to take $\mathrm{i}\phi_k$ when there '-1 case' occurs. The essential fact then follows from

Theorem 2.11 ([17])**.** *Let $\{u_n\}_{n=1}^{\infty}$ be a complete J-orthonormal system in $\mathcal{H}$, where J is an antilinear involution. Then following assertions are equivalent*
(i) for each $x \in \mathcal{H}$

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq M\|x\|^2, \tag{2.29}$$

(ii) $\{u_n\}_{n=1}^{\infty}$ is a Riesz basis with $m = M^{-1}$.

Hence, the set $\{\phi_n\}_{n=1}^{\infty}$ with all assumptions above is a Riesz basis.

It remains to verify if relation $\Theta A x = A^* \Theta x$ for all $x \in \mathrm{Dom}(A)$ holds. We consider $\mu \in \varrho(A)$ and rewrite the condition to following equivalent form

$$(A^* - \mu)^{-1}\Theta x = \Theta(A - \mu)^{-1} x \quad \forall x \in \mathcal{H}. \tag{2.30}$$

Now, the expansion of arbitrary $x \in \mathcal{H}$ into the Riesz basis $\{\psi_n\}_{n=1}^{\infty}$, relations

$$(A-\mu)^{-1}\psi_n = \frac{1}{\lambda_n - \mu}\psi_n, \quad (A^* - \mu)^{-1}\phi_n = \frac{1}{\lambda_n - \mu}\phi_n \tag{2.31}$$

and explicit form of Θ (2.23) are needed.

We may summarize the ideas above to the proposition.

Proposition 2.12. *Let $A \in \mathscr{L}(\mathcal{H})$ be an η-pseudo-Hermitian operator with antilinear symmetry C, where η and C are both involutive and commuting. Let the resolvent of A is compact for some $\mu \in \mathbb{C}$ and the spectrum of A is real. If*

$$\Theta = \operatorname*{s-lim}_{N\to\infty} \sum_{j=1}^{N} c_j \langle \phi_j, \cdot \rangle \phi_j, \tag{2.32}$$

where $\|\phi_j\| = 1$ are eigenvectors of A^* and c_j are positive constants satisfying (2.22), is an invertible bounded operator (i.e. $0 \notin \sigma_p(\Theta)$), then A is quasi-Hermitian with the metric Θ.

Typical example of operators satisfying the assumptions of the proposition are $\mathcal{P}$-pseudo-Hermitian $\mathcal{PT}$-symmetric Hamiltonians with real spectrum defined on finite interval (a, b). We present some examples of such operators in following section. Many more examples can be found in a literature dealing with $\mathcal{PT}$-symmetry. Let us stress particularly the carefully and rigorously treated model [22].

Chapter 3

Models

3.1 Chain models

An interesting class of finite dimensional models, so called chain-models, presented Znojil in [48, 46, 47, 45]. It refers to $N \times N$ tridiagonal matrices of the form

$$H^{(chain)} = \begin{pmatrix} 1-N & g_1 & 0 & 0 & \dots & 0 \\ -g_1 & 3-N & g_2 & 0 & \dots & 0 \\ 0 & -g_2 & 5-N & \ddots & \ddots & \vdots \\ 0 & 0 & -g_3 & \ddots & g_{N-2} & 0 \\ \vdots & \vdots & \ddots & \ddots & N-3 & g_{N-1} \\ 0 & 0 & \dots & 0 & -g_{N-1} & N-1 \end{pmatrix}, \tag{3.1}$$

where g_n are real parameters. For sake of simplicity the parameters are assumed to be up-down symmetric, i.e.

$$g_{N-k} = g_k \geq 0, \quad k \in \{1, 2, ..., J\}, \tag{3.2}$$

where $N = 2J$ or $N = 2J + 1$. From the physical point of view, these tridiagonal models may represent a nearest neighbor-interaction perturbation of a harmonic-oscillator-like systems with shifted original equidistant eigenvalues $\{1, 3, 5, ...\}$. It is obvious that the finite dimensional truncated Hamiltonian $H^{(chain)}$ is not Hermitian but $\mathcal{PT}$-symmetric, where $\mathcal{P}$ corresponds to the $N \times N$ matrix

$$\mathcal{P} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}. \tag{3.3}$$

In spite of the fact that the technical requirements for investigation of $H^{(chain)}$ increase rapidly with the dimension, several important fact have been shown in [48, 46, 47, 45].

For any dimension N there exist a J-dimensional domain $\mathcal{D}$ of the matrix elements for which the Hamiltonian is quasi-Hermitian. The domain $\mathcal{D}$ is contained in a bigger one $\mathcal{S}$ defined by following inequalities

$$\frac{N^3 - N}{2} \;\geq\; 2 \sum_{n=1}^{J-1} g_n^2 + \begin{cases} g_J^2, & N = 2J, \\ 2\, g_J^2, & N = 2J+1\,. \end{cases} \tag{3.4}$$

The set of intersections of $\mathcal{D}$ and $\partial\mathcal{S}$ is finite. These points are called extremely exceptional points, EEP, and it is possible to write their coordinates in a closed form

$$g_n^{(EEP)} = \sqrt{n\,(N-n)}\,, \qquad n = 1, 2, \ldots, J\,. \tag{3.5}$$

When we cross the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, some of the energies are complexified. The EEPs are special for complexification of all energies. Thus the presented systems may serve as a illustration of "quantum catastrophes", i.e. a perturbation may cause that we leave the domain $\mathcal{D}$ and some energies become complex. In the domaine $\mathcal{D}$ the Hamiltonian is quasi-Hermitian, i.e. physical - we may describe the system in a Hilbert space with modified scalar product (2.1).

Many numerical results, e.g. in [47], illustrates how energies may become complex, number of energies depends on the way out of the $\mathcal{D}$. Also two dimensional domain $\mathcal{D}^{(2)}$ for the Hamiltonian $H^{(3)}$ is presented,

$$H^{(3)} = \begin{pmatrix} -1 & a & 0 \\ -a & 1 & b \\ 0 & -b & 3 \end{pmatrix} \qquad \mathcal{P}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

The secular equation for $H^{(3)}$ reads

$$-E^3 + 3\,E^2 + \left(-a^2 + 1 - b^2\right) E - 3 + 3\,a^2 - b^2 = 0, \tag{3.7}$$

and EEPs have coordinates $(\pm\sqrt{2}, \pm\sqrt{2})$. The boundary $\partial\mathcal{D}$ of the star-like shape domain $\mathcal{D}$ may be parametrized in following way

$$a = a_\pm = \pm\sqrt{\frac{1}{2}\,(4 - 3\beta^2 - \beta^3)}, \qquad b = b_\pm = \pm\sqrt{\frac{1}{2}\,(4 - 3\beta^2 + \beta^3)}, \tag{3.8}$$

where parameter $\beta \in (-1, 1)$.

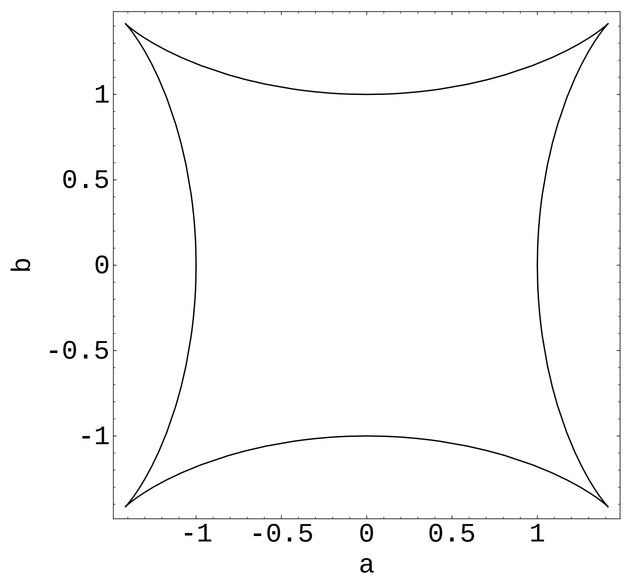


Figure 3.1: Boundary $\partial\mathcal{D}^{(2)}$

3.1.1 The domain $\mathcal{D}$ for $H^{(6)}$

We consider 6 dimensional model,

$$H^{(6)} = \begin{pmatrix} -5 & c & 0 & 0 & 0 & 0 \\ -c & -3 & b & 0 & 0 & 0 \\ 0 & -b & -1 & a & 0 & 0 \\ 0 & 0 & -a & 1 & b & 0 \\ 0 & 0 & 0 & -b & 3 & c \\ 0 & 0 & 0 & 0 & -c & 5 \end{pmatrix} \qquad (3.9)$$

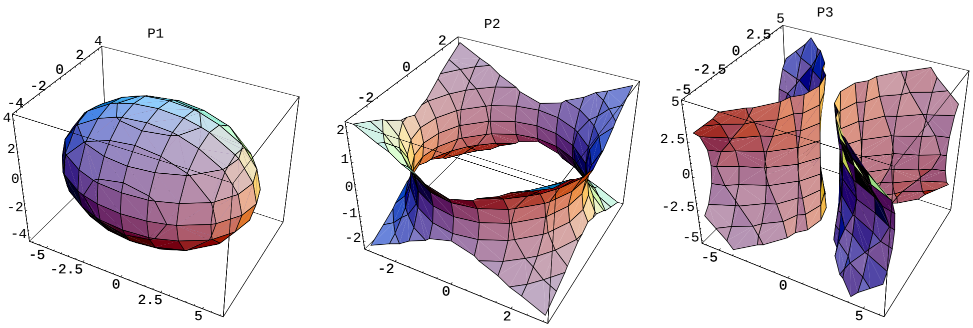
and try to find the domain $\mathcal{D}$. EEPs (3.5) are at $(a, b, c) = (\pm 3, \pm 2\sqrt{2}, \pm\sqrt{5})$. Secular equation for the Hamiltonian reads

$$\chi(a, b, c, s) \equiv \det\left(H^{(6)} - E\, I\right) = s^3 + P_1\, s^2 + P_2\, s + P_3 = 0\,, \qquad s = E^2, \quad (3.10)$$

where

$$\begin{aligned} P_1 &= a^2 + 2b^2 + 2c^2 - 35, \\ P_2 &= 259 - 34a^2 - 44b^2 + b^4 + 28c^2 + 2a^2c^2 + 2b^2c^2 + c^4, \qquad (3.11) \\ P_3 &= -225 + 225a^2 - 150b^2 - 25b^4 - 30c^2 + 30a^2c^2 - 10b^2c^2 - c^4 + a^2c^4. \end{aligned}$$

Elementary analysis of the roots of (3.10) (s must be non-negative for $E \in \mathbb{R}$) yields that the domain $\mathcal{D}^{(3)}$ is contained in the area determined by conditions $P_{1,2,3} \geq 0$, see figure 3.2. In order to fully determine the domain D we analyse the qualitative meaning of the parameter a, b, c, i.e. their impact on properties of the characteristic polynomial χ. $H^{(6)}$ is obviously Hermitian for $a = 0$, $b = 0$, $c = 0$. $E = s^2$, hence the complexification of energies E may occur for several reason:
(i) one or more roots s are negative,
(ii) two roots s are non-real, (complex conjugated to each other).
The second possibility may happen if the local minimum of χ is below zero (or the local maximum is above zero). Figure 3.3 illustrates, how the position of roots and local extremes is modified with respect to change of parameters a, b, c. Thus, the boundary of the domain $\mathcal{D}$ is defined by surface S_1 corresponding to the condition $P_3 = s_1.s_2.s_3 = 0$, where we denoted $s_{1,2,3}$ the roots of χ, and surfaces $S_{2,3}$ determined by the condition that the local extremes of χ are zero. Local extremes χ are



(a) $P_1 = 0$ (b) $P_2 = 0$ (c) $P_3 = 0$

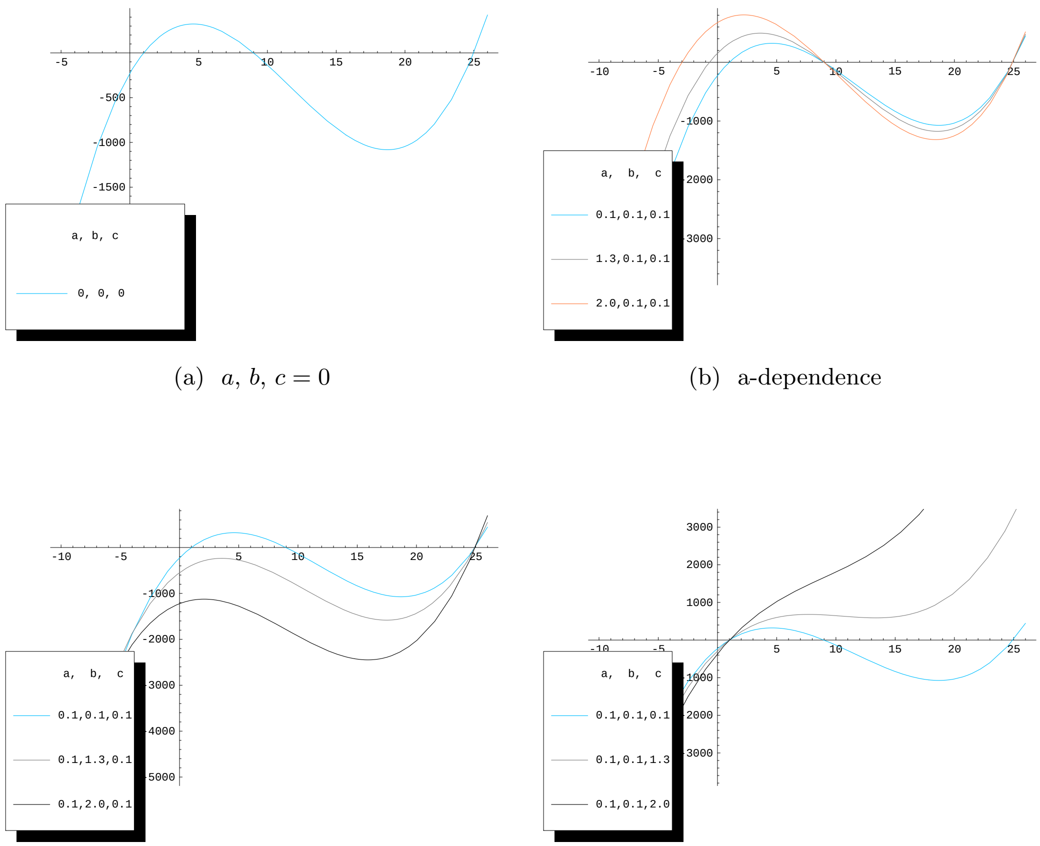
Figure 3.2: Areas determined by $P_i = 0$

situated at points $x_{1,2}$

$$x_{1,2} = \frac{1}{3}\Big(35 - a^2 - 2b^2 - 2c^2 \pm \\ \pm\sqrt{448 + 32a^2 + a^4 - 8b^2 + 4a^2b^2 + b^4 - 224c^2 - 2a^2c^2 + 2b^2c^2 + c^4}\Big). \quad (3.12)$$

Hence conditions $\chi(a, b, c, x_{1,2}) = 0$ determine the surfaces $S_{2,3}$, see figure 3.5, where the red points are EEPs. The intersection of the surfaces illustrates figure 3.4

Answer to the question how and how many energies are complexified provide figures 3.6,3.7, where the colors of the graphs of χ and areas correspond (white=black). Two energies are complexified in the orange area - there is one negative s root. Four energies are complexified in the yellow area - there are two negative s roots, and in the white area - there are two complex conjugated s roots. Finally, all six energies are complexified in the green area - there are one negative and two complex conjugated s roots.



(a) a, b, $c = 0$ (b) a-dependence

(c) b-dependence (d) c-dependence

Figure 3.3: Characteristic polynomial χ

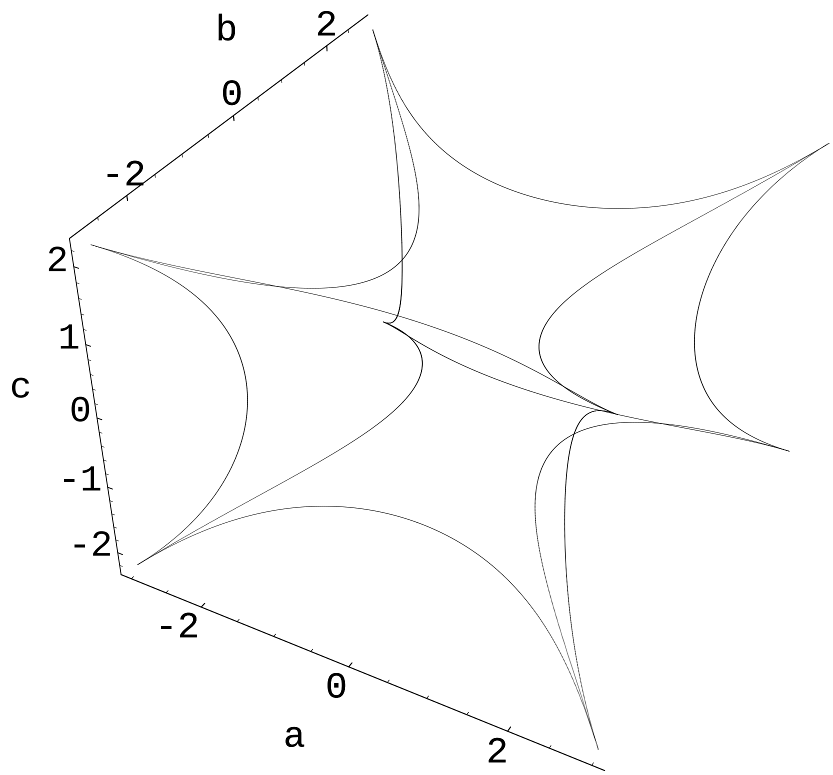
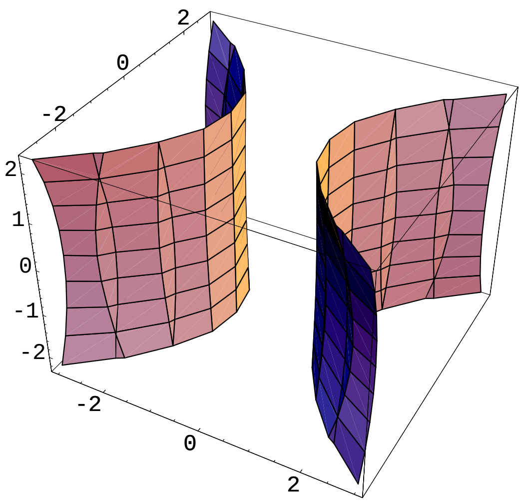
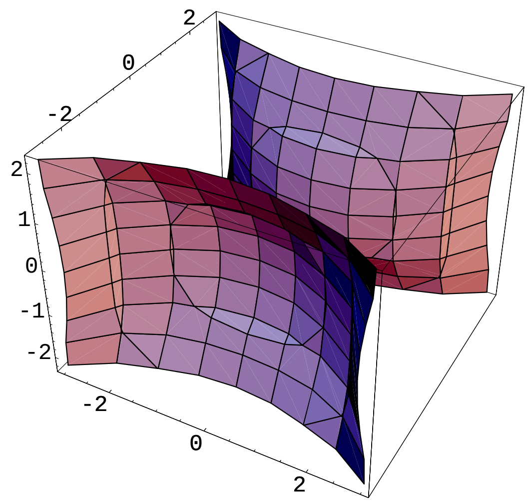


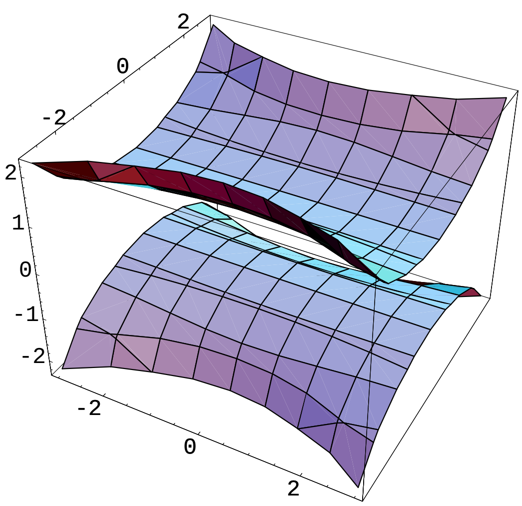
Figure 3.4: The intersection of $S_{1,2,3}$



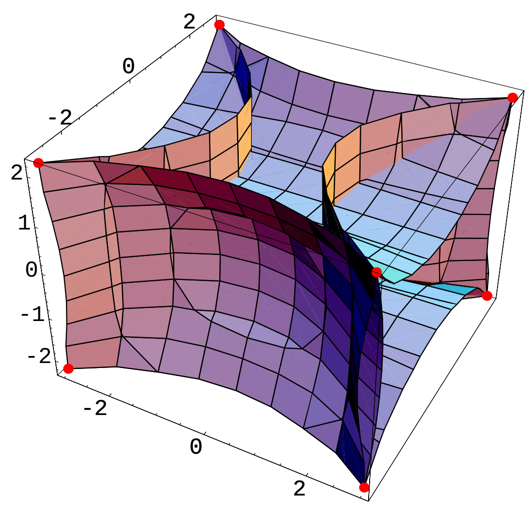
(a) Surface S_1



(b) Surface S_2



(c) Surface S_3



(d) The domain$\mathcal{D}$ with EEPs

Figure 3.5: Surfaces $S_{1,2,3}$ and the domain $\mathcal{D}$ with EEPs.

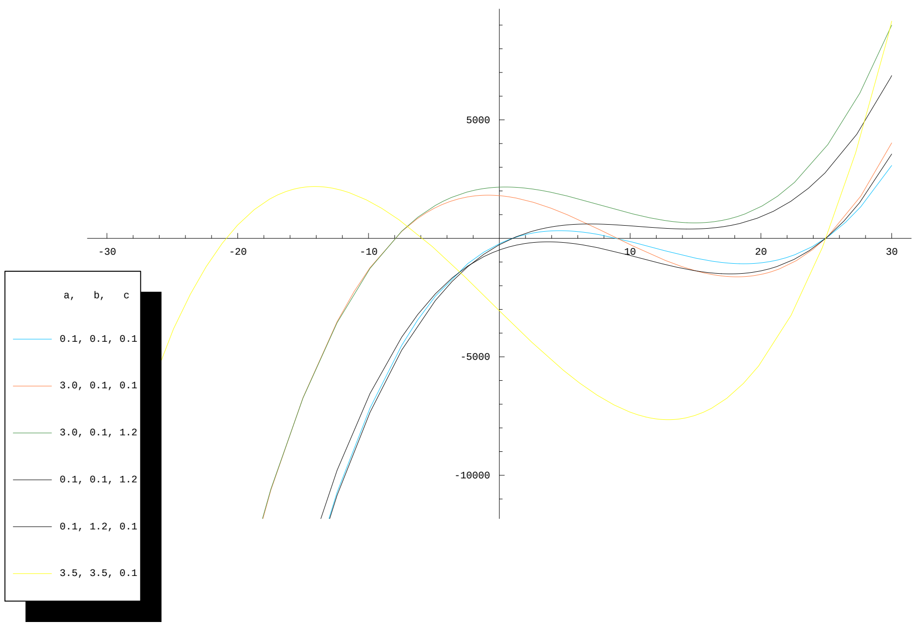


Figure 3.6: Characteristic polynomial χ

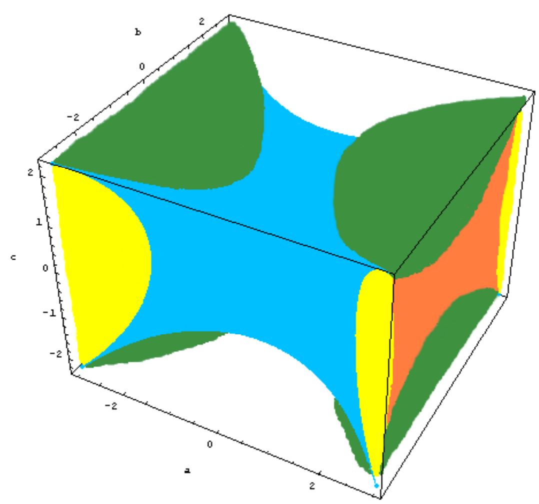


Figure 3.7: The domain $\mathcal{D}$ and areas of complexification

3.2 $\mathcal{PT}$-symmetric point interaction on a line

Typical one dimensional $\mathcal{PT}$-symmetric physical systems are Hamiltonians acting in $L_2(\mathbb{R})$ of the form

$$H = -\frac{d^2}{dx^2} + V(x), \tag{3.13}$$

where potential V is a complex function satisfying

$$V(-x) = \overline{V(x)}. \tag{3.14}$$

Although $\mathcal{PT}$-symmetry does not guarantee the reality of spectrum, many such systems were discovered and intensively studied. It is natural that systems with various types of potential may be very complicated and only little of them are exactly solvable. $\mathcal{PT}$-symmetry, in sense of non-self-adjointness, brings another difficulties. In order to obtain some suitable solvable models we use point interactions. The point interaction should be understood as any interaction which does not affect the function with the support separated from the point where interaction acts. Theory of standard Hermitian point interactions, which is based on the self-adjoint extensions of symmetric operators, is well described in several monographs, we mention one of them [2]. Some elementary facts about self-adjoint extensions of symmetric operators and point interactions in one dimension can be found in Appendix. The methods for self-adjoint operators are not applicable for $\mathcal{PT}$-symmetric systems and they has to be modified.

The most important initial work on $\mathcal{PT}$-symmetric point interactions is done in [1]. Another papers presents concrete systems on a line or finite interval [29, 49]. At first, we summarize the main principles and results. Then, we extend the results of [3] concerning the relation between systems with one point interaction on a line and quasi-Hermitian operators. Next section is devoted to supersymmetrical systems with two $\mathcal{PT}$-symmetric point interactions on a finite interval.

We consider one point interaction at the origin. More technically speaking, we take a second derivative operator $L_0 = -(d^2/dx^2)$ with the domain

$$\mathrm{Dom}(L_0) = C_0^\infty(\mathbb{R} \setminus \{0\}) \tag{3.15}$$

as a starting point. The adjoint L_0^* is again the second derivative $L_0^* = -(d^2/dx^2)$, however with the domain

$$\mathrm{Dom}(L_0^*) = AC^2(\mathbb{R} \setminus \{0\}). \tag{3.16}$$

We use the notation of [10], $\psi \in \mathrm{AC}^2(\Omega)$ if ψ, ψ' are absolutely continuous at Ω and $\psi'' \in L^2$. Any $\mathcal{PT}$-symmetric point interaction is represented by $\mathcal{PT}$-symmetric extension of L_0 (or restriction of L_0^*). Thus we describe the point interactions by boundary conditions.

Theorem 3.1 ([1])**.** *The family of $\mathcal{PT}$-symmetric second derivative operators with point interactions at the origin coincides with the set of restrictions of the second derivative operator $L_{\max} = -\frac{d^2}{dx^2}$, defined on $AC^2(\mathbb{R} \setminus \{0\})$, to the domain of functions satisfying the boundary conditions at the origin of one of the following two types*

1.
$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = B \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix} \tag{3.17}$$

with the matrix B equal to

$$B = e^{i\theta} \begin{pmatrix} \sqrt{1+bc}\,e^{i\phi} & b \\ c & \sqrt{1+bc}\,e^{-i\phi} \end{pmatrix} \tag{3.18}$$

with the real parameters $b \geq 0$, $c \geq -1/b$, $\theta, \phi \in [0, 2\pi)$

2.
$$h_0\psi'(0+) = h_1 e^{i\theta}\psi(0+) \tag{3.19}$$
$$h_0\psi'(0-) = -h_1 e^{-i\theta}\psi(0-) \tag{3.20}$$

with the real phase parameter $\theta \in [0, 2\pi)$ and with the parameter $\boldsymbol{h} = (h_0, h_1)$ taken from the (real) projective space $\boldsymbol{P}^1$.

The symbols $0\pm$ have usual meaning of limits, $\psi(0\pm) = \lim_{x\to 0\pm}\psi(x)$. Boundary conditions of the first type are called connected, conditions of the second type are called separated. Further we deal with connected boundary conditions only because the separated case can be decomposed to two separated systems on a half-line.

Although the original statement of [1] includes that the restricted operators are both $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian for the entire range of parameters b, c, θ, ϕ, we show that $\mathcal{P}$-pseudo-Hermiticity is ensured only for $\theta = 0$ (other ranges of parameters are preserved), see Appendix for the details. Nevertheless, the operators for $\theta \neq 0$ and $\theta = 0$ are unitary equivalent [1].

Spectrum of Hamiltonians with $\mathcal{PT}$-symmetric point interactions is described by following

Theorem 3.2 ([1])**.** *The spectrum of any $\mathcal{PT}$-symmetric second derivative operator with point interactions at the origin consists of the branch $[0, \infty)$ of the absolutely continuous spectrum and at most two (counting multiplicity) eigenvalues, which are real negative or are complex conjugated to each other.*

Proposition 3.3 ([1])**.** *The spectrum of the $\mathcal{PT}$-symmetric second derivative operator with connected point interaction at the origin is pure real if and only if the parameters appearing in theorem 3.1 satisfy in addition at least one of the following conditions*
(i) $bc \sin^2 \phi \leq \cos^2 \phi$,
(ii) $bc \sin^2 \phi \geq \cos^2 \phi$ and $\cos \phi \geq 0$.

3.2.1 Quasi-Hermitian $\mathcal{PT}$-symmetric point interactions

Our aim is to classify the Hamiltonians with one $\mathcal{PT}$-symmetric point interaction which have real spectrum according to their similarity to self-adjoint operator, i.e. quasi-Hermiticity. Analysis of a such type has been already done in [3] for special subclass of Hamiltonians with one $\mathcal{PT}$-symmetric point interaction corresponding to the potential composed of δ and δ' functions

$$V = a \langle \delta, \cdot \rangle \delta + b \langle \delta', \cdot \rangle \delta + c \langle \delta, \cdot \rangle \delta' + d \langle \delta', \cdot \rangle \delta', \tag{3.21}$$

where $a, d \in \mathbb{R}$ and $c = -\bar{b}$. The results show that the reality of spectrum itself does not guarantee quasi-Hermiticity.

Technically, we use the resolvent criterion, theorem 2.5, and necessary condition for similarity to self-adjoint operator, proposition 2.7. To be able to use these criteria we calculate resolvent $R_\lambda(H)$ of the Hamiltonian $H = -(d^2/dx^2)$ with the domain

$$\mathrm{Dom}(H) = \left\{ \psi \in \mathrm{AC}^2(\mathbb{R} \setminus \{0\}) \,\middle|\, \psi \text{ satisfies } (3.17, 3.18) \right\}. \tag{3.22}$$

We denote k the square root of λ and to obtain unique k pro every $\lambda \in \mathbb{C}$ we require $\operatorname{Im} k \geq 0$. We introduce two functions e_+ and e_-

$$e_+(x) = \vartheta(x)e^{\mathrm{i}kx}, \quad e_-(x) = \vartheta(-x)e^{-\mathrm{i}kx}, \tag{3.23}$$

where ϑ is a Heaviside step function. Then resolvent $R_\lambda(H)$ can be written for arbitrary $f \in L_2(\mathbb{R})$ in the form

$$g(x) \equiv R_\lambda(H)f(x) = (R_\lambda(H_0)f)(x) + C_+(f)e_+(x) + C_-(f)e_-(x), \tag{3.24}$$

where

$$R_\lambda(H_0)f(x) = \frac{\mathrm{i}}{2k}\int_{-\infty}^{\infty} e^{\mathrm{i}k|x-y|}f(y)\mathrm{d}y \tag{3.25}$$

is well-known formula for self-adjoint free particle Hamiltonian and $C_\pm$ are parameters to be calculated.

$$\begin{array}{rclcrcl} g(0+) &=& \frac{\mathrm{i}}{2k}(f_- + f_+) + C_+(f), && g'(0+) &=& -\frac{1}{2}(f_- - f_+) + \mathrm{i}kC_+(f), \\ g(0-) &=& \frac{\mathrm{i}}{2k}(f_- + f_+) + C_-(f), && g'(0-) &=& -\frac{1}{2}(f_- - f_+) - \mathrm{i}kC_-(f), \end{array} \tag{3.26}$$

where

$$f_\pm = \int_{\mathbb{R}_\pm} e^{\pm\mathrm{i}ky}f(y)\mathrm{d}y. \tag{3.27}$$

Function $g = R_\lambda(H)f$ must satisfy the boundary conditions (3.18) thus

$$\begin{pmatrix} e^{\mathrm{i}\theta}(\mathrm{i}bk - \sqrt{1+bc}\,e^{\mathrm{i}\phi}) & 1 \\ e^{\mathrm{i}\theta}(\mathrm{i}k\sqrt{1+bc}\,e^{-\mathrm{i}\phi} - c) & \mathrm{i}k \end{pmatrix} \cdot \begin{pmatrix} C_- \\ C_+ \end{pmatrix} =$$
$$\begin{pmatrix} \frac{1}{2k}\left(k(f_- - f_+) + \sqrt{1+bc}\,e^{\mathrm{i}(\theta-\phi)}k(-f_- + f_+) + \mathrm{i}ce^{\mathrm{i}\theta}(f_- + f_+)\right) \\ \frac{\mathrm{i}}{2}\left(\left(\sqrt{1+bc}\,e^{\mathrm{i}(\theta+\phi)} - 1\right)(f_- + f_+) + \mathrm{i}be^{\mathrm{i}\theta}k(f_- - f_+)\right) \end{pmatrix}. \tag{3.28}$$

This system of linear equations is solvable if and only if the determinant of the matrix is nonzero

$$e^{\mathrm{i}\theta}\left(bk^2 + 2k\mathrm{i}\sqrt{1+bc}\,\cos\phi - c\right) \neq 0. \tag{3.29}$$

Then the solutions can be written as

$$\begin{aligned} C_-(f) &= \frac{-1}{2kp(k)}\Big[2k\big(\sqrt{1+bc}\,f_+\cos\phi - f_+\cos\phi\big) + \mathrm{i}c(f_- + f_+) + \\ &\quad \mathrm{i}bk^2(f_- - f_+) + 2\mathrm{i}k(f_+\sin\phi + \sqrt{1+bc}\,f_-\sin\phi)\Big], \\ C_+(f) &= \frac{\mathrm{i}e^{-\mathrm{i}\phi}}{2kp(k)}\Big[k\Big(e^{\mathrm{i}\phi}(bk(f_- - f_+) + 2\sqrt{1+bc}\,(\mathrm{i}f_-\cos\phi + f_+\sin\phi)) + \\ &\quad 2f_-\big(\sin(\theta+\phi) - \mathrm{i}\cos(\theta+\phi)\big)\Big) - ce^{\mathrm{i}\phi}(f_- + f_+)\Big], \end{aligned} \tag{3.30}$$

where

$$p(k) = bk^2 + 2k\mathrm{i}\sqrt{1+bc}\cos\phi - c. \tag{3.31}$$

Proposition 3.4. *Let $H \in \mathscr{L}(L^2(\mathbb{R}))$ be a Hamiltonian corresponding to one $\mathcal{PT}$-symmetric point interaction at the origin (3.22). Let parameters b, c, ϕ satisfy $bc\sin^2\phi \le \cos^2\phi$ (i.e. spectrum of H is real, proposition 3.3). If in addition one of the following conditions for parameters is satisfied*

(i) $c > 0,\ \cos\phi < 0,\ b \ge 0,$

(ii) $c = 0,\ \cos\phi \le 0,\ b > 0,$

(iii) $c < 0,\ \cos\phi \ge 0,\ b > 0,$

(iv) $c < 0,\ \cos\phi > 0,\ b = 0,$

then H is not similar to any self-adjoint operator.

Proof. If H is similar to a self-adjoint operator then the inequality from the proposition 2.7 holds for every $g \in L_2(\mathbb{R})$. Further, $R_\lambda(H_0)$ (3.25) is resolvent of the self-adjoint operator, thus for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$

$$\|\left(R_\lambda(H) - R_\lambda(H_0)\right)g\|^2 \le \frac{M}{(\operatorname{Im}\lambda)^2}\|g\|^2, \tag{3.32}$$

where M is a positive constant independent of λ and g, is valid for each $g \in L_2(\mathbb{R})$, particularly for

$$g_0(x) = \vartheta(x)e^{-\mathrm{i}\bar{k}x}, \tag{3.33}$$

where ϑ is a Heaviside step function, and $k^2 = \lambda$ and $\operatorname{Im} k > 0$. We calculate the norms

$$\|g_0\|^2 = \frac{1}{2\operatorname{Im} k}, \quad \|e_\pm\|^2 = \frac{1}{2\operatorname{Im} k}, \tag{3.34}$$

and together with relations (3.30) and identities $(\operatorname{Im}\lambda)^2 = 4(\operatorname{Im} k)^2(\operatorname{Re} k)^2$ we receive condition (3.32) in the explicit form

$$\left(\left|\frac{\mathrm{i}}{2} + \frac{\mathrm{i}c + k\sqrt{1+bc}\,e^{-\mathrm{i}\phi}}{p(k)}\right|^2 + \left|-\frac{\mathrm{i}}{2} + \frac{ke^{-\mathrm{i}\theta}}{p(k)}\right|^2\right)\frac{(\operatorname{Re} k)^2}{|k|^2} \le M, \tag{3.35}$$

where $p(k)$ has been already defined (3.31). If we take into account that roots of $p(k)$ (for non degenerate case $b \ne 0$) read

$$k_{1,2} = -\frac{\mathrm{i}}{b}\cos\phi\left(\sqrt{1+bc} \pm \sqrt{\cos^2\phi - bc\sin^2\phi}\right), \tag{3.36}$$

we may easily prove that if parameters b, c, ϕ satisfy at least one of the conditions then left hand side of (3.35) tends to infinity in the neighborhood of $k_{1,2}$. $\square$

We intend to show the similarity to a self-adjoint operator for class of $\mathcal{PT}$-symmetric point interactions using criterion of the theorem 2.5. To make the proof more technically feasible we at first modify relations (3.30) for functions $f_{r,l}(x) := \vartheta(\pm x) f(x)$

$$
\begin{aligned}
C_-(f_l) &= \left(\frac{\mathrm{i}}{2} + \frac{\mathrm{i}c + k\sqrt{1+bc}\,e^{\mathrm{i}\phi}}{p(k)}\right)\frac{f_{l-}}{k}, \\
C_-(f_r) &= \left(-\frac{\mathrm{i}}{2} + \frac{ke^{-\mathrm{i}\theta}}{p(k)}\right)\frac{f_{r+}}{k}, \\
C_+(f_l) &= \left(-\frac{\mathrm{i}}{2} + \frac{ke^{\mathrm{i}\theta}}{p(k)}\right)\frac{f_{l-}}{k}, \\
C_+(f_r) &= \left(\frac{\mathrm{i}}{2} + \frac{\mathrm{i}c + k\sqrt{1+bc}\,e^{\mathrm{i}-\phi}}{p(k)}\right)\frac{f_{r+}}{k}.
\end{aligned} \tag{3.37}
$$

Proposition 3.5. *Let $H \in \mathscr{L}(L^2(\mathbb{R}))$ be a Hamiltonian corresponding to one $\mathcal{PT}$-symmetric point interaction at the origin (3.22). Let parameters b, c, ϕ satisfy*
(I) $bc\sin^2\phi \geq \cos^2\phi$ and $\cos\phi \geq 0$,
or
(II) $bc\sin^2\phi \leq \cos^2\phi$
with at least one of the following conditions in addition
(i) $c > 0,\ \cos\phi > 0,\ b > 0,$
(ii) $c > 0,\ \cos\phi = 0,\ b = 0,$
(iii) $c = 0,\ \cos\phi > 0,\ b \neq 0,$
(iv) $c = 0,\ \cos\phi \neq 0,\ b = 0,$
(v) $c < 0,\ \cos\phi = 0,\ b = 0,$
(vi) $c < 0,\ \cos\phi < 0,$
then H is similar to a self-adjoint operator.

Proof. Taking into account theorem 2.5, the existence of a constant M such that for all $g \in L_2(\mathbb{R})$

$$
\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|\left(R_\lambda(H) - R_\lambda(H_0)\right) g\|^2 \mathrm{d}\xi \leq M\|g\|^2, \tag{3.38}
$$

where $\lambda = \xi + \mathrm{i}\,\varepsilon$ and the resolvents $R_\lambda(H), R_\lambda(H_0)$ are defined above, already guarantee the similarity to self-adjoint operator. At first, we consider function g_r, where $g_{r,l}(x) := \vartheta(\pm x) g(x)$. Using relations (3.34, 3.37) and $\lambda = k^2$, $\mathrm{Im}\, k > 0$ we arrive at

$$\begin{aligned} &\| \left(R_\lambda(H) - R_\lambda(H_0)\right) g_r\|^2 = \\ &= \underbrace{\left(\left| \frac{\mathrm{i}}{2} + \frac{\mathrm{i}c + k\sqrt{1+bc}\, e^{-\mathrm{i}\phi}}{p(k)} \right|^2 + \left| -\frac{\mathrm{i}}{2} + \frac{k e^{-\mathrm{i}\theta}}{p(k)} \right|^2 \right)}_{M_r(k)} \frac{|g_{r+}|^2}{2|k|^2 \mathrm{Im}\, k}, \end{aligned} \tag{3.39}$$

where g_{r+} is defined by (3.27) and $p(k)$ by (3.31). Since we assumed (I), or (II) with at least one of the (i)-(vi), roots of $p(k)$ are located in the lower complex half-plane ($\mathrm{Im}\, k < 0$). Thus $M_r(k) < M_r$ for all $k \in \mathbb{C}$, $\mathrm{Im}\, k > 0$. (Little bit more delicate case $c = 0$ when one root is equal zero can be resolved by elementary analysis of $M_r(k)$, the estimate is valid as well.) Next we estimate the resting parts, using changing of integration variables, fact that g_{r+} is Fourier transformation of g_r and Carleson embedding theorem [21, 3],

$$\int_{-\infty}^{\infty} \frac{\varepsilon |g_{r+}(k)|^2}{|k|^2 \mathrm{Im}\, k} \mathrm{d}\xi = 4 \int_0^{\infty} |g_{r+}(k)|^2 d\mathrm{Re}\, k \le \frac{\pi}{2} M \|g_r\|^2, \tag{3.40}$$

where M is a constant independent of k and g_r. All together we have

$$\varepsilon \int_{-\infty}^{\infty} \| \left(R_\lambda(H)^{-1} - R_\lambda(H_0)\right) g_r\|^2 d\xi < \frac{\pi}{2} M_r M \|g_r\|^2. \tag{3.41}$$

We can use the similar procedure to g_l and we obtain almost the same result (constants can be different). Hence H is similar to self-adjoint operator. □

Using the theorem 2.5 and the same technique we may also classify all systems with one point interaction (not necessarily $\mathcal{PT}$-symmetric) on a line with real spectrum. It is possible to show that example of a complex delta interaction [29] is (in therein considered setting of the parameter) similar to self-adjoint operator. This result, in fact, justifies the effort to construct the metric for this system [29].

3.3 Supersymmetric $\mathcal{PT}$-symmetric point interactions on a loop

We consider a system on the finite interval $(-l, l)$ and two point interactions, at $x = 0$ and $x = \pm l$ (i.e. interaction at origin and between the two end points). $\mathcal{PT}$-symmetric point interaction are described by boundary conditions theorem 3.1, however, in order to show connection with self-adjoint case [33] we rewrite these conditions in the following form, using the same parameters b, c, θ, ϕ

$$(C - I)\Psi(a) + (C + I)\Psi'(a) = 0, \tag{3.42}$$

where

$$\Psi(a) = \begin{pmatrix} \psi(a+) \\ \psi(a-) \end{pmatrix}, \quad \Psi'(a) = \begin{pmatrix} \psi'(a+) \\ -\psi'(a-) \end{pmatrix}, \tag{3.43}$$

$$C = \begin{pmatrix} \frac{(b-c)e^{i\phi}+\sqrt{1+bc}(e^{2i\phi}-1)}{(b+c)e^{i\phi}+\sqrt{1+bc}(e^{2i\phi}+1)} & \frac{2e^{(\theta+\phi)}}{(b+c)e^{i\phi}+\sqrt{1+bc}(e^{2i\phi}+1)} \\ \frac{2e^{(-\theta+\phi)}}{(b+c)e^{i\phi}+\sqrt{1+bc}(e^{2i\phi}+1)} & \frac{(b-c)e^{i\phi}-\sqrt{1+bc}(e^{2i\phi}-1)}{(b+c)e^{i\phi}+\sqrt{1+bc}(e^{2i\phi}+1)} \end{pmatrix}. \tag{3.44}$$

Our aim is to find $\mathcal{PT}$-symmetric systems on a loop $(-l, l)$ with two point interactions (at 0 and l) which are supersymmetric. Various supersymmetric $\mathcal{PT}$-symmetric systems were intensively studied, e.g. [24, 4]. We would like to take advantage of usual simplicity (exact solvability in terms of elementary functions) of Hamiltonians with point interactions and moreover its special structure (SUSY) to find explicitly the spectra and metric Θ operators for these systems.

In order to obtain supersymmetric system with supercharges $Q_{1,2} \propto \frac{d}{dx}$,

$$\{Q_a, Q_b\} = H\delta_{ab}, \tag{3.45}$$

we have to restrict boundary conditions (3.42)-(3.44). We expect that we receive boundary conditions which connect values of functions and values of derivatives separately as well as in a self-adjoint case [33].

We recall briefly the procedure of finding suitable boundary conditions compatible with supersymmetry presented in [33] and we modify it to the $\mathcal{PT}$-symmetric case.

If φ is an eigenfunction of H then $Q\varphi$ is also eigenfunction of H corresponding to the same eigenvalue (or $Q\varphi = 0$). Since it is not guaranteed for general boundary conditions that $Q\varphi$ satisfies (3.42) although φ does, supercharges cannot be only derivatives multiplied by a scalar. We take an eigenfunction φ of H

$$H\varphi = E\varphi \tag{3.46}$$

and denote $\chi \equiv Q\varphi$. Since the supercharge is proportional to the derivative, boundary values of χ are related to those of φ'

$$\Psi_\chi(a) \equiv \begin{pmatrix} \chi(a+) \\ \chi(a-) \end{pmatrix} = M \begin{pmatrix} \varphi'(a+) \\ -\varphi'(a-) \end{pmatrix}, \tag{3.47}$$

where M is an invertible matrix. φ is an eigenfunction of H, hence φ'' is proportional to φ and

$$\Psi'_\chi(a) \equiv \begin{pmatrix} \chi'(a+) \\ -\chi'(a-) \end{pmatrix} = E\tilde{M} \begin{pmatrix} \varphi(a+) \\ \varphi(a-) \end{pmatrix}, \tag{3.48}$$

where $\tilde{M}$ is an invertible matrix again. When we combine (3.42),(3.47),(3.48) we arrive at

$$(C - I)\tilde{M}^{-1}\Psi'_\chi(0) + E(C + I)M^{-1}\Psi_\chi(0) = 0. \tag{3.49}$$

Boundary conditions have to be energy independent and $\Psi_\chi,\ \Psi'_\chi$ are not zero vectors simultaneously. Therefore $(C \pm I)$ must be singular matrices, i.e. eigenvalues of C are ± 1. This constraint restricts general form of C to two possibilities

$$C_\pm = \pm \begin{pmatrix} i\tan\phi & \frac{e^{\mathrm{i}\theta}}{\cos\phi} \\ \frac{e^{-i\theta}}{\cos\phi} & -\mathrm{i}\tan\phi \end{pmatrix}, \tag{3.50}$$

i.e. parameters b, c are equal to zero however the range of θ, ϕ is preserved.

After reparametrization of $C_\pm$ elements using new both $\vec{\beta}$ and $\vec{b}$ parameters

$$\beta_1 = b_1 = -\frac{\cos\theta}{\cos\phi},\ \beta_2 = b_2 = \frac{\sin\theta}{\cos\phi},\ \beta_3 = \mathrm{i}b_3 = -\mathrm{i}\tan\phi, \tag{3.51}$$

$$(\vec{\beta})^2 = b_1^2 + b_2^2 - b_3^2 = 1,\ \beta_{1,2} \in \mathbb{R},\ \beta_3 \in \mathrm{i}\mathbb{R},\ b_{1,2,3} \in \mathbb{R}, \tag{3.52}$$

we arrive at

$$C_\pm = \exp\left(\mathrm{i}\frac{\pi}{2}(I \pm \vec{\beta}.\vec{\sigma})\right), \tag{3.53}$$

where $\vec{\sigma}$ are the Pauli matrices.

We use parameters $\vec{\beta}$ in order to write following expressions in more elegant way and to show connection with the self-adjoint case, where parameters real $\vec{\alpha}$ are used, see (3.54, 3.55). However, we move to real parameters $\vec{b}$ in the following to avoid the tricky structure of $\vec{\beta}$, β_3 is not real.

We summarize results for self-adjoint case in following and adapt them to the $\mathcal{PT}$-symmetric case. Fortunately, the transition from the self-adjoint case turned out to be very easy, in fact a shift $\vec{\alpha} \mapsto \vec{\beta}$ is needed only. The direct connection can be found in the relation (3.53) because it is a slight generalization of the standard one (3.56).

Boundary conditions for the self-adjoint case are described by unitary matrix U and real parameter L_0 in equation (C.4) where Ψ, Ψ' are correspond to (3.43). We may use an exponential form of unitary matrix

$$U \equiv U_g(\theta_+, \theta_-) = \exp\left\{ \mathrm{i}\theta_+ P_g^+ + \mathrm{i}\theta_- P_g^- \right\}, \tag{3.54}$$

where $P_g^\pm$ are orthogonal projectors

$$\begin{gathered} P_g^\pm = \frac{1}{2}(I \pm g), \quad g = \vec{\alpha}.\vec{\sigma}, \quad \vec{\alpha} \in \mathbb{R}^3, \quad (\vec{\alpha})^2 = 1, \\ (P_g^\pm)^2 = P_g^\pm = (P_g^\pm)^*, \;\; P_g^\pm P_g^\mp = 0, \;\; P_g^+ + P_g^- = I. \end{gathered} \tag{3.55}$$

Supersymmetry restricts (proof in [33]) these general conditions to

$$U_g(\pi, 0) = \exp\left\{ \mathrm{i}\frac{\pi}{2}(I \pm \vec{\alpha}.\vec{\sigma}) \right\}. \tag{3.56}$$

Although we cannot use the exponential form for the general matrix C (3.44), both matrices $U_g(\pi, 0)$ and $C_\pm$ with restricted parameters may be written in the exponential form (3.53), (3.56). We note that the only difference between $U_g(\pi, 0)$ and $C_\pm$ is the structure of $\vec{\alpha}$ and $\vec{\beta}$. This fact allows us to obtain supercharges, eigenvalues and eigenfunctions of Hamiltonian very easily from the self-adjoint case.

In order to express boundary conditions in more convenient way we use operators $\mathcal{P}, \mathcal{Q}, \mathcal{R}$,

$$(\mathcal{P}\psi)(x) = \psi(-x), \quad (\mathcal{R}\psi)(x) = (\vartheta(x) - \vartheta(-x))\psi(x), \quad \mathcal{Q} = -\mathrm{i}\mathcal{R}\mathcal{P}, \tag{3.57}$$

where ϑ is a Heaviside step function. The operators are labeled in following way

$$\mathcal{P}_1 \equiv \mathcal{P}, \;\; \mathcal{P}_2 \equiv \mathcal{Q}, \;\; \mathcal{P}_3 \equiv \mathcal{R}. \tag{3.58}$$

The set of these operators forms an algebra of Pauli matrices, i.e.

$$\begin{aligned} [\mathcal{P}_l, \mathcal{P}_m] &= 2\mathrm{i}\varepsilon_{lmn}\mathcal{P}_n, \\ \{\mathcal{P}_l, \mathcal{P}_m\} &= 2\delta_{lm} I. \end{aligned} \tag{3.59}$$

Next, operator $\mathcal{G}$ associated to $g = \vec{\beta}.\vec{\sigma}$ is introduced

$$\mathcal{G} = \vec{\beta}.\vec{\mathcal{P}}, \tag{3.60}$$

obeying $\mathcal{G}^2 = I,\ \mathcal{G}^* \neq \mathcal{G},\ [\mathcal{G}, \mathcal{PT}] = 0$. (In the self-adjoint case is $\vec{\beta}$ replaced by $\vec{\alpha}$ and $\mathcal{G}$ is self-adjoint.) It allows us to decompose any function ψ into two eigenfunctions of $\mathcal{G}$

$$\psi_\pm = \frac{1}{2}(I \pm \mathcal{G})\psi,\ \psi = \psi_+ + \psi_-,\ \mathcal{G}\psi_\pm = \pm\psi_\pm. \tag{3.61}$$

Boundary conditions at $x = a$ corresponding to $C_\pm$ are now expressed in the form

$$\text{type} + :\ \psi_+(a+) = \psi'_-(a-) = 0, \quad \text{type} - :\ \psi'_+(a+) = \psi_-(a-) = 0. \tag{3.62}$$

Hence we study two types of models: $(++)$ and $(+-)$. $(++)$ denotes the interaction of the type $+$ at $x = 0$ and of the type $-$ at $x = l$ (at $x = l$ boundary conditions connect $x = -l$ and $x = l$). The other combinations provide equivalent models.

3.3.1 Model of the type $(++)$

We work in the Hilbert space $L^2(-l, l)$. The domain of definition of our Hamiltonian $H_1 \equiv H_{++} = -\frac{d^2}{dx^2}$ consists of functions $\psi \in \mathrm{AC}^2(\Omega)$, where $\Omega = (-l, 0) \cup (0, l)$, which obey boundary conditions $(++)$ at $x = 0$ and $x = \pm l$.

$$\begin{aligned} \mathrm{Dom}(H_1):\ & \psi \in \mathrm{AC}^2(\Omega), \\ & (b_1 + \mathrm{i}b_2)\psi(0+) + (1 - \mathrm{i}b_3)\psi(0-) = 0, \\ & (b_1 + \mathrm{i}b_2)\psi'(0+) + (1 + \mathrm{i}b_3)\psi'(0-) = 0, \\ & (b_1 + \mathrm{i}b_2)\psi(l) + (1 - \mathrm{i}b_3)\psi(-l) = 0, \\ & (b_1 + \mathrm{i}b_2)\psi'(l) + (1 + \mathrm{i}b_3)\psi'(-l) = 0, \\ & b_{1,2,3} \in \mathbb{R},\ \ b_1^2 + b_2^2 - b_3^2 = 1, \end{aligned} \tag{3.63}$$

where $b_{1,2,3} \in \mathbb{R}$ and $b_1^2 + b_2^2 - b_3^2 = 1$. Since the fractions $(1 \pm ib_3)/(b_1 + \mathrm{i}b_2)$ have absolute values equal to one, boundary conditions may be rewritten as

$$\psi(0+) = e^{\mathrm{i}\tau_1}\psi(0-),\ \ \psi'(0+) = e^{\mathrm{i}\tau_2}\psi'(0-),\ \ \tau_{1,2} \in \mathbb{R}, \tag{3.64}$$

for $x = \pm l$ similarly. Parameters $\tau_{1,2}$ are different if $b_3 \neq 0$, the case $b_3 = 0$ corresponds to the self-adjoint setting.

It is not difficult to find the adjoint operator H_1^* directly from the definition of adjoint operator A.3, i.e. using standard technique, integration by parts etc.

$$\begin{array}{rlcl} \mathrm{Dom}(H_1^*): & \psi \in \mathrm{AC}^2(\Omega), & & \\ & (b_1 + \mathrm{i}b_2)\psi(0+) + (1 + \mathrm{i}b_3)\psi(0-) & = & 0, \\ & (b_1 + \mathrm{i}b_2)\psi'(0+) + (1 - \mathrm{i}b_3)\psi'(0-) & = & 0, \\ & (b_1 + \mathrm{i}b_2)\psi(l) + (1 + \mathrm{i}b_3)\psi(-l) & = & 0, \\ & (b_1 + \mathrm{i}b_2)\psi'(l) + (1 - \mathrm{i}b_3)\psi'(-l) & = & 0, \\ & b_{1,2,3} \in \mathbb{R}, \quad b_1^2 + b_2^2 - b_3^2 = 1. & & \end{array} \tag{3.65}$$

Since H_1 is equal to $H_{-b_3}^*$ (we change a sign of b_3 in (3.63) and take the adjoint) it is closed. We remark that H_1 is $\mathcal{P}$-pseudo-Hermitian only for $b_2 = 0$. It is clear from the theorem A.8 that H_1 is operator with compact resolvent.

Eigenvalues of H_1 are the same as in self-adjoint case [33], eigenfunctions differ only in the substitution $\vec{\alpha} \mapsto \vec{b}$, i.e.

$$\begin{aligned} E_n &= \left(\frac{n\pi}{l}\right)^2, \\ \psi_{n+}(x) &= C_n \left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 + \mathrm{i}b_3}\right) \sin\frac{n\pi}{l}x, \\ \psi_{n-}(x) &= C_n \left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 - \mathrm{i}b_3}\right) \cos\frac{n\pi}{l}x, \; n \in \mathbb{N}_0, \end{aligned} \tag{3.66}$$

where $\vartheta(x)$ is a Heaviside step function and C_n are normalization constants. Eigenfunctions of Hamiltonian $\psi_{n\pm}$ are eigenfunctions of operator $\mathcal{G}$ (3.60) as well, corresponding to eigenvalues ± 1 (the generalization of the proof from the self-adjoint case is straightforward). Figures 3.8, 3.9 illustrate eigenfunctions $\psi_{5\pm}$, point interactions at the origin and end points rotate wavefunction in the complex plane.

Energy levels are doubly degenerate except the lowest one as we expected for the supersymmetric system. Supercharges $Q_{1,2}$ may be obtained from the self-adjoint case [33] easily again, by substitution $\vec{\alpha} \mapsto \vec{\beta}$

$$Q_{1,2} = \mathrm{i}\frac{\sqrt{2}}{2}\mathcal{G}_{1,2}\mathcal{P}_3\frac{d}{dx}, \tag{3.67}$$

where

$$\mathcal{G}_{1,2} = \vec{\gamma}_{1,2}.\vec{\mathcal{P}}, \quad (\vec{\gamma}_{1,2})^2 = 1 \quad \text{and} \quad \vec{\gamma}_{1,2}.\vec{\beta} = \vec{\gamma}_1.\vec{\gamma}_2 = 0. \tag{3.68}$$

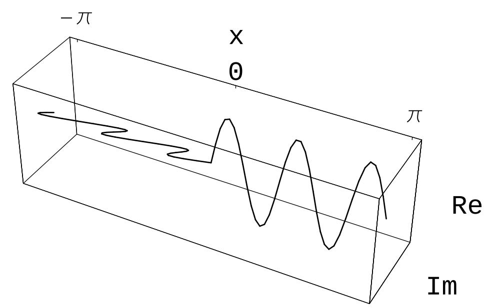


Figure 3.8: *Eigenfunction* ψ_{5+}, $\vec{b} = (10.000, 5.600, 11.417)$, $l = \pi$

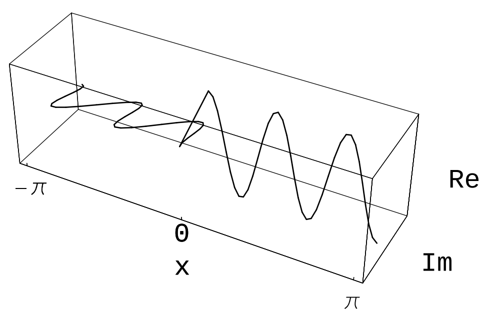


Figure 3.9: *Eigenfunction* ψ_{5-}, $\vec{b} = (10.000, 5.600, 11.417)$, $l = \pi$

Eigenfunctions of Hamiltonian have very simple form and this fact allows us to construct metric Θ operator using (2.23).

We denote $\phi_{n\pm}$ eigenfunctions of H^* and normalize them in a special way

$$
\begin{aligned}
\phi_{n+}(x) &= \sqrt{\frac{2}{l}}\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 - \mathrm{i}b_3}\right)\sin\frac{n\pi}{l}x, \\
\phi_{0-}(x) &= \frac{1}{\sqrt{l}}\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 + \mathrm{i}b_3}\right), \\
\phi_{n-}(x) &= \sqrt{\frac{2}{l}}\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 + \mathrm{i}b_3}\right)\cos\frac{n\pi}{l}x. \qquad (3.69)
\end{aligned}
$$

Sets $\{e_n^\pm\}_{n=1}^\infty$, $\{f_n^\pm\}_{n=0}^\infty$

$$\begin{aligned} e_n^\pm(x) &= \sqrt{\frac{2}{l}}\vartheta(\pm x)\sin\frac{n\pi}{l}x, \\ f_0^\pm(x) &= \frac{1}{\sqrt{l}}\vartheta(\pm x), \quad f_n^\pm(x) = \sqrt{\frac{2}{l}}\vartheta(\pm x)\cos\frac{n\pi}{l}x, \end{aligned} \tag{3.70}$$

form orthonormal bases of $L^2(-l,0)$ and $L^2(0,l)$. We express $\phi_{n\pm}$ in terms of $e_n^\pm$, $f_n^\pm$ and calculate

$$\begin{aligned} \langle\phi_{n+},\psi\rangle\,\phi_{n+} &= \langle e_n^+,\psi\rangle\, e_n^+ + \langle e_n^-,\psi\rangle\, e_n^- + \frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}\langle e_n^-,\mathcal{P}\psi\rangle\, e_n^- + \\ &\quad + \frac{b_1-\mathrm{i}b_2}{1+\mathrm{i}b_3}\langle e_n^+,\mathcal{P}\psi\rangle\, e_n^+, \\ \langle\phi_{n-},\psi\rangle\,\phi_{n-} &= \langle e_n^+,\psi\rangle\, f_n^+ + \langle f_n^-,\psi\rangle\, f_n^- - \frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\langle f_n^-,\mathcal{P}\psi\rangle\, f_n^- - \\ &\quad - \frac{b_1-\mathrm{i}b_2}{1-\mathrm{i}b_3}\langle f_n^+,\mathcal{P}\psi\rangle\, f_n^+. \end{aligned} \tag{3.71}$$

Finally, we calculate the sum (2.23)

$$\begin{aligned} \Theta &= \operatorname*{s-lim}_{N\to\infty}\frac{1}{2}\left(\sum_{n=1}^{N}\langle\phi_{n+},\cdot\rangle\,\phi_{n+} + \sum_{n=0}^{N}\langle\phi_{n-},\cdot\rangle\,\phi_{n-}\right) = \\ &= I - \frac{\mathrm{i}b_3}{b_1+\mathrm{i}b_2}P^+\mathcal{P} + \frac{\mathrm{i}b_3}{b_1-\mathrm{i}b_2}P^-\mathcal{P}, \end{aligned} \tag{3.72}$$

where $\mathcal{P}$ is a parity and $P^\pm$ are orthogonal projectors

$$(P^\pm\psi)(x) = \vartheta(\pm x)\psi(x), \quad (P^\pm)^2 = P^\pm = (P^\pm)^*, \quad P^+P^- = P^-P^+ = 0. \tag{3.73}$$

3.3.2 Model of the type $(+-)$

The domain of definition of Hamiltonian $H_2 \equiv H_{+-}$ reads

$$\begin{aligned} \mathrm{Dom}(H_2):\quad & \psi \in \mathrm{AC}^2(\Omega), \\ & (b_1+\mathrm{i}b_2)\psi(0+) + (1-\mathrm{i}b_3)\psi(0-) = 0, \\ & (b_1+\mathrm{i}b_2)\psi'(0+) + (1+\mathrm{i}b_3)\psi'(0-) = 0, \\ & (b_1+\mathrm{i}b_2)\psi(l) - (1+\mathrm{i}b_3)\psi(-l) = 0, \\ & (b_1+\mathrm{i}b_2)\psi'(l) - (1-\mathrm{i}b_3)\psi'(-l) = 0, \\ & b_{1,2,3}\in\mathbb{R}, \quad b_1^2+b_2^2-b_3^2 = 1. \end{aligned} \tag{3.74}$$

H_2 is also operator with compact resolvent, eigenvalues and eigenfunctions are

$$
\begin{aligned}
E_n &= \left(\frac{(2n-1)\pi}{2l}\right)^2, \\
\psi_{n+}(x) &= C_n\left(\vartheta(x)-\vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\right)\sin\frac{(n-1)\pi}{2l}x, \\
\psi_{n-}(x) &= C_n\left(\vartheta(x)-\vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}\right)\cos\frac{(n-1)\pi}{2l}x,\ n\in\mathbb{N}.
\end{aligned} \tag{3.75}
$$

Supercharges have exactly the same form as in previous case (3.67), however supersymmetric structure of this model is different because zero energy level is absent.

We use analogous procedure to obtain metric operator. We express eigenfunctions of H_2^* in terms of e_n, f_n

$$
\begin{aligned}
e_0(x)=\frac{1}{\sqrt{2l}},\quad e_{2k-1}(x)=\frac{1}{\sqrt{l}}\sin\frac{(2k-1)\pi}{2l}x,\quad e_{2k}(x)=\frac{1}{\sqrt{l}}\cos\frac{k\pi}{l}x, \\
f_{2k-1}(x)=\frac{1}{\sqrt{l}}\cos\frac{(2k-1)\pi}{2l}x,\quad f_{2k}(x)=\frac{1}{\sqrt{l}}\sin\frac{k\pi}{l}x,
\end{aligned} \tag{3.76}
$$

where sets $\{e_n\}_{n=0}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ form orthonormal bases of $L^2(-l,l)$. Summation (2.23) in the strong limit sense yields

$$
\begin{aligned}
\Theta &= P^+(O_1+O_2)P^+ + P^-(O_1+O_2)P^- - \frac{b_1-\mathrm{i}b_2}{1+\mathrm{i}b_3}P^+O_1P^- - \\
&\quad -\frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}P^-O_1P^+ - \frac{b_1-\mathrm{i}b_2}{1-\mathrm{i}b_3}P^+O_2P^- - \frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}P^-O_2P^+,
\end{aligned} \tag{3.77}
$$

where $O_{1,2}$ are orthogonal projectors

$$
\begin{aligned}
O_1e_{2k}=0,\quad O_1e_{2k-1}=e_{2k-1}, \\
O_2f_{2k}=0,\quad O_2f_{2k-1}=f_{2k-1}.
\end{aligned} \tag{3.78}
$$

This result is derived directly from the sum (2.23), nevertheless operators O_1, O_2 are projectors respectively on the odd and even part of the function, i.e.

$$
O_1=\frac{1}{2}(I-\mathcal{P}),\quad O_2=\frac{1}{2}(I+\mathcal{P}). \tag{3.79}
$$

Hence metric operator Θ has exactly the same form as in previous case (3.72).

3.3.3 Metric operator

According proposition 2.12 it is needed to show that $\mathrm{Ker}(\Theta) = \{0\}$. This is obvious from the following estimations.

$$\begin{aligned}\langle\psi, \Theta\psi\rangle &= \|\psi\|^2 - \frac{\mathrm{i}b_3}{b_1 + \mathrm{i}b_2} J + \frac{\mathrm{i}b_3}{b_1 - \mathrm{i}b_2} \overline{J} \geq \\ &\geq \|\psi\|^2 - \left|\frac{\mathrm{i}b_3}{b_1 + \mathrm{i}b_2}\right| \left|J\right| \left|1 - \frac{b_1 + \mathrm{i}b_2}{b_1 - \mathrm{i}b_2} \frac{\overline{J}}{J}\right|, \end{aligned} \tag{3.80}$$

where

$$J := \int_0^l \overline{\psi(x)} \psi(-x) \mathrm{d}x \tag{3.81}$$

$$\begin{aligned}|J| &\leq \int_0^l |\psi(x)||\psi(-x)| \mathrm{d}x \leq \frac{1}{2} \int_0^l |\psi(x)|^2 + |\psi(-x)|^2 \, \mathrm{d}x \leq \\ &\leq \frac{1}{2} \left(\|P^+\psi\|^2 + \|P^-\psi\|^2\right) \leq \frac{1}{2} \|\psi\|^2. \end{aligned} \tag{3.82}$$

These estimates yield all together

$$\langle\psi, \Theta\psi\rangle \geq \underbrace{\left(1 - \frac{|b_3|}{\sqrt{1 + |b_3|^2}}\right)}_{c_0} \|\psi\|^2 \geq 0. \tag{3.83}$$

In fact, this estimation shows more, $0 \notin \sigma(\Theta)$, thus $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$.

We can also easily directly verify [39] that Θ maps domains of definition of $H_{1,2}$ and $H_{1,2}^*$ correctly, i.e. $\Theta \mathrm{Dom}(H_{1,2}) = \mathrm{Dom}(H_{1,2}^*)$. Moreover, we can explicitly express the similarity transformation $\varrho = \sqrt{\Theta}$,

$$\varrho = a_1 I + a_2 P^+ \mathcal{P} + \overline{a_2} P^- \mathcal{P}, \tag{3.84}$$

where

$$a_1 > 0, \; a_1^2 = \frac{1}{2}\left(1 + \sqrt{1 - |k|^2}\right), \; a_2 = \frac{k}{2a_1}, \; k = -\frac{\mathrm{i}b_3}{b_1 + \mathrm{i}b_2}. \tag{3.85}$$

To verify that indeed $\varrho = \sqrt{\Theta}$, it suffices to show that $\varrho^2 = \Theta$, what is very easy with help of identities $\mathcal{P}P^\pm = P^\mp \mathcal{P}$ and $P^+ + P^- = I$, and that ϱ is positive. Slightly modified estimations (3.80)-(3.82) yield

$$\langle\psi, \varrho\psi\rangle \geq (a_1 - |a_2|)\|\psi\|^2, \tag{3.86}$$

and $a_1 - |a_2| > 0$.

The presented metric operator (3.72) is suitable also for the model $H_3 = -\frac{d^2}{dx^2}$ on $L_2(-l, l)$ [39].

$$\begin{array}{rlcl} \mathrm{Dom}(H_3): & \psi \in \mathrm{AC}^2(\Omega), & & \\ & (b_1 + \mathrm{i}b_2)\psi(0+) + (1 - \mathrm{i}b_3)\psi(0-) & = & 0, \\ & (b_1 + \mathrm{i}b_2)\psi'(0+) + (1 + \mathrm{i}b_3)\psi'(0-) & = & 0, \\ & \psi(-l) = \psi(l) = 0, & & \\ & b_{1,2,3} \in \mathbb{R}, \quad b_1^2 + b_2^2 - b_3^2 = 1. & & \end{array} \tag{3.87}$$

Eigenvalues and eigenfunctions of H_3 read

$$\begin{aligned} E_n &= \left(\frac{n\pi}{2l}\right)^2, \\ \psi_{2n}(x) &= C_{2n-1}\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 + \mathrm{i}b_3}\right)\sin\frac{n\pi}{l}x, \\ \psi_{2n+1}(x) &= C_{2n}\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 - \mathrm{i}b_3}\right)\cos\frac{(2n+1)\pi}{2l}x, n \in \mathbb{N}_0. \end{aligned} \tag{3.88}$$

Furthermore, if we consider Hilbert space $L^2(\mathbb{R})$ and $H_4 = -\frac{d^2}{dx^2}$

$$\begin{array}{rlcl} \mathrm{Dom}(H_4): & \psi \in \mathrm{AC}^2(\mathbb{R} \setminus \{0\}), & & \\ & (b_1 + \mathrm{i}b_2)\psi(0+) + (1 - \mathrm{i}b_3)\psi(0-) & = & 0, \\ & (b_1 + \mathrm{i}b_2)\psi'(0+) + (1 + \mathrm{i}b_3)\psi'(0-) & = & 0, \\ & b_{1,2,3} \in \mathbb{R}, \quad b_1^2 + b_2^2 - b_3^2 = 1. & & \end{array} \tag{3.89}$$

we may show that already found Θ (3.72) is the metric operator for H_4 which has empty point spectrum.

Chapter 4

Conclusions

The relations between $\mathcal{PT}$-symmetry and pseudo-Hermiticity are not simple even for bounded operators. We presented a bounded pseudo-Hermitian operator which cannot have any antilinear symmetry and also a bounded operator with antilinear symmetry which cannot be pseudo-Hermitian. Nevertheless, both these operators are not spectral. On the other hand, we extended the proof of equivalence of pseudo-Hermiticity and antilinear symmetry for spectral operators of scalar type. We would like to stress the importance of J-self-adjoint operators. It is possible to show the mentioned results on equivalence using this notion very easily. Moreover, it theory of J-self-adjoint operators turned out to be helpful also for construction of metric operator Θ.

We presented three types of models - finite dimensional chain models, $\mathcal{PT}$-symmetric point interaction on a line and supersymmetric point interactions on a loop. We find the domain $\mathcal{D}$ of quasi-Hermiticity for the chain model $H^{(6)}$, using the same procedure we can obtain also results for $H^{(7)}$. We corrected the range of parameters for which $\mathcal{PT}$-symmetric point interaction is $\mathcal{P}$-pseudo-Hermitian. We extended classification according to quasi-Hermiticity to all systems with one $\mathcal{PT}$-symmetric point interaction on a line with real spectrum. Finally, we found suitable boundary conditions for $\mathcal{PT}$-symmetric Hamiltonians with two point interactions compatible with supersymmetry. These boundary conditions turned out to be qualitatively the same as for self-adjoint systems - they connect values of functions and values of derivatives separately. We constructed metric operators together with its square root for these systems. Both operators can be written in a closed formula form. Constructed metric operator turned out to be applicable also for next, however no more supersymetric, systems on a loop and even on a line (with empty point spectrum).

Appendix A

Selected parts of the spectral theory

Definition A.1 ([10]). *Let A be a linear operator on a Banach space $\mathcal{X}$. A is said to be closed if for every sequence $\{x_n\} \subset \mathrm{Dom}(A)$, which is convergent $x_n \to x$ and $Ax_n \to y$, holds $x \in \mathrm{Dom}(A)$ and $y = Ax$.*

Definition A.2 ([10]). *Let A be a closed linear operator in a Banach space $\mathcal{X}$. A complex number λ is said to be in the resolvent set $\varrho(A)$ of A if $A - \lambda I$ is a bijection with a bounded inverse. $R_\lambda(A) = (A - \lambda I)^{-1}$ is called resolvent of A at λ. If $\lambda \notin \varrho(A)$, then λ is said to be in the spectrum $\sigma(A)$ of A. $\lambda \in \sigma(A)$ is in the*
(i) point spectrum $\sigma_p(A)$ of T, if $A - \lambda I$ is not injective,
(ii) continuous spectrum $\sigma_c(A)$ of A, if $A - \lambda I$ is injective and $\overline{\mathrm{Ran}(A - \lambda I)} = \mathcal{X}$,
(iii) residual spectrum $\sigma_r(A)$ of A, if $A - \lambda I$ is injective and $\overline{\mathrm{Ran}(A - \lambda I)} \neq \mathcal{X}$.
The spectrum is divided to three disjoint parts

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A). \tag{A.1}$$

Definition A.3 ([35]). *Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. Let*

$$\mathrm{Dom}(A^*) = \{\psi \in \mathcal{H} | (\exists \eta \in \mathcal{H})(\forall \varphi \in \mathrm{Dom}(A))(\langle \psi, A\varphi \rangle = \langle \eta, \varphi \rangle)\} \tag{A.2}$$

For each $\psi \in \mathrm{Dom}(A)$ we define $A^*\psi := \eta$. A^* is called adjoint of A.

Definition A.4 ([35]). *Let* $A \in \mathscr{L}(\mathcal{H})$ *be densely defined. We say that*
(i) A *is symmetric if* $A \subset A^*$ *or equivalently* $\langle \psi, A\varphi \rangle = \langle A\psi, \varphi \rangle$, $\forall \varphi, \psi \in \mathrm{Dom}(A)$,
(ii) A *is self-adjoint if* $A = A^*$.

Definition A.5 ([10]). *A symmetric operator* A *is called essentially self-adjoint if its closure* $\overline{A}$ *is self-adjoint.*

Theorem A.1 ([10]). *Let* $A \in \mathscr{L}(\mathcal{H})$ *be densely defined. Then* A^* *is closed.*

Theorem A.2 ([35]). *Let* $A \in \mathscr{L}(\mathcal{H})$ *be densely defined. Then* $\sigma(A^*) = \left\{ \lambda | \overline{\lambda} \in \sigma(A) \right\}$.

Theorem A.3 ([35]). *Let* $A \in \mathscr{C}(\mathcal{H})$ *be densely defined.*
(i) If $\lambda \in \sigma_r(A)$ *then* $\overline{\lambda} \in \sigma_p(A^*)$.
(ii) If $\lambda \in \sigma_p(A)$ *then* $\overline{\lambda} \in \sigma_p(A^*)$ *or* $\overline{\lambda} \in \sigma_r(A^*)$.

Theorem A.4 ([42]). *Let* $A \in \mathscr{L}(\mathcal{H})$ *be a self-adjoint operator. Then*

$$\|R_{\lambda_0}(A)\| \leq \frac{1}{|\mathrm{Im}\,\lambda_0|} \tag{A.3}$$

for all $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.

Definition A.6 ([10]). *Let* $A \in \mathscr{C}(\mathcal{H})$. *A complex number* λ *is said to be in the essential spectrum* $\sigma_{ess}(A)$ *of* A *if there exists a sequence* $\{x_n\} \subset \mathrm{Dom}(A)$ *of unit vectors for which* $\|(A - \lambda)x_n\| \to 0$ *and the set* $\{x_n\}$ *is not compact.*

Proposition A.5 ([10]). *Let* $A \in \mathscr{C}(\mathcal{H})$. *Then*

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_{ess}(A), \tag{A.4}$$

$$\sigma_c(A) = \sigma_{ess}(A) \setminus (\sigma_p(A) \cup \sigma_r(A)). \tag{A.5}$$

Remark A.6 ([10]). *The decomposition (A.5) is not disjoint in general, e.g. each eigenvalue of infinite multiplicity is included in* $\sigma_{ess}(A)$.

Definition A.7 ([20]). *Let* $A \in \mathscr{C}(\mathcal{H})$. *A complex number* λ *is said to be in the discrete spectrum* $\sigma_d(A)$ *of* A *if* λ *is an eigenvalue with finite multiplicity and it the isolated point of the spectrum. We say that* A *is an operator with pure discrete spectrum if* $\sigma(A) = \sigma_d(A)$.

Theorem A.7 ([20])**.** *Let $A \in \mathscr{C}(\mathcal{H})$ such that the resolvent $R_\lambda(A)$ exists and is compact for some λ. Then the spectrum of A is pure discrete and $R_\lambda(A)$ is compact for every $\lambda \in \varrho(A)$.*

Definition A.8 ([20])**.** *$T \in \mathscr{L}(\mathcal{H})$ is called a finite extension of $S \in \mathscr{L}(\mathcal{H})$ and S a finite restriction of T if $S \subset T$ and $\dim(\operatorname{Dom}(T)/\operatorname{Dom}(S)) = m < \infty$. m is the order of the extension or restriction.*

Theorem A.8 ([20])**.** *Let $T_1, T_2 \in \mathscr{C}(\mathcal{H})$ have non-empty resolvent sets. Let T_1, T_2 be either extensions of a common operator T_0 or restrictions of a common operator T, with order of extension or restriction for T_1 being finite. Then T_1 has compact resolvent if and only if T_2 has compact resolvent.*

Remark A.9 ([10])**.** *Let $A \in \mathscr{L}(\mathcal{H})$ is self-adjoint. Then*

$$\sigma_d(A) = \sigma(A) \setminus \sigma_{ess}(A). \tag{A.6}$$

Theorem A.10 ([10])**.** *Let $A \in \mathscr{L}(\mathcal{H})$ be self-adjoint. A has pure discrete spectrum if and only if for all $\mu \in \varrho(A)$ $(A-\mu)^{-1}$ is compact. Operators with pure discrete spectrum are called operators with compact resolvent.*

Theorem A.11 ([10])**.** *Let $\{A_n\}_{n=1}^{\infty}$ be a non decreasing sequence of bounded self-adjoint operators and B is bounded self-adjoint operator such that $A_n \leq B$ for all $n \in N$. Then there exists bounded self-adjoint operator A such that $A = \operatorname{s-lim}_{n\to\infty} A_n$.*

Appendix B

Spectral operators

Definition B.1 ([19, 15]).
(i) A spectral measure E in $\mathcal{H}$ is a homomorphic map of a σ-algebra $\mathcal{A}$ of sets into a Boolean algebra of projection operators in $\mathcal{H}$ such that the unit of $\mathcal{A}$ is mapped to identity I in $\mathcal{H}$. The spectral measure E is called bounded if for some $C > 0$, $\|E(\omega)\|_{\mathcal{B}(\mathcal{H})} \leq C$ for all $\omega \in \mathcal{A}$.
(ii) If $T \in \mathscr{C}(\mathcal{H}))$, then $\sigma \subseteq \sigma(T)$ is called a spectral set if σ is both open and closed in the topology of $\sigma(T)$.

We use $\Omega = \mathbb{C}$ and Σ is σ-algebra $\mathcal{B}_{\mathbb{C}}$ of Borel subsets of $\mathbb{C}$ in following.

Definition B.2 ([19, 15]). *Let $T \in \mathscr{B}(\mathcal{H}))$.*
(i) A projection-valued spectral measure E on $\mathcal{B}_{\mathbb{C}}$ is called a resolution of the identity (or a spectral resolution) for T if

$$E(\omega)T = TE(\omega), \quad \sigma\big(T\big|_{E(\omega)\mathcal{H}}\big) \subseteq \overline{\omega}, \quad \omega \in \mathcal{B}_{\mathbb{C}}. \tag{B.1}$$

(ii) A projection-valued spectral measure E in $\mathcal{H}$ defined on $\mathcal{B}_{\mathbb{C}}$ is called countably additive if for all $f, g \in \mathcal{H}$, $\langle f, E(\cdot)g\rangle$ is countably additive on $\mathcal{B}_{\mathbb{C}}$.
(iii) T is called a spectral operator if it has a countably additive resolution of the identity defined on $\mathcal{B}_{\mathbb{C}}$.

Basic properties of spectral operators are given by following lemma

Lemma B.1 ([19, 15]).
(i) Any countably additive projection-valued spectral measure E on $\mathcal{B}_{\mathbb{C}}$ is countably

additive in the strong operator topology and bounded.
(ii) Let $T \in \mathscr{B}(\mathcal{H})$ *be a spectral operator, then* $E(\sigma(T)) = I$.
(iii) Every bounded spectral operator has a uniquely defined countably additive resolution (denoted E_T*) of the identity defined on* $\mathcal{B}_{\mathbb{C}}$.

Definition B.3 ([19, 15])**.**
(i) Let $S \in \mathscr{B}(\mathcal{H})$ *be a spectral operator with spectral resolution* E_S *defined on* $\mathcal{B}_{\mathbb{C}}$*. Then* S *is said to be of scalar type (or a scalar spectral operator) if*

$$S = \int_{\mathbb{C}} \lambda \, \mathrm{d}E_S(\lambda). \tag{B.2}$$

(ii) $N \in \mathscr{B}(\mathcal{H})$ *is called quasi-nilpotent if* $\lim_{n\to\infty} \|N^n\|^{1/n} = 0$.

Lemma B.2 ([19, 15])**.**
(i) If E *is a countably additive projection-valued spectral measure on* $\mathcal{B}_{\mathbb{C}}$ *which vanishes outside a compact subset of* $\mathbb{C}$*, then*

$$S = \int_{\mathrm{supp}\,(\mathrm{d}E)} \lambda \, \mathrm{d}E(\lambda) \tag{B.3}$$

is a bounded spectral operator of scalar type whose spectral resolution is E.
(ii) $N \in \mathscr{B}(\mathcal{H})$ *is quasi-nilpotent if and only if* $\sigma(N) = \{0\}$.

Theorem B.3 ([19, 15])**.**
Let $T \in \mathscr{B}(\mathcal{H})$*. Then* T *is a spectral operator if and only if* $T = S + N$*, where* $S \in \mathscr{B}(\mathcal{H})$ *is a bounded spectral operator of scalar type and* N *is a quasi-nilpotent operator commuting with* S*. This decomposition is unique and*

$$\sigma(T) = \sigma(S). \tag{B.4}$$

Moreover, T *and* S *have the same resolution of the identity.*

Appendix C

Basics of point interactions

Definition C.1 ([10]). *Suppose that $A \in \mathscr{L}(\mathcal{H})$ is a symmetric operator. Let*

$$\mathcal{K}_+ := \mathrm{Ker}(\mathrm{i} - A^*) = \mathrm{Ran}(\mathrm{i} - A)^\perp,$$

$$\mathcal{K}_- := \mathrm{Ker}(\mathrm{i} + A^*) = \mathrm{Ran}(-\mathrm{i} + A)^{\perp.}$$

$\mathcal{K}_+$ and $\mathcal{K}_-$ are called the deficiency subspaces of A. The pair of numbers n_+, n_- given by $n_+(A) := \dim(\mathcal{K}_+), n_-(A) := \dim(\mathcal{K}_-)$ are called the deficiency indices of A.

Theorem C.1 ([10]). *Let $A \in \mathscr{L}(\mathcal{H})$ be a symmetric operator. Then the following assertions are equivalent*
(i) A is essentially self-adjoint,
(ii) $n_+ = n_- = 0$,
(iii) $\mathrm{Ran}(A \pm \mathrm{i})$ are dense.

Theorem C.2 ([10]). *Let $A \in \mathscr{L}(\mathcal{H})$ be a closed symmetric operator with deficiency indices n_+ and n_-. Then A has self-adjoint extension if and only if $n_+ = n_-$. There is one-to-one correspondence between self-adjoint extension of A and unitary maps from $\mathcal{K}_+$ onto $\mathcal{K}_-$. If U is such an map then the corresponding closed symmetric extension A_U has the domain*

$$\mathrm{Dom}(A_U) = \{\psi + \psi_+ + U\psi_+ | \psi \in \mathrm{Dom}(A), \psi_+ \in \mathcal{K}_+\}$$

and

$$A_U(\psi + \psi_+ + U\psi_+) = A\psi + \mathrm{i}\psi_+ - \mathrm{i}U\psi_+.$$

Definition C.2 ([10])**.** *An antilinear map* $C : \mathcal{H} \to \mathcal{H} : C(\alpha\varphi + \beta\psi) = \bar{\alpha}C\varphi + \bar{\beta}C\psi$ *is called a conjugation if it is norm-preserving and* $C^2 = I$*.*

Theorem C.3 (von Neumann's theorem, [10])**.** *Let* $A \in \mathscr{L}(\mathcal{H})$ *be a symmetric operator and suppose that there exists a conjugation* C *with* $C : \mathrm{Dom}(A) \to \mathrm{Dom}(A)$ *and* $AC = CA$*. Then* A *has equal deficiency indices and therefore has self-adjoint extensions.*

We consider one point interaction at $x = 0$. Corresponding Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$ and starting Hamiltonian reads

$$H_0 = -\frac{d^2}{dx^2} \tag{C.1}$$

with the domain

$$\mathrm{Dom}(H_0) = C_0^\infty(\mathbb{R} \setminus \{0\}). \tag{C.2}$$

H_0 is symmetric operator and

$$\mathrm{Dom}(H_0^*) = \left\{\psi \in \mathcal{H} \mid \psi \in AC^2(\mathbb{R} \setminus \{0\})\right\}. \tag{C.3}$$

It follows from von Neumann's theorem that deficiency indices are equal and explicit calculations yield $n_+ = n_- = 2$. $\overline{H}_0$, the closure of H_0, may be found easily with help of $\overline{A} = A^{**}$ (valid for every symmetric operator A). Theorem C.2 and vanishing limits of the functions from $\mathrm{Dom}(H_0)$ at $x = 0$ provide the result that all self-adjoint extensions of H_0, denoted by H_U, are restrictions of H_0^*

$$\mathrm{Dom}(H_U) = \left\{\psi \in \mathrm{Dom}(H_0^*) \mid (U - I)\Psi(0) + \mathrm{i}\, L_0(U + I)\Psi'(0) = 0\right\}, \tag{C.4}$$

where L_0 is arbitrary non-zero real constant, $U \in U(2)$ and

$$\Psi(0) = \begin{pmatrix} \psi(0+) \\ \psi(0-) \end{pmatrix}, \qquad \Psi'(0) = \begin{pmatrix} \psi'(0+) \\ -\psi'(0-) \end{pmatrix}. \tag{C.5}$$

Symbols $0\pm$ denote limits $\lim_{x\to 0\pm}$.

Appendix D

The correction of the ranges of parameters

Boundary conditions according to theorem 3.1, part 1 :

$$\begin{aligned} \varphi &\in \mathrm{Dom}(L) \\ \varphi(0+) &= e^{i(\theta+\phi)}\sqrt{1+bc}\;\varphi(0-) + e^{i\theta}b\;\varphi'(0-) \\ \varphi'(0+) &= e^{i\theta}c\;\varphi(0-) + e^{i\theta-\phi}\sqrt{1+bc}\;\varphi'(0-) \end{aligned} \tag{D.1}$$

Equality

$$\langle \psi, L\varphi\rangle = \langle L^*\psi, \varphi\rangle \tag{D.2}$$

holds for all $\psi \in \mathrm{Dom}(L^*)$ and $\varphi \in \mathrm{Dom}(L)$. We express (D.2) and use integration by parts

$$\begin{aligned} -\int_{\mathbb{R}} \overline{\psi}''\varphi &= -\int_{\mathbb{R}} \overline{\psi}\varphi'' \\ -\int_{\mathbb{R}} \overline{\psi}''\varphi &= -\lim_{a\to\infty}\left(\int_{-a}^{0}\overline{\psi}\varphi'' + \int_{0}^{a}\overline{\psi}\varphi''\right) = \\ &= -\lim_{a\to\infty}\left([\overline{\psi}\varphi']_{-a}^{0-} + [\overline{\psi}\varphi']_{0+}^{a} - [\overline{\psi'}\varphi]_{-a}^{0-} - [\overline{\psi'}\varphi]_{0+}^{a} + \int_{-a}^{a}\overline{\psi}''\varphi\right) = \\ &= -\overline{\psi}(0-)\varphi'(0-) + \overline{\psi}(0+)\varphi'(0+) + \overline{\psi'}(0-)\varphi(0-) - \overline{\psi'}(0+)\varphi(0+) - \\ &\quad -\int_{\mathbb{R}} \overline{\psi}''\varphi. \end{aligned} \tag{D.3}$$

Inserting the boundary conditions for $\varphi \in \mathrm{Dom}(L)$ into (D.4) yields

$$\begin{aligned} \overline{\varphi}'(0-)\left[\psi(0-) - \psi(0+)e^{i(\phi-\theta)}\sqrt{1+bc} + \psi'(0+)e^{-i\theta}b\right] + & \\ +\overline{\varphi}(0-)\left[-\psi'(0-) - \psi(0+)e^{-i\theta}c + \psi'(0+)e^{-i(\phi+\theta)}\sqrt{1+bc}\ \right] &= 0, \end{aligned} \tag{D.4}$$

where boundary values of φ at $x \to 0-$ may be chosen arbitrarily (we have already used (D.2)). We conclude that $\psi \in \mathrm{Dom}(L^*)$ must satisfy

$$\psi(0-) = \psi(0+)e^{i(\phi-\theta)}\sqrt{1+bc} - \psi'(0+)e^{-i\theta}b \tag{D.5}$$

$$\psi'(0-) = -\psi(0+)e^{-i\theta}c + \psi'(0+)e^{-i(\phi+\theta)}\sqrt{1+bc}. \tag{D.6}$$

If L is $\mathcal{P}$-pseudo-Hermitian then $L^* = \mathcal{P}L\mathcal{P}$ and hence $\mathrm{Dom}(L^*) = \mathcal{P}\mathrm{Dom}(L)$. Boundary conditions corresponding to $\mathcal{P}\mathrm{Dom}(L)$ read

$$\psi(0-) = \psi(0+)e^{i(\phi+\theta)}\sqrt{1+bc} - \psi'(0+)e^{i\theta}b \tag{D.7}$$

$$\psi'(0-) = -\psi(0+)e^{i\theta}c + \psi'(0+)e^{-i(\phi-\theta)}\sqrt{1+bc}. \tag{D.8}$$

Hence the equality of the domains is satisfied only if $\theta = 0$.

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