CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

DIPLOMA THESIS

SOLUTION OF CONTRACTION EQUATIONS FOR THE PAULI GRADING OF $sl(3, \mathbb{C})$

Jiří Hrivnák

Supervisor: Prof. Ing. J. Tolar, DrSc.

April 22, 2003

I declare that I wrote this diploma thesis independently using the listed references.

Jiří Hrivnák

Acknowledgements

I would like to thank prof. Miloslav Havlíček and doc. Edita Pelantová for advice and ideas on topics of Lie gradings and graded contractions, prof. Jiří Patera for advice, ideas and care during my study stay at Centre de recherches mathématiques, Université de Montréal, where this work was partially written.

Especially, I would like to thank prof. Jiří Tolar for his kind supervision during the past three years.

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Introduction

Lie algebra $A_2 = sl(3, \mathbb{C})$ is the second lowest dimensional among the classical Lie algebras A_n. It, as well as its real forms, have found numerous applications in physics. Also its subalgebras are of great interest. There are two simple subalgebras, $o(3)$ and $sl(2, \mathbb{C})$; Other subalgebras gained importance later and all of them were classified. The ubiquity of $sl(3, \mathbb{C})$ leads to an interesting question of its relation to other Lie algebras (excluding homomorphisms). One type of this relation are contractions, which were introduced by Wigner and Inönü in 1953. Here we are interested in contractions of $sl(3, \mathbb{C})$ which lead to Lie algebras of the same dimension 8. It turns out that the outcomes of various kinds of contractions are numerous, but at present not all of them are known, even in the case of $sl(3, \mathbb{C})$. The most general approach allowed classification of Lie algebras in dimensions 2, 3, 4, 5 and 6 in [13]. Thus, the set of all Lie algebras of dimension 8 is still unknown. The method of graded contractions allows us to partially fill this gap.

The goal of describing all graded contractions of $sl(3, \mathbb{C})$ has a lot of merit, and undoubtedly can be reached within a relatively short time. The starting point for achieving this goal are 4 fine gradings of $sl(3, \mathbb{C})$ which are known [7]. Well-known is the toroidal grading, which decomposes $sl(3, \mathbb{C})$ into 6 one-dimensional subspaces (root spaces) and one two-dimensional subspace (the Cartan subalgebra). All the graded contractions for this grading were found in [2] and more recently in [1]. The other 3 gradings and the corresponding graded contractions will undoubtedly yield many other Lie algebras which are non-isomorphic to those in [2]. Among three remaining gradings, the grading by generalized Pauli matrices [14] is considered in this work. It is distinguished from the others: it has very few coarsenings which are intermediate between the original $sl(3, \mathbb{C})$ and the finest Pauli grading. For that reason the solution of the system of contraction equation is the most difficult one of the four cases since the method in [1] is ineffective here. Another interesting outcome is the general result for gradings of Lie algebras $so(N + 1)$ in [9]. Unfortunately, we found a straightforward generalization impossible, even for the case $sl(3, \mathbb{C})$.

For the necessary explicit evaluation of the solutions the symmetry group of the Pauli grading [6] has been employed in this work. The method of using the symmetry group and reducing the case by case analysis is, however, developed generally and then applied to our concrete case. This method was already foreseen in [10], and since the symmetry group is in our case isomorphic to a finite matrix group, we make use of [12].

The facts and definitions of gradings and graded contractions are stated in the first and the second chapter, then the symmetry group and its action on the solutions and equations is introduced in the third chapter. The evaluation of the solutions is presented in the fourth chapter. The final results in Appendix will serve as an entry to a further analysis of desired Lie algebras which are graded contractions of $sl(3,\mathbb{C})$ corresponding to the Pauli grading.

Chapter 1

Lie gradings

1.1 Basic definitions

Let us first state the basic definitions of Lie gradings. We consider a Lie algebra $\mathcal{L}$ over the field of complex numbers $\mathbb{C}$. We shall focus on finite-dimensional cases, so let the dimension of $\mathcal{L}$ be finite. A decomposition of this algebra into a direct sum of its subspaces $\mathcal{L}_i, i \in I$

$$\mathcal{L} = \bigoplus_{i \in I} \mathcal{L}_i \tag{1.1}$$

is called a **grading** of Lie algebra $\mathcal{L}$, when the following property holds

$$(\forall i, j \in I)(\exists k \in I)([\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_k), \tag{1.2}$$

where I is an index set, and we denote

$$[\mathcal{L}_i, \mathcal{L}_j] := \left\{ [x, y] \middle| x \in \mathcal{L}_i, y \in \mathcal{L}_j \right\}. \tag{1.3}$$

Subspaces $\mathcal{L}_i, i \in I$ are then called **grading subspaces**.

Gradings of Lie algebra $\mathcal{L}$ are closely related to the group of automorphisms $\operatorname{Aut} \mathcal{L}$. Let us recall that a regular linear mapping g acting on $\mathcal{L}$, i.e. $g \in GL(\mathcal{L})$, is an **automorphism** of $\mathcal{L}$ if

$$g\,[X, Y] = [gX, gY] \tag{1.4}$$

holds for all $X, Y \in \mathcal{L}$. The relationship between two gradings can be described in the following way: if $\Gamma : \mathcal{L} = \bigoplus_{i \in I} \mathcal{L}_i$ is a grading of $\mathcal{L}$, then for an arbitrary automorphism $g \in \operatorname{Aut} \mathcal{L}$

$$\tilde{\Gamma} : \mathcal{L} = \bigoplus_{i \in I} g(\mathcal{L}_i)$$

is also a grading of $\mathcal{L}$. We call such gradings Γ and $\tilde{\Gamma}$ **equivalent**.

Grading $\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i$ is a **refinement** of grading $\tilde{\Gamma} : \mathcal{L} = \bigoplus_{j\in J} \tilde{\mathcal{L}}_j$ if for each $i \in I$ exists $j \in J$ such that $\mathcal{L}_i \subseteq \tilde{\mathcal{L}}_j$. Refinement is called **proper** if the cardinality of I is greater than the cardinality of the set J. Grading which cannot be properly refined is called **fine**. If all grading subspaces are one-dimensional then the grading is called **finest**.

The property (1.2) defines a binary operation on the set I. If $[\mathcal{L}_i, \mathcal{L}_j] = \{0\}$ holds, we can choose an arbitrary k. It is proved in [15] that this index set I with this operation can always be embedded into an *Abelian group* G; we are going to denote the operation additively as $+$ and we have

$$[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}\,, \quad \text{where } i, j, i+j \in G. \tag{1.5}$$

We say that the Lie algebra is graded by group G or it is ***G*-graded**. Group G is called a **grading group**.

1.2 The group $\operatorname{Aut}\mathcal{L}$ and gradings

This section describes the process of obtaining gradings and the correspondence between automorphisms and gradings. Let $g \in \operatorname{Aut}\mathcal{L}$ be a *diagonable* automorphism. Let X and Y be eigenvectors of g corresponding to (of course non–zero) eigenvalues λ and μ, e.g.

$$g\,X = \lambda X, \quad g\,Y = \mu Y.$$

Then we have

$$g\,[X, Y] = [gX, gY] = \lambda\mu[X, Y],$$

hence $[X, Y]$ is either eigenvector corresponding to the eigenvalue $\lambda\mu$ or zero vector. It follows that

$$[\mathcal{L}_\lambda, \mathcal{L}_\mu] \subseteq \mathcal{L}_{\lambda\mu}.$$

This means that the decomposition of $\mathcal{L}$ into the direct sum of eigenspaces of diagonable automorphism g is the grading of $\mathcal{L}$:

$$\mathcal{L} = \bigoplus_{i\in I} \operatorname{Ker}(g - \lambda_i\,\mathrm{id}), \tag{1.6}$$

where I is the set indexing all eigenvalues of g. If we take another diagonable automorphism h, which commutes with g, then there exist common eigenvectors (and eigenspaces) which determine the same or a finer grading. In this way every set of diagonable and mutually commuting automorphisms $g_1, g_2, \ldots, g_m \in \operatorname{Aut} \mathcal{L}$ determines some grading.

The maximal set of diagonable and mutually commuting automorphisms is in fact a subgroup of $\operatorname{Aut} \mathcal{L}$ called **MAD-group** (**m**aximal **A**belian group of **d**iagonable automorphisms). Conversely, each given grading (1.2) determines a subgroup $\operatorname{Diag} \Gamma \subset \operatorname{Aut} \mathcal{L}$ containing all automorphisms $g \in GL(\mathcal{L})$, which preserve Γ, $g(\mathcal{L}_i) = \mathcal{L}_i$, and are **diagonal**,

$$gx = \lambda_i x \quad \text{for all } x \in \mathcal{L}_i,\ i \in I,$$

where $\lambda_i \neq 0$ depends only on g and $i \in I$. In [15] an important theorem has been proved which for all simple Lie algebras classifies all their possible fine gradings.

Theorem 1.2.1. Let $\mathcal{L}$ be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero. Then the grading Γ is fine if and only if $\operatorname{Diag} \Gamma$ is equal to some MAD-group.

In this way, the problem of classification of all fine gradings of simple Lie algebras is converted to a classification of all MAD-groups in $\operatorname{Aut} \mathcal{L}$. Since classical simple Lie algebras are subalgebras of $gl(n, \mathbb{C})$, we will first investigate MAD-groups in $\operatorname{Aut} gl(n, \mathbb{C})$. Adding supplementary conditions one can obtain MAD-groups of other classical algebras.

Automorphisms of $gl(n, \mathbb{C})$ can be written as a combination of *inner* and *outer* automorphisms. For all $X \in gl(n, \mathbb{C})$,

inner automorphisms have the general form

$$\operatorname{Ad}_A X = A^{-1} X A \quad \text{where} \quad A \in GL(n, \mathbb{C}); \tag{1.7}$$

outer automorphisms have the general form

$$\operatorname{Out}_A X = -(A^{-1} X A)^T = \operatorname{Out}_I \operatorname{Ad}_A X, \quad \text{where} \quad A \in GL(n, \mathbb{C}). \tag{1.8}$$

We further convert the characteristics of automorphisms in MAD-groups to the characteristics of corresponding matrices in $GL(n, \mathbb{C})$. The properties of all inner and outer automorphisms are summarized in the following lemma [3]:

Proposition 1.2.2. Let $A, B, C \in GL(n, \mathbb{C})$. Then the following holds

(1) Ad_A is diagonable automorphism iff a matrix A is diagonable.

(2) Inner automorphisms commute, i.e. $\mathrm{Ad}_A\ \mathrm{Ad}_B = \mathrm{Ad}_B\ \mathrm{Ad}_A$, iff there exists $q \in \mathbb{C}$ such that

$$AB = qBA, \quad \text{where } q \text{ satisfies } \quad q^n = 1. \tag{1.9}$$

(3) Out_C is diagonable iff matrix $C(C^T)^{-1}$ is diagonable.

(4) Inner and outer automorphisms commute, i.e. $\mathrm{Ad}_A\ \mathrm{Out}_C = \mathrm{Out}_C\ \mathrm{Ad}_A$ iff

$$ACA^T = rC\,.$$

Since $\mathrm{Ad}_{\alpha A} = \mathrm{Ad}_A$ for $\alpha \neq 0$, the number r can be normalized to unity.

1.3 The Pauli grading of $sl(n, \mathbb{C})$

Since this section deals only with MAD-groups without outer automorphisms, let us assume that $\mathcal{G} \subset \mathrm{Aut}\, gl(n, \mathbb{C})$ is MAD-group without outer automorphism. Let us consider the set of corresponding matrices in $GL(n, \mathbb{C})$

$$G := \{A \in GL(n, \mathbb{C}) \mid \mathrm{Ad}_A \in \mathcal{G}\} \tag{1.10}$$

and we have indeed

$$\mathcal{G} = \mathrm{Ad}\, G := \{\mathrm{Ad}_A \mid A \in G\}\,. \tag{1.11}$$

According to (1) and (2) of lemma 1.2.2 G is a maximal set of diagonable matrices in $GL(n, \mathbb{C})$ such that $AB = q(A, B)BA$ for all $A, B \in G$. This leads us to the following definition: a subgroup of diagonable matrices $G \subset GL(n, \mathbb{C})$ is called **Ad-group** if

(i) For all $A, B \in G$ the commutator $q(A, B) = ABA^{-1}B^{-1}$ is a non zero multiple of identity matrix, i.e. it belongs to the center $Z = \{\alpha I_n | \alpha \in \mathbb{C} \setminus \{0\}\} \subset GL(n, \mathbb{C})$

(ii) G is maximal, $(\forall M \notin G)(\exists A \in G)(q(A, M) \notin Z)$.

To each MAD-group in $\mathrm{Aut}\, gl(n, \mathbb{C})$ without outer automorphisms there corresponds (according to (1.10)) an Ad-group in $GL(n, \mathbb{C})$ and similarly *vice versa*, according to the formula (1.11) to each Ad-group in $GL(n, \mathbb{C})$ corresponds a MAD-group without outer automorphisms.

In order to describe Ad-groups in $GL(n,\mathbb{C})$ we introduce the following notation. Subgroup in $GL(n,\mathbb{C})$ containing all regular diagonal matrices is denoted $D(n)$. We define also $k \times k$ matrices

$$Q_k = \mathrm{diag}(1, \omega_k, \omega_k^2, \dots, \omega_k^{k-1}), \tag{1.12}$$

where $\omega_k = \exp(2\pi i/k)$, and matrix

$$P_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{1.13}$$

For $k = 1$ we set $Q_1 = P_1 = (1)$. For matrices P_k, Q_k

$$P_k Q_k = \omega_k Q_k P_k \tag{1.14}$$

holds, hence they satisfy (1.9) with $n = k$, $q = \omega_k$. The finite subgroup of $GL(k,\mathbb{C})$ of order k^3 defined as

$$\Pi_k = \{\omega_k^l Q_k^i P_k^j \mid i,j,l = 0,1,\dots,k-1\} \tag{1.15}$$

is called the **Pauli group**. Ad-groups in $GL(n,\mathbb{C})$ are classified by the following theorem proved in [4].

Theorem 1.3.1. $G \subset GL(n,\mathbb{C})$ is an Ad-group if and only if G is conjugated to one of the finite groups

$$\Pi_{\pi_1} \otimes \cdots \otimes \Pi_{\pi_s} \otimes D(n/\pi_1 \dots \pi_s),$$

where $\pi_1, \dots, \pi_s$ are powers of primes and their product $\pi_1 \dots \pi_s$ divides n, with the exception of the case $\Pi_2 \otimes \cdots \otimes \Pi_2 \otimes D(1)$.

Remark 1. Subgroups $H_1, H_2 \subset G$ are conjugated if there exists $g \in G$ such that $H_1 = gH_2g^{-1} = \{ghg^{-1} \mid h \in H_2\}$. Since $\mathrm{Ad}_A = \mathrm{Ad}_g \mathrm{Ad}_B \mathrm{Ad}_g^{-1}$ iff $A = gBg^{-1}$ holds for any $A, B, g \in GL(n,\mathbb{C})$, we see that conjugated Ad-groups correspond to conjugated MAD-groups which give equivalent gradings.

We are interested in the case when an Ad-group is equal exactly to Π_n. The corresponding MAD-group in $\mathrm{Aut}\, gl(n,\mathbb{C})$ is clearly of order n^2 (matrices which differ only by a multiplier give equal automorphism due to (1.7))

$$\mathrm{Ad}\,\Pi_n = \{\mathrm{Ad}_{Q^i P^j} | (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n\}. \tag{1.16}$$

The fine grading of $gl(n,\mathbb{C})$ corresponding to this MAD-group is, according to [14], given by

$$gl(n,\mathbb{C}) = \bigoplus_{(r,s)\in\mathbb{Z}_n\times\mathbb{Z}_n} \mathcal{L}_{rs}, \tag{1.17}$$

where $\mathcal{L}_{rs} := \{X_{rs}\}_{lin}$ and

$$X_{rs} = {Q_n}^r {P_n}^s\,. \tag{1.18}$$

This grading is in fact finest, i.e. all $n^2 = \dim gl(n,\mathbb{C})$ subspaces are one-dimensional. We can easily check that (1.17) is indeed a grading by verification of the property (1.2); (henceforth the explicit notation of dimension P_n, Q_n and for algebraic operations *mod n* will be omitted)

$$[X_{rs}, X_{r's'}] = Q^r P^s Q^{r'} P^{s'} - Q^{r'} P^{s'} Q^r P^s = (\omega^{sr'} - \omega^{rs'}) X_{r+r',s+s'} \tag{1.19}$$

where relation

$$P^s Q^r = \omega^{sr} Q^r P^s \tag{1.20}$$

following from (1.14) was used. Hence we have that our grading group G is equal to the additive Abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$ with addition componentwise (*mod n*). For us it is important to notice that the result of the computation (1.19) is never the generator X_{00} for $(r,s) \neq (0,0)$ and $(r',s') \neq (0,0)$; namely $r + r' \;(mod\, n) = 0,\; s + s' \;(mod\, n) = 0$ lead to

$$[X_{rs}, X_{-r-s}] = 0 \cdot X_{00} = \{0\}.$$

Since for all matrices X_{rs} except X_{00}

$$\mathrm{tr}\, X_{rs} = 0, \;\; (r,s) \neq (0,0)$$

holds, we can state that these $n^2 - 1$ matrices yield a grading of $sl(n,\mathbb{C})$:

$$sl(n,\mathbb{C}) = \bigoplus_{(r,s)\in\mathbb{Z}_n\times\mathbb{Z}_n\setminus(0,0)} \mathcal{L}_{rs} \tag{1.21}$$

This grading of $sl(n,\mathbb{C})$ is called the **Pauli grading**.

1.4 Four gradings of $sl(3,\mathbb{C})$

According to [3] Aut $sl(3,\mathbb{C})$ has four non-conjugate MAD-groups and therefore four inequivalent fine gradings. They will be listed according to [7],[8]. First we list the grading group G and then the corresponding grading Γ. The symbol for linear span in the notation involving explicit matrices is omitted.

$G_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$

$$\begin{aligned}\Gamma_1 : sl(3,\mathbb{C}) &= \mathcal{L}_{00} \oplus \mathcal{L}_{10} \oplus \mathcal{L}_{01} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{-1-1} \oplus \mathcal{L}_{0-1} \oplus \mathcal{L}_{-10} \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \\ &\oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned} \tag{1.22}$$

$G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{aligned}\Gamma_2 : sl(3,\mathbb{C}) &= \mathcal{L}_{001} \oplus \mathcal{L}_{111} \oplus \mathcal{L}_{101} \oplus \mathcal{L}_{011} \oplus \mathcal{L}_{110} \oplus \mathcal{L}_{010} \oplus \mathcal{L}_{100} \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \oplus \\ &\oplus \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\end{aligned} \tag{1.23}$$

$G_3 = \mathbb{Z}_8$

$$\begin{aligned}\Gamma_3 : sl(3,\mathbb{C}) &= \mathcal{L}_{0} \oplus \mathcal{L}_{1} \oplus \mathcal{L}_{2} \oplus \mathcal{L}_{3} \oplus \mathcal{L}_{4} \oplus \mathcal{L}_{5} \oplus \mathcal{L}_{6} \oplus \mathcal{L}_{7} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \\ &\oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned} \tag{1.24}$$

$G_4 = \mathbb{Z}_3 \times \mathbb{Z}_3$

$$
\begin{aligned}
\Gamma_4 : sl(3,\mathbb{C}) &= \mathcal{L}_{01} \oplus \mathcal{L}_{02} \oplus \mathcal{L}_{10} \oplus \mathcal{L}_{20} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{22} \oplus \mathcal{L}_{12} \oplus \mathcal{L}_{21} \\
&= Q \oplus Q^2 \oplus P \oplus P^2 \oplus PQ \oplus P^2Q^2 \oplus PQ^2 \oplus P^2Q \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \\
&\oplus \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}
\end{aligned}
\tag{1.25}
$$

where $\omega = e^{\frac{2\pi i}{3}}$ and matrices P, Q are defined by the formulas (1.13) and (1.12) for the case $n = 3$. The grading Γ_1 is the Cartan decomposition of $sl(3,\mathbb{C})$ and Γ_4 is indeed the Pauli grading of $sl(3,\mathbb{C})$.

Chapter 2

Graded contractions of Lie algebras

2.1 Definitions and basic properties

Let us state the definition of a graded contraction. Suppose $\mathcal{L}$ is a Lie algebra graded by the group G, i.e. the relations (1.1) and (1.5) hold. We make use of the following notation determining which $[\mathcal{L}_i, \mathcal{L}_j]$ is zero or non-zero. This information is encoded in the matrix $\kappa := (\kappa_{ij})$ defined as

$$\begin{aligned} \kappa_{ij} &= 0 \quad \text{if} \quad [\mathcal{L}_i, \mathcal{L}_j] = \{0\}, \\ \kappa_{ij} &= 1 \quad \text{if} \quad [\mathcal{L}_i, \mathcal{L}_j] \neq \{0\}. \end{aligned} \tag{2.1}$$

The matrix κ is of course symmetric and of order $k \times k$, where k is the number of grading subspaces.

We define a bilinear mapping $[\ ,\ ]_\gamma$ on $\mathcal{L}$ (more precisely on the underlying vector space V) by the formula

$$[x, y]_\gamma := \gamma_{ij}[x, y] \quad \text{for all} \quad x \in \mathcal{L}_i,\ y \in \mathcal{L}_j,\ i, j \in I \text{ and } \gamma_{ij} \in \mathbb{C}. \tag{2.2}$$

Since we claim the bilinearity of $[\ ,\ ]_\gamma$, the condition (2.2) determines this mapping on the whole V. If we introduce the **contraction parameters** ε_{ij} via the equation (with no summation implied)

$$\varepsilon_{ij} := \gamma_{ij}\kappa_{ij}\ , \tag{2.3}$$

we can as well write for all $x \in \mathcal{L}_i,\ y \in \mathcal{L}_j,\ i, j \in I$

$$[x, y]_\gamma = [x, y]_\varepsilon = \varepsilon_{ij}[x, y] \tag{2.4}$$

because if the subspaces $\mathcal{L}_i$, $\mathcal{L}_j$ commute, $[\mathcal{L}_i, \mathcal{L}_j] = \{0\}$, then it is always $[\mathcal{L}_i, \mathcal{L}_j]_\gamma = \{0\}$ independently of γ_{ij}. Henceforth we will mostly deal with the variables ε which have zeros on the *irrelevant positions* such that $[\mathcal{L}_i, \mathcal{L}_j] = \{0\}$. If $\mathcal{L}^\varepsilon := (V, [\ ,\ ]_\varepsilon)$ is a Lie algebra, then it is called the **graded contraction** of the Lie algebra $\mathcal{L}$. Note that the contraction preserves a grading because it is also true that

$$\mathcal{L}^\varepsilon = \bigoplus_{i \in G} \mathcal{L}_i \tag{2.5}$$

is a grading of $\mathcal{L}^\varepsilon$.

The two conditions (and their solutions) which the parameters ε_{ij} must fulfill for $\mathcal{L}^\varepsilon$ to become a Lie algebra will be considered in the following sections. The first condition of antisymmetry of $[\ ,\ ]_\varepsilon$ immediately gives

$$\varepsilon_{ij} = \varepsilon_{ji} \tag{2.6}$$

Hence each such solution can be written in the form of a *symmetric* matrix $\varepsilon = (\varepsilon_{ij})$ which is called the **contraction matrix**. The validity of the Jacobi identity is the second condition and it is equivalent to the property: for all (unordered) triples $i, j, k \in I$

$$e(i\ j\ k): \ [x, [y, z]_\varepsilon]_\varepsilon + [z, [x, y]_\varepsilon]_\varepsilon + [y, [z, x]_\varepsilon]_\varepsilon = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k). \tag{2.7}$$

is satisfied. Each $e(i\ j\ k)$, $i, j, k \in I$ is then called a **contraction equation**. Using the Jacobi identity in $\mathcal{L}$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k) \tag{2.8}$$

one can rewrite (2.7) in the form

$$\begin{aligned} e(i\ j\ k) : &(\varepsilon_{i,j+k}\varepsilon_{jk} - \varepsilon_{k,i+j}\varepsilon_{ij})[x, [y, z]] + (\varepsilon_{j,k+i}\varepsilon_{ki} - \varepsilon_{k,i+j}\varepsilon_{ij})[y, [z, x]] = 0 \\ &(\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k). \end{aligned}$$

In special cases, when there exist for example $x' \in \mathcal{L}_i$, $y' \in \mathcal{L}_j$, $z' \in \mathcal{L}_k$ such that $[x', [y', z']]$ and $[y', [z', x']]$ are linearly independent then equation (2.7) is in this case equivalent to 2 two-term equations

$$e(i\ j\ k): \ \varepsilon_{i,j+k}\varepsilon_{jk} = \varepsilon_{k,i+j}\varepsilon_{ij} = \varepsilon_{j,k+i}\varepsilon_{ki}. \tag{2.9}$$

Example 1. Let us write the contraction equation $e((01)(10)(31))$ for the Pauli grading of $sl(5, \mathbb{C})$:

$$e((01)(10)(31)): \ [x, [y, z]_\varepsilon]_\varepsilon + [z, [x, y]_\varepsilon]_\varepsilon + [y, [z, x]_\varepsilon]_\varepsilon = 0 \quad \forall x \in \mathcal{L}_{01}\, \forall y \in \mathcal{L}_{10}\, \forall z \in \mathcal{L}_{31}.$$

Since the subspaces are one-dimensional, we have

$$e((01)(10)(31)):\ [X_{01},[X_{10},X_{31}]_\varepsilon]_\varepsilon+[X_{31},[X_{01},X_{10}]_\varepsilon]_\varepsilon+[X_{10},[X_{31},X_{01}]_\varepsilon]_\varepsilon=0,$$

and using (1.19), where $\omega=e^{\frac{2\pi i}{5}}$, we obtain a three-term equation

$$e((01)(10)(31)):\varepsilon_{(01)(10)}\varepsilon_{(11)(31)}(\omega-1)(\omega^3-\omega)+\varepsilon_{(10)(31)}\varepsilon_{(41)(01)}(1-\omega)(1-\omega^4)+\\ +\varepsilon_{(31)(01)}\varepsilon_{(32)(10)}(1-\omega^3)(\omega^2-1)=0.$$

2.2 Normalization of contraction matrices

Let us introduce the so called normalization process for contraction matrices. At first we introduce **componentwise matrix multiplication** $\bullet$, i.e. for two matrices $A=(A_{ij})$, $B=(B_{ij})$ we define the matrix $C:=(C_{ij})$ by the formula

$$C_{ij}:=A_{ij}B_{ij}\qquad \text{(no summation)} \tag{2.10}$$

and write $C=A\bullet B$.

For the given grading (1.1) we define also a matrix $\alpha:=(\alpha_{ij})$, where

$$\alpha_{ij}=\frac{a_ia_j}{a_{i+j}}\quad \text{for } i,j\in I \tag{2.11}$$

and $a_i\in\mathbb{C}\setminus\{0\}$ for all $i\in I$. The matrix α is then called a **normalization matrix**. Normalization is a process based on the following lemma:

Lemma 2.2.1. Let $\mathcal{L}^\varepsilon$ be a graded contraction of a graded Lie algebra $\mathcal{L}=\bigoplus_{i\in I}\mathcal{L}_i$. Then $\mathcal{L}^{\widetilde{\varepsilon}}$, where $\widetilde{\varepsilon}:=\alpha\bullet\varepsilon$, is for any normalization matrix α a graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^{\widetilde{\varepsilon}}$ and $\mathcal{L}^\varepsilon$ are isomorphic, $\mathcal{L}^{\widetilde{\varepsilon}}\simeq\mathcal{L}^\varepsilon$.

Proof. We define a diagonal regular linear mapping $h\in GL(\mathcal{L})$ corresponding to a normalization matrix $\alpha=\left(\frac{a_ia_j}{a_{i+j}}\right)$ by the formula

$$hx_i=a_ix_i\quad i\in I,\ x_i\in\mathcal{L}_i\,. \tag{2.12}$$

Then for all $x\in\mathcal{L}_i$, $y\in\mathcal{L}_j$ the bilinear mapping $[x,y]_{\widetilde{\varepsilon}}=\widetilde{\varepsilon}_{ij}[x,y]$ and the Lie bracket $[x,y]_\varepsilon=\varepsilon_{ij}[x,y]$ satisfy

$$[x,y]_{\widetilde{\varepsilon}}=\widetilde{\varepsilon}_{ij}[x,y]=\frac{a_ia_j}{a_{i+j}}\varepsilon_{ij}[x,y]=h^{-1}[hx,hy]_\varepsilon.$$

Hence $\mathcal{L}^{\widetilde{\varepsilon}}$ is a Lie algebra and h is an isomorphism between $\mathcal{L}^{\widetilde{\varepsilon}}$ and $\mathcal{L}^\varepsilon$. □

Practically, we set the matrix $\widetilde{\varepsilon}$ and then we try to normalize a given matrix ε to $\widetilde{\varepsilon}$, i.e. find such a normalization matrix α which satisfies the *normalization equations* $\widetilde{\varepsilon} = \alpha \bullet \varepsilon$. In most cases it is possible to normalize in this way the matrix ε to the matrix which consists of only 0's and 1's.

2.3 Contraction system for the Pauli grading of $sl(3,\mathbb{C})$

This section contains contraction equations for the Pauli grading of $sl(3,\mathbb{C})$, i.e. such equations for the variables ε_{ij} which must be fulfilled in order that $\mathcal{L}^\varepsilon$ be a Lie algebra. Generally, the set of all these contraction equations is called the **contraction system** $\mathcal{S}$, the set of its solutions will be denoted $\mathcal{R}(\mathcal{S})$; for the Pauli grading of $sl(n,\mathbb{C})$ we denote the contraction system $\mathcal{S}_n$. Let us take the Pauli grading in the form (1.25), i.e.

$$sl(3,\mathbb{C}) = \mathcal{L}_{01} \oplus \mathcal{L}_{02} \oplus \mathcal{L}_{10} \oplus \mathcal{L}_{20} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{22} \oplus \mathcal{L}_{12} \oplus \mathcal{L}_{21}\,, \tag{2.13}$$

where $\mathcal{L}_{ij} = \{X_{ij}\}_{lin}$ and $X_{01} = \{Q\}$, $X_{02} = \{Q^2\}$, $X_{10} = \{P\}$, $X_{20} = \{P^2\}$, $X_{11} = \{PQ\}$, $X_{22} = \{P^2Q^2\}$, $X_{12} = \{PQ^2\}$, $X_{21} = \{P^2Q\}$. The grading group is $\mathbb{Z}_3 \times \mathbb{Z}_3$; no subspace is labeled by $(0,0)$. Let us state the explicit form of matrix κ defined by (2.1); it is 8×8 symmetric matrix and we will order the indices as in formula (2.13), i.e. positions 11, 12, 13, ... are denoted (01)(01), (01)(02), (01)(10), ... and on each of these positions zero or unity is found depending on $[\mathcal{L}_{01}, \mathcal{L}_{01}]$, $[\mathcal{L}_{01}, \mathcal{L}_{02}]$, $[\mathcal{L}_{01}, \mathcal{L}_{10}]$, ... We have

$$\kappa = \begin{pmatrix} 0&0&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1\\ 1&1&0&0&1&1&1&1\\ 1&1&0&0&1&1&1&1\\ 1&1&1&1&0&0&1&1\\ 1&1&1&1&0&0&1&1\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&0&0 \end{pmatrix}. \tag{2.14}$$

We can see immediately that our task of finding all graded contractions for the Pauli grading of $sl(3,\mathbb{C})$ has 24 relevant contraction parameters. For $\varepsilon \in \mathcal{R}(\mathcal{S})$ the **number of zeros** among these 24 parameters is denoted by the symbol $\nu(\varepsilon)$. The symmetric contraction matrix ε with 24 variables is of the following general form

$$\varepsilon = \begin{pmatrix} 0 & 0 & \varepsilon_{(01)(10)} & \varepsilon_{(01)(20)} & \varepsilon_{(01)(11)} & \varepsilon_{(01)(22)} & \varepsilon_{(01)(12)} & \varepsilon_{(01)(21)} \\ 0 & 0 & \varepsilon_{(02)(10)} & \varepsilon_{(02)(20)} & \varepsilon_{(02)(11)} & \varepsilon_{(02)(22)} & \varepsilon_{(02)(12)} & \varepsilon_{(02)(21)} \\ \varepsilon_{(01)(10)} & \varepsilon_{(02)(10)} & 0 & 0 & \varepsilon_{(10)(11)} & \varepsilon_{(10)(22)} & \varepsilon_{(10)(12)} & \varepsilon_{(10)(21)} \\ \varepsilon_{(01)(20)} & \varepsilon_{(02)(20)} & 0 & 0 & \varepsilon_{(20)(11)} & \varepsilon_{(20)(22)} & \varepsilon_{(20)(12)} & \varepsilon_{(20)(21)} \\ \varepsilon_{(01)(11)} & \varepsilon_{(02)(11)} & \varepsilon_{(10)(11)} & \varepsilon_{(20)(11)} & 0 & 0 & \varepsilon_{(11)(12)} & \varepsilon_{(11)(21)} \\ \varepsilon_{(01)(22)} & \varepsilon_{(02)(22)} & \varepsilon_{(10)(22)} & \varepsilon_{(20)(22)} & 0 & 0 & \varepsilon_{(22)(12)} & \varepsilon_{(22)(21)} \\ \varepsilon_{(01)(12)} & \varepsilon_{(02)(12)} & \varepsilon_{(10)(12)} & \varepsilon_{(20)(12)} & \varepsilon_{(11)(12)} & \varepsilon_{(22)(12)} & 0 & 0 \\ \varepsilon_{(01)(21)} & \varepsilon_{(02)(21)} & \varepsilon_{(10)(21)} & \varepsilon_{(20)(21)} & \varepsilon_{(11)(21)} & \varepsilon_{(22)(21)} & 0 & 0 \end{pmatrix}. \quad (2.15)$$

The antisymmetry of the new product $[X_{kl}, X_{mn}]_N = \varepsilon_{(kl)(mn)}[X_{kl}, X_{mn}]$ for $sl(3,\mathbb{C})$ is granted due to the symmetry of the matrix ε. It is sufficient to verify the Jacobi identities on vectors of a basis; the vectors $X_{ij}, (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \setminus (0,0)$ form the basis of $sl(3,\mathbb{C})$ and the Jacobi identities are in the form

$$[X_{ij}, [X_{kl}, X_{mn}]_N]_N + [X_{mn}, [X_{ij}, X_{kl}]_N]_N + [X_{kl}, [X_{mn}, X_{ij}]_N]_N = 0. \quad (2.16)$$

These equations should hold for all possible triples of indices $(ij), (kl), (mn)$. It is clear that for a triple where two indices are identical the equation is automatically fulfilled; the equation also does not depend on the ordering of the triples. The number of equations is then equal to a the combination number $\binom{8}{3} = 56$. Modifying (2.16) we obtain

$$\varepsilon_{(ij)(k+m,l+n)}\varepsilon_{(kl)(mn)}[X_{ij}, [X_{kl}, X_{mn}]] + \quad \text{cyclically} \quad = 0, \quad (2.17)$$

where the word cyclically means that the two remaining terms are obtained from the first one by the substitutions: $(ij) \mapsto (mn)$, $(kl) \mapsto (ij)$, $(mn) \mapsto (kl)$, and $(ij) \mapsto (kl)$, $(kl) \mapsto (mn)$, $(mn) \mapsto (ij)$ respectively. The commutation relations (1.19) give the double commutator

$$[X_{ij}, [X_{kl}, X_{mn}]] = (\omega^{lm} - \omega^{kn})(\omega^{j(k+m)} - \omega^{i(l+n)})X_{i+k+m,j+l+n}\,. \quad (2.18)$$

If we substitute this into (2.17), we get

$$\left[\,\varepsilon_{(ij)(k+m,l+n)}\varepsilon_{(kl)(mn)}(\omega^{lm} - \omega^{kn})(\omega^{j(k+m)} - \omega^{i(l+n)}) + \quad \text{cyclically} \quad\right] X_{i+k+m,j+l+n} = 0. \quad (2.19)$$

From (2.19) we see that equations for which $i+k+m = 0 \wedge j+l+n = 0$ holds will also be automatically fulfilled. This situation arises in eight cases. Hence the contraction system consists of 48 equations.

Let us state an example of a computation of (2.19) for the given triple, e.g. (01)(02)(10). The result is

$$[\varepsilon_{(02)(10)}\varepsilon_{(01)(12)}(\omega^2-1)(\omega-1)+0+\varepsilon_{(10)(01)}\varepsilon_{(02)(11)}(1-\omega)(\omega^2-1)]X_{10}=0. \qquad (2.20)$$

The final form of the other equations is similar to this one. All equations for $sl(3,\mathbb{C})$ are in form $MN = PQ$. The whole contraction system $\mathcal{S}_3$ is presented in Table 1. When writing contraction equations we took into account that ε is a symmetric matrix; for example, instead of $\varepsilon_{(10)(01)}$ we equivalently write $\varepsilon_{(01)(10)}$. Note that for each equation the corresponding triple $(ij)(kl)(mn)$ of subspaces is given. The significance of matrices from $SL(2,\mathbb{Z}_3)$ will be explained in the next chapter in connection with the symmetry group of the Pauli grading.

Table 1: The system $\mathcal{S}_3$ of contraction equations for the Pauli grading of $sl(3,\mathbb{C})$ (1/2)

Equation number		Subspaces	$SL(2,\mathbb{Z}_3)$
1	$\varepsilon_{(02)(10)}\varepsilon_{(01)(12)} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0$	*(01)(02)(10)*	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
2	$\varepsilon_{(02)(20)}\varepsilon_{(01)(22)} - \varepsilon_{(01)(20)}\varepsilon_{(02)(21)} = 0$	*(01)(02)(20)*	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$
3	$\varepsilon_{(02)(11)}\varepsilon_{(01)(10)} - \varepsilon_{(01)(11)}\varepsilon_{(02)(12)} = 0$	*(01)(02)(11)*	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$
4	$\varepsilon_{(02)(22)}\varepsilon_{(01)(21)} - \varepsilon_{(01)(22)}\varepsilon_{(02)(20)} = 0$	*(01)(02)(22)*	$\left(\begin{smallmatrix}2&2\\0&2\end{smallmatrix}\right)$
5	$\varepsilon_{(02)(12)}\varepsilon_{(01)(11)} - \varepsilon_{(01)(12)}\varepsilon_{(02)(10)} = 0$	*(01)(02)(12)*	$\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$
6	$\varepsilon_{(02)(21)}\varepsilon_{(01)(20)} - \varepsilon_{(01)(21)}\varepsilon_{(02)(22)} = 0$	*(01)(02)(21)*	$\left(\begin{smallmatrix}2&1\\0&2\end{smallmatrix}\right)$
7	$\varepsilon_{(10)(21)}\varepsilon_{(01)(12)} - \varepsilon_{(10)(12)}\varepsilon_{(22)(21)} = 0$	*(10)(12)(21)*	$\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$
8	$\varepsilon_{(20)(21)}\varepsilon_{(11)(12)} - \varepsilon_{(20)(12)}\varepsilon_{(02)(21)} = 0$	*(12)(20)(21)*	$\left(\begin{smallmatrix}2&0\\1&2\end{smallmatrix}\right)$
9	$\varepsilon_{(10)(22)}\varepsilon_{(02)(11)} - \varepsilon_{(10)(11)}\varepsilon_{(22)(21)} = 0$	*(10)(11)(22)*	$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$
10	$\varepsilon_{(20)(22)}\varepsilon_{(11)(12)} - \varepsilon_{(20)(11)}\varepsilon_{(01)(22)} = 0$	*(11)(20)(22)*	$\left(\begin{smallmatrix}2&0\\2&2\end{smallmatrix}\right)$
11	$\varepsilon_{(01)(20)}\varepsilon_{(10)(21)} - \varepsilon_{(01)(10)}\varepsilon_{(20)(11)} = 0$	*(01)(10)(20)*	$\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)$
12	$\varepsilon_{(02)(20)}\varepsilon_{(10)(22)} - \varepsilon_{(02)(10)}\varepsilon_{(20)(12)} = 0$	*(02)(10)(20)*	$\left(\begin{smallmatrix}0&2\\1&0\end{smallmatrix}\right)$
13	$\varepsilon_{(01)(21)}\varepsilon_{(22)(12)} - \varepsilon_{(01)(12)}\varepsilon_{(10)(21)} = 0$	*(01)(12)(21)*	$\left(\begin{smallmatrix}0&1\\2&1\end{smallmatrix}\right)$
14	$\varepsilon_{(02)(21)}\varepsilon_{(20)(12)} - \varepsilon_{(11)(21)}\varepsilon_{(02)(12)} = 0$	*(02)(12)(21)*	$\left(\begin{smallmatrix}0&2\\1&2\end{smallmatrix}\right)$
15	$\varepsilon_{(01)(22)}\varepsilon_{(20)(11)} - \varepsilon_{(01)(11)}\varepsilon_{(22)(12)} = 0$	*(01)(11)(22)*	$\left(\begin{smallmatrix}0&1\\2&2\end{smallmatrix}\right)$
16	$\varepsilon_{(02)(22)}\varepsilon_{(11)(21)} - \varepsilon_{(02)(11)}\varepsilon_{(10)(22)} = 0$	*(02)(11)(22)*	$\left(\begin{smallmatrix}0&2\\1&1\end{smallmatrix}\right)$
17	$\varepsilon_{(20)(11)}\varepsilon_{(01)(10)} - \varepsilon_{(10)(11)}\varepsilon_{(20)(21)} = 0$	*(10)(11)(20)*	$\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)$
18	$\varepsilon_{(20)(22)}\varepsilon_{(10)(12)} - \varepsilon_{(10)(22)}\varepsilon_{(02)(20)} = 0$	*(10)(20)(22)*	$\left(\begin{smallmatrix}2&2\\1&0\end{smallmatrix}\right)$
19	$\varepsilon_{(20)(21)}\varepsilon_{(10)(11)} - \varepsilon_{(10)(21)}\varepsilon_{(01)(20)} = 0$	*(10)(20)(21)*	$\left(\begin{smallmatrix}2&1\\2&0\end{smallmatrix}\right)$
20	$\varepsilon_{(20)(12)}\varepsilon_{(02)(10)} - \varepsilon_{(10)(12)}\varepsilon_{(20)(22)} = 0$	*(10)(12)(20)*	$\left(\begin{smallmatrix}1&2\\1&0\end{smallmatrix}\right)$
21	$\varepsilon_{(11)(21)}\varepsilon_{(02)(12)} - \varepsilon_{(11)(12)}\varepsilon_{(20)(21)} = 0$	*(11)(12)(21)*	$\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$
22	$\varepsilon_{(22)(21)}\varepsilon_{(10)(12)} - \varepsilon_{(22)(12)}\varepsilon_{(01)(21)} = 0$	*(12)(21)(22)*	$\left(\begin{smallmatrix}2&2\\2&1\end{smallmatrix}\right)$
23	$\varepsilon_{(22)(12)}\varepsilon_{(01)(11)} - \varepsilon_{(11)(12)}\varepsilon_{(20)(22)} = 0$	*(11)(12)(22)*	$\left(\begin{smallmatrix}1&2\\2&2\end{smallmatrix}\right)$
24	$\varepsilon_{(22)(21)}\varepsilon_{(10)(11)} - \varepsilon_{(11)(21)}\varepsilon_{(02)(22)} = 0$	*(11)(21)(22)*	$\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$

Table 1: The system $\mathcal{S}_3$ of contraction equations for the Pauli grading of $sl(3,\mathbb{C})$ (2/2)

Equation number		Subspaces	$SL(2,\mathbb{Z}_3)$
25	$\varepsilon_{(10)(11)}\varepsilon_{(01)(21)} - \varepsilon_{(01)(11)}\varepsilon_{(10)(12)} = 0$	*(01)(10)(11)*	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$
26	$\varepsilon_{(20)(22)}\varepsilon_{(02)(12)} - \varepsilon_{(02)(22)}\varepsilon_{(20)(21)} = 0$	*(02)(20)(22)*	$\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)$
27	$\varepsilon_{(11)(12)}\varepsilon_{(01)(20)} - \varepsilon_{(01)(12)}\varepsilon_{(10)(11)} = 0$	*(01)(11)(12)*	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$
28	$\varepsilon_{(22)(21)}\varepsilon_{(02)(10)} - \varepsilon_{(02)(21)}\varepsilon_{(20)(22)} = 0$	*(02)(21)(22)*	$\left(\begin{smallmatrix}2&2\\0&2\end{smallmatrix}\right)$
29	$\varepsilon_{(10)(12)}\varepsilon_{(01)(22)} - \varepsilon_{(01)(10)}\varepsilon_{(11)(12)} = 0$	*(01)(10)(12)*	$\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$
30	$\varepsilon_{(20)(21)}\varepsilon_{(02)(11)} - \varepsilon_{(02)(20)}\varepsilon_{(22)(21)} = 0$	*(02)(20)(21)*	$\left(\begin{smallmatrix}2&1\\0&2\end{smallmatrix}\right)$
31	$\varepsilon_{(01)(21)}\varepsilon_{(10)(22)} - \varepsilon_{(01)(10)}\varepsilon_{(11)(21)} = 0$	*(01)(10)(21)*	$\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$
32	$\varepsilon_{(02)(20)}\varepsilon_{(22)(12)} - \varepsilon_{(02)(12)}\varepsilon_{(20)(11)} = 0$	*(02)(12)(20)*	$\left(\begin{smallmatrix}2&0\\1&2\end{smallmatrix}\right)$
33	$\varepsilon_{(11)(21)}\varepsilon_{(02)(10)} - \varepsilon_{(10)(21)}\varepsilon_{(01)(11)} = 0$	*(10)(11)(21)*	$\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$
34	$\varepsilon_{(22)(12)}\varepsilon_{(01)(20)} - \varepsilon_{(20)(12)}\varepsilon_{(02)(22)} = 0$	*(12)(20)(22)*	$\left(\begin{smallmatrix}2&0\\2&2\end{smallmatrix}\right)$
35	$\varepsilon_{(20)(21)}\varepsilon_{(01)(11)} - \varepsilon_{(01)(21)}\varepsilon_{(20)(22)} = 0$	*(01)(20)(21)*	$\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)$
36	$\varepsilon_{(10)(12)}\varepsilon_{(02)(22)} - \varepsilon_{(02)(12)}\varepsilon_{(10)(11)} = 0$	*(02)(10)(12)*	$\left(\begin{smallmatrix}0&2\\1&0\end{smallmatrix}\right)$
37	$\varepsilon_{(22)(21)}\varepsilon_{(01)(10)} - \varepsilon_{(01)(22)}\varepsilon_{(20)(21)} = 0$	*(01)(21)(22)*	$\left(\begin{smallmatrix}0&1\\2&1\end{smallmatrix}\right)$
38	$\varepsilon_{(11)(12)}\varepsilon_{(02)(20)} - \varepsilon_{(02)(11)}\varepsilon_{(10)(12)} = 0$	*(02)(11)(12)*	$\left(\begin{smallmatrix}0&2\\1&2\end{smallmatrix}\right)$
39	$\varepsilon_{(20)(22)}\varepsilon_{(01)(12)} - \varepsilon_{(01)(20)}\varepsilon_{(22)(21)} = 0$	*(01)(20)(22)*	$\left(\begin{smallmatrix}0&1\\2&2\end{smallmatrix}\right)$
40	$\varepsilon_{(10)(11)}\varepsilon_{(02)(21)} - \varepsilon_{(02)(10)}\varepsilon_{(11)(12)} = 0$	*(02)(10)(11)*	$\left(\begin{smallmatrix}0&2\\1&1\end{smallmatrix}\right)$
41	$\varepsilon_{(01)(20)}\varepsilon_{(11)(21)} - \varepsilon_{(01)(11)}\varepsilon_{(20)(12)} = 0$	*(01)(11)(20)*	$\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)$
42	$\varepsilon_{(02)(22)}\varepsilon_{(10)(21)} - \varepsilon_{(02)(10)}\varepsilon_{(22)(12)} = 0$	*(02)(10)(22)*	$\left(\begin{smallmatrix}2&2\\1&0\end{smallmatrix}\right)$
43	$\varepsilon_{(11)(21)}\varepsilon_{(02)(20)} - \varepsilon_{(20)(11)}\varepsilon_{(01)(21)} = 0$	*(11)(20)(21)*	$\left(\begin{smallmatrix}2&1\\2&0\end{smallmatrix}\right)$
44	$\varepsilon_{(22)(12)}\varepsilon_{(01)(10)} - \varepsilon_{(10)(22)}\varepsilon_{(02)(12)} = 0$	*(10)(12)(22)*	$\left(\begin{smallmatrix}1&2\\1&0\end{smallmatrix}\right)$
45	$\varepsilon_{(20)(12)}\varepsilon_{(02)(11)} - \varepsilon_{(20)(11)}\varepsilon_{(01)(12)} = 0$	*(11)(12)(20)*	$\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$
46	$\varepsilon_{(10)(22)}\varepsilon_{(02)(21)} - \varepsilon_{(10)(21)}\varepsilon_{(01)(22)} = 0$	*(10)(21)(22)*	$\left(\begin{smallmatrix}2&2\\2&1\end{smallmatrix}\right)$
47	$\varepsilon_{(01)(22)}\varepsilon_{(20)(12)} - \varepsilon_{(01)(12)}\varepsilon_{(10)(22)} = 0$	*(01)(12)(22)*	$\left(\begin{smallmatrix}1&2\\2&2\end{smallmatrix}\right)$
48	$\varepsilon_{(02)(21)}\varepsilon_{(20)(11)} - \varepsilon_{(02)(11)}\varepsilon_{(10)(21)} = 0$	*(02)(11)(21)*	$\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$

Chapter 3

Symmetries and graded contractions

3.1 Symmetry group of a grading

3.1.1 Definitions and statements

The contraction system for the Pauli grading of $sl(3,\mathbb{C})$ is the system of 48 quadratic polynomials in 24 variables. Employing the symmetries, the solution of such system could be less complicated. The system $\mathcal{S}$ contains transformed Jacobi identities; therefore we begin with the following consideration. We define the **symmetry group** $\mathrm{Aut}\,\Gamma$ of a grading

$$\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i \tag{3.1}$$

as such a subgroup of $\mathrm{Aut}\,\mathcal{L}$ which contains automorphisms g with the property

$$g\mathcal{L}_i = \mathcal{L}_{\pi_g(i)}, \tag{3.2}$$

where $\pi_g : I \to I$ is a permutation of the index set I. Thus, a permutation representation Δ_Γ of the group $\mathrm{Aut}\,\Gamma$ is given on the set I, defined as

$$\Delta_\Gamma(g) := \pi_g\,. \tag{3.3}$$

The kernel of this representation is the **stabilizer** of Γ in $\mathrm{Aut}\,\Gamma$,

$$\mathrm{Stab}\ \Gamma = \ker \Delta_\Gamma = \{g \in \mathrm{Aut}\ \mathcal{L} \mid g\mathcal{L}_i = \mathcal{L}_i\ \ \forall i \in I\}. \tag{3.4}$$

Hence the stabilizer is a normal subgroup of $\mathrm{Aut}\,\Gamma$ and, according to the isomorphism theorem for groups, we have

$$\mathrm{Aut}\ \Gamma/\mathrm{Stab}\ \Gamma \simeq \Delta_\Gamma\ \mathrm{Aut}\ \Gamma. \tag{3.5}$$

This permutation group Δ_Γ Aut Γ is crucial for solving the system $\mathcal{S}$. To determine it we now use relation (3.5). For fine gradings corresponding to the MAD-group $\mathcal{G}$ we have $\operatorname{Stab}\Gamma = \mathcal{G}$. We define the **normalizer** of a MAD-group $\mathcal{G}$ as a subgroup

$$\mathcal{N}(\mathcal{G}) := \left\{ h \in \operatorname{Aut}\mathcal{L} \mid h^{-1}\mathcal{G}h \subset \mathcal{G} \right\}. \tag{3.6}$$

Let us take $h \in \mathcal{N}(\mathcal{G})$ and the subspace $\mathcal{L}_i$ of the *fine* grading (3.1) corresponding to the MAD-group $\mathcal{G}$. Then for each $f \in \mathcal{G}$ there exists $g \in \mathcal{G}$ such that $h^{-1}fh = g$. Since $g\mathcal{L}_i = \mathcal{L}_i$ holds we see that $f(h\mathcal{L}_i) = h\mathcal{L}_i$. This means that $h\mathcal{L}_i$ is the eigenspace of $f \in \mathcal{G}$ and $h\mathcal{L}_i = \mathcal{L}_j$ must hold for some $j \in I$. We have shown the inclusion $\mathcal{N}(\mathcal{G}) \subset \operatorname{Aut}\Gamma$. *Vice versa*, $h\mathcal{L}_i = \mathcal{L}_{\pi_h(i)}$ holds for $h \in \operatorname{Aut}\Gamma$. Choosing arbitrary $f \in \mathcal{G}$ we see that

$$fh\mathcal{L}_i = f\mathcal{L}_{\pi_h(i)} = \mathcal{L}_{\pi_h(i)} = h\mathcal{L}_i$$

implies $h^{-1}fh\mathcal{L}_i = \mathcal{L}_i$ and $h^{-1}fh \in \mathcal{G}$, i.e. $h \in \mathcal{N}(\mathcal{G})$. Finally we have

$$\mathcal{N}(\mathcal{G}) = \operatorname{Aut}\Gamma \tag{3.7}$$

and we conclude due to (3.5):

Corollary 3.1.1.

$$\mathcal{N}(\mathcal{G})/\mathcal{G} \simeq \Delta_\Gamma \operatorname{Aut}\Gamma \tag{3.8}$$

3.1.2 Action of a symmetry group

We denote the set of **relevant unordered pairs of grading indices** as $\mathcal{I}$, i.e.

$$\mathcal{I} := \left\{ i\,j \,\middle|\, i,j \in I,\ [\mathcal{L}_i, \mathcal{L}_j] \neq \{0\} \right\}, \tag{3.9}$$

where $i\,j$ denotes an unordered pair. For the Pauli grading of $sl(n,\mathbb{C})$ we denote this set as $\mathcal{I}_n$. Analyzing relations (2.3), (2.1) and (1.19), we write explicitly

$$\mathcal{I}_n = \left\{ (ij)(kl) \,\middle|\, jk - il \neq 0 \,(\mathrm{mod}\ n),\ (ij),(kl) \in \mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(0,0)\} \right\}. \tag{3.10}$$

The set of **relevant contraction parameters** ε_{ij}, due to (2.6), can be written as $\mathcal{E} := \{\varepsilon_k,\ k \in \mathcal{I}\}$. For a permutation $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ and a contraction matrix $\varepsilon = (\varepsilon_{ij})$, the **action of π on a contraction matrix** $\varepsilon \mapsto \varepsilon^\pi$ is defined by

$$(\varepsilon^\pi)_{ij} := \varepsilon_{\pi(i)\pi(j)}. \tag{3.11}$$

We observe that **the action on variables** $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ is really the action on the set of relevant variables $\mathcal{E}$: if $\varepsilon_{ij} \in \mathcal{E}$, $[\mathcal{L}_i, \mathcal{L}_j] \neq \{0\}$ and $g \in \operatorname{Aut}\Gamma$, $\Delta_\Gamma(g) = \pi$, then $\{0\} \neq g[\mathcal{L}_i, \mathcal{L}_j] = [\mathcal{L}_{\pi(i)}, \mathcal{L}_{\pi(j)}]$ and $\varepsilon_{\pi(i)\pi(j)} \in \mathcal{E}$. Hence the matrix ε^π has zeros on the same irrelevant positions as the matrix ε.

Remark 2. Generally for a group G and a set $X \neq \emptyset$ the **left action** $\varphi : G \times X \mapsto X$ and the **right action** $\psi : G \times X \mapsto X$ of the group G on the set X satisfy

1. $\varphi(gh, x) = \varphi(g, \varphi(h, x)) \quad \varphi(e, x) = x \quad$ for all $x \in X$, $g, h \in G$
2. $\psi(gh, x) = \psi(h, \psi(g, x)) \quad \psi(e, x) = x \quad$ for all $x \in X$, $g, h \in G$,

where $e \in G$ is the unit element. For the action θ of the group G on the set X the relation $a \equiv b \Leftrightarrow (\exists g \in G)(\theta(g, a) = b)$ is an equivalence on X and the equivalence class corresponding to element $a \in X$

$$\{b \in X \mid (\exists g \in G)(b = \theta(g, a))\} \tag{3.12}$$

is called an **orbit** of $a \in X$. We will verify that (3.11) is a well-defined action of the group $\Delta_\Gamma \operatorname{Aut}\Gamma$ on the set $\mathcal{R}(\mathcal{S})$ of the contraction system solutions. However, we are going to make use of a different definition of equivalence on the solutions further on.

In order to verify now that we have defined a correct action on the set of contraction matrices, we state:

Lemma 3.1.2. Let $\mathcal{L}^\varepsilon$ be a graded contraction of a graded Lie algebra $\Gamma : \mathcal{L} = \bigoplus_{i \in I} \mathcal{L}_i$. Then $\mathcal{L}^{\varepsilon^\pi}$ is for any permutation $\pi \in \Delta_\Gamma$ Aut Γ a graded contraction of $\mathcal{L}$ and two Lie algebras $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$ are isomorphic, $\mathcal{L}^{\varepsilon^\pi} \simeq \mathcal{L}^\varepsilon$.

Proof. For given $\pi \in \Delta_\Gamma$ Aut Γ we take any $g \in \operatorname{Aut}\Gamma$ such that $\Delta_\Gamma(g) = \pi$. Consider

$$gx = z,\ gy = w \quad x \in \mathcal{L}_i,\ y \in \mathcal{L}_j,\ z \in \mathcal{L}_{\pi(i)},\ w \in \mathcal{L}_{\pi(j)},\ i, j \in I. \tag{3.13}$$

Then for all $x \in \mathcal{L}_i$, $y \in \mathcal{L}_j$ the bilinear mapping $[x, y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x, y]$ and the Lie bracket $[x, y]_\varepsilon = \varepsilon_{ij}[x, y]$ satisfy

$$[x, y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x, y] = \varepsilon_{\pi(i)\pi(j)} g^{-1}[z, w] = g^{-1}[gx, gy]_\varepsilon. \tag{3.14}$$

Hence $\mathcal{L}^{\varepsilon^\pi}$ is a Lie algebra and g is an isomorphism between $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$. $\square$

3.2 Symmetries of the contraction system

In other words, lemma 3.1.2 says that for a given contraction matrix ε it is possible to construct new contraction matrices ε^{π}, $\pi \in \Delta_{\Gamma} \operatorname{Aut} \Gamma$. Of course, the new matrices ε^{π} have to be the solutions of the contraction system. We obtained the substitution $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$, $\pi \in \Delta_{\Gamma} \operatorname{Aut} \Gamma$ under which the set of solutions of the contraction system is invariant. Now we can also define the **action of $\Delta_{\Gamma} \operatorname{Aut} \Gamma$ on the contraction system** $\mathcal{S}$: each equation of $\mathcal{S}$ is labeled by a triple of grading indices and we write $e(i\,j\,k) \in \mathcal{S}$ in the form

$$e(i\,j\,k) : \; [x, [y, z]_{\varepsilon}]_{\varepsilon} + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k); \tag{3.15}$$

then for each $\pi \in \Delta_{\Gamma} \operatorname{Aut} \Gamma$ we define the action

$$e(i\,j\,k) \;\mapsto\; e(\pi(i)\,\pi(j)\,\pi(k)). \tag{3.16}$$

Note that equation $e(\pi(i)\,\pi(j)\,\pi(k))$ can be written as

$$e(\pi(i)\,\pi(j)\,\pi(k)) : \; [gx, [gy, gz]_{\varepsilon}]_{\varepsilon} + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k), \tag{3.17}$$

where $g \in \operatorname{Aut} \Gamma$, $\Delta_{\Gamma}(g) = \pi \in \Delta_{\Gamma} \operatorname{Aut} \Gamma$. According to (3.14) this is equal to

$$g[x, [y, z]_{\varepsilon^{\pi}}]_{\varepsilon^{\pi}} + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k) \tag{3.18}$$

and (3.18) is satisfied if and only if

$$[x, [y, z]_{\varepsilon^{\pi}}]_{\varepsilon^{\pi}} + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k). \tag{3.19}$$

The equation (3.19) is precisely the equation (3.15) after the substitution $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$. In this way we have not only verified the invariance of the contraction system (up to equivalence of equations), but also have shown the method of its construction. Having a starting equation one can write a whole orbit of equations merely by substituting $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ till all $\pi \in \Delta_{\Gamma} \operatorname{Aut} \Gamma$ are exhausted. If we denote unordered k-tuple of grading indices $i_1, i_2, \ldots, i_k \in I$ as $i_1\, i_2 \ldots i_k$ and define the **action of $\Delta_{\Gamma} \operatorname{Aut} \Gamma$ on unordered k-tuples** as

$$i_1\, i_2\, \ldots i_k \;\mapsto\; \pi(i_1)\, \pi(i_2) \ldots \pi(i_k), \quad \pi \in \Delta_{\Gamma} \operatorname{Aut} \Gamma, \tag{3.20}$$

then it is clear that orbits of equations correspond to orbits of unordered triples of grading indices.

3.2.1 Equivalence of solutions

For the contraction system $\mathcal{S}$ of a graded Lie algebra (3.1) we denote the set of all its solutions as $\mathcal{R}(\mathcal{S})$. By combining lemma 2.2.1 and lemma 3.1.2 it is easy to see that an equivalence relation on the set $\mathcal{R}(\mathcal{S})$ naturally arises: two solutions $\varepsilon_1, \varepsilon_2 \in \mathcal{R}(\mathcal{S})$ are **equivalent**, $\varepsilon_1 \sim \varepsilon_2$, if there exists a normalization matrix α and $\pi \in \Delta_\Gamma \operatorname{Aut} \Gamma$ such that

$$\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi. \tag{3.21}$$

The reflexivity of the relation $\sim$ is clear. To check the symmetry and transitivity we note that if two solutions are equivalent there exist a diagonal mapping $h \in GL(V)$ defined via formula (2.12) and an automorphism $g \in \operatorname{Aut}\Gamma$, $\Delta_\Gamma(g) = \pi$ with the property (3.14) satisfying

$$gh[x,y]_{\varepsilon_1} = gh[x,y]_{\alpha\bullet\varepsilon_2^\pi} = g[hx,hy]_{\varepsilon_2^\pi} = [ghx,ghy]_{\varepsilon_2} \tag{3.22}$$

and *vice versa*, i.e. two solution $\varepsilon_1, \varepsilon_2$ are equivalent, $\varepsilon_1 \sim \varepsilon_2$, if and only if

$$gh[x,y]_{\varepsilon_1} = [ghx,ghy]_{\varepsilon_2} \tag{3.23}$$

holds for all $x \in \mathcal{L}_i$, $y \in \mathcal{L}_j$, $i,j \in I$. It is easy to see that

$$hg = gh^\pi \tag{3.24}$$

where h^π is the diagonal mapping defined by

$$h^\pi x = a_{\pi(i)} x \quad i \in I,\, x_i \in \mathcal{L}_i. \tag{3.25}$$

Modifying (3.23) we have

$$[h^{-1}g^{-1}x, h^{-1}g^{-1}y]_{\varepsilon_1} = h^{-1}g^{-1}[x,y]_{\varepsilon_2} \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j).$$

Using the relation $h^{-1}g^{-1} = g^{-1}(h^{\pi^{-1}})^{-1}$ which follows from (3.24) we have $\varepsilon_2 \sim \varepsilon_1$, i.e. we proved the symmetry of the relation $\sim$. The proof of transitivity can be carried out in a similar manner: if $\varepsilon_1 \sim \varepsilon_2$, $\varepsilon_2 \sim \varepsilon_3$, i.e. $\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi$, $\varepsilon_2 = \beta \bullet \varepsilon_3^\sigma$, then there exist diagonal mappings h, h' and automorphisms $g, g' \in \operatorname{Aut}\Gamma$ such that

$$\begin{aligned} gh[x,y]_{\varepsilon_1} &= [ghx,ghy]_{\varepsilon_2} \\ g'h'[x,y]_{\varepsilon_2} &= [g'h'x,g'h'y]_{\varepsilon_3} \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j). \end{aligned}$$

Hence using (3.24) we obtain

$$g'g(h')^{\pi}h[x,y]_{\varepsilon_1} = g'h'gh[x,y]_{\varepsilon_1} = [g'h'ghx, g'h'ghy]_{\varepsilon_3} = [g'g(h')^{\pi}hx, g'g(h')^{\pi}hy]_{\varepsilon_3}. \quad (3.26)$$

The diagonal mapping $(h')^{\pi}h$ and the automorphism $g'g$ then imply the equivalence $\varepsilon_1 \sim \varepsilon_3$.

We conclude with

Corollary 3.2.1. *Graded contractions corresponding to equivalent solutions are isomorphic.*

Proof. See (3.23). □

3.3 Symmetry group of the Pauli grading

In our case, when Γ is given by (1.21), the corresponding MAD-group is equal to $\mathrm{Ad}\,\Pi_n$. One can check the fact that $\mathrm{Stab}\,\Gamma = \mathrm{Ad}\,\Pi_n$ also directly,

$$\mathrm{Ad}_P\, X_{rs} = PQ^rP^sP^{-1} = \omega^r X_{rs}, \quad \mathrm{Ad}_Q\, X_{rs} = QQ^rP^sQ^{-1} = \omega^{-s} X_{rs}.$$

We introduce a finite matrix group

$$H_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z}_n,\ ad - bc = \pm 1 \bmod n \right\}. \quad (3.27)$$

This group contains the subgroup of matrices with determinant +1 called $SL(2,\mathbb{Z}_n)$. In [6] an important theorem is proved:

Theorem 3.3.1. The quotient group $\mathcal{N}(\mathrm{Ad}\,\Pi_n)/\mathrm{Ad}\,\Pi_n$ is isomorphic to the group H_n,

$$\mathcal{N}(\mathrm{Ad}\,\Pi_n)/\mathrm{Ad}\,\Pi_n \simeq H_n. \quad (3.28)$$

Denoting by π_A the permutation corresponding to the matrices $A \in H_n$, the action of π_A on the indices of the grading group $\mathbb{Z}_n \times \mathbb{Z}_n$ is given by

$$\pi_A\,(i\ j) = (i\ j)\,A, \quad (3.29)$$

where matrix multiplication modulo n is found on the right hand side.

Corollary 3.3.2. *The symmetry group of the Pauli grading of $sl(n,\mathbb{C})$ is isomorphic to the matrix group H_n.*

Chapter 4

Solution of $\mathcal{S}_3$

4.1 Simplification of $\mathcal{S}_3$

We have seen that in the case of the grading Γ_4 given explicitly by (1.25) the symmetry group $\Delta_\Gamma \operatorname{Aut}\Gamma$ is isomorphic to H_3. The matrix group H_3 has 48 elements and there exist exactly *two* 24-point orbits of grading indices triples. We can choose the triples *(01)(02)(10)* and *(01)(10)(11)* as representative elements of these orbits. Moreover, all 24 elements from each orbit can be obtained by the action of 24 elements from $SL(2,\mathbb{Z}_3) \subset H_3$ starting from an arbitrary point. Then for our choice of representative points our system $\mathcal{S}_3$ can be written elegantly as

$$\mathcal{S}_3^a : \varepsilon_{(02)(10)A}\varepsilon_{(01)(12)A} - \varepsilon_{(01)(10)A}\varepsilon_{(02)(11)A} = 0 \tag{4.1}$$

$$\mathcal{S}_3^b : \varepsilon_{(10)(11)A}\varepsilon_{(01)(21)A} - \varepsilon_{(01)(11)A}\varepsilon_{(10)(12)A} = 0 \qquad \forall A \in SL(2,\mathbb{Z}_3) \tag{4.2}$$

where we used abbreviation $\varepsilon_{(ij)(kl)A} := \varepsilon_{(ij)A,(kl)A}$. Equations 1-24 (**subsystem** $\mathcal{S}_3^a$) in Table 1 arose from (4.1) and equations 25-48 (**subsystem** $\mathcal{S}_3^b$) from (4.2). Table 1 of the system $\mathcal{S}_3$ clearly shows the correspondence between matrices from $SL(2,\mathbb{Z}_3)$ and their equations.

Looking closer at the system $\mathcal{S}_3^a$, we observe that adding equations 1 and 3 we obtain equation 5. Equation 5 is then satisfied automatically whenever 1 and 3 hold and is redundant in the system. The question whether or not there exist other similar triples of equations can be answered by the following consideration making use of (4.1): equation 1 can be written in the form

$$\varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0, \qquad X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

This is caused by the fact that [(02)(10)][(01)(12)] and [(01)(10)][(02)(11)] lie in the same orbit with respect to the action of $SL(2,\mathbb{Z}_3)$ (analogous to 3.20), where the pairs of indices in bracket [ ] and the pairs of these brackets are unordered. The equation generated from equation 1 by the matrix $A = X$

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2} - \varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X} = 0 \tag{4.4}$$

is also contained in $\mathcal{S}_3^a$, due to (4.1). Adding equations (4.3) and (4.4) we have

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0. \tag{4.5}$$

Since $X^3 = 1$ holds, equation (4.5) is generated from equation 1 by matrix $A = X^2$. Therefore we can conclude that the cosets of the left decomposition of the group $SL(2,\mathbb{Z}_3)$ with respect to the cyclic subgroup $\{1, X, X^2\}$ then generate just the triples of dependent equations. The number of these cosets according to the Lagrange's theorem is $24/3 = 8$. So we obtained 8 equations (one from each coset) which we eliminate from the system $\mathcal{S}_3^a$. On the other hand, we observe in $\mathcal{S}_3^b$ that the quadruples of indices [(10)(11)][(01)(21)] and [(01)(11)][(10)(12)] do *not* lie in the same orbit and in this way the equations are not dependent.

The dependent equations are listed below. Symbolically, for the equation numbers the following holds: 1 + 3 = 5, 2 + 6 = 4, 11 + 17 = 19, 13 + 7 = 22, 15 + 23 = 10, 12 + 20 = 18, 14 + 21 = 8, 9 + 16 = 24, this means precisely that adding equations 1 and 3 equation 5 is obtained etc. The equations 5, 4, 19, 22, 10, 18, 8, 24 then can be chosen as redundant and eliminated. Thus the number of equations of the system $\mathcal{S}_3$ is reduced to 40.

4.2 The algorithm of evaluation

Since a straightforward generalization of the method published in [9] turned out to be impossible for our case, we had to find another way. Of course, we are left with a laborious case by case analysis. We found a method simplifying this laborious analysis; it was even possible to compute the parametric solutions without relying on the computer. The method is based on the fact that under suitable assumptions the system $\mathcal{S}_3$ can be easily explicitly solved. But after leaving the assumption, that means putting zero on the position we had assumed non-zero before, the solution is far more complicated. This can be bypassed by our algorithm which is based on the following theorem.

Theorem 4.2.1. Let $\mathcal{R}(\mathcal{S})$ be the set of solutions and $\mathcal{I}$ the set of relevant pairs of unordered indices of the contraction system $\mathcal{S}$ of a graded Lie algebra $\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i$. For any $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} = \{k_1,\, k_2, \ldots, k_m\} \subset \mathcal{I}$ we denote

$$\mathcal{R}_0 := \left\{\varepsilon \in \mathcal{R}(\mathcal{S}) \middle| (\forall \varepsilon_1 \in \mathcal{Q})(\varepsilon \nsim \varepsilon_1)\right\}$$
$$\mathcal{R}_1 := \left\{\varepsilon \in \mathcal{R}_0 \,\middle|\, (\forall k \in \mathcal{P})(\varepsilon_k \neq 0)\right\}.$$

Then the solution $\varepsilon \in \mathcal{R}_0$ is non-equivalent to all solutions in $\mathcal{R}_1$ if and only if

$$\begin{aligned} \varepsilon_{\pi_1(k_1)}\varepsilon_{\pi_1(k_2)}\cdots\varepsilon_{\pi_1(k_m)} &= 0 \\ &\vdots \\ \varepsilon_{\pi_n(k_1)}\varepsilon_{\pi_n(k_2)}\cdots\varepsilon_{\pi_n(k_m)} &= 0 \end{aligned} \tag{4.6}$$

holds, where $\{\pi_1, \pi_2, \ldots, \pi_n\} = \Delta_\Gamma \operatorname{Aut}\Gamma$ is the symmetry group of the grading Γ.

Proof. For any $\varepsilon \in \mathcal{R}_0$ we have (see 3.21):

$$(\exists \varepsilon_1 \in \mathcal{R}_1)(\varepsilon \sim \varepsilon_1) \Leftrightarrow (\exists \varepsilon_1 \in \mathcal{R}_1)(\exists \alpha)(\exists \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\alpha \bullet \varepsilon^\pi = \varepsilon_1) \tag{4.7}$$
$$\Leftrightarrow (\exists \alpha)(\exists \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\alpha \bullet \varepsilon^\pi \in \mathcal{R}_0 \wedge (\alpha \bullet \varepsilon^\pi)_k \neq 0,\ \forall k \in \mathcal{P}) \tag{4.8}$$
$$\Leftrightarrow (\exists \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\forall k \in \mathcal{P})((\varepsilon^\pi)_k \neq 0). \tag{4.9}$$

The equivalence (4.7) is direct consequence of the definition (3.21), the equivalence (4.8) expresses the trivial fact that $(\exists \varepsilon_1 \in \mathcal{R}_1)(\alpha \bullet \varepsilon^\pi = \varepsilon_1) \Leftrightarrow (\alpha \bullet \varepsilon^\pi \in \mathcal{R}_1)$. Since $\alpha \bullet \varepsilon^\pi \in \mathcal{R}_0$ is for any $\varepsilon \in \mathcal{R}_0$ automatically fulfilled and $(\alpha \bullet \varepsilon^\pi)_k \neq 0 \Leftrightarrow (\varepsilon^\pi)_k \neq 0$, the equivalence (4.9) follows.

Negating (4.9) we obtain

$$(\forall \varepsilon_1 \in \mathcal{R}_1)(\varepsilon \nsim \varepsilon_1) \Leftrightarrow (\forall \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\exists k \in \mathcal{P})((\varepsilon^\pi)_k = 0)$$

and this is the statement of the theorem. □

We call the system of equations (4.6) corresponding to the sets $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} \subset \mathcal{I}$ a **non-equivalence system**.

Repeated use of the theorem leads us to the following algorithm for the evaluation of solutions:

1. we set $\mathcal{Q} = \emptyset$ and suppose we have a set of assumptions $\mathcal{P}^0 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S})$, we explicitly evaluate
$$\mathcal{R}^0 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}) \mid (\forall k \in \mathcal{P}^0)(\varepsilon_k \neq 0)\right\}$$
and write the non-equivalence system $\mathcal{S}^0$ of equations (4.6) corresponding to $\mathcal{Q} = \emptyset$, $\mathcal{P}^0$.

2. we set $\mathcal{Q} = \mathcal{R}^0$ and suppose we have the set $\mathcal{P}^1 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0)$, we explicitly evaluate
$$\mathcal{R}^1 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0) \mid (\forall k \in \mathcal{P}^1)(\varepsilon_k \neq 0)\right\}$$
and write the non-equivalence system $\mathcal{S}^1$ corresponding to $\mathcal{Q} = \mathcal{R}^0$, $\mathcal{P}^1$.

3. we set $\mathcal{Q} = \mathcal{R}^0 \cup \mathcal{R}^1$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0 \cup \mathcal{S}^1)$ and we continue till we have evaluated the whole $\mathcal{R}(\mathcal{S})$ up to equivalence, i.e. we have such $\mathcal{Q}$ that the corresponding set $\mathcal{R}_0$ is empty or trivial.

We observe that the algorithm crucially depends on the choice of the assumptions sets $\mathcal{P}^0$, $\mathcal{P}^1, \ldots$. Since the system $\mathcal{S}_3$ can be solved explicitly nicely assuming two of its variables non-zeros, we develop a theory for pairs from $\mathcal{I}$. For fixed $k \in \mathcal{I}$ we define an equivalence **relation** $\equiv^k$ on the set $\mathcal{I}^k := \mathcal{I} \setminus \{k\}$: for $i, j \in \mathcal{I}^k$
$$i \equiv^k j \quad \Leftrightarrow \quad (\exists \pi \in \Delta_\Gamma \operatorname{Aut} \Gamma)(\pi\,(i\ k) = (j\ k)), \tag{4.10}$$
where $(i\ j)$ denotes an unordered pair of $i, j \in \mathcal{I}$ and naturally $\pi\,(i\ k) := (\pi(i)\ \pi(k))$. The usage of this equivalence will be seen on our concrete evaluation. We will make use of the following example:

Example 2. The set of relevant indices $\mathcal{I}_3$ has 24 elements which are explicitly written in matrix (2.15). We choose the index $k = (01)(10)$ and in Table 2 we list nine equivalence classes $\mathcal{I}_1^k, \ldots, \mathcal{I}_9^k$ of the equivalence $\equiv^k$:

Table 2: The equivalence classes of $\equiv^{(01)(10)}$

$\mathcal{I}_1^k$	(11)(12), (11)(21), (22)(12), (22)(21)
$\mathcal{I}_2^k$	(01)(11), (10)(11), (01)(12), (10)(21)
$\mathcal{I}_3^k$	(02)(22), (20)(22), (02)(21), (20)(12)
$\mathcal{I}_4^k$	(01)(20), (02)(10)
$\mathcal{I}_5^k$	(01)(22), (10)(22)
$\mathcal{I}_6^k$	(01)(21), (10)(12)
$\mathcal{I}_7^k$	(02)(11), (20)(11)
$\mathcal{I}_8^k$	(02)(12), (20)(21)
$\mathcal{I}_9^k$	(02)(20)

4.3 Evaluation of $\mathcal{S}_3$ solutions

1. $\mathcal{R}^0 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}_3) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(22)(21)} \neq 0\right\}$

   $\mathcal{S}^0 :\ \varepsilon_{(01)(10)A}\varepsilon_{(22)(21)A} = 0 \quad \forall A \in H_3$

2. $\mathcal{R}^1 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}_3 \cup \mathcal{S}^0) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(10)(11)} \neq 0,\, \varepsilon_{(01)(22)} \neq 0\right\}$

   $\mathcal{S}^1 :\ \varepsilon_{(01)(10)A}\varepsilon_{(10)(11)A}\varepsilon_{(01)(22)A} = 0 \quad \forall A \in H_3$

3. $\mathcal{R}^2 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}_3 \cup \mathcal{S}^0 \cup \mathcal{S}^1) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(10)(11)} \neq 0\right\}$

   $\mathcal{S}^2 :\ \varepsilon_{(01)(10)A}\varepsilon_{(10)(11)A} = 0 \quad \forall A \in H_3$

4. $\mathcal{R}^3 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}_3 \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(02)(22)} \neq 0\right\}$

   $\mathcal{S}^3 :\ \varepsilon_{(01)(10)A}\varepsilon_{(02)(22)A} = 0 \quad \forall A \in H_3$

5. $\mathcal{R}^4 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}_3 \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3) \mid \varepsilon_{(01)(10)} \neq 0\right\}$

   $\mathcal{S}^4 :\ \varepsilon_{(01)(10)A} = 0 \quad \forall A \in H_3$

ad 1. We present the evaluation of $\mathcal{R}^0$ in detail. First note that equation 37 implies $\varepsilon_{(01)(22)} \neq 0$, $\varepsilon_{(20)(21)} \neq 0$. Now we solve explicitly. The equation number from which each equality follows is listed before the colon. The already evaluated unknowns are substituted into each equation.

$$
\begin{array}{lcl}
37 & : & \varepsilon_{(22)(21)} = \frac{\varepsilon_{(01)(22)}\varepsilon_{(20)(21)}}{\varepsilon_{(01)(10)}} \\
35 & : & \varepsilon_{(01)(11)} = \frac{\varepsilon_{(01)(21)}\varepsilon_{(20)(22)}}{\varepsilon_{(20)(21)}} \\
44 & : & \varepsilon_{(22)(12)} = \frac{\varepsilon_{(10)(22)}\varepsilon_{(02)(12)}}{\varepsilon_{(01)(10)}} \\
31 & : & \varepsilon_{(11)(21)} = \frac{\varepsilon_{(01)(21)}\varepsilon_{(10)(22)}}{\varepsilon_{(01)(10)}} \\
47 & : & \varepsilon_{(20)(12)} = \frac{\varepsilon_{(01)(12)}\varepsilon_{(10)(22)}}{\varepsilon_{(01)(22)}} \\
26 & : & \varepsilon_{(02)(22)} = \frac{\varepsilon_{(20)(22)}\varepsilon_{(02)(12)}}{\varepsilon_{(20)(21)}} \\
3 & : & \varepsilon_{(02)(11)} = \frac{\varepsilon_{(01)(11)}\varepsilon_{(02)(12)}}{\varepsilon_{(01)(10)}} = \frac{\varepsilon_{(01)(21)}\varepsilon_{(20)(22)}\varepsilon_{(02)(12)}}{\varepsilon_{(20)(21)}\varepsilon_{(01)(10)}} \\
21 & : & \varepsilon_{(11)(12)} = \frac{\varepsilon_{(02)(12)}\varepsilon_{(11)(21)}}{\varepsilon_{(20)(21)}} = \frac{\varepsilon_{(02)(12)}\varepsilon_{(01)(21)}\varepsilon_{(10)(22)}}{\varepsilon_{(20)(21)}\varepsilon_{(01)(10)}} \\
29 & : & \varepsilon_{(10)(12)} = \frac{\varepsilon_{(01)(10)}\varepsilon_{(11)(12)}}{\varepsilon_{(01)(22)}} = \frac{\varepsilon_{(01)(10)}\varepsilon_{(02)(12)}\varepsilon_{(01)(21)}\varepsilon_{(10)(22)}}{\varepsilon_{(01)(22)}\varepsilon_{(01)(10)}\varepsilon_{(20)(21)}} = \frac{\varepsilon_{(10)(22)}\varepsilon_{(02)(12)}\varepsilon_{(01)(21)}}{\varepsilon_{(20)(21)}\varepsilon_{(01)(22)}} \\
39 & : & \varepsilon_{(01)(20)} = \frac{\varepsilon_{(20)(22)}\varepsilon_{(01)(12)}\varepsilon_{(01)(10)}}{\varepsilon_{(01)(22)}\varepsilon_{(20)(21)}} \\
15 & : & \varepsilon_{(20)(11)} = \frac{\varepsilon_{(01)(11)}\varepsilon_{(22)(12)}}{\varepsilon_{(01)(22)}} = \frac{\varepsilon_{(01)(21)}\varepsilon_{(20)(22)}\varepsilon_{(10)(22)}\varepsilon_{(02)(12)}}{\varepsilon_{(20)(21)}\varepsilon_{(01)(10)}\varepsilon_{(01)(22)}} \\
17 & : & \varepsilon_{(10)(11)} = \frac{\varepsilon_{(01)(10)}\varepsilon_{(20)(11)}}{\varepsilon_{(20)(21)}} = \frac{\varepsilon_{(20)(22)}\varepsilon_{(10)(22)}\varepsilon_{(02)(12)}\varepsilon_{(01)(21)}}{\varepsilon_{(01)(22)}\varepsilon^2_{(20)(21)}} \\
30 & : & \varepsilon_{(02)(20)} = \frac{\varepsilon_{(20)(21)}\varepsilon_{(02)(11)}}{\varepsilon_{(22)(21)}} = \frac{\varepsilon_{(01)(21)}\varepsilon_{(20)(22)}\varepsilon_{(02)(12)}}{\varepsilon_{(20)(21)}\varepsilon_{(01)(22)}} \\
28 & : & \varepsilon_{(02)(10)} = \frac{\varepsilon_{(02)(21)}\varepsilon_{(20)(22)}\varepsilon_{(01)(10)}}{\varepsilon_{(01)(22)}\varepsilon_{(20)(21)}} \\
46 & : & \varepsilon_{(10)(21)} = \frac{\varepsilon_{(10)(22)}\varepsilon_{(02)(21)}}{\varepsilon_{(01)(22)}}
\end{array}
$$

Up to now we have used 15 out of 40 equations. But at this moment one can check that the rest of equations is either identically fulfilled or their fulfillment is a consequence of two equations, 1 and 13. After the substitution of already evaluated unknowns they are obtained in the form

$$
\begin{array}{lcl}
1 & : & \varepsilon_{(20)(22)}\varepsilon_{(01)(10)}\varepsilon_{(01)(12)}\varepsilon_{(02)(21)} = \varepsilon_{(20)(22)}\varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(12)} \\
13 & : & \varepsilon_{(10)(22)}\varepsilon_{(01)(10)}\varepsilon_{(01)(12)}\varepsilon_{(02)(21)} = \varepsilon_{(10)(22)}\varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(12)}.
\end{array}
$$

The discussion of these two equations is easy and here we state the result:

$$\mathcal{R}^0 = \{\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0\},$$

where

$$\varepsilon_1^0 = \begin{pmatrix} 0 & 0 & t_1 & 0 & 0 & t_4 & x_5 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{11} & x_{12} \\ t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{t_4\,t_{20}}{t_1} \\ x_5 & x_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & x_{12} & 0 & t_{20} & 0 & \frac{t_4\,t_{20}}{t_1} & 0 & 0 \end{pmatrix}$$

$$\varepsilon_2^0 = \begin{pmatrix} 0 & 0 & t_1 & \frac{x_{18}t_5t_1}{t_4t_{20}} & \frac{x_6x_{18}}{t_{20}} & t_4 & t_5 & x_6 \\ 0 & 0 & \frac{x_{11}x_6x_{18}}{t_5t_{20}} & \frac{x_{11}x_6x_{18}}{t_{20}t_4} & \frac{x_{11}x_6x_{18}}{t_{20}t_1} & \frac{x_{18}x_{11}}{t_{20}} & x_{11} & \frac{x_{11}x_6t_4}{t_5t_1} \\ t_1 & \frac{x_{11}x_6x_{18}}{t_5t_{20}} & 0 & 0 & \frac{x_{18}x_{14}x_{11}x_6}{t_4{t_{20}}^2} & x_{14} & \frac{x_{14}x_{11}x_6}{t_{20}t_4} & \frac{x_{14}x_{11}x_6}{t_5t_1} \\ \frac{x_{18}t_5t_1}{t_4t_{20}} & \frac{x_{11}x_6x_{18}}{t_{20}t_4} & 0 & 0 & \frac{x_{18}x_{14}x_{11}x_6}{t_1t_4t_{20}} & x_{18} & \frac{t_5x_{14}}{t_4} & t_{20} \\ \frac{x_6x_{18}}{t_{20}} & \frac{x_{11}x_6x_{18}}{t_{20}t_1} & \frac{x_{18}x_{14}x_{11}x_6}{t_4{t_{20}}^2} & \frac{x_{18}x_{14}x_{11}x_6}{t_1t_4t_{20}} & 0 & 0 & \frac{x_{14}x_{11}x_6}{t_1t_{20}} & \frac{x_6x_{14}}{t_1} \\ t_4 & \frac{x_{18}x_{11}}{t_{20}} & x_{14} & x_{18} & 0 & 0 & \frac{x_{14}x_{11}}{t_1} & \frac{t_4t_{20}}{t_1} \\ t_5 & x_{11} & \frac{x_{14}x_{11}x_6}{t_{20}t_4} & \frac{t_5x_{14}}{t_4} & \frac{x_{14}x_{11}x_6}{t_1t_{20}} & \frac{x_{14}x_{11}}{t_1} & 0 & 0 \\ x_6 & \frac{x_{11}x_6t_4}{t_5t_1} & \frac{x_{14}x_{11}x_6}{t_5t_1} & t_{20} & \frac{x_6x_{14}}{t_1} & \frac{t_4t_{20}}{t_1} & 0 & 0 \end{pmatrix}$$

$$\varepsilon_3^0 = \begin{pmatrix} 0 & 0 & t_1 & 0 & 0 & t_4 & 0 & 0 \\ 0 & 0 & \frac{x_{12}x_{18}t_1}{t_4t_{20}} & 0 & 0 & \frac{x_{18}x_{11}}{t_{20}} & x_{11} & x_{12} \\ t_1 & \frac{x_{12}x_{18}t_1}{t_4t_{20}} & 0 & 0 & 0 & x_{14} & 0 & \frac{x_{14}x_{12}}{t_4} \\ 0 & 0 & 0 & 0 & 0 & x_{18} & 0 & t_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_4 & \frac{x_{18}x_{11}}{t_{20}} & x_{14} & x_{18} & 0 & 0 & \frac{x_{14}x_{11}}{t_1} & \frac{t_4t_{20}}{t_1} \\ 0 & x_{11} & 0 & 0 & 0 & \frac{x_{14}x_{11}}{t_1} & 0 & 0 \\ 0 & x_{12} & \frac{x_{14}x_{12}}{t_4} & t_{20} & 0 & \frac{t_4t_{20}}{t_1} & 0 & 0 \end{pmatrix}$$

$$\varepsilon_4^0 = \begin{pmatrix} 0 & 0 & t_1 & 0 & \frac{x_6x_{18}}{t_{20}} & t_4 & 0 & x_6 \\ 0 & 0 & \frac{x_{12}x_{18}t_1}{t_4t_{20}} & 0 & 0 & 0 & 0 & x_{12} \\ t_1 & \frac{x_{12}x_{18}t_1}{t_4t_{20}} & 0 & 0 & 0 & x_{14} & 0 & \frac{x_{14}x_{12}}{t_4} \\ 0 & 0 & 0 & 0 & 0 & x_{18} & 0 & t_{20} \\ \frac{x_6x_{18}}{t_{20}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x_6x_{14}}{t_1} \\ t_4 & 0 & x_{14} & x_{18} & 0 & 0 & 0 & \frac{t_4t_{20}}{t_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & x_{12} & \frac{x_{14}x_{12}}{t_4} & t_{20} & \frac{x_6x_{14}}{t_1} & \frac{t_4t_{20}}{t_1} & 0 & 0 \end{pmatrix}.$$

In the notation of explicit matrices we use a convention that non-zero parameters are denoted by the letter t and others which can be zero by the letter x. Henceforth the same convention will be used.

ad 2. Note that the system of 48 equations $\mathcal{S}^0$ together with $\varepsilon_{(01)(10)} \neq 0$ enforces zeros on all positions from $\mathcal{I}_1^k$. *In general we can say that after solving with the assumption $\varepsilon_k \neq 0$, $\varepsilon_i \neq 0$ the corresponding non-equivalence system and the assumption $\varepsilon_k \neq 0$ will enforce zeros on all positions j, $j \equiv^k i$.* That is exactly the reason why we chose $(22)(21) \in \mathcal{I}_1^k$, $(10)(11) \in \mathcal{I}_2^k$, $(02)(22) \in \mathcal{I}_3^k$ for the evaluation of the sets $\mathcal{R}^0$, $\mathcal{R}^1$ and $\mathcal{R}^2$, $\mathcal{R}^3$ respectively. Moreover, the assumption $\varepsilon_{(10)(11)} \neq 0$ and $\mathcal{S}^0$ enforces further 4 zeros. Since the assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$, $\varepsilon_{(01)(22)} \neq 0$ gives us in some way a special solution we put it in a single set:

$$\mathcal{R}^1 = \{\varepsilon^1\}$$

$$\varepsilon^1 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & x_3 & t_4 & 0 & 0 \\ 0 & 0 & x_7 & 0 & 0 & x_{10} & 0 & 0 \\ t_1 & x_7 & 0 & 0 & t_{13} & x_{14} & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & x_{18} & 0 & 0 \\ x_3 & 0 & t_{13} & 0 & 0 & 0 & 0 & 0 \\ t_4 & x_{10} & x_{14} & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ad 3. The solutions with assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$ non-equivalent to those in $\mathcal{R}^1$ and $\mathcal{R}^0$ are listed below:

$$\mathcal{R}^2 = \{\varepsilon^2_1, \varepsilon^2_2, \varepsilon^2_3, \varepsilon^2_4, \varepsilon^2_5, \varepsilon^2_6, \varepsilon^2_7, \varepsilon^2_8\}$$

$$\varepsilon^2_1 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & t_{13} & x_{14} & 0 & x_{16} \\ x_2 & 0 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & x_{18} & x_{19} & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{14} & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{19} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon^2_2 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & t_{13} & x_{14} & t_{15} & x_{16} \\ x_2 & 0 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & 0 & x_{19} & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{15} & x_{19} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon^2_3 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_7 & 0 & 0 & 0 & 0 & 0 \\ t_1 & t_7 & 0 & 0 & t_{13} & x_{14} & 0 & x_{16} \\ x_2 & 0 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & x_{18} & 0 & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{14} & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon^2_4 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_7 & 0 & 0 & 0 & 0 & 0 \\ t_1 & t_7 & 0 & 0 & t_{13} & x_{14} & t_{15} & x_{16} \\ x_2 & 0 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & 0 & 0 & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon^2_5 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_7 & t_8 & 0 & 0 & 0 & 0 \\ t_1 & t_7 & 0 & 0 & t_{13} & \frac{x_{15} x_{18}}{t_8} & x_{15} & x_{16} \\ x_2 & t_8 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & x_{18} & \frac{x_{15} x_{18}}{t_7} & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_{15} x_{18}}{t_8} & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{15} & \frac{x_{15} x_{18}}{t_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon^2_6 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_8 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & t_{13} & 0 & 0 & x_{16} \\ x_2 & t_8 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & x_{18} & x_{19} & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{19} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon^2_7 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_8 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & t_{13} & 0 & t_{15} & x_{16} \\ x_2 & t_8 & 0 & 0 & \frac{x_2 x_{16}}{t_1} & 0 & x_{19} & \frac{x_2 x_{16}}{t_{13}} \\ 0 & 0 & t_{13} & \frac{x_2 x_{16}}{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{15} & x_{19} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{16} & \frac{x_2 x_{16}}{t_{13}} & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon^2_8 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & x_7 & x_8 & 0 & x_{10} & 0 & 0 \\ t_1 & x_7 & 0 & 0 & t_{13} & 0 & 0 & 0 \\ x_2 & x_8 & 0 & 0 & 0 & x_{18} & 0 & 0 \\ x_3 & 0 & t_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{10} & 0 & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ad 4. Now we can of course ignore the equations $\mathcal{S}^1$ because they are satisfied identically due to the system $\mathcal{S}^2$. We list the next set

$$\mathcal{R}^3 = \{\varepsilon^3_1, \varepsilon^3_2, \varepsilon^3_3, \varepsilon^3_4\}$$

$$\varepsilon^3_1 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_{10} & t_{11} & 0 \\ t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon^3_2 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_7 & 0 & 0 & t_{10} & 0 & 0 \\ t_1 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & x_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{10} & 0 & x_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_3^3 = \begin{pmatrix} 0 & 0 & t_1 & 0 & 0 & t_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_{10} & t_{11} & 0 \\ t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_4 & t_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon_4^3 = \begin{pmatrix} 0 & 0 & t_1 & 0 & 0 & t_4 & 0 & 0 \\ 0 & 0 & x_7 & 0 & 0 & t_{10} & 0 & 0 \\ t_1 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_4 & t_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ad 5. The systems $\mathcal{S}^0$, $\mathcal{S}^2$, $\mathcal{S}^3$ together with $\varepsilon_{(01)(10)} \neq 0$ give us 12 zeros and further 20 non-trivial conditions. Adding two zeros following from $\mathcal{S}_3$ we have:

$$\mathcal{R}^4 = \{\varepsilon_1^4, \varepsilon_2^4, \varepsilon_3^4\}$$

$$\varepsilon_1^4 = \begin{pmatrix} 0 & 0 & t_1 & 0 & 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & x_8 & 0 & 0 & x_{11} & 0 \\ t_1 & 0 & 0 & 0 & 0 & 0 & x_{15} & 0 \\ 0 & x_8 & 0 & 0 & 0 & 0 & 0 & x_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{11} & x_{15} & 0 & 0 & 0 & 0 & 0 \\ x_6 & 0 & 0 & x_{20} & 0 & 0 & 0 & 0 \end{pmatrix} \varepsilon_2^4 = \begin{pmatrix} 0 & 0 & t_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_7 & x_8 & 0 & 0 & 0 & 0 \\ t_1 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_3^4 = \begin{pmatrix} 0 & 0 & t_1 & 0 & 0 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & 0 & x_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & x_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since all pairs of relevant indices lie in *one* orbit - the whole set $\mathcal{I}_3$, the system $\mathcal{S}^4 : \varepsilon_k = 0, \forall k \in \mathcal{I}_3$ enforces zeros on all 24 positions. This precisely means that *only trivial zero solution is non-equivalent to solutions in* $\mathcal{R}^0$, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$, $\mathcal{R}^4$, *i.e. we evaluated the whole* $\mathcal{R}(\mathcal{S}_3)$ *up to equivalence.*

Our goal is to compute a set of non-equivalent normalized solutions. It will be used as an input to a further analysis - the identification of resulting Lie algebras. We take each solution matrix and discuss all possible combinations of zero or non-zero parameters like in the example below.

Example 3. We take the matrix ε_2^0 and let all its parameters are non-vanishing. Our question is whether or not it is possible to normalize it to the matrix (2.14). Then the resulting graded contractions would be isomorphic to the algebra $sl(3,\mathbb{C})$ for arbitrary non-zero values of parameters in ε_2^0. We have the normalization matrix α in the form:

$$\alpha = \begin{pmatrix} 0 & 0 & \frac{a_{01}a_{10}}{a_{11}} & \frac{a_{01}a_{20}}{a_{21}} & \frac{a_{01}a_{11}}{a_{12}} & \frac{a_{01}a_{22}}{a_{20}} & \frac{a_{01}a_{12}}{a_{10}} & \frac{a_{01}a_{21}}{a_{22}} \\ 0 & 0 & \frac{a_{02}a_{10}}{a_{12}} & \frac{a_{02}a_{20}}{a_{22}} & \frac{a_{02}a_{11}}{a_{10}} & \frac{a_{02}a_{22}}{a_{21}} & \frac{a_{02}a_{12}}{a_{11}} & \frac{a_{02}a_{21}}{a_{20}} \\ \frac{a_{01}a_{10}}{a_{11}} & \frac{a_{02}a_{10}}{a_{12}} & 0 & 0 & \frac{a_{10}a_{11}}{a_{21}} & \frac{a_{10}a_{22}}{a_{02}} & \frac{a_{10}a_{12}}{a_{22}} & \frac{a_{10}a_{21}}{a_{01}} \\ \frac{a_{01}a_{20}}{a_{21}} & \frac{a_{02}a_{20}}{a_{22}} & 0 & 0 & \frac{a_{20}a_{11}}{a_{01}} & \frac{a_{20}a_{22}}{a_{12}} & \frac{a_{20}a_{12}}{a_{02}} & \frac{a_{20}a_{21}}{a_{11}} \\ \frac{a_{01}a_{11}}{a_{12}} & \frac{a_{02}a_{11}}{a_{10}} & \frac{a_{10}a_{11}}{a_{21}} & \frac{a_{20}a_{11}}{a_{01}} & 0 & 0 & \frac{a_{11}a_{12}}{a_{20}} & \frac{a_{11}a_{21}}{a_{02}} \\ \frac{a_{01}a_{22}}{a_{20}} & \frac{a_{02}a_{22}}{a_{21}} & \frac{a_{10}a_{22}}{a_{02}} & \frac{a_{20}a_{22}}{a_{12}} & 0 & 0 & \frac{a_{22}a_{12}}{a_{01}} & \frac{a_{22}a_{21}}{a_{10}} \\ \frac{a_{01}a_{12}}{a_{10}} & \frac{a_{02}a_{12}}{a_{11}} & \frac{a_{10}a_{12}}{a_{22}} & \frac{a_{20}a_{12}}{a_{02}} & \frac{a_{11}a_{12}}{a_{20}} & \frac{a_{22}a_{12}}{a_{01}} & 0 & 0 \\ \frac{a_{01}a_{21}}{a_{22}} & \frac{a_{02}a_{21}}{a_{20}} & \frac{a_{10}a_{21}}{a_{01}} & \frac{a_{20}a_{21}}{a_{11}} & \frac{a_{11}a_{21}}{a_{02}} & \frac{a_{22}a_{21}}{a_{10}} & 0 & 0 \end{pmatrix}$$

We only state that the system of 24 equations which was created from the matrix equality $\varepsilon_2^0 \bullet \; \alpha = \kappa$ has a general solution in $\mathbb{C}$. The matrix ε_2^0 with non-zero parameters is then equivalent to the trivial solution κ and the corresponding graded contraction is isomorphic to $sl(3, \mathbb{C})$.

Example 4. a) Let us take ε_2^0 and put $x_{14} = 0$, others parameters non-zero. The resulting matrix is denoted as ε_9 and $\nu(\varepsilon_9) = 9$ holds. The system $\varepsilon_9 \bullet \alpha = \varepsilon_{9,I}$, where

$$\varepsilon_{9,I} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

has a solution in $\mathbb{C}$ and so ε_9 can be normalized to $\varepsilon_{9,I}$. To see how the whole orbit looks like we apply all permutations corresponding to $A \in H_3$ on $\varepsilon_{9,I}$. The set $E_9 := \left\{ (\varepsilon_{9,I})^{\pi_A} \;\middle|\; A \in H_3 \right\}$ has 8 elements:

$$\varepsilon_{9,I} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \varepsilon_{9,II} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_{9,III} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_{9,IV} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_{9,V} = \begin{pmatrix} 0&0&0&1&1&1&1&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0\\ 1&0&0&0&1&1&1&0\\ 1&1&1&1&0&0&1&1\\ 1&1&1&1&0&0&1&1\\ 1&0&0&1&1&1&0&0\\ 0&0&0&0&1&1&0&0 \end{pmatrix} \qquad \varepsilon_{9,VI} = \begin{pmatrix} 0&0&0&0&1&1&0&0\\ 0&0&1&0&1&1&0&1\\ 0&1&0&0&1&1&0&1\\ 0&0&0&0&1&1&0&0\\ 1&1&1&1&0&0&1&1\\ 1&1&1&1&0&0&1&1\\ 0&0&0&0&1&1&0&0\\ 0&1&1&0&1&1&0&0 \end{pmatrix}$$

$$\varepsilon_{9,VII} = \begin{pmatrix} 0&0&0&0&0&0&1&1\\ 0&0&0&1&1&0&1&1\\ 0&0&0&0&0&0&1&1\\ 0&1&0&0&1&0&1&1\\ 0&1&0&1&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&0&0 \end{pmatrix} \qquad \varepsilon_{9,VIII} = \begin{pmatrix} 0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&1&1\\ 1&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1\\ 1&0&1&0&0&0&1&1\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&0&0 \end{pmatrix}$$

b) Next we take again ε_2^0 and substitute $x_{18} = 0$ in it and let all other parameters be non-zero; the result is denoted as $\varepsilon_{9,2}$ and $\nu(\varepsilon_{9,2}) = 9$. But the matrix $\varepsilon_{9,2}$ can be normalized to $\varepsilon_{9,VIII}$ and thus we have discovered that *the solutions ε_9 and $\varepsilon_{9,2}$ are equivalent.* In our sets $\mathcal{R}^0, \ldots$ there exists no other solution ε with the property $\nu(\varepsilon) = 9$. Therefore we conclude that every solution in $\mathcal{R}(\mathcal{S}_3)$ with $\nu(\varepsilon) = 9$ is equivalent to $\varepsilon_{9,I}$ and thus only this matrix will appear on the final list of solutions as representative of solutions with 9 zeros.

Example 5. The matrix $\varepsilon^1 \in \mathcal{R}^1$ contains 6 parameters x, these can be zero or non-zero, and so there are $2^6 = 64$ cases to analyze. We take the first one and assume that all the parameters contained in it are non-zero. We have the matrix

$$\varepsilon_{15} = \begin{pmatrix} 0&0&t_1&t_2&t_3&t_4&0&0\\ 0&0&t_7&0&0&t_{10}&0&0\\ t_1&t_7&0&0&t_{13}&t_{14}&0&0\\ t_2&0&0&0&0&t_{18}&0&0\\ t_3&0&t_{13}&0&0&0&0&0\\ t_4&t_{10}&t_{14}&t_{18}&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

and $\nu(\varepsilon_{15}) = 15$ holds. We try to normalize it to the matrix

$$\varepsilon_{15n} = \begin{pmatrix} 0 & 0 & a & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & b & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further on, it will be clear why we have chosen $a \neq 0$, $b \neq 0$ to appear in the matrix ε_{15n}. The $24 - 15 = 9$ normalization equations are in the form:

$$\begin{array}{ccc} \frac{a_{01}a_{10}}{a_{11}} t_1 = a & \frac{a_{01}a_{20}}{a_{21}} t_2 = 1 & \frac{a_{01}a_{11}}{a_{12}} t_3 = 1 \\ \frac{a_{01}a_{22}}{a_{20}} t_4 = 1 & \frac{a_{02}a_{10}}{a_{12}} t_7 = 1 & \frac{a_{02}a_{22}}{a_{21}} t_{10} = 1 \\ \frac{a_{10}a_{11}}{a_{21}} t_{13} = b & \frac{a_{10}a_{22}}{a_{02}} t_{14} = 1 & \frac{a_{20}a_{22}}{a_{12}} t_{18} = 1 \end{array}$$

and their solution is

$$\begin{aligned} a_{01} &= q \\ a_{02} &= q^2 \frac{t_2 t_4}{t_{10}} \\ a_{10} &= \frac{q^2}{p} \frac{t_2 t_4}{t_{10} t_{14}} \\ a_{20} &= q p t_4 \\ a_{11} &= p^2 \frac{t_4 t_{18}}{t_3} \\ a_{22} &= p, \ p \neq 0 \\ a_{12} &= q p^2 t_4 t_{18} \\ a_{21} &= q^2 p t_2 t_4 \\ a &= \frac{t_1 t_3 t_{10}}{t_2 t_4 t_7} \\ b &= \frac{t_4 t_{13} t_{18}}{t_3 t_{10} t_{14}} \end{aligned}$$

where $q = \sqrt[3]{\frac{p^3 t_{10}^2 t_{14} t_{18}}{t_2^2 t_4 t_7}}$ and $p \neq 0$ is a parameter. We see that in general it is not possible to normalize the matrix ε_{15} to ε_{15n}, $a = 1$, $b = 1$ because the parameters t are independent. The maximal result of normalization is a two-parameter solution ε_{15n}.

We will continue in a similar vein as in the examples above - it is the most laborious task of our problem - till we exhaust all 316 combinations in our 20 matrices contained in the sets $\mathcal{R}^0$, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$ and $\mathcal{R}^4$.

Conclusion

Using the symmetry group of the Pauli grading of $sl(3,\mathbb{C})$, we have evaluated the set of all solutions of the corresponding contraction system up to equivalence. For the solution of the normalization equations and for the explicit evaluation of orbits of solutions we used the computer program Maple 8 at Centre de recherches mathématiques, Montréal. We proposed a sophisticated method based on Theorem (4.2.1) which enabled us to check all solutions in the sets $\mathcal{R}^0$, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$ and $\mathcal{R}^4$ also by hands. We remark that we did not touch the problem of distinguishing between so called continuous and discrete graded contractions [11]. It is interesting to note:

$$\min\left\{\nu(\varepsilon)\in\{1,2,3,\dots\}\;\middle|\;\varepsilon\in\mathcal{R}(\mathcal{S}_3)\right\}=9$$

It turned out that there are no solutions with less than 9 zeros (excluding the trivial solution κ). Moreover, there are no solutions with 10, 11, 13 or 14 zeros. The complete list of solutions is placed in the Appendix.

Appendix

The complete list of non-equivalent solutions of $\mathcal{S}_3$ has 180 elements. The list is divided into sections according to the number of zeros among 24 relevant parameters $\nu(\varepsilon)$ of the solution ε. In the whole list $a \neq 0$, $b \neq 0$ holds.

First trivial solution

$$\begin{pmatrix} 0&0&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1\\ 1&1&0&0&1&1&1&1\\ 1&1&0&0&1&1&1&1\\ 1&1&1&1&0&0&1&1\\ 1&1&1&1&0&0&1&1\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&0&0 \end{pmatrix}$$

Solutions with 9 zeros

$$\begin{pmatrix} 0&0&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1\\ 1&1&0&0&0&0&0&0\\ 1&1&0&0&0&1&0&1\\ 1&1&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&1\\ 1&1&0&0&0&0&0&0\\ 1&1&0&1&0&1&0&0 \end{pmatrix}$$

Solutions with 12 zeros

$$\begin{pmatrix} 0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&0&0&0&1&1&1\\ 1&0&0&0&0&0&0&1\\ 1&0&1&1&0&0&0&1\\ 1&0&0&1&0&0&0&0\\ 1&0&0&1&1&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&a&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&b&b&1\\ a&1&0&0&a&1&b&a\\ 0&0&1&a&0&0&0&0\\ 0&0&b&1&0&0&0&0\\ 0&0&b&b&0&0&0&0\\ 0&0&1&a&0&0&0&0 \end{pmatrix}$$

Solutions with 15 zeros

$$\begin{pmatrix} 0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 1&0&0&1&0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&1&1&0\\ 0&0&0&0&0&1&1&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&1\\ 1&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&1&1&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&0&1&1&0&0&0&1\\ 1&0&0&1&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&1&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&1&1\\ 0&1&0&0&0&1&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&a&1&1&1&0&0\\ 0&0&1&0&0&1&0&0\\ a&1&0&0&b&1&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&b&0&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&a&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&1&1&0\\ a&1&0&0&0&1&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&a&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ a&1&0&0&1&1&1&1\\ 0&1&0&0&0&1&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix}$$

Solutions with 16 zeros

$$\begin{pmatrix} 0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&1&a\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&0&0\\ 1&a&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&a&1&1&1&0&0\\ 0&0&1&0&0&1&0&0\\ a&1&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&a&1&1&1&0&0\\ 0&0&1&0&0&1&0&0\\ a&1&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&1&0&0&0\\ 0&0&1&1&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 1&1&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&0&0&1\\ 1&1&0&0&1&0&0&1\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&1&1&0\\ 0&1&0&0&0&1&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&1\\ 1&0&0&0&1&1&0&1\\ 0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0 \end{pmatrix}$$

Solutions with 17 zeros

$$\begin{pmatrix} 0&0&1&1&0&1&1&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&1&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&1\\ 1&0&0&1&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&0&0\\ 1&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 1&1&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&1&0\\ 0&0&0&0&0&0&1&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&0&0\\ 0&1&0&1&0&1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&a&1&1&1&0&0\\ 0&0&1&0&0&1&0&0\\ a&1&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&1&1&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&1&1&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&0&1&0&0\\ 0&0&1&0&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&a&1&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ a&1&0&0&1&1&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&1&0&0&0\\ 0&0&1&0&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&1&0&0&0\\ 0&0&1&1&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 1&1&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&1\\ 1&0&0&0&1&0&0&1\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0 \end{pmatrix}$$

Solutions with 18 zeros

$$\begin{pmatrix} 0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 1&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&1&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&0&1\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 1&0&0&1&0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&1&1&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix}$$

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\begin{pmatrix}0&0&1&1&1&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&1&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\1&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&1&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&0&1&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&1&0&0&0\\1&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\1&0&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&0&1&0&1&0&0\\1&0&0&0&1&0&0&0\\1&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&1&0&0&0\\1&1&0&0&0&1&0&0\\1&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&1&0&0&0\\0&0&0&1&0&1&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\1&0&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&a&0&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&1&0&1&0\\a&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&1&0&0&0\\1&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&1&0&1&1\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&1&0&0&1\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&a&0&0&0&0&0\\1&a&0&0&1&1&1&1\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&a&0&0&0&0&0\\1&a&0&0&1&1&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&1&0&1&0\\1&1&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&1&0&0&1\\0&1&0&0&0&1&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&a&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&1&0\\a&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&1\\1&0&0&0&1&0&0&1\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&1&0&0&1&0\\1&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\end{pmatrix}
$$

Solutions with 19 zeros

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\begin{pmatrix}0&0&1&0&0&1&1&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0\\0&0&0&1&0&1&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&1\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\1&0&0&1&0&1&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&1\\0&1&0&0&0&0&0&0\\0&0&0&1&0&1&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&1\\0&0&0&0&0&0&0&0\\1&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&1&0&1&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&1&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&0&1&1&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&1&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&1&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\1&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\1&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&1&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&1&0&0&0\\1&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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$$\begin{pmatrix} 0&0&1&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 0&1&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 1&1&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&1\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 1&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 1&1&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&1&0\\ 1&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}$$

Solutions with 20 zeros

$$\begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&1&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&1&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

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\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&1&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&1&1&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
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$$
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&1&1&0&0&0&0\\1&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&1&0&0&1&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&1&0&0&0&0\\1&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&1&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&1&0&0&0&0\end{pmatrix}
$$

Solutions with 21 zeros

$$
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&1&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
$$

$$
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&1&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&1\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
$$

$$
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&1&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&1&0&0\\1&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
$$

$$
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&1&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{pmatrix}
$$

Solutions with 22 zeros

$$
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&1&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&1\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}
$$

$$
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}
$$

Solutions with 23 zeros

$$\begin{pmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Second trivial solution

$$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

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