Solvable Quasi-Hermitian Models

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Prague 2007

Acknowledgement

I would like to thank to my supervisor Miloslav Znojil for support and consultations.

Abstract

All types of models with two $\mathcal{PT}$-symmetric point interactions compatible with supersymmetry are presented. Positive, bounded metric operator is constructed for these models. Method of reference modes is used to find self-adjoint extensions of the Hamiltonian with Scarf I potential.

Contents

1 Introduction **4**

2 Self-adjoint extensions of symmetric operators **5**

2.1 Point interactions in one dimension 6

2.2 Singular potentials 7

3 $\mathcal{PT}$-symmetry **10**

3.1 $\mathcal{PT}$-symmetry, pseudo-Hermiticity, quasi-Hermiticity 10

3.2 $\mathcal{PT}$-symmetric point interactions in one dimension 11

4 Point interactions with SUSY **14**

4.1 Self-adjoint point interactions with SUSY 14

4.1.1 Model of the type (A,A) 16

4.1.2 Model of the type (A,B) 17

4.2 $\mathcal{PT}$-symmetric point interactions with SUSY 18

4.2.1 Model of the type (A,A) 19

4.2.2 Model of the type (A,B) 21

5 Scarf I potential **23**

5.1 Case $n_\pm = 0$ 24

5.2 Case $n_\pm = 1$ 25

5.3 Case $n_\pm = 2$ 26

6 Conclusions **29**

1 Introduction

Quasi-Hermitian models are of the great importance in a framework of $\mathcal{PT}$-symmetric Quantum Mechanics because of the possible probabilistic interpretation in a modified Hilbert space [1]. More mathematically oriented discussion of quasi-Hermicity, i.e. the formal definition and stressing the danger in a naive approach to the construction of metric operators, is presented in [2]. Closed formula for the metric operator Θ is found for a solvable quasi-Hermitian model with one $\mathcal{PT}$-symmetric point interaction in [3]. All essential properties of quasi-Hermitian operator are proved carefully here. Our first aim is to investigate possible solvable quasi-Hermitian models with two $\mathcal{PT}$-symmetric point interactions allowing supersymmetry. Systems with two point interactions compatible with supersymmetry in a self-adjoint case are found in [4]. In order to generalize these results to the $\mathcal{PT}$-symmetric case we need the description of $\mathcal{PT}$-symmetric point interactions provided by [5]. Our second aim is to add missing discussion on domains of definition of the Hamiltonians with Scarf I potential and to help understanding of 'problematic' settings of this potential mentioned in [6]. Method of reference modes [7] should provide a suitable tool for the description of self-adjoint extensions.

2 Self-adjoint extensions of symmetric operators

Observables in the Quantum Mechanics are represented by self-adjoint operators. We recall the formal definitions of the adjoint of operator [8].

Definition 1. Let A be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Let

$$\mathrm{Dom}(A^*) = \{\psi \in \mathcal{H} | (\exists \eta \in \mathcal{H})(\forall \varphi \in \mathrm{Dom}(A))(\langle \psi, A\varphi \rangle = \langle \eta, \varphi \rangle)\}$$

For each $\psi \in \mathrm{Dom}(A^*)$ we define $A^*\psi := \eta$. A^* is called adjoint of A.

Definition 2. A densely defined linear operator A on a Hilbert space $\mathcal{H}$ is called symmetric if $A \subset A^*$, that is, if $\mathrm{Dom}(A) \subset \mathrm{Dom}(A^*)$ and $A\psi = A^*\psi$ for all $\psi \in \mathrm{Dom}(A)$. Equivalently, A is symmetric if and only if $\langle \psi, A\varphi \rangle = \langle A\psi, \varphi \rangle$ for all $\varphi, \psi \in \mathcal{H}$. A is called self-adjoint if $A = A^*$, that is, if and only if A is symmetric and $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$.

Usual starting point for the definition of physical observable is symmetric operator acting on a suitable domain such as C_0^∞. Extension of this domain may lead to self-adjoint operator. Existence of the self-adjoint extension is not guaranteed for every symmetric operator and there may exist more self-adjoint extensions as well. Every self-adjoint extension corresponds to the different physical situation. Problem of finding self-adjoint extensions of symmetric operators is well described in mathematical literature [9, 10]. Basic criterion for self-adjointness may be given with the help of indices of deficiency.

Definition 3. Suppose that A is a symmetric operator. Let

$$\mathcal{K}_+ := \mathrm{Ker}(i - A^*) = \mathrm{Ran}(i - A)^{\perp}$$

$$\mathcal{K}_- := \mathrm{Ker}(i + A^*) = \mathrm{Ran}(-i + A)^{\perp}$$

$\mathcal{K}_+$ and $\mathcal{K}_-$ are called the deficiency subspaces of A. The pair of numbers n_+, n_- given by $n_+(A) := \dim(\mathcal{K}_+), n_-(A) := \dim(\mathcal{K}_-)$ are called the deficiency indices of A.

Theorem 4. Let A be a closed symmetric operator with deficiency indices n_+ and n_-. Then A has self-adjoint extension if and only if $n_+ = n_-$. There is one-to-one correspondence between self-adjoint extension of A and unitary maps from $\mathcal{K}_+$ onto $\mathcal{K}_-$. If U is such an map then the corresponding closed symmetric extension A_U has the domain

$$\mathrm{Dom}(A_U) = \{\psi + \psi_+ + U\psi_+ | \psi \in \mathrm{Dom}(A), \psi_+ \in \mathcal{K}_+\}$$

and

$$A_U(\psi + \psi_+ + U\psi_+) = A\psi + i\psi_+ - iU\psi_+.$$

Definition 5. A symmetric operator A is called essentially self-adjoint if its closure $\overline{A}$ is self-adjoint.

Theorem 6. Let A be a symmetric operator. Then the following are equivalent:

1. A is essentially self-adjoint
2. $n_+ = n_- = 0$
3. $\mathrm{Ran}(A \pm i)$ are dense.

Equality of deficiency indices is necessary for existence of self-adjoint extensions. Following von Neumman's theorem provides a simple criterion.

Definition 7. An antilinear map $C : \mathcal{H} \to \mathcal{H} : C(\alpha\varphi + \beta\psi) = \bar{\alpha}C\varphi + \bar{\beta}C\psi$ is called a conjugation if it is norm-preserving and $C^2 = I$.

Theorem 8. (von Neumann's theorem) Let A be a symmetric operator and suppose that there exists a conjugation C with $C : \mathrm{Dom}(A) \to \mathrm{Dom}(A)$ and $AC = CA$. Then A has equal deficiency indices and therefore has self-adjoint extensions.

2.1 Point interactions in one dimension

We consider one point interaction at $x = 0$. Corresponding Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$ and starting Hamiltonian reads

$$H_0 = -\frac{d^2}{dx^2} \tag{1}$$

with the domain

$$\mathrm{Dom}(H_0) = C_0^\infty(\mathbb{R} - \{0\}). \tag{2}$$

H_0 is symmetric operator and it is not difficult to show that

$$\mathrm{Dom}(H_0^*) = \left\{\psi \in \mathcal{H} \mid \psi \in AC^2(\mathbb{R} - \{0\})\right\}. \tag{3}$$

It follows From von Neumann's theorem that deficiency indices are equal and explicit calculations give $n_+ = n_- = 2$. $\overline{H_0}$, the closure of H_0, may be found easily with regard of the property $\overline{A} = A^{**}$ (valid for every symmetric operator A). Theorem 4 and vanishing limits of the functions from $\mathrm{Dom}(H_0)$ at $x = 0$ provide the result that all self-adjoint extensions of H_0, denoted by H_U, are restrictions of H_0^*

$$\mathrm{Dom}(H_U) = \left\{\psi \in \mathrm{Dom}(H_0^*) \mid (U - I)\Psi(0) + iL_0(U + I)\Psi'(0) = 0\right\}, \tag{4}$$

where L_0 is arbitrary non zero real constant, $U \in U(2)$ and

$$\Psi(0) = \begin{pmatrix} \psi(0+) \\ \psi(0-) \end{pmatrix}, \quad \Psi'(0) = \begin{pmatrix} \psi'(0+) \\ -\psi'(0-) \end{pmatrix}. \tag{5}$$

Symbols $0\pm$ denote limits $\lim_{x\to 0\pm}$.

For later convenience, we parametrize matrix U in a following way

$$U_g(\theta_+, \theta_-) = \exp\left\{i\theta_+ P_g^+ + i\theta_- P_g^-\right\}, \tag{6}$$

where

$$\begin{gathered} P_g^{\pm} = \tfrac{1}{2}(I \pm g), g = \vec{\alpha}.\vec{\sigma}, \\ \vec{\alpha} \in \mathbb{R}^3,\ \vec{\alpha}^2 = 1,\ \vec{\sigma} \text{ are the Pauli matrices.} \end{gathered} \tag{7}$$

It is easy to show that $P_g^{\pm}$ are projectors satisfying

$$(P_g^{\pm})^2 = P_g^{\pm},\ P_g^{\pm} P_g^{\mp} = 0,\ P_g^+ + P_g^- = I. \tag{8}$$

2.2 Singular potentials

We consider differential expression

$$H = -\frac{d^2}{dx^2} + V(x) \tag{9}$$

on interval (a, b), where $-\infty \leq a < b \leq \infty$ and V is real, locally integrable function, $V \in L^1_{\mathrm{loc}}(a, b)$. The classification of these expressions is given by following definition [10].

Definition 9. Differential expression (9) is called regular if $b - a < \infty$ and $V \in L^1(a, b)$, it is called singular otherwise. The endpoint a is called regular if $a > -\infty$ and if there exists $c > a$ such that $V \in L^1(a, c)$, it is called singular otherwise. The endpoint b is classified similarly.

We define operators $H_{\min}$ and $H_{\max}$ in $\mathcal{H} = L^2(a,b)$ acting as differential expression (9).

$$\mathrm{Dom}(H_{\max}) = \left\{\psi \in L^2(a,b) \mid \psi \in AC^2 \text{ and } H\psi \in L^2(a,b)\right\} \tag{10}$$

$$\mathrm{Dom}(H_{\min}) = \{\varphi \in \mathrm{Dom}(H_{max}) \mid (\forall \psi \in \mathrm{Dom}(H_{max}))(W[\overline{\varphi},\psi](b) - W[\overline{\varphi},\psi](a) = 0)\}\,, \tag{11}$$

where W denotes a Wronskian

$$W[\varphi,\psi](x) = \varphi(x)\psi'(x) - \varphi'(x)\psi(x) \tag{12}$$

and expressions $W[\varphi,\psi](a), W[\varphi,\psi](b)$ are understood as appropriate limits. Both these limits are finite even for singular H. Integration by parts yields

$$\int_c^d (H\overline{\varphi}\ \psi - \overline{\varphi}\ H\psi)\mathrm{d}x = W[\overline{\varphi},\psi](c) - W[\overline{\varphi},\psi](d), \tag{13}$$

for every $c > a, d < b$ and $\varphi, \psi \in \mathrm{Dom}(H_{\max})$. Since $\overline{\varphi}H\psi \in L^1(a,b)$ limit $c \to a+$ at the left hand side of (13) is finite and similarly for $d \to b-$.

Relation between $H_{\min}$ and $H_{\max}$ is described by following theorem [10]

Theorem 10. Let H be a differential expression (9) on interval (a,b), where $-\infty \le a < b \le \infty$. Then $H_{\min}$ is unique closed, symmetric operator obeying $H^*_{\min} = H_{\max}$. Indices of deficiency for operator are $n_+ = n_- = n$, $0 \le n \le 2$. For the regular H $n = 2$ and

$$\mathrm{Dom}(H_{\min}) = \{\varphi \in \mathrm{Dom}(H_{\max}) \mid \psi(a) = \psi'(a) = \psi(b) = \psi'(b) = 0\}. \tag{14}$$

Although the theorem 4 yields all self-adjoint extensions H_U of $H_{\min}$, description of its domains is not very suitable, particularly for a singular H. The domain of H_U may be specified by boundary conditions for every regular H [10], however functions and its derivatives in the domain of H_{U} may have infinite limits at a, b for singular H. General results of [7] provide more convenient way how to characterize $\mathrm{Dom}(H_{\mathrm{U}})$. We recall another terminology for behaviour of potential V [9].

Definition 11. We say that $V(x)$ is in the limit circle case at a (respectively at b) if for some, and therefore all, λ , all solutions of

$$-\varphi''(x) + V(x)\varphi(x) = \lambda\varphi(x) \tag{15}$$

are square integrable at a (respectively at b). If $V(x)$ is not in the limit circle case at a (respectively at b), it is said to be in the limit point case.

At a limit point case no boundary condition is needed for ensuring self-adjointness of the Hamiltonian [7], however in a limit circle point we need to describe the self-adjoint domain by some conditions. We assume singular endpoint $a = 0$ in a limit circle case further. We use reference modes, i.e. two independent, real solutions $R_{1,2}$ of the Schrödinger equation for the same real eigenvalue

$$HR_{1,2} = ER_{1,2}. \tag{16}$$

Wronskian of these reference modes $W[R_1, R_2]$ is constant and non-zero. Real eigenvalue E is chosen arbitrarily and reference modes are not required to obey any boundary condition. Essential fact is that limits $W[\psi, R_{1,2}]_{\pm 0}$ are finite. We generalize the boundary vectors Ψ, Ψ' (5) and denote them Φ, Φ'

$$\Phi = \begin{pmatrix} W[\psi, R_1](0+) \\ \\ W[\psi, R_1](0-) \end{pmatrix}, \quad \Phi' = \begin{pmatrix} W[\psi, R_2](0+) \\ \\ -W[\psi, R_1](0-) \end{pmatrix}. \tag{17}$$

It is proved that the condition

$$(U - I)\Phi + iL_0(U + I)\Phi' = 0,\ U \in U(2) \tag{18}$$

provides required self-adjoint extensions [7].

3 $\mathcal{PT}$-symmetry

3.1 $\mathcal{PT}$-symmetry, pseudo-Hermiticity, quasi-Hermiticity

Since 1998, when numerical studies of the Hamiltonian

$$H = -\frac{d^2}{dx^2} + ix^3 \tag{19}$$

showed that its spectrum is discrete, real, positive and bounded from below [12], the $\mathcal{PT}$-symmetric Hamiltonians have been intensively investigated. Operator $\mathcal{P}$ represents a parity and T a complex conjugation

$$(\mathcal{P}\psi)(x) = \psi(-x), \quad (\mathcal{T}\psi)(x) = \overline{\psi}(x). \tag{20}$$

$\mathcal{PT}$-symmetric operator A satisfies

$$[A, \mathcal{PT}]\psi = 0 \quad \text{for all } \psi \in \mathrm{Dom}(A). \tag{21}$$

It is straightforward and easy to show that eigenvalues of a $\mathcal{PT}$-symmetric operator are real or they form complex conjugate pairs.

$\mathcal{PT}$-symmetric operators have often properties called pseudo-Hermiticity [13] and quasi-Hermiticity [1].

Definition 12. Densely defined operator A acting on a Hilbert space $\mathcal{H}$ is called pseudo-Hermitian, if there exists an operator η with properties

1. $\eta \in B(\mathcal{H})$
2. $\eta^* = \eta$
3. $\eta A = A^* \eta$.

Definition 13. Densely defined operator A acting on a Hilbert space $\mathcal{H}$ is called quasi-Hermitian, if there exists an operator Θ with properties

1. $\Theta \in B(\mathcal{H})$
2. $\Theta > 0$
3. $\Theta A = A^* \Theta$.

Relations between the terms $\mathcal{PT}$-symmetry, pseudo- and quasi-Hermiticity are not fully described yet. It is obvious that every quasi-Hermitian operator is pseudo-Hermitian, however a converse is not true. The class of diagonalizable $\mathcal{PT}$-symmetric operators is composed of pseudo-Hermitian, which are often $\mathcal{P}$-pseudo-Hermitian, i.e. $\eta = \mathcal{P}$ in the definition. However, not every $\mathcal{P}$-pseudo-Hermitian operator is $\mathcal{PT}$-symmetric [13].

Quasi-Hermiticity is important property, because it allows us to define new scalar product

$$\langle \cdot, \cdot \rangle_\Theta := \langle \cdot, \Theta \cdot \rangle \tag{22}$$

and the quasi-Hermitian operator is a symmetric operator in this new scalar product. Moreover the quasi-Hermitian operator may be self-adjoint (or extended to the self-adjoint) and it may be considered as an observable.

Operator Θ is often called a 'metric' in a physical literature. The problem of finding and constructing of the Θ operator is partly solved for the class of operators having a discrete, real and nondegenerate spectrum and the eigenfunctions form a biorthonormal complete set [13]. If A is an operator of this class then metric operator may be found as a sum

$$\Theta := \sum_n c_n \langle \varphi_n, \cdot \rangle \varphi_n, \tag{23}$$

where c_n are arbitrary positive numbers and φ_n are eigenfunctions of A^*. The convergency of the sum (23) is not obvious and it must be proved for particular case, as well as the properties of Θ stated in the definition 13. Example of the closed formula of metric obtained by (23) is given in [3].

3.2 $\mathcal{PT}$-symmetric point interactions in one dimension

We consider one $\mathcal{PT}$-symmetric point interaction at $x = 0$. Different point interactions are described by boundary conditions at the origin. Specification of allowed boundary conditions are given by following theorem [5].

Theorem 14. The family of $\mathcal{PT}$-symmetric second derivative operators with point interactions at the origin coincides with the set of restricitions of the second derivative operator $L_{\max} = -\frac{d^2}{dx^2}$, defined on $AC^2(\mathbb{R} - \{0\})$, to the domain of functions satisfying the boundary conditions at the origin of one of the following two types

1.
$$\begin{pmatrix} \psi(0+) \\ \\ \psi'(0+) \end{pmatrix} = B \begin{pmatrix} \psi(0-) \\ \\ \psi'(0-) \end{pmatrix} \tag{24}$$

with the matrix B equal to

$$B = e^{i\theta} \begin{pmatrix} \sqrt{1+bc}e^{i\phi} & b \\ \\ c & \sqrt{1+bc}e^{-i\phi} \end{pmatrix} \tag{25}$$

with the real parameters $B \geq 0$, $c \geq -1/b^3$, $\theta, \phi \in \langle 0, 2\pi)$

2.

$$\begin{aligned} h_0\psi'(0+) &= h_1 e^{i\theta}\psi(0+) \\ h_0\psi'(0-) &= -h_1 e^{-i\theta}\psi(0-) \end{aligned} \tag{26}$$

with the real phase parameter $\theta \in \langle 0, 2\pi)$ and with the parameter $\mathbf{h}= (h_0, h_1)$ taken from the (real) projective space $\mathbf{P}^1$.

Boundary conditions of the first type are called connected, conditions of the second type are called separated. We will deal only with connected boundary conditions further.

Although the authors [5] stated that these restricted operators are both $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian for the entire range of parameters, we show that $\mathcal{P}$-pseudo-Hermiticity is ensured only for $\theta = 0$ (other ranges of parameters are preserved).

If we take ψ from $\mathrm{Dom}(L^*)$ (L is a restriction of $L_{\max}$), then with regard of the definition of the adjoint operator the equality

$$\langle \psi, L\varphi \rangle = \langle L^*\psi, \varphi \rangle \tag{27}$$

must be satisfied for all $\varphi \in \mathrm{Dom}(L)$. When we express rhs and use integration by parts and boundary conditions describing $\mathrm{Dom}(L)$ (24) we arrive at

$$\begin{aligned} &\overline{\varphi}'(0-) \left[\psi(0-) - \psi(0+)e^{i(\phi-\theta)}\sqrt{1+bc} - \psi'(0+)e^{-i\theta}b\right] + \\ &+\overline{\varphi}(0-) \left[-\psi'(0-) - \psi(0+)e^{-i\theta}c + \psi'(0+)e^{-i(\phi+\theta)}\sqrt{1+bc}\ \right] = 0. \end{aligned} \tag{28}$$

Since boundary values of φ at $x \to 0-$ may be chosen arbitrarily, we conclude that ψ must fulfill

$$\begin{aligned} \psi(0-) &= \psi(0+)e^{i(\phi-\theta)}\sqrt{1+bc} - \psi'(0+)e^{-i\theta}b \\ \psi'(0-) &= -\psi(0+)e^{-i\theta}c + \psi'(0+)e^{-i(\phi+\theta)}\sqrt{1+bc}. \end{aligned} \tag{29}$$

If L is $\mathcal{P}$-pseudo-Hermitian then $L^* = \mathcal{P}L\mathcal{P}$ and hence $\mathrm{Dom}(L^*) = \mathcal{P}\mathrm{Dom}(L)$. Boundary conditions corresponding to $\mathcal{P}\mathrm{Dom}(L)$ read

$$\begin{aligned} \psi(0-) &= \psi(0+)e^{i(\phi+\theta)}\sqrt{1+bc} - \psi'(0+)e^{i\theta}b \\ \psi'(0-) &= -\psi(0+)e^{i\theta}c + \psi'(0+)e^{-i(\phi-\theta)}\sqrt{1+bc}. \end{aligned} \tag{30}$$

Hence, the equality of the domains is satisfied only if $\theta = 0$.

We rewrite boundary conditions (24) using boundary vectors Ψ, Ψ' (5) so that the description of $\mathcal{PT}$-symmetric point interaction has the compatible form with self-adjoint case

$$(U_{\mathcal{PT}} - I)\Psi + (U_{\mathcal{PT}} + I)\Psi' = 0, \tag{31}$$

where $U_{\mathcal{PT}}$ corresponds to the original matrix B (25)

$$U_{\mathcal{PT}} = \begin{pmatrix} \frac{(b-c)e^{i\varphi}+\sqrt{1+bc}(e^{2i\varphi}-1)}{(b+c)e^{i\varphi}+\sqrt{1+bc}(e^{2i\varphi}+1)} & \frac{2e^{(\theta+\varphi)}}{(b+c)e^{i\varphi}+\sqrt{1+bc}(e^{2i\varphi}+1)} \\ \frac{2e^{(-\theta+\varphi)}}{(b+c)e^{i\varphi}+\sqrt{1+bc}(e^{2i\varphi}+1)} & \frac{(b-c)e^{i\varphi}-\sqrt{1+bc}(e^{2i\varphi}-1)}{(b+c)e^{i\varphi}+\sqrt{1+bc}(e^{2i\varphi}+1).} \end{pmatrix} \tag{32}$$

4 Point interactions with SUSY

We investigate $\mathcal{PT}$-symmetric model on finite interval $(-l, l)$ with two point interactions and determine a class of models compatible with supersymmetry. At first we summarize results of [4] dealing with self-adjoint setting.

4.1 Self-adjoint point interactions with SUSY

We consider two point interactions at $x = 0$ and $x = l$. Point interactions are specified by (4) for $\Psi(0), \Psi(l)$. We use the parametrization of U in a form (6)

$$U_g(\theta_+, \theta_-) \text{ at } x = 0, \quad U_g(\bar{\theta}_+, \bar{\theta}_-) \text{ at } x = l. \tag{33}$$

In order to express boundary conditions at $x = \pm 0, \pm l$ in more convenient way we introduce operators $\mathcal{P}, \mathcal{Q}, \mathcal{R}$

$$(\mathcal{P}\psi)(x) := \psi(-x), \quad (\mathcal{R}\psi)(x) := (\vartheta(x) - \vartheta(-x))\psi(x), \quad \mathcal{Q} := -i\mathcal{R}\mathcal{P}, \tag{34}$$

where ϑ is a Heaviside step function. We label the operators $\mathcal{P}_1 \equiv \mathcal{P},\ \mathcal{P}_2 \equiv \mathcal{Q},\ \mathcal{P}_3 \equiv \mathcal{R}$. The set of these operators forms an algebra of Pauli matrices, i.e.

$$\begin{aligned} [\mathcal{P}_l, \mathcal{P}_m] &= 2i\varepsilon_{lmn}\mathcal{P}_n, \\ \{\mathcal{P}_l, \mathcal{P}_m\} &= 2\delta_{lm}I. \end{aligned} \tag{35}$$

Next we introduce operator $\mathcal{G}$ associated to $g = \vec{\alpha}.\vec{\sigma}$

$$\mathcal{G} := \vec{\alpha}.\vec{\mathcal{P}}, \tag{36}$$

obeying $\mathcal{G}^2 = I,\ \mathcal{G}^* = \mathcal{G}$. It allows us to decompose function ψ into two eigenfunctions of $\mathcal{G}$

$$\psi_\pm := \frac{1}{2}(I \pm \mathcal{G})\psi, \ \psi = \psi_+ + \psi_-, \ \mathcal{G}\psi_\pm = \pm\psi_\pm. \tag{37}$$

We may show by straightforward calculations that boundary conditions (4) at $x = 0$ and $x = l$ for different (θ_+, θ_-), $(\bar{\theta}_+, \bar{\theta}_-)$, read

$$
\begin{aligned}
\sin\tfrac{\theta_+}{2}\psi_+(0+) + L_0\cos\tfrac{\theta_+}{2}\psi'_+(0+) = 0, &\quad \sin\tfrac{\theta_-}{2}\psi_-(0+) + L_0\cos\tfrac{\theta_-}{2}\psi'_-(0+) = 0,\\
\sin\tfrac{\theta_+}{2}\psi_+(0-) - L_0\cos\tfrac{\theta_+}{2}\psi'_+(0-) = 0, &\quad \sin\tfrac{\theta_-}{2}\psi_-(0-) - L_0\cos\tfrac{\theta_-}{2}\psi'_-(0-) = 0,\\
\sin\tfrac{\bar{\theta}_+}{2}\psi_+(+l) + L_0\cos\tfrac{\bar{\theta}_+}{2}\psi'_+(+l) = 0, &\quad \sin\tfrac{\bar{\theta}_-}{2}\psi_-(+l) + L_0\cos\tfrac{\bar{\theta}_-}{2}\psi'_-(+l) = 0,\\
\sin\tfrac{\bar{\theta}_+}{2}\psi_+(-l) - L_0\cos\tfrac{\bar{\theta}_+}{2}\psi'_+(-l) = 0, &\quad \sin\tfrac{\bar{\theta}_-}{2}\psi_-(-l) - L_0\cos\tfrac{\bar{\theta}_-}{2}\psi'_-(-l) = 0.
\end{aligned}
\tag{38}
$$

We restrict values of (θ_+, θ_-), $(\bar{\theta}_+, \bar{\theta}_-)$ to those which allows supersymmetric Hamiltonian,i.e. it is written in terms of supercharge Q, $H = 2Q^2$. Hence supercharge Q is expected to be proportional to the derivative, $Q \propto \frac{d}{dx}$. The basic property of supercharge is that if φ is an eigenfunction of H then $Q\varphi$ is also eigenfunction of H corresponding to the same eigenvalue (or $Q\varphi = 0$). However, for general boundary conditions (38) it is not guaranteed that $Q\varphi$ satisfies (38) although ϕ does.

We take an eigenfunction φ of H

$$H\varphi = E\varphi \tag{39}$$

and denote $\chi \equiv Q\varphi$. Since supercharge is proportional to derivative boundary values of χ are related to those of φ'

$$\Psi_\chi(0) \equiv \begin{pmatrix} \chi(0+) \\ \chi(0-) \end{pmatrix} = M \begin{pmatrix} \varphi'(0+) \\ -\varphi'(0-) \end{pmatrix}, \tag{40}$$

where M is an invertible matrix. φ is an eigenfunction of H, hence φ'' is proportional to φ and

$$\Psi'_\chi(0) \equiv \begin{pmatrix} \chi'(0+) \\ -\chi'(0-) \end{pmatrix} = E\tilde{M} \begin{pmatrix} \varphi(0+) \\ \varphi(0-) \end{pmatrix}, \tag{41}$$

where $\tilde{M}$ is an invertible matrix again. When we combine (4),(40),(41) we arrive at

$$(U - I)\tilde{M}^{-1}\Psi'_\chi(0) + iEL_0(U + I)M^{-1}\Psi_\chi(0) = 0. \tag{42}$$

Boundary conditions have to be energy independent and Ψ_χ, Ψ'_χ are not zero vectors simultaneously. Therefore $(U \pm I)$ must be singular matrix, i.e. eigenvalues of U are ± 1. This constraint restricts general form of U (6) to

$$U = U_g(\pi, 0) = \exp\left\{ i\frac{\pi}{2}(I + \vec{\alpha}.\vec{\sigma}) \right\}, \quad \vec{\alpha}^2 = 1. \tag{43}$$

Thus connection conditions may be expressed explicitly for $\psi_\pm$ (38)

$$\begin{array}{rcccc} \text{type A}: \; \psi_+(0+) & = & \psi'_-(0-) & = & 0 \\ \psi_+(l) & = & \psi'_-(-l) & = & 0 \end{array} \qquad \begin{array}{rcccc} \text{type B}: \; \psi'_+(0+) & = & \psi_-(0-) & = & 0 \\ \psi'_+(l) & = & \psi_-(-l) & = & 0 \end{array} \tag{44}$$

where type B conditions are given by replacing $\vec{\alpha}$ by $-\vec{\alpha}$.

We consider two nonequivalent models (A,A) and (A,B), i.e. boundary conditions of the type A at both $x = 0$ and $x = l$ for (A,A) and of the type A at $x = 0$ and of the type B at $x = l$ for (A,B). Remaining possibilities are equivalent to the previous ones [4].

4.1.1 Model of the type (A,A)

Connection conditions are given by

$$\psi_+(0+) = \psi'_-(0-) = \psi_+(l) = \psi'_-(-l) = 0. \tag{45}$$

The eigenfunctions of H are found as

$$\begin{aligned} \psi_{n+}(x) &= C_n \left(\vartheta(x) - \vartheta(-x)\tfrac{\alpha_1 + i\alpha_2}{1+\alpha_3} \right) \sin(\tfrac{n\pi}{l}x), \; n \in \mathbb{N} \\ \psi_{n-}(x) &= C_n \left(\vartheta(x) - \vartheta(-x)\tfrac{\alpha_1 + i\alpha_2}{1-\alpha_3} \right) \cos(\tfrac{n\pi}{l}x), \; n \in \mathbb{N}_0, \end{aligned} \tag{46}$$

where C_n are normalization constants. Since H commutes with $\mathcal{G}$ these eigenfunctions where found as eigenfunctions of $\mathcal{G}$ as well

$$\mathcal{G}\psi_\pm = \pm\psi_\pm. \tag{47}$$

Energy eigenvalues read

$$E_n = \left(\frac{n\pi}{l}\right)^2, \quad n \in \mathbb{N}_0. \tag{48}$$

Energy levels are doubly degenerate except of E_0.

4.1.2 Model of the type (A,B)

Connection conditions are given by

$$\psi_+(0+) = \psi'_-(0-) = \psi'_+(l) = \psi_-(-l) = 0. \tag{49}$$

The eigenfunctions of both H and $\mathcal{G}$ are found as

$$\begin{aligned}\psi_{n+}(x) &= C_n\left(\vartheta(x) - \vartheta(-x)\tfrac{\alpha_1+i\alpha_2}{1+\alpha_3}\right)\sin(\tfrac{(n-1)\pi}{2l}x),\ n\in\mathbb{N}\\ \psi_{n-}(x) &= C_n\left(\vartheta(x) - \vartheta(-x)\tfrac{\alpha_1+i\alpha_2}{1-\alpha_3}\right)\cos(\tfrac{(n-1)\pi}{2l}x),\ n\in\mathbb{N},\end{aligned} \tag{50}$$

where C_n are normalization constants. Energy eigenvalues are doubly degenerate for all levels (unlike for the (A,A) type)

$$E_n = \left(\frac{(n-1)\pi}{2l}\right)^2, \quad n\in\mathbb{N}. \tag{51}$$

Degeneracy of levels was expected with regard of requiring connection conditions to be compatible with supersymmetry. The existence of supersymmetry (however broken in the (A,B) type) is confirmed by following construction of supercharges satisfying

$$\begin{aligned}\{Q_a, Q_b\} &= H\delta_{ab}\\ Q_a^* &= Q_a, \quad a,b\in\{1,2\}.\end{aligned} \tag{52}$$

Supercharge Q_a should be proportional to derivative $\frac{d}{dx}$ and map $Q_a\psi_\pm \propto \psi_\mp$, i.e. it should exchange eigenfunctions of $\mathcal{G}$ and therefore anticommute with $\mathcal{G}$

$$\{Q_a, \mathcal{G}\} = 0, \quad a,b\in\{1,2\}. \tag{53}$$

It can be easily shown [4] that supercharges Q_a have form

$$Q_a = i\frac{\sqrt{2}}{2}\mathcal{G}_a\mathcal{P}_3\frac{d}{dx}, \quad a,b\in\{1,2\}, \tag{54}$$

where

$$\mathcal{G}_a = \vec{\gamma}_a.\vec{\mathcal{P}}, \quad (\vec{\gamma}_a)^2 = 1 \ \text{ and } \ \vec{\gamma}_a.\vec{\alpha} = \vec{\gamma}_1.\vec{\gamma}_2 = 0. \tag{55}$$

4.2 $\mathcal{PT}$-symmetric point interactions with SUSY

We consider a similar model as previous one with two point interactions at $x = 0$ and $x = l$, however the point interactions are $\mathcal{PT}$-symmetric. $\mathcal{PT}$-symmetric point interactions are given by boundary conditions (31) which are similar to the self-adjoint case, nevertheless matrix U (32) is not unitary. In order to receive a system compatible with supersymmetry we follow almost the same procedure (40,41,42) as before and obtain suitable boundary conditions. General U matrix (32) with parameters b, c, Θ, φ is restricted to

$$U_{\mathcal{PT}} = \begin{pmatrix} i\tan\varphi & \frac{e^{i\theta}}{\cos\varphi} \\ \frac{e^{-i\theta}}{\cos\varphi} & -i\tan\varphi \end{pmatrix}. \tag{56}$$

We denote

$$\beta_1 := -\frac{\cos\theta}{\cos\varphi}, \quad \beta_2 := \frac{\sin\theta}{\cos\varphi}, \quad \beta_3 := -i\tan\varphi \tag{57}$$

and we see

$$\vec{\beta}^2 = 1, \quad \beta_{1,2} \in \mathbb{R}, \quad \beta_3 \in i\mathbb{R}. \tag{58}$$

The $U_{\mathcal{PT}}$ matrix has very similar form as U matrix in self-adjoint case fortunately(43) (although these matrices are very different without SUSY constraint),

$$U_{\mathcal{PT}} = \begin{pmatrix} -\beta_3 & -\beta_1 + i\beta_2 \\ -\beta_1 - i\beta_2 & \beta_3 \end{pmatrix}, \quad U = \begin{pmatrix} -\alpha_3 & -\alpha_1 + i\alpha_2 \\ -\alpha_1 - i\alpha_2 & \alpha_3 \end{pmatrix}. \tag{59}$$

This fact allows us to generalize previous results of self-adjoint case to the $\mathcal{PT}$-symmetric one

$$U_{\mathcal{PT}} = \exp\left\{i\frac{\pi}{2}(I + \vec{\beta}.\vec{\sigma}\right\}), \quad \mathcal{G} := \vec{\beta}.\vec{\mathcal{P}}. \tag{60}$$

Properties of $\mathcal{G}$ are slightly changed

$$\mathcal{G}^2 = I, \mathcal{G}^* \neq \mathcal{G}, [G, \mathcal{PT}] = 0, \tag{61}$$

however we may decompose any function ψ to $\psi_\pm$ as before (37). Boundary conditions expressed in terms of $\psi_\pm$ have very similar explicit form as (44). Hence we arrive at two nonequivalent models of the type (A,A) and (A,B). We prove that both models are quasi-Hermitian.

4.2.1 Model of the type (A,A)

Eigenvaules of H are the same as in self-adjoint case, eigenfunctions differ only in substituting $\alpha \to \beta$, i.e.

$$\begin{aligned} \psi_{n+}(x) &= C_n \left(\vartheta(x) - \vartheta(-x) \tfrac{\beta_1 + i\beta_2}{1+\beta_3} \right) \sin(\tfrac{n\pi}{l} x), \; n \in \mathbb{N} \\ \psi_{n-}(x) &= C_n \left(\vartheta(x) - \vartheta(-x) \tfrac{\beta_1 + i\beta_2}{1-\beta_3} \right) \cos(\tfrac{n\pi}{l} x), \; n \in \mathbb{N}_0. \end{aligned} \tag{62}$$

Supercharges are given by (54) again,

$$Q_a = \frac{\sqrt{2}}{2} \mathcal{G}_a \mathcal{P}_3 i \frac{d}{dx}, \quad a, b \in \{1, 2\}, \tag{63}$$

where

$$\mathcal{G}_a = \vec{\gamma}_a . \vec{\mathcal{P}}, \quad (\vec{\gamma}_a)^2 = 1 \quad \text{and} \quad \vec{\gamma}_a . \vec{\beta} = \vec{\gamma}_1 . \vec{\gamma}_2 = 0. \tag{64}$$

Metric operator for this model reads

$$\Theta = I - \frac{\beta_3}{\beta_1 + i\beta_2} P^+ \mathcal{P} + \frac{\beta_3}{\beta_1 - i\beta_2} P^- \mathcal{P}, \tag{65}$$

where $P^\pm$ are orthogonal projectors

$$(P^\pm \psi)(x) = \vartheta(\pm x) \psi(x), \quad (P^\pm)^2 = P^\pm = (P^\pm)^*, \quad P^+ P^- = P^- P^+ = 0. \tag{66}$$

We prove all properties of a metric operator (definition 13. Θ is a bounded operator.

$$\|\Theta\| \leq \|I\| + \left| \frac{\beta_3}{\beta_1 + i\beta_2} \right| \|P^+\| \|\mathcal{P}\| + \left| \frac{\beta_3}{\beta_1 - i\beta_2} \right| \|P^-\| \|\mathcal{P}\| \leq 1 + 1 + 1, \tag{67}$$

$$\left| \frac{\beta_3}{\beta_1 + i\beta_2} \right| < 1. \tag{68}$$

Θ is self-adjoint. When we take the adjoint

$$\Theta^* = I - \frac{-\beta_3}{\beta_1 - i\beta_2} \mathcal{P} P^+ + \frac{-\beta_3}{\beta_1 + i\beta_2} \mathcal{P} P^-, \tag{69}$$

we arrive at $\Theta^* = \Theta$ since

$$\mathcal{P} P^\pm = P^\mp \mathcal{P}, \quad \beta_{1,2} \in \mathbb{R}, \quad \beta_3 \in i\mathbb{R}, \quad (\vec{\beta})^2 = 1. \tag{70}$$

Θ is positive.

$$\begin{aligned}\langle\psi,\Theta\psi\rangle = \|\psi\|^2 - \tfrac{\beta_3}{\beta_1+i\beta_2}\textstyle\int_0^l \overline{\psi(x)}\psi(-x)dx + \tfrac{\beta_3}{\beta_1-i\beta_2}\int_{-l}^0 \overline{\psi(x)}\psi(-x)dx \geq \\ \geq \|\psi\|^2 - \left|\tfrac{\beta_3}{\beta_1+i\beta_2}\right|\left|Int\right|\left|1-\tfrac{\beta_1-i\beta_2}{\beta_1+i\beta_2}\tfrac{\overline{Int}}{Int}\right|,\end{aligned} \tag{71}$$

where

$$Int := \int_0^l \overline{\psi(x)}\psi(-x)dx \tag{72}$$

$$|Int| \leq \int_0^l |\psi(x)||\psi(-x)|dx \leq \frac{1}{2}\int_0^l |\psi(x)|^2 + |\psi(-x)|^2 \; dx \leq \frac{1}{2}\left(\|P^+\psi\|^2 + \|P^-\psi\|^2\right) \leq \frac{1}{2}\|\psi\|^2. \tag{73}$$

These estimates yield all together

$$\langle\psi,\Theta\psi\rangle \geq \underbrace{\left(1-\frac{|\beta_3|}{\sqrt{1+|\beta_3|^2}}\right)}_{c_0} \|\psi\|^2 \geq 0. \tag{74}$$

Moreover

$$\langle\psi,\Theta\psi\rangle \geq c_0\|\psi\|^2,\ c_0>0 \ \Rightarrow\ 0\notin\sigma(\Theta), \tag{75}$$

hence Θ is invertible and Θ^{-1} is bounded.

Finally, it is necessary to verify equality of the domains $\Theta A = A^*\Theta$, i.e. $\Theta\mathrm{Dom}(H) = \mathrm{Dom}(H^*)$. Equality of the action of operators is obvious. $\mathrm{Dom}(H)$ is described by boundary conditions (45) expressed in terms of $\psi_\pm$. Boundary conditions for $\mathrm{Dom}(H^*)$ may be easily obtained from the definition of adjoint operator (1), explicitly

$$\begin{array}{llll} \mathrm{Dom}(H): & & & \\ (\beta_1+i\beta_2)\psi(0+)+(1-\beta_3)\psi(0-) = 0, & & (\beta_1+i\beta_2)\psi(l)+(1-\beta_3)\psi(-l) = 0, & \\ (\beta_1+i\beta_2)\psi'(0+)+(1+\beta_3)\psi'(0-) = 0, & & (\beta_1+i\beta_2)\psi'(l)+(1+\beta_3)\psi'(-l) = 0, & \\ \mathrm{Dom}(H^*): & & & \\ (\beta_1+i\beta_2)\psi(0+)+(1+\beta_3)\psi(0-) = 0, & & (\beta_1+i\beta_2)\psi(l)+(1+\beta_3)\psi(-l) = 0, & \\ (\beta_1+i\beta_2)\psi'(0+)+(1-\beta_3)\psi'(0-) = 0, & & (\beta_1+i\beta_2)\psi'(l)+(1-\beta_3)\psi'(-l) = 0. & \end{array} \tag{76}$$

It is straightforward to calculate limits (for $\pm l$ analogously)

$$\begin{aligned}(\Theta\psi)(0+) &= \psi(0+) - \tfrac{\beta_3}{\beta_1+i\beta_2}\psi(0-)\\(\Theta\psi)(0-) &= \psi(0-) + \tfrac{\beta_3}{\beta_1-i\beta_2}\psi(0+)\end{aligned} \tag{77}$$

and check that $\Theta\psi$ satisfies boundary conditions for $\mathrm{Dom}(H^*)$.

Construction of Θ operator is based on decomposition of a vector into the orthonormal basis and Parseval equality. When we express sum (23), we may identify particular parts with appropriate operators (65) since

$$\begin{gathered}\left\{A_n \sin\left(\tfrac{n\pi}{l}x\right)\right\}_{n=1}^{\infty}, \left\{A_n \cos\left(\tfrac{n\pi}{l}x\right)\right\}_{n=0}^{\infty},\\A_0 = \tfrac{1}{\sqrt{2l}}, \quad A_n = \tfrac{1}{\sqrt{l}} \text{ for } n \in \mathbb{N}\end{gathered} \tag{78}$$

form the orthonormal bases of both $L^2(-l,0)$ and $L^2(0,l)$.

All requirements imposed on metric operator (definition 13) are satisfied and therefore Hamiltonian H is quasi-Hermitian. Since Θ is invertible it is not difficult to show that H is a self-adjoint operator in a Hilbert space with $\langle\cdot,\Theta\cdot\rangle$ scalar product.

4.2.2 Model of the type (A,B)

We summarize analogous results for the model of the type (A,B). Eigenfunctions and eigenvalues read

$$\begin{aligned}\psi_{n+}(x) &= C_n\left(\vartheta(x) - \vartheta(-x)\tfrac{\beta_1+i\beta_2}{1+\beta_3}\right)\sin\left(\tfrac{(n-1)\pi}{2l}x\right),\\\psi_{n-}(x) &= C_n\left(\vartheta(x) - \vartheta(-x)\tfrac{\beta_1+i\beta_2}{1-\beta_3}\right)\cos\left(\tfrac{(n-1)\pi}{2l}x\right),\\E_n &= \left(\tfrac{(2n-1)\pi}{2l}\right)^2, \; n \in \mathbb{N}.\end{aligned} \tag{79}$$

Supercharges have the same form as in previous model (63), however zero energy state is missing. Hamiltonian is quasi-Hermitian again, metric operator has more complicated form

$$\Theta = P^+(O_1+O_2)P^+ + P^-(O_1+O_2)P^- - \tfrac{\beta_1 - i\beta_2}{1+\beta_3} P^+ O_1 P^- - \tfrac{\beta_1 + i\beta_2}{1-\beta_3} P^- O_1 P^+ - \\ - \tfrac{\beta_1 - i\beta_2}{1-\beta_3} P^+ O_2 P^- - \tfrac{\beta_1 + i\beta_2}{1+\beta_3} P^- O_2 P^+, \tag{80}$$

where $P^{\pm}$ are the same projectors as in previous model (66) and $O_{1,2}$ are projectors given by action on orthonormal bases $\{e_n\}_{n=0}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ of $L^2(-l,l)$

$$e_0(x) = \tfrac{1}{\sqrt{2l}}, \quad e_{2k-1}(x) = \tfrac{1}{\sqrt{l}} \sin \tfrac{(2k-1)\pi}{2l} x, \quad e_{2k}(x) = \tfrac{1}{\sqrt{l}} \cos \tfrac{k\pi}{l} x \\ f_{2k-1}(x) = \tfrac{1}{\sqrt{l}} \cos \tfrac{(2k-1)\pi}{2l} x, \quad f_{2k}(x) = \tfrac{1}{\sqrt{l}} \sin \tfrac{k\pi}{l} x \tag{81}$$

$$O_1 e_{2k} = 0, \quad O_1 e_{2k-1} = e_{2k-1}, \\ O_2 f_{2k} = 0, \quad O_2 f_{2k-1} = f_{2k-1}, \\ O_1^2 = O_1 = O_1^*, \quad O_2^2 = O_2 = O_2^*. \tag{82}$$

5 Scarf I potential

We consider Hamiltonian on finite interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ with so called Scarf I potential [6], with both singular ends

$$V(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\frac{1}{\cos^2 x} + \frac{\alpha^2-\beta^2}{2}\frac{\sin x}{\cos^2 x}, \quad \alpha, \beta \geq 0. \tag{83}$$

We define usual operators $H_{\max}$ and $H_{\min}$ acting on $\mathcal{H} = L^2(-\frac{\pi}{2}, \frac{\pi}{2})$ with the domains $\mathrm{Dom}(H_{\max})$, $\mathrm{Dom}(H_{\min})$ (10,11). We describe $\mathrm{Ker}(H^*_{\min} - i)$ in order to specify the deficiency indices depending on parameters α, β. The Schrödinger equation $H\psi = E\psi$ reads

$$-z(1-z)\,\psi''(z) - \frac{1-2z}{2}\,\psi'(z) + \frac{1+4(z-1)\alpha^2 - 4z\beta^2}{16z(z-1)}\,\psi(z) = k^2\psi(z), \tag{84}$$

after the substitution

$$z = \frac{1-\sin x}{2},\ E = k^2. \tag{85}$$

Another substitution

$$\psi(z) = z^{\frac{1}{4}-\frac{\alpha}{2}}(1-z)^{\frac{1}{4}+\frac{\beta}{2}}\varphi(z) \tag{86}$$

leads to

$$4z(z-1)\varphi''(z) + 4\left[\alpha - 1 + z(2-\alpha+\beta)\right]\varphi'(z) + \left[(1-\alpha+\beta)^2 - 4k^2\right]\varphi(z) = 0. \tag{87}$$

Two linear independent solutions φ_1, φ_2 are given in the form of hypergeometric functions

$$\begin{aligned} \varphi_1(z) &= F\left(\tfrac{-\alpha+\beta+1}{2} - k, \tfrac{-\alpha+\beta+1}{2} + k; 1-\alpha, z\right) \\ \varphi_2(z) &= z^\alpha F\left(\tfrac{\alpha+\beta+1}{2} - k, \tfrac{\alpha+\beta+1}{2} + k; 1+\alpha, z\right), \end{aligned} \tag{88}$$

and for the previous function ψ

$$\begin{aligned} \psi_1(z) &= z^{\frac{1}{4}-\frac{\alpha}{2}}(1-z)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\tfrac{-\alpha+\beta+1}{2} - k, \tfrac{-\alpha+\beta+1}{2} + k; 1-\alpha, z\right) \\ \psi_2(z) &= z^{\frac{1}{4}+\frac{\alpha}{2}}(1-z)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\tfrac{\alpha+\beta+1}{2} - k, \tfrac{\alpha+\beta+1}{2} + k; 1+\alpha, z\right). \end{aligned} \tag{89}$$

A behaviour of the solutions ψ_i at $z = 0$, i.e. $x = \frac{\pi}{2}$, is given only by the prefactors, however the hypergeometric function has to be taken into the consideration at $z = 1$,

	$x=-\frac{\pi}{2}$		$x=\frac{\pi}{2}$		$x=\pm\frac{\pi}{2}$	
	Finite	Dom$(H_{\max})$	Finite	Dom$(H_{\max})$	Finite	Dom$(H_{\max})$
ψ_1	$0<\beta<\frac{1}{2}$	$0<\beta<1$	$0<\alpha<\frac{1}{2}$	$0<\alpha<1$	$0<\alpha<\frac{1}{2}$ $0<\beta<\frac{1}{2}$	$0<\alpha<1$ $0<\beta<1$
ψ_2	$0<\beta<\frac{1}{2}$	$0<\beta<1$	$0<\alpha$	$0<\alpha$	$0<\alpha$ $0<\beta<\frac{1}{2}$	$0<\alpha$ $0<\beta<1$

Table 1: The behaviour of solutions at endpoints

i.e. $x=-\frac{\pi}{2}$, as well. We summarize the properties of the solutions for the case that hypergeometric functions do not degenerate to polynomial.

The special values of parameters $(\alpha, \beta \in \{0, \frac{1}{2}, 1\})$ are excluded. Since the case $k^2 = i$ is characterized by these non polynomial solutions, we may determine the deficiency indices easily.

$n_\pm$		
0	$0<\alpha, \beta>1,$	
1	$\alpha>1,\ 0<\beta<1$	$\psi_2 \in$ Dom$(H_{\max})$
2	$0<\alpha<1,\ 0<\beta<1$	$\psi_{1,2} \in$ Dom$(H_{\max})$

Table 2: Deficiency indices

5.1 Case $n_\pm = 0$

Operator $H_{\min}$ is essentially self-adjoint in the range of parameters $0<\alpha,\ \beta>1$, both ends are in a limit point case. Moreover, $H^*_{\min} = H_{\max}$ and $H_{\min}$ is closed in the entire range of parameters. Therefore

$$H_{\min} = \overline{H}_{\min} = H^*_{\min} = H_{\max} \equiv H_0 \tag{90}$$

in this case. Every eigenfunction for real eigenvalue has to belong to Dom$(H_{\min})$ = Dom$(H_{\max}) \equiv$ Dom(H_0). Solutions of Schrödinger equation $\psi_{1,2}$ are square integrable only if the hypergeometric functions degenerates to the polynomials, i.e.

$$k = \frac{1-\alpha+\beta}{2} + n, \qquad k = \frac{1+\alpha+\beta}{2} + n, \qquad n \in \mathbb{N}_0. \tag{91}$$

The eigenfunctions $\psi_n^{(1,2)}$ belonging to energies $E_n^{(1,2)}$ may be expressed in the terms of Jacobi polynomials

$$\begin{aligned} \psi_n^{(1)}(z) &= C_n^{(1)} z^{\frac{1}{4}-\frac{1}{\alpha}} (1-z)^{\frac{1}{4}+\frac{1}{\beta}} P_n^{(-\alpha,\beta)}(1-2z) \\ \psi_n^{(2)}(z) &= C_n^{(2)} z^{\frac{1}{4}+\frac{1}{\alpha}} (1-z)^{\frac{1}{4}+\frac{1}{\beta}} P_n^{(\alpha,\beta)}(1-2z) \end{aligned} \tag{92}$$

$$E_n^{(1)} = \left(\frac{1-\alpha+\beta}{2} + n\right)^2, \qquad E_n^{(2)} = \left(\frac{1+\alpha+\beta}{2} + n\right)^2, \qquad n \in \mathbb{N}_0, \tag{93}$$

where $C_n^{(1,2)}$ are normalization constants, $P_n^{(\pm\alpha,\beta)}$ are Jacobi polynomials. We see that the square integrability of $\psi_n^{(1)}$ is ensured only for $\alpha < 1$.

	E_n	ψ_n
$0 < \alpha < 1$	$E_n^{(1)},\ E_n^{(2)}$	$\psi_n^{(1)},\ \psi_n^{(2)} \in \mathrm{Dom}(H_0)$
$1 < \alpha$	$E_n^{(2)}$	$\psi_n^{(2)} \in \mathrm{Dom}(H_0)$

Table 3: Eigenvalues and eigenfunctions for $n_\pm = 0$ case

Eigenfunctions $\psi_n^{(2)}$ are regular for all considered values of parameters α, β and they vanish at the both ends, i.e. $z = 0, 1$. However, $\psi_n^{(1)}$ may go to infinity at $z = 0$ end for $\alpha > \frac{1}{2}$.

5.2 Case $n_\pm = 1$

Although a method of reference modes can be applied in this case (endpoint $x = \pi/2$ is in a limit point case, $x = -\pi/2$ in a limit circle case) we refer to the theorem 4 only. Deficiency subspaces $\mathrm{Ker}(H^*_{\min} \mp i)$ are characterized by the functions $\Phi^{(\pm)}$ that correspond to the solutions ψ_2 (89) and $k^2 = \pm i$.

$$\begin{aligned} \phi^{(+)}(z) &= z^{\frac{1}{4}+\frac{\alpha}{2}} (1-z)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\tfrac{\alpha+\beta+1}{2} - \sqrt{i}, \tfrac{\alpha+\beta+1}{2} + \sqrt{i}; 1+\alpha, z\right) \\ \phi^{(-)}(z) &= z^{\frac{1}{4}+\frac{\alpha}{2}} (1-z)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\tfrac{\alpha+\beta+1}{2} - \sqrt{-i}, \tfrac{\alpha+\beta+1}{2} + \sqrt{-i}; 1+\alpha, z\right). \end{aligned} \tag{94}$$

It follows from $\overline{\Phi}^{(+)} = \Phi^{(-)}$ that the norms of these functions are equal

$$\|\phi^{(+)}\| = \|\phi^{(-)}\|. \tag{95}$$

Operator $U : \mathrm{Ker}(H^*_{\min} - i) \to \mathrm{Ker}(H^*_{\min} + i) : \Phi^{(+)} \mapsto e^{i\tau}\Phi^{(-)}, \tau \in \langle 0; 2\pi)$ is therefore norm preserving and it describes the domain self-adjoint extensions of H_{min}

$$\mathrm{Dom}(H_U) = \{\psi + C\phi^{(+)} + e^{i\Theta}C\phi^{(-)} \mid \psi \in \mathrm{Dom}(H_{\min}), C \in \mathbb{C}.\} \tag{96}$$

5.3 Case $n_\pm = 2$

Both endpoints are in a limit circle case. We apply the reference modes method. Reference modes are chosen as the solutions of (16) with $E = E_1^{(1)}$ (92, 89), i.e.

$$\begin{aligned} R_1(x) &= (1-\sin x)^{\frac{1}{4}-\frac{\alpha}{2}}(1+\sin x)^{\frac{1}{4}+\frac{\beta}{2}} \\ R_2(x) &= (1-\sin x)^{\frac{1}{4}+\frac{\alpha}{2}}(1+\sin x)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\alpha, 1+\beta; 1+\alpha, \tfrac{1-\sin x}{2}\right). \end{aligned} \tag{97}$$

Their Wronskian $W[R_1, R_2](x)$ may be determined explicitly

$$W[R_1, R_2](x) = -2^{1+\beta}\alpha \tag{98}$$

what is obviously a non-zero constant as it should be. We calculate generalized boundary vectors Φ, Φ' (17) for general linear combination of solutions (89) and selected reference modes.

$$\begin{aligned} \psi(x) &= C_1\psi_1(x) + C_2\psi_2(x), \\ \psi_1(x) &= (1-\sin x)^{\frac{1}{4}-\frac{\alpha}{2}}(1+\sin x)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\tfrac{-\alpha+\beta+1}{2} - k, \tfrac{-\alpha+\beta+1}{2} + k; 1-\alpha, \tfrac{1-\sin x}{2}\right), \\ \psi_2(x) &= (1-\sin x)^{\frac{1}{4}+\frac{\alpha}{2}}(1+\sin x)^{\frac{1}{4}+\frac{\beta}{2}} F\left(\tfrac{+\alpha+\beta+1}{2} - k, \tfrac{+\alpha+\beta+1}{2} + k; 1+\alpha, \tfrac{1-\sin x}{2}\right). \end{aligned} \tag{99}$$

Explicit forms of Wronskians $W[\psi, R_{1,2}]$ expressed in the z variable are

$$\begin{aligned} W[\psi, R_1](z) &= C_1 W_{11}(z) + C_2 W_{12}(z), \\ W[\psi, R_2](z) &= C_1 W_{21}(z) + C_2 W_{22}(z), \end{aligned} \tag{100}$$

$$
\begin{aligned}
W_{11}(z) &= \tfrac{2^{\beta-\alpha-1}}{1-\alpha}(1-z)^{\beta+1}z^{1-\alpha}\left((1-\alpha+\beta)^2-4k^2\right)\times \\
&\quad \times F\left(\tfrac{3-\alpha+\beta}{2}-k,\tfrac{3-\alpha+\beta}{2}+k;2-\alpha,z\right), \\
W_{12}(z) &= \tfrac{2^{\beta-1}}{1+\alpha}(1-z)^{\beta+1}[4\alpha(1+\alpha)F\left(\tfrac{1+\alpha+\beta}{2}-k,\tfrac{1+\alpha+\beta}{2}+k;1+\alpha,z\right)+ \\
&\quad +z\left((1+\alpha+\beta)^2-4k^2\right)F\left(\tfrac{3+\alpha+\beta}{2}-k,\tfrac{3+\alpha+\beta}{2}+k;2+\alpha,z\right)] \\
W_{21}(z) &= -2^{\beta+1}\alpha F\left(\tfrac{1-\alpha+\beta}{2}-k,\tfrac{1-\alpha+\beta}{2}+k;1-\alpha,z\right)+ \\
&\quad +\tfrac{2^{\beta-1}}{1-\alpha}(1-z)^{\beta+1}z\left((1-\alpha+\beta)^2-4k^2\right)F\left(\alpha,1+\beta;1+\alpha,z\right)\times \\
&\quad \times F\left(\tfrac{3-\alpha+\beta}{2}-k,\tfrac{3-\alpha+\beta}{2}+k;2-\alpha,z\right) \\
W_{22}(z) &= 2^{\alpha+\beta-1}z^{\alpha}[4\alpha\left((1-z)^{\beta+1}F\left(\alpha,1+\beta;1+\alpha,z\right)-1\right)\times \\
&\quad \times F\left(\tfrac{1+\alpha+\beta}{2}-k,\tfrac{1+\alpha+\beta}{2}+k;1+\alpha,z\right)+\tfrac{1}{1+\alpha}(1-z)^{\beta+1}z\left((1+\alpha+\beta)^2-4k^2\right)\times \\
&\quad \times F\left(\alpha,1+\beta;1+\alpha,z\right)F\left(\tfrac{3+\alpha+\beta}{2}-k,\tfrac{3+\alpha+\beta}{2}+k;2+\alpha,z\right)].
\end{aligned} \tag{101}
$$

Limits at $z \to 0+$ are easier to calculate and results are simple

$$
\begin{aligned}
W[\psi, R_1](0+) &= 2^{1+\beta}\alpha C_2 \\
W[\psi, R_2](0+) &= -2^{1+\beta}\alpha C_2.
\end{aligned} \tag{102}
$$

On the other hand, calculation of the limits at $z \to 1-$ is much more difficult and the results are much more complicated as well

$$
\begin{aligned}
W[\psi, R_1](1-) &= C_1 W_{11}(1-) + C_2 W_{12}(1-) \\
W[\psi, R_2](1-) &= C_1 W_{21}(1-) + C_2 W_{22}(1-),
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
W_{11}(1-) &= \frac{2^{1-\alpha+\beta}\Gamma[1-\alpha]\Gamma[1+\beta]}{\Gamma[\frac{1-\alpha+\beta}{2}-k]\Gamma[\frac{1-\alpha+\beta}{2}+k]}, \\
W_{12}(1-) &= \frac{2^{1+\beta}\Gamma[1+\alpha]\Gamma[1+\beta]}{\Gamma[\frac{1+\alpha+\beta}{2}-k]\Gamma[\frac{1+\alpha+\beta}{2}+k]}, \\
W_{21}(1-) &= 2^{1+\beta}\alpha\left(\frac{\Gamma[1-\alpha]\Gamma[-\beta]}{\Gamma[\frac{1-\alpha-\beta}{2}-k]\Gamma[\frac{1-\alpha-\beta}{2}+k]} + \frac{\pi^2}{\sin(\pi\alpha)\sin(\pi\beta)\Gamma[\alpha-\beta]\Gamma[\frac{1-\alpha+\beta}{2}-k]\Gamma[\frac{1-\alpha+\beta}{2}+k]}\right), \\
W_{22}(1-) &= \frac{2^{2+\beta}\Gamma[2+\alpha]\Gamma[1-\beta]}{(1+\alpha)^2\beta}\left(\frac{\big((1+\alpha+\beta)^2-4k^2\big)\Gamma[2+\alpha]\Gamma[1+\beta]}{\Gamma[\alpha-\beta]\Gamma[\frac{1-\alpha+3\beta}{2}-k]\Gamma[\frac{1-\alpha+3\beta}{2}+k]} - \frac{4\alpha(1+\alpha)}{\Gamma[\frac{1+\alpha-\beta}{2}-k]\Gamma[\frac{1+\alpha-\beta}{2}+k]}\right).
\end{aligned} \tag{104}
$$

If we insert the generalized boundary vectors into the condition (18) and search for non-trivial solutions, we receive the condition for eigenvalues.

6 Conclusions

In the first part of our results we found models with two $\mathcal{PT}$-symmetric point interactions compatible with supersymmetry. Supersymmetry allows only 'very weak' interactions both in self-adjoint and $\mathcal{PT}$-symmetric case. The $\mathcal{PT}$-symmetric models are exactly solvable and their properties turned out to be close to the self-adjoint case - same eigenvalues and only slightly changed eigenfunctions. Moreover, we showed that both of $\mathcal{PT}$-symmetric models are quasi-Hermitian and we constructed positive and bounded metric operator Θ (in the closed formula form). Possible generalization to higher numbers of interactions seems to be straightforward, although the feasibility of used method to construct the metric depends significantly on the structure of eigenfunctions of the Hamiltonian. If the 'almost separated' form (62) is preserved (particularly for the (A,A) types of models) then other examples of not very complicated formulas for metric operators may be found analogously to presented results. During our calculations we had to correct the result of [5] concerning the $\mathcal{P}$-pseudo-Hermiticity of a Hamiltonian with one $\mathcal{PT}$-symmetric point interaction.

In the second part we classified the endpoints of singular Scarf I potential, we determined dependence of indices of deficiency on parameters α, β giving different settings potential and we described self-adjoint extensions of the Hamiltonian for all three possible cases. The results show that prescribing of boundary conditions for the functions in the self-adjoint domain may be correct only in the essentially self-adjoint case. The domains for the other cases may be specified by boundary conditions given for limits of Wronskians (with selected reference modes). The explicit calculations containing the condition for eigenvalues are presented for $n_{\pm} = 2$ case. Since the self-adjoint case has already showed the non-trivial structure of the domains the $\mathcal{PT}$-symmetric extension have to be described carefully. This part of the work should serve as the initial step to answering the questions stated in [6] for both self-adjoint and $\mathcal{PT}$-symmetric case.

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