CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

DIPLOMA THESIS

SEMICLASSICAL ANALYSIS OF TIME DEPENDENT QUANTUM SYSTEMS

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5.5. 2002

I declare that I wrote this diploma thesis independently using the listed references.

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1. Preface

During three years I am considering with mathematical physics, I met with six functional analytic methods:
1) KAM algorithm [D1];
application of
2) WKB in Feynman integral and Geometric asymptotics [Sch], [GS]
3) Dyson expansion and
4) Egorov theorem [BR], [Lev]
on quantum ball;
5) Egorov theorem for time dependent case [Lev],
6) Matrix elements investigated with WKB.
I decided to involve in this thesis three of them. Matrix elements investigated with WKB, because it is my last work, Egorov theorem for time dependent case, because new improvements appeared and finally KAM, although not semiclassical method it was inspiration for starting semiclassical analysis of matrix elements. As the consequence of this decision this thesis consists of three more or less independent parts.
The last part contains the results of paper [D2] and application of this improved KAM technique on two models. The quantity V_{knm} defined by (79), in fact matrix element of perturbation V in eigen-basis of unperturbed Floquet Hamiltonian, is crucial to Theorem 4.1 be applicable.
Section 3 gives new statement of Egorov theorem for time dependent systems, based on ideas of [BR], [Lev] and Joachim Asch, who showed direct approach to time dependent generalization. Also assumptions on the form and growth of time dependent Hamiltonian are weaken.
The heart of thesis, section 2, contains application of WKB (although nonstandard one, see 2.1. Introduction) in investigation of asymptotic behavior of

$$W_{n,\,l} := < W\,\Psi_n,\ \Psi_l >,$$

for large $n,\ l$, where Ψ_n is normalised Dirichlet $L^2(\mathbb{R}_+)$ solution of

$$(-\Delta + x^\alpha)\Psi_n(x) = E_n\,\Psi_n(x)\ ,\quad \alpha > 1.$$

Asymptotic behavior is important for finiteness of quantities like $\sup_n \sum_l W_{n,\,l}$ or $\sup_n \sum_l \frac{W_{n,\,l}}{E_n - E_l}$, which are important in KAM, Dyson or adiabatic analysis of time dependent systems.
The result of this method is precise in computations of leading terms of normalization constant and diagonal of matrix $W_{n,\,l}$, where the numerical values were given, but less precise for offdiagonal elements, which are only estimated from above (see Theorem 2.9). Unfortunately, there is no sufficient condition yet for W to be Shur-Holmgren ($\sup_n \sum_l W_{n,\,l} < \infty$).
This is the beginning of the work, which started 3 month ago within consultations with prof. Pierre Duclos and Dr. Michel Vittot during my stage in CPT, Marseille and continued in Prague within meetings with prof. Pavel Šťovíček, my supervisor.
Possible continuation could be improvement of estimate (67), computing further terms in (13), (16), generalization to asymptotically homogeneous potentials, taking Neumann condition instead of Dirichlet and use it to treat true symmetric well on whole line.

Acknowledgments
I would like to thank to Dr. Joachim Asch (CPT, Marseille) for consultations and ideas in Egorov theorem problematic,
prof. Pierre Duclos and Dr. Michel Vittot for consultations and ideas in WKB problematic and their care during my stage in CPT, Marseille.
Finally, I would like to thank very much to prof. Pavel Šťovíček for his perfect leadership during all three years.

2.1. Introduction.

WKB analysis is well-known method in quantum mechanics, which allows us to construct wave functions in semiclassical regime. This method exists also in mathematics, under name Liouville-Green method. For details for further discussion one can consult [Ol] Chapters 6 and 11.

Let us consider equation

$$(-\Delta + x^\alpha - E)\Psi(x) = 0$$

in $E \to \infty$ regime. Using scaling trick (17) this equation becomes equivalent to

$$(-\hbar^2\Delta + x^\alpha - 1)\Phi(x) = 0 \ ,$$

where $\hbar \to 0$.

Standard WKB yields the solution

$$\begin{aligned} \Phi(x) = \quad & \frac{1}{\sqrt[4]{1-x^\alpha}} \sin\left(\frac{1}{\hbar}\int_0^x \sqrt{1-y^\alpha}\, dy\right) \quad \text{for } x < 1 \\ & \frac{1}{\sqrt[4]{x^\alpha - 1}} \exp\left(-\frac{1}{\hbar}\int_1^x \sqrt{y^\alpha - 1}\, dy\right) \quad \text{for } x > 1 \ , \end{aligned}$$

which is diverging in turning point 1.

The problem is that standard WKB is in fact unitary transformation between the solutions of

$$(-\hbar^2\Delta + V)\Phi = 0$$

and

$$(-\hbar^2\Delta + C)\zeta = 0 \ .$$

Constant C may be chosen $+1$, where $V > 0$, or -1, where $V < 0$, i.e. in classically forbidden or permitted region. The solution is there exponentially decaying or oscillating respectively. The divergence of standard WKB solution is caused by the fact that the unitary mapping between solutions of equations

$$(-\hbar^2\Delta + V)\Phi = 0$$

and

$$(-\hbar^2\Delta + v)\zeta = 0 \ ,$$

where v is some function, conserves the sign of V.

Because our problem $V(x) = x^\alpha - 1$ changes the sign, it looks more naturally trying to map the equation

$$(-\hbar^2\Delta + x^\alpha - 1)\Psi(x) = 0$$

to

$$(-\hbar^2\Delta + x - 1)\zeta(x) = 0 \ .$$

If we proceed this mapping, Airy functions Ai and Bi appear. Because we are interested in $L^2(\mathbb{R}_+)$ solutions, we exclude Bi. Now our new solution has no singularity! Other advantage is that error is estimated uniformly on $\mathbb{R}_+$. We see that this approach is convenient for our problematic.

Now if we ask for Dirichlet boundary condition at origin, we will obtain Bohr-Sommerfeld quantization condition and Energy E become labeled by a natural number n.

So at the moment we have two Dirichlet $L^2(\mathbb{R}_+)$ solutions and we would like to compute matrix elements

$$\frac{< W\,\Psi_n,\,\Psi_l >}{\sqrt{<\Psi_n,\,\Psi_n><\Psi_l,\,\Psi_l>}},$$

for some perturbation W. This expression looks very complicated, but numerically calculated examples $\alpha = 1, 2$ showed us the way, the most important thing is the behavior in the neighborhood of the origin.
The section is divided into several parts: computation of normalization constant, diagonal and offdiagonal matrix elements. The approach is given in details, what is interesting, is that in fact only mathematical tool, which is used is integration by parts. The results are formulated in the beginning of each subsection and at the end of this section is given summary of all results, stated as Theorem 2.9.

Convention: We will use (for simplicity) for all positive constants, which depend only on α, the letter K, although they may change.
This section is dealing only with $\alpha > 1$ case.

2.2. **Results of F.W.J. Olver.**
Due to the Theorem 3.1 , Chapter 11 of [Ol] the only $L^2(\mathbb{R}^+)$ solution of equation ($E > 0$)

$$(-\Delta + x^\alpha)\Psi(x) = E\Psi(x) \tag{1}$$

can be written in the form

$$\Psi(x) = \left(\frac{s(x)}{x^\alpha - E}\right)^{\frac{1}{4}} (\mathrm{Ai}(s(x)) + \varepsilon(x)), \tag{2}$$

where

$$s(x) := \begin{cases} (\frac{3}{2}\int_{E^{\frac{1}{\alpha}}}^{x} \sqrt{y^\alpha - E}\;dy)^{\frac{2}{3}}, & \text{for } \mathrm{x} \geq \mathrm{E}^{\frac{1}{\alpha}} \\ -\;(\frac{3}{2}\int_{x}^{E^{\frac{1}{\alpha}}} \sqrt{E - y^\alpha}\;dy)^{\frac{2}{3}}, & \text{for } \mathrm{x} < \mathrm{E}^{\frac{1}{\alpha}}. \end{cases}$$

ε is an error, which is controlled by inequality

$$\frac{|\varepsilon(x)|}{M(s(x))},\; \frac{|\varepsilon'(x)|}{N(s(x))}\sqrt{\frac{s(x)}{x^\alpha - E}} \leq \frac{1}{G(s(x))}\left(\exp(\frac{8}{\pi^2}E^{-\frac{\alpha+2}{2\alpha}}\mathcal{V}(x,\infty)) - 1\right), \tag{3}$$

where $\mathcal{V}$ is given by formula

$$\mathcal{V}(x,y) := \int_x^y \frac{1}{\sqrt{|\tilde{s}(z)|}}\left|\frac{4VV' - 5V'^2}{16V^{\frac{5}{2}}} + \frac{5\sqrt{|V|}}{16|\tilde{s}|^3}\right|(z)\;dz = \int_x^y \left|\frac{2\tilde{s}'(z)\tilde{s}'''(z) - 3\tilde{s}''^2(z)}{4\sqrt{|\tilde{s}(z)|}\tilde{s}'^3(z)}\right| dz.$$

We used short notation

$$V(x) := x^\alpha - 1,\;\; \tilde{s}(x) := \begin{cases} (\frac{3}{2}\int_1^x \sqrt{y^\alpha - 1}\;dy)^{\frac{2}{3}}, & \text{for } x \geq 1 \\ -\;(\frac{3}{2}\int_x^1 \sqrt{1 - y^\alpha}\;dy)^{\frac{2}{3}}, & \text{for } x < 1. \end{cases} \tag{4}$$

To investigate possible boundness of $\mathcal{V}$ it is useful to notice that $\tilde{s}\tilde{s}'^2 = V$.
If we take into account the behavior of V and $\tilde{s}$

$$\tilde{s}(x) \overset{\infty}{\approx} x^{\frac{\alpha+2}{3}},\quad \tilde{s}(x) \overset{1}{\sim} x - 1,$$

we conclude (using the fact that $\tilde{s} \in C^\infty$) that $\mathcal{V}$ is bounded function on $\mathbb{R}^+$.
Let c be negative root of the equation

$$\mathrm{Ai}(c) = \mathrm{Bi}(c)$$

of the smallest absolute value ($c = -0.36605$).
The functions G, M, N are defined by prescriptions

$$M := \sqrt{2\,\mathrm{Ai}\,\mathrm{Bi}} \quad \text{for} \quad x \geq c\,, \qquad M := \sqrt{\mathrm{Ai}^2 + \mathrm{Bi}^2} \quad \text{for} \quad x \leq c$$

$$G := \sqrt{\frac{\mathrm{Bi}}{\mathrm{Ai}}} \quad \text{for} \quad x \geq c\,, \qquad G := 1 \quad \text{for} \quad x \leq c$$

$$N := \sqrt{\frac{\mathrm{Ai}'^2\,\mathrm{Bi}^2 + \mathrm{Ai}^2\,\mathrm{Bi}'^2}{\mathrm{Ai}\,\mathrm{Bi}}} \quad \text{for} \quad x \geq c\,, \qquad N := \sqrt{\mathrm{Ai}'^2 + \mathrm{Bi}'^2} \quad \text{for} \quad x \leq c$$

By graphical analysis with Mathematica P. Duclos and M. Vittot obtained the following bounds of Airy functions and their derivatives, which are valid on $\mathbb{R}$.

$$|\,\mathrm{Ai}\,| \leq 2\frac{1}{\sqrt{\pi}A\,\omega} \quad , \quad |\,\mathrm{Bi}\,| \leq 2\frac{A}{\sqrt{\pi}\omega}$$
$$|\,\mathrm{Ai}'\,| \leq \frac{\omega}{\sqrt{\pi}A} \quad , \quad |\,\mathrm{Bi}'\,| \leq \frac{A\,\omega}{\sqrt{\pi}} \tag{5}$$

Auxiliary functions A and ω are defined by

$$A(s) := e^{\frac{2}{3}s_+^{\frac{3}{2}}} \quad , \quad \omega(s) := |s|^{\frac{1}{4}} + \frac{1}{2}$$

$s_+ := \max(s, 0)$.
Putting definitions of G, M, N, estimates above (5) and estimates of error (3) together we obtain

$$e^{\frac{2}{3}s_+(x)^{\frac{2}{3}}} \left(|s(x)|^{\frac{1}{4}} + \frac{1}{2} \right) |\varepsilon(x)| \leq \frac{2}{\sqrt{\pi}} \left(\exp(\frac{8}{\pi^2} E^{-\frac{\alpha+2}{2\alpha}} \mathcal{V}(x,\infty)) - 1 \right)$$

$$e^{\frac{2}{3}s_+(x)^{\frac{2}{3}}} \left(|s(x)|^{\frac{1}{4}} + \frac{1}{2} \right)^{-1} |\varepsilon'(x)| \leq \frac{2}{\sqrt{\pi}} \sqrt{\frac{x^\alpha - E}{s(x)}} \left(\exp(\frac{8}{\pi^2} E^{-\frac{\alpha+2}{2\alpha}} \mathcal{V}(x,\infty)) - 1 \right).$$

Finally using the boundness of $\mathcal{V}$ on $\mathbb{R}^+$, we conclude the following estimates of error

$$|\varepsilon(x)| \leq K E^{-\frac{\alpha+2}{2\alpha}} |s(x)|^{-\frac{1}{4}} e^{-\frac{2}{3}s_+(x)^{\frac{3}{2}}} \tag{6}$$

$$|\varepsilon'(x)| \leq K E^{-\frac{\alpha+2}{2\alpha}} \sqrt{\frac{x^\alpha - E}{s(x)}} \left(\frac{1}{2} + |s(x)|^{\frac{1}{4}} \right) e^{-\frac{2}{3}s_+(x)^{\frac{3}{2}}} \tag{7}$$

2.3. **Bohr-Sommerfeld.**
If we want (2) to be Dirichlet solution on $\mathbb{R}^+$, i.e.

$$\Psi(0) = 0 \tag{8}$$

we have to satisfy the condition

$$\mathrm{Ai}(s(0)) + \varepsilon(0) = 0$$

Using well-known asymptotics of Ai

$$\mathrm{Ai}(-z) = z^{-\frac{1}{4}}\frac{1}{\sqrt{\pi}}\cos(\frac{2}{3}z^{\frac{2}{3}} - \frac{\pi}{4})\left(1 + \mathcal{O}(z^{-\frac{3}{2}})\right) \tag{9}$$

$$\mathrm{Ai}(z) = \frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}z^{\frac{3}{2}}}\left(1 + \mathcal{O}(z^{-\frac{3}{2}})\right) \tag{10}$$

and (6), we obtain

$$\frac{1}{\sqrt{\pi}|s(0)|^{\frac{1}{4}}}\cos\left(\frac{2}{3}|s(0)|^{\frac{3}{2}} - \frac{\pi}{4}\right)\left(1 + \mathcal{O}(E^{-\frac{\alpha+2}{2\alpha}})\right) = -\varepsilon(0)$$

$$\iff \sin\left(\frac{2}{3}|s(0)|^{\frac{3}{2}} + \frac{\pi}{4}\right) = -\sqrt{\pi}|s(0)|^{\frac{1}{4}}\varepsilon(0)\left(1 + \mathcal{O}(E^{-\frac{\alpha+2}{2\alpha}})\right)$$

$$\iff \frac{2}{3}|s(0)|^{\frac{3}{2}} + \frac{\pi}{4} = \pi n + \mathcal{O}(E^{-\frac{\alpha+2}{2\alpha}})$$

what yields Bohr-Sommerfeld quantization condition

$$\int_0^{E^{\frac{1}{\alpha}}}\sqrt{E - x^\alpha}\,dx = \pi(n - \frac{1}{4}) + \mathcal{O}(E^{-\frac{\alpha+2}{2\alpha}}), \tag{11}$$

or equivalently

$$\int_0^{E^{\frac{1}{\alpha}}}\sqrt{E - x^\alpha}\,dx = \pi(n - \frac{1}{4}) + \mathcal{O}(\frac{1}{n}).$$

Remark 2.1. *This formula, well-known as Bohr-Sommerfeld quantization condition, may be seen as the defining relationship between quantum mechanical Energy E and a natural number n.*
What is interesting is that this formula is here rigorously proven for the class of homogeneous potentials x^α, using Liouville-Green method.
This result can be similarly proven to much more general class of potentials.

To satisfy (8), we define the Energy behavior by

$$E_n = \left(\frac{\pi}{\int_0^1\sqrt{1 - x^\alpha}dx}(n - \frac{1}{4})\right)^{\frac{2\alpha}{\alpha+2}} + \mathcal{O}\left((\frac{1}{n})^{\frac{2\alpha}{\alpha+2}}\right). \tag{12}$$

2.4. **Normalization constant and diagonal elements.**
In this section we will compute the leading terms of normalization constant of the solution (2) and of diagonal matrix elements of some regular perturbation with respect to functions (2). Firstly we will formulate the result of this subsection, which will be proved at the end of it.

2.4.1. *Result.*

Theorem 2.2. *Let $\alpha > 1$ and function Ψ is defined by (2). This means that Ψ is the only $L^2(\mathbb{R}+)$ solution of equation*

$$(-\Delta + x^\alpha)\Psi(x) = E\ \Psi(x).$$

Then for normalization constant holds

$$C_E^{-2} := <\Psi,\ \Psi> = \int_0^\infty|\Psi|^2 = \frac{E^{\frac{2-\alpha}{2\alpha}}}{2\pi}\int_0^1\frac{dx}{\sqrt{1 - x^\alpha}} + \mathcal{O}(E^{\frac{16-10\alpha}{18\alpha}}). \tag{13}$$

Suppose moreover that a perturbation W satisfies the following decay properties ($< x >= \sqrt{1+x^2}$).

$$W, W' \in L^1(\mathbb{R}^+), \; < x >^r W \in L^\infty(\mathbb{R}^+)\,, \quad \text{for some r} > 1 \tag{14}$$

Then it holds true that

$$\frac{< W\,\Psi,\,\Psi >}{< \Psi,\,\Psi >} = E^{-\frac{1}{\alpha}} \left(\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} \right)^{-1} \left(\int_{\mathbb{R}_+} W + \mathcal{O}\left(E^{-\min\{\frac{\alpha+2}{18\alpha}, \frac{r-1}{\alpha}\}} \right) \right) \tag{15}$$

If we want Ψ to satisfy Dirichlet boundary condition (8), then Energy has to be restricted by Bohr-Sommerfeld quantization condition (12) and then the quantity (15) becomes diagonal matrix element

$$\frac{< W\,\Psi_n,\,\Psi_n >}{< \Psi_n,\,\Psi_n >} = \left(\frac{\pi(n-\frac{1}{4})}{\int_0^1 \sqrt{1-x^\alpha}dx} \right)^{-\frac{2}{\alpha+2}} \left(\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} \right)^{-1} \left(\int_{\mathbb{R}_+} W + \mathcal{O}\left(n^{-t}\right) \right), \tag{16}$$

where $t := \min\{\frac{1}{9}, \frac{2(r-1)}{\alpha+2}\} > 0$

Corollary 2.3. *Straightforward consequence of (13) is that the quantum density of probability*

$$p_Q(x) := \frac{1}{\sqrt{\int_{\mathbb{R}+} |\Psi|^2}} |\Psi(x)|^2$$

tends to the classical one

$$p_C(x) := \frac{1}{\int_0^1 (1-y^\alpha)^{-\frac{1}{2}} dy} \frac{\chi_{[0,1]}(x)}{\sqrt{1-x^\alpha}}$$

in $L^1(\mathbb{R}_+)$ limit, or equivalently

$$\int_{\mathbb{R}_+} |p_Q - p_C| \overset{E\to\infty}{\longrightarrow} 0$$

Remark 2.4. It is interesting that both integrals in (16) are important classical quantities. (This is the reason, why we didn't write their explicit numerical values, although they are known.)
The first one

$$\int_0^1 \sqrt{1-x^\alpha}dx$$

is the volume of area $\{(q,p) | E(q,p) \leq 1\}$ in phase space.
The second one

$$\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}}$$

is proportional to the period of the classical motion with Energy $E = 1$.

Because there is only one index n in this subsection, which labels the Energy E_n according to Bohr-Sommerfeld condition (12), we will use scaling to replace E_n, which is assumed to be large, by small parameter $\hbar$.
The reason of this scaling is only comfort which comes with manipulation with $\hbar$. It is of course possible to pass everything without scaling, like in the part dedicated to offdiagonal elements where it is necessary, because of the presence of both indexes n, l.

2.4.2. *Scaling.*
Consider equation

$$(-\Delta + x^\alpha - E)\Psi(x) = 0.$$

This is for any λ equivalent to the equation

$$(-\lambda^{-2-\alpha}\Delta + x^\alpha - \lambda^{-\alpha}E)\Psi(\lambda x) = 0$$

We define the scaling by relations

$$\lambda^{-\alpha}E := 1\ , \quad \lambda^{-2-\alpha} := \hbar^2\ , \quad \Phi(x) := \lambda^{\frac{1}{2}}\Psi(\lambda x)$$

what means that our problem (1) is transformed to

$$(-\hbar^2\Delta + x^\alpha - 1)\Phi(x) = 0\ , \quad E = \hbar^{-\frac{2\alpha}{2+\alpha}}\ , \quad \Phi(x) = \hbar^{-\frac{\beta}{2}}\Psi(\hbar^{-\beta}x)\ , \quad \beta := \frac{2}{\alpha+2}\ . \tag{17}$$

We see that investigation of the behavior with respect to large parameter E is equivalent to the one with respect to small parameter $\hbar$.
By rescaling (2), (6) we conclude that exact $L^2(\mathbb{R}+)$ solution of equation

$$(-\hbar^2\Delta + V)\Phi = 0 \tag{18}$$

can be written in the following form (we remind the definition of V and $\tilde{s}$ (4)):

$$\Phi(x) = \hbar^{\frac{\alpha-4}{3(\alpha+2)}}\left(\frac{\tilde{s}(x)}{V(x)}\right)^{\frac{1}{4}}(\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{s}(x)) + \varepsilon(\hbar, x)), \tag{19}$$

where the error term ε is controlled by inequality

$$|\varepsilon(\hbar, x)| \le K\hbar^{\frac{7}{6}}|\tilde{s}(x)|^{-\frac{1}{4}}\exp(-\frac{2}{3}\tilde{s}_+(x)^{\frac{3}{2}}) \tag{20}$$

(Again we defined $\tilde{s}_+ := \max(\tilde{s}, 0)$).
For simplicity, we define new function

$$\tilde{\Phi}(x) := \left(\frac{\tilde{s}(x)}{V(x)}\right)^{\frac{1}{4}}(\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{s}(x)) + \varepsilon(\hbar, x)) \implies \Phi = \hbar^{\frac{\alpha-4}{3(\alpha+2)}}\tilde{\Phi} \tag{21}$$

The quantities we are interested in, are transformed by

$$<\Psi,\ \Psi> = <\Phi,\ \Phi> = \hbar^{\frac{2(\alpha-4)}{3(\alpha+2)}} <\tilde{\Phi},\ \tilde{\Phi}> = E^{\frac{4-\alpha}{3\alpha}} <\tilde{\Phi},\ \tilde{\Phi}> \tag{22}$$

$$\frac{<W\,\Psi,\ \Psi>}{<\Psi,\ \Psi>} = \frac{<W(\frac{\cdot}{\hbar^\beta})\,\Phi,\ \Phi>}{<\Phi,\ \Phi>} = \frac{<W(\frac{\cdot}{\hbar^\beta})\,\tilde{\Phi},\ \tilde{\Phi}>}{<\tilde{\Phi},\ \tilde{\Phi}>} \tag{23}$$

In this section we will not need the estimate of ε'.

2.4.3. *Normalization constant.*

Theorem 2.5. *For the function $\tilde{\Phi}$ given by (21) is true*

$$\int_0^\infty |\tilde{\Phi}|^2 = \frac{\hbar^{\frac{1}{3}}}{2\pi}\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} + \mathcal{O}(\hbar^{\frac{4}{9}}).$$

Proof. Let us define

$$\phi(x) := (\frac{\tilde{s}(x)}{V(x)})^{\frac{1}{4}}\,\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{s}(x)) \tag{24}$$

We would like to compute leading term of integral

$$I(\hbar) := \int_0^\infty |\phi(x)|^2 dx. \tag{25}$$

This expression looks very complicated, for example for $\alpha = 2$ is the function $\tilde{s}$ equal to

$$\tilde{s}(x) = -\left(\frac{3}{4}(\arcsin x + x\sqrt{1-x^2})\right)^{\frac{2}{3}}, \text{ for } x \leq 1.$$

What is important, is that the function $\tilde{s}$ maps the point 1 to the origin. But point 1 is nothing, but the turning point (of classical problem) of our rescaled potential $V = x^\alpha - 1$. Now because we are interested in $\hbar \to 0$ regime, we conclude that there are three imported area of very different behavior of $\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{s})$ (we suppose that $0 < \gamma < \frac{2}{3}$).

1) for $x \in [0, 1-\hbar^\gamma]$ is the argument of Airy function $\hbar^{-\frac{2}{3}}\tilde{s}(x)$ mapped to the "neighborhood" of $-\infty$, $[-\hbar^{-\frac{2}{3}}K, -\hbar^{-\frac{2}{3}+\gamma}K]$. We use asymptotic (9), Ai is fast oscillating.

2) $x \in [1-\hbar^\gamma, 1+\hbar^\gamma]$. This small window of "regular" regime of Ai, where the function is only bounded, is in fact mapped to the whole line. But the influence to integrals like $I(\hbar)$ is controlled by $\hbar^\gamma$ and can be suppressed.

3) $x \in [1-\hbar^\gamma, \infty)$ the argument of Airy function, $\hbar^{-\frac{2}{3}}\tilde{s}(x)$ is mapped to the neighborhood of $+\infty$, $[\hbar^{-\frac{2}{3}+\gamma}K, +\infty)$. We use asymptotic (10), Ai is exponentially decaying.

Because of the exponential decay of Ai in the area 3) and because of the fact, we can control the influence of the "window" (area 2) by choosing γ, we will see that the main contribution of integral (25) will come from the oscillating area 1).
This strategy will be used in all computation, but the use of exponential decay will be necessary only in part dealing with off-diagonal matrix elements. Where it is not necessary we will omit it to simplify the text.
The "regular" behavior of function ϕ in the neighborhood of turning point is the reason, why we chose this type of model solution instead of usual WKB. It is well-known that usual WKB (see [Ol] Chapter 6) is diverging in the turning point. Choosing our type of model solution we can omit this fact and using explained strategy we obtain in fact usual WKB as asymptotic of Ai (see (9), (10)).

To compute the leading term of normalization constant it is sufficient to split integral $I(\hbar)$ into following three parts (as we said we will omit exponential decay in the area 3)).

$$I(\hbar) = \int_1^\infty + \int_{1-\hbar^\gamma}^1 + \int_0^{1-\hbar^\gamma},$$

where $\gamma > 0$ will be defined later.
Let us introduce auxiliary function

$$g_\alpha(x) := \frac{\tilde{s}(x)}{V(x)}. \tag{26}$$

It is easy to see that g_α is bounded on $\mathbb{R}^+$ for $\alpha > 1$. So we may estimate (using $\tilde{s}\,\tilde{s}'^2 = V$)

$$\begin{aligned}
\int_1^\infty |\phi(x)|^2 dx &= \int_1^\infty (\frac{\tilde{s}(x)}{V(x)})^{\frac{1}{2}} \mathrm{Ai}^2(\hbar^{-\frac{2}{3}}\tilde{s}(x))dx \\
&= \int_1^\infty (\frac{\tilde{s}(x)}{V(x)}) \mathrm{Ai}^2\left(\hbar^{-\frac{2}{3}}\tilde{s}(x)\right)\tilde{s}'(x)dx \\
&\leq \sup_{[1,+\infty)} g_\alpha \int_1^\infty \mathrm{Ai}^2\left(\hbar^{-\frac{2}{3}}\tilde{s}(x)\right)\tilde{s}'(x)dx \\
&= \hbar^{\frac{2}{3}} \sup_{[1,+\infty)} g_\alpha \int_0^\infty \mathrm{Ai}^2(z)dz.
\end{aligned} \tag{27}$$

Because of the boundness of g_α and Airy function Ai we may easily conclude

$$\int_{1-\hbar^\gamma}^{1} |\phi(x)|^2 dx = \mathcal{O}(\hbar^\gamma). \tag{28}$$

Remaining (and dominant) is this term

$$\int_0^{1-\hbar^\gamma} (\frac{\tilde{s}(x)}{V(x)})^{\frac{1}{2}} \operatorname{Ai}^2(\hbar^{-\frac{2}{3}}\tilde{s}(x))dx.$$

We will follow the explained strategy to the area 1). An easy computation gives

$$\tilde{s}(1-\hbar^\gamma) = -(\frac{3}{2}\int_{1-\hbar^\gamma}^{1} \sqrt{1-y^\alpha}dy)^{\frac{2}{3}} = -(\frac{3}{2}\sqrt{\alpha})^{\frac{2}{3}}\hbar^\gamma + \mathcal{O}(\hbar^{\frac{4\gamma}{3}}).$$

So as we said, the argument lies in fact in the "neighborhood" of $-\infty$. As was announced, we will use asymptotics (9) of Ai.

This applied yields

$$\begin{aligned}
&\int_0^{1-\hbar^\gamma} (\frac{\tilde{s}(x)}{V(x)})^{\frac{1}{2}} \operatorname{Ai}^2(\hbar^{-\frac{2}{3}}\tilde{s}(x))dx \\
&= \frac{\hbar^{\frac{1}{3}}}{\pi}\int_0^{1-\hbar^\gamma} \frac{\cos^2(\frac{1}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy - \frac{\pi}{4})}{\sqrt{1-x^\alpha}}(1+\mathcal{O}(\hbar^{1-\frac{3}{2}\gamma}))dx \\
&= \frac{\hbar^{\frac{1}{3}}}{\pi}\int_0^{1-\hbar^\gamma} \frac{1}{\sqrt{1-x^\alpha}}\cos^2(\frac{1}{\hbar}\int_x^1 \sqrt{1-y^\alpha}dy - \frac{\pi}{4})\; dx + \\
&+ \mathcal{O}(\hbar^{1-\frac{3}{2}\gamma})\;\hbar^{\frac{1}{3}}\int_0^{1-\hbar^\gamma}\frac{dx}{\sqrt{1-x^\alpha}}
\end{aligned} \tag{29}$$

The second term is of order $\hbar^{\frac{1}{3}}\mathcal{O}(\hbar^{1-\frac{3}{2}\gamma})$.

Because the argument of $\cos^2$ is large, it causes fast oscillating. We will use identity $\cos^2(a) = \frac{1}{2}(1+\cos(2a))$ to compute "average" and the rest, which is negligible thanks to the fast oscillation.

$$\begin{aligned}
&\int_0^{1-\hbar^\gamma}\frac{dx}{\sqrt{1-x^\alpha}}\cos^2(\frac{1}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy - \frac{\pi}{4}) \\
&= \frac{1}{2}\int_0^{1-\hbar^\gamma}\frac{dx}{\sqrt{1-x^\alpha}} + \frac{1}{2}\int_0^{1-\hbar^\gamma}\frac{dx}{\sqrt{1-x^\alpha}}\cos(\frac{2}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy - \frac{\pi}{2})
\end{aligned}$$

From the first integral will come the leading term

$$\int_0^{1-\hbar^\gamma}\frac{dx}{\sqrt{1-x^\alpha}} = \int_0^1\frac{dx}{\sqrt{1-x^\alpha}} - \int_{1-\hbar^\gamma}^1\frac{dx}{\sqrt{1-x^\alpha}} = \int_0^1\frac{dx}{\sqrt{1-x^\alpha}} + \mathcal{O}(\hbar^{\frac{\gamma}{2}}) \tag{30}$$

To suppress the second one, we will integrate by parts

$$\begin{aligned}
&\int_0^{1-\hbar^\gamma}\frac{dx}{\sqrt{1-x^\alpha}}\sin(\frac{2}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy) = \int_0^{1-\hbar^\gamma}\sin(\frac{2}{\hbar}f(x))\frac{f'(x)}{(f'(x))^2}dx \\
&= -\frac{\hbar}{2}\left[\frac{\cos(\frac{2f}{\hbar})}{(f'^2)}\right]_0^{1-\hbar^\gamma} + \frac{\hbar}{2}\int_0^{1-\hbar^\gamma}\cos(\frac{2}{\hbar}f(x))\frac{f''(x)}{(f'(x))^3}dx
\end{aligned}$$

We defined auxiliary function

$$f(x) := \int_x^1\sqrt{1-y^\alpha}dy. \tag{31}$$

The fact that
$$\frac{f''(x)}{(f'(x))^3} = -\frac{\alpha x^{\alpha-1}}{2(1-x^\alpha)^2},$$
allows us to see
$$\int_0^{1-\hbar^\gamma} \frac{dx}{\sqrt{1-x^\alpha}} \cos\left(\frac{2}{\hbar}\int_x^1 \sqrt{1-y^\alpha}dy - \frac{\pi}{4}\right) = \mathcal{O}(\hbar^{1-\gamma}) + \mathcal{O}(\hbar^{1-2\gamma}) \tag{32}$$
Collecting all these informations (27) - (32) yields
$$I(\hbar) = \frac{\hbar^{\frac{1}{3}}}{2\pi}\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} + \mathcal{O}(\hbar^{\frac{2}{3}}) + \mathcal{O}(\hbar^\gamma) + \hbar^{\frac{1}{3}}(\mathcal{O}(\hbar^{\frac{\gamma}{2}}) + \mathcal{O}(\hbar^{1-2\gamma}))$$
Now choosing $\gamma := \frac{4}{9}$, we finally obtain the leading term of normalization constant
$$I(\hbar) = \int_0^\infty \left(\frac{\tilde{s}(x)}{V(x)}\right)^{\frac{1}{2}} \mathrm{Ai}^2(\hbar^{-\frac{2}{3}}\tilde{s}(x))dx = \frac{\hbar^{\frac{1}{3}}}{2\pi}\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} + \mathcal{O}(\hbar^{\frac{4}{9}}) \tag{33}$$

To finish the proof we have to show that the contribution of error to normalization constant for exact solution of (18) is negligible.
$$\int_0^\infty |\tilde{\Phi}|^2 = \int_0^\infty |\phi|^2 + \int_0^\infty \left(\frac{\tilde{s}}{V}\right)^{\frac{1}{2}}|\varepsilon|^2 + 2\int_0^\infty \left(\frac{\tilde{s}}{V}\right)^{\frac{1}{2}} Re(\overline{\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{s})}\ \varepsilon)$$
Thanks to the boundness of Airy function and g_α (see (26)), we have easily (with help of (20))
$$\int_0^\infty \left(\frac{\tilde{s}}{V}\right)^{\frac{1}{2}}|\varepsilon|^2 = \mathcal{O}(\hbar^{\frac{7}{3}}) \tag{34}$$
$$\int_0^\infty \left(\frac{\tilde{s}}{V}\right)^{\frac{1}{2}} Re(\overline{\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{s})}\ \varepsilon) = \mathcal{O}(\hbar^{\frac{7}{6}}). \tag{35}$$
So finally, using the result (33) we have this expression of normalization constant for arbitrary $\alpha > 1$
$$\int_0^\infty |\tilde{\Phi}|^2 = \frac{\hbar^{\frac{1}{3}}}{2\pi}\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} + \mathcal{O}(\hbar^{\frac{4}{9}}). \tag{36}$$
This is nothing but the statement of Theorem 2.5 . ∎

2.4.4. *Diagonal matrix elements.*
In this section we will follow again the strategy explained in the beginning of the Proof of Theorem 2.5 to compute leading term of diagonal matrix elements.
Because the order of the leading term will be $\hbar^{\frac{1}{3}+\frac{2}{2+\alpha}}$, what is more then in the case of normalization constant, we will have to assume some decay properties of the perturbation W, to estimate better the rest.
First of all let us formulate the result of this subsection.

Theorem 2.6. *Suppose that the perturbation W satisfies decay properties (14). Then for "rescaled diagonal matrix elements" holds true that*
$$< W\left(\frac{\cdot}{\hbar^\beta}\right)\tilde{\Phi}, \tilde{\Phi} >= \frac{\hbar^{\frac{1}{3}+\beta}}{2\pi}\left(\int_0^\infty W + \mathcal{O}\left(\hbar^{\min\{\beta(r-1),\, \frac{\alpha}{2+\alpha}\}}\right)\right) \tag{37}$$

Under assumptions (14) the main contribution of (37) comes from the neighborhood of origin.

To make the proof more easy to follow we split the computation into several parts.

Tail estimate.
In this part we will estimate the contribution of tail ($a<1$)

$$\begin{aligned}\int_a^\infty W(\frac{x}{\hbar^\beta})|\tilde\Phi(x)|^2dx &\le \sup\{W(\frac{x}{\hbar^\beta}),\ x>a\}\int_a^\infty|\tilde\Phi|^2\\ &\le C_\hbar^{-2}\sup\{W(x),\ x>\frac{a}{\hbar^\beta}\}\\ &\le C_\hbar^{-2}\sup\{<x>^{-r}, x>\frac{a}{\hbar^\beta}\}\|<x>^r W\|_\infty\\ &= C_\hbar^{-2}\frac{\hbar^{\beta r}}{(a^2+\hbar^{2\beta})^{\frac{r}{2}}}\|<x>^r W\|_\infty\end{aligned}$$

We define normalization constant

$$C_\hbar^{-2}:=\int_0^\infty|\tilde\Phi|^2$$

So the result is

$$\int_a^\infty W(\frac{x}{\hbar^\beta})|\tilde\Phi(x)|^2dx\le\frac{C_\hbar^{-2}}{a^r}\hbar^{\beta r}\|<x>^r W\|_\infty \tag{38}$$

Error contribution.
We know that $L^2(\mathbb{R}_+)$ solution of (18) can be expressed in the form (19), with error $\varepsilon(\hbar,x)$ controlled by (20).
Taking into account the definition (21) we will first of all use estimate (38) to get

$$\begin{aligned}<W(\frac{\cdot}{\hbar^\beta})\,\tilde\Phi,\ \tilde\Phi> &= \int_0^a W(\frac{x}{\hbar^\beta})|\tilde\Phi(x)|^2dx+\int_a^\infty W(\frac{x}{\hbar^\beta})|\tilde\Phi(x)|^2dx\\ &= \int_0^a W(\frac{x}{\hbar^\beta})|\tilde\Phi(x)|^2dx+\mathcal{O}(\hbar^{\beta r})C_\hbar^{-2}\end{aligned}$$

Now we will show that the influence of error to the quantity $<W(\frac{\cdot}{\hbar^\beta})\,\tilde\Phi,\ \tilde\Phi)>$ is of order $\hbar^{\frac{13}{6}}$ (ϕ is defined by (24)).

$$\begin{aligned}\int_0^a W(\frac{\cdot}{\hbar^\beta})|\tilde\Phi|^2 &= \int_0^a W(\frac{\cdot}{\hbar^\beta})|\phi|^2+\int_0^a W(\frac{\cdot}{\hbar^\beta})(\frac{\tilde s}{V})^{\frac12}|\varepsilon|^2\\ &\quad+2\int_0^a W(\frac{\cdot}{\hbar^\beta})(\frac{\tilde s}{V})^{\frac12}\ Re(\overline{\mathrm{Ai}(\hbar^{-\frac23}\tilde s)}\ \varepsilon)\end{aligned}$$

Similarly to (34), using the boundness of W

$$\int_0^a W(\frac{\cdot}{\hbar^\beta})(\frac{\tilde s}{V})^{\frac12}|\varepsilon|^2=\mathcal{O}(\hbar^{\frac73}).$$

For the third integral is the power $\frac76$ of $\hbar$ in (35) not sufficient. We will do the estimate more carefully

$$\begin{aligned}&\left|\int_0^a W(\frac{\cdot}{\hbar^\beta})(\frac{\tilde s}{V})^{\frac12}\ Re(\overline{\mathrm{Ai}(\hbar^{-\frac23}\tilde s)}\ \varepsilon)\right|\le K\hbar^{\frac76}\int_0^a|\tilde s(x)|^{-\frac14}|\,\mathrm{Ai}(\hbar^{-\frac23}\tilde s(x))|dx\\ &=K\hbar^{\frac76}\hbar^{\frac16}\int_0^a|\tilde s(x)|^{-\frac12}\left|\cos(\frac1\hbar\int_x^1\sqrt{-V}-\frac\pi4)\right|dx\ \ (1+\mathcal{O}(\hbar))=\mathcal{O}(\hbar^{\frac43}).\end{aligned}$$

So we yield

$$\int_0^a W(\frac{\cdot}{\hbar^\beta})|\tilde{\Phi}(x)|^2 = \int_0^a W(\frac{\cdot}{\hbar^\beta})|\phi|^2 + \mathcal{O}(\hbar^{\frac{4}{3}}) \tag{39}$$

Leading term of $< W(\frac{\cdot}{\hbar^\beta})\,\tilde{\Phi},\ \tilde{\Phi}) >$.

Now we will concentrate to the most important term, from which the leading term will come. We will use the same tricks like in (29) - (32).

$$\begin{aligned}
&\int_0^a W(\frac{x}{\hbar^\beta})(\frac{\tilde{s}(x)}{V(x)})^{\frac{1}{2}}\,\mathrm{Ai}^2(\hbar^{-\frac{2}{3}}\tilde{s}(x))dx \\
&= \frac{\hbar^{\frac{1}{3}}}{\pi}\int_0^a W(\frac{x}{\hbar^\beta})\frac{\cos^2(\frac{1}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy-\frac{\pi}{4})}{\sqrt{1-x^\alpha}}(1+\mathcal{O}(\hbar))dx \\
&= \frac{\hbar^{\frac{1}{3}}}{2\pi}\int_0^a W(\frac{x}{\hbar^\beta})\frac{dx}{\sqrt{1-x^\alpha}} \\
&+\frac{\hbar^{\frac{1}{3}}}{2\pi}\int_0^a W(\frac{x}{\hbar^\beta})\frac{dx}{\sqrt{1-x^\alpha}}\sin(\frac{2}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy)+\mathcal{O}(\hbar^{\frac{4}{3}}).
\end{aligned} \tag{40}$$

The first integral extends the window into whole line

$$\begin{aligned}
&\int_0^a W(\frac{x}{\hbar^\beta})\frac{dx}{\sqrt{1-x^\alpha}} = \hbar^\beta\int_0^\infty W - \int_a^\infty W(\frac{x}{\hbar^\beta})dx + \int_0^a W(\frac{x}{\hbar^\beta})\left(\frac{1}{\sqrt{1-x^\alpha}}-1\right)dx \\
&= \hbar^\beta\int_0^\infty W - K\|W<x>^r\|_\infty\,\hbar^{\beta\, r} \\
&\quad +K\|W<x>^u\|_\infty\int_0^a\frac{x^\alpha}{(1+(\hbar^{-\beta}x)^2)^{\frac{u}{2}}}\frac{dx}{(1+\sqrt{1-x^\alpha})\sqrt{1-x^\alpha}} \\
&= \hbar^\beta\int_0^\infty W(x)dx + \mathcal{O}\left(\hbar^{\beta\, r}\right) + \mathcal{O}\left(\hbar^{\beta\, u}\right) = \mathcal{O}\left(\hbar^{\beta\, u}\right).
\end{aligned} \tag{41}$$

We chose

$$u := \min\{r, 1+\frac{\alpha}{2}\}. \tag{42}$$

The second term in (40) we will integrate by parts

$$\begin{aligned}
&\int_0^a W(\frac{x}{\hbar^\beta})\frac{dx}{\sqrt{1-x^\alpha}}\sin(\frac{2}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy) = \int_0^a W(\frac{x}{\hbar^\beta})\sin(\frac{2}{\hbar}f(x))\frac{f'(x)}{(f'(x))^2}dx \\
&= -\frac{\hbar}{2}\left[W(\frac{\cdot}{\hbar^\beta})\frac{\cos(\frac{2f}{\hbar})}{(f'^2)}\right]_0^a + \hbar\int_0^a\cos(\frac{2}{\hbar}f(x))\frac{f''(x)}{(f'(x))^3}W(\frac{x}{\hbar^\beta})dx \\
&\quad -\hbar^{-\beta}\frac{\hbar}{2}\int_0^a\cos(\frac{2}{\hbar}f(x))\frac{W'(\frac{x}{\hbar^\beta})}{(f'(x))^2}dx
\end{aligned} \tag{43}$$

Auxiliary function f is again defined by (31).
We estimate roughly, (using (14))

$$\left|\hbar^{-\beta}\int_0^a\cos(\frac{2}{\hbar}f(x))\frac{W'(\frac{x}{\hbar^\beta})}{(f'(x))^2}dx\right| \le K\hbar^{-\beta}\int_0^a\left|W'(\frac{x}{\hbar^\beta})\right|dx = K\int_0^{a\hbar^{-\beta}}|W'(x)|\,dx = K$$

Now because all functions in (43) are bounded on $[0,a]$ we have

$$\int_0^a W(\frac{x}{\hbar^\beta})\frac{dx}{\sqrt{1-x^\alpha}}\cos(\frac{2}{\hbar}\int_x^1\sqrt{1-y^\alpha}dy-\frac{\pi}{4})=\mathcal{O}(\hbar) \tag{44}$$

Putting (40), (41), (42) and (44) together yields

$$\begin{aligned}
&\int_0^a W(\frac{x}{\hbar^\beta})(\frac{\tilde{s}(x)}{V(x)})^{\frac{1}{2}}\operatorname{Ai}^2(\hbar^{\frac{-2}{3}}\tilde{s}(x))dx\\
&=\frac{\hbar^{\frac{1}{3}}}{2\pi}\left(\hbar^\beta\int_0^\infty W+\mathcal{O}\left(\hbar^{\beta\, u}\right)\right)+\mathcal{O}(\hbar^{\frac{4}{3}})\\
&=\frac{\hbar^{\beta+\frac{1}{3}}}{2\pi}\left(\int_0^\infty W+\mathcal{O}\left(\hbar^{\min\{\beta(r-1),\,\frac{\alpha}{\alpha+2}\}}\right)\right)
\end{aligned} \tag{45}$$

Proof of the Theorem 2.6.

Proof. If we collect all results (36), (38), (39) and (45), we obtain what we wanted to prove

$$\begin{aligned}
< W(\frac{\cdot}{\hbar^\beta})\,\tilde{\Phi},\ \tilde{\Phi}> &= \int_0^a W(\frac{\cdot}{\hbar^\beta})\,|\tilde{\Phi}|^2+\int_a^\infty W(\frac{\cdot}{\hbar^\beta})\,|\tilde{\Phi}|^2\\
&= \int_0^a W(\frac{\cdot}{\hbar^\beta})|\phi|^2+\mathcal{O}(\hbar^{\frac{4}{3}})+C_\hbar^{-2}\mathcal{O}(\hbar^{\beta r})\\
&= \frac{\hbar^{\frac{1}{3}+\beta}}{2\pi}\left(\int_0^\infty W+\mathcal{O}\left(\hbar^{\min\{\beta(r-1),\,\frac{\alpha}{2+\alpha}\}}\right)\right)
\end{aligned}$$

∎

2.4.5. *Proof of the Theorem 2.2.*

Now if we collect proven statements of Theorems 2.5 and 2.6 and use (22), (23), the relationship (17) between E and $\hbar$, we may conclude that Theorem 2.2 is proven. ∎

2.5. **Offdiagonal elements.**
To understand the behavior of all elements in matrix is our precise analysis of diagonal not much important, because the diagonal is only "set of zero measure".
There are several main directions in our matrix, the first one is parallel to the diagonal, the second one is normal to the diagonal and the third one is to go through columns with a fixed row.
Our guess is that the behavior of elements close to the diagonal is more or less the same like on the diagonal. We will formulate this result with help of Cauchy-Schwartz inequality. This estimate will be valid for all elements, but elements far from the diagonal decay much faster, so we will use other, direct method to improve the decay estimate for elements far from the diagonal.

2.5.1. *Elements close to the diagonal.*
By Cauchy-Schwartz inequality we mean

$$|F(a,b)|^2 \leq q_F(a) \cdot q_F(b),$$

where $F(.,.)$ is a sesquilinear form and $q_F(a) := F(a,a)$ is its corresponding quadratic form. If we apply this inequality to the result of diagonal case we obtain the estimate

Theorem 2.7. *Let $\alpha > 1$, Ψ_n, Ψ_l are two $L^2(\mathbb{R}_+)$ solutions of (1) with corresponding Energies E_n and E_l, which depends on natural numbers n, l according to the prescription (12).*
Suppose moreover that a function W satisfies decay properties (14).
Then all matrix elements can be estimated by

$$W_{n,l} := \frac{< W\, \Psi_n,\, \Psi_l >}{\sqrt{< \Psi_n,\, \Psi_n >}\sqrt{< \Psi_l,\, \Psi_l >}} \leq K(n\, l)^{-\frac{1}{\alpha+2}}$$

2.5.2. *Elements far from to the diagonal.*
We will investigate elements for which

$$n > \delta\, l\,, \quad \delta > 0.$$

directly, by estimating integral $< W\, \Psi_n,\, \Psi_l >$.
Because of the presence of different Energies E_n and E_l is our scaling trick not possible, but we lose by it only comfort.
Similarly to the part dealing with diagonal elements we will divide the area of $\int_{\mathbb{R}_+} W \overline{\Psi}_n \Psi_l$ into two parts, window and tail. By choosing the boundary of them between the turning points of functions Ψ_n , Ψ_l we yield $\mathcal{O}(E_n^{-\infty})$ estimate of the tail! This is caused by exponential decay of function Ψ_l above its turning point. Again only mathematical tool we will use is integration by parts.
Let us choose (without loss of generality) the boundary between the window and the tail by

$$a_n := (\frac{E_n}{2})^{\frac{1}{\alpha}}$$

and consider such n and l for which

$$E_n > 4E_l. \tag{46}$$

Of course, instead of 4 we can choose $\delta^{\frac{2\alpha}{\alpha+2}}$. Again we will first of all formulate the result.

Result.

Theorem 2.8. *Consider the case $n > \delta\, l$ for some positive number δ. $\alpha > 1$.
Assume that the perturbation W satisfies (14) and in addition that W'' is defined and bounded almost everywhere on $\mathbb{R}_+$.
Then matrix elements $W_{n,l}$ can be estimated from above by*

$$W_{n,l} := \frac{\int_{\mathbb{R}_+} W\,\overline{\Psi}_n \Psi_l}{\sqrt{\int_{\mathbb{R}_+} |\Psi_n|^2}\sqrt{\int_{\mathbb{R}_+} |\Psi_l|^2}} = \mathcal{O}\left(n^{-\frac{\alpha+1}{\alpha+2}} l^{\frac{\alpha-1}{\alpha+2}}\right) = \mathcal{O}\left((n\,l)^{-\frac{1}{\alpha+2}}\,(\frac{l}{n})^{\frac{\alpha}{\alpha+2}}\right)$$

Where Ψ_n is Dirichlet $L^2(\mathbb{R}_+)$ solution of equation

$$(-\Delta + x^\alpha)\Psi_n(x) = E_n \Psi_n(x)\ ,$$

so the relationship between energy E_n and natural number n is given by Bohr-Sommerfeld quantization condition (12).

The proof of this theorem will be again given through following text and finished in the very last part of this subsection.
First of all we show that the main contribution comes from window.

Tail estimate.
Set

$$\phi_n(x) := (\frac{s_n(x)}{x^\alpha - E_n})^{\frac{1}{4}}\,\mathrm{Ai}(s_n(x)).$$

In this part we will show that the contribution of the tail

$$\int_{a_n}^\infty W(x)\phi_n(x)\phi_l(x)dx$$

is negligible.
Using (10) we obtain

$$\begin{aligned}
&\int_{a_n}^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha - E_l)(x^\alpha - E_n)}\right)^{\frac{1}{4}} \mathrm{Ai}(s_l(x))\,\mathrm{Ai}(s_n(x))dx \\
&= \int_{a_n}^\infty W(x)\left(\frac{s_n(x)}{x^\alpha - E_n}\right)^{\frac{1}{4}} \frac{\mathrm{Ai}(s_n(x))}{2\sqrt{\pi}\sqrt[4]{|E_l - x^\alpha|}} \exp\left(-\int_{E_l^{\frac{1}{\alpha}}}^x \sqrt{y^\alpha - E_l}\;dy\right) dx \\
&\qquad \times \left(1 + \mathcal{O}(E_n^{-\frac{\alpha+2}{2\alpha}})\right) \\
&= \mathcal{O}(E_n^{-\infty}) \qquad (47)
\end{aligned}$$

Because with help of (46) we can estimate

$$\int_{E_l^{\frac{1}{\alpha}}}^{a_n} \sqrt{y^\alpha - E_l}\;dy = E_l^{\frac{\alpha+2}{2\alpha}} \int_1^{(\frac{E_n}{2E_l})^{\frac{1}{\alpha}}} \sqrt{y^\alpha - 1}\;dy \geq K E_n^{-\frac{\alpha+2}{2\alpha}} \qquad (48)$$

Error estimate.
The results (6), (7) will be often used in this part. The technique we will apply will be

similar like in the tail estimate paragraph.
We shall treat integral

$$\int_0^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha - E_l)(x^\alpha - E_n)}\right)^{\frac{1}{4}}\left(\bar{\varepsilon}_l(x)\operatorname{Ai}(s_n(x)) + \varepsilon_n(x)\operatorname{Ai}(s_l(x)) + \bar{\varepsilon}_l(x)\varepsilon_n(x)\right)dx$$

The last term is the smallest

$$\begin{aligned}
&\left|\int_0^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha - E_l)(x^\alpha - E_n)}\right)^{\frac{1}{4}}\bar{\varepsilon}_l(x)\varepsilon_n(x)dx\right| \qquad (49)\\
&\le K(E_lE_n)^{-\frac{\alpha+2}{2\alpha}}\int_0^\infty |W(x)|\frac{e^{-\frac{2}{3}s_{l+}(x)^{\frac{3}{2}}}e^{-\frac{2}{3}s_{n+}(x)^{\frac{3}{2}}}}{\sqrt[4]{|x^\alpha - E_l||x^\alpha - E_n|}}dx\\
&\le K(E_lE_n)^{-\frac{\alpha+2}{2\alpha}}\int_0^{2a_l}|W(x)|\frac{dx}{\sqrt[4]{|x^\alpha - E_l||x^\alpha - E_n|}} + \mathcal{O}(E_n^{-\frac{3}{4}-\frac{1}{\alpha}}E_l^{-\infty})\\
&\le K\|W\|_\infty(E_lE_n)^{-\frac{\alpha+2}{2\alpha}}E_n^{-\frac{1}{4}}\int_0^{2a_l}\frac{dx}{\sqrt[4]{|x^\alpha - E_l|}} + \mathcal{O}(E_n^{-\frac{3}{4}-\frac{1}{\alpha}}E_l^{-\infty})\\
&\le K\|W\|_\infty E_n^{-\frac{3}{4}-\frac{1}{\alpha}}E_l^{-\frac{1}{2}-\frac{1}{\alpha}-\frac{1}{4}+\frac{1}{\alpha}}\int_0^2\frac{dx}{\sqrt[4]{|x^\alpha - 1|}} + \mathcal{O}(E_n^{-\frac{3}{4}-\frac{1}{\alpha}}E_l^{-\infty})\\
&= \mathcal{O}\left((E_nE_l)^{-\frac{3}{4}}E_n^{-\frac{1}{\alpha}}\right)
\end{aligned}$$

$\mathcal{O}(E^{-\infty})$ results are obtained using exponential decay of ε (see (6)) and Ai (see (10)) and with help of (48). The second integral can be similarly estimated by

$$\begin{aligned}
&\left|\int_0^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha - E_l)(x^\alpha - E_n)}\right)^{\frac{1}{4}}\varepsilon_n(x)\operatorname{Ai}(s_l(x))dx\right|\\
&\le KE_n^{-\frac{\alpha+2}{2\alpha}}\int_0^{a_n}|W(x)|\left(\frac{s_l(x)}{(x^\alpha - E_l)|x^\alpha - E_n|}\right)^{\frac{1}{4}}|\operatorname{Ai}(s_l(x))|dx + \mathcal{O}(E_n^{-\infty})\\
&\le KE_n^{-\frac{3}{4}-\frac{1}{\alpha}}\left(\int_0^{a_l}\frac{|W(x)|dx}{\sqrt[4]{|x^\alpha - E_l|}} + \|\operatorname{Ai}\|_\infty\sup_{[a_l,\,2a_l]}|W(x)|\int_{a_l}^{2a_l}\left(\frac{s_l(x)}{x^\alpha - E_l}\right)^{\frac{1}{4}}dx + \mathcal{O}(E_l^{-\infty})\right)\\
&\quad+\mathcal{O}(E_n^{-\infty})\\
&= E_n^{-\frac{3}{4}-\frac{1}{\alpha}}\left(\mathcal{O}\left(E_l^{-\frac{1}{4}}\right) + \mathcal{O}\left(E_l^{-\frac{r}{\alpha}-\frac{1}{6}+\frac{7}{6\alpha}}\right)\right) + \mathcal{O}(E_n^{-\infty})\ , \qquad (50)
\end{aligned}$$

where we used the fact that

$$\frac{s_l(x)}{x^\alpha - E_l} = E_l^{-\frac{2}{3}+\frac{2}{3\alpha}}u\left(\frac{x}{E_l^{\frac{1}{\alpha}}}\right)$$

for a continuous function u.
To obtain sufficient estimate of the first integral, we will have to integrate by parts, using

again estimates (6), (7) and the relation $s(x)s'^2(x) = x^\alpha - E$.

$$\begin{aligned}
&\int_0^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha - E_l)(x^\alpha - E_n)}\right)^{\frac14} \varepsilon_l(x)\,\mathrm{Ai}(s_n(x))dx \\
&= -\int_0^{a_n} W(x)\left(\frac{s_l(x)}{(x^\alpha - E_l)(E_n - x^\alpha)^3}\right)^{\frac14} \varepsilon_l(x)\cos(f(x))f'(x)dx + \mathcal{O}(E_n^{-\infty}) \\
&= -\left[\frac{W(x)\sin(f(x))\varepsilon_l(x)}{\sqrt{s_l'(x)}\sqrt[4]{E_n - x^\alpha}^3}\right]_0^{a_n} + \int_0^{a_n} \frac{W'(x)\sin(f(x))\varepsilon_l(x)}{\sqrt{s_l'(x)}\sqrt[4]{E_n - x^\alpha}^3}dx \\
&\quad + \int_0^{a_n} \frac{W(x)\sin(f(x))\varepsilon_l(x)}{\sqrt{s_l'(x)}\sqrt[4]{E_n - x^\alpha}^3}\left(\frac{\varepsilon_l'(x)}{\varepsilon_l(x)} - \frac{s_l''(x)}{2s_l'(x)} + \frac{3\alpha x^{\alpha-1}}{4(E_n - x^\alpha)}\right)dx + \mathcal{O}(E_n^{-\infty})
\end{aligned}$$

Auxiliary function f is now defined by

$$f(x) := \int_x^{E_n^{\frac{1}{\alpha}}} \sqrt{E_n - y^\alpha}dy - \frac{\pi}{4}\,, \quad f'(x) = -\sqrt{E_n - x^\alpha}$$

Let us investigate all terms

$$-\left[\frac{W(x)\sin(f(x))\varepsilon_l(x)}{\sqrt{s_l'(x)}\sqrt[4]{E_n - x^\alpha}^3}\right]_0^{a_n} = \frac{W(0)\sin(f(0))\varepsilon_l(0)}{\sqrt{s_l'(0)}E_n^{\frac34}} + \mathcal{O}(E_n^{-\infty}) = \mathcal{O}(E_n^{-\frac34}E_l^{-\frac34-\frac1\alpha})\,, \qquad (51)$$

where we used again (6).
The second term is estimated by (like in (49))

$$\begin{aligned}
&\left|\int_0^{a_n} \frac{W'(x)\sin(f(x))\varepsilon_l(x)}{\sqrt{s_l'(x)}\sqrt[4]{E_n - x^\alpha}^3}dx\right| \le E_n^{-\frac34}E_l^{-\frac12-\frac1\alpha}\left(K\int_0^{2a_l}\frac{|W'(x)|dx}{\sqrt[4]{|E_l - x^\alpha|}} + \mathcal{O}(E_l^{-\infty})\right) \\
&= \mathcal{O}\left(E_n^{-\frac34}E_l^{-\frac34-\frac{1}{2\alpha}}\right) \qquad (52)
\end{aligned}$$

Because using Hölder inequality we may estimate

$$\int_0^{2a} |W'(x)g(\frac{x}{a})|dx \le \sqrt{\|W'\|_\infty \|W'\|_{L^1(\mathbb{R}_+)}}\sqrt{a\int_0^2 |g(x)|^2dx}$$

To treat the third integral it is useful to remark that (again with help of $s(x)s'^2(x) = x^\alpha - E$)

$$\begin{aligned}
s(x) &= E^{\frac13+\frac{2}{3\alpha}}\ \tilde{s}'(\frac{x}{E^{\frac1\alpha}}) \\
s'(x) &= E^{\frac13-\frac{1}{3\alpha}}\ \tilde{s}'(\frac{x}{E^{\frac1\alpha}}) \qquad (53)\\
s''(x) = \frac{V' - s'^3}{2ss'} &= E^{\frac13-\frac{4}{3\alpha}}\ \tilde{s}''(\frac{x}{E^{\frac1\alpha}})
\end{aligned}$$

So using the same tricks and (7) again we conclude

$$\begin{aligned}
&\int_0^{a_n} \frac{W(x)\sin(f(x))\varepsilon_l(x)}{\sqrt{s_l'(x)}\sqrt[4]{E_n - x^\alpha}^3}\left(\frac{\varepsilon_l'(x)}{\varepsilon_l(x)} - \frac{s_l''(x)}{2s_l'(x)} + \frac{3\alpha x^{\alpha-1}}{4(E_n - x^\alpha)}\right)dx \\
&= E_n^{-\frac34}\left(\mathcal{O}(E_l^{-\frac14-\frac1\alpha}) + \mathcal{O}(E_l^{-\frac34-\frac{3}{2\alpha}}) + \mathcal{O}(E_nE_l^{\frac14-\frac{3}{2\alpha}})\right) = \mathcal{O}(E_n^{-\frac34}E_l^{-\frac14-\frac1\alpha}) \qquad (54)
\end{aligned}$$

If we collect these informations (51), (52), (54) we yield

$$\int_0^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha-E_l)(x^\alpha-E_n)}\right)^{\frac14}\varepsilon_l(x)\,\mathrm{Ai}(s_n(x))dx$$
$$=\mathcal{O}(E_n^{-\frac34}E_l^{-\frac34-\frac1\alpha})+\mathcal{O}(E_n^{-\frac34}E_l^{-\frac34-\frac{1}{2\alpha}})+\mathcal{O}(E_n^{-\frac34}E_l^{-\frac14-\frac1\alpha})$$
$$=\mathcal{O}(E_n^{-\frac34}E_l^{-\frac14-\frac{1}{2\alpha}}) \tag{55}$$

Putting (49), (50), (55) together we see that under assumptions $W''\in L^\infty(\mathbb{R}_+)$, (14) on the perturbation W the contribution of the error may be estimated by

$$\int_0^\infty W(x)\left(\frac{s_l(x)s_n(x)}{(x^\alpha-E_l)(x^\alpha-E_n)}\right)^{\frac14}\Big(\bar\varepsilon_l(x)\,\mathrm{Ai}(s_n(x))+\varepsilon_n(x)\,\mathrm{Ai}(s_l(x))+\bar\varepsilon_l(x)\varepsilon_n(x)\Big)dx$$
$$=\mathcal{O}\left(E_n^{-\frac34-\frac1\alpha}E_l^{-\frac{r}{\alpha}-\frac16+\frac{7}{6\alpha}}\right)+\mathcal{O}\left(E_n^{-\frac34}E_l^{-\frac14-\frac{1}{2\alpha}}\right) \tag{56}$$

Window.
We will compute the contribution of

$$\int_0^{a_n} W(x)\phi_n(x)\phi_l(x)dx=$$
$$\int_0^{a_n} W(x)\left(\frac{s_l(x)}{(E_n-x^\alpha)(x^\alpha-E_l)}\right)^{\frac14}\cos\left(\int_x^{E_n^{\frac1\alpha}}\sqrt{E_n-y^\alpha}dy-\frac\pi4\right)\frac{dx}{\sqrt\pi}\left(1+\mathcal{O}(E_n^{-\frac{\alpha+2}{2\alpha}})\right)$$

The main part is proportional to

$$\int_0^{a_n} W(x)\left(\frac{s_l(x)}{(E_n-x^\alpha)(x^\alpha-E_l)}\right)^{\frac14}\cos\left(\int_x^{E_n^{\frac1\alpha}}\sqrt{E_n-y^\alpha}dy-\frac\pi4\right)dx$$
$$=-\left[\frac{W(x)\sin(f(x))\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3}\left(\frac{s_l(x)}{x^\alpha-E_l}\right)^{\frac14}\right]_0^{a_n}+\int_0^{a_n}\frac{W'(x)\sin(f(x))\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}}dx$$
$$+\int_0^{a_n}W(x)\sin(f(x))\left(\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}}\right)'dx$$
$$=-\left[\frac{W(x)\sin(f(x))\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3}\left(\frac{s_l(x)}{x^\alpha-E_l}\right)^{\frac14}\right]_0^{a_n}+\left[\frac{W'(x)\cos(f(x))\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^5\sqrt{s_l'(x)}}\right]_0^{a_n}$$
$$+\int_0^{a_n}W(x)\sin(f(x))\left(\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}}\right)'dx$$
$$-\int_0^{a_n}\cos(f(x))\left(\frac{W'(x)\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^5\sqrt{s_l'(x)}}\right)'dx, \tag{57}$$

where f is again defined by

$$f(x):=\int_x^{E_n^{\frac1\alpha}}\sqrt{E_n-y^\alpha}dy-\frac\pi4,\quad f'(x)=-\sqrt{E_n-x^\alpha}$$

Let us investigate all terms individually.
The first two are very small

$$\left[\frac{W(x)\sin(f(x))\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3}\left(\frac{s_l(x)}{x^\alpha-E_l}\right)^{\frac14}\right]_0^{a_n} = \mathcal{O}(E_n^{-\frac34}E_l^{-\frac34-\frac1\alpha}) \tag{58}$$

$$\left[\frac{W'(x)\cos(f(x))\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^5\sqrt{s_l'(x)}}\right]_0^{a_n} = \mathcal{O}(E_n^{-\frac74-\frac1\alpha}E_l^{-\frac34-\frac1\alpha}) \tag{59}$$

from the same reason like (51), namely with help of (9), (11), (53).
To investigate the third term

$$\int_0^{a_n} W(x)\sin(f(x))\left(\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}}\right)'dx \tag{60}$$

of (57) we firstly compute

$$\left(\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^k\sqrt{s_l'(x)}}\right)' = \left(\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^k\sqrt{s_l'(x)}}\right)\left[\frac{\mathrm{Ai}'(s_l(x))}{\mathrm{Ai}(s_l(x))}s_l'(x)-\frac{s_l''(x)}{2s_l'(x)}+\frac{k\alpha x^{\alpha-1}}{4(E_n-x^\alpha)}\right]$$

Let us estimate the third expression (using similar tricks like in (50))

$$\begin{aligned}&\left|\int_0^{a_n} W(x)\sin(f(x))\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}}\frac{x^{\alpha-1}}{E_n-x^\alpha}dx\right|\\ &\leq E_n^{-\frac74}\left(\int_0^{a_l}\frac{|W(x)|\,x^{\alpha-1}}{\sqrt[4]{E_l-x^\alpha}}dx+K\sup_{[a_l,\,2a_l]}|W(x)|\int_{a_l}^{2a_l}x^{\alpha-1}\left(\frac{s_l(x)}{x^\alpha-E_l}\right)^{\frac14}dx+\mathcal{O}(E_l^{-\infty})\right)\\ &= E_n^{-\frac74}\left(\mathcal{O}(E_l^{\frac34-\frac1\alpha})+\mathcal{O}(E_l^{\frac{5\alpha+1-6r}{6\alpha}})\right)\end{aligned} \tag{61}$$

The second part of the integral above is

$$\begin{aligned}&\left|\int_0^{a_n}\frac{W(x)\sin(f(x))\,\mathrm{Ai}(s_l(x))s_l''(x)}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}^3}dx\right| \leq E_n^{-\frac34}\left(K\int_0^{2a_l}\frac{|W(x)s_l''(x)|}{\sqrt{s_l'(x)}^3}dx+\mathcal{O}(E_l^{-\infty})\right)\\ &= \mathcal{O}\left(E_n^{-\frac34}E_l^{-\frac16-\frac{1}{3\alpha}}\right)\end{aligned} \tag{62}$$

The first part of the third term from (57) can be estimated with help of (5)

$$\int_0^{a_n}\frac{W(x)\sin(f(x))\,\mathrm{Ai}'(s_l(x))\sqrt{s_l'(x)}}{\sqrt[4]{E_n-x^\alpha}^3}dx = \mathcal{O}\left(E_n^{-\frac34}E_l^{\frac14}\right) \tag{63}$$

The results of (61), (62), (63) allows us to say that the third integral (60) in (57)

$$\int_0^{a_n} W(x)\sin(f(x))\left(\frac{\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^3\sqrt{s_l'(x)}}\right)'dx = \mathcal{O}\left(E_n^{-\frac34}E_l^{\frac14}\right) \tag{64}$$

The very last term in (57) is

$$\int_0^{a_n}\cos(f(x))\left(\frac{W'(x)\,\mathrm{Ai}(s_l(x))}{\sqrt[4]{E_n-x^\alpha}^5\sqrt{s_l'(x)}}\right)'dx = \mathcal{O}\left(E_n^{-\frac74}E_l^{\frac14}\right) \tag{65}$$

by the same tricks like for the previous case (60).

The main result of this subsection is obtained by collecting (58), (59), (64) and (65).

$$\int_0^{a_n} W(x)\phi_n(x)\phi_l(x)dx =$$
$$= \mathcal{O}(E_n^{-\frac{3}{4}}E_l^{-\frac{3}{4}-\frac{1}{\alpha}}) + \mathcal{O}(E_n^{-\frac{7}{4}-\frac{1}{\alpha}}E_l^{-\frac{3}{4}-\frac{1}{\alpha}})$$
$$+\mathcal{O}\left(E_n^{-\frac{3}{4}}E_l^{\frac{1}{4}}\right) + \mathcal{O}\left(E_n^{-\frac{7}{4}}E_l^{\frac{1}{4}}\right),$$

what yields

$$\int_0^{a_n} W(x)\phi_n(x)\phi_l(x)dx = \mathcal{O}\left(E_n^{-\frac{3}{4}}E_l^{\frac{1}{4}}\right) \tag{66}$$

Proof of the Theorem 2.8.

Now we are ready to state upper bound of non-diagonal matrix elements.

Proof. Taking (47), (56) and (66) together we yield

$$\int_{\mathbb{R}_+} W\Psi_n\Psi_l = \mathcal{O}(E_n^{-\infty}) + \mathcal{O}\left(E_n^{-\frac{3}{4}-\frac{1}{\alpha}}E_l^{-\frac{r}{\alpha}-\frac{1}{6}+\frac{7}{6\alpha}}\right) + \mathcal{O}\left(E_n^{-\frac{3}{4}}E_l^{-\frac{1}{4}-\frac{1}{2\alpha}}\right) + \mathcal{O}\left(E_n^{-\frac{3}{4}}E_l^{\frac{1}{4}}\right)$$
$$= \mathcal{O}\left(E_n^{-\frac{3}{4}}E_l^{\frac{1}{4}}\right)$$

Because $E_n > 4E_l$.

We use the knowledge about normalization constant (13)

$$\int_{\mathbb{R}_+} \Psi_n^2 = \frac{1}{2\pi}E_n^{-\frac{1}{2}+\frac{1}{\alpha}} \int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} + \mathcal{O}\left(E_n^{\frac{16-10\alpha}{18\alpha}}\right)$$

to get

$$W_{n,\,l} := \frac{\int_{\mathbb{R}_+} W\Psi_n\Psi_l}{\sqrt{\int_{\mathbb{R}_+}\Psi_n^2}\sqrt{\int_{\mathbb{R}_+}\Psi_l^2}} = \mathcal{O}\left(E_n^{-\frac{1}{2}-\frac{1}{2\alpha}}E_l^{\frac{1}{2}-\frac{1}{2\alpha}}\right),$$

what is just the statement of the Theorem 2.8. ∎

2.6. **Summary.**

If we sum up all results we can state the main theorem about asymptotic behavior of matrix elements.

Theorem 2.9. *Let Ψ_n is Dirichlet $L^2(\mathbb{R}_+)$ solution of the equation*

$$(-\Delta + x^\alpha)\Psi_n(x) = E_n\Psi_n(x)\,,$$

where $\alpha > 1$.

Dirichlet boundary condition implies that there is the relationship between energy E_n and a natural number n given by Bohr-Sommerfeld quantization condition

$$E_n = \left(\frac{\pi}{\int_0^1\sqrt{1-x^\alpha}dx}(n-\frac{1}{4})\right)^{\frac{2\alpha}{\alpha+2}} + \mathcal{O}\left((\frac{1}{n})^{\frac{2\alpha}{\alpha+2}}\right).$$

Assume that a perturbation W satisfies the following decay properties ($<x>=\sqrt{1+x^2}$).

$$W, W' \in L^1(\mathbb{R}^+),\ <x>^r W \in L^\infty(\mathbb{R}^+)\,,\quad \text{for some } r>1$$

and in addition that W'' is defined and bounded almost everywhere on $\mathbb{R}_+$.
Then for matrix elements $W_{n,l}$ holds true that

$$W_{n,n} := \frac{< W\,\Psi_n,\, \Psi_n >}{< \Psi_n,\, \Psi_n >} = \left(\frac{\pi(n-\frac{1}{4})}{\int_0^1 \sqrt{1-x^\alpha}dx} \right)^{-\frac{2}{\alpha+2}} \left(\int_0^1 \frac{dx}{\sqrt{1-x^\alpha}} \right)^{-1} \left(\int_{\mathbb{R}_+} W + \mathcal{O}\left(n^{-t}\right) \right),$$

(where $t := \min\{\frac{1}{9}, \frac{2(r-1)}{\alpha+2}\} > 0$) for the diagonal elements. Other elements are estimated from above by

$$W_{n,l} := \frac{< W\,\Psi_n,\, \Psi_l >}{\sqrt{< \Psi_n,\, \Psi_n >}\sqrt{< \Psi_l,\, \Psi_l >}} \leq K(n\,l)^{-\frac{1}{\alpha+2}},$$

or if moreover $n > \delta\, l$, for some $\delta > 0$, by better estimate

$$W_{n,l} \leq K(n\,l)^{-\frac{1}{\alpha+2}}\,(\frac{l}{n})^{\frac{\alpha}{\alpha+2}}. \tag{67}$$

3.1. **Introduction.**
This section is dealing with Egorov theorem, mathematically rigorous tool for approximation of quantum time evolution with help of the classical one. It is proven that the difference between exact quantum evolution and approximation is proportional to $\hbar^2$ in the sense of the statement of Theorem 3.7.
Physical interpretation of such result is clear, when observables of given physical system with the same dimension as $\hbar$ like action are much larger then numerical value of $\hbar$ (i.e. is possible limit $\hbar \to 0$), then quantum evolution of an observable tends to the classical one. The theory of this type uses so called Weyl-quantization, as powerful tool to construct quantum observables (i.e. symmetric or even essentially self-adjoint operators) according to the principal of correspondence.
The result of this section, formulated in Theorem 3.7, is improvement of [Lev]. It is again based on [BR], but the generalization to time dependent systems is led directly using the trick (78). This is idea of Joachim Asch, I would like to thank him for it.
The statement of Theorem 3.7 is improved with respect to [Lev] in several ways. The first is arbitrary dimension, then general time dependent Hamiltonian despite the Schrödinger one is included. The assumptions on the growth of $H(\tau)$ are also weaken.

3.2. **Notation and conventions.**
Let us denote by $X = \mathbb{R}^n$ the configuration space of a classical mechanical system with n degrees of freedom. The corresponding phase space Z is identified with $\mathbb{R}^{2n}$ equipped with the symplectic form σ defined by

$$\sigma(z; z') = \langle Jz, z' \rangle. \tag{68}$$

where $\langle \ , \ \rangle$ is the Euclidean scalar product and J is the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

A generic point in Z is denoted z and its coordinates by (q, p) where $q, p \in \mathbb{R}^n$. We will again use notation $\langle x \rangle = (1 + |x|^2)^{1/2}$.
A classical time dependent Hamiltonian is a time dependent real function, defined on the phase space $H : [s, t] \times Z \to \mathbb{R}$. Basic example is $H(\tau, q, p) = \frac{\|p\|^2}{2m} + V(\tau, q)$, $(m > 0)$ where $\|p\|^2 =< p, p >$ and V is time dependent potential.
In what follows we put emphasis on the case $X = \mathbb{R}^n$.
The motion of the classical system is determined by the system of Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H(t)}{\partial p}(q, p), \quad \frac{dp}{dt} = -\frac{\partial H(t)}{\partial q}(q, p). \tag{69}$$

We will suppose that the equations (69) generate a flow $\Phi(t, s)$ on the phase space Z, defined by

$$\Phi(t, s, q, p) = (q(t, s), p(t, s)) \ , \ \Phi(s, s) = \mathbb{1}.$$

We define Poisson bracket by

$$\{A, B\} = \partial_q A \cdot \partial_p B - \partial_p A \cdot \partial_q B.$$

Here we have used the notation $\partial_q = \frac{\partial}{\partial q}$.

3.3. **Weyl quantization.**
First of all we define a class of classical observables, which are quantizable by Weyl quantization, introduced after.

Definition 3.1. *$A \in \mathcal{O}(m)$, $m \in \mathbb{R}$, if and only if $Z \overset{A}{\to} \mathbb{C}$ is C^∞ in Z and for every multi-index $\gamma \in \mathbb{N}^{2n}$ there exists $C > 0$ such that*

$$|\partial_z^\gamma A(z)| \leq C\langle z\rangle^m, \ \forall z \in Z.$$

Definition 3.2. *Let $A \in \mathcal{O}(m)$. We will define $\hat{A}$ as operator from $\mathcal{S}(X)$ to itself, called $\hbar$-Weyl quantization of A, by the following formula, with $\psi \in \mathcal{S}(X)$,*

$$\hat{A}\psi(x) = (2\pi\hbar)^{-n}\int_X\left(\int_X A\left(\frac{x+y}{2},p\right)\mathrm{e}^{i\hbar^{-1}\langle x-y,p\rangle}\psi(y)dy\right)dp.$$

Example: Let us focus for example to observable $A := x.p$. To compute $\hat{A}$ we will use

$$p\exp(\frac{i}{\hbar}(x-y)p) = -\frac{\hbar}{i}\partial_y\exp(\frac{i}{\hbar}(x-y)p)$$

and after integration by parts and using inverse Fourier transform we yield
$\hat{A} = \frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})$.
For $\hat{A} \in \mathcal{S}(Z)$ we will compute, (with help of integral kernel) inverse prescription of Weyl symbol if operator is given as integral operator.

$$(\hat{A}\psi)(x) = (2\pi\hbar)^{-n}\iint_Z A\left(\frac{x+y}{2},p\right)\mathrm{e}^{i\hbar^{-1}\langle x-y,p\rangle}\psi(y)dydp = \int_{\mathbb{R}^n}\mathcal{K}_A(x,y)\psi(y)dy$$

$$\mathcal{K}_A(x,y) := (2\pi\hbar)^{-n}\int_{\mathbb{R}^n}A\left(\frac{x+y}{2},p\right)\mathrm{e}^{i\hbar^{-1}\langle x-y,p\rangle}dp$$

After inverse Fourier transform.

$$A(x,\xi) = \int_{\mathbb{R}^n}\mathcal{K}_A\left(x+\frac{v}{2},x-\frac{v}{2}\right)\mathrm{e}^{-\frac{i}{\hbar}v\cdot\xi}dv \tag{70}$$

The class $\mathcal{O}(m)$ has nice properties. For example, if $A \in \mathcal{O}(m)$ is real-valued function, then $\hat{A}$ is symmetric operator in $L^2(X)$, with domain $\mathcal{S}(X)$. Further there exists a product formula. For the proof one can see [Ro1], [Fo] or [Ho].

3.4. **Product formula.**

Theorem 3.3. *Let $A \in \mathcal{O}(m)$, $B \in \mathcal{O}(p)$, then there exists $C \in \mathcal{O}(m+p)$, such that*

$$\hat{A}\cdot\hat{B} = \hat{C}$$

And there exists following asymptotic expansion of Weyl symbol of this operator

$$C(q,p) = \exp\left(\frac{i\hbar}{2}\sigma(D_q,D_p;D_{q'},D_{p'})\right)A(q,p)B(q',p')|_{(q,p)=(q',p')},$$

where σ is the symplectic bilinear form (68) and $D = i^{-1}\nabla$. By expanding the exponential term, we get

$$C(q,p) = \sum_{j\geq 0}\frac{\hbar^j}{j!}(\frac{i}{2}\sigma(D_q,D_p;D_{q'},D_{p'}))^j A(q,p)B(q',p')|_{(q,p)=(q',p')}.$$

So that $C(q,p)$ is a formal power series in $\hbar$ with coefficients given by

$$C_j(q,p) = \frac{1}{2^j}\sum_{|\alpha+\beta|=j}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(D_q^\beta\partial_p^\alpha A).(D_q^\alpha\partial_p^\beta B)(q,p). \tag{71}$$

Proof. We will show this for $A, B \in \mathcal{S}(Z)$

$$\left(\hat{A}\cdot\hat{B}\psi\right)(x) = \int_{\mathbb{R}^n}\mathcal{K}_{A\cdot B}(x,y)\psi(y)dy$$

where

$$\mathcal{K}_{A\cdot B}(x,y) = \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{3n}} A\left(\frac{x+z}{2},p\right)B\left(\frac{z+y}{2},\eta\right)\mathrm{e}^{-\frac{i}{\hbar}\langle x-z,p\rangle+\langle z-y,\eta\rangle}dzdpd\eta$$

Let's define

$$\hat{C} := \hat{A}\cdot\hat{B}$$

then Weyl symbol of $\hat{C}$ can be computed with help of formula (70).

$$\begin{aligned} C(x,\xi) &= \int_{\mathbb{R}^n}\mathcal{K}_{A\cdot B}\left(x+\frac{v}{2},x-\frac{v}{2}\right)\mathrm{e}^{-\frac{i}{\hbar}v\cdot\xi}dv \\ &= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} A\left(\frac{x+z-\frac{v}{2}}{2},p\right)B\left(\frac{z+y-\frac{v}{2}}{2},\eta\right)\times \\ &\quad \times\mathrm{e}^{-\frac{i}{\hbar}\left(\langle x+\frac{v}{2}-z,p\rangle+\langle z-x+\frac{v}{2},\eta\rangle-\langle v,\xi\rangle\right)}dzdpd\eta dv \end{aligned}$$

We apply the change of variables

$$s := \frac{1}{2}(x+z)+\frac{1}{4}v, \quad t := \frac{1}{2}(x+z)-\frac{1}{4}v \Rightarrow z = s+t-x, \quad v = 2(s-t), \quad dzdv = 4^n dsdt$$

to get

$$C(x,\xi) = \frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} A(s,p)\,B(t,\eta)\,\mathrm{e}^{-\frac{2i}{\hbar}(\langle x-s,\xi-\eta\rangle-\langle x-t,\xi-p\rangle)}dsdpdtd\eta$$

Now using Taylor formula

$$\begin{aligned} A(s,p)B(t,\eta) &= \sum_{|\alpha+\beta|<N}\partial_x^\alpha A(x,p)\;\partial_x^\beta B(x,\eta)\frac{(s-x)^\alpha}{\alpha!}\frac{(t-x)^\beta}{\beta!} \\ &+ \sum_{|\alpha+\beta|=N}\int_0^1(1-\sigma)^N\partial_x^\alpha A(x+\sigma(s-x),p)\;\partial_x^\beta B(x+\sigma(t-x),\eta)\frac{(s-x)^\alpha}{\alpha!}\frac{(t-x)^\beta}{\beta!}d\sigma \end{aligned}$$

we obtain

$$\begin{aligned} C(x,\xi) &= \frac{1}{(\pi\hbar)^{2n}}\sum_{|\alpha+\beta|<N}\int_{\mathbb{R}^{4n}}\partial_x^\alpha A(x,p)\;\partial_x^\beta B(x,\eta)\frac{(s-x)^\alpha}{\alpha!}\frac{(t-x)^\beta}{\beta!}\times \\ &\quad\times\mathrm{e}^{\frac{2i}{\hbar}(\langle x-s,\xi-\eta\rangle-\langle x-t,\xi-p\rangle)}dsdpdtd\eta + R_{N-1}(A,B;z;\hbar), \end{aligned}$$

where the remainder is defined by:

$$C(z) - \sum_{0\le j\le N}\hbar^j C_j(z) =: R_N(A,B;z;\hbar). \tag{72}$$

$$z = (x,\xi)\in Z$$

Using

$$(s-x)^\alpha(t-x)^\beta\mathrm{e}^{\frac{2i}{\hbar}(\langle x-s,\xi-\eta\rangle-\langle x-t,\xi-p\rangle)} = \left(\frac{\hbar}{2i}\right)^{|\alpha|+|\beta|}(-1)^{|\beta|}\;\partial_\eta^\alpha\partial_p^\beta\;\mathrm{e}^{\frac{2i}{\hbar}(\langle x-s,\xi-\eta\rangle-\langle x-t,\xi-p\rangle)}$$

and integrating by parts we get

$$C(x,\xi) = \frac{1}{(\pi\hbar)^{2n}} \sum_{|\alpha+\beta|<N} \left(\frac{\hbar}{2i}\right)^{|\alpha|+|\beta|} (-1)^{|\alpha|} \int_{\mathbb{R}^{4n}} \frac{\partial_p^\beta \partial_x^\alpha A(x,p)}{\alpha!} \frac{\partial_\eta^\alpha \partial_x^\beta B(x,\eta)}{\beta!} \times$$

$$\times\ \mathrm{e}^{\frac{2i}{\hbar}(\langle x-s,\xi-\eta\rangle - \langle x-t,\xi-p\rangle)} dsdpdtd\eta \ + R_{N-1}(A,B;z;\hbar).$$

Finally

$$C(x,\xi) = \sum_{|\alpha+\beta|<N} \left(\frac{\hbar}{2i}\right)^{|\alpha|+|\beta|} (-1)^{|\alpha|} \frac{\partial_\xi^\beta \partial_x^\alpha A(x,\xi)}{\alpha!} \frac{\partial_\xi^\alpha \partial_x^\beta B(x,\xi)}{\beta!} + \\ + \ R_{N-1}(A,B;z;\hbar).$$

Remainder (72) can be also expressed as

$$R_{N-1}(A,B;z;\hbar) := \frac{1}{(\pi\hbar)^{2n}} \sum_{|\alpha+\beta|=N} \int_0^1 (1-\sigma)^N \partial_x^\alpha A(x+\sigma(s-x),p)\ \partial_x^\beta B(x+\sigma(t-x),\eta) \times$$

$$\times \frac{(s-x)^\alpha}{\alpha!} \frac{(t-x)^\beta}{\beta!} \mathrm{e}^{\frac{2i}{\hbar}(\langle x-s,\xi-\eta\rangle - \langle x-t,\xi-p\rangle)} d\sigma dsdpdtd\eta$$

After the change of variables

$$\sigma(s-x) =: s',\ \ \sigma(t-x) =: t',\ \ \eta-\xi =: \eta',\ \ p-\xi =: p'$$

and using the same trick with integrating by parts (and in what follows omitting primes) we finally get

$$R_{N-1}(A,B;z;\hbar) := \left(\frac{\hbar}{2i}\right)^N \int_0^1 \frac{(1-\sigma)^N}{(\pi\hbar\sigma)^{2n}} \sum_{|\alpha+\beta|=N} \frac{(-1)^\alpha}{\alpha!\beta!} \partial_p^\beta \partial_x^\alpha A(x+s,p+\xi)\ \partial_\eta^\alpha \partial_x^\beta B(x+t,\eta+\xi) \times$$

$$\times \exp\left(\frac{2i}{\hbar\sigma}(\langle s,\eta\rangle - \langle t,p\rangle)\right) d\sigma dsdpdtd\eta$$

$$= \left(\frac{\hbar}{2i}\right)^N \int_0^1 \frac{(1-\tau)^N}{(\pi\hbar\tau)^{2n}} \int_{Z\times Z} \exp(-\frac{2i}{\tau\hbar}\sigma(u,v))\sigma^N(D_u,D_v)A(u+z)B(v+z)dudvd\tau.$$

∎

Now we estimate this remainder and his derivatives.

Theorem 3.4. *Suppose that* $\lambda > 1$.
For every $m \in \mathbb{N}$, $m > 2n$, *for every* $s > 4n$ *there exists a constant* $\rho_{n,m,s}$ *such that for every* $N \geq 1$, *for every multi-index* γ, *there exists* $K_{n,N,|\gamma|} > 0$, *such that the following estimate holds, for every* $A, B \in \mathcal{S}(Z)$, $z \in Z$,

$$|\partial_z^\gamma \left(C(z) - \sum_{0\leq j\leq N} \hbar^j C_j(z)\right)| \leq \hbar^{N+1} \rho_{n,m,s} K_{n,N,|\gamma|} \times \\ \times \sup_{(*)} \left[(1+u^2+v^2)^{(s-m)/2} |\partial_u^{(\alpha,\beta)+\mu} A(u+z)||\partial_v^{(\beta,\alpha)+\nu} B(v+z)|\right] \quad (73)$$

where $\sup\limits_{(*)}$ *means that the supremum holds under the conditions*
$u,v \in Z,\ |\mu|+|\nu| \leq m+|\gamma|,\ |\alpha|+|\beta| = N+1\ \ (\mu,\nu \in \mathbb{N}^{2n},\ \alpha,\beta \in \mathbb{N}^n)$.

Proof : We shall use the following lemma to estimate $R_N(A,B;z,\hbar)$.

Lemma 3.5. *Let us consider $F \in \mathcal{S}(Z\times Z)$ and the integral ($\lambda > 1$)*

$$I(\lambda) = \lambda^{2n}\int_{Z\times Z}\exp[-i\lambda\sigma(u,v)]F(u,v)dudv.$$

Then for every real number $s > 4n$ and every integer $m > 2n$ there exists $\kappa(n,s,m) > 0$ depending only on n,s,m (but independent of F) such that the following estimate holds

$$|I(\lambda)| \leq \kappa(n,s,m)\sup_{\substack{u,v\in Z,\\ |\mu|+|\nu|\leq m}}(1+u^2+v^2)^{(s-m)/2}|\partial_u^\mu\partial_v^\nu F(u,v)|.$$

Proof : Let us introduce a cut-off χ_0, C^∞ on $\mathbb{R}$, $\chi_0(x) = 1$ for $|x| \leq 1/2$ and $\chi_0(x) = 0$ for $|x| \geq 1$.
We split $I(\lambda)$ into three pieces

$$I_0(\lambda) = \lambda^{2n}\int_{Z\times Z}\exp[-i\lambda\sigma(u,v)]\chi_0(u^2+v^2)\chi_0(\lambda(u^2+v^2))F(u,v)dudv,$$

$$I_1(\lambda) = \lambda^{2n}\int_{Z\times Z}\exp[-i\lambda\sigma(u,v)]\left(1-\chi_0(\lambda(u^2+v^2))\right)\chi_0(u^2+v^2)F(u,v)dudv,$$

$$I_2(\lambda) = \lambda^{2n}\int_{Z\times Z}\exp[-i\lambda\sigma(u,v)](1-\chi_0(u^2+v^2))F(u,v)dudv.$$

For $I_0(\lambda)$, we easily have

$$|I_0(\lambda)| \leq \omega_{4n}\sup_{u^2+v^2\leq 1}|F(u,v)|,$$

where ω_{4n} is the volume of the unit ball in Z^2. For $I_1(\lambda)$ and $I_2(\lambda)$, we integrate by parts with the differential operator

$$L = \frac{i}{u^2+v^2}\left(Ju\frac{\partial}{\partial v} - Jv\frac{\partial}{\partial u}\right),$$

where J is the matrix associated to the symplectic form ($\sigma(u,v) =< Ju, v >$). It holds true that

$$L\left(\exp[-i\lambda\sigma(u,v)]\right) = \lambda\exp[-i\lambda\sigma(u,v)].$$

Performing $2n$ integrations by parts, we can see that it exists a constant c_n such that

$$|I_1(\lambda)| \leq c_n\sup_{\substack{u^2+v^2\leq 1\\ |\mu|+|\nu|\leq 4n}}|\partial_u^\mu\partial_v^\nu F(u,v)|.$$

Similarly, performing m integrations by parts,

$$\begin{aligned} |I_2(\lambda)| &\leq \left|\int_{Z\times Z}\exp[-i\sigma(u,v)]\left(1-\chi_0(\frac{1}{\lambda}(u^2+v^2)\right)F(\frac{u}{\sqrt{\lambda}},\frac{v}{\sqrt{\lambda}})\,dudv\right| \\ &\leq c_m\int_{Z\times Z}(\frac{|u|+|v|}{u^2+v^2})^m|\partial_u^\mu\partial_v^\nu\left((1-\chi_0(\frac{1}{\lambda}(u^2+v^2))F(\frac{u}{\sqrt{\lambda}},\frac{v}{\sqrt{\lambda}})\right)|dudv. \end{aligned}$$

Using the existence of integral

$$\int_{Z\times Z}(\frac{|u|+|v|}{u^2+v^2})^m(1+u^2+v^2)^{\frac{m-s}{2}}dudv$$

we get for a constant $c(n,s,m)$,

$$|I_2(\lambda)| \leq c(n,s,m) \sup_{\substack{u,v\in Z\\ |\mu|+|\nu|\leq m}} (1+u^2+v^2)^{(s-m)/2} |\partial_u^\mu \partial_v^\nu F(u,v)|.$$

∎

Now we can complete the proof of the theorem by using Lemma 3.5, the Leibniz formula and the following elementary estimate, using in Z the coordinates $u=(x,\xi)$, $v=(y,\eta)$,

$$|\sigma^N(\partial_x,\partial_\xi;\partial_y,\partial_\eta))A(x,\xi)B(y,\eta)| \leq (2n)^N \sup_{|\alpha|+|\beta|=N} |\partial_x^\alpha \partial_\xi^\beta A(x,\xi)\partial_y^\beta \partial_\eta^\alpha B(y,\eta)|.$$

∎

Remark 3.6. *We can easily extend the estimate (73) for observables A, B with polynomial growth at infinity, by choosing m large enough to get a finite r.h.s. Let us assume that $A\in\mathcal{O}(\mu_A)$, $B\in\mathcal{O}(\mu_B)$, where $\mu_A,\mu_B\in\mathbb{R}$. Then we can apply (73) to $A_\varepsilon(u)=\mathrm{e}^{-\varepsilon u^2}A(u)$ and $B_\varepsilon(v)=\mathrm{e}^{-\varepsilon v^2}B(v)$ for $\varepsilon>0$ and pass to the limit $\varepsilon\to 0$ with $m-s\geq\mu_A+\mu_B$.*

3.5. Main theorem.

Theorem 3.7. *Let $[t,s]$ is compact interval. Let us consider time dependent Hamiltonian function $H(\tau)\in\mathcal{O}(c)$ and an observable $A\in\mathcal{O}(d)$ for some c,d, satisfying*

$$\begin{aligned} |\partial_z^\gamma H(\tau,z)| &\leq C_\gamma, \quad \text{for} \quad 3\leq|\gamma|\leq 4n+[\frac{n}{2}]+5\,,\ \tau\in[s,t] \\ |\partial_z^\gamma A(z)| &\leq C_\gamma, \quad \text{for} \quad 1\leq|\gamma|\leq 4n+[\frac{n}{2}]+5 \\ |\partial_z^\gamma \Phi(t,\tau,z)| &\leq C_\gamma, \quad \text{for} \quad 1\leq|\gamma|\leq 4n+[\frac{n}{2}]+5,\ \tau\in[s,t], \end{aligned} \tag{74}$$

where the classical flow $\Phi(t,s)$ is given by classical Hamilton equations (69) and is assumed to be unique.
We know that $\widehat{H(\tau)},\hat{A}$ are symmetric operators on Schwartz space $\forall\tau\in[s,t]$. We assume moreover that A is essentially self-adjoint operator in $L^2(X)$, with core $\mathcal{S}(X)$.
Finally we suppose that quantum evolution given by Schrödinger equation is defined on Schwartz space by the propagator U

$$i\hbar\, d_\tau U(\tau,s)\psi = \widehat{H(\tau)}U(\tau,s)\psi, \tag{75}$$

which conserves Schwartz space and which satisfies initial condition

$$U(\tau,\tau)=\mathbb{1}\,,\ \forall\tau\in[s,t] \tag{76}$$

valid also on Schwartz space. Then the quantum evolution of observable $\hat{A}$ can be approximated by the operator with the Weyl symbol

$$A_0(t,s,z)=A(\Phi(t,s,z))$$

in the following sense:

$$\|\widehat{A}(t,s)-\widehat{A_0(t,s)}\|_{L^2(X)} \leq \hbar^2 C_n(t,s)$$

Proof. For any $B(t)\in\mathcal{O}(m)$, $\psi\in\mathcal{S}(X)$ it holds true that

$$\frac{d}{dt}\widehat{B(t)}\psi = (\widehat{\frac{d}{dt}B(t)})\psi \tag{77}$$

On Schwartz space holds equation

$$
\begin{aligned}
U(s,t)\hat{A}U(t,s)-\widehat{A_0}(t,s) &= \int_s^t \frac{d}{d\tau}\left(U(s,\tau)\widehat{A\circ\Phi(t,\tau)}U(\tau,s)\right)\,\mathrm{d}\tau \\
&= \int_s^t U(s,\tau)\left(\frac{i}{\hbar}[\widehat{H(\tau)},\widehat{A\circ\Phi(t,\tau)}]+\widehat{\frac{d}{d\tau}A\circ\Phi(t,\tau)}\right)U(\tau,s)\,\mathrm{d}\tau \\
&= \int_s^t U(s,\tau)\left(\frac{i}{\hbar}[\widehat{H(\tau)},\widehat{A\circ\Phi(t,\tau)}]-\widehat{\{H(\tau),A\circ\Phi(t,\tau)\}}\right)U(\tau,s)\,\mathrm{d}\tau, \qquad (78)
\end{aligned}
$$

where we used (77), Schrödinger equation (75), with initial condition (76).
Differentiating basic property of the propagator $U(\tau,s)U(s,\tau)\psi=\psi$ yields

$$d_\tau U(s,\tau) = -U(s,\tau)\Big(d_\tau U(\tau,s)\Big)U(s,\tau).$$

And finally it was used

$$\frac{d}{d\tau}A\circ\Phi(t,\tau) = -\{H(\tau),A\circ\Phi(t,\tau)\},$$

what can be obtained by differentiating $\Phi(t,\tau,y)=y$ with respect to t (see (69)) and then putting $y:=\Phi(t,\tau,x)$.
With help of (71) we compute that the Weyl symbol of the expression

$$\widehat{C} := \frac{i}{\hbar}[\hat{H},\hat{A}]$$

is

$$C = 0+\{H,A\}+0+2\frac{i}{\hbar}R_2(H,A;z,\hbar)\ ,$$

where

$$R_2(H,A;z,\hbar) = \left(\frac{\hbar}{2i}\right)^3\int_0^1\frac{(1-\tau)^3}{(\pi\hbar\tau)^{2n}}\int_{Z\times Z}\exp(-\frac{2i}{\tau\hbar}\sigma(u,v))\sigma^3(D_u,D_v)H(u+z)A(v+z)dudvd\tau.$$

Applying L^2 norm to (78) we obtain

$$\|U(s,t)\hat{A}\,U(t,s)-\widehat{A_0(t,s)}\|_{L^2} \le 2\frac{|t-s|}{\hbar}\|\widehat{R_2}(H,A;z,\hbar)\|_{L^2}.$$

But there is a Calderon-Vaillancourt Theorem with improvement by A. Boulkhemair [Bo], which estimates the norm of Weyl-quantized operator

$$\|\hat{B}\|_{L^2} \le \gamma_n \sup_{\substack{|\alpha|,|\beta|\le[n/2]+1\\ z\in Z}} |\partial_z^{\alpha,\beta}B(z)|.$$

So we need to estimate

$$\sup_{\substack{|\alpha|,|\beta|\le[n/2]+1\\ z\in Z}} |\partial_z^{\alpha,\beta}R_2(H,A;z,\hbar)|.$$

This is exactly what we have prepared in Theorem 3.4. Using (74) we deduce that for $s=m:=4n+1$ there exist $K_n(t,s)$, such that:

$$K_n(t,s) := \sup_{(*)}\left[|\partial_u^{(\alpha,\beta)+\mu}H(\tau,u+z)||\partial_v^{(\beta,\alpha)+\nu}A\Big(\Phi(t,\tau,u+z)\Big)|\right] < \infty$$

where $\sup\limits_{(*)}$ means that the supremum holds under the conditions
$u,v,z\in Z, \tau\in[s,t]$, $|\mu|+|\nu|\le 5n+[n/2]+1$, $|\alpha|+|\beta|=3$ $(\mu,\nu\in\mathbb{N}^{2n},\ \alpha,\beta\in\mathbb{N}^n)$.

We have also used following estimate on derivatives of composition of functions

$$|\partial^\alpha (f \circ g)| \leq K_{n,|\alpha|} \sup_{\substack{\beta,\gamma \neq 0 \\ |\beta|,|\gamma| \leq |\alpha|}} |\partial^\beta f||\partial^\gamma g|$$

Combining all these facts we may finish the proof. ∎

4.1. **Introduction.**

The problem we address in this part concerns spectral analysis of so called Floquet Hamiltonians. The study of stability of non autonomous quantum dynamical systems is an effective tool to understand most of quantum problems which involve a small number of particles. When these systems are periodic the spectral analysis of the evolution operator over one period can give a fairly good information on this stability, see e.g. [EV]. In fact this type of result generalizes the celebrated RAGE theorem concerned with time-independent systems (one can consult [RS] for a summary). As shown in [How] and [Ya] the spectral analysis of the evolution operator over one period (so called monodromy operator or Floquet operator) is equivalent to the spectral analysis of the corresponding Floquet Hamiltonian (also called operator of quasi-energy). This is also what we are aiming for in this part. More precisely, we analyze periodic quantum systems which are weakly regular in time and "space" in the sense of an appropriately chosen norm, and give sufficient conditions to insure that the Floquet Hamiltonians has a pure point spectrum.
Another generalization is that in the present result (Theorem 4.1) we allow degenerate eigenvalues of the unperturbed Hamilton operator (denoted H in what follows). The degeneracy of eigenvalues h_m of H can grow arbitrarily fast with m provided the time-dependent perturbation is sufficiently regular. To our knowledge this is a new feature in this context. Previously two conditions were usually imposed, namely bounded degeneracy and a growing gap condition on eigenvalues h_m, reducing this way the scope of applications of this theory to one dimensional confined systems. Owing to the generalization to degenerate eigenvalues we are able to consider also some models in higher dimensions, for example the N-dimensional quantum top, i.e., the N-dimensional version of the pulsed rotor. A short description of this model is also given in subsection 4.3.
Because the proof of Main Theorem 4.1 is quite long we will address reader to [D2]. Further generalization to unbounded perturbations is treated in [D3].

4.2. **Main theorem.**

The central object we wish to study in this part is a self-adjoint operator of the form $\mathbf{K}+\mathbf{V}$ acting in the Hilbert space

$$\mathcal{K} = L^2([\,0,T\,],\mathrm{dt}) \otimes \mathcal{H} \cong L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$$

where $T = 2\pi/\omega$, ω is a positive number (a frequency) and $\mathcal{H}$ is a fixed separable Hilbert space. The operator $\mathbf{K}$ is self-adjoint and has the form

$$\mathbf{K} = -i\,\partial_t \otimes 1 + 1 \otimes H$$

where the differential operator $-i\partial_t$ acts in $L^2([\,0,T\,],\mathrm{dt})$ and represents the self-adjoint operator characterised by periodic boundary conditions. This means that the eigenvalues of $-i\partial_t$ are $k\omega$, $k \in \mathbb{Z}$, and the corresponding normalised eigenvectors are $\chi_k(t) = T^{-1/2}\exp(ik\omega t)$. H is a self-adjoint operator in $\mathcal{H}$ and is supposed to have a discrete spectrum. Finally, $\mathbf{V}$ is a bounded Hermitian operator in $\mathcal{K}$ determined by a measurable operator-valued function $t \mapsto V(\omega t) \in \mathcal{B}(\mathcal{H})$ such that $\sup_{t\in\mathbb{R}} \|V(t)\| < \infty$, $V(t)$ is 2π-periodic, and for almost all $t \in \mathbb{R}$, $V(t)^* = V(t)$. Naturally, $(\mathbf{V}\psi)(t) = V(\omega t)\psi(t)$ in $\mathcal{K} \cong L^2([\,0,T\,],\mathcal{H},\mathrm{dt})$.
Let

$$\sum_{k\in\mathbb{Z}} k\omega\, P_k$$

be the spectral decomposition of $-i\partial_t$ in $L^2([0,T],\mathrm{dt})$ and let

$$H=\sum_{m\in\mathbb{N}}h_m\,Q_m$$

be the spectral decomposition of H in $\mathcal{H}$. Thus we can write

$$\mathcal{H}=\sum_{m\in\mathbb{N}}^{\oplus}\mathcal{H}_m$$

where $\mathcal{H}_m=\operatorname{Ran}Q_m$ are the eigenspaces. We suppose that the multiplicities are finite,

$$M_m=\dim\mathcal{H}_m<\infty,\ \forall m\in\mathbb{N}.$$

Hence the spectrum of $\mathbf{K}$ is pure point and its spectral decomposition reads

$$\mathbf{K}=\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}(k\omega+h_m)\,P_k\otimes Q_m,$$

implying a decomposition of $\mathcal{K}$ into a direct sum,

$$\mathcal{K}=\sum_{(k,m)\in\mathbb{Z}\times\mathbb{N}}^{\oplus}\operatorname{Ran}(P_k\otimes Q_m)\,.$$

Here is some additional notation. Set

$$V_{knm}=\frac{1}{T}\int_0^T e^{-ik\omega t}\,Q_nV(\omega t)\,Q_m\,\mathrm{dt}=\frac{1}{2\pi}\int_0^{2\pi}e^{-ikt}\,Q_nV(t)\,Q_m\,\mathrm{dt}\in\mathcal{B}(\mathcal{H}_m,\mathcal{H}_n)\,. \qquad (79)$$

Further,

$$\Delta_{mn}=h_m-h_n,$$

and

$$\Delta_0=\inf_{m\neq n}|\Delta_{mn}|.$$

Finally we set

$$\mu_{mn}=M_mM_n.$$

Now we are able to formulate our main result. Though not indicated explicitly in the notation the operator $\mathbf{K}+\mathbf{V}$ is considered as depending on the parameter ω.

Theorem 4.1. *Fix $J>0$ and set $\Omega_0:=[\frac{8}{9}J,\frac{9}{8}J]$. Assume that $\Delta_0>0$ and that there exists $\sigma>0$ such that*

$$\Delta_\sigma(J):=J^\sigma\sum_{\substack{m,n\in\mathbb{N}\\ \Delta_{mn}>J/2}}\frac{\mu_{mn}}{(\Delta_{mn})^\sigma}<\infty\,.$$

Then for every $r>\sigma+\frac{1}{2}$ there exist positive constants (depending, as indicated, on σ, r, Δ_0 and J but independent of V), $\epsilon_\star(r,\Delta_0,J)$ and $\delta_\star(\sigma,r,J)$, with the property: if

$$\epsilon_V:=\sup_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\|V_{knm}\|\,\max\{|k|^r,1\}<\min\left\{\epsilon_\star(r,\Delta_0,J),\frac{|\Omega_0|}{\delta_\star(\sigma,r,J)}\right\}$$

(here $|\Omega_*|$ stands for the Lebesgue measure of Ω_*) then there exists a measurable subset $\Omega_\infty\subset\Omega_0$ such that

$$|\Omega_\infty|\geq|\Omega_0|-\delta_\star(\sigma,r,J)\,\epsilon_V \qquad (80)$$

and the operator $\mathbf{K}+\mathbf{V}$ has a pure point spectrum for all $\omega\in\Omega_\infty$

Remark.

1) It is sometimes necessary to consider potentials $\mathbf{V}$ *which depend on the frequency* ω *in a more elaborate way. Suppose that* $V : \mathbb{R} \times \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ *is a bounded measurable function, which is* 2π *periodic with respect to the first variable and such that for almost all* $t \in \mathbb{R}$ *and* $\omega \in \mathbb{R}_+$, $V(t,\omega)^\star = V(t,\omega)$*; then* $(\mathbf{V}\psi)(t) = V(\omega t, \omega)\psi(t)$ *defines an operator family which is uniformly bounded on* $\mathcal{K}$ *with respect to the variable* ω*. Let now*

$$\epsilon_V := \sup_{\omega,\omega' \in \Omega_0} \sup_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}, m \in \ N} (\|V_{knm}(\omega)\| + J\|\partial V_{knm}(\omega,\omega')\|) \max\{|k|^r, 1\}$$

where

$$V_{knm}(\omega) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} Q_n V(t,\omega) Q_m dt$$

and where the symbol ∂ *is the discrete derivative in* ω*,*

$$\partial V_{knm}(\omega,\omega') := \frac{V_{knm}(\omega) - V_{knm}(\omega')}{\omega - \omega'}.$$

It is not difficult to check that Theorem 1 applies in exactly the same conditions.

2) In the course of the proof is shown even more. Namely, for all $\omega \in \Omega_\infty$ *and any eigenvalue of* $\mathbf{K} + \mathbf{V}$ *the corresponding eigen-projector* P *belongs to the Banach algebra with the norm*

$$\|P\| = \sup_{n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \|P_{knm}\| \ \max\{|k|^{r-\sigma-\frac{1}{2}}, 1\}.$$

This shows that P *is* $(r - \sigma - 1/2)$*-differentiable as a map from* $[\,0, T\,]$ *to the space of bounded operators in* $\mathcal{H}$*.*

3) The constants $\epsilon_\star(r, \Delta_0, J)$ *and* $\delta_\star(\sigma, r, J)$ *are in fact known quite explicitly .*

$$\epsilon_\star(r, \Delta_0, J) = \frac{\min\{4\Delta_0,\ J\}}{270\, e^3},$$

and

$$\begin{aligned} \delta_\star(\sigma, r, J) &= \frac{1440}{\pi} e^5 2^{\sigma+1} \left(\frac{2\sigma+1}{\left(1 - e^{-\frac{2}{r}}\right) e} \right)^{\sigma+\frac{1}{2}} \left(\sum_{s=1}^{\infty} s^2 e^{-\frac{2}{r}(r-\sigma-\frac{1}{2})s} \right) \Delta_\sigma(J) \\ &= \frac{1440}{\pi} \left(\frac{2\sigma+1}{\left(1 - e^{-\frac{2}{r}}\right) e} \right)^{\sigma+\frac{1}{2}} 2^{\sigma+1} e^{3+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)} \frac{1 + e^{-2+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)}}{\left(1 - e^{-2+\frac{2}{r}\left(\sigma+\frac{1}{2}\right)}\right)^3} \Delta_\sigma(J) \end{aligned}$$

where we used the identity

$$\sum_{s=1}^{\infty} s^2 e^{-2xs} = \frac{\cosh(x)}{4\,\sinh(x)^3}$$

4.3. **Two models.**

We conclude this subsection with a brief description of two models illustrating the effectiveness of Theorem 4.1. In the first model we set $\mathcal{H} = L^2([\,0,1\,], \mathrm{dx})$, $H = -\partial_x^2$ with Dirichlet boundary conditions, and $V(t) = z(t)x^2$ where $z(t)$ is a sufficiently regular 2π-periodic function. As shown in [Š] the spectral analysis of this simple model is essentially equivalent to the analysis of the so called quantum Fermi accelerator. The particularity of the latter model is that the underlying Hilbert space itself is time-dependent, $\mathcal{H}_t = L^2([\,0, a(t)\,], \mathrm{dx})$ where $a(t)$ is a strictly positive periodic function. The time-dependent Hamiltonian is $-\partial_x^2$ with

Dirichlet boundary conditions. Using a convenient transformation one can pass from the Fermi accelerator to the former model getting the function $z(t)$ expressed in terms of $a(t)$, $a'(t)$ and $a''(t)$. But let us return to the analysis of our model. Eigenvalues of H are non-degenerate, $h_m = m^2\pi^2$ for $m \in \mathbb{N}$, with normalised eigenfunctions equal to $\sqrt{2}\sin(m\pi x)$. A straightforward calculation gives

$$V_{knm} = z_k \times \begin{cases} \frac{8(-1)^{m+n}mn}{(m^2-n^2)^2\pi^2} & \text{if } m \neq n, \\ \frac{1}{3} - \frac{1}{2m^2\pi^2} & \text{if } m = n, \end{cases}$$

where $z_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}z(t)\,dt$ is the Fourier coefficient of $z(t)$. Hence one derives that

$$\epsilon_V = \sup_{n\in\mathbb{N}}\left(\frac{1}{3} + \frac{2}{n^2\pi^2} + \frac{4}{\pi^2}\sum_{j=1}^{n-1}\frac{1}{j^2}\right)\sum_{k\in\mathbb{Z}}|z_k|\max\{|k|^r,1\} = \sum_{k\in\mathbb{Z}}|z_k|\max\{|k|^r,1\}.$$

For any $J > 0$, $\Delta_\sigma(J)$ is finite if and only if $\sigma > 1$. On the other hand, to have ϵ_V finite it is sufficient that $z(t) \in C^s$ where $s > r + 1 > \sigma + \frac{1}{2} + 1 > \frac{5}{2}$. So $z(t) \in C^3$ suffices for the theory to be applicable. This may be compared to an older result in [D1], §4.2, giving a much worse condition, namely $z(t) \in C^{17}$.

The second model is the pulsed rotator in N dimensions. In this case $\mathcal{H} = L^2(S^N, d\mu)$, with $S^N \subset \mathbb{R}^{N+1}$ being the N-dimensional unit sphere with the standard (rotationally invariant) Riemann metric and the induced normalised measure $d\mu$, and $H = -\Delta_{LB}$ is the Laplace-Beltrami operator on S^N. The spectrum of H is well known, [Ne]: $\mathrm{Spec}(H) = \{h_m\}_{m=0}^{\infty}$, where

$$h_m = m(m+N-1)$$

and the multiplicities are

$$M_m = \binom{m+N}{N} - \binom{m+N-2}{N}.$$

The time-dependent operator $V(t)$ in $\mathcal{H}$ acts via multiplication, $(V(t)\varphi)(x) = v(t,x)\varphi(x)$, where $v(t,x)$ is a real measurable bounded function on $\mathbb{R} \times S^N$ which is 2π-periodic in the variable t. Consequently, $\mathcal{K} \cong L^2([\,0,T\,] \times S^N, dt\,d\mu)$ and $(\mathbf{V}\psi)(t,x) = v(\omega t, x)\psi(t,x)$. Note that the asymptotic behavior of the eigenvalues and the multiplicities, as $m \to \infty$, is $h_m \sim m^2$, $M_m \sim (2/(N-1)!)\,m^{N-1}$. So $\Delta_\sigma(J)$ is finite, for any $J > 0$, if and only if

$$\sum_{m^2-n^2>J/2}\frac{(mn)^{N-1}}{(m^2-n^2)^\sigma} < \infty\,.$$

To ensure this condition we require that $\sigma > 2(N-1)+1$. Let us assume that there exist $s, u \in \mathbb{Z}_+$ such that, for any system of local (smooth) coordinates $(y_1,\ldots,y_N)$ on S^N, the derivatives $\partial_t^\alpha\partial_{y_1}^{\beta_1}\ldots\partial_{y_N}^{\beta_N}v(t,y_1,\ldots,y_N)$ exist and are continuous for all α, β, $\alpha \leq s$ and $\beta_1 + \ldots + \beta_N \leq u$. If $u \geq 4$ then $[\,H,[\,H,V(t)\,]]$ is a well defined second order differential operator with continuous coefficient functions and the operator $[\,H,[\,H,V(t)\,]](1+H)^{-1}$ is bounded. Clearly,

$$\frac{(h_m-h_n)^2}{1+h_m}Q_nV(t)Q_m = Q_n[\,H,[\,H,V(t)\,]](1+H)^{-1}Q_m.$$

Using this relation one derives an estimate on V_{knm},

$$\|V_{knm}\| \le \text{const}\, \frac{1+\min\{h_n,h_m\}}{|k|^s (h_m-h_n)^2}\,,$$

valid for $k \neq 0$ and $m \neq n$. The number

$$\sup_{n\in\mathbb{Z}_+} \sum_{m\in\mathbb{Z}_+,\, m\neq n} \frac{1+\min\{h_n,h_m\}}{(h_m-h_n)^2}$$

is finite. To see it one can employ the asymptotics of h_m and the fact that the sequence

$$a_n = \sum_{m\in\mathbb{Z}_+,\, m\neq n} \frac{1+\min\{n^2,m^2\}}{(m^2-n^2)^2} = \left(1+\frac{1}{n^2}\right)\frac{\pi^2}{12} - \frac{3}{16n^2} + \frac{5}{16n^4} - \frac{1}{2n}\sum_{m=1}^{2n-1}\frac{1}{m}\,,$$

$n = 1,2,3,\ldots$, is bounded. It follows that the norm ϵ_V is finite if $s > r+1 > \sigma + \frac{1}{2} + 1 > 2(N-1) + 1 + \frac{3}{2} = 2N + \frac{1}{2}$. Thus the theory is applicable provided $u \ge 4$ and $s > 2N + \frac{1}{2}$. The same example has also been treated by adiabatic methods in [Ne]. In that case the assumptions are weaker. It suffices that $v(t,x)$ be $(N+1)$-times differentiable in t with all derivatives $\partial_t^\alpha v(t,x)$, $0 \le \alpha \le N+1$, uniformly bounded. However the conclusion is somewhat weaker as well. Under this assumption $\mathbf{K}+\mathbf{V}$ has no absolutely continuous spectrum but nothing is claimed about the singular continuous spectrum.

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