1 Groups, Transformations, Differential Operators and Identities

A **group** is an ordered pair $(G, \circ)$, where G is a non-empty set and $\circ : G \times G \to G$ a mapping (operation called multiplication) such that:

1. $\forall g_1, g_2, g_3 \in G \quad g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$ associativity
2. $\exists e \in G \; \forall g \in G \quad e \circ g = g \circ e = g$ identity element
3. $\forall g \in G \; \exists_1 g^{-1} \in G \quad g \circ g^{-1} = g^{-1} \circ g = e$ inverse element

If the multiplication operation in the group $(G, \circ)$ is obvious from the context, we omit its symbol and simply talk about the group G. A group G is called **abelian** (commutative) if its multiplication is commutative. A subset $H \subset G$ of the group $(G, \circ)$ is called a **subgroup** if $(H, \circ|_{H\times H})$ is a group, i.e. if H contains the identity element and is closed with respect to multiplication and inversion.

If $(G, \circ)$ and $(H, *)$ are groups, then a mapping $f : G \to H$ is called a **group homomorphism** if for all $g_1, g_2 \in G$ it holds that $f(g_1 \circ g_2) = f(g_1) * f(g_2)$. Let $GL(V)$ denote the group of all linear regular operators on a vector space V with the operation of mapping composition. A group homomorphism $\rho : G \to GL(V)$ is called a **representation of group** G on the vector space V. The dimension of the representation ρ is the dimension of V.

A group homomorphism $f : G \to G$, which is a bijection (i.e. injective and surjective), is called a **group automorphism** of G, and the set of all automorphisms of the group G is denoted $Aut(G)$. Let $\varphi : H \to Aut(G)$ be a group homomorphism, then a **semidirect product of groups** $(G, \circ)$ and $(H, *)$ is a group $(G \times H, \bullet_\varphi)$ denoted $G \rtimes H$ with the operation defined by the rule $(g_1, h_1) \bullet_\varphi (g_2, h_2) = (g_1 \circ \varphi(h_1)g_2, h_1 * h_2)$. If the homomorphism φ is trivial, i.e. $\varphi(h) = e_G, \forall h \in H$, the product of groups G and H is called **direct** and denoted $G \times H$.

A **transformation** of a set M is a bijection (one-to-one correspondence) of the set M onto itself. The set of all transformations of the set M forms a group. If the set M is equipped with some structure (topology/algebraic operations), we usually are interested in such transformations (homeomorphisms/automorphisms) that preserve this structure. Such transformations will be called **symmetries**. The set of all symmetries again forms a group.

For example, the real vector space V of dimension $n \in \mathbb{N}$ has a linear structure (vector addition and multiplication by a number) and hence we require its transformations to be linear. Thus, the symmetries of the vector space are regular linear operators (group $GL(n)$). If a scalar product $\langle ., . \rangle$ is additionally defined on V, we also require its preservation, and symmetries of $(V, \langle ., . \rangle)$ will be only the orthogonal operators (group $O(n)$). All linear transformations $(V, \langle ., . \rangle)$, which were symmetries of the vector space V itself, are sometimes briefly called transformations of $(V, \langle ., . \rangle)$. [1] [2]

A **symmetry** of an object is a transformation that does not change the properties of this object, and conversely, an object is called **invariant** with respect to a transformation if this transformation is its symmetry. For example, a symmetry S of a subset $N \subset M$ will be a transformation of M

[1] Another example could be the phase space of a closed system. It has the structure of a symplectic variety (Γ, ω), which is a smooth variety Γ equipped with a symplectic form ω (a closed nondegenerate differential form of degree 2) represented by the Poisson bracket $\{., .\}$. Its transformations are smooth diffeomorphisms, and symmetries are only those among them that preserve the symplectic form ω (the Poisson bracket), hence canonical transformations. The requirement for transformations to be smooth diffeomorphisms (smooth bijections whose inverse is also smooth) follows from the structure of a smooth variety.

[2] The vector space V with a scalar product can be viewed as a smooth variety. From the scalar product, we can inductively induce a norm, metric, and topology on V. Thus, we should require transformations to be smooth, although this already automatically follows from linearity in finite-dimensional spaces.

satisfying $S(N) = \{S(m) \,|m \in M\} = N$ and thus the subset N will be invariant with respect to S. The set of all symmetries of a given object again forms a group.

Since the mapping $f : A \to B$ is by definition a subset $f \subset A \times B$ of the Cartesian product of sets A, B satisfying that each element from A has at most one image in B, we can define its symmetry as a transformation of $A \times B$ with respect to which this subset f is invariant. For a function $f : \mathbb{R} \to \mathbb{R}$, we get that its symmetries are symmetries of its graph as a subset in $\mathbb{R}^2$.

In specific cases, we may limit ourselves to certain types of symmetries. For example, from the symmetries of a function f, we may consider only those corresponding to transformations $S : Dom(f) \subset A \to Dom(f)$ of its domain preserving its value $f(S(x)) = f(x),\ \forall x \in Dom(f)$. This type of symmetries was used for Lagrangian functions in Noether's theorem and will be encountered in the case of scalar fields.

For equations (algebraic, trigonometric or differential) or systems of equations, symmetries are defined as such transformations of the domain that map every solution to a solution.

Space transformations mentioned so far are referred to as **active**, i.e. those that "move" points of the given space. If we define a coordinate system on our space (for example, on the vector space V_n linear coordinates defined by the coordinate isomorphism $(.)_{\mathcal{E}} : V_n \to \mathbb{R}^n$ given by the choice of basis $\mathcal{E}$ in V_n), we can examine transitions between different choices of these coordinate systems (corresponding to different choices of base $\mathcal{E}$ in V_n), which we call **passive** transformations.

Active transformation - rotation of vector $\vec{v}$ counter-clockwise	$\longleftarrow$ rotation by angle φ $\longrightarrow$	Passive transformation - rotation of coordinates by φ clockwise
$(\vec{v}\,')_{\mathcal{E}} = {}^{\mathcal{E}}\!\mathbb{A}(\varphi)^{\mathcal{E}}(\vec{v})_{\mathcal{E}}$	$\vec{v}\,' = \mathbb{A}(\varphi)\vec{v}$	$(\vec{v})_{\mathcal{E}'} = {}^{\mathcal{E}}\mathrm{Id}^{\mathcal{E}'}(\vec{v})_{\mathcal{E}}$
components of new $\vec{v}\,'$ and old vector $\vec{v}$ in the same basis $\mathcal{E}$	$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$	components of the same vector $\vec{v}$ in new $\mathcal{E}'$ and old basis $\mathcal{E}$

Passive transformations thus represent only a change in the description of objects when changing coordinate systems and cannot in principle change their properties (e.g. lengths of vectors), but can change the formulas by which these properties are calculated from coordinates. In contrast, active transformations can change the properties of objects but do not change the formulas by which these properties are calculated from coordinates, as they describe the change of objects relative to a fixed coordinate system.

Exercise 1.1 *Provide examples of abelian groups you have already encountered. Specify what the operation, identity element and inverse element are in them.*

Exercise 1.2 *Prove that the following sets form a group with respect to the operation of matrix multiplication*

$$GL(n) = \{\mathbb{A} \in \mathbb{R}^{n,n} \,|\det \mathbb{A} \neq 0\}$$

$$SL(n) = \{\mathbb{A} \in \mathbb{R}^{n,n} \,|\det \mathbb{A} = 1\}$$

$$O(n) = \left\{\mathbb{A} \in \mathbb{R}^{n,n} \,\middle|\mathbb{A}^T\mathbb{A} = \mathbb{A}\mathbb{A}^T = \mathbb{1}_n\right\}$$

$$SO(n) = \left\{\mathbb{A} \in \mathbb{R}^{n,n} \,\middle|\mathbb{A}^T\mathbb{A} = \mathbb{A}\mathbb{A}^T = \mathbb{1}_n,\ \det \mathbb{A} = 1\right\}$$

$$Sp(2s) = \left\{\mathbb{A} \in \mathbb{R}^{2s,2s} \,\middle|\mathbb{A}^T\mathbb{J}\mathbb{A} = \mathbb{J}\right\}, \quad \mathbb{J} = \left(\begin{smallmatrix} 0 & \mathbb{1}_s \\ -\mathbb{1}_s & 0 \end{smallmatrix}\right)$$

Exercise 1.3 *Calculate the determinant for any element of groups $O(n)$ and $Sp(2s)$.*

Exercise 1.4 *What is the mutual relationship between the groups from exercise 1.2?*

Exercise 1.5 *Determine what the inverse element looks like in groups $O(n)$ and $Sp(2s)$.*

Exercise 1.6 *What geometric significance do the elements of the group $O(n)$, $SL(n)$, and $SO(n)$ have if interpreted as active/passive transformations of the Euclidean space E^n? What operations do they preserve in the case of $n = 3$?*

Exercise 1.7 *Write the condition for elements of the group $O(n)$ using matrix elements and Einstein summation.*

Exercise 1.8 *Calculate $\varepsilon_{ijk}\delta_{jk}$ and show that the same result is obtained for any expression of type $A_{ij}S_{ij}$ where $S_{ij} = S_{ji}$ is symmetric and $A_{ij} = -A_{ji}$ antisymmetric.*

Exercise 1.9 *Using Einstein's summation convention, prove the following identities for scalar fields $\varphi(\vec{x})$ and vector fields $\vec{A}(\vec{x})$, $\vec{B}(\vec{x})$*

$$\operatorname{rot}(\operatorname{grad}\varphi) = 0$$

$$\operatorname{div}(\vec{A} \times \vec{B}) = \vec{B} \cdot \operatorname{rot}\vec{A} - \vec{A} \cdot \operatorname{rot}\vec{B}$$

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$

Exercise 1.10 *Prove that for a spherically symmetric scalar field $\varphi = \varphi(r)$, where $r = |\vec{r}| = \sqrt{\sum x_l^2}$ it holds that*

$$\Delta\varphi(r) = \varphi''(r) + \frac{2}{r}\varphi'(r) = \frac{1}{r}\frac{d^2}{dr^2}(r\varphi(r)) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right)$$

Exercise 1.11 *Prove the following for scalar $\varphi(\vec{x})$ and vector fields $\vec{A}(\vec{x})$, $\vec{B}(\vec{x})$*

$$\operatorname{div}(\operatorname{rot}\vec{A}) = 0$$

$$\operatorname{rot}\operatorname{rot}\vec{A} = \operatorname{grad}\operatorname{div}\vec{A} - \Delta\vec{A}$$

$$\operatorname{rot}(\vec{A} \times \vec{B}) = (\vec{B} \cdot \operatorname{grad})\vec{A} - (\vec{A} \cdot \operatorname{grad})\vec{B} + \vec{A}\operatorname{div}\vec{B} - \vec{B}\operatorname{div}\vec{A}$$

$$\operatorname{div}(\varphi\vec{A}) = \varphi\operatorname{div}\vec{A} + \vec{A} \cdot \operatorname{grad}\varphi$$

$$\operatorname{rot}(\varphi\vec{A}) = \varphi\operatorname{rot}\vec{A} + \operatorname{grad}\varphi \times \vec{A}$$

Exercise 1.12 *Prove the following for scalar fields $\varphi(\vec{x})$ and $\psi(\vec{x})$*

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$$

$$\Delta(\varphi\psi) = \varphi\Delta\psi + 2\nabla\varphi \cdot \nabla\psi + \psi\Delta\varphi$$

Exercise 1.13 *Prove the following identities, where $r = |\vec{r}|$, $\vec{p}$ is a constant vector and $\varphi(r)$ a scalar field*

$$\operatorname{grad}\varphi(r) = \varphi'(r)\frac{\vec{r}}{r}$$

$$\operatorname{grad}(\vec{p} \cdot \vec{r}) = \vec{p}$$

$$\operatorname{div}\frac{\vec{p}}{r} = -\frac{\vec{p} \cdot \vec{r}}{r^3}$$

2 Tensors

A **tensor T of type $\binom{p}{q}$** (contravariant of order $p \in \mathbb{N}_0$ and covariant of order $q \in \mathbb{N}_0$) on a real vector space V of dimension $n \in \mathbb{N}$ has, with respect to a basis $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$ of this space, n^{p+q} components $T^{i_1,\dots,i_p}_{j_1,\dots,j_q} \in \mathbb{R}$, which transform upon a change of basis $\vec{\tilde{e}}_j = \vec{e}_i S^i_{\ j}$ given by the transition matrix $\mathbb{S} \in GL(n)$ according to the relationship

$$\widetilde{T}^{i_1,\dots,i_p}_{j_1,\dots,j_q} = (\mathbb{S}^{-1})^{i_1}_{\ \ k_1} \dots (\mathbb{S}^{-1})^{i_p}_{\ \ k_p} T^{k_1,\dots,k_p}_{l_1,\dots,l_q} S^{l_1}_{\ \ j_1} \dots S^{l_q}_{\ \ j_q}.$$

A **tensor density $\mathcal{T}$ of weight $\lambda \in \mathbb{Z}$ and type $\binom{p}{q}$** on V has the same number of components as tensor T, but they transform upon a change of basis according to the relationship

$$\widetilde{\mathcal{T}}^{i_1,\dots,i_p}_{j_1,\dots,j_q} = (\det \mathbb{S})^{\lambda} (\mathbb{S}^{-1})^{i_1}_{\ \ k_1} \dots (\mathbb{S}^{-1})^{i_p}_{\ \ k_p} \mathcal{T}^{k_1,\dots,k_p}_{l_1,\dots,l_q} S^{l_1}_{\ \ j_1} \dots S^{l_q}_{\ \ j_q}.$$

A tensor or tensor density is called **invariant** with respect to a chosen group of transformations if its components do not change during these transformations. The **order of a tensor** of type $\binom{p}{q}$ is the number $p+q$. Indices that are upper are called **contravariant** and lower indices are called **covariant**. A tensor is **symmetric (antisymmetric)** in two indices of the same type if its components do not change (change sign) upon their interchange. If a tensor of type $\binom{p}{0}$ or $\binom{0}{q}$ is symmetric (antisymmetric) with respect to all pairs of indices, it is called **completely symmetric (completely antisymmetric)**. A non-degenerate bilinear symmetric form $g \in T^0_2(V)$ called a **(pseudo)metric tensor** allows for the canonical identification of elements V and its dual space $V^{\#}$ which corresponds to **lowering** $T^{i}_{\cdot jk} = g_{jl} T^{il}_{\cdot\cdot k}$ **and raising** $T^{il}_{\cdot\cdot k} = g^{lj} T^{i}_{\cdot jk}$ **indices**. Where (g^{ij}) are elements of the inverse matrix to the matrix (g_{ij}). When raising and lowering indices, it is necessary to maintain the same relative order of indices among upper and lower indices. Dots are used to denote the order of indices. **Contraction of a tensor** T refers to a tensor whose components are derived from the components $T^{i_1\dots i_p}_{j_1\dots j_p}$ of the tensor T by summing over a pair of the same indices, one of which is upper and the other lower. Contraction in indices i_k and j_l corresponds to multiplying tensor components by the Kronecker $\delta^{j_l}_{i_k}$ and performing Einstein summation.

Exercise 2.1 *Based on the values of quantities* **A, B** *in basis $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_3)$ and $\widetilde{\mathcal{E}} = (\vec{\tilde{e}}_1, \dots, \vec{\tilde{e}}_3)$, determine whether they are vectors or covectors.*

quantity	basis ε	basis $\tilde{\varepsilon}$
A	$(1,2,3)$	$(2,0,-1)$
B	$(4,5,6)$	$(15,5,-2)$

$\vec{\tilde{e}}_1 = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$
$\vec{\tilde{e}}_2 = \vec{e}_1 - \vec{e}_2 + \vec{e}_3$
$\vec{\tilde{e}}_3 = \vec{e}_1 - \vec{e}_3$

Exercise 2.2 *Let the components of quantity* **C** *in basis $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_3)$ have values $(1,1,1)$ and in the basis $\widetilde{\mathcal{E}}$ from the previous exercise also have values $(1,1,1)$. Is the quantity* **C** *a covariant or contravariant vector?*

Exercise 2.3 *Consider a quantity T defined on V_n, whose components with respect to any basis of V_n are equal to the products $u^i v^j$ of components of vectors $\vec{u}, \vec{v} \in V_n$. How does this quantity transform upon a change of basis $\vec{\tilde{e}}_j = \vec{e}_i S^i_{\ j}$, $\mathbb{S} \in GL(n)$? What type of quantity is it? Write the relationship between T and $\vec{u}, \vec{v}$ a) using matrices b) using tensor operations.*

Exercise 2.4 *What are the components of tensors $\underleftarrow{\alpha} \otimes \vec{v}$, $\vec{v} \otimes \underleftarrow{\alpha}$, $\hat{1} \otimes \vec{v}$, $\vec{v} \otimes \hat{1}$ in basis $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$.*

Exercise 2.5 *Using matrices, write the relationships for transformations of components for all types of 2nd order tensors on V_n.*

Exercise 2.6 *Let $\mathbb{F} \in GL(n)$ be the transition matrix from an orthonormal basis $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$ to a generally non-orthonormal basis $\widetilde{\mathcal{E}} = (\vec{\tilde{e}}_1, \dots, \vec{\tilde{e}}_n)$. Show that the scalar product $\vec{a} \cdot \vec{b}$ in basis $\widetilde{\mathcal{E}}$ has the form $\vec{a} \cdot \vec{b} = a^i b^j g_{ij}$, where $g_{ij} := F^k_{\ i} F^k_{\ j}$ is the Gram matrix of the set $(\vec{\tilde{e}}_1, \dots, \vec{\tilde{e}}_n)$.*

Exercise 2.7 *Show that the elements of the matrix g_{ij} defining the scalar product in a general basis can be considered as components of a covariant (so-called metric) tensor.*

Exercise 2.8 *Show that $\gamma = \det(g_{ij})$, where (g_{ij}) is the matrix defining the scalar product in a general basis, is a scalar density of degree 2.*

Exercise 2.9 *Define a quantity T, whose components in any basis of the vector space V_n are given by $T^i_j = \delta^i_j$, show that it is a $GL(n)$–invariant tensor of type $\binom{1}{1}$, namely, the identity tensor.*

Exercise 2.10 *Define a quantity T, whose components in any orthonormal basis of the vector space V_n are given by $T_{ij} = \delta_{ij}$. Show that T behaves as an invariant tensor of type $\binom{0}{2}$ under orthogonal transformations. What would its components look like in any basis if T was indeed a tensor?*

Exercise 2.11 *Define a tensor T of type $\binom{0}{3}$ such that its components in any right-handed orthonormal basis of the vector space $\mathbb{R}^3$ are given by $T_{ijk} = \varepsilon_{ijk}$, what will the components look like in any basis? (some authors call such a quantity the Levi-Civita tensor).*

Exercise 2.12 *Determine what type of quantity is represented by components which are given in any basis as $T^{ijk} = \varepsilon^{ijk} \equiv \varepsilon_{ijk}$.*

Exercise 2.13 *Decompose the tensor $(F_{ij}) = \left(\begin{smallmatrix} 1 & 7 \\ -3 & 4 \end{smallmatrix}\right)$ into its symmetric part F^S and antisymmetric part F^A. For the symmetric part, find the polar basis, i.e., a basis in which F^S is diagonal, and find the components of F^A and F with respect to this basis.*

Exercise 2.14 *How many independent components does a 2nd order tensor of type $\binom{p}{0}$ have on spaces V_n, V_3, and V_4 if it is a) completely antisymmetric b) completely symmetric?*

Exercise 2.15 *Perform the contraction of a tensor $T \in T^1_1(V_3)$ whose components are in a given basis $\mathcal{E}$ as $\mathbb{T} = (T^i_j) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{smallmatrix}\right)$. Does the result depend on the given basis $\mathcal{E}$?*

Exercise 2.16 *Using the (pseudo)metric tensor $g = (g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$, lower the indices of vector $\mathbf{X}$ and contravariant second-order tensor $\mathbf{F}$. Indices μ, ν range from 0 to 3.*

Exercise 2.17 *Show that the only $GL(n)$–invariant tensors of odd order are zero tensors.*

Exercise 2.18 *Show that under orthogonal transformations, the components of vectors and covectors transform in the same way.*

Exercise 2.19 *How does the dual basis of the dual vector space transform?*

Exercise 2.20 *How many independent components does a tensor of type $\binom{p}{0}$ have on space V_n, if it is a) completely antisymmetric b) completely symmetric?*

Exercise 2.21 *Show that the totally antisymmetric Levi-Civita symbols in dimension n $\varepsilon_{i_1,i_2,\ldots i_n} \in \{-1, 0, 1\}$, $\varepsilon_{1,2,\ldots,n} = 1$, can be considered as components of a $GL(n)$–invariant covariant tensor density of weight -1 or invariant contravariant tensor density of weight +1. Use the definition of the determinant of a matrix. (Levi-Civita tensor densities)*

Exercise 2.22 *Show that for any tensor T of type $\binom{1}{1}$, the coefficients of the characteristic polynomial $\det(S - \lambda\hat{1}) = -\lambda^3 + \vartheta_1\lambda^2 - \vartheta_2\lambda + \vartheta_3 = 0$ are $GL(n)$–invariant quantities:*
$\vartheta_1 = \delta^i_j T^j_i = T^i_i = Tr(T)$
$\vartheta_2 = \frac{1}{2}(\varepsilon_{ijk}\varepsilon^{lmk} T^i_l T^j_m) = \frac{1}{2}(T^i_i T^j_j - T^i_j T^j_i)$
$\vartheta_3 = \frac{1}{3!}(\varepsilon_{ijk}\varepsilon^{lmn} T^i_l T^j_m T^k_n) = \frac{1}{3!}(\varepsilon_{ijk}\varepsilon^{ijk}\det(T)) = \det(T)$.

3 Affine Space, Tensor Fields, Galilean Transformations

An **affine (rectilinear) coordinate system** $\langle o, \mathcal{E}\rangle$ in an affine space A_n of dimension $n \in \mathbb{N}$ is given by choice of an origin $o \in A_n$ and a basis $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$ of the associated vector space $\vec{A}_n$. The coordinates of a point $b \in A_n$ in this system are $x^i(b) = \overleftarrow{e}^{\,i}(b - o)$ for all $i \in \hat{n}$.

A **tensor field** T **of type** $\binom{p}{q}$ on an affine space A_n is a mapping $T : A \to T^p_q(\vec{A}_n)$, which assigns to each point of A_n a tensor of type $\binom{p}{q}$ defined on its associated vector space $\vec{A}_n$. The components of the tensor field transform upon coordinate transformation of the affine space $x^i = \hat{x}^i(\widetilde{x}), \forall i \in \hat{n}$ according to the relationship

$$\widetilde{T}^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\widetilde{x}) = (\mathbb{S}^{-1})^{i_1}{}_{k_1} \dots (\mathbb{S}^{-1})^{i_p}{}_{k_p} T^{k_1,\dots,k_p}_{l_1,\dots,l_q}(x) \mathbb{S}^{l_1}{}_{j_1} \dots \mathbb{S}^{l_q}{}_{j_q}, \quad \text{where } \mathbb{S}^i_j = \frac{\partial \hat{x}^i}{\partial \widetilde{x}^j}$$

are elements of the Jacobian matrix of the transformation. A coordinate transformation represents a **symmetry of the tensor field** T if it holds that $\widetilde{T}^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\widetilde{x}) = T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(\widetilde{x})$ i.e., its components are the same function before and after the transformation.

An affine space (A_n, g) with a scalar product $g(\vec{a}, \vec{b}) = \vec{a} \cdot \vec{b}$ is called **Euclidean** and denoted as E_n. An affine coordinate system $\langle o, \mathcal{E}\rangle$ in a Euclidean affine space E_n is called a **Cartesian coordinate system** (CCS) if the basis $\mathcal{E}$ is orthonormal. In CCS, covariant and contravariant components are the same (under $O(3)$ transformations between CCS, they transform in the same way) and thus in CCS we write all the indices as subscripts. A CCS firmly connected with a reference (rigid) body and clocks (method of time measurement) forms a **reference frame** (RF). Physical quantities related to moving bodies (such as position vector, instantaneous velocity and acceleration, kinetic energy, momentum, angular momentum) are defined with respect chosen RF. Therefore, it is necessary to distinguish between transformations among coordinates, whether Cartesian or general within one reference frame, changing only the mathematical description without deeper physical content (as in winter semester) from transformations between reference frames (Galilean principle of relativity).

Galilean transformations are coordinate transformations between CCS $\langle o, \mathcal{E}\rangle, t$ and CCS $\langle \tilde{o}(t) = o + \vec{W}t + \vec{x}_0, \widetilde{\mathcal{E}} = \mathcal{E}\mathbb{S}\rangle, \tilde{t} = t - t_0$ in na Euclidean affine space A, where $\vec{W}, \vec{x}_0 \in \vec{A}$ and $\mathbb{S} \in SO(3)$. Galilean transformations are transformations between inertial RF (represented by CCS).[3]

Exercise 3.1 *How do affine coordinates $x^i(b)$ of a point $b \in A$ in an affine space transform from one coordinate system $\langle o, (\vec{e}_1, \dots, \vec{e}_n)\rangle$ to another $\langle \widetilde{o}, (\vec{\widetilde{e}}_1, \dots, \vec{\widetilde{e}}_n)\rangle$?*

Exercise 3.2 *Derive the inverse relation to $\tilde{x}^j(b) = (\mathbb{S}^{-1})^j_i(x^i(b) - x^i(\tilde{o}))$, i.e., $x^i(b)$ as a function of $\tilde{x}^i(b)$ and $\tilde{x}^i(o)$.*

Exercise 3.3 *Show that general momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$ transform as components of a covector when changing general coordinates $q^i = \hat{q}^i(Q, t)$, $\forall i \in \hat{s}$.*

Exercise 3.4 *Show that for any tensor field $T_{ij}(x)$ on a Euclidean affine space E_n, its divergence $\frac{\partial T_{ij}}{\partial x_j}$ transforms as a vector field under orthogonal coordinate transformations (transformations between CCS).*

Exercise 3.5 *Show that for any vector field $\vec{F}(x)$ on E_n, the quantity* $\operatorname{rot}\vec{F}(x)$ *transforms as a vector field under proper orthogonal coordinate transformations (transformations between right-handed CCS).*

[3]Sometimes $\mathbb{S} \in O(3)$ is considered and sometimes even a change in the direction of time flow $\tilde{t} = -t$ is allowed.

Exercise 3.6 *Transform the intensity of the electric field $\vec{E}(x)$ induced by a point charge Q located at the origin of CCS $\langle o, \mathcal{E} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\rangle$ into*
a) CCS $\langle \tilde{o} = o + \vec{a}, \widetilde{\mathcal{E}} = \mathcal{E}\rangle$ whose origin is shifted by $a = (a_1, a_2, a_3) \in \mathbb{R}^3$
b) CCS $\langle \tilde{o} = o, \widetilde{\mathcal{E}} = \mathcal{E}\mathbb{S}\rangle$, where $\mathbb{S} \in SO(3)$
c) [H.W.] CCS whose axes are rotated around the 3rd axis by an angle $\frac{\pi}{6}$.
Which of these transformations represent a symmetries of the vector field $\vec{E}(x)$?

Exercise 3.7 *Compose Galilean coordinate transformations given by elements $(\mathbb{S}, W, x_0)$ and $(\mathbb{S}', W', x_0')$, where $\mathbb{S}, \mathbb{S}' \in O(3)$ and $W, W', x_0, x_0' \in \mathbb{R}^3$, i.e., write the result as $(\mathbb{S}'', W'', x_0'')$.*

Exercise 3.8 *Write the Galilean transformation $\widetilde{x} = \mathbb{S}^T(x - Wt - x_0)$, $\tilde{t} = t$ of coordinates (passive) using matrices – matrix representation of the Galilean group. Using this notation, compose two successive Galilei transformations.*

Exercise 3.9 *Show that the only single-particle Galilean invariant system is a free particle.*

Exercise 3.10 *How does the kinetic energy of a free particle transform when transforming coordinates $x_i = \widetilde{x}_i + w_i t, \forall i = 1, 2, 3$, where w_i are constants, if this transformation is a) a change of coordinates within one reference frame b) a transition between reference frames?*

Exercise 3.11 *Find the transformation relation for the total angular momentum $\vec{L} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha$ of a system of N mass points when transitioning from an inertial RF $\langle o, \mathcal{E}\rangle$ to RF $\langle o'(t), \mathcal{E}' = \mathcal{E}\rangle$ moving relative to it, such that it is*

a) non-inertial

b) inertial i.e., $\vec{r}(o') = \vec{W}t + \vec{a}_0$, where $W, a_0 \in \mathbb{R}^3$ are constants. Show that in this case for an isolated system $\vec{L}' = constant$.

c) the center of mass frame i.e., $\vec{x}(o') = \vec{R}$. Show that the second law of motion has the same form $\dot{\vec{L}} = \vec{N}^{(e)}$ in the center of mass frame as in the inertial reference frame.

Exercise 3.12 *Show that for any scalar field $U(x)$ on E_n, the quantity* $\operatorname{grad} U$ *transforms as a vector field under orthogonal transformations.*

Exercise 3.13 *Show that for any vector field $\vec{F}(x)$ on E_n, the quantity* $\operatorname{div} \vec{F}$ *transforms as a scalar field under orthogonal coordinate transformations.*

Exercise 3.14 *Transform the scalar field $\phi(x_1, x_2, x_3) = -\frac{GM}{\sqrt{x_1^2+x_2^2+x_3^2}}$ describing the gravitational potential of a mass point of mass M at the origin of CCS $\langle o, (\vec{e}_1, \vec{e}_2, \vec{e}_3)\rangle$ into CCS $\langle \tilde{o}, (\tilde{\vec{e}}_1 = \vec{e}_1, \tilde{\vec{e}}_2 = \frac{1}{\sqrt{2}}(\vec{e}_2 + \vec{e}_3), \tilde{\vec{e}}_3 = \frac{1}{\sqrt{2}}(\vec{e}_2 - \vec{e}_3))\rangle$ whose origin $\tilde{o}$ has coordinates $x(\tilde{o}) = (a_1, a_2, 0)$ in the original system.*

Exercise 3.15 *Transform the components of the vector field $E_i(x_1, x_2, x_3) = \frac{1}{4\pi\varepsilon_0} \frac{Qx_i}{(x_1^2+x_2^2+x_3^2)^{3/2}}$ describing the intensity of the electric field induced by a charge Q located at the origin of CCS $\langle o, (\vec{e}_1, \vec{e}_2, \vec{e}_3)\rangle$ into CCS $\langle \tilde{o} = o, (\tilde{\vec{e}}_1 = \vec{e}_2, \tilde{\vec{e}}_2 = -\vec{e}_1, \tilde{\vec{e}}_3 = \vec{e}_3)\rangle$.*

Exercise 3.16 *Scalar field transformation. Write functions describing the scalar electric potential of a system of two electrons separated by a distance l in the reference frame:*
a) $\langle o, \mathcal{E}\rangle$, where o is the midpoint of the line segment connecting the electrons and $\mathcal{E}$ is an orthonormal basis such that $\vec{e}_1$ is along the line segment connecting both electrons and $\vec{e}_2, \vec{e}_3$ are perpendicular to it?
b) $\langle \tilde{o}, \widetilde{\mathcal{E}}\rangle$, where $\tilde{o}$ is the vertex of an isosceles right triangle, whose hypotenuse is formed by the line segment connecting the electrons, and $\widetilde{\mathcal{E}}$ is an orthonormal basis such that $\vec{\tilde{e}}_1, \vec{\tilde{e}}_2$ are in the direction of each electron and $\vec{\tilde{e}}_3$ is perpendicular to them?
Realize that although functions φ and $\widetilde{\varphi}$ have different forms, they describe the same electric field!

Exercise 3.17 *Convert Newton's equations for a single mass point into general coordinates. For which choices of general coordinates will these equations have the same form as in Cartesian coordinates in the case of a free particle without forces i.e., which of the transformations will be their symmetries in the case of a null force field?*

Exercise 3.18 *Determine what restrictions arise for forces acting in a closed two-particle system from the requirement of Galilean invariance.*

Exercise 3.19 *Prove that central isotropic forces $\vec{F}(\vec{r}) = f(|\vec{r}\,|)\vec{r}$ are potential.*

4 Rotation, Rigid Body

Consider the mutual rotational motion of two reference frames represented by Cartesian coordinate systems $\langle o, \mathcal{E}\rangle$ and $\langle o, \widetilde{\mathcal{E}}(t) = \mathcal{E}\mathbb{S}(t)\rangle$ with a common origin, where $t \mapsto \mathbb{S}(t) \in SO(3)$ is a continuously differentiable mapping, from the perspective of the RF $\langle o, \mathcal{E}\rangle$. The **angular velocity pseudovector** $\vec{\Omega}$ of the rotation of the RF $\langle o, \widetilde{\mathcal{E}}(t)\rangle$ relative to the RF $\langle o, \mathcal{E}\rangle$ is defined using the **angular velocity tensor** ω, whose components with respect to the basis $\widetilde{\mathcal{E}}$ are given by $\widetilde{\omega}_{ij} = -(\mathbb{S}^T\dot{\mathbb{S}})_{ij}$. In the right-handed (orthonormal) basis $\widetilde{\mathcal{E}}$, the components of the pseudovector $\vec{\Omega}$ are equal to $\widetilde{\Omega}_i = \frac{1}{2}\varepsilon_{ijk}\widetilde{\omega}_{jk} = -\frac{1}{2}\varepsilon_{ijk}(\mathbb{S}^T\dot{\mathbb{S}})_{jk}$.

To describe the motion of a rigid body relative to the inertial (laboratory) RF $\langle o, \mathcal{E}\rangle$, we connect it with the so-called **body-fixed** RF $\langle \tilde{o}(t), \widetilde{\mathcal{E}}(t)\rangle$, whose origin $\tilde{o}(t) = o + \vec{R}(t)$ is located at its center of mass and whose axes are immobile relative to the body. The translational motion of the rigid body is then given by the resultant of external forces $\vec{F}^{(e)}$ acting on the body and the 1st momentum theorem in the laboratory frame $\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}^{(e)}$, where M is the total mass of the body.

The distribution of mass in a continuous body of volume V is described by the **moment of inertia tensor** I, whose components in the respective RF are given by $I_{jk} = \int_V \rho(\delta_{jk}x_l x_l - x_j x_k)dV$, $\forall i, j = 1, 2, 3$. In the laboratory RF and the RF of the center of mass, these components are time-dependent, hence they are usually calculated in the body-fixed RF. The non-diagonal elements of I are called deviational moments. The moment of inertia tensor is symmetric ($I_{jk} = I_{kj}$) and therefore the axes of the body-fixed RF can always be chosen so that its matrix will be diagonal $\mathbb{I} = diag(I_1, I_2, I_3)$. Such axes are called **principal axes of inertia** and the corresponding moments I_j **principal moments of inertia**. The rotational motion of the rigid body is obtained by solving the **Euler equations of motion** $\tilde{I}_{ij}\dot{\widetilde{\Omega}}_j + \varepsilon_{ijk}\widetilde{\Omega}_j\tilde{I}_{kl}\widetilde{\Omega}_l = \widetilde{N}_i^{(e)}$, $\forall i = 1, 2, 3$, representing the 2nd momentum theorem $\dot{\vec{L}}' = \vec{N}'^{(e)}$ in the center of mass frame written in components in the body-fixed frame, and then calculating the elements of the matrix $\mathbb{S}(t)$ from the equations $\dot{\mathbb{S}}_{ij} = -\varepsilon_{jkl}\widetilde{\Omega}_k\mathbb{S}_{il}$, $\forall i = 1, 2, 3$.

The kinetic energy of a rigid body in the laboratory frame $T = \frac{1}{2}M\dot{R_i}^2 + \frac{1}{2}\tilde{I}_{jk}\widetilde{\Omega}_j\widetilde{\Omega}_k$ is the sum of the kinetic energy of translational motion of the center of mass and the kinetic energy of rotational motion with respect to the center of mass (kinetic energy in the center of mass frame). In the case of rotation about a "fixed"axis passing through the body's center of mass, whose direction in the laboratory frame is given by a constant unit vector $\vec{n}$, we have $\vec{\Omega} = |\vec{\Omega}|\vec{n}$ and the kinetic energy in the center of mass frame will be $T' = \frac{1}{2}I_{jk}n_jn_k|\vec{\Omega}|^2 = \frac{1}{2}I_{\vec{n}}|\vec{\Omega}|$, where $I_{\vec{n}}$ is called the **moment of inertia of the body with respect to axis** $\vec{n}$.

Exercise 4.1 *Show that* $\omega_{ij} = -(\dot{\mathbb{S}}\mathbb{S}^T)_{ij}$ *and* $\widetilde{\omega}_{ij} = -(\mathbb{S}^T\dot{\mathbb{S}})_{ij}$ *are components of the same angular velocity tensor* ω *of the rotation of RF* $\langle\widetilde{o}, \widetilde{\mathcal{E}} = \mathcal{E}\mathbb{S}(t)\rangle$ *relative to RF* $\langle o, \mathcal{E}\rangle$ *and* Ω_i *and* $\widetilde{\Omega}_i$ *are components of the same pseudovector* $\vec{\Omega}$ *in bases* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}}(t) = \mathcal{E}\mathbb{S}(t)$*, where* $t \mapsto \mathbb{S}(t) \in SO(3)$ *is a continuously differentiable mapping.*

Exercise 4.2 *Find the components* $\widetilde{\Omega}(t)$ *of the angular velocity pseudovector* $\vec{\Omega}(t)$ *of the rotation of RF* $\langle o, \widetilde{\mathcal{E}}(t) = \mathcal{E}\mathbb{S}(t)\rangle$ *relative to RF* $\langle o, \mathcal{E}\rangle$*, if*

$$\mathbb{S}(t) = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) & 0 \\ \sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4.3 *Time change of a vector in a rotating frame. The time change of a vector* $\vec{y} = y_i\vec{e}_i$ *relative to the RF* $\langle o, \mathcal{E}\rangle$ *is defined by* $\frac{d}{dt}\vec{y} = \dot{y}_i\vec{e}_i$*. Consider two reference frames with a common origin* $\langle o, \mathcal{E}\rangle$ *and* $\langle o, \widetilde{\mathcal{E}}(t) = \mathcal{E}\mathbb{S}(t)\rangle$, $\mathbb{S}(t) \in SO(3)$*. Show that the time changes of vectors relative to the first of these frames can be expressed using* $\dot{\widetilde{\vec{e}}}_i = \frac{d}{dt}\widetilde{\vec{e}}_i = \widetilde{\omega}_{ij}\widetilde{\vec{e}}_j$ *for basis vectors and* $\dot{\vec{y}} = \frac{d}{dt}\vec{y} = \frac{\widetilde{d}}{dt}\vec{y} + \vec{\Omega} \times \vec{y}$*, where* $\frac{\widetilde{d}}{dt}$ *denotes the derivative with respect to RF* $\langle o, \widetilde{\mathcal{E}}\rangle$*. What relationships apply to the components of the vector* $\vec{y}(t) = y_i(t)\vec{e}_i = \widetilde{y}_j(t)\widetilde{\vec{e}}_j(t)$*?*

Exercise 4.4 *Determine the components of the moment of inertia tensor in the principal axes of inertia and the moment of inertia with respect to any axis passing through the center of mass for a homogeneous rectangular cuboid with edge lengths* a, b, c*.*

Exercise 4.5 *Show that for a body with constant mass density in a homogeneous gravitational field, the moment of forces with respect to the body's center of mass is zero.*

Exercise 4.6 *Solve the Euler equations of motion for a free symmetric gyroscope. What motions can this gyroscope perform?*

Exercise 4.7 *A homogeneous cylinder with principal components of the moment of inertia* $I_1 = I_2$, I_3 *that experiences no forces is rotating at time* t_0 *with an angular velocity of 10 rad/s around the 2nd axis. Around which axis and at what speed will it be rotating 10 seconds later?*

Exercise 4.8 *Stability of the rotation of a force-free gyroscope: We say that rotation is stable when a small change in* $\vec{\Omega}$ *from the free axis remains permanently small. Show that rotation around axes with the maximum and minimum principal moments of inertia is stable, whereas rotation around the axis with the intermediate moment of inertia is unstable, a small deviation grows, and the rotation "flips".*

Exercise 4.9 *Determine the magnitude of the force exerted by a misaligned wheel on the shaft bearing. Consider the misaligned wheel as a homogeneous hoop of radius* R*, which rotates with a constant angular velocity of magnitude* $|\vec{\Omega}|$ *around an axis (shaft) passing through its center, where the plane of the hoop forms an angle* ε *with this axis. The shaft is supported (in the bearing) on both sides of the*

hoop at a distance a *from the center of the hoop.*

Exercise 4.10 *Show that the derivatives* $\dot{\omega}_{ij}$ *and* $\dot{\widetilde{\omega}}_{ij}$ *are components of the same tensor in bases* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}} = \mathcal{E}\mathbb{S}(t)$*, where* $\mathbb{S}(t) \in SO(3)$*.*

Exercise 4.11 *Determine the moment of inertia tensor for a homogeneous cylinder.*

Exercise 4.12 *The principal moments of inertia of a homogeneous cylinder of height* h*, radius* r*, and mass* m *are* $\frac{1}{12}m(3r^2+h^2)$ *and* $\frac{1}{2}mr^2$*. Determine the moment of inertia with respect to any axis passing through the center of mass of the cylinder.*

Exercise 4.13 *Determine the moment of inertia tensor and the moment of inertia with respect to any axis passing through the center for a homogeneous cube.*

Exercise 4.14 *Solve the Euler equations of motion for a free spherical gyroscope.*

Exercise 4.15 *Find the rotation matrix* $\mathbb{S}(t)$ *for a homogeneous sphere on which no forces act.*

Exercise 4.16 *Determine the motion of a point on the surface of a homogeneous sphere in a homogeneous force field.*

Exercise 4.17 *Show that for any* $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) \in \mathbb{R}^3$ *it holds*

$$(\mathbb{S}^T\dot{\mathbb{S}})_{ij}\widetilde{y}_j = -\widetilde{\omega}_{ij}\widetilde{y}_j = (\widetilde{\Omega} \times \widetilde{y})_i, \quad (\widetilde{\omega}^2)_{ik}\widetilde{y}_k = (\widetilde{\Omega} \times (\widetilde{\Omega} \times \widetilde{y}))_i.$$

5 Continuum

Forces acting on an elementary volume dV of a continuum are divided into volume (body) and surface (contact) forces. **Volume forces** are external forces acting on the entire body, which depend on the volume dV of the chosen part of the body and the corresponding density (e.g., mass or charge). These forces are expressed using the force density $\vec{f}$ as $d\vec{F}^{(o)} = \vec{f}dV$. Surface forces describe the transmission of force interaction within the body. They are the forces with which neighboring elementary volumes act on the volume dV. **Surface forces** $d\vec{F}^{(p)}(\vec{r}, t, d\vec{S})$ at a given point $\vec{r}$ of the body depend on time t and the chosen elementary area $d\vec{S} = \vec{n}\,dS$, where dS is the size of this area and $\vec{n}$ its normal with external orientation (i.e., pointing out of volume dV). Using Taylor expansion of the components of the surface force by the area on which the force acts, we get $dF_i(\vec{r}, t, d\vec{S}) = dF_i(\vec{r}, t, \vec{0}) + \frac{\partial F_i}{\partial S_j}\big|_{d\vec{S}=\vec{0}}\, n_j dS + O(dS^2) \doteq 0 + \frac{\partial F_i}{\partial S_j}\big|_{d\vec{S}=\vec{0}}\, n_j dS$. The expression $\sigma_{ij}(\vec{r}, t) = \frac{\partial F_i}{\partial S_j}\big|_{d\vec{S}=\vec{0}}(\vec{r}, t)$ represents the components of a tensor field called the **stress tensor** at point $\vec{r}$ and time t. The surface force $d\vec{F}^{(p)} = \vec{T}dS$ referred to the size dS of the area $d\vec{S} = \vec{n}\,dS$, on which it acts, is called the **stress vector** $\vec{T}(\vec{r}, t)$. For the stress vector, it holds $T_i = \sigma_{ij}n_j$. Diagonal components of the stress tensor represent normal stresses to coordinate planes, off-diagonal components represent shear stresses. For forces obeying the 3rd Newton's law, the stress tensor is symmetric, and thus, in a suitably chosen Cartesian Coordinate System (CCS) at a given point in the continuum, the corresponding matrix is diagonal. The stress tensor σ can be divided into purely volumetric (tensile-pressure) $\sigma^{(o)} = \frac{1}{N}\,\mathrm{Tr}(\sigma)\hat{1}$ and purely shear $\sigma^{(s)} = \sigma - \sigma^{(o)}$ parts, where $\mathrm{Tr}(\sigma) = \sum \sigma_{ii}$ is the trace of the tensor σ (denoted as $-Np$) and $N = \mathrm{Tr}(\hat{1})$ is the trace of the unit tensor $\hat{1}$.

When examining the statics of the continuum, we substitute zeros for the accelerations of the continuum's points a_i into the continuum's motion equations $\varrho a_i = f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$ and obtain three equations $0 = f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$ for six unknown components of the stress tensor, which must be supplemented by boundary conditions specifying the distribution of stress $\sigma_{ij} n_j\big|_{\partial V} = T_i\big|_{\partial V}$ or displacement $\vec{u}$ on the surface of the continuum.

Ideal fluid is a fluid in which no shear stresses act, i.e., its stress tensor has only a volumetric part $\sigma = -p\hat{1}$. Ideal fluids include **ideal liquids**, which are perfectly incompressible ($\varrho = const.$) and **ideal gases**, which are, on the contrary, perfectly compressible and expandable. The mechanical behavior of an ideal fluid is described using the vector field of fluid flow velocity $\vec{v}(\vec{r}, t)$ and the scalar fields of density $\varrho(\vec{r}, t)$ and pressure $p(\vec{r}, t)$ of the ideal fluid. These fields must satisfy the **Euler hydrodynamic equation** for an ideal fluid $(\nabla \cdot \vec{v})\vec{v} + \frac{\partial \vec{v}}{\partial t} = \frac{1}{\varrho}(\vec{f} - \nabla p)$.

Consider an active transformation of elastic continuum – displacement of continuum points $\vec{r}' = \vec{r} + \vec{u}$ given by the displacement vector field $\vec{u}(\vec{r})$. If this displacement leads to a change in the mutual distances of the continuum's points, we call it deformation. The **deformation (strain) tensor** is a tensor ϵ, whose components are
$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$. Its part $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is called the **small deformation tensor**. Both these tensors are symmetric and thus diagonalizable by choosing a suitable CCS; moreover, they can be divided, just like the stress tensor, into purely volumetric and shear parts. Diagonal components of the deformation tensor represent the relative elongation in the direction of individual axes, off-diagonal components represent shear deformations. The trace of the small deformation tensors is called the **bulk strain** $\vartheta = \mathrm{Tr}(e)$ and represents the relative change in volume.

Using the deformation tensor, we particularly describe solid-state bodies, where changes in mutual positions of "neighboring" continuum particles are typically small compared to other states of matter. If we require the continuum to remain continuous after deformation, the components of the small deformation tensor must satisfy compatibility equations $\varepsilon_{ijm}\varepsilon_{kln}\frac{\partial^2 e_{ik}}{\partial x_m \partial x_n} = 0, \forall j, l$, provable under the assumption that the components of the displacement vector field are of class C^3.

Hooke's law $\sigma_{ij} = C_{ijkl} e_{kl}$ represents the relationship between the small deformation tensor and the stress tensor and is given by the **tensor of elastic coefficients**, whose (due to symmetries) only 21 independent components represent constants (or functions in the case of a non-homogeneous continuum) specified by the particular continuum. For a homogeneous isotropic continuum, the number of these constants is reduced to two: λ, μ called **Lamé's coefficients**, and Hooke's law takes the form $\sigma = \lambda\,\mathrm{Tr}(e)\hat{1} + 2\mu e$. Dividing both tensors into volumetric and shear parts, we can rewrite Hooke's law in the form $\sigma^{(o)} = 3K e^{(o)}$ and $\sigma^{(e)} = 2\mu e^{(s)}$, where $K = \lambda + \frac{2}{3}\mu$ is the so-called **compressibility modulus**, and μ is the **shear modulus**. Besides constants λ, μ, K, the **Young's modulus of elasticity** $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ expressing the ratio between stress and relative elongation, and **Poisson's ratio** $\bar{\sigma} = \frac{\lambda}{2(\lambda + \mu)}$ expressing the ratio between transverse shortening and longitudinal elongation are commonly defined.

Exercise 5.1 *Consider a part of a continuum in the shape of a cube with edge a where the stress tensor is constant and with respect to axes formed by the cube's edges has components σ_{ij}. Determine the normal stress to a plane surface whose edges are a) axes xy b) parallel diagonal lines of two opposite sides of the cube c) two diagonal lines going from the vertex at which the origin is located.*

Exercise 5.2 *Calculate the strain tensor and the small strain tensor in the plane for displacement vectors:*
a) $\vec{u} = (Ax, By)$ b) $\vec{u} = (By, Ax)$ c) $\vec{u} = (-Ay, Ax)$.

Exercise 5.3 *For the tensors from exercise 5.2 calculate the bulk strain ϑ and find their decomposition into purely volumetric and purely shear parts.*

Exercise 5.4 *Derive the continuity equation representing the conservation of mass for fluids.*

Exercise 5.5 *Hydrostatics: From Euler's hydrodynamic equations for an ideal fluid, derive the pressure dependence on depth in a stationary* $\vec{v}(\vec{r},t) = 0$ *ideal (i.e., incompressible* $\varrho = const.$ *and without internal friction) liquid in a homogeneous gravitational field* $\vec{g}$*.*

Exercise 5.6 *From the Euler hydrodynamic equation, derive Bernoulli's equation for stationary* $\frac{\partial \vec{v}}{\partial t} = 0$ *irrotational* $\nabla \times \vec{v} = 0$ *flow of an ideal liquid in a homogeneous gravitational field* $\vec{g}$*.*

Exercise 5.7 *Find the small deformation tensor for an elastic homogeneous isotropic continuum filling a cuboid cavity, which is compressed from one side of the cuboid of area* S *by a force* F *perpendicular to this side. Assume that you know the compressibility modulus* K *and the shear modulus* μ *for the continuum.*

Exercise 5.8 *A cuboid of height* l*, made from a homogeneous isotropic elastic material with density* ϱ*, stands on one of its square bases with edge* a *in a homogeneous gravitational field of intensity* $\vec{g}$*. Find the stress tensor, the small deformation tensor, and determine how the height of the cuboid changes due to the influence of the gravitational field. *Determine how the cuboid will deform, i.e., what the displacement vector field will look like.*

Exercise 5.9 *Express Young's modulus of elasticity* $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ *and Poisson's ratio* $\bar{\sigma} = \frac{\lambda}{2(\lambda+\mu)}$ *using the compressibility modulus* $K = \lambda + \frac{2}{3}\mu$ *and the shear modulus* μ*.*

Exercise 5.10 *Find the small deformation tensor for an elastic homogeneous isotropic continuum in the shape of a cuboid with edges* a, b, c*, which is compressed in the direction of the edge* a *by force* $\vec{F}$*. Assume the cuboid is inserted between two fixed walls perpendicular to the edge* b *and that for the continuum you know the compressibility modulus* K *and the shear modulus* μ*.*

Exercise 5.11 *Show that for the vector field of fluid flow velocity* $\vec{v}(\vec{r},t)$*, the relation* $(\vec{v}\cdot\nabla)\vec{v} = (\nabla\times\vec{v})\times\vec{v} + \frac{1}{2}\nabla(\vec{v}\cdot\vec{v})$ *holds.*

Exercise 5.12 *Derive the Navier-Stokes equation. The stress tensor of a non-ideal fluid has components* $\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}\,\mathrm{div}\,\vec{v} + 2\eta\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$*, where* λ^*, η *are viscosity coefficients,* η *is dynamic viscosity,* $\nu = \frac{\eta}{\varrho}$ *is kinematic viscosity,* $\zeta = \lambda^* + \frac{2}{3}\eta$ *is the second viscosity.*

Exercise 5.13 *Show that the quantity defined by* $\sigma_{ij}(x) = \frac{\partial F_i(x)}{\partial S_j}$ *transforms as a tensor field when transitioning between CCS.*

Exercise 5.14 *Show that the quantity defined by* $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$*, where* $\vec{u}(x)$ *is a vector field, transforms as a 2nd order tensor when transitioning between CCS.*

Exercise 5.15 *What tensile and compressive forces act at a location in the continuum where the stress tensor has the form* $\sigma = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$*.*

Exercise 5.16 *Derive the compatibility equations for deformations.*

Exercise 5.17 *Show that the only isotropic i.e., SO(3)-invariant tensors of order 1,2,3 are tensors whose components in any CCS are equal to* $0, A\delta_{ij}$ *and* $A\varepsilon_{ijk}$*, where* $A \in \mathbb{R}$*.*

Exercise 5.18 *Find the general form for components of an isotropic 4th order tensor in CCS.*

6 Special Theory of Relativity, Lorentz Transformations

The special Lorentz transformation (boost) in the direction of the x axis, representing the transition between two inertial reference frames (IRFs) with parallel axes, where IRF$'$ moves relative to IRF with velocity $\vec{V}$ in the direction of the x axis, is given by the relations

$$x' = \gamma(x - Vt), \quad t' = \gamma(t - \tfrac{V}{c^2}x), \quad y' = y, \quad z' = z, \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

is the Lorentz factor and $\beta = \frac{V}{c}$.

The coordinates of an event in Minkowski spacetime, equipped with a pseudo-metric tensor $(g_{\mu\nu}) = (g^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$, are given by the contravariant components of the position four-vector $(x^\mu) = (ct, x, y, z)$. The matrix of the boost in the direction of the x axis is

$$\mathbb{A} = (\alpha^\mu{}_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For a body (particle), we define the proper time τ and the proper (rest) length l_0 as the time and length in its instantaneous rest frame $d\tau = \frac{1}{\gamma}dt$.

Exercise 6.1 *Derive the relativistic law of composition for parallel velocities by composing two special Lorentz transformations. Are special Lorentz transformations along the x axis commutative—does their order matter?*

Exercise 6.2 *Verify that Lorentz transformations can be written in compact vector form $\vec{r}\,' = \vec{r} + (\frac{\gamma-1}{V^2}\vec{V}\cdot\vec{r} - \gamma t)\vec{V}$, $t' = \gamma(t - \frac{\vec{V}\cdot\vec{r}}{c^2})$. Hint: decompose vector $\vec{r}$ into two components, one parallel to $\vec{V}$ and one perpendicular to $\vec{V}$. The components then transform similarly to the coordinates x and y respectively.*

Exercise 6.3 *Find the matrix of the special Lorentz transformation (boost) in arbitrary direction. What property does this matrix have?*

Exercise 6.4 *Derive the relativistic law of composition of velocities for any mutual orientation of the two velocities. Specialise the result for the case of parallel and perpendicular velocities. How is the formula for $V << c$ simplified? What will be the magnitude of the resulting velocity?*

Exercise 6.5 *Determine the difference in angles between an incoming light beam and the x axis as observed in two inertial reference frames, which move relative to each other with a velocity of magnitude V in the direction of the x axis. Use this result to explain the phenomenon of stellar aberration, which is the effect of stars tracing small ellipses, circles or line segments of about $41''$ angular size across the sky over the course of a year due to the motion of the Earth around the Sun at a speed of $V = 30km/s$.*

Exercise 6.6 *Fizeau's experiment (1859). Using an interferometer, Fizeau measured the speed of light v in fluids flowing (at velocity $\pm V$) in and against the direction of propagation of light, and found the dependence $v = \frac{c}{n} \pm V(1 - \frac{1}{n^2})$, where n is the refractive index of the fluid. Derive this empirical relation using the law of velocity composition. Hint: Use a first order Taylor expansion in $\frac{V}{c}$.*

Exercise 6.7 *Consider two inertial systems S and S' connected by a special Lorentz transformation. Determine the velocity V of system S' relative to system S so that*
a) an event with coordinates $(ct, x, 0, 0)$ satisfying $(ct)^2 - x^2 = \Delta s^2 < 0$ in system S is simultaneous in system S' with an event at coordinates $(0, 0, 0, 0)$
b) an event with coordinates $(ct, x, 0, 0)$ satisfying $(ct)^2 - x^2 = \Delta s^2 > 0$ in system S is colocated in system S' with an event at coordinates $(0, 0, 0, 0)$.

Exercise 6.8 *Show that the phase of a plane light wave can be written as a product of the position four-vector and so-called wave four-vector and use the transformation of the wave four-vector to derive the Doppler effect for a plane light wave.*

Exercise 6.9 *A π^0 meson of rest mass m_0 moving at velocity v decays into two identical gamma radiation quanta (photons). Find the angle φ between the directions of motion of the photons.*

Exercise 6.10 *Show that in the absence of an external field, a photon cannot be transformed into an electron-positron pair.*

Exercise 6.11 *Compton Effect: a photon with energy $h\nu_0$ strikes an electron, which is at rest in the laboratory frame. Find the dependence of the photon's energy after the collision on the angle φ, which is the angle between the direction of the photon after the collision and its direction before the collision.*

Exercise 6.12 *Write the Lagrangian and Hamiltonian function for a relativistic charged particle (m_0, q) in an electromagnetic field determined by the scalar potential $\varphi(\vec{r}, t)$ and vector potential $\vec{A}(\vec{r}, t)$. Find the integrals of motion in the case that $\varphi = 0$ and $\vec{A} = (0, 0, A(x, y))$.*

Exercise 6.13 *Motion of a charged relativistic particle in a magnetic field - cyclotron frequency. Determine how a charged particle will move in a homogeneous magnetic field of intensity $\vec{B} = const.$ in the non-relativistic and relativistic case.*

Exercise 6.14 *The relative velocity of two particles v_{rel} is defined as the velocity of one of them in the frame where the other particle is at rest. Determine the square v_{rel}^2, if in some inertial reference frame the particles have velocities $\vec{v}_1$, $\vec{v}_2$.*

Exercise 6.15 *A muon moving with velocity $v = 0.99c$ was created in the upper layer of the atmosphere. Before decaying, it managed to travel a distance $l = 5km$ (in the Earth-connected frame - EF). a) What is the lifetime of the muon observed in the EF? b) What was the muon's lifetime in its own rest frame? c) How thick was the layer of atmosphere that passed around the muon in its rest frame?*

Exercise 6.16 *Derive formulas for time dilation and length contraction from the invariance of the interval.*

Exercise 6.17 *The velocity $\vec{v}$ (in system S) lies in the xy plane and forms an angle θ with the x axis, similarly the angle θ' is defined for the velocity $\vec{v}'$ in system S'. Derive the relation between θ and θ', if the systems S and S' are connected by a special Lorentz transformation. Determine $\tan\theta'$ using θ.*

Exercise 6.18 *A rod of proper length l' is at rest in system S', where it forms an angle θ' with the x' axis. Determine the length l and angle θ (with the x axis) in system S, if the systems S and S' are connected by a special Lorentz transformation in the direction of the x axis.*

Exercise 6.19 *Verify that the transformation formulas for four-velocity lead to the relativistic formulas for the composition of velocities. What does the transformation of zero components give?*

Exercise 6.20 *Determine the relationship between the energy and momentum of an ultrarelativistic particle, i.e., a particle moving at a velocity close to the speed of light.*

Exercise 6.21 *Cosmic rays contain protons with an energy $E = 10^{10}\mathrm{GeV}$. How long will it take them to travel through our galaxy, which has a diameter of 10^5 light-years, from the perspective of our and their rest frames?*

Exercise 6.22 *Show that in the Minkowski spacetime, the four-acceleration is perpendicular to the four-velocity, i.e., $u_\mu w^\mu = 0$.*

7 Field Theory

Exercise 7.1 *Write the Lagrangian density for a string with constant linear density* ϱ *and fixed ends under tension* T *along the* z *axis, which can only undergo transverse vibrations in the* xz *plane. Find the equation of motion for the string.*

Exercise 7.2 *Derive the equation of motion for fields described by functions* ψ, θ *in two-dimensional spacetime* (t, x)*, given its Lagrangian density* $\mathcal{L} = \frac{1}{2}\theta_x\theta_t + \frac{\alpha}{6}\theta_x^3 + \theta_x\psi_x + \frac{1}{2}\psi^2$*. Simplify the resulting equations of motion using the substitution* $\varphi = \theta_x$ *to the form corresponding to the Korteweg–de Vries equation describing waves in shallow water.*

Exercise 7.3 *Derive the equation of motion for a real scalar field* $\varphi = \varphi(t, \vec{x})$ *in Minkowski spacetime, given its Lagrangian density* $\mathcal{L} = \frac{1}{2}(\varphi,_\mu \varphi,^\mu - \kappa^2\varphi^2)$*. (Klein–Gordon)*

Exercise 7.4 *Find the solution to Maxwell's equations in a homogeneous electrically anisotropic soft dielectric, where permeability* μ *is constant and permittivity* ε_{ij} *is a symmetric second-order tensor. Look for the solution in the form of a plane wave.*

Exercise 7.5 *Consider an electromagnetic wave described by potentials* $\vec{A} = \vec{a}\cos(\vec{k}\cdot\vec{r} - \omega t)$*,* $\varphi = b\cos(\vec{k}\cdot\vec{r} - \omega t)$*. Show that the magnetic field* $\vec{B}$ *is automatically transverse, while the transversality of the electric field* $\vec{E}$ *requires fulfilling the condition* $b = \omega\frac{\vec{k}\cdot\vec{a}}{k^2}$*. Show that under this condition, vector* $\vec{B}$ *will be perpendicular to vector* $\vec{E}$*.*

Exercise 7.6 *Determine the gauge transformation that transforms the potentials from the previous example fulfilling the condition* $b = \omega\frac{\vec{k}\cdot\vec{a}}{k^2}$ *to the form* $\vec{A}' = \vec{a}'\cos(\vec{k}\cdot\vec{r} - \omega t)$*,* $\varphi' = 0$ *corresponding to the Coulomb gauge, where* $\vec{a}'\cdot\vec{k} = 0$*.*

Exercise 7.7 *What physical meaning does the quantity* $\Gamma = \oint_l \vec{A}\,\mathrm{d}\vec{l}$ *have? Is this quantity gauge invariant?*

Exercise 7.8 *Two different vector potentials, which describe a constant homogeneous magnetic field* $\vec{B} = (0, 0, B)$*. Show that these vector potentials are related by a gauge transformation, i.e., find the function* $\Lambda(\vec{r}, t)$*.*

Exercise 7.9 *Using the four-potential and the electromagnetic field tensor, derive the transformation relations for the scalar* φ *and vector* $\vec{A}$ *potential and the fields* $\vec{E}$ *and* $\vec{B}$ *during a special Lorentz transformation.*

Exercise 7.10 *Find the scalar* φ *and vector potential* $\vec{A}$ *of the electromagnetic field generated in vacuum by a point charge* e *moving uniformly straight at velocity* $\vec{V}$ *in the direction of the* x *axis in an inertial frame. What shape do the equipotential surfaces* $\varphi = const.$ *have?*

Exercise 7.11 *Decide whether the quantity* $A^\mu A_\mu$ *formed from the four-potential* A^μ *is a) Lorentz invariant b) gauge invariant.*

Exercise 7.12 *Compute the energy-momentum tensor for a scalar field described by the Lagrangian function* $\mathscr{L} = \frac{1}{2}\rho\psi_t^2 - \frac{1}{2}T\psi_x^2$ *in two-dimensional spacetime* $\mathbb{R}^2(t, x)$ *corresponding to transverse vibrations of a string. Write the corresponding conservation laws corresponding to the conserved Noether currents forming* $\mathcal{T}$*, i.e., here zero double divergence from biproducts.*

Exercise 7.13 *From the Lagrangian density for the electromagnetic field with sources* $\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu(x^\lambda)$ *derive the Maxwell–Lorentz equations.*

Exercise 7.14 *Derive the field equations of motion in two-dimensional spacetime* (t, x) *from the given Lagrangian densities. The indices* t, x *denote partial derivatives with respect to* t, x*.*
a) $\mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4}\lambda\varphi^4$ *where* $\mu^2, \lambda > 0$.
b) sine–Gordon $\mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + (\cos\varphi - 1)$.

Exercise 7.15 *The energy density of elastic deformation occurring slowly (isothermal deformations) for an isotropic homogeneous elastic medium is given by the relation* $w = \left(\frac{1}{2}\lambda + \mu\right)\vartheta^2 + 2\mu\vartheta_2$*, where* λ, μ *are constants (Lamé's coefficients) and* $\vartheta = Tr(e) = e_{ii}$*,* $\vartheta_2 = \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ij})$ *invariants of the tensor of small deformations* $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)$*. Write the Lagrangian density* $\mathcal{L}$ *for an isotropic homogeneous elastic body on which no external forces act and use it to derive the equations of motion for the vector field of body point displacements* $\vec{u}(t, \vec{r})$*. Compare the result with Lamé's equation.*

Exercise 7.16 *Determine the components of the vector potentials from example* **??** *in spherical and cylindrical coordinates.*

Exercise 7.17 *Show that the Lagrangian density for the electromagnetic field with sources* $\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$ *changes only by the four-divergence of some four-vector field during a gauge transformation* $\widetilde{A}_\mu = A_\mu - \partial_\mu\Lambda$ *(i.e., it is quasi-symmetric).*

Exercise 7.18 *From the Lagrangian density* $\mathcal{L}(A_\mu, A_{\mu,\nu}, x^\lambda) = -\frac{1}{4\mu_0}\left(F_{\mu\nu}F^{\mu\nu} - 2m^2 A_\mu A^\mu\right)$*, where* m *is a constant, derive the field equations of motion.*