ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ
Fakulta jaderná a fyzikálně inženýrská
katedra matematiky

DIPLOMOVÁ PRÁCE

Studium vlastností Fairlieho algebry

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Školní rok: 1997/98

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Master Thesis

Matrix rereprestations of Fairlie algebra

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School year: 1997/98

Prohlašuji, že jsem diplomovou práci vypracoval samostatně a uvedl všechnu použitou literaturu.

V praze 24. 4. 1998

I state that this thesis was done exclusively
by myself and the bibliography is complete.

Prague, April 24, 1998

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1. Introduction

The main aim of this work is to explicitly express all irreducible finite matrix representations of the associative algebra $\mathrm{U_q}(\mathrm{so}_3)$ what is called the Fairlie's quantum deformation of so_3, and point out some other properties of this algebra (Poincare-Birkhoff-Witt property, homomorphism into $\mathrm{U_q}(\mathrm{sl}_2)$ and its connection to the representations, tensor product of representations, infinite dimensional representations, center, ...). It is also pointed out the big differences between the cases when parameter q is/is not root of unity.

$\mathrm{U_q}(\mathrm{so}_3)$ is a infinitedimensional quotient algebra, it is obtained by q-deformation of the standard commutation relations

$$[I_1,I_2]=I_3,\quad [I_2,I_3]=I_1,\quad [I_3,I_1]=I_2 \tag{1.1}$$

of the Lie algebra so_3. So, $U_q(\mathrm{so}_3)$ is defined as the complex associative algebra with unit element generated by the elements I_1, I_2, I_3 satisfying the defining relations

$$[I_1,I_2]_q:=q^{1/2}I_1I_2-q^{-1/2}I_2I_1=I_3, \tag{1.2}$$

$$[I_2,I_3]_q:=q^{1/2}I_2I_3-q^{-1/2}I_3I_2=I_1, \tag{1.3}$$

$$[I_3,I_1]_q:=q^{1/2}I_3I_1-q^{-1/2}I_1I_3=I_2. \tag{1.4}$$

$\mathrm{U_q}(\mathrm{so}_3)$ can also be given (more precisely is isomorphic to the algebra generated) by these three relations:

$$\begin{aligned} J_1 &= J_3J_2 - qJ_2J_3,\\ J_2 &= J_3J_1 - q^{-1}J_1J_3,\\ J_3 &= J_2J_1 - qJ_1J_2. \end{aligned} \tag{1.5}$$

Together with results of [2] the complete list of all irreducible finite matrix *-representations for all possible q is reached. However, we do not restrict to *-representations, the case $q\neq\sqrt[p]{1}$ is solved completely. We also provide the proof of Poincare-Birkhoff-Witt theorem for this algebra & partial discussion about PBW feature of quadratic algebras generated by 3 relations without quadratic terms which leads into this type of deformation.

Quantum deformations were first discovered by theoretical physicists to occur as symmetries of integrable 1+1-dimensional systems, particularly through the quantum inverse scattering mechanism. The strong relations of quantum deformations to the Yang-Baxter equation and transfer matrices in statistical mechanical models was also crucial in their development (see for example R. J. Baxter: *Exactly solved models in statistical mechanics*, 1st ed., Academic Press, London 1982). Between these deformations some of them are especially important, namely Drinfeld and Jimbo's 1-parameter deformations of universal enveloping algebras of semi-simple Lie algebras which are Hopf algebras; Faddeev, Reshetikhin and Takhajan developed the theory of matrix quantum groups; Woronowicz independently initiated the study of quantum groups from a C^*-algebra point of view and so on.

However, quantum deformations are not always seen from the right point of view (see for example [10], [11]). It should be realized that using these deformations is not only introducing some new free parameters which can be derived from experiments and then properly adjusted -- the main reason are the new kinds representations using them one can derive usually generally new results which are not "continuously" related to the classical ones.

More recently it has been realized that quantum deformations occur as symmetries of a large number of systems in mathematical physics (see for example A. Tsuchiya, E.

Eguchi, and M. Jimbo, Infinite analysis: *Proccedings the R. I. M. S. research project*, June-August 1991, Advanced series in mathematical physics, vol. 16, Kyoto, R. I. M. S., World Scientific, Singapore, 1992, also published in Int. J. Mod. Physics A7 1992, proceedings supplement 1 A, B <or> T. I. Curtright, D. B. Fairlie, and C. K. Zachos (eds.), *Proceedings of the Argonne Workshop: Quantum groups*, ANL, Illinois, World Scientific, Singapore, 1991). Notably quantum enveloping deformed algebras occur as symmetries in quantum spin chains and solvable lattice models (see for example M. Jimbo and T. Miwa, *Algebraic analysis of solvable lattice models*, preprint RIMS-981, R. I. M. S. Kyoto May 1994, appeared also in a book in the CBMS Regional Conference Series in Mathematics), in two dimensional conformal field theory (see for instance J. Fuchs, *Affine Lie algebras and quantum groups*, Cambridge University Press, Cambridge 1992), and massive integrable systems (see F. A. Smirnov, *Form factors in completely integrable models of quantum field theory, World Scientific*, Singapore, 1992). Quantum deformations also play an important role in the theory of link invariants and knots (see C. N. Yang and M. L. Ge (eds.), *Braid groups, knot theory and statistical mechanics*, Advanced series in math physics, vol. 9, World Scientific, Singapore 1989).
One can say that by now the use of quantum deformations in theoretical and mathematical physics has become so wide-spread that almost every day new preprints appear describing their structure and applications.

Let us briefly compare results of this paper with results reached by other authors. The first mention about representations of $\mathrm{U_q}(\mathrm{so}_3)$ appears in [4], one concrete irreducible matrix representation for each dimension is constructed according to analogy with non-deformed case. One concrete kind of irreducible representation appears in [6], in [5] there is constructed so called "Fock" representation (it is representation of more general case quadratic algebra and one represetation constructed in this paper is a special case of representation constructed there). Classification of *-representations is completely done in [2] (however *-representation is a special case of representations considered here), some *-representations are obtained another way in [15]. Irreducible representations which can be obtained using algebra homomorphism into $\mathrm{U_q}(\mathrm{sl}_2)$ (see chapter 9) are described in [7].
The results in chapters 2-7 are quite original, sum up and finish partial results of previous papers and are ready for publication, the results in chapters 8-16 have been reached in close cooperation with M. Havlíček and A. U. Klimyk and submitted into Journal Physics A.

Let us do the brief look on the contents of each chapter. In chapter 1 we start with basic preliminary, set up notation and give basic definitions which are valid to the rest of the work.
The chapter 2 gives a standard approach to the Poincare-Birkhoff-Witt theorem for universal envelopping algebras of simple Lie algebras.
This approach is needed in chapter 4 where it is applied on quantum generalization of Lie algebra - quotient algebra of tensor algebra generated by two-sided ideal which consists of the relations used to define $\mathrm{U_q}(\mathrm{so}_3)$. It is pointed out that from this approach is clear that this and only this type of deformation in the wide class of even multi-parameter deformation leads to the algebra which is PBW-type i. e. which has the basis generated by all well ordered monomials. This fact can be used to give reasons why this type of deformation of the algebra so_3 can be considered as more important than the others because the PBW feature is also important in physical aplications.

In the chapter 3 there is a detailed look on the definition of $\mathrm{U_q}(\mathrm{so}_3)$.
The chapter 5 explains the procedure how to reach all irreducible matrix representations of this algebra for the case when q not root of unity. It is also clarified why this proccess is not successful for the case when q is root of unity. The rest of the chapter contains important theorems about using the derived representations for the case when q is root of unity. Everything in the chapter is done constructive way (i. e. reducibility of representations is shown as decomposing the reducible representation into irreducible ones or finding concrete invariant subspace).
The chapters 6 and 7 connect the work to the related topics - Hopf and quadratic algebras. However it is not clear yet if this algebra has or has not Hopf structure. Hopf algebra structure is not known on $U_q(\mathrm{so}_3)$, however, it can be embedded into the Hopf algebra $U_q(\mathrm{sl}_3)$ as a Hopf coideal[9]. This embedding is very important for the possible application in spectroscopy.
The chapter 8 gives alternative way how to obtain all representations of the $\mathrm{U_q}(\mathrm{so}_3)$ when q is root of unity. It contanis some information about the center of the algebra, about the inner automorphisms and also gives list for some low dimensions but the complete list for any n is not reached yet.
In the chapter 9 an algebra homomorphism ψ from the algebra $U_q(\mathrm{so}_3)$ to the extension $\widehat{U}_q(\mathrm{sl}_2)$ of the Hopf algebra $U_q(\mathrm{sl}_2)$ is constructed. Not all irreducible representations of $U_q(\mathrm{sl}_2)$ can be extended to representations of $\widehat{U}_q(\mathrm{sl}_2)$. Composing the homomorphism ψ with irreducible representations of $\widehat{U}_q(\mathrm{sl}_2)$ we obtain representations of $U_q(\mathrm{so}_3)$. Not all of these representations of $U_q(\mathrm{so}_3)$ are irreducible. Reducible representations of $U_q(\mathrm{so}_3)$ are decomposed in the next chapters into irreducible components. In this way we obtain all irreducible representations of $U_q(\mathrm{so}_3)$ when q is not a root of unity. A part of these representations turn into irreducible representations of the Lie algebra so_3 when $q\to 1$.
One of exciting aspects of this quantum deformation is the presence of new types of representations that have no classical analogues. It seems that for the case when q is root of unity the number of representations with no classical analogue is increasing rapidly.
Using the homomorphism ψ it is shown in the next chapter how to construct tensor products of finite dimensional representations of $U_q(\mathrm{so}_3)$. Irreducible representations of $U_q(\mathrm{so}_3)$ when q is a root of unity are constructed. Part of them are obtained from irreducible representations of $\widehat{U}_q(\mathrm{sl}_2)$ by means of the homomorphism ψ.

2. Basic preliminary

We start by recalling basic definitions of some elementary algebraic structures used in the following parts: associative algebras, free algebras, lie algebras, universal enveloping algebras to set up my notation.
Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all integers, whole numbers, real numbers and complex numbers, resp. Let $\widehat{n}$ denote the set $\{1,2,...,n\}$.
2.1. **Definition.** *Ring.* A Ring $(R,+,\cdot)$ is a non-empty set R with 2 operations: an addition $+$ and a multiplication operation $\cdot$, such that
1. $(R,+)$ is a commutative group,
2. the multiplication is associative,
3. the multiplication and addition are distributive, i. e.

$$\forall a,b,c\in R:(a+b)\cdot c=a\cdot c+b\cdot c\wedge a\cdot(b+c)=a\cdot b+a\cdot c.$$

The identity element of $(R,+)$ is usually denoted as 0. If a ring R has a unity element 1 (such that $1\cdot x = x = x\cdot 1$) then R is called a ring with unity $(R,+,\cdot,1)$. A ring is said to be commutative if $ab = ba$ holds for each $a,b \in R$.

2.2. **Definition.** *Module.* Let R be a ring. A left-module $(M,R,+,\cdot)$ over the ring R (or R-module) is an abelian group $(M,+)$ and an operation $\cdot : R\times M \to M$) such that

$$(a+b)\cdot u = a\cdot u + b\cdot u \wedge a\cdot(u+v) = a\cdot u + a\cdot v \ \forall a,b \in R, u,v \in M.$$

2.3. **Definition.** *Algebra.* Let R be a commutative ring. An algebra (A,R,m) over R (or R-algebra) is an R-module A with a bilinear product map $A\times A \to A$. An algebra is said to be associative if

$$m(x,m(y,z)) = m(m(x,y),z) \ \forall x,y,z \in A.$$

If A has an element 1 suchthat $m(x,1) = x = m(1,x)$ for all x, then the algebra is called an R-algebra with unity.

2.4. **Definition.** *Free algebra.* Let $n \in \mathbb{N}$, $X = x_1,...,x_n$ be set of n distinct letters. Consider the set of finite (noncommutative) monomials (or, more precisely, finite sequences) in the elements of $X \cup 1$ (strings of letters in the alphabet X). The set of (finite) linear combinations of these monomials with coefficients in a ring R (noncommutative polynomials) $R[[x_1,...,x_n]]$ can be given the structure of an associative algebra $(R[[X]],R,\cdot)$ by defining the multiplication of any pair of monomials $x_{i_1},...,x_{i_r}$ and $x_{j_1},...,x_{j_s}$ in an obvious way to be

$$(x_{i_1}...x_{i_r})\cdot(x_{j_1}...x_{j_s}) = x_{i_1}...x_{i_r}\cdot x_{j_1}...x_{j_s},$$
$$x_{i_1}...x_{i_r}\cdot 1 = x_{i_1}...x_{i_r},$$
$$1\cdot x_{i_1}...x_{i_r} = x_{i_1}...x_{i_r}.$$

This R-algebra is called free R-algebra on the generators in the set X.

2.5. **Definition.** *Ideal.* Let A be an algebra. A subset $B \subset A$ is called a subalgebra of A if B also forms an algebra. A subalgebra I of A is called a left (resp. right) ideal, if $\forall x \in I, A\cdot x \subset I$ (resp. $x\cdot A \subset I$). A subalgebra which is both a left and a right ideal, is called a two sided ideal. If $I \neq A$ then I is called proper ideal of A. If $I \neq \{\}$ then I is nontrivial ideal of A.

2.6. **Lemmma.** *Quotient algebra.* Let A be an R-algebra with a two-sided ideal I. The quotient set A/I which contains of all classes denoted by equivalence $x \sim y \Leftrightarrow x \Leftrightarrow y \in I$ is an algebra called the quotient algebra of A by I.

2.7. **Definition.** *Centre.* Let A be an R-algebra. The centre $Z(A)$ of A is defined to be the set $\{x \in A | x\cdot y = y\cdot x \ \forall y \in A\}$.

2.8. **Definition.** *Tensor algebra.* Let V be a vector space on field0 $\mathbb{C}$. For each integer n let us define $T^n(V) = \bigotimes_{i=1}^{n} V$. Define $T^0(V) = \mathbb{C}$. Because the tensor product is associative, there is a bilinear map $T^r(V)\times T^s(V) \to T^{r+s}(V)$ using this we can obtain the space

$$T(V) = \bigoplus_{n=0}^{+\infty} T^n(V) = \mathbb{C} \oplus (V) \oplus (V\otimes V) \oplus ....$$

This space forms structure of $\mathbb{C}$-algebra called tensor algebra $T(V)$ of V. The product in $T(V)$ is again denoted by $\otimes$.

and and

2.9. **Definition.** *Lie algebra.* A Lie algebra $(g,[.,.])$ is a $\mathbb{C}$-algera with a $\mathbb{C}$-bilinear product map $g\times g \to g$ satisfying antisymmetry and Jacobi identity

$[x,x]=0\ \forall x\in g,$
$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0\ \forall x,y,z\in g.$

2.10. **Definition.** *Universal enveloping algebra.* Let $(g,[.,.])$ be a Lie algebra. Let $T(g)$ be the tensor algebra of the $\mathbb{C}$-vector space g. Let us consider then the following ideal in $T(g)$:

$$I=\text{lin}\{a\otimes(x\otimes y\Leftrightarrow y\otimes x\Leftrightarrow[x,y])\otimes b|x,y\in g,a,b\in T(g)\}.$$

Let's consider the quotient set $U(g)=T(g)/I$, denote classes of equivalence of $x\in T(g)$ by $i(x)$. Then $U(g)$ with multiplication

$$i(x)\otimes i(y)=i(x\otimes y)\forall x,y\in T(g).$$

is called the universal enveloping algebra of g.

2.11. **Lemmma.** $U(g)$ is an associative infinite dimensional $\mathbb{C}$-algebra with unity.

Proof. Let $x,x',y,y'\in U(g)$, let $i(x)=i(x')$, $i(y)=i(y')$. Then $\exists k_1,k_2\in I{:}x=x'+k_1\wedge y=y'+k_2$. According to 2.10 we have

$$i(x')\cdot i(y')=i(x'\cdot y')=i((x+k_1)\cdot(y+k_2))=i(x\cdot y+x\cdot k_2+k_1\cdot y+k_1\cdot k_2)=$$
$$=i(x\cdot y+\text{something}\in K)=i(x\cdot y).$$

Multiplication in $U(g)$ is associative because multiplication in $T(g)$ is associative.

3. PBW theorem for universal enveloping algebras

3.1. **Lemmma.** Let $(A,\cdot)$ be an associative $\mathbb{C}$-algebra. Defining the commutator $[.,.]{:}A\times A\to A$

$$[x,y]=x\cdot y\Leftrightarrow y\cdot x,$$

we obtain Lie algebra $L(A)=(A,[.,.])$.

This result can be reversed in some sense:

3.2. **Theorem.** PBW theorem about Lie algebras. For each Lie algebra g there exists an associative algebra A (over the same field) such that g is issomorphic to some sub-algebra of $L(A)$.

We need following definitons & lemma's for the proof.

3.3. **Lemmma.** Defining a Lie bracket in natural way

$$[i(x),i(y)]=i(x)\otimes i(y)\Leftrightarrow i(y)\otimes i(x),$$

$U(g)$ can be considered as a Lie algebra.

3.4. **Definition.** Let g be a finite dimensional Lie algebra with basis $(e_1,\ldots,e_n)$. Any $x\in T(g)$ we call monomial $\Leftrightarrow$ $\exists k\in\mathbb{N}\ \exists i_1,\ldots,i_k\in\widehat{n}{:}x=e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_k}$.

A monomial is ordered $\Leftrightarrow$ when $i_1\leqq\ldots\leqq i_k$ holds.

3.5. **Definition.** Let $(i_1,\ldots,i_k)\in\mathbb{N}^k$. Every pair $(a,b)\in\widehat{k}^2$ where $i_a>i_b$ is called inverse in permutation $(i_1,\ldots,i_k)$.

3.6. **Lemmma.** Each class in $U(g)$ contains at least 1 element which is linear combination of ordered monomials.

Proof. It is sufficient to show that each monomial $e_{i_1}\otimes\ldots\otimes e_{i_k}$ is able to be written as ordered monomial + something from ideal.

We will prove lemma by induction according to monomial length and (then) according to number of inverses in permutation $(i_1,\ldots,i_k)$:

Lemma is certainly valid for ordered monomials. Now let's take any monomial

$$e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{j+1}}\otimes\ldots\otimes e_{i_k},$$

where $i_j>i_{j+1}$. Then

$$e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{j+1}}\otimes\ldots\otimes e_{i_k}=$$
$$=e_{i_1}\otimes\ldots\otimes(e_{i_j}\otimes e_{i_{j+1}}\Leftrightarrow[e_{i_j},e_{i_{j+1}}]\Leftrightarrow e_{i_{j+1}}\otimes e_{i_j})\otimes\ldots\otimes e_{i_k}+$$
$$+e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes e_{i_k}+e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_k},$$

the first term is from ideal, we can use induction hypothesis acting on the second (it is shorter) and the third (it contains certainly less inverses). Lemma is fulfilled by induction.

3.7. **Lemmma.** Any set of classes in $U(g)$ generated by different ordered monomials is linear independent.

Proof. We will prove it by contradiction. Let

$$\alpha_1 i(e_{i_{1,1}}\otimes\ldots\otimes e_{i_{1,k_1}})+\alpha_2 i(e_{i_{2,1}}\otimes\ldots\otimes e_{i_{2,k_2}})+\ldots+\alpha_m i(e_{i_{m,1}}\otimes\ldots\otimes e_{i_{m,k_m}})=i(\theta).$$

We want to show

$$\alpha_1=\ldots=\alpha_m=0.$$

Let B denote span of all ordered monomials. Clearly it is a subspace of $T(g)$.

We will construct linear mapping $\sigma: T(g)\to B$ for which following is true:

i) $\forall m\in\mathbb{N}\ \forall i_1\leqq\ldots\leqq i_m\in\widehat{n}:\sigma(e_{i_1}\otimes\ldots\otimes e_{i_m})=e_{i_1}\otimes\ldots\otimes e_{i_m}$

ii) $\forall m\in\mathbb{N}\ \forall i_1,\ldots,i_m\in\widehat{n}, i_j\geqq i_{j+1}:\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{j+1}}\otimes\ldots\otimes e_{i_m})=$
$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_m})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes e_{i_m})$.

If σ exists, it is obviously constant on all classes, it means

$$\forall x\in i(y)\in T(g)/_I:\sigma(x)=\sigma(y).$$

Consequently it induces mapping $\tilde{\sigma}: T(g)/_I\to B$ defined by

$$\forall x\in T(g):\tilde{\sigma}(i(x))=\sigma(x).$$

$\tilde{\sigma}$ is linear, in addition

$$\forall m\in\mathbb{N}\ \forall i_1\leqq\ldots\leqq i_m\in\widehat{n}:\tilde{\sigma}(i(e_{i_1}\otimes\ldots\otimes e_{i_m}))=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_m})=e_{i_1}\otimes\ldots\otimes e_{i_m},$$

applying $\tilde{\sigma}$ on original linear combination yields

$$\theta=\tilde{\sigma}(i(\theta))=\tilde{\sigma}(\alpha_1 i(e_{i_{1,1}}\otimes\ldots\otimes e_{i_{1,k_1}})+\alpha_2 i(e_{i_{2,1}}\otimes\ldots\otimes e_{i_{2,k_2}})+$$
$$+\ldots+\alpha_m i(e_{i_{m,1}}\otimes\ldots\otimes e_{i_{m,k_m}}))=\alpha_1 e_{i_{1,1}}\otimes\ldots\otimes e_{i_{1,k_1}}+\alpha_2 e_{i_{2,1}}\otimes\ldots\otimes e_{i_{2,k_2}}+$$
$$+\ldots+\alpha_m e_{i_{m,1}}\otimes\ldots\otimes e_{i_{m,k_m}}$$

which is contradiction (any set of different ordered monomials is linear independent).

Now We will show that mapping σ fulfilling i) and ii) can be constructed.

σ is well defined on ordered monomials by i). Let's take any monomial

$$e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{j+1}}\otimes\ldots\otimes e_{i_k},$$

where $i_j>i_{j+1}$.

Assume σ is well defined on a subspace of $T(g)$ consisting of span of all monomials shorter than this one and of all monomials which have equal length but less inverses. Now define

$$\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{j+1}}\otimes\ldots\otimes e_{i_k})=$$
$$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes e_{i_k})$$

(we know how to apply σ on last two monomials: the first has less inverses and the second is shorter).

We have to make sure that the definition is correct. Original monomial could contain another inverse $i_l>i_{l+1}$. We must consider 2 cases:

1) $j+1<l$ (or $l+1<j$) (inverses are farther),

2) $j+1=l$ (or $l+1=j$) (one element is common to both inverses).

(Let's consider first alternatives in both cases, proof of the other is similar.) In both cases 1), 2) we need to show

$$\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes e_{i_k})=$$
$$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k}).$$

Case 1). We know

$$\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_l}\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes e_{i_l}\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k})=$$
$$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{j+1}}\otimes e_{i_j}\otimes\ldots\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{j+1}}]\otimes\ldots\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k}),$$

and likewise for the second expression.

Case 2). We need to show

$$\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k})=$$
$$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_l}\otimes e_{i_j}\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_l}]\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k}).$$

Left side can be written as

$$\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k})=$$
$$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{l+1}}\otimes e_{i_j}\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{l+1}}]\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k})=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes e_{i_j}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{l+1}}\otimes[e_{i_j},e_{i_l}]\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_{l+1}}]\otimes e_{i_l}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_j}\otimes[e_{i_l},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k}),$$

right side as

$$\sigma(e_{i_1}\otimes\ldots\otimes e_{i_l}\otimes e_{i_j}\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_l}]\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k})=$$
$$=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_l}\otimes e_{i_{l+1}}\otimes e_{i_j}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_l}\otimes[e_{i_j},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_l}]\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k})=\sigma(e_{i_1}\otimes\ldots\otimes e_{i_{l+1}}\otimes e_{i_l}\otimes e_{i_j}\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_l},e_{i_{l+1}}]\otimes e_{i_j}\otimes\ldots\otimes e_{i_k})+\sigma(e_{i_1}\otimes\ldots\otimes e_{i_l}\otimes[e_{i_j},e_{i_{l+1}}]\otimes\ldots\otimes e_{i_k})+$$
$$+\sigma(e_{i_1}\otimes\ldots\otimes[e_{i_j},e_{i_l}]\otimes e_{i_{l+1}}\otimes\ldots\otimes e_{i_k}).$$

The first terms are identical. In next 3 terms brackets can be combined from base elements:

$$[e_{i_j},e_{i_l}]=\sum_{q=1}^{n}\alpha_q e_q,$$
$$[e_{i_j},e_{i_{l+1}}]=\sum_{q=1}^{n}\beta_q e_q,$$
$$[e_{i_l},e_{i_{l+1}}]=\sum_{q=1}^{n}\gamma_q e_q.$$

If we put these sums into left and right sides, for each $q\in\widehat{n}$ we can from induction hypothesis reduce inverses on both sides. For example, if $q\geqq i_{l+1}$ then we have on the left side

$$\sigma(\ldots\otimes e_{i_{l+1}}\otimes e_q\otimes\ldots)=\sigma(\ldots\otimes e_q\otimes e_{i_{l+1}}\otimes\ldots)+\sigma(\ldots\otimes[e_{i_{l+1}},e_q]\otimes\ldots),$$

the first term is equal with that one on the right side, the second term remains.
If we now put all terms on the left side, we have

$$\sigma(\ldots\otimes[e_{i_{l+1}},[e_{i_j},e_{i_l}]]\otimes\ldots)+\sigma(\ldots\otimes[[e_{i_j},e_{i_{l+1}}],e_{i_l}]\otimes\ldots)+\sigma(\ldots\otimes[e_{i_j},[e_{i_l},e_{i_{l+1}}]]\otimes\ldots),$$

but it is zero because of Jacobi identity.

3.8. **Lemmma.** $U(g)$ and the subset B of $T(g)$ consisting of all ordered monomials are isomorphic.

Proof. It can be easily seen that $\tilde{\sigma}$ defined in previous proof is bijection: It is onto -- for each linear combination of ordered monomials $x=\sum_{j=1}^{k}\alpha_j x_j$ there exists a class $\tilde{x}=\sum_{j=1}^{k}\alpha_j i(x_j)$ such that $\tilde{\sigma}(\tilde{x})=x$. It is injection -- for any class $i(x)$ we know that x can be considered as linear combination of ordered monomials $x=\sum_{j=1}^{k}\alpha_j x_j$ but $i(x)=\theta$ implies $\alpha_j=0$ for each j.

Now we can finish the proof of PBW theorem.

Proof. If we take a subset B of $T(g)$ consisting of all ordered monomials, we can easy define associative multiplication on B as in previous lemma. Now let's take all elements of B corresponding to base elements of g ⇔there exists a natural isomorphism between this subspace and g.

4. Fairlie's quantum deformation of so(3)

It is well-known that the Lie algebras sl_2 and so_3 of the Lie groups $SL(2,\mathbb{C})$ and $SO(3)$, respectively, are isomorphic. But these algebras differ from each other if we consider their embedding to the wider Lie algebra sl_3. There is no automorphism of sl_3 which transfers the embedding $\mathrm{sl}_2\subset\mathrm{sl}_3$ to the embedding $\mathrm{so}_3\subset\mathrm{sl}_3$. Note that the embedding $\mathrm{so}_3\subset\mathrm{sl}_3$ is of great importance for nuclear physics: it is used in spectroscopy.

The definition of the q-analogue of the universal enveloping algebra $U(\mathrm{sl}_2)$ is well-known. It is the quantum algebra $U_q(\mathrm{sl}_2)$ which is a Hopf algebra. If we wish to have a q-analogue of the universal enveloping algebra so_3 such that at $q\to 1$ we obtain the classical embedding $\mathrm{so}_3\subset\mathrm{sl}_3$, then the algebra sl_2 is not appropriate for this role. By other words, an algebra $U_q(\mathrm{so}_3)$ must differ from $U_q(\mathrm{sl}_2)$. This algebra $U_q(\mathrm{so}_3)$ is well. It is the associative algebra generated by three elements I_1, I_2 and I_3 satisfying the relations

$$q^{1/2}I_1I_2\Leftrightarrow q^{-1/2}I_2I_1=I_3, \quad (4.1)$$
$$q^{1/2}I_2I_3\Leftrightarrow q^{-1/2}I_3I_2=I_1, \quad (4.2)$$
$$q^{1/2}I_3I_1\Leftrightarrow q^{-1/2}I_1I_3=I_2. \quad (4.3)$$

Such (and more general) deformation of the commutator $[I_i,I_j]=I_iI_j\Leftrightarrow I_jI_i$ was defined at 1967 by R. Santilli (see also Refs. [13]) under studying a generalization of the Lie theory. Afterwards (in 1990), the algebra $U_q(\mathrm{so}_3)$ with commutation relations (4.1)--(4.3) was determined by D. Fairlie (see [4]). An algebra which can be reduced to $U_q(\mathrm{so}_3)$ was defined in 1986 by M. Odesski (see [14]).

The algebra $U_q(\mathrm{so}_3)$ can be also defined using slightly different relations (1.5):

4.1. **Definition.** Let's take three dimensional vector space $g=\mathrm{lin}\{I_1,I_2,I_3\}$. Let $q\neq 0$. Then take following subspace of tensor algebra $T(g)$:

$$I=\mathrm{lin}(\{x\otimes(I_1\Leftrightarrow I_3\otimes I_2+qI_2\otimes I_3)\otimes y|x,y\in T(g)\}\cup$$
$$\cup\{x\otimes(I_2\Leftrightarrow I_3\otimes I_1+q^{-1}I_1\otimes I_3)\otimes y|x,y\in T(g)\}\cup$$
$$\cup\{x\otimes(I_3\Leftrightarrow I_2\otimes I_1+qI_1\otimes I_2)\otimes y|x,y\in T(g)\}).$$

I is clearly ideal in $T(g)$. A quotient algebra $T(g)/I$ is what we call quantum deformation of so(3) (the algebra $U_q(\mathrm{so}_3)$).

Fairlie gave finite dimensional irreducible representations of the algebra $U_q(\mathrm{so}_3)$ which at $q\to 1$ give the well-known finite dimensional irreducible representations of the Lie algebra so_3. These representations are given by integral or half-integral non-negative numbers. Odesski also gave some classes of irreducible representations.

It was shown (see [2]) that the algebra $U_q(\mathrm{so}_3)$ has irreducible finite dimensional representations which have no classical analogue (that is, which do not admit the limit $q\to 1$). It was not clear why such strange representations of the algebra $U_q(\mathrm{so}_3)$ appear. Construction of a homomorphism from $U_q(\mathrm{so}_3)$ to the algebra $\hat{U}_q(\mathrm{sl}_2)$ gives clear answer to this question.

We construct a homomorphism from $U_q(\mathrm{so}_3)$ to the algebra $\hat{U}_q(\mathrm{sl}_2)$ which is an extension of the well-known quantum algebra $U_q(\mathrm{sl}_2)$ (note that there is no homomorphism from $U_q(\mathrm{so}_3)$ to $U_q(\mathrm{sl}_2)$). Irreducible finite dimensional representations of $U_q(\mathrm{sl}_2)$ (but not all) can be extended to finite dimensional representations of the algebra $\hat{U}_q(\mathrm{sl}_2)$. Composing a homomorphism $U_q(\mathrm{so}_3)\to\hat{U}_q(\mathrm{sl}_2)$ with these representations of $\hat{U}_q(\mathrm{sl}_2)$, we obtain representations of the algebra $U_q(\mathrm{so}_3)$. But some of irreducible representations of $\hat{U}_q(\mathrm{sl}_2)$ lead to reducible representations of the algebra $U_q(\mathrm{so}_3)$. Decomposing these reducible representations of $U_q(\mathrm{so}_3)$ we obtain irreducible representations of this algebra which have no analogue for the Lie algebra so_3. If q is not a root of unity, then in this way we obtain all finite dimensional irreducible representations of $U_q(\mathrm{so}_3)$. But there are infinite dimensional irreducible representations of $U_q(\mathrm{so}_3)$ which cannot be obtained in this way.

Existence of the homomorphism $U_q(\mathrm{so}_3)\to\hat{U}_q(\mathrm{sl}_2)$ allows us to define tensor products of representations of the algebra $U_q(\mathrm{so}_3)$ which is not a Hopf algebra.

Using the homomorphism $U_q(\mathrm{so}_3)\to\hat{U}_q(\mathrm{sl}_2)$ and irreducible representations of $\hat{U}_q(\mathrm{sl}_2)$ we obtain representations of $U_q(\mathrm{so}_3)$ when q is a root of unity. Taking irreducible representations of $U_q(\mathrm{so}_3)$ obtained in this way and decomposing reducible representations, we obtain several series of irreducible representations of $U_q(\mathrm{so}_3)$. In addition, we construct irreducible representations of $U_q(\mathrm{so}_3)$ which cannot be derived from $\hat{U}_q(\mathrm{sl}_2)$.

When q is not a root of unity, then each irreducible (finite or infinite dimensional) representation of $U_q(\mathrm{so}_3)$ is equivalent to one of the representations constructed in the next chapters. For the irreducible representations of $U_q(\mathrm{so}_3)$ when q is a root of unity we have no proof of similar assertion. The reason of this is that in this case there are many classes of irreducible representations and a proof of completeness of irreducible representations becomes very tedious.

Let us remark that in Ref. [14] there were constructed irreducible finite dimensional representations of $U_q(\mathrm{so}_3)$ when q is not a root of unity and a part of irreducible infinite dimensional representations. In Refs. [2] and [15], there were constructed irreducible representations of $U_q(\mathrm{so}_3)$ which satisfy the conditions of $*$-representations (that is, such that $T(I_j^*)=\Leftrightarrow T(I_j)$, $j=1,2$). These $*$-representations are a part of irreducible representations of $U_q(\mathrm{so}_3)$ constructed in this paper. Also remark irreducible representations of $U_q(\mathrm{so}_3)$ for q a root of unity in Ref. [7], where a part of irreducible representations for this case were constructed.

Note that in all Refs. above there are no relations of representations of $U_q(\mathrm{so}_3)$ to representations of $\hat{U}_q(\mathrm{sl}_2)$. This relation makes representations of $U_q(\mathrm{so}_3)$ clear and understandable.

Now we can ask the first important question: if PBW holds for this type of algebra, it means if any set of classes in $T(g)/I$ generated by different ordered monomials is linear independent. The answer is yes, we can say even more:

Let $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3$ are non-zero complex numbers. Let us consider following set:

$$I_1 = \mathrm{lin}(\{x\otimes(\alpha_1 I_1 \Leftrightarrow \beta_1 I_3\otimes I_2 + \gamma_1 I_2\otimes I_3)\otimes y | x,y\in T(g)\}\cup$$
$$\cup\{x\otimes(\alpha_2 I_2 \Leftrightarrow \beta_2 I_3\otimes I_1 + \gamma_2 I_1\otimes I_3)\otimes y | x,y\in T(g)\}\cup$$
$$\cup\{x\otimes(\alpha_3 I_3 \Leftrightarrow \beta_3 I_2\otimes I_1 + \gamma_3 I_1\otimes I_2)\otimes y | x,y\in T(g)\}).$$

Without loss of generality we can assume $\alpha_1=\alpha_2=\alpha_3=1$ (otherwise we could divide all three expressions by α_1, α_2, α_3 resp. and the span wouldn't change).

Another simplification can be achieved considering new basis (J_1,J_2,J_3):

$$J_1 = \frac{1}{\sqrt{\alpha_2\alpha_3}} I_1,$$
$$J_2 = \frac{1}{\sqrt{\alpha_1\alpha_3}} I_2,$$
$$J_3 = \frac{1}{\sqrt{\alpha_1\alpha_2}} I_3.$$

Then the set I_1 is given by formula

$$I_1 = \mathrm{lin}(\{x\otimes(J_1 \Leftrightarrow J_3\otimes J_2 + \beta_{23} J_2\otimes J_3)\otimes y | x,y\in T(g)\}\cup$$
$$\cup\{x\otimes(J_2 \Leftrightarrow J_3\otimes J_1 + \beta_{13} J_1\otimes J_3)\otimes y | x,y\in T(g)\}\cup$$
$$\cup\{x\otimes(J_3 \Leftrightarrow J_2\otimes J_1 + \beta_{12} J_1\otimes J_2)\otimes y | x,y\in T(g)\}),$$

where $\beta_{13}=\gamma_2, \beta_{12}=\gamma_3, \beta_{23}=\beta_1$.

4.2. **Lemmma.** Every class $T(g)/_{I_1}$ contains at least one element which is linear combination of ordered monomials.

Proof. Very simliar to the proof of 3.6

It is sufficient to show that each monomial $J_{i_1}\otimes\ldots\otimes J_{i_k}$ is able to be written as ordered monomial + something from ideal.

We will prove lemma by induction according to monomial length and (then) according to number of inverses in permutation $(i_1,\ldots,i_k)$:

Lemma is certainly valid for ordered monomials. Now let's take any monomial

$$J_{i_1}\otimes\ldots\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes\ldots\otimes J_{i_k},$$

where $i_j > i_{j+1}$. Then

$$J_{i_1}\otimes\ldots\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes\ldots\otimes J_{i_k} =$$
$$= J_{i_1}\otimes\ldots\otimes(J_{i_j}\otimes J_{i_{j+1}} \Leftrightarrow [J_{i_j},J_{i_{j+1}}] \Leftrightarrow \beta_{i_{j+1}i_j} J_{i_{j+1}}\otimes J_{i_j})\otimes\ldots\otimes J_{i_k} +$$
$$+ J_{i_1}\otimes\ldots\otimes[J_{i_j},J_{i_{j+1}}]\otimes\ldots\otimes J_{i_k} + \beta_{i_{j+1}i_j} J_{i_1}\otimes\ldots\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes\ldots\otimes J_{i_k},$$

the first term is from ideal, we can use induction hypothesis acting on the second (it is shorter) and the third (it contains certainly less inverses). Lemma is fulfilled by induction.

4.3. **Theorem.** Any set of classes in $T(g)/I_1$ generated by different ordered monomials is linear independent $\Leftrightarrow \exists q\in\mathbb{C}, q\neq 0$ such that $q=\gamma_1=\gamma_3=\dfrac{1}{\gamma_2}$.

Proof. $\Rightarrow$: Because the tensor product in $T(g)/_{I_1}$ is associative, following must hold:

$$i(J_3)\cdot(i(J_2)\cdot i(J_1)) = (i(J_3)\cdot i(J_2))\cdot i(J_1).$$

Reordering the left side we have

$$i(J_3)\cdot(i(J_2)\cdot i(J_1)) = i(J_3)\cdot i(\beta_{12}J_1\otimes J_2 + J_3) =$$
$$= \beta_{12}i((J_3\otimes J_1)\otimes J_2) + i(J_3\otimes J_3) = \beta_{12}(i(\beta_{13}J_1\otimes J_3 + J_2)\cdot i(J_2)) + i(J_3\otimes J_3) =$$
$$= \beta_{12}\beta_{13}i(J_1\otimes(J_3\otimes J_2)) + \beta_{12}i(J_2\otimes J_2) + i(J_3\otimes J_3) =$$
$$= \beta_{12}\beta_{13}(i(J_1)\cdot i(\beta_{23}J_2\otimes J_3 + J_1)) + \beta_{12}i(J_2\otimes J_2) + i(J_3\otimes J_3) =$$
$$= \beta_{12}\beta_{13}\beta_{23}i(J_1\otimes J_2\otimes J_3) + \beta_{12}\beta_{13}i(J_1\otimes J_1) + \beta_{12}i(J_2\otimes J_2) + i(J_3\otimes J_3).$$

On the right side we have

$$(i(J_3)\cdot i(J_2))\cdot i(J_1) = i(\beta_{23}J_2\otimes J_3 + J_1)\cdot i(J_1) =$$
$$= \beta_{23}i(J_2\otimes(J_3\otimes J_1)) + i(J_1\otimes J_1) = \beta_{23}(i(J_2)\cdot i(\beta_{13}J_1\otimes J_3 + J_2)) + i(J_1\otimes J_1) =$$
$$= \beta_{23}\beta_{13}i((J_2\otimes J_1)\otimes J_3) + \beta_{23}i(J_2\otimes J_2) + i(J_1\otimes J_1) =$$
$$= \beta_{23}\beta_{13}(i(\beta_{12}J_1\otimes J_2 + J_3)\cdot i(J_3)) + \beta_{23}i(J_2\otimes J_2) + i(J_1\otimes J_1) =$$
$$= \beta_{23}\beta_{13}\beta_{12}i(J_1\otimes J_2\otimes J_3) + \beta_{23}\beta_{13}i(J_3\otimes J_3) + \beta_{23}i(J_2\otimes J_2) + i(J_1\otimes J_1).$$

According to assumption the classes generated by different ordered monomials are linear independent so that the corresponding coefficients on both sides must be equal.

$$\begin{aligned} \beta_{12}\beta_{13}\beta_{23} &= \beta_{23}\beta_{13}\beta_{12} \\ \beta_{12}\beta_{13} &= 1 \\ \beta_{12} &= \beta_{23} \\ 1 &= \beta_{23}\beta_{13}. \end{aligned}$$

This system of equations (all coef. are non-zero) is fulfilled if and only if there is a $q\in C, q\neq 0$ such that

$$\begin{aligned} \beta_{23} &= q \\ \beta_{13} &= q^{-1} \\ \beta_{12} &= q. \end{aligned}$$

$\Leftarrow$: Similar to the proof of the 3.7 We will prove it by contradiction. Let

$$\alpha_1 i(J_{i_{1,1}}\otimes\ldots\otimes J_{i_{1,k_1}}) + \alpha_2 i(J_{i_{2,1}}\otimes\ldots\otimes J_{i_{2,k_2}}) + \ldots + \alpha_m i(J_{i_{m,1}}\otimes\ldots\otimes J_{i_{m,k_m}}) = i(\theta).$$

We want to show $\alpha_1 = \ldots = \alpha_m = 0$.

Let B denote span of all ordered monomials. Clearly it is a subspace of $T(g)$.

We will construct linear mapping $\sigma: T(g)\to B$ for which following is true:

i) $\forall m\in N\ \forall i_1\leqq\ldots\leqq i_m\in\widehat{n}: \sigma(J_{i_1}\otimes\ldots\otimes J_{i_m}) = J_{i_1}\otimes\ldots\otimes J_{i_m}$

ii) $\forall m\in N\ \forall i_1,\ldots,i_m\in\widehat{n}, i_j\geqq i_{j+1}: \sigma(J_{i_1}\otimes\ldots\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes\ldots\otimes J_{i_m}) =$
$= \beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes\ldots\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes\ldots\otimes J_{i_m}) + \sigma(J_{i_1}\otimes\ldots\otimes[J_{i_j}, J_{i_{j+1}}]\otimes\ldots\otimes J_{i_m})$.

If σ exists, it is obviously constant on all classes, it means

$$\forall x\in i(y)\in T(g)/_{I_1}: \sigma(x) = \sigma(y).$$

Consequently it induces mapping $\tilde{\sigma}: T(g)/_{I_1}\to B$ defined by

$$\forall x\in T(g): \tilde{\sigma}(i(x)) = \sigma(x).$$

$\tilde{\sigma}$ is linear, in addition

$$\forall m\in N\ \forall i_1\leqq\ldots\leqq i_m\in\widehat{n}: \tilde{\sigma}(i(J_{i_1}\otimes\ldots\otimes J_{i_m})) = \sigma(J_{i_1}\otimes\ldots\otimes J_{i_m}) = J_{i_1}\otimes\ldots\otimes J_{i_m},$$

applying $\tilde{\sigma}$ on original linear combination yields

$$\theta = \tilde{\sigma}(i(\theta)) = \tilde{\sigma}(\alpha_1 i(J_{i_{1,1}}\otimes\ldots\otimes J_{i_{1,k_1}}) + \alpha_2 i(J_{i_{2,1}}\otimes\ldots\otimes J_{i_{2,k_2}}) + \ldots +$$
$$+ \alpha_m i(J_{i_{m,1}}\otimes\ldots\otimes J_{i_{m,k_m}})) = \alpha_1 J_{i_{1,1}}\otimes\ldots\otimes J_{i_{1,k_1}} + \alpha_2 J_{i_{2,1}}\otimes\ldots\otimes J_{i_{2,k_2}} + \ldots +$$
$$+ \alpha_m J_{i_{m,1}}\otimes\ldots\otimes J_{i_{m,k_m}}$$

which is contradiction (any set of different ordered monomials is linear independent).
Now we will show that mapping σ fulfilling i) and ii) can be constructed.
σ is well defined on ordered monomials by i). Let's take any monomial

$$J_{i_1}\otimes\ldots\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes\ldots\otimes J_{i_k},$$

where $i_j > i_{j+1}$.
Assume σ is well defined on a subspace of $T(g)$ consisting of span of all monomials shorter than this one and of all monomials which have equal length but less inverses.
Now define

$$\sigma(J_{i_1}\otimes...\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes...\otimes J_{i_k})=\beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes J_{i_k}),$$

(we know how to apply σ on last two monomials: the first has less inverses and the second is shorter).
We have to make sure that the definition is correct. Original monomial could contain another inverse $i_l > i_{l+1}$. We must consider 2 cases:
1) $j+1 < l$ (or $l+1 < j$) (inverses are farther),
2) $j+1 = l$ (or $l+1 = j$) (one element is common to both inverses).
(Let's consider first alternatives in both cases, proof of the other is similar.) In both cases 1), 2) we need to show

$$\beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes J_{i_k})+\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes J_{i_k})=$$
$$=\beta_{i_{l+1}i_l}\sigma(J_{i_1}\otimes...\otimes J_{i_{l+1}}\otimes J_{i_l}\otimes...\otimes J_{i_k})+\sigma(J_{i_1}\otimes...\otimes[J_{i_l},J_{i_{l+1}}]\otimes...\otimes J_{i_k}).$$

Case 1). We know

$$\beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes J_{i_l}\otimes J_{i_{l+1}}\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes J_{i_l}\otimes J_{i_{l+1}}\otimes...\otimes J_{i_k})=$$
$$=\beta_{i_{j+1}i_j}\beta_{i_{l+1}i_l}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes J_{i_{l+1}}\otimes J_{i_l}\otimes...\otimes J_{i_k})+$$
$$+\beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes[J_{i_l},J_{i_{l+1}}]\otimes...\otimes J_{i_k})+$$
$$+\beta_{i_{l+1}i_l}\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes J_{i_{l+1}}\otimes J_{i_l}\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes[J_{i_l},J_{i_{l+1}}]\otimes...\otimes J_{i_k}),$$

and likewise for the second expression:

$$\beta_{i_{l+1}i_l}\sigma(J_{i_1}\otimes...\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes...\otimes J_{i_{l+1}}\otimes J_{i_l}\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes J_{i_j}\otimes J_{i_{j+1}}\otimes...\otimes[J_{i_l},J_{i_{l+1}}]\otimes...\otimes J_{i_k})=$$
$$=\beta_{i_{l+1}i_l}\beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes J_{i_{l+1}}\otimes J_{i_l}\otimes...\otimes J_{i_k})+$$
$$+\beta_{i_{l+1}i_l}\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes J_{i_{l+1}}\otimes J_{i_l}\otimes...\otimes J_{i_k})+$$
$$+\beta_{i_{j+1}i_j}\sigma(J_{i_1}\otimes...\otimes J_{i_{j+1}}\otimes J_{i_j}\otimes...\otimes[J_{i_l},J_{i_{l+1}}]\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes[J_{i_j},J_{i_{j+1}}]\otimes...\otimes[J_{i_l},J_{i_{l+1}}]\otimes...\otimes J_{i_k}).$$

Case 2). Corollary $i_j=3, i_{j+1}=i_l=2, i_{l+1}=1$. We need to show

$$\beta_{12}\sigma(J_{i_1}\otimes...\otimes J_3\otimes J_1\otimes J_2\otimes...\otimes J_{i_k})+\sigma(J_{i_1}\otimes...\otimes J_3\otimes[J_2,J_1]\otimes...\otimes J_{i_k})=$$
$$=\beta_{23}\sigma(J_{i_1}\otimes...\otimes J_2\otimes J_3\otimes J_1\otimes...\otimes J_{i_k})+\sigma(J_{i_1}\otimes...\otimes[J_3,J_2]\otimes J_1\otimes...\otimes J_{i_k}).$$

At this moment we're applying assumption $\beta_{12}=q, \beta_{13}=q^{-1}, \beta_{23}=q$ on the left side...

$$\beta_{12}\sigma(J_{i_1}\otimes...\otimes J_3\otimes J_1\otimes J_2\otimes...\otimes J_{i_k})+\sigma(J_{i_1}\otimes...\otimes J_3\otimes[J_2,J_1]\otimes...\otimes J_{i_k})=$$
$$=\beta_{12}\beta_{13}\sigma(J_{i_1}\otimes...\otimes J_1\otimes J_3\otimes J_2\otimes...\otimes J_{i_k})+\beta_{12}\sigma(J_{i_1}\otimes...\otimes[J_3,J_1]\otimes J_2\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes J_3\otimes[J_2,J_1]\otimes...\otimes J_{i_k})=\beta_{12}\beta_{13}\beta_{23}\sigma(J_{i_1}\otimes...\otimes J_1\otimes J_2\otimes J_3\otimes...\otimes J_{i_k})+$$
$$+\beta_{12}\beta_{13}\sigma(J_{i_1}\otimes...\otimes J_1\otimes[J_3,J_2]\otimes...\otimes J_{i_k})+\beta_{12}\sigma(J_{i_1}\otimes...\otimes[J_3,J_1]\otimes J_2\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes J_3\otimes[J_2,J_1]\otimes...\otimes J_{i_k})=q\sigma(J_{i_1}\otimes...\otimes J_1\otimes J_2\otimes J_3\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes J_1\otimes J_1\otimes...\otimes J_{i_k})+q\sigma(J_{i_1}\otimes...\otimes J_2\otimes J_2\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes J_3\otimes J_3\otimes...\otimes J_{i_k}),$$

and similarly on the right side...

$$\beta_{23}\sigma(J_{i_1}\otimes...\otimes J_2\otimes J_3\otimes J_1\otimes...\otimes J_{i_k})+\sigma(J_{i_1}\otimes...\otimes[J_3,J_2]\otimes J_1\otimes...\otimes J_{i_k})=$$
$$=\beta_{23}\beta_{13}\sigma(J_{i_1}\otimes...\otimes J_2\otimes J_1\otimes J_3\otimes...\otimes J_{i_k})+\beta_{23}\sigma(J_{i_1}\otimes...\otimes J_2\otimes[J_3,J_1]\otimes...\otimes J_{i_k})+$$
$$+\sigma(J_{i_1}\otimes...\otimes[J_3,J_2]\otimes J_1\otimes...\otimes J_{i_k})=\beta_{23}\beta_{13}\beta_{12}\sigma(J_{i_1}\otimes...\otimes J_1\otimes J_2\otimes J_3\otimes...\otimes J_{i_k})+$$
$$+\beta_{23}\beta_{13}\sigma(J_{i_1}\otimes...\otimes[J_2,J_1]\otimes J_3\otimes...\otimes J_{i_k})+\beta_{23}\sigma(J_{i_1}\otimes...\otimes J_2\otimes[J_3,J_1]\otimes...\otimes J_{i_k})+$$

$+\sigma(J_{i_1}\otimes\ldots\otimes[J_3,J_2]\otimes J_1\otimes\ldots\otimes J_{i_k})=q\sigma(J_{i_1}\otimes\ldots\otimes J_1\otimes J_2\otimes J_3\otimes\ldots\otimes J_{i_k})+$
$+\sigma(J_{i_1}\otimes\ldots\otimes J_3\otimes J_3\otimes\ldots\otimes J_{i_k})+q\sigma(J_{i_1}\otimes\ldots\otimes J_2\otimes J_2\otimes\ldots\otimes J_{i_k})+$
$+\sigma(J_{i_1}\otimes\ldots\otimes J_1\otimes J_1\otimes\ldots\otimes J_{i_k})$.

Both sides are identical, theorem holds.

4.4. **Theorem.** If $\exists q\in C, q\neq 0$ such that $\beta_{23}=q\wedge\beta_{13}=q^{-1}\wedge\beta_{12}=q$ then $U(g)$ and the subspace B containing all ordered monomials from $T(g)$ are isomorphic.

Proof. See 3.8

The finishing of the proof of PBW theorem for deformation case is identical to non-deformed one.

5. Matrix representations not at root of unity

5.1. **Definition.** *Representation.* Let V be a $\mathbb{C}$-vector space and let A be $\mathbb{C}$-algebra. Denote set of all endomorphisms on V by $\mathrm{End}(V)$. If $\varphi: A\to\mathrm{End}(V)$ is a homomorphism, that means

$$\varphi(x\cdot y)=\varphi(x)\varphi(y)\forall x,y\in A,$$

then it is called a (linear) representation of A on V. The vector space V is also called A-module.

5.2. **Definition.** *Submodule.* Let V be an A-module. If V contains a proper linear subspace W such that W is closed under the action of A, that is $\varphi(A)W\subset W$, then W is said to carry a subrepresentation of A and W is a A-submodule of V.

5.3. **Definition.** *Irreducibility.* If an A-module has no proper, non-trivial submodules, it is called irreducible, otherwise it is called reducible.

The main result of this section is covered in the following theorem:

5.4. **Theorem.** Let $q\in\mathbb{C}$, $q^k\neq 1\ \forall k\in\mathbb{N}$. Let us consider following matrices of dimension n:

$$J_3=\begin{pmatrix}\alpha_0 & & & & \\ & \ddots & & & \\ & & \alpha_k & & \\ & & & \ddots & \\ & & & & \alpha_{n-1}\end{pmatrix},\quad J_1=\begin{pmatrix}0 & \beta_0 & & & \\ \gamma_0 & \ddots & \beta_k & & \\ & \gamma_k & \ddots & & \\ & & & \ddots & \beta_{n-2} \\ & & & \gamma_{n-2} & K\end{pmatrix},$$

$$J_2=J_3J_1\Leftrightarrow q^{-1}J_1J_3,$$

where α_k, β_k, γ_k and K are in this table:

	α_k	β_k	γ_k	K
1)	$\left[\Leftrightarrow\frac{n-1}{2}+k\right]_q$	$\frac{-q^2(q^{2(k+1-n)}-1)(q^{2k+2}-1)}{(q^{2k-n+3}+1)(q^2-1)^2}$	$\frac{1}{q^{2k+1-n}+1}$	0
2)	$+i\frac{1+q^{2n-1-2k}}{q^{n-\frac{3}{2}-k}(q^2-1)}$	$\frac{q^{1-2n}(q^{2(2n-1-k)}-1)(q^{2(k+1)}-1)}{(q^{2n-3-2k}-1)(q^2-1)^2}$	$\frac{q^{2n-1-2k}}{q^{2n-1-2k}-1}$	$+i\frac{q^{\frac{3}{2}-n}(q^{2n}-1)}{(q-1)(q^2-1)}$
3)	$+i\frac{1+q^{2n-1-2k}}{q^{n-\frac{3}{2}-k}(q^2-1)}$	$\frac{q^{1-2n}(q^{2(2n-1-k)}-1)(q^{2(k+1)}-1)}{(q^{2n-3-2k}-1)(q^2-1)^2}$	$\frac{q^{2n-1-2k}}{q^{2n-1-2k}-1}$	$\Leftrightarrow i\frac{q^{\frac{3}{2}-n}(q^{2n}-1)}{(q-1)(q^2-1)}$
4)	$\Leftrightarrow i\frac{1+q^{2n-1-2k}}{q^{n-\frac{3}{2}-k}(q^2-1)}$	$\frac{q^{1-2n}(q^{2(2n-1-k)}-1)(q^{2(k+1)}-1)}{(q^{2n-3-2k}-1)(q^2-1)^2}$	$\frac{q^{2n-1-2k}}{q^{2n-1-2k}-1}$	$+i\frac{q^{\frac{3}{2}-n}(q^{2n}-1)}{(q-1)(q^2-1)}$
5)	$\Leftrightarrow i\frac{1+q^{2n-1-2k}}{q^{n-\frac{3}{2}-k}(q^2-1)}$	$\frac{q^{1-2n}(q^{2(2n-1-k)}-1)(q^{2(k+1)}-1)}{(q^{2n-3-2k}-1)(q^2-1)^2}$	$\frac{q^{2n-1-2k}}{q^{2n-1-2k}-1}$	$\Leftrightarrow i\frac{q^{\frac{3}{2}-n}(q^{2n}-1)}{(q-1)(q^2-1)}$

Then following theorem holds: Every finite dimensional irreducible matrix representation is equivalent to one of listed above.
To proof this theorem we need following (helping) theorems & lemmas.

5.5. **Definition.**. Let $q\in\mathbb{C}\Leftrightarrow\{0,\Leftrightarrow 1,1\}$, $\nu\in\mathbb{C}$. We put $[\nu]_q=\dfrac{q^\nu\Leftrightarrow q^{-\nu}}{q\Leftrightarrow q^{-1}}$.

5.6. **Lemmma.** Let $n\in\mathbb{N}$, $q\in\mathbb{C}\Leftrightarrow\{0,\Leftrightarrow 1,1\}$, V be finite dimensional complex vector space, let the operators $J_1,J_2,J_3:V\to V$ fulfil commutation relations

$$\begin{aligned} J_2J_1\Leftrightarrow q\quad J_1J_2 &= J_3,\\ J_3J_1\Leftrightarrow q^{-1}J_1J_3 &= J_2,\\ J_3J_2\Leftrightarrow q\quad J_2J_3 &= J_1; \end{aligned} \tag{5.1}$$

let $x\in V$ and $\nu\in\mathbb{C}$ such that $J_3x=[\nu]_qx$.
Then

$$J_3(J_1\pm q^{\pm\nu}J_2)x=[\nu\pm 1]_q(J_1\pm q^{\pm\nu}J_2)x.$$

Proof.

$$\begin{aligned} &J_3J_1x\pm q^{\pm\nu}J_3J_2x=(J_2+q^{-1}J_1[\nu]_q)x\pm q^{\pm\nu}(J_1+qJ_2[\nu]_q)x=\\ &=(1\pm q^{\pm\nu+1}[\nu]_q)J_2+(\pm q^{\pm\nu}+q^{-1}[\nu]_q)J_1x=([\nu\pm 1]_q\cdot(\pm q^{\pm\nu}))J_2+([\nu\pm 1]_q)J_1x. \end{aligned}$$

5.7. **Note.** In [9] the commutation relations have slightly different form:

$$\begin{aligned} \tilde{q}XY\Leftrightarrow\frac{1}{\tilde{q}}YX &= Z\\ \tilde{q}YZ\Leftrightarrow\frac{1}{\tilde{q}}ZY &= X\\ \tilde{q}ZX\Leftrightarrow\frac{1}{\tilde{q}}XZ &= Y. \end{aligned} \tag{5.2}$$

It can be easily seen that transformation between our relations and those presented in [9] is

$$\begin{aligned} &J_3=iX, J_2=\tilde{q}Y, J_1=iZ,\\ &q=\tilde{q}^{-2}. \end{aligned}$$

5.8. **Lemmma.** Let $\lambda\in\mathbb{C}$, $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$. Pak $\exists\nu\in\mathbb{C}:[\nu]_q=\lambda$.
Proof. We need to solve equation

$$\frac{q^{\nu}\Leftrightarrow q^{-\nu}}{q\Leftrightarrow q^{-1}} = \lambda$$

$$q^{2\nu}\Leftrightarrow 1\Leftrightarrow\lambda q^{\nu}(q\Leftrightarrow q^{-1}) = 0$$

$$q^{\nu} = \underbrace{\frac{\lambda(q\Leftrightarrow q^{-1})\pm\sqrt{\lambda^2(q\Leftrightarrow q^{-1})^2+4}}{2}}_{\neq 0}$$

$$\nu = \frac{\ln\left(\frac{\lambda(q\Leftrightarrow q^{-1})\pm\sqrt{\lambda^2(q\Leftrightarrow q^{-1})^2+4}}{2}\right)}{\ln q}$$

Any of signs can be choosen. General power q^{ν} as function of ν maps onto $\mathbb{C}\Leftrightarrow\{0\}$ for any $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$ so that ν always exists.

5.9. **Lemmma.** $\forall\mu,\nu,k,l\in\mathbb{C}$:

1) $[\mu]_q=[\nu]_q \Leftrightarrow q^{\nu}=q^{\mu}\vee q^{\mu}=\Leftrightarrow q^{-\nu}$,

2) $[\mu]_q=[\mu+k]_q \Leftrightarrow q^k=1\vee q^{2\mu+k}=\Leftrightarrow 1$,

3) $[\mu+k]_q=[\mu+l]_q \Leftrightarrow q^{l-k}=1\vee q^{2\mu+k+l}=\Leftrightarrow 1$.

Proof. 1)

$$\begin{aligned}
[\mu]_q &= [\nu]_q \Leftrightarrow \\
q^{\nu}\Leftrightarrow q^{-\nu} &= q^{\mu}\Leftrightarrow q^{-\mu}\Leftrightarrow \\
q^{\nu}\Leftrightarrow q^{\mu} &= q^{-\nu}\Leftrightarrow q^{-\mu}\Leftrightarrow \\
q^{\nu}(1\Leftrightarrow q^{\mu-\nu}) &= \Leftrightarrow q^{-\mu}(1\Leftrightarrow q^{\mu-\nu})\Leftrightarrow \\
q^{\nu}=q^{\mu}\vee q^{\mu} &= \Leftrightarrow q^{-\nu}
\end{aligned}$$

2) according to 1)

$$[\mu]_q=[\mu+k]_q\Leftrightarrow q^{\mu}=q^{\mu+k}\vee q^{\mu}=q^{-\mu-k}\Leftrightarrow q^k=1\vee q^{2\mu+k}=\Leftrightarrow 1.$$

3) directly from 2).

5.10. **Lemmma.** $\forall q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$ $\forall c\in\mathbb{C}$:

1) $[c]_q=q^{c-1}+\frac{1}{q}[c\Leftrightarrow 1]_q$,

2) $[c]_q=q^{-(c-1)}+q[c\Leftrightarrow 1]_q$,

3) $(q[c]_q\Leftrightarrow[c+1]_q)(\frac{1}{q}[c]_q\Leftrightarrow[c+1]_q)=1$,

Proof. By inspection.

5.11. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $q^k\neq 1$ $\forall k\in\mathbb{N}$. Let $\mu\in\mathbb{C}$, $k_1,k_2\in Z$, $k_1\neq k_2$, $[\mu+k_1]_q=[\mu+k_2]_q$. Then

$$\forall k_3,k_4\in\mathbb{Z}:[\mu+k_3]_q=[\mu+k_4]_q\Leftrightarrow k_3+k_4=k_1+k_2.$$

Proof.

$$q^{2(\mu+k_3)+k_4-k_3}=\Leftrightarrow 1\Leftrightarrow q^{2\mu+k_1+k_2+k_3+k_4-(k_1+k_2)}=\Leftrightarrow 1\Leftrightarrow q^{k_3+k_4-(k_1+k_2)}=1\Leftrightarrow$$
$$\Leftrightarrow k_3+k_4\Leftrightarrow(k_1+k_2)=0.$$

5.12. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $q^k\neq 1$ $\forall k\in\mathbb{N}$. Let $\mu\in\mathbb{C}$. Then $\exists\nu\in\mathbb{C}:[\mu]_q=[\nu]_q$ and the numbers $[\nu+k]_q$ are for all $k\in\mathbb{N}_0$ mutually different. (Similar lemma is valid for the numbers $[\nu\Leftrightarrow k]_q$, $k\in\mathbb{N}_0$.)

Proof. If the condition is not met for μ, then there exist $k_1,k_2\in\mathbb{N}_0$ such that $[\mu+k_1]_q=[\mu+k_2]_q$. According to 5.11 it is sufficient to put $\nu=\mu+k_1+k_2$.

5.13. **Theorem.** Let $n\in\mathbb{N}$, $q\in\mathbb{C}\setminus\{0\}, q^k\neq 1\ \forall k\in\mathbb{N}$, $\nu\in\mathbb{C}, \alpha\in\mathbb{C}$.
Let

$$\left(q^{4\nu}=q^{-2n+2}\wedge\alpha=0\right)\vee\left(\alpha^2=\frac{-[n+1]_q[n-1]_q-1}{q}\wedge[\nu+n]_q=[\nu+n-1]_q\right).$$

Then the matrices (we write non-zero elements only)

$$J_3=\begin{pmatrix}[\nu]_q &&&& \\ &\ddots&&& \\ &&[\nu+k]_q&& \\ &&&\ddots& \\ &&&&[\nu+n-1]_q\end{pmatrix},$$

$$J_1=\begin{pmatrix}0 & \frac{-q^{\nu}[2\nu]_q[1]_q}{q^{\nu+1}+q^{-(\nu+1)}} &&&&& \\ \frac{1}{q^{2\nu}+1} & 0 &&&&& \\ &&\ddots&&&& \\ &&&0&\frac{-q^{\nu+k}[2\nu+k]_q[k+1]_q}{q^{\nu+k+1}+q^{-(\nu+k+1)}}&& \\ &&&\frac{1}{q^{2(\nu+k)}+1}&0&& \\ &&&&&\ddots& \\ &&&&&0&\frac{-q^{\nu+n-2}[2\nu+n-2]_q[n-1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}} \\ &&&&&\frac{1}{q^{2(\nu+n-2)}+1}&\frac{\alpha}{q^{2(\nu+n-1)}+1}\end{pmatrix},$$

$$J_2=\begin{pmatrix}0 & \frac{q^{-1}[2\nu]_q[1]_q}{q^{\nu+1}+q^{-(\nu+1)}} &&&&& \\ \frac{1}{q^{\nu}+q^{-\nu}} & 0 &&&&& \\ &&\ddots&&&& \\ &&&0&\frac{q^{-1}[2\nu+k]_q[k+1]_q}{q^{\nu+k+1}+q^{-(\nu+k+1)}}&& \\ &&&\frac{1}{q^{\nu+k}+q^{-(\nu+k)}}&0&& \\ &&&&&\ddots& \\ &&&&&0&\frac{q^{-1}[2\nu+n-2]_q[n-1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}} \\ &&&&&\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}&\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\end{pmatrix},$$

fulfil comutation relations (5.1)

Proof. Let's verify the relation $J_3J_1-q^{-1}J_1J_3=J_2$. It is sufficient to compute elements above and below diagonal and in the last column. The others are clearly equal to zero.
$-$Elements above the diagonal, row $i+1$ $(i=0,\ldots,n-2)$, column i:

$$[\nu+i+1]_q\frac{1}{q^{2(\nu+i)}+1}-q^{-1}\frac{1}{q^{2(\nu+i)}+1}[\nu+i]_q=\frac{q^{\nu+i}}{q^{2(\nu+i)}+1}=\frac{1}{q^{\nu+i}+q^{-(\nu+i)}}.\ \text{OK}$$

$-$Elements above the diagonal, row i, column $i+1$, $i=0,\ldots,n-3$:

$$[\nu+i]_q\frac{-q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}-q^{-1}\frac{-q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}[\nu+i+1]_q=$$
$$=\frac{-q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}([\nu+i]_q-q^{-1}[\nu+i+1]_q)=$$

$$= \frac{\Leftrightarrow q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}(\Leftrightarrow q^{-1}q^{-(\nu+i)}) = \frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}. \text{ OK}$$

⇔Elements in the last column $n \Leftrightarrow 1$, row i, $i=0,\ldots,n \Leftrightarrow 1$:
If $i \neq n \Leftrightarrow 1$, element is clearly zero, if $i = n \Leftrightarrow 1$, we have

$$[\nu+n\Leftrightarrow 1]_q \frac{\alpha}{q^{2(\nu+n-1)}+1} \Leftrightarrow q^{-1}\frac{\alpha}{q^{2(\nu+n-1)}+1}[\nu+n\Leftrightarrow 1]_q =$$
$$= \frac{\alpha}{q^{2(\nu+n-1)}+1}([\nu+n\Leftrightarrow 1]_q \Leftrightarrow q^{-1}[\nu+n\Leftrightarrow 1]_q) =$$
$$= \frac{\alpha}{q^{2(\nu+n-1)}+1}([\nu+n\Leftrightarrow 1]_q \Leftrightarrow [\nu+n]_q + q^{\nu+n-1}).$$

It is fulfilled if the following is true:

$$\alpha = 0 \vee [\nu+n\Leftrightarrow 1]_q = [\nu+n]_q,$$

but this yields from assumption. OK.

Now let's take the relation $J_3J_2 \Leftrightarrow qJ_2J_3 = J_1$. It suffices to verify elements above and below the diagonal and in the last column. The others are clearly equal to zero.
⇔Elements below the diagonal, row $i+1$ ($i=0,\ldots,n\Leftrightarrow 2$), column i:

$$[\nu+i+1]_q \frac{1}{q^{\nu+i}+q^{-(\nu+i)}} \Leftrightarrow q\frac{1}{q^{\nu+i}+q^{-(\nu+i)}}[\nu+i]_q = \frac{q^{-(\nu+i)}}{q^{\nu+i}+q^{-(\nu+i)}} =$$
$$= \frac{1}{q^{2(\nu+i)}+1}. \text{ OK}$$

⇔Elements above the diagonal except the last one, row i, column $i+1$, $i=0,\ldots,n\Leftrightarrow 3$:

$$[\nu+i]_q \frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}} \Leftrightarrow q\frac{\Leftrightarrow q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}[\nu+i+1]_q =$$
$$= \frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}([\nu+i]_q \Leftrightarrow q[\nu+i+1]_q) = \frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}(\Leftrightarrow q q^{\nu+i}) =$$
$$= \frac{\Leftrightarrow q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}. \text{ OK}$$

⇔Elements in the last column, column $n\Leftrightarrow 1$, row i, $i=0,\ldots,n\Leftrightarrow 1$:
If $i \neq n\Leftrightarrow 1$ the element is trivially zero, if $i = n\Leftrightarrow 1$ then we have

$$[\nu+n\Leftrightarrow 1]_q \frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}} \Leftrightarrow q\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}[\nu+n\Leftrightarrow 1]_q =$$
$$= \frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}([\nu+n\Leftrightarrow 1]_q \Leftrightarrow q[\nu+n\Leftrightarrow 1]_q) =$$
$$= \frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}([\nu+n\Leftrightarrow 1]_q \Leftrightarrow [\nu+n]_q + q^{\nu+n-1}). \text{ OK}$$

This is true if following condition holds:

$$\alpha = 0 \vee [\nu+n\Leftrightarrow 1]_q = [\nu+n]_q,$$

but this follows from the assumption (see previous relation ↑).

The last relation $J_2J_1 \Leftrightarrow qJ_1J_2 = J_3$. If we multiply two matrices with non-zero elements above and under the diagonal and in the right lower corner non-zero elements can appear theoretically at these places:

$$\begin{pmatrix} \star & 0 & \star & 0 & 0 & & 0 & 0 & 0 \\ 0 & \star & 0 & \star & 0 & & 0 & 0 & 0 \\ \star & 0 & \star & 0 & \star & & 0 & 0 & 0 \\ 0 & \star & 0 & \star & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & \star & & \star & 0 & 0 \\ 0 & 0 & 0 & \star & 0 & \ddots & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & \star & & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & & \star & \star & \star \end{pmatrix}.$$

$\Leftrightarrow$Elements under the diagonal, row i, column $i\Leftrightarrow 2$, where $i=2,3,\dots,n\Leftrightarrow 1$. Element containing α is out of interest here.

$$\frac{1}{q^{\nu+i-1}+q^{-(\nu+i-1)}}\cdot\frac{1}{q^{2(\nu+i-2)}+1}+\frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}\cdot 0\Leftrightarrow$$
$$\Leftrightarrow q\left(\frac{1}{q^{2(\nu+i-1)}+1}\cdot\frac{1}{q^{\nu+i-2}+q^{-(\nu+i-2)}}+\frac{\Leftrightarrow q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}\cdot 0\right)=$$
$$=\frac{q^{\nu+i-1}\Leftrightarrow q q^{\nu+i-2}}{(q^{2(\nu+i-1)}+1)(q^{2(\nu+i-2)}+1)}=0.\ \text{OK}$$

$\Leftrightarrow$Elements above the diagonal, row i, column $i+2$, $i=0,\dots,n\Leftrightarrow 3$:
If $i\neq n\Leftrightarrow 4$ then we have

$$\frac{1}{q^{\nu+i-1}+q^{-(\nu+i-1)}}\cdot 0+\frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}\cdot\frac{\Leftrightarrow q^{\nu+i+1}[2\nu+i+1]_q[i+2]_q}{q^{\nu+i+2}+q^{-(\nu+i+2)}}+$$
$$+\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot 0\Leftrightarrow q\Big(\tfrac{1}{q^{2(\nu+i-1)}+1}\cdot 0+$$
$$+\frac{\Leftrightarrow q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}\cdot\frac{q^{-1}[2\nu+i+1]_q[i+2]_q}{q^{\nu+i+2}+q^{-(\nu+i+2)}}+\frac{\alpha}{q^{2(\nu+n-1)}+1}\cdot 0\Big)=$$
$$=\frac{q^{-1}[2\nu+i]_q[i+1]_q[2\nu+i+1]_q[i+2]_q(\Leftrightarrow q^{\nu+i+1}\Leftrightarrow q(\Leftrightarrow q^{\nu+i}))}{(q^{\nu+i+1}+q^{-(\nu+i+1)})(q^{\nu+i+2}+q^{-(\nu+i+2)})}=0.\ \text{OK}$$

If $i=n\Leftrightarrow 4$ we will get the same plus the term containing α:

$$\dots+\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\frac{1}{q^{2(\nu+n-2)}+1}\Leftrightarrow q\Big(\dots+\frac{\alpha}{q^{2(\nu+n-1)}+1}\cdot$$
$$\cdot\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}\Big)=0+\frac{\alpha(q^{\nu+n-1}\Leftrightarrow q q^{\nu+n-2})}{(q^{2(\nu+n-1)}+1)(q^{2(\nu+n-2)}+1)}=0.\ \text{OK}$$

$\Leftrightarrow$Diagonal elements. For $i=0$ we have

$$\frac{q^{-1}[2\nu]_q[1]_q}{q^{\nu+1}+q^{-(\nu+1)}}\cdot\frac{1}{q^{2\nu}+1}\Leftrightarrow q\frac{\Leftrightarrow q^{\nu}[2\nu]_q[1]_q}{q^{\nu+1}+q^{-(\nu+1)}}\cdot\frac{1}{q^{\nu}+q^{-\nu}}=$$
$$=\frac{[2\nu]_q[1]_q(q^{-1}\Leftrightarrow q(\Leftrightarrow q^{\nu})q^{\nu})}{(q^{\nu+1}+q^{-(\nu+1)})(q^{2\nu}+1)}=\frac{q^{\nu}[2\nu]_q[1]_q(q^{2\nu+2}+1)}{(q^{2\nu+2}+1)(q^{2\nu}+1)}=\frac{q^{\nu}[2\nu]_q[1]_q}{(q^{2\nu}+1)}=$$
$$=\frac{q^{-\nu}(q^{4\nu}\Leftrightarrow 1)}{(q^{2\nu}+1)(q\Leftrightarrow q^{-1})}=[\nu]_q.\ \text{OK}$$

For $i=1,2,\dots,n\Leftrightarrow 3$ we have

$$\frac{1}{q^{\nu+i-1}+q^{-(\nu+i-1)}}\cdot\frac{\Leftrightarrow q^{\nu+i-1}[2\nu+i\Leftrightarrow 1]_q[i]_q}{q^{\nu+i}+q^{-(\nu+i)}}+\frac{q^{-1}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}\cdot\frac{1}{q^{2(\nu+i)}+1}\Leftrightarrow$$

$$\Leftrightarrow q\Big(\frac{1}{q^{2(\nu+i-1)}+1}\cdot\frac{q^{-1}[2\nu+i\Leftrightarrow 1]_q[i]_q}{q^{\nu+i}+q^{-(\nu+i)}}+\frac{\Leftrightarrow q^{\nu+i}[2\nu+i]_q[i+1]_q}{q^{\nu+i+1}+q^{-(\nu+i+1)}}\cdot\frac{1}{q^{\nu+i}+q^{-(\nu+i)}}+$$

$$+\frac{\alpha}{q^{2(\nu+n-1)}+1}\cdot 0\Big)=\frac{(q^{\nu+i}q^{\nu+i-1}(\Leftrightarrow q^{\nu+i-1})\Leftrightarrow qq^{-1}q^{\nu+i})[2\nu+i\Leftrightarrow 1]_q[i]_q}{(q^{2(\nu+i-1)}+1)(q^{2(\nu+i)}+1)}+$$

$$+\frac{(q^{\nu+i+1}q^{-1}\Leftrightarrow qq^{\nu+i+1}(\Leftrightarrow q^{\nu+i})q^{\nu+i})[2\nu+i]_q[i+1]_q}{(q^{2(\nu+i-1)}+1)(q^{2(\nu+i)}+1)}=$$

$$=\frac{q^{\nu+i+2}}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}\Big(\frac{(q^{2\nu+i-1}\Leftrightarrow q^{-(2\nu+i-1)})(q^i\Leftrightarrow q^{-i})(\Leftrightarrow q^{2(\nu+i-1)}\Leftrightarrow 1)}{q^{2(\nu+i-1)}+1}\Big)+$$

$$+\frac{q^{\nu+i+2}}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}\Big(\frac{(q^{2\nu+i}\Leftrightarrow q^{-(2\nu+i)})(q^{i+1}\Leftrightarrow q^{-(i+1)})(1+q^{2(\nu+i+1)})}{q^{2(\nu+i+1)}+1}\Big)=$$

$$=\frac{q^{\nu+i+2}}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}\cdot$$

$$\cdot\Big((q^{2\nu+i-1}\Leftrightarrow q^{-(2\nu+i-1)})(q^i\Leftrightarrow q^{-i})(\Leftrightarrow 1)(q^{2\nu+i}\Leftrightarrow q^{-(2\nu+i)})(q^{i+1}\Leftrightarrow q^{-(i+1)})\Big)=$$

$$=\frac{\Leftrightarrow q^{3\nu+3i+1}+q^{-\nu+3+i}+q^{3\nu+i+1}\Leftrightarrow q^{-\nu-i+3}}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}+$$

$$+\frac{q^{3\nu+3i+3}\Leftrightarrow q^{3\nu+i+1}\Leftrightarrow q^{-\nu+i+3}+q^{-i-\nu+1}}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}=$$

$$=\frac{q^{3\nu+3i+1}(q^2\Leftrightarrow 1)+(q^2\Leftrightarrow 1)(\Leftrightarrow q^{-\nu-i+1})}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}=\frac{q(q^2\Leftrightarrow 1)q^{-(\nu+i)}(q^{4(\nu+i)}\Leftrightarrow 1)}{(q^{2(\nu+i)}+1)(q^2\Leftrightarrow 1)^2}=$$

$$=\frac{qq^{-(\nu+i)}(q^{2(\nu+i)}\Leftrightarrow 1)}{q^2\Leftrightarrow 1}=[\nu+i]_q.\ \text{OK}$$

For $i=n\Leftrightarrow 2$ we can use calculation $\uparrow$ for $i=1,\ldots,n\Leftrightarrow 3$ and the fact that

$$\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\frac{1}{q^{2(\nu+n-2)}+1}\Leftrightarrow q\frac{\alpha}{q^{2(\nu+n-1)}+1}\cdot\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}=$$

$$=\frac{\alpha(q^{\nu+n-1}\Leftrightarrow qq^{\nu+n-2})}{(q^{2(\nu+n-1)}+1)(q^{2(\nu+n-2)}+1)}=0.\ \text{OK}$$

Diagonal element when $i=n\Leftrightarrow 1$:

$$\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}\cdot\frac{\Leftrightarrow q^{\nu+n-2}[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}+\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot$$

$$\cdot\frac{\alpha}{q^{2(\nu+n-1)}+1}\Leftrightarrow q\Big(\frac{1}{q^{2(\nu+n-2)}+1}\cdot\frac{q^{-1}[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}+\frac{\alpha}{q^{2(\nu+n-1)}+1}\cdot$$

$$\cdot\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\Big)=\frac{1}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\Big(\frac{(\Leftrightarrow 1)[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q}{1}+$$

$$+\frac{\alpha^2(1\Leftrightarrow q)}{q^{2(\nu+n-1)}+1}\Big),$$

now we must consider two cases: if $\alpha\neq 0$, it implies $\alpha=\dfrac{\Leftrightarrow [n\Leftrightarrow 1]_q[n+1]_q\Leftrightarrow 1}{q}$ and furthermore we know $[\nu+n\Leftrightarrow 1]_q=[\nu+n]_q\Leftrightarrow q^{2\nu+2n-1}+1=0\Leftrightarrow q^{-2\nu-n+2}=$ $=\Leftrightarrow q^{n+1}\Leftrightarrow[2\nu+n\Leftrightarrow 2]_q=[n+1]_q$, hence

$$\ldots=\frac{1}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\underbrace{\left(\Leftrightarrow[n+1]_q[n\Leftrightarrow1]_q+\frac{(\Leftrightarrow[n\Leftrightarrow1]_q[n+1]_q\Leftrightarrow1)(1\Leftrightarrow q)}{q(\Leftrightarrow q^{-1}+1)}\right)}_{=1}=$$

$$=\frac{1}{q^{\nu+n-1}+q^{-(\nu+n-1)}}=\frac{q^{\nu+n-1}}{q^{2(\nu+n-1)}+1}=\frac{q^{\nu+n-1}}{\Leftrightarrow q^{-1}+1}=\frac{q^{\nu+n-1}(1+q^{-1}}{(1\Leftrightarrow q^{-1})(1+q^{-1})}=$$

$$=\frac{q^{\nu+n-1}+q^{\nu+n-2}}{(1\Leftrightarrow q^{-2})}=\frac{q^{\nu+n}+q^{\nu+n-1}}{(q\Leftrightarrow q^{-1})}=\frac{q^{\nu+n}\Leftrightarrow q^{-(\nu+n)}}{(q\Leftrightarrow q^{-1})}=[\nu+n]_q=$$

$$=[\nu+n\Leftrightarrow1]_q.\text{ OK}$$

If $\alpha=0$, then from the assumption we know $q^{4\nu}=q^{-2n+2}$, hence

$$\ldots=\frac{(\Leftrightarrow1)[2\nu+n\Leftrightarrow2]_q[n\Leftrightarrow1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}=\frac{\Leftrightarrow(q^{2\nu+n-2}\Leftrightarrow q^{-(2\nu+n-2)})(q^{n-1}\Leftrightarrow q^{-(n-1)})}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}=$$

$$=\frac{q^{-2\nu+1}+q^{2\nu-1}\Leftrightarrow q^{2\nu+2n-3}\Leftrightarrow q^{-2\nu-2n+3}}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}=$$

$$=\frac{q^{-2\nu+1}+q^{2\nu-1}+\overbrace{(\Leftrightarrow q^{2\nu+2n-1}\Leftrightarrow q^{-2\nu-2n+1}+q^{2\nu+2n-1}+q^{-2\nu-2n+1})}^{\text{inserted }0}}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}\Leftrightarrow$$

$$\Leftrightarrow\frac{q^{2\nu+2n-3}+q^{-2\nu-2n+3}}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}=$$

$$=\frac{q^{-2\nu}(q+q^{4\nu-1}\Leftrightarrow q^{4\nu+2n-1}\Leftrightarrow q^{-2n+1})+(q^{2\nu+2n-1}+q^{-2\nu-2n+1}}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}\Leftrightarrow$$

$$\Leftrightarrow\frac{q^{2\nu+2n-3}+q^{-2\nu-2n+3})}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}=$$

$$=\frac{q^{-2\nu}(q^{4\nu}(q^{-1}\Leftrightarrow q^{2n-1})\Leftrightarrow q^{-2n+2}(q^{-1}\Leftrightarrow q^{2n-1}))}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}+$$

$$+\frac{(q^{2\nu+2n-2}\Leftrightarrow q^{-2\nu-2n+2})(q\Leftrightarrow q^{-1})}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}=\frac{q^{-2\nu}\overbrace{(q^{4\nu}\Leftrightarrow q^{-2n+2})}^{=0\text{ from assump.}}(q^{-1}\Leftrightarrow q^{2n-1})}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}+$$

$$+\frac{(q^{\nu+n-1}\Leftrightarrow q^{-(\nu+n-1)})(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}=[\nu+n\Leftrightarrow1]_q.\text{ OK}$$

$\Leftrightarrow$Column $n\Leftrightarrow2$, row $n\Leftrightarrow1$:

$$\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}\cdot0+\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\frac{1}{q^{2(\nu+n-2)}+1}\Leftrightarrow$$

$$\Leftrightarrow q\left(\frac{1}{q^{2(\nu+n-2)}+1}\cdot0+\frac{\alpha}{q^{2(\nu+n-1)}+1}\cdot\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}\right)=$$

$$=\frac{\alpha(q^{\nu+n-1}\Leftrightarrow qq^{\nu+n-2})}{(q^{2(\nu+n-1)}+1)(q^{2(\nu+n-2)}+1)}=0.\text{ OK}$$

$\Leftrightarrow$Column $n\Leftrightarrow1$, row $n\Leftrightarrow2$:

$$\frac{q^{-1}[2\nu+n\Leftrightarrow2]_q[n\Leftrightarrow1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\frac{\alpha}{q^{2(\nu+n-1)}+1}\Leftrightarrow$$

$$\Leftrightarrow q\frac{\Leftrightarrow q^{\nu+n-2}[2\nu+n\Leftrightarrow2]_q[n\Leftrightarrow1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\frac{\alpha}{q^{\nu+n-1}+q^{-(\nu+n-1)}}=$$

$$=\alpha\,[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q\Big(\frac{q^{-1}+qq^{\nu+n-2}q^{\nu+n-1}}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q^{2(\nu+n-1)}+1)}\Big)=$$
$$=\alpha\,[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q q^{-1}\Big(\frac{1+q^{2\nu+2n-1}}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q^{2(\nu+n-1)}+1)}\Big)=0,$$

because either $\alpha=0$ or $1+q^{2\nu+2n-1}=0\Leftrightarrow[\nu+n]_q=[\nu+n\Leftrightarrow 1]_q$.
The proof is finished.

5.14. **Theorem.** Let $n\in\mathbb{N}$, $q\in\mathbb{C}\Leftrightarrow\{0\}$, $q^k\neq 1\ \forall k\in\mathbb{N}$. Let φ n-dimensional representation $U(g)$ on a complex vector space V $(\dim V=n)$. Denote the operators resp. their matrices $\varphi(J_1),\varphi(J_2),\varphi(J_3)$ simply as J_1,J_2,J_3.
Then there exist a basis $X=(x_0,\ldots,x_{n-1})$ of the space V such that matrices of operators J_1,J_2,J_3 have the form from 5.4

Proof. We are in complex case that implies there exists at least one eigenvalue of J_3, denote it λ. From 5.8 and 5.12 implies that exists $\mu\in\mathbb{C}$ such that $\lambda=[\mu]_q$ and the numbers $[\mu\Leftrightarrow k]_q$, $k\in\mathbb{N}_0$ are mutually different.

Denote y_0 eigenvector corresponding to λ. Let $y_{i+1}=(J_1\Leftrightarrow q^{-(\mu-i)}J_2)y_i\ \forall i\in\mathbb{N}_0$. From 5.6 follows that $\exists i_0\in\mathbb{N}_0{:}y_i=\theta$ for $i>i_0$ and $y_0,\ldots,y_{i_0}$ are eigenvectors of J_3 belonging to different eigenvalues which implies they are linear independent.
Now let $\nu=\mu\Leftrightarrow i_0$, $x_0=y_{i_0}$, $x_{i+1}=(J_1+q^{\nu+i}J_2)x_i\ \forall i\in\mathbb{N}_0$. From the formula for y_{i_0} follows that $(J_1\Leftrightarrow q^{-\nu}J_2)x_0=\theta$.
Let us search in the sequence $x_0,x_1,\ldots$ the first vector x_{m+1} $(m\in\mathbb{N}_0)$ such that it is a linear combination of previous ones i. e. $x_{m+1}=\sum_{i=0}^{m}\alpha_i x_i$.

We need to show that $m=n\Leftrightarrow 1$. The case $m>n\Leftrightarrow 1$ is impossible (it implies in V there are more than n linear independent vectors); the case $m<n\Leftrightarrow 1$ is impossible too because we will now show that the vectors $x_0,\ldots,x_m$ would compose an invariant subspace of all three operators J_1,J_2,J_3 (contradiction with irreducibility of representation).
Let $C_3=(q\Leftrightarrow q^{-1})J_1J_2J_3+(J_1^2\Leftrightarrow q^{-1}J_2^2+J_3^2)$. Operator C_3 commutes with the operators J_1,J_2,J_3. According to Schur lemma $\exists\alpha\in\mathbb{C}\ \forall x{:}C_3x=\alpha x$.
The following is true

$$(J_1\Leftrightarrow q^{-(\nu+k+1)}J_2)x_{k+1}=\Leftrightarrow(q^{-(\nu+k+1)}[\nu+k]_q+[\nu+k]_q^2\Leftrightarrow\alpha)x_k\forall k\in N_0 \qquad (5.3)$$

because

$$\theta=C_3x_k\Leftrightarrow\alpha x_k=(q\Leftrightarrow q^{-1})J_1J_2[\nu+k]_qx_k+(J_1^2\Leftrightarrow q^{-1}J_2^2)x_k+[\nu+k]_q^2x_k\Leftrightarrow\alpha x_k$$

and

$$(J_1\Leftrightarrow q^{-(\nu+k+1)}J_2)x_{k+1}=(J_1\Leftrightarrow q^{-(\nu+k+1)}J_2)(J_1+q^{\nu+k}J_2)x_k=$$
$$=(J_1^2\Leftrightarrow q^{-1}J_2^2)x_k\Leftrightarrow q^{-(\nu+k+1)}(J_3+qJ_1J_2)x_k+q^{\nu+k}J_1J_2x_k=$$
$$=(J_1^2\Leftrightarrow q^{-1}J_2^2)x_k\Leftrightarrow q^{-(\nu+k+1)}[\nu+k]_qx_k+(\Leftrightarrow q^{-(\nu+k)}+q^{\nu+k})J_1J_2x_k=$$
$$=(J_1^2\Leftrightarrow q^{-1}J_2^2)x_k\Leftrightarrow q^{-(\nu+k+1)}[\nu+k]_qx_k+(q\Leftrightarrow q^{-1})[\nu+k]_qJ_1J_2x_k,$$

it yields

$$(q\Leftrightarrow q^{-1})J_1J_2[\nu+k]_qx_k+(J_1^2\Leftrightarrow q^{-1}J_2^2)x_k=$$
$$=(J_1\Leftrightarrow q^{-(\nu+k+1)}J_2)x_{k+1}+q^{-(\nu+k+1)}[\nu+k]_qx_k$$

which implies

$$\theta=(J_1\Leftrightarrow q^{-(\nu+k+1)}J_2)x_{k+1}+q^{-(\nu+k+1)}[\nu+k]_qx_k+[\nu+k]_q^2x_k\Leftrightarrow\alpha x_k,$$

which is (5.3).
The next equation we need is

$$\theta=(q^{\nu-1}[\nu]_q-[\nu]_q^2+\alpha)x_0. \tag{5.4}$$

The proof is similar..

$$\theta=(J_1+q^{\nu-1}J_2)\theta=(J_1+q^{\nu-1}J_2)(J_1-q^{-\nu}J_2)x_0,$$

$$\theta=C_3x_0-\alpha x_0=(q-q^{-1})J_1J_2[\nu]_q x_0+(J_1^2-q^{-1}J_2^2)x_0+[\nu]_q^2 x_0-\alpha x_0$$

and

$$\begin{aligned}&(J_1+q^{\nu-1}J_2)(J_1-q^{-\nu}J_2)x_0=\\ &=(J_1^2-q^{-1}J_2^2)x_0+q^{\nu-1}(J_3+qJ_1J_2)x_0-q^{-\nu}J_1J_2x_0=\\ &=(J_1^2-q^{-1}J_2^2)x_0+q^{\nu-1}[\nu]_q x_0+(q^{\nu}-q^{-\nu})J_1J_2x_0=\\ &=(J_1^2-q^{-1}J_2^2)x_0+q^{\nu-1}[\nu]_q x_0+(q-q^{-1})[\nu]_q J_1J_2x_0,\end{aligned}$$

it yields

$$\begin{aligned}&(q-q^{-1})[\nu]_q J_1J_2x_0+(J_1^2-q^{-1}J_2^2)x_0=\\ &=(J_1+q^{\nu-1}J_2)(J_1-q^{-\nu}J_2)x_0-q^{\nu-1}[\nu]_q x_0,\end{aligned}$$

hence

$$\theta=(J_1+q^{\nu-1}J_2)(J_1-q^{-\nu}J_2)x_0-q^{\nu-1}[\nu]_q x_0+[\nu]_q^2x_0-\alpha x_0.$$

From definition of x_i we have

$$(J_1+q^{\nu+k}J_2)x_k=x_{k+1}\ \forall k\in N_0. \tag{5.5}$$

Put $x_{-1}=\theta$. Then considering (5.3) and (5.4) we have $\forall k\in N_0$ two equations

$$(q^{\nu+k}+q^{-(\nu+k)})J_2x_k=x_{k+1}+(q^{-(\nu+k)}[\nu+k-1]_q+[\nu+k-1]_q^2-\alpha)x_{k-1}, \tag{5.6}$$

$$\begin{aligned}&(q^{-(\nu+k)}+q^{(\nu+k)})J_1x_k=\\ &=q^{-(\nu+k)}x_{k+1}-q^{\nu+k}\cdot(q^{-(\nu+k)}[\nu+k-1]_q+[\nu+k-1]_q^2-\alpha)x_{k-1}.\end{aligned} \tag{5.7}$$

For $k\in\{0,1,2,\dots,m-1\}$ we can both equations (5.6) and (5.7) divide by constant $q^{\nu+k}+q^{-(\nu+k)}$, it is not equal to zero (if it is the right side would be equal to θ but there is a nontrivial linear combination (coefficient by x_{k+1} is 1), so that it would imply $x_{k+1}=\theta$ but it is not true because x_{k+1} is eigenvector of J_3). (The vectors $x_0,\dots,x_{m-1}$ map to $\text{lin}\{x_0,\dots,x_m\}$ for all three operators.)
For $k=m$ it is clear that we can divide both sides only when at least one α_i is non-zero. For all $\alpha_i=0$ we must show another way that the coefficient we're dividing by is not equal to zero (on the right side there is not nontrivial combination more).
Let us consider all $\alpha_i=0$. It implies $x_m=\theta$.
For $k=m$ we have $x_{m+1}=\theta$. We need to show $q^{\nu+m}+q^{-(\nu+m)}\neq 0$.
From (5.3) for $k=m$ we have

$$(J_1-q^{-(\nu+m+1)}J_2)x_{m+1}=\theta=-(q^{-(\nu+m+1)}[\nu+m]_q+[\nu+m]_q^2-\alpha)x_m,$$

therefore

$$\alpha=q^{-(\nu+m+1)}[\nu+m]_q+[\nu+m]_q^2.$$

From (5.4) we have

$$\alpha=-q^{\nu-1}[\nu]_q+[\nu]_q^2.$$

Apllying definition of $[\nu]_q$ we get

$$q^{-(\nu+m+1)}\frac{q^{\nu+m}\Leftrightarrow q^{-(\nu+m)}}{q\Leftrightarrow q^{-1}}+\frac{q^{2(\nu+m)}\Leftrightarrow 2+q^{-2(\nu+m)}}{(q\Leftrightarrow q^{-1})^2}=$$
$$=\frac{q^{2\nu}\Leftrightarrow 2+q^{-2\nu}}{(q\Leftrightarrow q^{-1})^2}\Leftrightarrow q^{\nu-1}\frac{q^{\nu}\Leftrightarrow q^{-\nu}}{q\Leftrightarrow q^{-1}},$$

hence

$$q^{-2m}(1\Leftrightarrow q^{2+2m})=q^{4\nu}(1\Leftrightarrow q^{2m+2}),$$

i. e.

$$q^{2+2m}=1\vee q^{2(2\nu+m)}=1.$$

From the assumption we know $q^k\neq 0\ \forall k\in\mathbb{N}$, it yields $q^{2+2m}=1$ is not true. So let's consider the second case $q^{2(2\nu+m)}=1$.
Now if $q^{\nu+m}+q^{-(\nu+m)}=0$ we can square it $\Rightarrow q^{2(2\nu+m)+2m}=1$, and applying previous fact $q^{2(2\nu+m)}=1$ we have $q^{2m}=1$ which is contradiction.
Thus, $q^{\nu+m}+q^{-(\nu+m)}\neq 0$ is always true, we can divide (5.6) and (5.7) by this constant and we see all vectors map back to the span $\mathrm{lin}\{x_0,\dots,x_m\}$.
Therefore $\mathrm{lin}\{x_0,\dots,x_m\}$ is invariant subspace of operators J_1,J_2,J_3.

From the equations (5.6) and (5.7) we can get for $k=0,\dots,m=n\Leftrightarrow 1$ matrices of operators J_1,J_2. Matrix of J_3 is trivial; if we show $\alpha_0=\dots=\alpha_{n-2}=0$ the proof will be finished.
The matrices must hold commutation relations. From the relation $J_3J_1\Leftrightarrow q^{-1}J_1J_3=J_2$ we will get comparing the first column $\forall k\in\widehat{n}\Leftrightarrow 1$ the equations

$$\frac{\alpha_k}{q^{2(\nu+n-1)}+1}[\nu+k]_q\Leftrightarrow q^{-1}\frac{\alpha_k}{q^{2(\nu+n-1)}+1}[\nu+n\Leftrightarrow 1]_q=\frac{\alpha_k q^{\nu+n-1}}{q^{2(\nu+n-1)}+1},$$

they are fulfilled if and only if $\alpha_k=0$ or

$$[\nu+k]_q\Leftrightarrow q^{-1}[\nu+n\Leftrightarrow 1]_q=q^{\nu+n-1}$$

i. e.

$$[\nu+k]_q=[\nu+n]_q.$$

Thus for $\alpha_0,\dots,\alpha_{n-1}$ the following condition holds:

$$\forall k\in\widehat{n}\Leftrightarrow 1:\alpha_k=0\vee[\nu+k]_q=[\nu+n]_q.$$

If all $\alpha_k=0$ the proof is finished. Let us assume that for any $k\in\widehat{n}\Leftrightarrow 1$ is $\alpha_k\neq 0$. Then we have $[\nu+n]_q=[\nu+k]_q$.
It implies $\alpha_i=0$ for the others $i\in\widehat{n}\Leftrightarrow 1$, $i\neq k$. (If for any $i_0\in\widehat{n}\Leftrightarrow 1$, $i_0\neq k$ was $\alpha_{i_0}\neq 0$ then $[\nu+n]_q=[\nu+i_0]_q$, 5.11 would yield $i_0+n=k+n\Rightarrow i_0=k$ which is contradiction.)
Thus only one from α_i is non-zero.
From the first relation $J_2J_1\Leftrightarrow qJ_1J_2=J_3$ we will get comparing elements in the rows $i=1,2,\dots,n\Leftrightarrow 4$ resp. $i=n\Leftrightarrow 2$

$$\frac{1}{q^{\nu+i-1}+q^{-(\nu+i-1)}}\cdot\frac{\alpha_{i-1}}{q^{2(\nu+n-1)}+1}\Leftrightarrow q\frac{1}{q^{2(\nu+i-1)}+1}\cdot\frac{\alpha_{i-1}}{q^{\nu+n-1}+q^{-(\nu+n-1)}}=$$
$$=\alpha_{i-1}\left(\frac{q^{\nu+i-1}\Leftrightarrow q q^{\nu+n-1}}{(q^{2(\nu+n-1)}+1)(q^{2(\nu+i-1)}+1)}\right)\overset{\text{should be}}{=}0,$$

it can't be true for $\alpha_{i-1}\neq 0$ because $q^{\nu+i-1}=q^{\nu+n}\Leftrightarrow q^{i-1}=q^n\Leftrightarrow i\Leftrightarrow 1=n$, which is not true. Thus $\alpha_j=0$ for $j=0,\dots,n\Leftrightarrow 5$, $j=n\Leftrightarrow 3$.
We must examine α_j for $j=n\Leftrightarrow 4$ a $j=n\Leftrightarrow 2$:
All α_{n-2k}, $k=1,2,\dots$ are equal to zero, if this was not true the following would hold

$$[\nu+n\Leftrightarrow 2k]_q=[\nu+n]_q \Leftrightarrow q^{2(\nu+n-k)}=\Leftrightarrow 1 \Leftrightarrow q^{(\nu+n-k)}+q^{-(\nu+n-k)}=0,$$

but this is a number we were dividing by in the first part of the proof -- it is non-zero. Thus the only non-zero is α_{n-1}.
Let's compare right lower element $(n\Leftrightarrow 1, n\Leftrightarrow 1)$ in the first relation $J_2J_1\Leftrightarrow qJ_1J_2=J_3$. It will give us the condition on α_{n-1}.

$$\frac{1}{q^{\nu+n-2}+q^{-(\nu+n-2)}}\cdot\frac{\Leftrightarrow q^{\nu+n-2}[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}+$$
$$+\frac{\alpha_{n-1}}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\cdot\frac{\alpha_{n-1}}{q^{2(\nu+n-1)}+1}\Leftrightarrow$$
$$\Leftrightarrow q\Big(\frac{1}{q^{2(\nu+n-2)}+1}\cdot\frac{q^{-1}[2\nu+n\Leftrightarrow 2]_q[n\Leftrightarrow 1]_q}{q^{\nu+n-1}+q^{-(\nu+n-1)}}+$$
$$+\frac{\alpha_{n-1}}{q^{2(\nu+n-1)}+1}krat\frac{\alpha_{n-1}}{q^{\nu+n-1}+q^{-(\nu+n-1)}}\Big)\stackrel{?}{=}[\nu+n\Leftrightarrow 1]_q.$$

Left side is equal to

$$=\frac{\Leftrightarrow\overbrace{[2\nu+n\Leftrightarrow 2]_q}^{[n+1]_q}[n\Leftrightarrow 1]_q}{(q^{\nu+n-1}+q^{-(\nu+n-1)})}+\frac{\alpha_{n-1}^2(1\Leftrightarrow q)}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q^{2(\nu+n-1)}+1)}=$$
$$=\frac{\Leftrightarrow(\overbrace{q^{2\nu+n-2}}^{-q^{-n-1}}\Leftrightarrow q^{-2\nu-n+2})(q^{n-1}\Leftrightarrow q^{-n+1})}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(q\Leftrightarrow q^{-1})^2}+$$
$$+\frac{\alpha_{n-1}^2(1\Leftrightarrow q)}{(q^{\nu+n-1}+q^{-(\nu+n-1)})(\underbrace{q^{2(\nu+n-1)}}_{-q^{-1}}+1)}=$$
$$=\frac{q^{\nu+n-1}}{(\underbrace{q^{2(\nu+n-1)}}_{-q^{-1}}+1)}\Big(\Leftrightarrow\frac{(\Leftrightarrow q^{-n-1}\Leftrightarrow(\Leftrightarrow q^{n+1}))(q^{n-1}\Leftrightarrow q^{-n+1})}{(q\Leftrightarrow q^{-1})^2}+\frac{\alpha_{n-1}^2(1\Leftrightarrow q)}{(\Leftrightarrow q^{-1}+1)}\Big)=$$
$$=\frac{q^{\nu+n-1}}{1\Leftrightarrow q^{-1}}\Big(\Leftrightarrow[n+1]_q[n\Leftrightarrow 1]_q\Leftrightarrow{\alpha_{n-1}}^2q\Big),$$

Therefore following must be true

$$\frac{q^{\nu+n-1}}{1\Leftrightarrow q^{-1}}\Big(\Leftrightarrow[n+1]_q[n\Leftrightarrow 1]_q\Leftrightarrow{\alpha_{n-1}}^2q\Big)=[\nu+n\Leftrightarrow 1]_q=[\nu+n]_q=\frac{\overbrace{q^{\nu+n}\Leftrightarrow q^{-(\nu+n)}}^{q^{\nu+n-1}}}{q\Leftrightarrow q^{-1}}$$
$$\text{i. e. }\Big(\Leftrightarrow[n+1]_q[n\Leftrightarrow 1]_q\Leftrightarrow{\alpha_{n-1}}^2q\Big)=\frac{q\Leftrightarrow 1}{q}\frac{q+1}{q\Leftrightarrow q^{-1}}=1$$
$$\Leftrightarrow[n+1]_q[n\Leftrightarrow 1]_q=1+\alpha_{n-1}^2q$$
$$\Rightarrow\ \frac{\Leftrightarrow[n+1]_q[n\Leftrightarrow 1]_q\Leftrightarrow 1}{q}=\alpha_{n-1}^2.$$

The proof is finished.
5.15. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.
Let $c^2q^{-n+1+2k}+1\neq 0$ for $k\in\{0,1,\ldots,n\Leftrightarrow 1\}$.
Let $q^\nu=cq^{\frac{-n+1}{2}}$. Then the matrices from theorem 5.13 have following form:

$$J_3=\begin{pmatrix} k=0 & & & & \\ & \ddots & & & \\ & & \frac{c^2q^{-n+1+2k}-1}{cq^{\frac{-n+1}{2}+k-1}(q^2-1)} & & \\ & & & \ddots & \\ & & & & k=n\Leftrightarrow 1 \end{pmatrix},$$

$$J_1=\begin{pmatrix} 0 & k=0 & & & \\ k=0 & \ddots & \frac{-q^2(q^{2(k+1-n)}-1)(q^{2k+2}-1)}{(c^2q^{2k-n+3}+1)(q^2-1)^2} & & \\ & \frac{1}{c^2q^{2k+1-n}+1} & & & \\ & & & \ddots & k=n\Leftrightarrow 2 \\ & & & k=n\Leftrightarrow 2 & 0 \end{pmatrix},$$

$$J_2=J_3J_1\Leftrightarrow q^{-1}J_1J_3=$$

$$=\begin{pmatrix} 0 & k=0 & & & \\ k=0 & \ddots & \frac{q^{\frac{n+1}{2}-k}(q^{2(k+1-n)}-1)(q^{2k+2}-1)}{c(c^2q^{2k-n+3}+1)(q^2-1)^2} & & \\ & \frac{cq^{\frac{-n+1}{2}+k}}{c^2q^{2k+1-n}+1} & & & \\ & & & \ddots & k=n\Leftrightarrow 2 \\ & & & k=n\Leftrightarrow 2 & 0 \end{pmatrix}.$$

Proof. The conditions are because of there must be non-zero elements in denominators.

5.16. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.

Let $c^2=\Leftrightarrow 1$, n odd. Then the matrices written above in 5.15 don't have sense.

Proof. $c^2q^{2k+1-n}\neq\Leftrightarrow 1\forall k\in\{0,\ldots,n\Leftrightarrow 1\}\Leftrightarrow q^{1-n},q^{3-n},\ldots,q^{n-3},q^{n-1}\neq\frac{-1}{c^2}=1$ now, but this is always not tru for n odd (it runs accross $q^0=1$).

5.17. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.

Let n odd, $c^2=1$ i. e. $c=\pm 1$.

Let $q^2,q^4,\ldots,q^{n-1}\neq\Leftrightarrow 1$.

Then the matrices written in 5.15 have sense and form

$$J_3=c\begin{pmatrix} \left[\Leftrightarrow\frac{n-1}{2}\right]_q & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \left[\frac{n-1}{2}\right]_q \end{pmatrix},$$

$$J_1=\begin{pmatrix} 0 & k=0 & & & \\ k=0 & \ddots & \frac{-q^2(q^{2(k+1-n)}-1)(q^{2k+2}-1)}{(q^{2k-n+3}+1)(q^2-1)^2} & & \\ & \frac{1}{q^{2k+1-n}+1} & & & \\ & & & \ddots & k=n\Leftrightarrow 2 \\ & & & k=n\Leftrightarrow 2 & 0 \end{pmatrix}.$$

Proof. $c^2q^{2k+1-n}=q^{2k+1-n}\neq\Leftrightarrow 1\Leftrightarrow q^{1-n},q^{3-n},\ldots,q^{n-3},q^{n-1}\neq$
$=\Leftrightarrow 1\Leftrightarrow 1,q^2,\ldots,q^{n-3},q^{n-1}\neq\Leftrightarrow 1$ OK.

5.18. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.

Let $c^2q^{-n+2},c^2q^{-n+3},\ldots,c^2q^{n-2}\neq\Leftrightarrow 1\wedge q^1,q^2,\ldots,q^{n-1}\neq 1$.

Then J_3 written in 5.15 has sense and mutually different eigenvalues on the diagonal.

Proof.

$$[\nu+k]_q=[\nu+l]_q\ \forall k>l,k,l\in\{0,...,n\Leftrightarrow 1\}\Leftrightarrow q^{2\nu+k+l}=\Leftrightarrow 1\vee q^{k-l}=$$
$$=1\ \forall k>l,k,l\in\{0,...,n\Leftrightarrow 1\}\Leftrightarrow c^2q^{-n+1+k+l}=\Leftrightarrow 1\vee q^{k-l}=$$
$$=1\ \forall k>l,k,l\in\{0,...,n\Leftrightarrow 1\}\Leftrightarrow \mathrm{the condition in assignment}.$$

5.19. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.
Let $c^2=\Leftrightarrow 1, n\geqq 2$. Then J_3 written in 5.15 has sense and does NOT have mutually different eigenvalues on the diagonal.
Proof. From previous lemma we know the condition $c^2q^{-n+1+k+l}=\Leftrightarrow 1$ so that we choose $k=\frac{n}{2}\Leftrightarrow 1$, $l=\frac{n}{2}$ for n even and $k=\left[\frac{n}{2}\right]\Leftrightarrow 1$, $l=\left[\frac{n}{2}\right]+1$ for n odd.
Then we have $c^2q^{-n+1+k+l}=c^2\cdot 1=\Leftrightarrow 1\Rightarrow[\nu+k]_q=[\nu+l]_q$.

5.20. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$. Let $c^2=1$.
Let $q^1,q^2,...,q^{n-2}\neq\Leftrightarrow 1\wedge q^1,q^2,...,q^{n-1}\neq 1$.
Then J_3 written 5.15 has sense and mutually different eigenvalues on the diagonal.
Proof. From 5.18 for $c^2=1$.

5.21. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.
Let $c^2=1$.
Let $q^2,q^4,...,q^{n-1}\neq\Leftrightarrow 1$ for n odd resp. let $q^1,q^3,...,q^{n-1}\neq\Leftrightarrow 1$ for n even.
Then the matrices J_1, J_2, J_3 written in 5.15 have sense.
Proof. Clearly from 5.15

5.22. **Lemmma.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.
Let $c^2=1$.
Let $q^1,q^2,...,q^{n-1}\neq\Leftrightarrow 1\wedge q^1,q^2,...,q^{n-1}\neq 1$.
Then J_1, J_2, J_3 written in 5.15 have sense and J_3 has mutually different eigenvalues on the diagonal.
Proof. From 5.18 for $c^2=1$.

5.23. **Theorem.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.
Let $c^2=1$.
Let $q^1,q^2,...,q^{n-1}\neq\Leftrightarrow 1\wedge q^1,q^2,...,q^{n-1}\neq 1$.
Then J_1, J_2, J_3 written in 5.15 have sense and it is an irreducible representation of Fairlie algebra.
Proof. From 5.22 we have: J_1, J_2, J_3 written in 5.15 have sense and J_3 has mutually different eigenvalues on the diagonal.
We will prove equivalent proposition that every non-zero vector is cyclic:

$\forall x\neq\theta\forall y\exists$ polynomial P in variables J_1,J_2,J_3 such that $Px=y$.

Let $x\neq\theta$. Let $x=\sum_{j=0}^{n-1}\alpha_j e_j$.
Without loss of generality we can choose $y=e_{i_0}$ for any appropriate $i_0\in\widehat{n}\Leftrightarrow 1$.
Let $j_0\in\widehat{n}\Leftrightarrow 1$ such that $\alpha_{j_0}\neq 0$ (because of $x\neq\theta$ it always exists).
Then obviously

$$\frac{1}{\alpha_{j_0}}\prod_{\substack{k=0\\k\neq j_0}}^{n-1}\frac{J_3\Leftrightarrow c\left[\Leftrightarrow\frac{n\Leftrightarrow 1}{2}+k\right]_q}{c\left[\Leftrightarrow\frac{n\Leftrightarrow 1}{2}+j_0\right]_q\Leftrightarrow c\left[\Leftrightarrow\frac{n\Leftrightarrow 1}{2}+k\right]_q}x=\frac{1}{\alpha_{j_0}}\alpha_{j_0}e_{j_0}=e_{j_0}.$$

If $i_0=j_0$ we have finished, otherwise it is sufficient to show that

$$\forall j\in\{0,\dots,n\Leftrightarrow 2\}\exists\alpha,\beta\in\mathbb{C},\beta\neq 0\colon (J_1+\alpha J_2)e_j=\beta e_{j+1}\wedge$$
$$\forall j\in\{1,\dots,n\Leftrightarrow 1\}\exists\alpha,\beta\in\mathbb{C},\beta\neq 0\colon (J_1+\alpha J_2)e_j=\beta e_{j-1}.$$

(After showing this we can easily go from e_{j_0} to e_{i_0}.)
For any $\alpha\in\mathbb{C}$ is

$$(J_1+\alpha J_2)e_j=\begin{pmatrix} 0\\ \vdots\\ 0\\ \frac{(-q^2+\alpha c q^{\frac{n+1}{2}-j+1})(q^{2(j-n)}-1)(q^{2j}-1)}{(q^{2j-n+1}+1)(q^2-1)^2}\\ 0\\ \frac{1+\alpha c q^{\frac{-n+1}{2}+j}}{q^{2j+1-n}+1}\\ 0\\ \vdots\\ 0 \end{pmatrix},$$

so that if we choose α properly we can achieve zero at least on one of these two places. Will the other stay non-zero?
First of all it is clear that $(q^{2(j-n)}\Leftrightarrow 1)(q^{2j}\Leftrightarrow 1)\neq 0$ for any $j=1,\dots,n\Leftrightarrow 1$. These conditions mean

$$q^2,q^4,\dots,q^{2n-2}\neq 1\wedge$$
$$q^{2(1-n)},q^{2(2-n)},\dots,q^{2(-1)}\neq 1,$$

the second line follows from the first one and the first one from $q^1\neq\Leftrightarrow 1,\dots,\ q^{n-1}\neq\Leftrightarrow 1$ squared.
For the index-growing $j=1,2,\dots,n\Leftrightarrow 2$ (for $j=0$ we can choose $\alpha=0$) we want
$q^2=\alpha c q^{\frac{n+1}{2}-j+1}$ i. e. $\alpha=\frac{1}{c}q^{2-\frac{n+1}{2}+j-1}=cq^{1+j-\frac{n+1}{2}}$ $(c=\frac{1}{c})$.
Now we can make a test if the second one is $\neq 0$:

$$1+\alpha c q^{\frac{-n+1}{2}+j}=1+c^2q^{1+j-\frac{n+1}{2}+\frac{-n+1}{2}+j}=1+q^{1+2j-n}\neq 0?$$

I. e. $q^{3-n},q^{5-n},\dots,q^{n-3}\neq\Leftrightarrow 1$?
It is true because it is condition in the assumption.
For decreasing index $j=1,2,\dots,n\Leftrightarrow 2$ (for $j=n\Leftrightarrow 1$ we can choose $\alpha=0$) we choose α such that

$$1+\alpha c q^{\frac{-n+1}{2}+j}=0 \text{ i. e. } \alpha=\Leftrightarrow c q^{\frac{n-1}{2}-j}.$$

Then $\Leftrightarrow q^2+\alpha c q^{\frac{n+1}{2}-j+1}=\Leftrightarrow q^2\Leftrightarrow c^2 q^{\frac{n+1}{2}-j+1+\frac{n-1}{2}-j}=\Leftrightarrow q^2\Leftrightarrow q^{n+1-2j}=$
$=\Leftrightarrow q^2(1+q^{n-1-2j})\neq 0$ similarly as in previous case -- from the assumption follows that this is true.
The result is: it is really irreducible representation.
5.24. **Theorem.** Let $q\in\mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$, $c\in\mathbb{C}$, $c^4=1$.
Let $c^2=1$ tj. $c=\pm 1$. Let $q^1,q^2,\dots,q^{n-1}\neq\Leftrightarrow 1\wedge q^1,q^2,\dots,q^{n-1}\neq 1$.
Then J_3, J_1, J_2 written in 5.15 resp. in 5.17 have sense and they are two equivalent ireducible representations of Fairlie algebra.
Proof. Let

$$R=PD=\begin{pmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}\begin{pmatrix}\alpha_0 & & \\ & \alpha_1 & \\ & & \ddots \\ & & & \alpha_{n-1}\end{pmatrix}=\begin{pmatrix} & & \alpha_{n-1}\\ & \cdot^{\cdot^{\cdot}} & \\ \alpha_1 & & \\ \alpha_0 & & \end{pmatrix},$$

we can easily compute

$$R^{-1} = \begin{pmatrix} & & & \frac{1}{\alpha_0} \\ & & \cdot^{\cdot^{\cdot}} & \\ & \frac{1}{\alpha_{n-2}} & & \\ \frac{1}{\alpha_{n-1}} & & & \end{pmatrix}.$$

It is immediatelly clear that

$$RJ_3^{(+1)}R^{-1} = PDJ_3^{(+1)}D^{-1}P^{-1} = PJ_3^{(+1)}DD^{-1}P^{-1} = PJ_3^{(+1)}P^T = J_3^{(-1)}.$$

Let's take

$$X = \begin{pmatrix} 0 & a_0 & & & \\ b_0 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & a_{n-2} \\ & & & b_{n-2} & 0 \end{pmatrix}.$$

Then

$$RXR^{-1} = \begin{pmatrix} 0 & b_{n-2}\frac{\alpha_{n-1}}{\alpha_{n-2}} & & & & \\ a_{n-2}\frac{\alpha_{n-2}}{\alpha_{n-1}} & 0 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 0 & b_1\frac{\alpha_2}{\alpha_1} & \\ & & & a_1\frac{\alpha_1}{\alpha_2} & 0 & b_0\frac{\alpha_1}{\alpha_0} \\ & & & & a_0\frac{\alpha_0}{\alpha_1} & 0 \end{pmatrix}.$$

$\Rightarrow$ we have equations for α_k, $k = 0,\ldots,n \Leftrightarrow 2$:
If we compare elements above diagonal we will get condition ($k = 0,\ldots,n \Leftrightarrow 2$)

$$b_k \frac{\alpha_{k+1}}{\alpha_k} = a_{n-2-k} \tag{5.8}$$

i. e.

$$\frac{1}{q^{2k+1-n}+1}\frac{\alpha_{k+1}}{\alpha_k} = \frac{\Leftrightarrow q^2(q^{2(n-2-k+1-n)} \Leftrightarrow 1)(q^{2(n-2-k)+2} \Leftrightarrow 1)}{(q^{2(n-2-k)+3-n}+1)(q^2 \Leftrightarrow 1)^2},$$

comparing elements under the diagonal gives us the second condition ($k = 0,\ldots,n \Leftrightarrow 2$)

$$a_k \frac{\alpha_k}{\alpha_{k+1}} = b_{n-2-k} \tag{5.9}$$

i. e.

$$\frac{\Leftrightarrow q^2(q^{2(k+1-n)} \Leftrightarrow 1)(q^{2k+2} \Leftrightarrow 1)}{(q^{2k+3-n}+1)(q^2 \Leftrightarrow 1)^2}\frac{\alpha_k}{\alpha_{k+1}} = \frac{1}{q^{2(n-2-k)+1-n}+1}.$$

From (5.8) we have

$$\frac{\alpha_{k+1}}{\alpha_k} = (q^{2k+1-n}+1)\frac{\Leftrightarrow q^2(q^{-2-2k} \Leftrightarrow 1)(q^{2n-2-2k} \Leftrightarrow 1)}{(q^{n-1-2k}+1)(q^2 \Leftrightarrow 1)^2},$$

from (5.9) we have

$$\frac{\alpha_k}{\alpha_{k+1}} = \frac{1}{q^{n-3-2k}+1}\frac{(q^{2k+3-n}+1)(q^2 \Leftrightarrow 1)^2}{\Leftrightarrow q^2(q^{2(k+1-n)} \Leftrightarrow 1)(q^{2k+2} \Leftrightarrow 1)}.$$

It is consistent i. e. it is true that

$$1 = \frac{\alpha_{k+1}}{\alpha_k}\frac{\alpha_k}{\alpha_{k+1}} = q^{2n-2k-2-2k-2-n+1+2k-n+3+2k} = 1.$$

It is sufficient to solve (5.8), (5.9) is automatically fulfilled. (5.8) is set of equations for $k=0,\dots,n\Leftrightarrow 2$ that means it is undetermined, we put $\alpha_0 =$any number $\neq 0$,
$\alpha_{k+1}=\dfrac{\alpha_k a_{n-2-k}}{b_k}$ ($b_k \neq 0$ OK).

5.25. **Lemmma.** Let $q\in \mathbb{C}\Leftrightarrow\{0,1,\Leftrightarrow 1\}$, $n\in\mathbb{N}$.
Let us consider the matrices written in 5.13 for the case when following condition is fulfilled:

$$[\nu+n\Leftrightarrow 1]_q=[\nu+n]_q \wedge \alpha^2=\Leftrightarrow\frac{[n\Leftrightarrow 1]_q[n+1]_q+1}{q}$$

Then the matrices have sense $\Leftrightarrow q^{2n-1-2k}\neq 1$ for $k=0,1,\dots,n\Leftrightarrow 1$ i. e. $q^3\neq 1, q^5\neq 1,\dots,q^{2n-1}\neq 1$ and they look like this:

$$J_3=\pm i\begin{pmatrix} \frac{1+q^{2n-1}}{q^{n-\frac{3}{2}}(q^2-1)} & & & & \\ & \ddots & & & \\ & & \frac{1+q^{2n-1-2k}}{q^{n-\frac{3}{2}-k}(q^2-1)} & & \\ & & & \ddots & \\ & & & & \frac{1+q}{q^{-\frac{1}{2}}(q^2-1)}=\frac{q^{\frac{1}{2}}}{q-1} \end{pmatrix},$$

$$J_1=\begin{pmatrix} 0 & \beta_0 & & & \\ \gamma_0 & \ddots & \beta_k & & \\ & \gamma_k & \ddots & & \\ & & & \ddots & \beta_{n-2} \\ & & & \gamma_{n-2} & K \end{pmatrix},$$

where

$$\beta_k=\frac{q^{1-2n}(q^{2(2n-1-k)}-1)(q^{2(k+1)}-1)}{(q^2-1)^2(q^{2n-3-2k}-1)}, \gamma_k=\frac{q^{2n-1-2k}}{q^{2n-1-2k}-1}, K=\pm i\frac{q^{\frac{3}{2}-n}(q^{2n}-1)}{(q-1)(q^2-1)};$$

$$J_2=\begin{pmatrix} 0 & \beta'_0 & & & \\ \gamma'_0 & \ddots & \beta'_k & & \\ & \gamma'_k & \ddots & & \\ & & & \ddots & \beta'_{n-2} \\ & & & \gamma'_{n-2} & K' \end{pmatrix},$$

where

$$\beta'_k=\frac{q^{-\frac{1}{2}-n-k}(q^{2(2n-1-k)}-1)(q^{2(k+1)}-1)}{(q^2-1)^2(q^{2n-3-2k}-1)}, \gamma'_k=\frac{q^{n-\frac{1}{2}-k}}{q^{2n-1-2k}-1}, K'=\pm i\frac{q^{1-n}(q^{2n}-1)}{(q-1)(q^2-1)}.$$

Proof. The conditions in 5.13 say that

$$[\nu+n\Leftrightarrow 1]_q=[\nu+n]_q \Leftrightarrow q=1\vee q^{2\nu+2n-1}=\Leftrightarrow 1\Leftrightarrow q^{2\nu+2n-1}=\Leftrightarrow 1\Leftrightarrow q^\nu= =\pm i q^{-n+\frac{1}{2}},$$

$$\alpha^2=\Leftrightarrow\frac{[n-1]_q[n+1]_q+1}{q}\Leftrightarrow \alpha^2=\Leftrightarrow\frac{q^{1-2n}(q^{4n}-2q^{2n}+1)}{(q^2-1)^2}=\Leftrightarrow\frac{q^{1-2n}(q^{2n}-1)^2}{(q^2-1)^2}\Leftrightarrow\alpha= =\pm i\frac{q^{\frac{1}{2}-n}(q^{2n}-1)}{q^2-1}.$$

5.26. **Lemmma.** The same assumptions as in 5.25 Then J_3 has mutually different eigenvalues on the diagonal $\Leftrightarrow q^2\neq 1, q^4\neq 1,\dots,q^{2n-4}\neq 1$.

Proof. We must test condition $1+q^{2n-1-2k}=1+q^{2n-1-2l}$ for $l,k=0,\dots,n\Leftrightarrow 1$.
WLOG take $0\leqq l<k\leqq n\Leftrightarrow 1$. Then

$$q^{-2k}=q^{-2l}\Leftrightarrow q^{-2k+2l}=1\Leftrightarrow q^{2(l-k)}=1\Leftrightarrow q^{2(k-l)}=1 \tag{5.10}$$

and because of $0\leqq l<k\leqq n\Leftrightarrow 1$ we know that

$$k\Leftrightarrow l\in\{1,2,\dots,n\Leftrightarrow 2\},$$

and sufficient for fulfilling (5.10) is to ensure

$$q^{2(1)}\neq 1, q^{2(2)}\neq 1,\dots,q^{2(n-2)}\neq 1$$
$$\Leftrightarrow q^{2}\neq 1,\dots,q^{2n-4}\neq 1.$$

5.27. **Lemmma.** The same assumptions as in 5.25 If the matrices exists (conditions in 5.25 are met) and

$$q^2=1\vee q^4=1\vee\dots\vee q^{2(2n-1)}=1$$

then the representation is reducible.
Proof. There is obviously zero above the diagonal on certain (but same) place in matrices J_1,J_2,J_3. Thus $\mathrm{lin}\{e_{i_0},\dots,e_{n-1}\}$ is an invariant subspace for some $i_0\in\widehat{n\Leftrightarrow 1}$.
5.28. **Lemmma.** The same assumptions as in 5.25 If the matrices exists (conditions in 5.25 are met) and

$$q^2\neq 1\wedge q^4\neq 1\wedge\dots\wedge q^{2(2n-1)}\neq 1$$

then the representation is irreducible.
Proof. We will prove that every $x\neq\theta$ is cyclic vector. Lemma 5.26 says that J_3 has mutually different eigenvalues on the diagonal. Thus there exists some polynomial P in variable J_3 and $i_0\in\widehat{n\Leftrightarrow 1}$ such that $Px=e_{i_0}$.
J_1 has everywhere above and below the diagonal non-zero elements, thus for any $i_0\in\{0,\dots,n\Leftrightarrow 2\}$ there is a polynomial Q in variables J_3 and J_1 and some $\alpha\neq 0$ such that

$$Qe_{i_0}=\alpha e_{i_0+1}.$$

For any $i_0\in\{1,\dots,n\Leftrightarrow 1\}$ the similar is true: there is some Q and α such that

$$Qe_{i_0}=\alpha e_{i_0-1}.$$

Hence now can be seen that for any given $x\neq\theta$ and for given y there exists polynomial S in variables J_1,J_3 such that $Sx=y$.
5.29. **Lemmma.** The same assumptions as in 5.25 If the matrices exists (conditions in 5.25 are met) and the condition in 5.28 is met (representations are irreducible) then all four representations (for all combinations of $\pm$) are mutually nonequivalent.
Proof. One can easily see that either eigenvalues of J_3 are different or trace of J_1 is different.
5.30. **Lemmma.** Let's consider the same assumptions as in 5.15 Let $c^2=\Leftrightarrow 1$ i. e. $c=\pm i$. Then

$$J_1=\begin{pmatrix} 0 & \beta_0 & & & \\ \gamma_0 & \ddots & \beta_k & & \\ & \gamma_k & & & \\ & & & \ddots & \beta_{n-2} \\ & & & \gamma_{n-2} & 0 \end{pmatrix},$$

where

$$\beta_k=\frac{-q^2(q^{2(k+1-n)}-1)(q^{2k+2}-1)}{(1-q^{2k-n+3})(q^2-1)^2},\gamma_k=\frac{1}{1-q^{2k+1-n}},$$

and $\forall k\in\{0,\dots,n\Leftrightarrow 2\}$: $\dfrac{\beta_{n-2-k}}{\gamma_k}=\dfrac{\beta_k}{\gamma_{n-2-k}}$.

Proof.

$$\frac{\beta_{n-2-k}}{\gamma_k} = \frac{\Leftrightarrow q^2(q^{2(-1-k)} \Leftrightarrow 1)(q^{2n-2-2k} \Leftrightarrow 1)(1 \Leftrightarrow q^{2k+1-n})}{(1 \Leftrightarrow q^{n-1-2k})(q^2 \Leftrightarrow 1)^2} =$$
$$= \frac{q^{-2k-n-1}(q^{2(k+1)} \Leftrightarrow 1)(q^{2(k+1)} \Leftrightarrow q^{2n})}{(q^2 \Leftrightarrow 1)^2} =$$
$$= \frac{\Leftrightarrow q^2(q^{2(k+1-n)} \Leftrightarrow 1)(q^{2k+2} \Leftrightarrow 1)(1 \Leftrightarrow q^{n-3-2k})}{(1 \Leftrightarrow q^{2k-n+3})(q^2 \Leftrightarrow 1)^2} = \frac{\beta_k}{\gamma_{n-2-k}}.$$

5.31. **Lemmma.** Let's consider the same assumptions as in 5.15 Let $c^2 = \Leftrightarrow 1$ i. e. $c = \pm i$. Then representation is completely reducible.

Proof. We know n is even (otherwise matrices don't have sense). Let $r_0, r_1, \ldots, r_{\frac{n}{2}-1}$ be non-zero complex numbers. Let

$$X = \begin{pmatrix} r_0 & & & & & \Leftrightarrow r_0 \\ & \ddots & & & \iddots & \\ & & r_{\frac{n}{2}-1} & \Leftrightarrow r_{\frac{n}{2}-1} & & \\ & & 1 & 1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & 1 \end{pmatrix}.$$

Then

$$X^{-1} = \frac{1}{2} \begin{pmatrix} \frac{1}{r_0} & & & & & 1 \\ & \ddots & & & \iddots & \\ & & \frac{1}{r_{\frac{n}{2}-1}} & 1 & & \\ & & \Leftrightarrow\frac{1}{r_{\frac{n}{2}-1}} & 1 & & \\ & \iddots & & & \ddots & \\ \Leftrightarrow\frac{1}{r_0} & & & & & 1 \end{pmatrix},$$

$$X^{-1}J_1 = \frac{1}{2} \begin{pmatrix} & \frac{\beta_0}{r_0} & & & & & & & \gamma_{n-2} & \\ \frac{\gamma_0}{r_1} & & \ddots & & & & & \iddots & & \beta_{n-2} \\ & \ddots & & & \frac{\beta_{\frac{n}{2}-2}}{r_{\frac{n}{2}-2}} & \gamma_{\frac{n}{2}} & & & \iddots & \\ & & & \frac{\gamma_{\frac{n}{2}-2}}{r_{\frac{n}{2}-1}} & \gamma_{\frac{n}{2}-1} & \frac{\beta_{\frac{n}{2}-1}}{r_{\frac{n}{2}-1}} & \beta_{\frac{n}{2}} & & & \\ & & & \Leftrightarrow\frac{\gamma_{\frac{n}{2}-2}}{r_{\frac{n}{2}-1}} & \gamma_{\frac{n}{2}-1} & \Leftrightarrow\frac{\beta_{\frac{n}{2}-1}}{r_{\frac{n}{2}-1}} & \beta_{\frac{n}{2}} & & & \\ & \iddots & & & \Leftrightarrow\frac{\beta_{\frac{n}{2}-2}}{r_{\frac{n}{2}-2}} & \gamma_{\frac{n}{2}} & & & \ddots & \\ \Leftrightarrow\frac{\gamma_0}{r_1} & & \iddots & & & & & \ddots & & \beta_{n-2} \\ & \Leftrightarrow\frac{\beta_0}{r_0} & & & & & & & \gamma_{n-2} & \end{pmatrix},$$

finally

$$X^{-1}J_1X=\frac{1}{2}\begin{pmatrix} & a_{10} & & & & & a_{20} & \\ a_{30} & & \ddots & & & \iddots & & a_{40} \\ & \ddots & & a_{1m} & a_{2m} & & \iddots & \\ & & a_{3m} & A & B & a_{4m} & & \\ & & a_{4m} & \Leftrightarrow B & \Leftrightarrow A & a_{3m} & & \\ & \iddots & & a_{2m} & a_{1m} & & \ddots & \\ a_{40} & & \iddots & & & \ddots & & a_{30} \\ & a_{20} & & & & & a_{10} & \end{pmatrix},$$

where $m=\frac{n}{2}\Leftrightarrow 2$ and

$$a_{1k}=\frac{\beta_k r_{k+1}+\gamma_{n-2-k}r_k}{r_k},\quad a_{2k}=\frac{-\beta_k r_{k+1}+\gamma_{n-2-k}r_k}{r_k},$$
$$a_{3k}=\frac{\beta_{n-2-k}r_{k+1}+\gamma_k r_k}{r_{k+1}},\quad a_{4k}=\frac{\beta_{n-2-k}r_{k+1}-\gamma_k r_k}{r_{k+1}}.$$
$$A=\frac{\beta_{\frac{n}{2}-1}+\gamma_{\frac{n}{2}-1}r^2_{\frac{n}{2}-1}}{r_{\frac{n}{2}-1}},$$
$$B=\frac{\beta_{\frac{n}{2}-1}-\gamma_{\frac{n}{2}-1}r^2_{\frac{n}{2}-1}}{r_{\frac{n}{2}-1}}.$$

If we now choose r_k to satisfy conditions (see 5.30)

$$r^2_{\frac{n}{2}-1}=\frac{\beta_{\frac{n}{2}-1}}{\gamma_{\frac{n}{2}-1}},$$
$$\frac{r_k}{r_{k+1}}=\frac{\beta_{n-2-k}}{\gamma_k}\ \forall k\in\{0,1,\ldots,\tfrac{n}{2}\Leftrightarrow 2\},$$

we see that $X^{-1}J_1X$ has block-diagonal form. Because J_3 is diagonal and J_2 is uniquely determined by J_1 and J_3, proof is finished.

In the next two paragraphs we pose two open questions which are together with detailed examination of representations for the case $q^k=1$ of my interest in the future.

6. Fairlie algebra as a special case of quadratic algebra

6.1. **Definition.** *Homogeneous quadratic algebra.* Let V be a vector space and $T(V)$ its tensor algebra. Fix a subspace $R\subset T^2(V)=V\otimes V$ and consider the two-sided ideal I generated by R:

$$I=\mathrm{lin}\{x\otimes r\otimes y|r\in R,x,y\in T(V)\}.$$

The quotient algebra $U=T(V)/I$ is called a (homogeneous) quadratic algebra.

6.2. **Definition.** *Nonhomogeneous quadratic algebra.* Similarly define nonhomogeneous quadratic algebra. Consider the natural filtration $F^i(V)=\bigoplus_{n=0}^{i}T^n(V)$.

Then fix a subspace $P\subset F^2(V)=\mathbb{C}\oplus(V)\oplus(V\otimes V)$. Let I be a two-sided ideal generated by P:

$$I=\mathrm{lin}\{x\otimes p\otimes y|p\in P,x,y\in T(V)\}.$$

Then the quotient algebra $U=T(V)/I$ is called a nonhomogeneous quadratic algebra.

6.3. **Lemmma.**airlies deformation of SU(2) is nonhomogeneous quadratic algebra.

Proof. The subspace

$$P=\mathrm{lin}\{I_1\Leftrightarrow I_3\otimes I_2+qI_2\otimes I_3,I_2\Leftrightarrow I_3\otimes I_1+q^{-1}I_1\otimes I_3,I_3\Leftrightarrow I_2\otimes I_1+qI_1\otimes I_2\}.$$

Open question. In [8] there is introduced special class of quadratic algebras (so called Kozsul type) for which can be formulated a condition which completely determines if some algebra is of PBW type or not. The open question is if Fairlie algebra is of Kozsul type.

7. Hopf algebras

7.1. **Definition.** *Coalgebra.* Let R be a ring. A coalgebra $(C,+,R,\triangle)$ over R (R-coalgebra) is a R-module C with a linear map $\triangle$: $C \to C \otimes C$ called the coproduct (or comultiplication map). If

$$((\mathrm{id} \times \triangle) \circ \triangle)(x) = ((\triangle \times \mathrm{id}) \circ \triangle)(x) \forall x \in C$$

then the coproduct $\triangle$ and the coalgebra is called coassociative. If there exists a map ε: $C \to R$ such that

$$((\varepsilon \times \mathrm{id}) \circ \triangle)(x) = ((\mathrm{id} \times \varepsilon) \circ \triangle)(x) \forall x \in C$$

then the map ε is called a counit map and the coalgebra is called a coalgebra with counit. A coalgebra C is called cocommutative if

$$(\sigma \circ \triangle)(x) = \triangle(x) \forall x \in C,$$

where σ: $C \otimes C \to C \otimes C$ is a twist map, i. e.

$$\sigma(x \otimes y) = y \otimes x \forall x, y \in C.$$

7.2. **Definition.** *Bialgebra.* Let B be an R-algebra. B has the structure of a bialgebra $(B,+,m,\triangle)$ if $(B,+,m)$ is an R-algebra and $\triangle$ is a coproduct map that is an algebra homomorphism.

7.3. **Definition.** *Antipode.* Let $(B,+,R,m,\triangle,\mu,\varepsilon)$ be a bialgebra with unity μ and counit ε homomorphisms. An algebra antiautomorphism S: $B \to B$ which satisfies

$$(S \circ m)(x \otimes y) = m(S(y) \otimes S(x)) \forall x, y \in B$$
$$(m \circ (S \times \mathrm{id}) \circ \triangle)(a) = (\mu \circ \varepsilon)(a) = (m \circ (\mathrm{id} \times S) \circ \triangle)(a) \forall a \in B$$

is called antipode map.

7.4. **Definition.** *Hopf algebra.* A Hopf algebra $(H,+,R,m,\triangle,\mu,\varepsilon,S)$ is a bialgebra with a unity μ: $R \to H$, counit ε: $H \to R$ and an antipode antiautomorphism S.

7.5. **Lemmma.** Let g be a $\mathbb{C}$-Lie algebra and $U(g)$ its universal enveloping algebra. Then $U(g)$ has the structure of a Hopf algebra by extending the maps

$$\begin{aligned}
&\triangle(x) = x \otimes 1 + 1 \otimes x, \\
&\triangle(\alpha \cdot 1) = \alpha \cdot 1 \otimes 1, \\
&\mu(\alpha) = \alpha 1, \\
&\varepsilon(x) = 0, \qquad \forall x \in g, \alpha \in \mathbb{C} \\
&S(x) = \Leftrightarrow x, \\
&\varepsilon(\alpha \cdot 1) = \alpha, \\
&S(1) = 1
\end{aligned}$$

homomorphically to all elements of $U(g)$.

Proof. Clearly all the Hopf algebra maps are algebra homomorphisms. For the coproduct following holds:

$$\begin{aligned}
&\triangle(x \cdot y \Leftrightarrow y \cdot x) = \triangle(x) \cdot \triangle(y) \Leftrightarrow \triangle(y) \cdot \triangle(x) = \\
&= (x \cdot y) \otimes 1 + x \otimes y + y \otimes x + 1 \otimes (x \cdot y) \Leftrightarrow (y \cdot x) \otimes 1 \Leftrightarrow y \otimes x \Leftrightarrow x \otimes y \Leftrightarrow 1 \otimes (y \cdot y) = \\
&= (x \cdot y \Leftrightarrow y \cdot x) \otimes 1 + 1 \otimes (x \cdot y \Leftrightarrow y \cdot x) = \triangle([x, y]).
\end{aligned}$$

Open Question. Can be Fairlie algebra given the structure of a Hopf algebra by suitable coproduct map?

8. Extension of sl(2) and its representations

It follows from the relations (4.1)--(4.3) and we have shown that for the algebra $U_q(\mathrm{so}_3)$ the Poincaré-Birkhoff-Witt theorem is true and this theorem can be formulated as:
The elements $I_1^k I_2^m I_3^n$, $k,m,n=0,1,2,\cdots$, *form a basis of the linear space* $U_q(\mathrm{so}_3)$.
Indeed, by using the relations (4.1)--(4.3) any product $I_{j_1} I_{j_2} \cdots I_{j_s}$, $j_1,j_2,\cdots,j_s=1,2,3$, can be reduced to a sum of the elements $I_1^k I_2^m I_3^n$ with complex coefficients.
Note that by (4.3) the element I_3 is not independent: it is determined by the elements I_1 and I_2. Thus, the algebra $U_q(\mathrm{so}_3)$ is generated by I_1 and I_2, but now instead of quadratic relations (4.1)--(4.3) we must take the relations

$$I_1 I_2^2 - (q+q^{-1}) I_2 I_1 I_2 + I_2^2 I_1 = -I_1, \qquad (8.1)$$
$$I_2 I_1^2 - (q+q^{-1}) I_1 I_2 I_1 + I_1^2 I_2 = -I_2, \qquad (8.2)$$

which are obtained if we substitute the expression (4.3) for I_3 into (4.1) and (4.2). The equation $I_3 = q^{1/2} I_1 I_2 - q^{-1/2} I_2 I_1$ and the relations (8.1) and (8.2) restore the relations (4.1)--(4.3).
Remark that the definition of $U_q(\mathrm{so}_3)$ by means of relations (8.1) and (8.2) was used for the embedding of $U_q(\mathrm{so}_3)$ to $U_q(\mathrm{sl}_3)$. The relations (8.1) and (8.2) differ from Serre's relations in the definition of quantum algebras by V. Drinfeld and M. Jimbo by appearance of non-vanishing right hand sides.
The algebra $U_q(\mathrm{so}_3)$ is closely related to (but not coincides with) the quantum algebra $U_q(\mathrm{sl}_2)$. The last algebra is generated by the elements q^H, q^{-H}, E, F satisfying the relations

$$q^H q^{-H} = q^{-H} q^H = 1, \quad q^H E q^{-H} = qE, \quad q^H F q^{-H} = q^{-1} F, \qquad (8.3)$$
$$[E,F] := EF - FE = \frac{q^{2H} - q^{-2H}}{q-q^{-1}}.$$

Note that $U_q(\mathrm{sl}_2)$ is the associative algebra equipped with a Hopf algebra structure (a comultiplication, a counit and an antipode). In particular, the comultiplication Δ is determined by the formulas

$$\Delta(q^{\pm H}) = q^{\pm H} \otimes q^{\pm H}, \quad \Delta(E) = E \otimes q^H + q^{-H} \otimes E,$$
$$\Delta(F) = F \otimes q^H + q^{-H} \otimes F.$$

In order to relate the algebras $U_q(\mathrm{so}_3)$ and $U_q(\mathrm{sl}_2)$ we need to extend $U_q(\mathrm{sl}_2)$ by the elements $(q^k q^H + q^{-k} q^{-H})^{-1}$ in the sense of Ref. 10. We denote by $\widehat{U}_q(\mathrm{sl}_2)$ the associative algebra with unit element generated by the elements

$$q^H, \quad q^{-H}, \quad E, \quad F, \quad (q^k q^H + q^{-k} q^{-H})^{-1}, \quad k \in \mathbf{Z},$$

satisfying the defining relations of the algebra $U_q(\mathrm{sl}_2)$ and the following natural relations:

$$(q^k q^H + q^{-k} q^{-H})^{-1} (q^k q^H + q^{-k} q^{-H}) = (q^k q^H + q^{-k} q^{-H})(q^k q^H + q^{-k} q^{-H})^{-1} =$$
$$= 1, \qquad (8.4)$$
$$q^{\pm H} (q^k q^H + q^{-k} q^{-H})^{-1} = (q^k q^H + q^{-k} q^{-H})^{-1} q^{\pm H}, \qquad (8.5)$$
$$(q^k q^H + q^{-k} q^{-H})^{-1} E = E (q^{k+1} q^H + q^{-k-1} q^{-H})^{-1}, \qquad (8.6)$$
$$(q^k q^H + q^{-k} q^{-H})^{-1} F = F (q^{k-1} q^H + q^{-k+1} q^{-H})^{-1}. \qquad (8.7)$$

Note that the algebra $U_q(\mathrm{sl}_2)$ has finite dimensional irreducible representations $T_l \equiv T_l^{(1)}$, $T_l^{(-1)}$, $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$, acting on the vector spaces H_l with base vectors x_m, $m = -l, -l+1, \cdots, l$.
These representations are given by the formulas

$$T_l^{(1)}(q^H) x_m = q^m x_m, \quad T_l^{(1)}(E) x_m = [l-m] x_{m+1}, \qquad (8.8)$$

$$T_l^{(1)}(F)x_m = [l+m]x_{m-1}, \tag{8.9}$$

where a number in square brackets means a q-number, defined by the formula

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}},$$

and by the formulas

$$T_l^{(-1)}(q^H)x_m = -q^m x_m, \quad T_l^{(-1)}(E) = T_l^{(1)}(E), \quad T_l^{(-1)}(F) = T_l^{(1)}(F), \tag{8.10}$$

$$T_l^{(\mathrm{i})}(q^H)x_m = \mathrm{i}q^m x_m, \quad T_l^{(\mathrm{i})}(E) = T_l^{(1)}(E), \quad T_l^{(\mathrm{i})}(F) = -T_l^{(1)}(F), \tag{8.11}$$

$$T_l^{(-\mathrm{i})}(q^H)x_m = -\mathrm{i}q^m x_m, \quad T_l^{(-\mathrm{i})}(E) = T_l^{(1)}(E), \quad T_l^{(-\mathrm{i})}(F) = -T_l^{(1)}(F). \tag{8.12}$$

The representations $T_l^{(1)}$, $T_l^{(-1)}$, $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$, are pairwise non-equivalent, and any finite dimensional irreducible representation of $U_q(\mathrm{sl}_2)$ is equivalent to one of these representations (see, for example, Ref. [17], Chapter 3).

Now we wish to extend these representations of $U_q(\mathrm{sl}_2)$ to the representations of $\hat{U}_q(\mathrm{sl}_2)$ by using the relation

$$T((q^k q^H + q^{-k} q^{-H})^{-1}) := (q^k T(q^H) + q^{-k} T(q^{-H}))^{-1}. \tag{8.13}$$

Clearly, only those irreducible representations T of $U_q(\mathrm{sl}_2)$ can be extended to $\hat{U}_q(\mathrm{sl}_2)$ for which the operators $q^k T(q^H) + q^{-k} T(q^{-H})$ are invertible. From formulas (8.8)--(8.9) it is clear that these operators are always invertible for the representations $T_l^{(1)}$, $T_l^{(-1)}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$, and for the representations $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. For the representations $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l = 0, 1, 2, \cdots$, some of these operators are not invertible since they have zero eigenvalue. Denoting the extended representations by the same symbols, we can formulate the following statement:

8.1. **Theorem.** The algebra $\hat{U}_q(\mathrm{sl}_2)$ has the irreducible finite dimensional representations $T_l^{(1)}$, $T_l^{(-1)}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$, and $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. Any irreducible finite dimensional representation of $\hat{U}_q(\mathrm{sl}_2)$ is equivalent to one of these representations.

9. Algebra homomorphism from deformation of so(3) into deformation of sl(2)

The aim of this chapter is to give (in an explicit form) the homomorphism of the algebra $U_q(\mathrm{so}_3)$ to $\hat{U}_q(\mathrm{sl}_2)$. This homomorphism is described by the following proposition:

9.1. **Theorem.** There exists a unique algebra homomorphism ψ: $U_q(\mathrm{so}_3) \to \hat{U}_q(\mathrm{sl}_2)$ such that

$$\psi(I_1) = \frac{\mathrm{i}}{\mathrm{q} - \mathrm{q}^{-1}}(q^H - q^{-H}), \tag{9.1}$$

$$\psi(I_2) = (E - F)(q^H + q^{-H})^{-1}, \tag{9.2}$$

$$\psi(I_3) = (\mathrm{i}q^{H-1/2}E + \mathrm{i}q^{-H-1/2}F)(q^H + q^{-H})^{-1}, \tag{9.3}$$

where $q^{H+a} := q^H q^a$ for $a \in \mathbf{C}$.

Proof. In order to prove this proposition we have to show that

$$\begin{aligned} q^{1/2}\psi(I_1)\psi(I_2) - q^{-1/2}\psi(I_2)\psi(I_1) &= \psi(I_3), \\ q^{1/2}\psi(I_2)\psi(I_3) - q^{-1/2}\psi(I_3)\psi(I_2) &= \psi(I_1), \\ q^{1/2}\psi(I_3)\psi(I_1) - q^{-1/2}\psi(I_1)\psi(I_3) &= \psi(I_2). \end{aligned} \tag{9.4}$$

Let us prove the first relation. (Other relations are proved similarly.) Substituting the expressions (9.1)--(9.3) for $\psi(I_i)$, $i=1,2,3$, into the first relation we have (after multiplying both sides of equality by (q^H+q^{-H}) on the right) the relation

$$q(E\Leftrightarrow F)Eq^H(qq^H+q^{-1}q^{-H})^{-1}+q(E\Leftrightarrow F)Fq^{-H}(q^{-1}q^H+qq^{-H})^{-1}\Leftrightarrow$$
$$\Leftrightarrow qE^2q^H(qq^H+q^{-1}q^{-H})^{-1}\Leftrightarrow q^{-1}FEq^{-H}(qq^H+q^{-1}q^{-H})^{-1}+$$
$$+q^{-1}EFq^H(q^{-1}q^H+qq^{-H})^{-1}+qF^2q^{-H}(q^{-1}q^H+qq^{-H})^{-1}=\mathrm{i}\frac{q^{2H}-q^{-2H}}{q-q^{-1}}.$$

The formula (9.4) is true if and only if this relation is correct. We multiply both its sides by $(qq^H+q^{-1}q^{-H})(q^{-1}q^H+qq^{-H})$ on the right and obtain the relation in the algebra $U_q(\mathrm{sl}_2)$ (that is, without the expressions $(q^kq^H+q^{-k}q^{-H})^{-1}$). This relation is easily verified by using the defining relations of the algebra $U_q(\mathrm{sl}_2)$. Proposition is proved.

10. Finite dimensional representations obtained using algebra homomorphism

We assume in this and following three chapters that q is not a root of unity.

If T is a representation of the algebra $\hat{U}_q(\mathrm{sl}_2)$ on a linear space V, then the mapping R: $U_q(\mathrm{so}_3)\to V$ defined as the composition $R=T\circ\psi$, where ψ is the homomorphism from theorem 9.1, is a representation of $U_q(\mathrm{so}_3)$.

Let us consider the representations

$$R_l^{(1)}=T_l^{(1)}\circ\psi,\quad R_l^{(-1)}=T_l^{(-1)}\circ\psi,\quad R_l^{(\mathrm{i})}=T_l^{(\mathrm{i})}\circ\psi,\quad R_l^{(-\mathrm{i})}=T_l^{(-\mathrm{i})}\circ\psi$$

of $U_q(\mathrm{so}_3)$, where $T_l^{(1)}$, $T_l^{(-1)}$, $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$ are the irreducible representations of $\hat{U}_q(\mathrm{sl}_2)$ from theorem 8.1.

Using formulas for the representations $T_l^{(\pm 1)}$ of $U_q(\mathrm{sl}_2)$ and the expressions (9.1)--(9.3) for $\psi(I_j)$, $j=1,2,3$, we find that

$$R_l^{(1)}(I_1)x_m=\mathrm{i}[m]x_m,$$
$$R_l^{(1)}(I_2)x_m=\frac{1}{q^m+q^{-m}}([l\Leftrightarrow m]x_{m+1}\Leftrightarrow[l+m]x_{m-1}),$$
$$R_l^{(1)}(I_3)x_m=\frac{\mathrm{i}q^{1/2}}{q^m+q^{-m}}(q^m[l\Leftrightarrow m]x_{m+1}+q^{-m}[l+m]x_{m-1})$$

for the representation $R_l^{(1)}$ and

$$R_l^{(-1)}(I_1)x_m=\Leftrightarrow\mathrm{i}[m]x_m,\quad R_l^{(-1)}(I_2)=\Leftrightarrow R_l^{(1)}(I_2),\quad R_l^{(-1)}(I_3)=R_l^{(1)}(I_3).$$

Denoting the vectors x_m by x_{-m} for the representations $R_l^{(-1)}$ we easily find that the matrices of the representation $R_l^{(-1)}$ in the basis containing vectors x_{-m}, $m=\Leftrightarrow l,\Leftrightarrow l+1,\cdots,l$, coincide with the corresponding matrices of the representation $R_l^{(1)}$. Thus, the non-equivalent representations $T_l^{(1)}$ and $T_l^{(-1)}$ of the algebra $\hat{U}_q(\mathrm{sl}_2)$ lead to equivalent representations of $U_q(\mathrm{so}_3)$.

For the representations $R_l^{(\mathrm{i})}$ and $R_l^{(-\mathrm{i})}$ we have

$$R_l^{(\mathrm{i})}(I_1)x_m=\Leftrightarrow\frac{q^m+q^{-m}}{q-q^{-1}}x_m,$$
$$R_l^{(\mathrm{i})}(I_2)x_m=\mathrm{i}\frac{[l-m]}{q^m-q^{-m}}x_{m+1}+\mathrm{i}\frac{[l+m]}{q^m-q^{-m}}x_{m-1},$$
$$R_l^{(\mathrm{i})}(I_3)x_m=\Leftrightarrow\frac{\mathrm{i}q^{m+1/2}[l-m]}{q^m-q^{-m}}x_{m+1}\Leftrightarrow\frac{\mathrm{i}q^{-m+1/2}[l+m]}{q^m-q^{-m}}x_{m-1}$$

and

$R_l^{(-\mathrm{i})}(I_1)x_m = \frac{q^m+q^{-m}}{q-q^{-1}} x_m,$
$R_l^{(-\mathrm{i})}(I_2)x_m = -\mathrm{i}\frac{[l-m]}{q^m-q^{-m}} x_{m+1} - \mathrm{i}\frac{[l+m]}{q^m-q^{-m}} x_{m-1},$
$R_l^{(-\mathrm{i})}(I_3)x_m = -\frac{\mathrm{i}q^{m+1/2}[l-m]}{q^m-q^{-m}} x_{m+1} - \frac{\mathrm{i}q^{-m+1/2}[l+m]}{q^m-q^{-m}} x_{m-1}.$

10.1. **Theorem.** The representations $R_l^{(1)}$ of $U_q(\mathrm{so}_3)$ are irreducible. The representations $R_l^{(\mathrm{i})}$ and $R_l^{(-\mathrm{i})}$ are reducible.

Proof. To prove the first part of the proposition we first note that since q is not a root of unity, the eigenvalues $\mathrm{i}[m]$, $m=-l,-l+1,\cdots,l$, of the operator $R_l^{(1)}(I_1)$ are pairwise different.

Let V be an invariant subspace of the space H_l of the representation $R_l^{(1)}$, and let $v \equiv \sum_{m_i} \alpha_i x_{m_i} \in V$, where x_{m_i} are eigenvectors of $R_l^{(1)}(I_1)$. Then $x_{m_i} \in V$. We prove this for the case when $v=\alpha_1 x_{m_1}+\alpha_2 x_{m_2}$. (The case of more number of summands is proved similarly.) We have $R_l^{(1)}(I_1)v = \mathrm{i}\alpha_1[m_1]x_{m_1}+\mathrm{i}\alpha_2[m_2]x_{m_2}$. Since

$$v=\alpha_1 x_{m_1}+\alpha_2 x_{m_2}\in V, \quad v' \equiv \mathrm{i}\alpha_1[m_1]x_{m_1}+\mathrm{i}\alpha_2[m_2]x_{m_2}\in V$$

one derives that

$$\mathrm{i}[m_1]v - v' = \mathrm{i}\alpha_2([m_1]-[m_2])x_{m_2}\in V.$$

Since $[m_1]\neq[m_2]$, then $x_{m_2}\in V$ and hence $x_{m_1}\in V$.

In order to prove that $V=H_l$ we obtain from the above formulas for $R_l^{(1)}(I_2)x_m$ and $R_l^{(1)}(I_3)x_m$ that

$$(R_l^{(1)}(I_3) - \mathrm{i}q^{m+1/2}R_l^{(1)}(I_2))x_m = \mathrm{i}q^{1/2}x_{m-1},$$
$$(R_l^{(1)}(I_3) + \mathrm{i}q^{-m+1/2}R_l^{(1)}(I_2))x_m = \mathrm{i}q^{1/2}x_{m+1}.$$

Since V contains at least one basis vector x_m, it follows from these relations that V contains the vectors x_{m-1} , x_{m-2} ,$\cdots$,x_{-l} and the vectors x_{m+1} ,x_{m+2} ,$\cdots$,x_l. This means that $V=H_l$ and the representation $R_l^{(1)}$ is irreducible.

Let us show that the representations $R_l^{(\mathrm{i})}$ are reducible. The eigenvalues of the operator $R_l^{(\mathrm{i})}(I_1)$ are

$$-\frac{q^m+q^{-m}}{q-q^{-1}}, \quad m=-l,-l+1,\cdots,l,$$

that is, every spectral point has multiplicity 2. Namely, the pairs of vectors x_m and x_{-m} are of the same eigenvalue. Let V_1 be the subspace of the representation space H_l spanned by the vectors

$$x_{\frac{1}{2}}+\mathrm{i}x_{-\frac{1}{2}},\ x_{\frac{3}{2}}-\mathrm{i}x_{-\frac{3}{2}},\ x_{\frac{5}{2}}+\mathrm{i}x_{-\frac{5}{2}},\ x_{\frac{7}{2}}-\mathrm{i}x_{-\frac{7}{2}},\ \cdots, \tag{10.1}$$

and let V_2 be the subspace spanned by the vectors

$$x_{\frac{1}{2}}-\mathrm{i}x_{-\frac{1}{2}},\ x_{\frac{3}{2}}+\mathrm{i}x_{-\frac{3}{2}},\ x_{\frac{5}{2}}-\mathrm{i}x_{-\frac{5}{2}},\ x_{\frac{7}{2}}+\mathrm{i}x_{-\frac{7}{2}},\ \cdots. \tag{10.2}$$

We denote the vectors (10.1) by

$$x'_{\frac{1}{2}},\ x'_{\frac{3}{2}},\ x'_{\frac{5}{2}},\ x'_{\frac{7}{2}},\ \cdots \tag{10.3}$$

and the vectors (10.2) by

$$x''_{\frac{1}{2}},\ x''_{\frac{3}{2}},\ x''_{\frac{5}{2}},\ x''_{\frac{7}{2}},\ \cdots. \tag{10.4}$$

Then

$$R_l^{(\mathrm{i})}(I_1)x'_m = -\frac{q^m+q^{-m}}{q-q^{-1}}x'_m, \quad R_l^{(\mathrm{i})}(I_1)x''_m = -\frac{q^m+q^{-m}}{q-q^{-1}}x''_m.$$

We also have

$$R_l^{(\mathrm{i})}(I_2)x'_{\frac12} = \mathrm{i}\frac{[l-\frac12]}{q^{1/2}-q^{-1/2}}x_{\frac32} + \mathrm{i}\frac{[l+\frac12]}{q^{1/2}-q^{-1/2}}x_{-\frac12} + \frac{[l+\frac12]}{q^{1/2}-q^{-1/2}}x_{\frac12} + \frac{[l-\frac12]}{q^{1/2}-q^{-1/2}}x_{-\frac32} =$$
$$= \frac{[l+\frac12]}{q^{1/2}-q^{-1/2}}x'_{\frac12} + \mathrm{i}\frac{[l-\frac12]}{q^{1/2}-q^{-1/2}}x'_{\frac32}.$$

We derive similarly that

$$R_l^{(\mathrm{i})}(I_2)x''_{\frac12} = -\frac{[l+\frac12]}{q^{1/2}-q^{-1/2}}x''_{\frac12} + \mathrm{i}\frac{[l-\frac12]}{q^{1/2}-q^{-1/2}}x''_{\frac32}$$

and that

$$R_l^{(\mathrm{i})}(I_2)x'_m = \mathrm{i}\frac{[l-m]}{q^m-q^{-m}}x'_{m+1} + \mathrm{i}\frac{[l+m]}{q^m-q^{-m}}x'_{m-1}, \quad m > \tfrac12,$$
$$R_l^{(\mathrm{i})}(I_2)x''_m = \mathrm{i}\frac{[l-m]}{q^m-q^{-m}}x''_{m+1} + \mathrm{i}\frac{[l+m]}{q^m-q^{-m}}x''_{m-1}, \quad m > \tfrac12.$$

Thus, the subspaces V_1 and V_2 are invariant with respect to the operators $R_l^{(\mathrm{i})}(I_1)$ and $R_l^{(\mathrm{i})}(I_2)$. This means that they are invariant with respect to the representation $R_l^{(\mathrm{i})}$. It is proved similarly that the subspace V_1 of the space H_l of the representation $R_l^{(-\mathrm{i})}$ spanned by the vectors (10.1) and the subspace V_2 of H_l spanned by the vectors (10.2) are invariant with respect to the operators $R_l^{(-\mathrm{i})}(I_1)$ and $R_l^{(-\mathrm{i})}(I_2)$. That is, the representation $R_l^{(-\mathrm{i})}$ is also reducible. Proposition is proved.

Let $R_n^{(\mathrm{i},+)}$ and $R_n^{(\mathrm{i},-)}$, $n = l+\frac12 = \dim V_1 = \dim V_2$, be the representations of $U_q(\mathrm{so}_3)$ which are restrictions of $R_l^{(\mathrm{i})}$ to the subspaces V_1 and V_2, respectively. Denoting the vectors (10.3) of the subspace V_1 by

$$x_1, \quad x_2, \quad x_3, \quad x_4, \cdots, x_n \equiv x_{l+\frac12}, \tag{10.5}$$

respectively, we have

$$R_n^{(\mathrm{i},+)}(I_1)x_k = -\frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}x_k,$$
$$R_n^{(\mathrm{i},+)}(I_2)x_1 = \frac{[n]}{q^{1/2}-q^{-1/2}}x_1 + \mathrm{i}\frac{[n-1]}{q^{1/2}-q^{-1/2}}x_2,$$
$$R_n^{(\mathrm{i},+)}(I_2)x_k = \mathrm{i}\frac{[n-k]}{q^{k-1/2}-q^{-k+1/2}}x_{k+1} + \mathrm{i}\frac{[n+k-1]}{q^{k-1/2}-q^{-k+1/2}}x_{k-1}, \quad k \ne 1.$$

For the operator $R_n^{(\mathrm{i},+)}(I_3)$ we have

$$R_n^{(\mathrm{i},+)}(I_3)x_1 = -\frac{[n]}{q^{1/2}-q^{-1/2}}x_1 - \mathrm{i}\frac{q[n-1]}{q^{1/2}-q^{-1/2}}x_2,$$
$$R_n^{(\mathrm{i},+)}(I_3)x_k = -\mathrm{i}\frac{q^k[n-k]}{q^{k-1/2}-q^{-k+1/2}}x_{k+1} - \mathrm{i}\frac{q^{-k+1}[n+k-1]}{q^{k-1/2}-q^{-k+1/2}}x_{k-1}, \quad k \ne 1.$$

Denoting the vectors (10.4) of the subspace V_2 by the symbols (10.5), respectively, we obtain

$$R_n^{(\mathrm{i},-)}(I_1)x_k = -\frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}x_k,$$
$$R_n^{(\mathrm{i},-)}(I_2)x_1 = -\frac{[n]}{q^{1/2}-q^{-1/2}}x_1 + \mathrm{i}\frac{[n-1]}{q^{1/2}-q^{-1/2}}x_2,$$
$$R_n^{(\mathrm{i},-)}(I_2)x_k = R_n^{(\mathrm{i},+)}(I_2)x_k, \quad k \ne 1.$$

For the operator $R_l^{(\mathrm{i},-)}(I_3)$ we find that

$$R_n^{(\mathrm{i},-)}(I_3)x_1 = \frac{[n]}{q^{1/2}-q^{-1/2}}x_1 - \mathrm{i}\frac{q[n-1]}{q^{1/2}-q^{-1/2}}x_2,$$

$$R_n^{(\mathrm{i},-)}(I_3)x_k = R_n^{(\mathrm{i},+)}(I_3)x_k, \quad k \neq 1.$$

Let now $R_n^{(-\mathrm{i},+)}$ and $R_n^{(-\mathrm{i},-)}$, $n = l + \frac{1}{2}$, be the representations of $U_q(\mathrm{so}_3)$ which are restrictions of the representation $R_l^{(-\mathrm{i})}$ to the subspaces V_1 and V_2, respectively. Introducing the vectors similar to the vectors (10.5), for the representation $R_n^{(-\mathrm{i},+)}$ we have

$$R_n^{(-\mathrm{i},+)}(I_1)x_k = \frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}x_k, \quad R_n^{(-\mathrm{i},+)}(I_2) = \Leftrightarrow R_n^{(\mathrm{i},+)}(I_2),$$
$$R_n^{(-\mathrm{i},+)}(I_3)x_1 = \frac{[n]}{q^{1/2}-q^{-1/2}}x_1 + \mathrm{i}\frac{q[n-1]}{q^{1/2}-q^{-1/2}}x_2,$$
$$R_n^{(-\mathrm{i},+)}(I_3)x_k = \mathrm{i}\frac{q^k[n-k]}{q^{k-1/2}-q^{-k+1/2}}x_{k+1} + \mathrm{i}\frac{q^{-k+1}[n+k-1]}{q^{k-1/2}-q^{-k+1/2}}x_{k-1}, \quad k \neq 1.$$

For the representation $R_l^{(-\mathrm{i},-)}$ we obtain

$$R_n^{(-\mathrm{i},-)}(I_1)x_k = \frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}x_k, \quad R_n^{(-\mathrm{i},-)}(I_2) = \Leftrightarrow R_n^{(\mathrm{i},-)}(I_2),$$
$$R_n^{(-\mathrm{i},-)}(I_3)x_1 = \Leftrightarrow \frac{[n]}{q^{1/2}-q^{-1/2}}x_1 + \mathrm{i}\frac{q[n-1]}{q^{1/2}-q^{-1/2}}x_2,$$
$$R_n^{(-\mathrm{i},-)}(I_3)x_k = R_n^{(-\mathrm{i},+)}(I_3)x_k.$$

Thus, we constructed the representations $R_n^{(\mathrm{i},+)}$, $R_n^{(\mathrm{i},-)}$, $R_n^{(-\mathrm{i},+)}$ and $R_n^{(-\mathrm{i},-)}$ of the algebra $U_q(\mathrm{so}_3)$. The following theorem characterizes them.

10.2. **Theorem.** The representations $R_n^{(\mathrm{i},+)}$, $R_n^{(\mathrm{i},-)}$, $R_n^{(-\mathrm{i},+)}$ and $R_n^{(-\mathrm{i},-)}$ are irreducible and pairwise nonequivalent. For any l the representation $R_l^{(1)}$ is not equivalent to any of these representations.

Proof. The irreducibility is proved exactly in the same way as in 10.1. Equivalence relations may exist only for irreducible representations of the same dimension. That is, we have to show that under fixed n no pair of the representations $R_n^{(\mathrm{i},+)}$, $R_n^{(\mathrm{i},-)}$, $R_n^{(-\mathrm{i},+)}$ and $R_n^{(-\mathrm{i},-)}$ is equivalent. It follows from the above formulas that the operators $R_n^{(\mathrm{i},+)}(I_1)$ and $R_n^{(\mathrm{i},-)}(I_1)$, as well as the operators $R_n^{(-\mathrm{i},+)}(I_1)$ and $R_n^{(-\mathrm{i},-)}(I_1)$, have the same set of eigenvalues. Moreover, the spectrum of the first pair of operators differs from that of the second pair. Hence, no of representations $R_n^{(\mathrm{i},+)}$ and $R_n^{(\mathrm{i},-)}$ is equivalent to $R_n^{(-\mathrm{i},+)}$ or $R_n^{(-\mathrm{i},-)}$. The representations $R_n^{(\mathrm{i},+)}$ and $R_n^{(\mathrm{i},-)}$ are not equivalent since the operators $R_n^{(\mathrm{i},+)}(I_2)$ and $R_n^{(\mathrm{i},-)}(I_2)$ have different traces (for equivalent representations these operators must have the same trace). For the same reason, the representations $R_n^{(-\mathrm{i},+)}$ and $R_n^{(-\mathrm{i},-)}$ are not equivalent. The last assertion of the theorem follows from the fact that the spectrum of the operator $R_l^{(1)}(I_1)$ differs from the spectra of the operators $R_n^{(\mathrm{i},+)}(I_1)$, $R_n^{(\mathrm{i},-)}(I_1)$, $R_n^{(-\mathrm{i},+)}(I_1)$ and $R_n^{(-\mathrm{i},-)}(I_1)$. Theorem is proved.

Clearly, the reducible representations $R_n^{(\mathrm{i})}$ and $R_n^{(-\mathrm{i})}$ decomposes into irreducible components as

$$R_n^{(\mathrm{i})} = R_n^{(\mathrm{i},+)} \oplus R_n^{(\mathrm{i},-)}, \quad R_n^{(-\mathrm{i})} = R_n^{(-\mathrm{i},+)} \oplus R_n^{(-\mathrm{i},-)}. \tag{10.6}$$

10.3. **Theorem.** Every irreducible finite dimensional representation of $U_q(\mathrm{so}_3)$ is equivalent to one of the representations $R_l^{(1)}$, $R_n^{(\mathrm{i},+)}$, $R_n^{(\mathrm{i},-)}$, $R_n^{(-\mathrm{i},+)}$, $R_n^{(-\mathrm{i},-)}$. That is, these representations exhaust, up to equivalence, all irreducible finite dimensional representations of $U_q(\mathrm{so}_3)$.

Proof. It can be easily seen that representations $R_l^{(1)}$, $R_n^{(\mathrm{i},+)}$, $R_n^{(\mathrm{i},-)}$, $R_n^{(-\mathrm{i},+)}$, $R_n^{(-\mathrm{i},-)}$ are equivalent to the representations described in the theorem 5.4. Thus the proof follows from this theorem.

11. Tensor product of representations

As mentioned above, no Hopf algebra structure is known for the algebra $U_q(\mathrm{so}_3)$. Therefore, we cannot construct tensor product of finite dimensional representations of $U_q(\mathrm{so}_3)$ by using a comultiplication as we do in the case of the quantum algebra $U_q(\mathrm{sl}_2)$. However, we may construct some tensor product representations by using the algebra homomorphism of 9.1.

First we determine which tensor products of irreducible representations of $U_q(\mathrm{sl}_2)$ can be extended to representations of the algebra $\widehat{U}_q(\mathrm{sl}_2)$. Verifying for which tensor products $T=T'\otimes T''$ of irreducible representations of $U_q(\mathrm{sl}_2)$ the operators

$$q^kT(q^H)+q^{-k}T(q^{-H}),\quad k\in\mathbb{Z},$$

are invertible, we conclude that only the tensor products

$$\begin{aligned}
&T_l^{(\pm1)}\otimes T_{l'}^{(\pm1)},\ T_l^{(\pm1)}\otimes T_{l'}^{(\mp1)},\ \ l,l'=0,\tfrac12,1,\tfrac32,\cdots,\\
&T_l^{(\pm1)}\otimes T_{l'}^{(\pm\mathrm{i})},\ T_l^{(\pm1)}\otimes T_{l'}^{(\mp\mathrm{i})},\ \ l=0,1,2,\cdots,\ l'=\tfrac12,\tfrac32,\tfrac52,\cdots,\\
&T_l^{(\pm\mathrm{i})}\otimes T_{l'}^{(\pm1)},\ T_l^{(\pm\mathrm{i})}\otimes T_{l'}^{(\mp1)},\ \ l=\tfrac12,\tfrac32,\tfrac52,\cdots,\ l'=0,1,2,\cdots,\\
&T_l^{(\pm\mathrm{i})}\otimes T_{l'}^{(\pm\mathrm{i})},\ T_l^{(\pm\mathrm{i})}\otimes T_{l'}^{(\mp\mathrm{i})},\ l,l'=\tfrac12,\tfrac32,\tfrac52,\cdots,
\end{aligned}$$

can be extended to the algebra $\widehat{U}_q(\mathrm{sl}_2)$. Taking into account the decompositions of tensor products of irreducible representations of $U_q(\mathrm{sl}_2)$ (see, for example, the end of Subsection 3.2.1 and Proposition 3.22 in [17]) we find that

$$T_l^{(\omega)}\otimes T_{l'}^{(\omega')}\simeq T_{l+l'}^{(\omega\omega')}\oplus T_{l+l'-1}^{(\omega\omega')}\oplus\cdots\oplus T_{|l+l'|}^{(\omega\omega')},\tag{11.1}$$

$$T_l^{(\omega)}\otimes T_{l'}^{(\pm\mathrm{i})}\simeq T_{l+l'}^{(\pm\omega\mathrm{i})}\oplus T_{l+l'-1}^{(\pm\omega\mathrm{i})}\oplus\cdots\oplus T_{|l+l'|}^{(\pm\omega\mathrm{i})},\tag{11.2}$$

$$T_l^{(\pm\mathrm{i})}\otimes T_{l'}^{(\omega)}\simeq T_{l+l'}^{(\pm\omega\mathrm{i})}\oplus T_{l+l'-1}^{(\pm\omega\mathrm{i})}\oplus\cdots\oplus T_{|l+l'|}^{(\pm\omega\mathrm{i})},\tag{11.3}$$

$$T_l^{(\omega\mathrm{i})}\otimes T_{l'}^{(\omega'\mathrm{i})}\simeq T_{l+l'}^{(-\omega\omega')}\oplus T_{l+l'-1}^{(-\omega\omega')}\oplus\cdots\oplus T_{|l+l'|}^{(-\omega\omega')},\tag{11.4}$$

where $\omega,\omega'=\pm1$.

Now we define tensor products of representations of $U_q(\mathrm{so}_3)$ corresponding to the above tensor product representations of $\widehat{U}_q(\mathrm{sl}_2)$ as

$$R\otimes R'=(T\otimes T')\circ\psi,$$

where $R=T\circ\psi$ and $R'=T'\circ\psi$. Taking into account the definitions of tensor products of representations of $U_q(\mathrm{sl}_2)$ by means of the comultiplication and the definition of the mapping ψ we have

$$(R\otimes R')(I_1)=(T\otimes T')\circ\psi(I_1)=\tfrac{\mathrm{i}}{\mathrm{q}-\mathrm{q}^{-1}}\left(T(q^H)\otimes T'(q^H)\Leftrightarrow T(q^{-H})\otimes T'(q^{-H})\right).$$

Similarly,

$$\begin{aligned}
&(R\otimes R')(I_2)=(T(E)\otimes T'(q^H)+T(q^{-H})\otimes T'(E)\Leftrightarrow T(F)\otimes T'(q^H)\Leftrightarrow\\
&\Leftrightarrow T(q^{-H})\otimes T'(F))\cdot(T(q^H)\otimes T'(q^H)+T(q^{-H})\otimes T'(q^{-H}))^{-1}.
\end{aligned}$$

Composing both sides of the relations (11.1)--(11.4) with the mapping ψ of 9.1, we find the decomposition into representations of $U_q(\mathrm{so}_3)$ for the tensor products

$$R_l^{(1)}\otimes R_{l'}^{(1)},\quad R_l^{(1)}\otimes R_{l'}^{(\pm \mathrm{i})},\quad R_{l'}^{(\pm \mathrm{i})}\otimes R_l^{(1)},\quad R_l^{(\pm \mathrm{i})}\otimes R_{l'}^{(\pm \mathrm{i})},\quad R_l^{(\pm \mathrm{i})}\otimes R_{l'}^{(\mp \mathrm{i})},$$

where the second and the third tensor products are defined only for $l=0,1,2,\cdots$. (Note that the representations $R_l^{(\pm \mathrm{i})}$ are defined only for $l=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots$.) We have

$$R_l^{(1)}\otimes R_{l'}^{(1)}\simeq R_{l+l'}^{(1)}\oplus R_{l+l'-1}^{(1)}\oplus\cdots\oplus R_{|l-l'|}^{(1)},$$
$$R_l^{(1)}\otimes R_{l'}^{(\pm \mathrm{i})}\simeq R_{l+l'}^{(\pm \mathrm{i})}\oplus R_{l+l'-1}^{(\pm \mathrm{i})}\oplus\cdots\oplus R_{|l-l'|}^{(\pm \mathrm{i})},$$
$$R_l^{(\pm \mathrm{i})}\otimes R_{l'}^{(1)}\simeq R_{l+l'}^{(\pm \mathrm{i})}\oplus R_{l+l'-1}^{(\pm \mathrm{i})}\oplus\cdots\oplus R_{|l-l'|}^{(\pm \mathrm{i})},$$
$$R_l^{(\omega \mathrm{i})}\otimes R_{l'}^{(\omega' \mathrm{i})}\simeq R_{l+l'}^{(1)}\oplus R_{l+l'-1}^{(1)}\oplus\cdots\oplus R_{|l-l'|}^{(1)}.$$

In these formulas the representations $R_l^{(\pm \mathrm{i})}$ are reducible. Unfortunately, our definition of tensor products of representations of $U_q(\mathrm{so}_3)$ does not allow to determine the tensor products containing the irreducible representations $R_n^{(\pm \mathrm{i},\pm)}$ and $R_n^{(\pm \mathrm{i},\mp)}$.

12. Infinite dimensional representations obtained using algebra homomorphism

By using the homomorphism ψ: $U_q(\mathrm{so}_3)\to \widehat{U}_q(\mathrm{sl}_2)$ from 9.1 and infinite dimensional irreducible representations of the algebra $\widehat{U}_q(\mathrm{sl}_2)$ we can construct infinite dimensional irreducible representations of the algebra $U_q(\mathrm{so}_3)$.

Let us first describe irreducible infinite dimensional representations of the algebra $U_q(\mathrm{sl}_2)$. Note that by an infinite dimensional representation T of $U_q(\mathrm{sl}_2)$ we mean a homomorphism of $U_q(\mathrm{sl}_2)$ into the algebra of linear operators (bounded or unbounded) on a Hilbert space, defined on an everywhere dense invariant subspace D, such that the operator $T(q^H)$ can be diagonalized, has a discrete spectrum and its eigenvectors belong to D. Infinite dimensional representations T of $U_q(\mathrm{so}_3)$ are described in the same way replacing the operator $T(q^H)$ by $T(I_1)$.

Two representations T and T' of $U_q(\mathrm{sl}_2)$ on spaces H and H', respectively, are called (algebraically) equivalent if there exist everywhere dence invariant subspaces $V\subset H$ and $V'\subset H'$ and a one-to-one linear operator A: $V\to V'$ such that $AT(a)v=T'(a)Av$ for all $a\in U_q(\mathrm{sl}_2)$ and $v\in V$. Equivalence of infinite dimensional representations of $U_q(\mathrm{so}_3)$ is defined in the same way.

Let ϵ be a fixed complex number such that $0\le \mathrm{Re}\,\epsilon<1$, and let H_ϵ be a complex Hilbert space with the orthonormal basis

$$x_m,\quad m=n+\epsilon,\ n=0,\pm 1,\pm 2,\cdots. \tag{12.1}$$

For every complex number a we construct the representation $T_{a\epsilon}$ on the Hilbert space H_ϵ defined by

$$T_{a\epsilon}(q^H)x_m=q^m x_m,\ T_{a\epsilon}(E)x_m=[a-m]x_{m+1},\ T_{a\epsilon}(F)x_m=[a+m]x_{m-1},$$

where $[a\pm m]$ is the q-number (see, for example, Ref. [18]). The equivalence relations in the set of the representations $T_{a\epsilon}$ can be extracted from Ref. [18].

Note that the representation $T_{a\epsilon}$ is irreducible if and only if $a\ne\pm\epsilon(\mathrm{mod}\,\mathbb{Z})$.

All the representations $T_{a\epsilon}$ can be extended to representations of the algebra $\widehat{U}_q(\mathrm{sl}_2)$ except for the case when $\epsilon=\pm \mathrm{i}\pi/2\tau$, where $q=e^\tau$. (We suppose below that $\epsilon\ne\pm \mathrm{i}\pi/2\tau$.) We denote these extended representations by the same symbols $T_{a\epsilon}$.

The formula $R_{a\epsilon}=T_{a\epsilon}\circ\psi$ associates with every irreducible representation $T_{a\epsilon}$, $\epsilon\ne\pm \mathrm{i}\pi/2\tau$, of $\widehat{U}_q(\mathrm{sl}_2)$ a representation of the algebra $U_q(\mathrm{so}_3)$.

Let $\epsilon \neq \pm \mathrm{i}\pi/2\tau$ and $\epsilon \neq \pm \mathrm{i}\pi/2\tau+\frac{1}{2}$. Then for the representations $R_{a\epsilon}$ of $U_q(\mathrm{so}_3)$ we have

$$R_{a\epsilon}(I_1)x_m = \mathrm{i}[m]x_m, \quad (12.2)$$

$$R_{a\epsilon}(I_2)x_m = \frac{1}{q^m+q^{-m}}\{[a \Leftrightarrow m]x_{m+1} \Leftrightarrow [a+m]x_{m-1}\}, \quad (12.3)$$

$$R_{a\epsilon}(I_3)x_m = \frac{\mathrm{i}q^{1/2}}{q^m+q^{-m}}\{q^m[a \Leftrightarrow m]x_{m+1}+q^{-m}[a+m]x_{m-1}\}. \quad (12.4)$$

If $\epsilon = \mathrm{i}\pi/2\tau+\frac{1}{2}$, then denoting the basis elements x_m, $m=n+\epsilon$, $n\in\mathbb{Z}$, by $x_{n+\frac{1}{2}}$, $n\in\mathbb{Z}$, respectively, we obtain

$$R_{a\epsilon}(I_1)x_k = \Leftrightarrow \frac{q^k+q^{-k}}{q-q^{-1}}x_k,$$

$$R_{a\epsilon}(I_2)x_k = \mathrm{i}\frac{[a'-k]}{q^k-q^{-k}}x_{k+1}+\mathrm{i}\frac{[a'+k]}{q^k-q^{-k}}x_{k-1},$$

$$R_{a\epsilon}(I_3)x_k = \Leftrightarrow \frac{\mathrm{i}q^{k+1/2}[a'-k]}{q^k-q^{-k}}x_{k+1} \Leftrightarrow \frac{\mathrm{i}q^{-k+1/2}[a'+k]}{q^k-q^{-k}}x_{k-1},$$

where $a' = a+\mathrm{i}\pi/2\tau$ and $k=n+\frac{1}{2}$. If $\epsilon = \Leftrightarrow \mathrm{i}\pi/2\tau+\frac{1}{2}$, then using the same notations for basis elements we obtain

$$R'_{a\epsilon}(I_1)x_k = \frac{q^k+q^{-k}}{q-q^{-1}}x_k,$$

$$R'_{a\epsilon}(I_2)x_k = \Leftrightarrow \mathrm{i}\frac{[a'-k]}{q^k-q^{-k}}x_{k+1} \Leftrightarrow \mathrm{i}\frac{[a'+k]}{q^k-q^{-k}}x_{k-1},$$

$$R'_{a\epsilon}(I_3)x_k = \Leftrightarrow \frac{\mathrm{i}q^{k+1/2}[a'-k]}{q^k-q^{-k}}x_{k+1} \Leftrightarrow \frac{\mathrm{i}q^{-k+1/2}[a'+k]}{q^k-q^{-k}}x_{k-1}$$

(to distinquish these representations from the previous ones we supplied $R_{a\epsilon}$ by prime).

12.1. **Theorem.** The representations $R_{a\epsilon}$ of $U_q(\mathrm{so}_3)$ are irreducible for irreducible representations $T_{a\epsilon}$, $\epsilon \neq \pm\mathrm{i}\pi/2\tau+\frac{1}{2}$, of $\widehat{U}_q(\mathrm{sl}_2)$. The representations $R_{a\epsilon}$, $\epsilon = \mathrm{i}\pi/2\tau+\frac{1}{2}$, and $R'_{a\epsilon}$, $\epsilon = \Leftrightarrow \mathrm{i}\pi/2\tau+\frac{1}{2}$, are reducible.

Proof. is given in the same way as in the case of 10.1.

As in the case of finite dimensional representations in previous chapter, decomposing the representations $R_{a\epsilon}$, $\epsilon = \pm\mathrm{i}\pi/2\tau+\frac{1}{2}$, we obtain irreducible infinite dimensional representations of $U_q(\mathrm{so}_3)$ which will be denoted by $R^{(\mathrm{i},\pm)}_{a'}$ and $R^{(-\mathrm{i},\pm)}_{a'}$, $a' = a+\mathrm{i}\pi/2\tau$. In the basis

$$x_n, \quad n = 1,2,3,\cdots,$$

they are given by the formulas

$$R^{(\mathrm{i},\pm)}_{a'}(I_1)x_k = \Leftrightarrow \frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}x_k,$$

$$R^{(\mathrm{i},\pm)}_{a'}(I_2)x_1 = \pm\frac{[a']}{q^{1/2}-q^{-1/2}}x_1+\mathrm{i}\frac{[a'-1]}{q^{1/2}-q^{-1/2}}x_2,$$

$$R^{(\mathrm{i},\pm)}_{a'}(I_2)x_k = \mathrm{i}\frac{[a'-k]}{q^{k-1/2}-q^{-k+1/2}}x_{k+1}+\mathrm{i}\frac{[a'+k-1]}{q^{k-1/2}-q^{-k+1/2}}x_{k-1}, \quad k\neq 1.$$

$$R^{(\mathrm{i},\pm)}_{a'}(I_3)x_1 = \mp\frac{[a']}{q^{1/2}-q^{-1/2}}x_1 \Leftrightarrow \mathrm{i}\frac{q[a'-1]}{q^{1/2}-q^{-1/2}}x_2,$$

$$R^{(\mathrm{i},\pm)}_{a'}(I_3)x_k = \Leftrightarrow \mathrm{i}\frac{q^k[a'-k]}{q^{k-1/2}-q^{-k+1/2}}x_{k+1} \Leftrightarrow \mathrm{i}\frac{q^{-k+1}[a'+k-1]}{q^{k-1/2}-q^{-k+1/2}}x_{k-1}, \quad k\neq 1.$$

and by the formulas

$$R^{(-\mathrm{i},\pm)}_{a'}(I_1)x_k = \frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}x_k, \quad R^{(-\mathrm{i},\pm)}_{a'}(I_2) = \Leftrightarrow R^{(\mathrm{i},\pm)}_{a'}(I_2),$$

$$R^{(-\mathrm{i},\pm)}_{a'}(I_3)x_1 = \pm\frac{[a']}{q^{1/2}-q^{-1/2}}x_1+\mathrm{i}\frac{q[a'-1]}{q^{1/2}-q^{-1/2}}x_2,$$

$$R^{(-\mathrm{i},\pm)}_{a'}(I_3)x_k = \mathrm{i}\frac{q^k[a'-k]}{q^{k-1/2}-q^{-k+1/2}}x_{k+1}+\mathrm{i}\frac{q^{-k+1}[a'+k-1]}{q^{k-1/2}-q^{-k+1/2}}x_{k-1}, \quad k\neq 1.$$

12.2. **Theorem.** The representations $R^{(\mathrm{i},\pm)}_{a'}$ $R^{(-\mathrm{i},\pm)}_{a'}$ are irreducible and pairwise

nonequivalent. For any a the irreducible representation $R_{a\epsilon}$ is not equivalent to some of these representations.
Proof. is given in the same way as in the finite dimensional case (see the proof of 10.2).
The algebra $U_q(\mathrm{sl}_2)$ has also irreducible infinite dimensional representations with highest weights or with lowest weights. They are classified in Ref. [18]. All of these representations T can be extended to the algebra $\widehat{U}_q(\mathrm{sl}_2)$. Using the composition $R = T\circ\psi$ we obtain the corresponding representations R of $U_q(\mathrm{so}_3)$. As above, it can be easily proved that to nonequivalent representations T of $\widehat{U}_q(\mathrm{sl}_2)$ with highest or lowest weight there correspond nonequivalent irreducible representations of $U_q(\mathrm{so}_3)$. We give a list of these representations.
Let $l = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$. We denote by R_l^+ the representation of $U_q(\mathrm{so}_3)$ acting on the Hilbert space H_l with the orthonormal base vectors x_m, $m = l, l+1, l+2, \cdots$, and given by formulas (12.2)--(12.4) with $a = \Leftrightarrow l$. By R_l^- we denote the representation of $U_q(\mathrm{so}_3)$ acting on the Hilbert space $\widehat{H}_l$ with the orthonormal base vectors x_m, $m = \Leftrightarrow l, \Leftrightarrow l \Leftrightarrow 1, \Leftrightarrow l \Leftrightarrow 2, \cdots$, and given by formulas (12.2)--(12.4) with $a = l$.
Now let $a \ne 0 \ (\mathrm{mod}\mathbb{Z})$ and $a \ne \frac{1}{2} \ (\mathrm{mod}\mathbb{Z})$. We denote by H_a the Hilbert space with the orthonormal basis x_m, $m = \Leftrightarrow a, \Leftrightarrow a+1, \Leftrightarrow a+2, \cdots$. On this space the representation R_a^+ acts which is given by formulas (12.2)--(12.4). On the Hilbert space $\widehat{H}_a$ with the orthonormal basis x_m, $m = a, a \Leftrightarrow 1, a \Leftrightarrow 2, \cdots$, the representation R_a^- acts which is given by formulas (12.2)--(12.4).
12.3. **Theorem.** The above representations $R_l^{\pm}$ and $R_a^{\pm}$ are irreducible and pairwise nonequivalent.
Proof. of this proposition is contained in Ref. [9].

13. Other infinite dimensional representations

The algebra $U_q(\mathrm{so}_3)$ has also irreducible infinite dimensional representations which cannot be obtained from representations of $\widehat{U}_q(\mathrm{sl}_2)$. We describe these representations in this section.
Let H be the infinite dimensional vector space with the basis x_m, $m = 0, \pm 1, \pm 2, \cdots$, and let $\lambda = q^\tau$ be a nonzero complex number such that $0 \le \mathrm{Re}\,\tau < 1$. Then a direct calculation shows that the operators $Q_\lambda^+(I_1)$ and $Q_\lambda^+(I_2)$ given by the formulas

$$Q_\lambda^+(I_1)x_m = \frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}} x_m, \qquad (13.1)$$
$$Q_\lambda^+(I_2)x_m = \frac{1}{q-q^{-1}} x_{m+1} + \frac{1}{q-q^{-1}} x_{m-1}$$

satisfy the relations (8.1) and (8.2) and hence determine a representation of $U_q(\mathrm{so}_3)$ which will be denoted by Q_λ^+. Similarly, the operators $Q_\lambda^-(I_1)$ and $Q_\lambda^-(I_2)$ given on the space H by

$$Q_\lambda^-(I_1)x_m = \Leftrightarrow \frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}} x_m, \quad Q_\lambda^-(I_2) := Q_\lambda^+(I_2)$$

determine a representation of $U_q(\mathrm{so}_3)$ which is denoted by Q_λ^-. The operators $Q_\lambda^{\pm}(I_3)$ can be calculated by means of formula (1.2).

13.1. **Theorem.** If $\lambda \ne 1$ and $\lambda \ne q^{1/2}$, then the representations Q_λ^+ and Q_λ^- are irreducible. The representations $Q_1^{\pm}$ and $Q_{\sqrt{q}}^{\pm}$ are reducible.
Proof. The first part is proved in the same way as that of 10.1. Let us prove the second part. The representations $Q_1^{\pm}$ and $Q_{\sqrt{q}}^{\pm}$ are the only representations in the set $\{Q_\lambda^{\pm}\}$ for

which the operator $Q_\lambda^\pm(I_1)$ has not a simple spectrum. The operators $Q_1^\pm(I_1)$ has the spectrum

$$\cdots,\ q^{-2}+q^2,\ q^{-1}+q,\ 2,\ q+q^{-1},\ q^2+q^{-2},\ \cdots.$$

Thus, only the spectral point 2 has multiplicity 1. All other points have multiplicity 2. Let V_1 and V_2 be the vector subspaces of H with the bases

$$x_0,\ \ x'_m = x_m \Leftrightarrow x_{-m},\ \ m=1,2,\cdots,$$

and

$$x''_m = x_m + x_{-m},\ \ m=1,2,\cdots,$$

respectively. These basis vectors are eigenvectors of the operator $Q_1^\pm(I_1)$:

$$Q_1^\pm(I_1)x'_m = \pm\tfrac{q^m+q^{-m}}{q-q^{-1}}x'_m,\quad Q_1^\pm(I_1)x''_m = \pm\tfrac{q^m+q^{-m}}{q-q^{-1}}x''_m,$$

and

$$Q_1^\pm(I_2)x_0 = \tfrac{1}{q-q^{-1}}x'_1,\quad Q_1^\pm(I_2)x''_1 = \tfrac{1}{q-q^{-1}}x''_2,$$
$$Q_1^\pm(I_2)x'_m = \tfrac{1}{q-q^{-1}}x'_{m+1} + \tfrac{1}{q-q^{-1}}x'_{m-1},\quad m>0,$$
$$Q_1^\pm(I_2)x''_m = \tfrac{1}{q-q^{-1}}x''_{m+1} + \tfrac{1}{q-q^{-1}}x''_{m-1},\quad m>1.$$

Thus, the subspaces V_1 and V_2 are invariant with respect to the representation Q_1^+ (and the representation Q_1^-). We denote the subrepresentations of $Q_1^\pm$ realized on V_1 and V_2 by $Q_1^{1,\pm}$ and $Q_1^{2,\pm}$, respectively.
The eigenvalues of the operators $Q_{\sqrt{q}}^\pm(I_1)$ are

$$\cdots,\ q^{-3/2}+q^{3/2},\ q^{-1/2}+q^{1/2},\ q^{1/2}+q^{-1/2},\ q^{3/2}+q^{-3/2},\ \cdots.$$

Thus, every spectral point has multiplicity 2. We denote by W_1 and W_2 the vector subspaces of H spanned by the basis vectors

$$x'_{1/2} = x_0 \Leftrightarrow x_{-1},\ x'_{3/2} = x_1 \Leftrightarrow x_{-2},\cdots, x'_{m+\frac12} = x_m \Leftrightarrow x_{-m-1},\cdots \tag{13.2}$$

and

$$x''_{1/2} = x_0 + x_{-1},\ x''_{3/2} = x_1 + x_{-2},\cdots, x''_{m+\frac12} = x_m + x_{-m-1},\cdots, \tag{13.3}$$

respectively. These basis vectors are eigenvectors of the operator $Q_{\sqrt{q}}^\pm(I_1)$:

$$Q_{\sqrt{q}}^\pm(I_1)x'_{m+\frac12} = \pm\frac{q^{m+1/2}+q^{-m-1/2}}{q-q^{-1}}x'_{m+\frac12}, \tag{13.4}$$
$$Q_{\sqrt{q}}^\pm(I_1)x''_{m+\frac12} = \pm\frac{q^{m+1/2}+q^{-m-1/2}}{q-q^{-1}}x''_{m+\frac12} \tag{13.5}$$

and

$$Q_{\sqrt{q}}^\pm(I_2)x'_{\frac12} = \Leftrightarrow\tfrac{1}{q-q^{-1}}x'_{\frac12} + \tfrac{1}{q-q^{-1}}x'_{\frac32}, \tag{13.6}$$
$$Q_{\sqrt{q}}^\pm(I_2)x'_{m+\frac12} = \tfrac{1}{q-q^{-1}}x'_{m+\frac32} + \tfrac{1}{q-q^{-1}}x'_{m-\frac12},\quad m>0, \tag{13.7}$$
$$Q_{\sqrt{q}}^\pm(I_2)x''_{\frac12} = \tfrac{1}{q-q^{-1}}x''_{\frac12} + \tfrac{1}{q-q^{-1}}x''_{\frac32}, \tag{13.8}$$
$$Q_{\sqrt{q}}^\pm(I_2)x''_{m+\frac12} = \tfrac{1}{q-q^{-1}}x''_{m+\frac32} + \tfrac{1}{q-q^{-1}}x''_{m-\frac12},\quad m>0. \tag{13.9}$$

Thus, the subspaces W_1 and W_2 are invariant with respect to the representations $Q_{\sqrt{q}}^\pm$. We denote the subrepresentations of $Q_{\sqrt{q}}^\pm$ realized on W_1 and W_2 by $Q_{\sqrt{q}}^{1,\pm}$ and $Q_{\sqrt{q}}^{2,\pm}$, respectively. Proposition is proved.
13.2. **Theorem.** The representations $Q_1^{1,\pm}$, $Q_1^{2,\pm}$, $Q_{\sqrt{q}}^{1,\pm}$ and $Q_{\sqrt{q}}^{2,\pm}$ are irreducible and

pairwise nonequivalent. For any admissible value of λ the representation Q_λ^+ (as well as the representation Q_λ^-) is not equivalent to some of these representations.
Proof. Proof is similar to that of 10.2 if to take into account spectra of the operators $Q_1^{1,\pm}(I_1)$, $Q_1^{2,\pm}(I_1)$, $Q_{\sqrt{q}}^{1,\pm}(I_1)$, $Q_{\sqrt{q}}^{2,\pm}(I_1)$, $Q_\lambda^{\pm}(I_1)$ and traces of the operators $Q_1^{1,\pm}(I_2)$, $Q_1^{2,\pm}(I_2)$, $Q_{\sqrt{q}}^{1,\pm}(I_2)$, $Q_{\sqrt{q}}^{2,\pm}(I_2)$.

14. Finite dimensional representations at root of unity

Everywhere below q is a root of unity, that is, there is a smallest positive integer p such that $q^p=1$. We suppose that $p\neq 1,2$. We introduce the number p' setting $p'=p$ if p is odd and $p'=p/2$ if p is even.
As in the case of the algebra $U_q(\mathrm{sl}_2)$ (see Ref. [17], Chapter 3), if q is a root of unity, then we claim $U_q(\mathrm{so}_3)$ is a finite dimensional vector space over the center of $U_q(\mathrm{so}_3)$. If q is a primitive root of unity, then this assertion is stated in Ref. [14]. If q is any root of unity, then this assertion may be proved in the following way. If $q^p=1$, then we claim that the center C of $U_q(\mathrm{so}_3)$ contains the elements

$$P_p=I_j^p+aI_j^{p-2}+bI_j^{p-4}+\cdots+dI_j^r,\quad j=1,2,3,$$

where $r=0$ if p is even and $r=1$ if p is odd and $a,b,\cdots,d$ are certain fixed complex numbers expressed in terms of q. (They are the polynomials P defined in Ref. [14] if q is a primitive root of unity. Unfortunately, we could not find the explicit expressions for the coefficients $a,b,\cdots,d$. But note that $P_3=I_j^3+I_j$, $P_4=I_j^4+I_j^2$ and $P_5=I_j^5+(1+(q+q^{-1})^{-1}I_j^3+(q+q^{-1})^{-1}I_j$.) Therefore, I_j^s, $s>n$, can be reduced to the linear combination of I_j^i, $i<n$, with coefficients from the center C. Now our assertion follows from this and from Poincaré--Birkhoff--Witt theorem for $U_q(\mathrm{so}_3)$. Thus, following seems to be true:
14.1. **Theorem.** If q is a root of unity, then any irreducible representation of $U_q(\mathrm{so}_3)$ is finite dimensional.
Proof. (using the claim) Let T be an irreducible representation of $U_q(\mathrm{so}_3)$. Then T maps central elements into scalar operators. Since according to our claim the linear space $U_q(\mathrm{so}_3)$ seems to be finite dimensional over the center C with the basis $I_1^kI_2^mI_3^n$, $k,m,n<p$, then for any $a\in U_q(\mathrm{so}_3)$ we have $T(a)=\sum_{k,m,n<p} T(I_1^kI_2^mI_3^n)$. Hence, if $\mathbf{v}$ is a nonzero vector of the representation space $\mathcal{V}$, then $T(U_q(\mathrm{so}_3))\mathbf{v}=\mathcal{V}$ and $\mathcal{V}$ is finite dimensional. If our claim is valid, theorem is proved.
Taking into account 14.1, below we consider only finite dimensional representations of $U_q(\mathrm{so}_3)$.
In order to find irreducible representations of $U_q(\mathrm{so}_3)$ for q a root of unity, we use the same method as before, that is, we apply the homomorphism ψ from 9.1 and irreducible representations of the algebra $\hat{U}_q(\mathrm{sl}_2)$ for q a root of unity.

Let us find irreducible representations of $\hat{U}_q(\mathrm{sl}_2)$ for q a root of unity. The quantum algebra $U_q(\mathrm{sl}_2)$ for q a root of unity has the following irreducible representations (see Ref. [17], Subsection 3.3.2):

(a) The representations $T_l^{(1)}$, $T_l^{(-1)}$, $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $2l<p'$, given by the formulas (8.8)--(8.12).
(b) The representations $T_{ab\lambda}$, $a,b,\lambda\in\mathbb{C}$, $\lambda\neq 0$, acting on a p'-dimensional vector space H with the basis x_j, $j=0,1,2,\cdots,p'-1$, and given by the formulas

$$T_{ab\lambda}(q^H)x_i = q^{-i}\lambda x_i, \quad T_{ab\lambda}(F)x_{p'-1} = bx_0, \tag{14.1}$$
$$T_{ab\lambda}(F)x_i = x_{i+1},\ i < p' \Leftrightarrow 1, \quad T_{ab\lambda}(E)x_0 = ax_{p'-1}, \tag{14.2}$$
$$T_{ab\lambda}(E)x_i = \left(ab + [i]\frac{\lambda^2 q^{1-i} - \lambda^{-2}q^{i-1}}{q-q^{-1}}\right)x_{i-1},\ \ i>0. \tag{14.3}$$

The representations $T_{ab\lambda}$ with $(a,b)=(0,0)$ and $\lambda=\pm q^n$, $n=0,1,2,\cdots,p'\Leftrightarrow 2$, are reducible and must be taken out from this set.
(c) The representations $T'_{0b\lambda}$, $b,\lambda\in\mathbb{C}$, $\lambda\neq 0$, acting on a p'-dimensional vector space H with the basis x_j, $j=0,1,2,\cdots,p'\Leftrightarrow 1$, and given by the formulas

$$T'_{0b\lambda}(q^H)x_i = q^i\lambda^{-1}x_i, \quad T'_{0b\lambda}(E)x_{p'-1} = bx_0, \tag{14.4}$$
$$T'_{0b\lambda}(E)x_i = x_{i+1},\ i<p'\Leftrightarrow 1, \quad T_{0b\lambda}(F)x_0 = 0, \tag{14.5}$$
$$T'_{0b\lambda}(F)x_i = [i]\frac{\lambda^2 q^{1-i} - \lambda^{-2}q^{i-1}}{q-q^{-1}}x_{i-1},\ \ i>0. \tag{14.6}$$

The representations $T'_{00\lambda}$ with $\lambda=\pm q^n$, $n=0,1,2,\cdots,p'\Leftrightarrow 2$, are reducible and must be taken out from this set.
14.2. **Note.** In the set of representations (a)--(c) there exist equivalent representations (see, for example, Propositions 3.17 and 3.18 in Ref. [17]).
14.3. **Note.** In Ref. [17], Subsection 3.3.2, irreducible representations of the algebra generated by the elements $E,F,K:=q^{2H},K^{-1}:=q^{-2H}\in U_q(\mathrm{sl}_2)$ are given. Clearly, this algebra is a subalgebra in $U_q(\mathrm{sl}_2)$. It is easy to generalize the results of Subsection 3.3.2 in Ref. [17] for $U_q(\mathrm{sl}_2)$. Let us note that the algebra $U_q(\mathrm{sl}_2)$ has a unique automorphism φ such that $\varphi(q^H)=\mathrm{i}q^H$, $\varphi(E)=\Leftrightarrow E$ and $\varphi(F)=F$. (If q is not a root of unity, then this automorphism transforms the representations $T_l^{(1)}$ to the representations $T_l^{(\mathrm{i})}$, respectively.) Therefore, the mapping $\tilde{T}_{-a,b,\lambda}=T_{ab\lambda}\circ\varphi$ is also a representation of $U_q(\mathrm{sl}_2)$. We have

$$\tilde{T}_{ab\lambda}(q^H)x_i = \mathrm{i}q^{-i}\lambda x_i, \quad \tilde{T}_{ab\lambda}(F)x_{p'-1} = bx_0, \tag{14.7}$$
$$\tilde{T}_{ab\lambda}(F)x_i = x_{i+1},\ i<p'\Leftrightarrow 1, \quad \tilde{T}_{ab\lambda}(E)x_0 = ax_{p'-1}, \tag{14.8}$$
$$\tilde{T}_{ab\lambda}(E)x_i = \left(ab \Leftrightarrow [i]\frac{\lambda^2 q^{1-i} - \lambda^{-2}q^{i-1}}{q-q^{-1}}\right)x_{i-1},\ \ i>0. \tag{14.9}$$

However, it is easy to see by comparing (14.1)--(14.3) with (14.7)--(14.9) that the representation $\tilde{T}_{ab\lambda}$ is equivalent to $T_{a,b,\mathrm{i}\lambda}$. This means that for q a root of unity we do not obtain new representations of $U_q(\mathrm{sl}_2)$ from $T_{ab\lambda}$ applying the automorphism φ as in the case of the representations $T_l^{(1)}$.
We have described irreducible representations of the algebra $U_q(\mathrm{sl}_2)$. Now we wish to extend these representations to obtain representations of the algebra $\widehat{U}_q(\mathrm{sl}_2)$ by using the relation

$$T((q^kq^H+q^{-k}q^{-H})^{-1}) := (q^kT(q^H)+q^{-k}T(q^{-H}))^{-1}.$$

Clearly, only those irreducible representations T of $U_q(\mathrm{sl}_2)$ can be extended to $\widehat{U}_q(\mathrm{sl}_2)$ for which the operators $q^kT(q^H)+q^{-k}T(q^{-H})$ are invertible. From formulas (8.8)--(8.12) it is clear that these operators are always invertible for the irreducible representations $T_l^{(1)}$, $T_l^{(-1)}$, $l=0,\frac{1}{2},1,\frac{3}{2},\cdots,\frac{p'-1}{2}$, and for the irreducible representations $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots,\frac{p'-1}{2}$ (or $\frac{p'-2}{2}$). (For the representations $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l=0,1,2,\cdots$, some of these operators are not invertible since they have zero eigenvalue.) We denote the extended representations by the same symbols $T_l^{(1)}$, $T_l^{(-1)}$, $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, respectively. Similarly, the representation $T_{ab\lambda}$ (and the representation $T'_{0b\lambda}$) can be extended to a representation of the algebra $\widehat{U}_q(\mathrm{sl}_2)$ if and only if $\lambda \neq \pm \mathrm{i} q^k$, $k \in \mathbb{Z}$.

14.4. **Theorem.** The algebra $\widehat{U}_q(\mathrm{sl}_2)$ for q a root of unity has the irreducible representations $T_l^{(1)}$, $T_l^{(-1)}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{p'-1}{2}$, the irreducible representations $T_l^{(\mathrm{i})}$, $T_l^{(-\mathrm{i})}$, $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \frac{p'-1}{2}$ (or $\frac{p'-2}{2}$), and the irreducible representations $T_{ab\lambda}$, $T'_{0b\lambda}$, $\lambda \neq \pm \mathrm{i} q^k$, $k \in \mathbb{Z}$. Any irreducible representation of $\widehat{U}_q(\mathrm{sl}_2)$ for q a root of unity is equivalent to one of these representations.

15. Representations at root of unity obtained using homomorphism

As in previous sections, we shall obtain representations of $U_q(\mathrm{so}_3)$ for q a root of unity by applying the homomorphism ψ from 9.1. Namely, if T is a representation of $\widehat{U}_q(\mathrm{sl}_2)$, then

$$R = T \circ \psi \qquad (15.1)$$

is a representation of $U_q(\mathrm{so}_3)$. As in previous sections, application of this method to the pair of the irreducible representations $T_l^{(1)}$ and $T_l^{(-1)}$ of $\widehat{U}_q(\mathrm{sl}_2)$ leads to the same representation of $U_q(\mathrm{so}_3)$ which will be denoted by $R_l^{(1)}$. Applying the formula (15.1) to the irreducible representations $T_l^{(\mathrm{i})}$ and $T_l^{(-\mathrm{i})}$ of $\widehat{U}_q(\mathrm{sl}_2)$ give the representations of $U_q(\mathrm{so}_3)$ which will be denoted by $R_l^{(\mathrm{i})}$ and $R_l^{(-\mathrm{i})}$, respectively.

vetaproposition8 The representations $R_l^{(1)}$ of $U_q(\mathrm{so}_3)$ are irreducible. The representations $R_l^{(\mathrm{i})}$ and $R_l^{(-\mathrm{i})}$ are reducible.

Proof. of this proposition is the same as that of 10.1.

Repeating word-by-word the reasoning of previous sections, we decompose the representations $R_l^{(\mathrm{i})}$ and $R_l^{(-\mathrm{i})}$ into the direct sums of representations of $U_q(\mathrm{so}_3)$ which are denoted by $R_n^{(\pm\mathrm{i},+)}$ and $R_n^{(\pm\mathrm{i},-)}$:

$$R_l^{(\mathrm{i})} = R_n^{(\mathrm{i},+)} \oplus R_n^{(\mathrm{i},-)}, \quad R_l^{(-\mathrm{i})} = R_n^{(-\mathrm{i},+)} \oplus R_n^{(-\mathrm{i},-)}, \quad n = l + \tfrac{1}{2}.$$

Moreover, the representations $R_n^{(\pm\mathrm{i},+)}$ and $R_n^{(\pm\mathrm{i},-)}$ are given in the appropriate bases x_1, x_2, $\cdots$, x_n by the corresponding formulas of previous sections.

15.1. **Theorem.** The representations $R_n^{(\mathrm{i},+)}$, $R_n^{(\mathrm{i},-)}$, $R_n^{(-\mathrm{i},+)}$, $R_n^{(-\mathrm{i},-)}$, $n = 1, 2, 3, \cdots, \frac{p'}{2}$ (or $\frac{p'-1}{2}$) are irreducible and pairwise nonequivalent. For any l, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{p'-1}{2}$, the representation $R_l^{(1)}$ is not equivalent to some of these representations.

Proof. is the same as that of 10.2.

Now we apply formula (15.1) to the representations $T_{ab\lambda}$ and $T'_{0b\lambda}$. As a result, we obtain the representations

$$R_{ab\lambda} = T_{-a,b,-\mathrm{i}\lambda} \circ \psi, \quad R'_{0b\lambda} = T'_{0,b,-\mathrm{i}\lambda}$$

given in the bases x_j, $j = 0, 1, 2, \cdots, p' \Leftrightarrow 1$, by the formulas

$$R_{ab\lambda}(I_1) x_i = \frac{-1}{q - q^{-1}} (q^{-i}\lambda + q^i \lambda^{-1}) x_i, \qquad (15.2)$$

$$R_{ab\lambda}(I_2) x_0 = \frac{\mathrm{i}}{\lambda - \lambda^{-1}} (a x_{p'-1} + x_1), \qquad (15.3)$$

$$R_{ab\lambda}(I_2) x_{p'-1} = \frac{\mathrm{i}}{q^{-p'+1}\lambda - q^{p'-1}\lambda^{-1}} \Big\{ b x_0 +$$

$$+\left(ab+[p'\Leftrightarrow 1]\frac{q^{-p'+2}\lambda^2-q^{p'-2}\lambda^{-2}}{q-q^{-1}}\right)x_{p'-2}\Big\},\tag{15.4}$$

$$R_{ab\lambda}(I_2)x_i=\frac{\mathrm{i}}{q^{-i}\lambda-q^i\lambda^{-1}}\Big\{\left(ab+[i]\frac{q^{-i+1}\lambda^2-q^{i-1}\lambda^{-2}}{q-q^{-1}}\right)x_{i-1}+$$

$$+x_{i+1}\Big\},\quad 0<i<p'\Leftrightarrow 1.\tag{15.5}$$

and by the formulas

$$R'_{0b\lambda}(I_1)x_i=\frac{1}{q-q^{-1}}(q^{-i}\lambda+q^i\lambda^{-1})x_i,\quad R'_{0b\lambda}(I_2)x_0=\frac{-\mathrm{i}}{\lambda-\lambda^{-1}}x_1,$$

$$R'_{0b\lambda}(I_2)x_{p'-1}=\frac{-\mathrm{i}}{q^{-p'+1}\lambda-q^{p'-1}\lambda^{-1}}\left(bx_0+[p'\Leftrightarrow 1]\frac{q^{-p'+2}\lambda^2-q^{p'-2}\lambda^{-2}}{q-q^{-1}}x_{p'-2}\right),$$

$$R'_{0b\lambda}(I_2)x_i=\frac{-\mathrm{i}}{q^{-i}\lambda-q^i\lambda^{-1}}\left(x_{i+1}+[i]\frac{q^{-i+1}\lambda^2-q^{i-1}\lambda^{-2}}{q-q^{-1}}x_{i-1}\right),\quad 0<i<p'\Leftrightarrow 1.$$

The operators $R_{ab\lambda}(I_3)$ and $R'_{0b\lambda}(I_3)$ can be calculated by means of the relation

$$R(I_3)=q^{1/2}R(I_1)R(I_2)\Leftrightarrow q^{-1/2}R(I_2)R(I_1).$$

Recall that the representations $R_{ab\lambda}$ and $R'_{0b\lambda}$ are determined for $\lambda\neq 0$ and $\lambda\neq\pm q^k$, $k\in\mathbb{Z}$.

It is seen from the above formulas that

$$R'_{0b\lambda}(I_1)=R_{0,b,-\lambda}(I_1),\quad R'_{0b\lambda}(I_2)=R_{0,b,-\lambda}(I_2),$$

that is, the representations $R_{0,b,-\lambda}$ and $R'_{0b\lambda}$ are equivalent. For this reason, we consider below only the representations $R_{ab\lambda}$.

In order to study the representations $R_{ab\lambda}$ of $U_q(\mathrm{so}_3)$ we consider the spectrum of the operator $R_{ab\lambda}(I_1)$. It coincides with the set of points

$$\Leftrightarrow\frac{\lambda+\lambda^{-1}}{q-q^{-1}},\ \Leftrightarrow\frac{q^{-1}\lambda+q\lambda^{-1}}{q-q^{-1}},\ \Leftrightarrow\frac{q^{-2}\lambda+q^2\lambda^{-1}}{q-q^{-1}},\cdots,\Leftrightarrow\frac{q^{1-p'}\lambda+q^{p'-1}\lambda^{-1}}{q-q^{-1}}.\tag{15.6}$$

It is easy to see that there exist coinciding points in this set if and only if λ is equal to one of the numbers

$$\pm q^{1/2},\ \pm q^{3/2},\ \pm q^{5/2},\cdots,\pm q^{(p'-1)/2}\ (\text{or } \pm q^{(p'-2)/2}).$$

(Here we have to take $\pm q^{(p'-1)/2}$ if p' is even and $\pm q^{(p'-2)/2}$ if p' is odd.) Moreover, the set (15.6) splits into pairs of coinciding points if and only if $\lambda=\pm q^{(p'-1)/2}$. In all other cases there exists at least one spectral point which coincides with no other point. In particular, if $\lambda=\pm q^{(p'-2)/2}$, then in this set there exists only one eigenvalue with multiplicity 1. In all other cases there are more than one eigenvalues with multiplicity 1.

15.2. **Theorem.** If $\lambda\neq\pm q^{(p'-1)/2}$ for even p' and $\lambda\neq\pm q^{(p'-2)/2}$ for odd p', then the representation $R_{ab\lambda}$ is irreducible.

Proof. Let $\lambda\neq\pm q^{(p'-1)/2}$ for even p' and $\lambda\neq\pm q^{(p'-2)/2}$ for odd p'. We distinguish two cases: when the spectrum of the operator $R_{ab\lambda}(I_1)$ is simple and when there exists at list one spectral point of this operator having multiplicity 2. In the first case the proof is the same as the first part of the proof of 10.1. For the second case, we give a proof only for $\lambda=q^{1/2}$. (Proofs for other values of q are similar.) Then in the set (15.6) there are only two coinciding points $\Leftrightarrow\frac{\lambda+\lambda^{-1}}{q-q^{-1}}$ and $\Leftrightarrow\frac{q^{-1}\lambda+q\lambda^{-1}}{q-q^{-1}}$ corresponding to the eigenvectors x_0 and x_1. Let V be an invariant subspace of the representation space H. As in the proof of 10.1, it is shown that V is a linear span of eigenvectors of the operator $R_{ab\lambda}(I_1)$, that is, a certain part of the vectors x_i, $i\neq 0,1$, $\alpha_0x_0+\alpha_1x_1$, $\beta_0x_0+\beta_1x_1$

constitutes a basis of V. Let V contain some basis vector x_j. Then as in the proof of 10.1, acting successively upon x_j by certain linear combinations of the operators $R_{ab\lambda}(I_2)$ and $R_{ab\lambda}(I_3)$ we generate all the vectors x_i, $i=0,1,\cdots,\frac{1}{2}(p'\Leftrightarrow 1)$. This means that $V=H$ and the representation $R_{ab\lambda}$ is irreducible. If V contains no vector x_j, $j\neq 0,1$, then some linear combination $\alpha_0 x_0+\alpha_1 x_1$ belongs to V. Then the vector $v=R_{ab\lambda}(I_2)(\alpha_0 x_0+\alpha_1 x_1)$ belongs to V. Since v contains the summand αx_2 with nonzero coefficient α, then $x_2\in V$. This is a contradiction. Hence, the representation $R_{ab\lambda}$ is irreducible. Proposition is proved.
Let p' be even. Let us study the representations $R_{ab\lambda}$ for $\lambda=\pm q^{(p'-1)/2}$. For $\lambda=q^{(p'-1)/2}$ we have

$$R_{ab\lambda}(I_1)x_i=\frac{-1}{q-q^{-1}}(q^{-i+(p'-1)/2}+q^{i-(p'-1)/2})x_i, \qquad (15.7)$$

$$R_{ab\lambda}(I_2)x_0=c_{(p'-1)/2}(ax_{p'-1}+x_1), \qquad (15.8)$$

$$R_{ab\lambda}(I_2)x_{p'-1}=\Leftrightarrow c_{(p'-1)/2}\big((ab+[p'\Leftrightarrow 1]^2)x_{p'-2}+bx_0\big), \qquad (15.9)$$

$$R_{ab\lambda}(I_2)x_i=c_{-i+(p'-1)/2}\big((ab+[i]^2)x_{i-1}+x_{i+1}\big), \qquad (15.10)$$

where

$$c_j=\frac{\mathrm{i}}{\mathrm{q}^{\mathrm{j}}-\mathrm{q}^{-\mathrm{j}}}.$$

The operator $R_{a,b,(p'-1)/2}(I_1)$ has the spectrum

$$\frac{-1}{q-q^{-1}}(q^{-i+(p'-1)/2}+q^{i-(p'-1)/2}),\quad i=0,1,2,\cdots,p'\Leftrightarrow 1,$$

that is, if p' is even, then all spectral points are of multiplicity 2.
We assume that $ab\neq\Leftrightarrow[j]^2$, $j=0,1,\cdots,p'\Leftrightarrow 1$, and go over from the basis contaning vectors x_i to the basis contaning vectors x_i°, where

$$x_i^\circ=\prod_{j=0}^{i}(ab+[j]^2)^{-1/2}x_i,\quad i=0,1,2,\cdots,p'\Leftrightarrow 1.$$

Then the formula (15.7) does not change and the formulas (15.8)--(15.10) turn into

$$R_{ab\lambda}(I_2)x_0^\circ=c_{(p'-1)/2}\left(a\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}x_{p'-1}^\circ+(ab+1)^{1/2}x_1^\circ\right),$$

$$R_{ab\lambda}(I_2)x_{p'-1}^\circ=\Leftrightarrow c_{(p'-1)/2}\left((ab+1)^{1/2}x_{p'-2}^\circ+\frac{b}{\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}}x_0^\circ\right),$$

$$R_{ab\lambda}(I_2)x_i^\circ=c_{-i+(p'-1)/2}\big((ab+[i]^2)^{1/2}x_{i-1}^\circ+(ab+[i+1]^2)^{1/2}x_{i+1}^\circ\big).$$

We split the representation space H into the direct sum of two linear subspaces H_1 and H_2 spanned by the basis vectors x'_j, $j=0,1,2,\cdots,\frac{1}{2}(p'\Leftrightarrow 2)$, and x''_j, $j=0,1,2,\cdots,\frac{1}{2}(p'\Leftrightarrow 2)$, where

$$x'_j=x_j^\circ+\mathrm{i}(\Leftrightarrow 1)^{-j-1+p'/2}x_{p'-j-1}^\circ,\quad x''_j=x_j^\circ+\mathrm{i}(\Leftrightarrow 1)^{-j+p'/2}x_{p'-j-1}^\circ.$$

Then as in previous sections, we derive

$$R_{a,b,(p'-1)/2}(I_1)x'_j=\frac{-1}{q-q^{-1}}(q^{-j+(p'-1)/2}+q^{j-(p'-1)/2})x'_j,$$

$$R_{a,b,(p'-1)/2}(I_1)x''_j=\frac{-1}{q-q^{-1}}(q^{-j+(p'-1)/2}+q^{j-(p'-1)/2})x''_j$$

for the operator $R_{a,b,(p'-1)/2}(I_1)$ and

$$R_{a,b,(p'-1)/2}(I_2)x'_j=c_{-j+(p'-1)/2}\big((ab+[j+1]^2)^{1/2}x'_{j+1}+(ab+[j]^2)^{1/2}x'_{j-1}\big),$$

$$R_{a,b,(p'-1)/2}(I_2)x''_j = c_{-j+(p'-1)/2}\big((ab+[j+1]^2)^{1/2}x''_{j+1}+(ab+[j]^2)^{1/2}x''_{j-1}\big),$$

where $j\neq 0, \frac{p'}{2}\Leftrightarrow 1$,

$$R_{a,b,(p'-1)/2}(I_2)x'_{\frac{p'}{2}-1} = \frac{1}{q^{1/2}-q^{-1/2}}(ab+[\tfrac{p'}{2}]^2)^{1/2}x'_{\frac{p'}{2}-1}+$$
$$+\frac{\mathrm{i}}{\mathrm{q}^{1/2}-\mathrm{q}^{-1/2}}(ab+[\tfrac{p'}{2}\Leftrightarrow 1]^2)^{1/2}x'_{\frac{p'}{2}-2},$$
$$R_{a,b,(p'-1)/2}(I_2)x''_{\frac{p'}{2}-1} = \Leftrightarrow\frac{1}{q^{1/2}-q^{-1/2}}(ab+[\tfrac{p'}{2}]^2)^{1/2}x''_{\frac{p'}{2}-1}+$$
$$+\frac{\mathrm{i}}{\mathrm{q}^{1/2}-\mathrm{q}^{-1/2}}(ab+[\tfrac{p'}{2}\Leftrightarrow 1]^2)^{1/2}x''_{\frac{p'}{2}-2},$$
$$R_{a,b,(p'-1)/2}(I_2)x'_0 = c_{(p'-1)/2}\left(a\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}x^\circ_{p'-1}+(ab+1)^{1/2}x^\circ_1\right)\Leftrightarrow$$
$$\Leftrightarrow \mathrm{i}(\Leftrightarrow 1)^{(p'-2)/2}c_{(p'-1)/2}\left((ab+1)^{1/2}x^\circ_{p'-2}+\frac{b}{\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}}x^\circ_0\right).$$

When

$$a\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2} = \frac{b}{\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}}, \qquad (15.11)$$

then the last relation reduces to

$$R_{a,b,(p'-1)/2}(I_2)x'_0 = \frac{(-1)^{(p'-2)/2}}{q^{(p'-1)/2}-q^{-(p'-1)/2}}a\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}x'_0\Leftrightarrow$$
$$\Leftrightarrow c_{(p'-1)/2}(ab+1)^{1/2}x'_1.$$

Similarly, if the condition (15.11) is fulfilled, then

$$R_{a,b,(p'-1)/2}(I_2)x''_0 = \frac{(-1)^{p'/2}}{q^{(p'-1)/2}-q^{-(p'-1)/2}}a\prod_{j=1}^{p'-1}(ab+[j]^2)^{1/2}x''_0+$$
$$+c_{(p'-1)/2}(ab+1)^{1/2}x''_1.$$

Thus, the subspaces H_1 and H_2 are invariant with respect to the representation $R_{a,b,(p'-1)/2}$ if the condition (15.11) is fulfilled. We denote the corresponding subrepresentations by $R^{1,+}_{a,b,(p'-1)/2}$ and $R^{2,+}_{a,b,(p'-1)/2}$, respectively.

Similarly, if $\lambda = \Leftrightarrow q^{(p'-1)/2}$, then

$$R_{a,b,-(p'-1)/2}(I_1) = \Leftrightarrow R_{a,b,(p'-1)/2}(I_1), \quad R_{a,b,-(p'-1)/2}(I_2) =$$
$$= \Leftrightarrow R_{a,b,(p'-1)/2}(I_2)$$

and the subspaces H_1 and H_2 are invariant with respect to the representation $R_{a,b,-(p'-1)/2}$ if the condition (15.11) is fulfilled. We denote the corresponding subrepresentations by $R^{1,-}_{a,b,-(p'-1)/2}$ and $R^{2,-}_{a,b,-(p'-1)/2}$, respectively.

15.3. **Theorem.** Let the condition (15.11) is satisfied. Then the representations $R^{i,+}_{a,b,(p'-1)/2}$ and $R^{i,-}_{a,b,-(p'-1)/2}$, $i=1,2$, of the algebra $U_q(\mathrm{so}_3)$ are irreducible and pairwise nonequivalent. If the condition (15.11) is not satisfied, then the

representations $R_{a,b,(p'-1)/2}$ and $R_{a,b,-(p'-1)/2}$ are irreducible.
Proof. is similar to that of the previous propositions and we omit it.
Remark that the representations $R^{i,+}_{a,b,(p'-1)/2}$ and $R^{i,-}_{a,b,-(p'-1)/2}$, $i=1,2$, have two nonzero diagonal matrix elements $x^{\#}_{\frac{p'}{2}-1}(Rx_{\frac{p'}{2}-1})$ and $x^{\#}_0(Rx_0)$.

Let now p' be odd and $\lambda=q^{(p'-2)/2}$. For this value of λ we have

$$R_{ab\lambda}(I_1)x_i=\tfrac{-1}{q-q^{-1}}(q^{-i+(p'-2)/2}+q^{i-(p'-2)/2})x_i,$$
$$R_{ab\lambda}(I_2)x_0=c_{(p'-2)/2}(ax_{p'-1}+x_1),$$
$$R_{ab\lambda}(I_2)x_{p'-1}=-c_{p'/2}((ab+\epsilon[p'-1][p'])x_{p'-2}+bx_0),$$
$$R_{ab\lambda}(I_2)x_i=c_{-i+(p'-2)/2}((ab+\epsilon[i][i+1])x_{i-1}+x_{i+1}),$$

where $\epsilon=1$ for $p'=p/2$, $\epsilon=-1$ for $p'=p$ and c_j is such as in (15.7)--(15.10). The operator $R_{a,b,(p'-2)/2}(I_1)$ has the spectrum

$$\tfrac{-1}{q-q^{-1}}(q^{-i+(p'-2)/2}+q^{i-(p'-2)/2}),\quad i=0,1,2,\cdots,p'-1,$$

that is, all spectral points are of multiplicity 2 except for the point $-(q^{p'/2}+q^{-p'/2})/(q-q^{-1})$ which is of multiplicity 1.
We assume that $ab\neq-\epsilon[j][j+1]$, $j=0,1,\cdots,p'-1$, and go over from the basis containing vectors x_i to the basis containing vectors x_i°, where

$$x_i^\circ=\prod_{j=0}^{i}(ab+\epsilon[j][j+1])^{-1/2}x_i,\quad i=0,1,2,\cdots,p'-1.$$

Then

$$R_{ab\lambda}(I_1)x_i^\circ=\tfrac{-1}{q-q^{-1}}(q^{-i+(p'-2)/2}+q^{i-(p'-2)/2})x_i^\circ,$$
$$R_{ab\lambda}(I_2)x_0^\circ=c_{(p'-2)/2}\left(a\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}x^\circ_{p'-1}+(ab+\epsilon[2])^{1/2}x_1^\circ\right),$$
$$R_{ab\lambda}(I_2)x^\circ_{p'-1}=-c_{p'/2}((ab+\epsilon[p'-1][p'])^{1/2}x^\circ_{p'-2}+$$
$$+b\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{-1/2}x_0^\circ),$$
$$R_{ab\lambda}(I_2)x_i^\circ=c_{-i+(p'-2)/2}((ab+\epsilon[i][i+1])^{1/2}x^\circ_{i-1}+(ab+\epsilon[i+1][i+2])^{1/2}x^\circ_{i+1}),$$

where $\lambda=q^{(p'-2)/2}$. Let H_1 and H_2 be two linear subspaces of the representation space H spanned by the basis vectors

$$x'_j=x_j^\circ+\mathrm{i}(-1)^j x^\circ_{p'-j-2},\quad j=0,1,2,\cdots,\tfrac{p'-3}{2},$$

and the basis vectors

$$x''_j=x_j^\circ+\mathrm{i}(-1)^{j+1} x^\circ_{p'-j-2},\quad j=0,1,2,\cdots,\tfrac{p'-3}{2},$$

respectively. Then the operator $R_{a,b,(p'-2)/2}(I_1)$ acts on the basis elements x'_j and x''_j as on the vectors x_j and

$$R_{a,b,(p'-1)/2}(I_2)x'_j=c_{-j+(p'-2)/2}\cdot$$
$$\cdot\big((ab+\epsilon[j+1][j+2])^{1/2}x'_{j+1}+(ab+\epsilon[j][j+1])^{1/2}x'_{j-1}\big),$$
$$R_{a,b,(p'-2)/2}(I_2)x''_j=c_{-j+(p'-2)/2}\cdot$$
$$\cdot\big((ab+\epsilon[j+1][j+2])^{1/2}x''_{j+1}+(ab+\epsilon[j][j+1])^{1/2}x''_{j-1}\big),$$

where $j \neq 0, \frac{p'-3}{2}$,

$$R_{a,b,(p'-2)/2}(I_2)x'_{\frac{p'-3}{2}} = \frac{(-1)^{(p'-3)/2}}{q^{1/2}-q^{-1/2}}(ab+\epsilon[\tfrac{p'-1}{2}][\tfrac{p'+1}{2}])^{1/2}x'_{\frac{p'-3}{2}}+$$
$$+\frac{\mathrm{i}}{q^{1/2}-q^{-1/2}}(ab+\epsilon[\tfrac{p'-1}{2}][\tfrac{p'-3}{2}])^{1/2}x'_{\frac{p'-5}{2}},$$
$$R_{a,b,(p'-2)/2}(I_2)x''_{\frac{p'-3}{2}} = \Leftrightarrow\frac{(-1)^{(p'-3)/2}}{q^{1/2}-q^{-1/2}}(ab+\epsilon[\tfrac{p'-1}{2}][\tfrac{p'+1}{2}])^{1/2}x''_{\frac{p'-3}{2}}+$$
$$+\frac{\mathrm{i}}{q^{1/2}-q^{-1/2}}(ab+\epsilon[\tfrac{p'-1}{2}][\tfrac{p'-3}{2}])^{1/2}x''_{\frac{p'-5}{2}},$$
$$R_{a,b,(p'-2)/2}(I_2)x'_0 = c_{(p'-2)/2}\Bigg(a\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}x^\circ_{p'-1}+$$
$$+(ab+\epsilon[2])^{1/2}x^\circ_1 \Leftrightarrow \mathrm{i}(ab+\epsilon[2])^{1/2}x^\circ_{p'-1} \Leftrightarrow \mathrm{i}(ab+\epsilon[p'\Leftrightarrow 2][p'\Leftrightarrow 1])^{1/2}x^\circ_{p'-3}\Bigg),$$
$$R_{a,b,(p'-2)/2}(I_2)x''_0 = c_{(p'-2)/2}\Bigg(a\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}x^\circ_{p'-1}+$$
$$+(ab+\epsilon[2])^{1/2}x^\circ_1 + \mathrm{i}(ab+\epsilon[2])^{1/2}x^\circ_{p'-1} + \mathrm{i}(ab+\epsilon[p'\Leftrightarrow 2][p'\Leftrightarrow 1])^{1/2}x^\circ_{p'-3}\Bigg).$$

If

$$a\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}+\mathrm{i}(ab+\epsilon[2])^{1/2}=0, \tag{15.12}$$

$$(ab+\epsilon[2])^{1/2}\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}=\mathrm{i}b, \tag{15.13}$$

then

$$R_{a,b,(p'-2)/2}(I_2)x_{p'-1} = \frac{-bc_{p'/2}}{\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}}x'_0,$$
$$R_{a,b,(p'-2)/2}(I_2)x'_0 = \frac{\mathrm{i}(ab+\epsilon[2])^{1/2}}{q^{(p'-2)/2}-q^{-(p'-2)/2}}x'_1 + cx'_{p'-1},$$
$$R_{a,b,(p'-2)/2}(I_2)x''_0 = \frac{\mathrm{i}(ab+\epsilon[2])^{1/2}}{q^{(p'-2)/2}-q^{-(p'-2)/2}}x''_1,$$

where c is a nonzero coefficient easily determined from the above formulas. Hence, the subspaces $H_1+\mathbb{C}x_{p'-1}$ and H_2 of the representation space are invariant with respect to the representation $R_{a,b,(p'-2)/2}$ (we denote these subrepresentations by $R^1_{a,b,(p'-2)/2}$ and $R^2_{a,b,(p'-2)/2}$, respectively). Remark that

$$\dim\, H_1+\mathbb{C}x_{p'-1}=\tfrac{1}{2}(p'+1), \quad \dim\, H_2=\tfrac{1}{2}(p'\Leftrightarrow 1).$$

If

$$a\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}\Leftrightarrow \mathrm{i}(ab+\epsilon[2])^{1/2}=0, \tag{15.14}$$

$$(ab+\epsilon[2])^{1/2}\prod_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}=\Leftrightarrow \mathrm{i}b, \tag{15.15}$$

then

$$R_{a,b,(p'-2)/2}(I_2)x_{p'-1} = \frac{-bc_{p'/2}}{\prod\limits_{j=1}^{p'-1}(ab+\epsilon[j][j+1])^{1/2}}x''_0,$$

$$R_{a,b,(p'-2)/2}(I_2)x'_0 = \frac{\mathrm{i}(ab+\epsilon[2])^{1/2}}{q^{(p'-2)/2}-q^{-(p'-2)/2}}x'_1,$$

$$R_{a,b,(p'-2)/2}(I_2)x''_0 = \frac{\mathrm{i}(ab+\epsilon[2])^{1/2}}{q^{(p'-2)/2}-q^{-(p'-2)/2}}x''_1 + cx_{p'-1},$$

where c is a nonzero coefficient. Hence, now the subspaces H_1 and $H_2+\mathbb{C}x_{p'-1}$ of the representation space are invariant. We denote the subrepresentations on these subspaces by $\hat{R}^1_{a,b,(p'-2)/2}$ and $\hat{R}^2_{a,b,(p'-2)/2}$, respectively). Note that the representation $\hat{R}^1_{a,b,(p'-2)/2}$ is not equivalent to $R^2_{a,b,(p'-2)/2}$ (and the representation $\hat{R}^2_{a,b,(p'-2)/2}$ is not equivalent to $R^1_{a,b,(p'-2)/2}$) since the parameters a and b determining these representations satisfy different equations.

If a and b do not satisfy the relations (15.12) and (15.13) or the relations (15.14) and (15.15), then the representation $R_{a,b,(p'-2)/2}$ is irreducible.

Let now p' be odd and $\lambda = \Leftrightarrow q^{(p'-2)/2}$. In this case, the representation $R_{a,b,-(p'-2)/2}$ is irreducible if a and b do not satisfy the relations (15.12) and (15.13) or the relations (15.14) and (15.15). If a and b satisfy the relations (15.12) and (15.13), then $R_{a,b,-(p'-2)/2}$ is a reducible representation and decomposes into the direct sum of two subrepresentations acting on the subspaces $H_1+\mathbb{C}x_{p'-1}$ and H_2. These subrepresentations are denoted by $R^1_{a,b,-(p'-2)/2}$ and $R^2_{a,b,-(p'-2)/2}$, respectively, and are determined as

$$R^i_{a,b,-(p'-2)/2}(I_1) = \Leftrightarrow R^i_{a,b,(p'-2)/2}(I_1), \quad R^i_{a,b,-(p'-2)/2}(I_2) =$$
$$= \Leftrightarrow R^i_{a,b,(p'-2)/2}(I_2), \quad i = 1,2.$$

Similarly, if a and b satisfy the relations (15.14) and (15.15), then $R_{a,b,-(p'-2)/2}$ is a reducible representation and decomposes into the direct sum of two subrepresentations acting on the subspaces H_1 and $H_2+\mathbb{C}x_{p'-1}$. These subrepresentations are denoted by $\hat{R}^1_{a,b,-(p'-2)/2}$ and $\hat{R}^2_{a,b,-(p'-2)/2}$, respectively, and are determined as

$$\hat{R}^i_{a,b,-(p'-2)/2}(I_1) = \Leftrightarrow \hat{R}^i_{a,b,(p'-2)/2}(I_1), \quad \hat{R}^i_{a,b,-(p'-2)/2}(I_2) =$$
$$= \Leftrightarrow \hat{R}^i_{a,b,(p'-2)/2}(I_2), \quad i = 1,2.$$

15.4. **Theorem.** Let the conditions (15.12) and (15.13) are satisfied. Then the representations $R^1_{a,b,(p'-2)/2}$, $R^2_{a,b,(p'-2)/2}$, $R^1_{a,b,-(p'-2)/2}$ and $R^2_{a,b,-(p'-2)/2}$ are irreducible and pairwise nonequivalent. If the conditions (15.14) and (15.15) are satisfied, the representations $\hat{R}^1_{a,b,(p'-2)/2}$, $\hat{R}^2_{a,b,(p'-2)/2}$, $\hat{R}^1_{a,b,-(p'-2)/2}$ and $\hat{R}^2_{a,b,-(p'-2)/2}$ are irreducible and pairwise nonequivalent.

Proof. is similar to that of the previous propositions and we omit it.

16. Other representations at root of unity

In the previous section we described irreducible representations of $U_q(\mathrm{so}_3)$ obtained from irreducible representations of the algebra $\hat{U}_q(\mathrm{sl}_2)$ for q a root of unity. However, at q a root of unity the algebra $U_q(\mathrm{so}_3)$ has irreducible representations which cannot be

derived from those of $\widehat{U}_q(\mathrm{sl}_2)$. They are obtained as irreducible components of the representations Q_λ from Section VII when one put q equal to a root of unity. We describe these representations of $U_q(\mathrm{so}_3)$ in this section.
Let $\lambda = q^\tau$ be a nonzero complex number such that $0 \le \mathrm{Re}\tau < 1$ and let H be the p'-dimensional complex vector space with basis

$$x_m, \quad m = 0,1,2,\cdots,p' \Leftrightarrow 1.$$

We define on this space the operators $Q'_\lambda(I_1)$ and $Q'_\lambda(I_2)$ determined by the formulas

$$Q'_\lambda(I_1)x_m = \frac{\lambda q^m + \lambda^{-1}q^{-m}}{q-q^{-1}}x_m,$$
$$Q'_\lambda(I_2)x_0 = \frac{1}{q-q^{-1}}x_1 + \frac{1}{q-q^{-1}}x_{p'-1},$$
$$Q'_\lambda(I_2)x_{p'-1} = \frac{1}{q-q^{-1}}x_{p'-2} + \frac{1}{q-q^{-1}}x_0,$$
$$Q'_\lambda(I_2)x_m = \frac{1}{q-q^{-1}}x_{m-1} + \frac{1}{q-q^{-1}}x_{m+1}, \quad m \ne 0, p' \Leftrightarrow 1.$$

A direct computation shows that these operators satisfy the relations (8.1) and (8.2) and hence determine a representation of $U_q(\mathrm{so}_3)$ which will be denoted by Q'_λ.

16.1. **Theorem.** If $\lambda \ne 1$ and $\lambda \ne q^{1/2}$, then the representation Q'_λ is irreducible.
Proof. of this proposition is the same as that of the first part of 10.1.
The representations Q'_1 and $Q'_{\sqrt{q}}$ are studied in the same way as the representations Q_1 and $Q_{\sqrt{q}}$ in previous sections. This study leads to the irreducible representations of $U_q(\mathrm{so}_3)$ which are described below. (Note that the description of these representations for p' even and for p' odd is deferent.)
Let p' be odd. We denote by H_r and H_s, $r = \frac12(p'+1)$, $s = \frac12(p' \Leftrightarrow 1)$, the complex vector spaces with the bases

$$x_0,\, x_1,\, x_2,\, \cdots,\, x_{\frac12(p'-1)} \quad \text{and} \quad x_1,\, x_2,\, \cdots,\, x_{\frac12(p'-1)},$$

respectively. Four representations $Q_1^{\pm,\pm}$ act on the space H_r and are given by the formulas

$$Q_1^{+,\pm}(I_1)x_m = \frac{q^m+q^{-m}}{q-q^{-1}}x_m, \quad m = 0,1,2,\cdots,\tfrac12(p' \Leftrightarrow 1), \tag{16.1}$$
$$Q_1^{+,\pm}(I_2)x_{\frac12(p'-1)} = \pm\frac{1}{q-q^{-1}}x_{\frac12(p'-1)} + \frac{1}{q-q^{-1}}x_{\frac12(p'-3)}, \tag{16.2}$$
$$Q_1^{+,\pm}(I_2)x_m = \frac{1}{q-q^{-1}}x_{m+1} + \frac{1}{q-q^{-1}}x_{m-1}, \quad m < \tfrac12(p' \Leftrightarrow 1), \tag{16.3}$$

and by the formulas

$$Q_1^{-,\pm}(I_1)x_m = \Leftrightarrow\frac{q^m+q^{-m}}{q-q^{-1}}x_m, \quad m = 0,1,2,\cdots,\tfrac12(p' \Leftrightarrow 1), \tag{16.4}$$
$$Q_1^{-,\pm}(I_2) := Q_1^{+,\pm}(I_2). \tag{16.5}$$

Note that the upper sign corresponds to the representations $Q_1^{+,+}$ and $Q_1^{-,+}$ and the lower sign to the representations $Q_1^{+,-}$ and $Q_1^{-,-}$.
On the space H_s, four representations $\hat{Q}_1^{\pm,\pm}$ act by the corresponding formulas (16.1)--(16.5), but now m runs over the values $1,2,3,\cdots,\frac12(p' \Leftrightarrow 1)$.
Let now H'_r and H'_s, $r = \frac12(p'+1)$, $s = \frac12(p' \Leftrightarrow 1)$, be the complex vector spaces with the bases

$$x_{m+\frac12},\ m = 0,1,2,\cdots,\tfrac12(p' \Leftrightarrow 1), \quad \text{and} \quad x_{m+\frac12},\ m = 0,1,2,\cdots,\tfrac12(p' \Leftrightarrow 3),$$

respectively. The four representations $Q_{\sqrt{q}}^{\pm,\pm}$ act on the space H'_r and are given by the formulas

$$Q^{+,\pm}_{\sqrt{q}}(I_1)x_{m+\frac12}=\frac{q^{m+1/2}+q^{-m-1/2}}{q-q^{-1}}x_{m+\frac12},\;\; m=0,1,2,\cdots,\tfrac12(p'\Leftrightarrow 1), \tag{16.6}$$
$$Q^{+,\pm}_{\sqrt{q}}(I_2)x_{\frac12}=\pm\frac{1}{q-q^{-1}}x_{\frac12}+\frac{1}{q-q^{-1}}x_{\frac32}, \tag{16.7}$$
$$Q^{+,\pm}_{\sqrt{q}}(I_2)x_{m+\frac12}=\frac{1}{q-q^{-1}}x_{m+\frac32}+\frac{1}{q-q^{-1}}x_{m-\frac12},\quad m\neq 0, \tag{16.8}$$

where $x_{m+\frac32}\equiv 0$ if $m=\frac12(p'\Leftrightarrow 1)$, and by the formulas

$$Q^{-,\pm}_{\sqrt{q}}(I_1)x_{m+\frac12}=\Leftrightarrow\frac{q^{m+1/2}+q^{-m-1/2}}{q-q^{-1}}x_{m+\frac12},\;\; m=0,1,2,\cdots,\tfrac12(p'\Leftrightarrow 1), \tag{16.9}$$
$$Q^{-,\pm}_{\sqrt{q}}(I_2):=Q^{+,\pm}_{\sqrt{q}}(I_2). \tag{16.10}$$

On the space H'_s, four representations $\check{Q}^{\pm,\pm}_{\sqrt{q}}$ act by the corresponding formulas (16.6)--(16.10), but now m runs through the values $0,1,2,\cdots,\frac12(p'\Leftrightarrow 3)$.

Let now p' be even. We denote by H_r and H_s, $r=\frac12(p'+2)$, $s=\frac12(p'\Leftrightarrow 2)$, the complex vector spaces with the bases

$$x_0,\, x_1,\, x_2,\, \cdots,\, x_{\frac12 p'} \quad \text{and} \quad x_1,\, x_2,\, \cdots,\, x_{\frac12(p'-2)},$$

respectively. The representations $Q^{1,\pm}_1$ and $Q^{2,\pm}_1$ act on H_r and H_s, respectively, which are given by the formulas

$$Q^{i,\pm}_1(I_1)x_m=\pm\frac{q^m+q^{-m}}{q-q^{-1}}x_m,\quad i=1,2,$$
$$Q^{i,\pm}_1(I_2)x_m=\frac{1}{q-q^{-1}}x_{m+1}+\frac{1}{q-q^{-1}}x_{m-1},\quad i=1,2,$$

where x_{m+1} or x_{m-1} must be put equal to 0 if the corresponding vector does not exist.

Let $H_{p'/2}$ be the complex vector space with the basis

$$x_{m+\frac12},\quad m=0,1,2,\cdots,\tfrac12(p'\Leftrightarrow 2).$$

Four representations $\hat{Q}^{\pm,\pm}_{\sqrt{q}}$ act on this space which are given by the formulas

$$\hat{Q}^{+,\pm}_{\sqrt{q}}(I_1)x_{m+\frac12}=\frac{q^{m+1/2}+q^{-m-1/2}}{q-q^{-1}}x_{m+\frac12},$$
$$\hat{Q}^{+,\pm}_{\sqrt{q}}(I_2)x_{\frac12}=\pm\frac{1}{q-q^{-1}}x_{\frac12}+\frac{1}{q-q^{-1}}x_{\frac32},$$
$$\hat{Q}^{+,\pm}_{\sqrt{q}}(I_2)x_{\frac12(p'-2)}=\pm\frac{1}{q-q^{-1}}x_{\frac12(p'-2)}+\frac{1}{q-q^{-1}}x_{\frac12(p'-4)},$$
$$\hat{Q}^{+,\pm}_{\sqrt{q}}(I_2)x_{m+\frac12}=\frac{1}{q-q^{-1}}x_{m-\frac12}+\frac{1}{q-q^{-1}}x_{m+\frac32},\;\; m\neq \tfrac12, x_{\frac12(p'-2)},$$

and by the formulas

$$\hat{Q}^{-,\pm}_{\sqrt{q}}(I_1)x_{m+\frac12}=\Leftrightarrow\frac{q^{m+1/2}+q^{-m-1/2}}{q-q^{-1}}x_{m+\frac12},$$
$$\hat{Q}^{-,\pm}_{\sqrt{q}}(I_2)=\hat{Q}^{+,\pm}_{\sqrt{q}}(I_2).$$

Let us mention peculiarities of the representations described above. The operators $Q^{\pm,\pm}_1(I_2)$, $Q^{\pm,\pm}_{\sqrt{q}}(I_2)$, $\check{Q}^{\pm,\pm}_{\sqrt{q}}$ and $\hat{Q}^{\pm,\pm}_{\sqrt{q}}(I_2)$ have nonzero diagonal matrix elements and nonzero traces. Moreover, the operators $\hat{Q}^{\pm,\pm}_{\sqrt{q}}(I_2)$ have two such diagonal elements. Spectra of the operators $Q^{\pm,\pm}_1(I_1)$, $Q^{\pm,\pm}_{\sqrt{q}}(I_1)$, $Q^{1,\pm}_1(I_1)$, $Q^{2,\pm}_1(I_1)$ and $\hat{Q}^{\pm,\pm}_{\sqrt{q}}(I_1)$ are not symmetric with respect to the zero point.

16.2. **Theorem.** The representations $Q^{\pm,\pm}_1$, $Q^{\pm,\pm}_{\sqrt{q}}$, $\check{Q}^{\pm,\pm}_{\sqrt{q}}$, $Q^{1,\pm}_1$, $Q^{2,\pm}_1$, $\hat{Q}^{\pm,\pm}_{\sqrt{q}}$ are irreducible and pairwise nonequivalent. No representation Q'_λ is equivalent to any of these representations.

Proof. is the same as that of 10.1.

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