1 Einstein summation notation, symbols and derivatives

Cvičení 1.1 *Determine which indices are free and which are summation (dummy) indices in the following expression and add the summation marks* $A_{ij}x_j + B_{ij}C_{jk}y_k + D_{ll}x_k y_k z_i + \frac{\partial T_{ki}}{\partial x_k} + x_i y_j^2$

Cvičení 1.2 *Sum* $\delta_{ij}x_j$, δ_{ii}, $\delta_{ij}\delta_{jk}$ *and* $\delta_{jk}\epsilon_{ijk}$.

Cvičení 1.3 *Write the formulas for the product of matrices* $(\mathbb{A}\mathbb{B})_{ij}$, *the scalar product* $\vec{a}\cdot\vec{b}$ *and the cross product* $(\vec{a}\times\vec{b})_i$ *of vectors* $\vec{a},\vec{b}\in\mathbb{R}^3$ *and the determinant of a matrix* $\mathbb{A}\in\mathbb{R}^{3,3}$ *using Einstein's summation rule and symbols.*

Cvičení 1.4 *Prove the formula* $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

Cvičení 1.5 *Prove the identity* $\mathrm{curl}(\mathrm{curl}\,\vec{A}) = \mathrm{grad}(\mathrm{div}\,\vec{A}) - \Delta\vec{A}$

Cvičení 1.6 *Derivation of a composite function of several variables (chain rule). Explain in detail the schematic formula*

$$\frac{\partial f}{\partial x_i} = \sum_j \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_i}$$

and use it to calculate the derivatives of the composite function $h = f\circ g$, *where the function* $f:\mathbb{R}^2\to\mathbb{R}$ *is given by rule* $f(\vec{y}) = f(y_1,y_2) = y_1 y_2^2$ *and the mapping* $\vec{g}:\mathbb{R}^2\to\mathbb{R}^2$ *by formula* $g_1(\vec{x}) = g_1(x_1,x_2) = x_1^2 + 3x_2$ *a* $g_2(\vec{x}) = g_2(x_1,x_2) = x_1 x_2^2$.

Cvičení 1.7 *Prove Euler's theorem for homogeneous functions of degree* $k\in\mathbb{N}$

$$\sum_{i=1}^{n}\frac{\partial f(\vec{x})}{\partial x_i}x_i = kf(\vec{x}).$$

A function $f:\mathbb{R}^n\to\mathbb{R}$ *is called homogeneous of degree* $k\in\mathbb{N}$ *if for any* $\lambda\in\mathbb{R}\setminus\{0\}$ *it holds* $f(\lambda\vec{x}) = \lambda^k f(\vec{x})$, $\forall\vec{x}\in\mathbb{R}^n$.

Cvičení 1.8 *Show that for a spherically symmetric scalar field* $\varphi = \varphi(r)$ *the Laplace operator can be expressed by the formula*

$$\Delta\varphi(r) = \varphi''(r) + \frac{2}{r}\varphi'(r) = \frac{1}{r}\frac{d^2}{dr^2}(r\varphi(r)) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right),$$

where $r = |\vec{r}\,| = \sqrt{\sum_{l=1}^{3} x_l^2}$.

Cvičení 1.9 *Solve the ordinary differential equations*
(1) $y' = f(x)$, *(2)* $y' = f(y)$, *(3)* $y'' = f(x)$, *(4)* $y'' = f(y')$, *(5)* $y'' = f(y)$,
where $y = y(x)$ *is unknown function and* f *is arbitrary continuous function.*

Cvičení 1.10 **Prove that every function* $f:\mathbb{R}\to\mathbb{R}$ *whose primitive function* $F:\mathbb{R}\to\mathbb{R}$ *is bounded has mean* $\langle f\rangle = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau f(t)\,\mathrm{d}t$ *equal to zero.*

Cvičení 1.11 **Prove* **the theorem about the virial (1870 Rudolf Clausius)***: For a system of N particles with masses m_α, $\alpha \in \widehat{N}$ denote*

$$G = \sum_{\alpha=1}^{N} \vec{p}_\alpha \cdot \vec{r}_\alpha, \qquad T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha^2,$$

then for each solution of Newton's equations of motion $m_\alpha \ddot{\vec{r}}_\alpha = \vec{F}_\alpha$ $\forall \alpha \in \widehat{N}$ paltí

$$\left\langle \frac{\mathrm{d}G}{\mathrm{d}t} \right\rangle = 0 \Rightarrow \langle T \rangle = -\frac{1}{2}\left\langle \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle,$$

where $\langle\ \rangle$ denotes the time mean. We call the virial the right-hand side of the equation.

Cvičení 1.12 **What the virial theorem says for a system of particles moving in a bounded part of space with bounded velocities, if all the forces acting in the system are potential and their potentials are homogeneous functions of degree k.*

Cvičení 1.13 **Viral theorem for magnetic fields: Derive the relation between the time-averaged kinetic and potential energies for a system of charged particles in a homogeneous magnetic field of induction $\vec{B}$. Assume that the motion of the particles takes place in a bounded region of space with bounded velocities, the particles have the same mass m, the same charge q, and the potential energy U is a homogeneous function of the degree k in the coordinates.*

Cvičení 1.14 **What does the virial theorem say for a linear harmonic potential and for a Coulombic field.*

Cvičení 1.15 **Prove the relation $[\vec{A}\times(\vec{B}\times\vec{C})]_i = A_j B_i C_j - A_j B_j C_i$, consider that the components of the vectors $\vec{A}, \vec{B}, \vec{C}$ do not commute.*

Cvičení 1.16 **Prove that for any $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ holds.*

2 Lagrange function

Cvičení 2.1 *Show that the Lorentz force $\vec{F} = \vec{F}(\vec{x},\vec{v},t) = e\left(\vec{E}(\vec{x},t) + \vec{v}\times\vec{B}(\vec{x},t)\right)$ can be obtained from the generalized potential $U(\vec{x},\vec{v},t) = e\left(\varphi(\vec{x},t) - \vec{v}\cdot\vec{A}(\vec{x},t)\right)$, where φ and $\vec{A}$ are the electromagnetic field potentials, for which the*

$$\vec{E} = -grad\,\varphi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = rot\,\vec{A}.$$

Cvičení 2.2 *Find the velocity components in spherical and cylindrical coordinates. Calculate the corresponding Jacobians*

$$det\,\frac{\partial \hat{x}_j}{\partial q_k}$$

for the transition to spherical coordinates $\hat{x}_j = \hat{x}_j(r,\theta,\varphi)$ and cylindrical coordinates $\hat{x}_j = \hat{x}_j(R,\varphi,z)$ in $\mathbb{R}^3$. What does the (non)nullity of the Jacobian say?

Cvičení 2.3 *Write the Lagrange function of a free (no constraint) force-free (no force) mass point in (a) Cartesian (b) spherical (c) cylindrical coordinates.*

Cvičení 2.4 *Write the Lagrange function of a free mass point subject to a homogeneous gravitational field and an elastic central isotropic force. Use it to construct the equations of motion.*

Cvičení 2.5 *Using the Lagrange function, derive the equations of motion for a mathematical pendulum with an elastic suspension of stiffness k and length l (the length of the unloaded spring). Investigate the limit $k/m \longrightarrow +\infty$ as a transition to an ideal holonomic constraint.*

Cvičení 2.6 **Learn how Lagrange functions differ for a charged mass point in an electromagnetic field for electromagnetic potentials differing by the calibration transformation ($\vec{A}' = \vec{A} + \nabla\Lambda(\vec{r},t)$, $\varphi' = \varphi - \frac{\partial\Lambda(\vec{r},t)}{\partial t}$) and show that this difference does not affect the Lagrange equations.*

3 Constraints, constraint forces, degrees of freedom, and generalized coordinates

Cvičení 3.1 *Single-sided non-retaining constraint: a mass point is placed on a vertical circle in close proximity to the highest point of the circle. From there it starts to slide (without friction) with zero initial velocity due to gravity. When does this point leave the circle?*
Instructions: write the Lagrange equations of the 1st kind, derive the constraint twice, express the Lagrange multiplier, substitute in the velocity from the ZZE, and find when the multiplier is zero.

Cvičení 3.2 *Determine the configuration space and generalized coordinates of a double planar mathematical pendulum with suspension lengths l_1 and l_2.*

Cvičení 3.3 *Two points in space are connected by a massless rod of varying length $l = l(t)$. Write down this constraint. Determine the number of degrees of freedom and find the generalized coordinates for this system. Determine the constraint forces. Calculate the velocities using the generalized velocities and the generalized coordinates and verify the dot cancellation rule $\frac{\partial \dot{\hat{x}}_i}{\partial \dot{q}_j} = \frac{\partial \hat{x}_i}{\partial q_j}$.*

Cvičení 3.4 *Determine the number of degrees of freedom and find the generalized coordinates for a mass point constrained to an ellipse that rotates about its minor axis with constant velocity ω. Instructions: the mass point is subject to constraints $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$ and $x\sin(\omega t) - ycos(\omega t) = 0$.*

Cvičení 3.5 **Show that for rheonomic holonomic constraint $f(\vec{x}, t) = 0$, the following holds*

$$\frac{\partial f}{\partial x_i}\frac{\partial \hat{x}_i}{\partial t} + \frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial x_i}\frac{\partial \hat{x}_i}{\partial q_j} = 0.$$

Cvičení 3.6 **Show that a thin disk rolling without slipping and without tilting along the horizontal plane is subject to a nonholonomic constraint*

4 Lagrange function and Lagrange's equations

Cvičení 4.1 *A mass point m slides without friction on a circular cone standing vertically on a tip in a homogeneous gravitational field of intensity $\vec{g}$. Construct its Lagrange function and the corresponding Lagrange equations. *Discuss the case of general holonomic constraint to a rotating surface $R = R(z)$.*

Cvičení 4.2 *A body of mass m_1 can slide along the axis x without friction. This body is connected by a massless rod of length l to a body of mass m_2, which by the action of gravity performs an oscillating motion in the vertical plane x, y. Find the Lagrange function in generalized coordinates and calculate the Lagrange equations. *Show that the body m_2 is moving along an ellipse and calculate the period of oscillation T of this elliptical pendulum for small amplitudes.*

Cvičení 4.3 *Two points of equal mass are constrained to move on a parabola of equation $y+x^2=0$ and connected by a massless rope of length $l>1$ that passes through the focus of the parabola and is always stretched. The system is in a homogeneous gravitational field of intensity $\vec{g}=(0,-g)$. Construct the Lagrange equations in generalized coordinates.*

Cvičení 4.4 *A homogeneous cylinder of radius R and mass M can move along the horizontal plane without friction. A homogeneous rod of mass m and length l rests on the cylinder so that the rod is perpendicular to the axis of the cylinder and one of its ends lies on the horizontal plane. The system is placed in a homogeneous gravitational field of intensity $\vec{g}$. Determine the generalized coordinates for this system and the Lagrange function, assuming that the rod is tangent to the cylinder and there is no friction between them.*

Cvičení 4.5 *Find generalized coordinates and a Lagrange function for the systems in the figure. The systems consist of two homogeneous rigid rods of length l that are connected to each other by a joint. The end of the first rod is fixed at the origin and the end of the second rod lies on the axis x and connected by an ideal spring to a stationary point on the axis x lying at a distance d from the origin.*

Cvičení 4.6 **Derive the equation of motion for a mathematical pendulum whose suspension length increases linearly with time according to the relation $l(t)=l_0(1+kt)$, where l_0, k are positive constants*

Cvičení 4.7 **A mass point in the plane (x,y) is bound to a circle of radius R whose center makes an oscillatory motion along the axis y with amplitude R, i.e., the point is subject to a constraint*

$$x^2+(y-R\cos\Omega t)^2-R^2=0,$$

where Ω and R are constants. The mass point is freely movable along the circle k and no real force acts on it. Use the Lagrange function to derive its equation of motion.

Cvičení 4.8 **Show that the form of Lagrange's equations of the 2nd kind is invariant to substitution of general coordinates, i.e. if the*

$$\frac{\hat{d}}{dt}\left(\frac{\partial}{\partial\dot{q}_j}\hat{L}(q,\dot{q},t)\right)-\frac{\partial}{\partial q_j}\hat{L}(q,\dot{q},t)=Q_j^{(o)}(q,\dot{q},t):=\sum_{i=1}^{3N}F_i^{(o)}\frac{\partial\hat{x}_i}{\partial q_j}, \tag{1}$$

and $q_j=\hat{q}_j(q'_1,\dots,q'_s,t)$ then (1), where $q\mapsto q'$, $\dot{q}\mapsto\dot{q}'$,

Cvičení 4.9 **Show that the form of the Lagrange equations of the 2nd kind does not change if the Lagrange function is changed by the total derivative of the coordinate function and time, i.e., if $\hat{L}'(q,\dot{q},t)=\hat{L}(q,\dot{q},t)+G(q,\dot{q},t)$, where $G(q,\dot{q},t)=\frac{\hat{d}}{dt}g(q,t)$.*

Cvičení 4.10 **Find an expression for the generalized momentum and generalized energy of a charged particle in an electromagnetic field from the Lagrange function $L(\vec{x},\dot{\vec{x}},t)=\frac{1}{2}m\dot{\vec{x}}^2-q[\varphi(\vec{x},t)-\dot{\vec{x}}\cdot\vec{A}(\vec{x},t)]$*

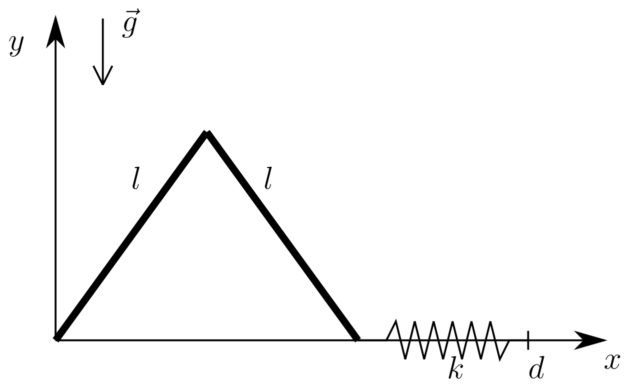
Cvičení 4.11 **How the generalized momentum and generalized energy change when the Lagrange function is changed by $\frac{\hat{d}}{dt}g(q,t)$?*

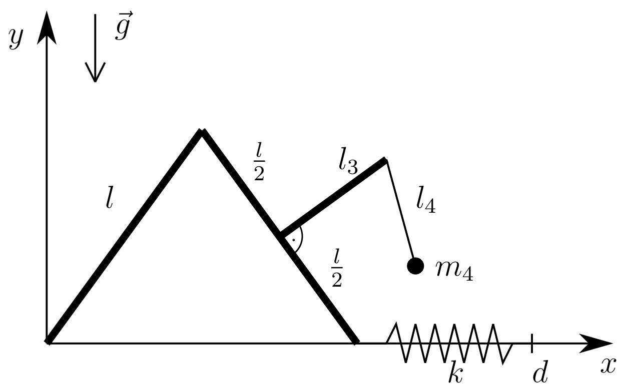
5 Small oscillations

Cvičení 5.1 *A circular disc of radius R and mass M can roll without sliding on a horizontal plane. The center of mass of the disc T lies at a distance e from its centre. The moment of inertia of the disc with respect to the axis perpendicular to its plane and passing through the center of mass is I_T. If we move the disc out of the equilibrium position, it performs a periodic motion around it due to gravity. Determine the period of this motion at small deflections.*

Cvičení 5.2 *Determine the oscillations of a system of two linear harmonic oscillators connected by a weak ($0 < \alpha << \omega_0^2$) bilinear coupling described by the Lagrange function $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2(x^2 + y^2) + \alpha xy$.*

Cvičení 5.3 *Find the angular frequencies of small oscillations of a double planar physical pendulum formed by two homogeneous rigid rods with lengths l_1, l_2 and masses m_1, m_2. *Find the normal coordinates.*





6 Integrals of motion, Noether's Theorem

Cvičení 6.1 *Prove that the functions $F_1(x, \dot{x}, t) = \dot{x} + gt$ and $F_2(x, \dot{x}, t) = \dot{x}^2 + 2gx$ are the first integrals of the equation $\ddot{x} + g = 0$, where $g = konst$. Use them to calculate the trajectory $x = x(t)$.*

Cvičení 6.2 *Find the cyclic coordinates and integrals of motion for the system described by the Lagrange function $L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - mgx_3$. Rewrite this function in cylindrical coordinates and again find the cyclic coordinates and integrals of motion. Show that the found integral of motion is the third component of angular momentum written in cylindrical coordinates*

Cvičení 6.3 *Find the integrals of motion for a charged particle with charge e and mass m in a homogeneous magnetic field of induction $\vec{B} = (0, 0, B)$ with vector potential*

(a) $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$

(b) $\vec{A} = (0, Bx_1, 0)$.

Compare the results found and explain any discrepancy.

Cvičení 6.4 *Which components of the total momentum $\vec{P}$ and total angular momentum $\vec{L}$ of a system of particles are conserved in a force field $U = U(x, y, z)$ whose equipotential surfaces are*

a) planes perpendicular to the axis z

b) *cylindrical surfaces with the axis z*

c) *spherical surfaces centered at the origin*

d) $U = U_1 + U_2$, *where* U_1, U_2 *have property c), but with different centres of symmetry lying on the axis z.*

Cvičení 6.5 *Show that when a particle moves in a field* $U = \frac{\alpha}{r}$ *where* $\alpha \neq 0$, *there is a vector integral of motion* $\vec{A} = \vec{v} \times \vec{L} + \alpha\frac{\vec{r}}{r}$, *called the Runge–Lenz vector, which is specific to that field*

Cvičení 6.6 *Use the integrals of motion to find the shape of the particle trajectory in the field* $U = \frac{\alpha}{r}$.

Cvičení 6.7 ** Prove that the functions* $F_1(x, \dot{x}, t) = -\omega t + \operatorname{arctg}(\frac{\omega x}{\dot{x}})$ *and* $F_2(x, \dot{x}, t) = \frac{\dot{x}^2}{\omega^2} + x^2$, *are integrals of motion for a system with equation of motion* $\ddot{x} + \omega^2 x = 0$, *where* $\omega > 0$ *is a constant. Use them to calculate the trajectory* $x = x(t)$.

Cvičení 6.8 **Show that if the Lagrange function does not depend on time (isolated system), the generalized energy*

$$E = E(q, \dot{q}) := \sum_{j=1}^{s} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}(q, \dot{q}) - L(q, \dot{q}).$$

is the integral of the motion.

Cvičení 6.9 **Let the Lagrange function have the form*

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U\left((x_1^2 + x_2^2), x_3\right).$$

Show that the vector field $\vec{Y}(\vec{x}) = (-x_2, x_1, 0)$ *satisfies*

$$\sum_{j=1}^{s}\left[\frac{\partial L}{\partial x_j}Y_j + \frac{\partial L}{\partial \dot{x}_j}\left(\sum_{k=1}^{s}\frac{\partial Y_j}{\partial x_k}\dot{x}_k + \frac{\partial Y_j}{\partial t}\right)\right] = 0,$$

Write the corresponding conserving quantity.

Cvičení 6.10 **Let the Lagrange function have the form (free mass point on a line)*

$$L = \frac{1}{2}m\dot{x}^2.$$

Show that the equations of motion are invariant to "scaling" $x \mapsto x\, e^{\alpha}$, $\alpha \in \mathbb{R}$, *What does the vector field Y look like for this group of transformations? Is* $F(x, \dot{x}, t)$ *given by the relation*

$$F(x, \dot{x}, t) = \sum_{j=1}^{s} Y_j(x, t)\frac{\partial}{\partial \dot{x}_j}L(x, \dot{x}, t)$$

the integral of motion?

7 Principle of virtual work and d'Alembert principle

Cvičení 7.1 *Using the principle of virtual work , derive the conditions for equilibrium on the lever.*

Cvičení 7.2 *Using the virtual work principle, find the equilibrium position for a homogeneous rigid rod of length l and mass m in a homogeneous gravitational field of intensity $\vec{g}$, which is supported in the vertical plane by a table edge and is leaning against a vertical wall. The wall is spaced from the table edge by $a < l/2$. Consider the constraints as ideal.*

Cvičení 7.3 **Determine the conditions for real and virtual displacements for a mass point moving along tri-axial ellipsoid whose axes are time dependent*

Cvičení 7.4 **How a particle on which no forces act will move in the xy plane if subjected to nonholonomic constraint $\alpha t\dot{x} - \dot{y} = 0$*

8 Calculus of variations

Cvičení 8.1 *Along what path between two points in the vertical plane xy does a bee move to reach its destination in the shortest possible time? Assume that its speed is proportional to its height, $v = ky,\ k > 0,\ y > 0,\ x_1 \neq x_2$.*

Cvičení 8.2 *Determine the position of a heavy homogeneous fiber under the influence of gravity. Among all plane curves of length $\int_{x_0}^{x_1} \sqrt{1 + y'^2}dx = l$ whose ends lie at given points $(x_0, y_0), (x_1, y_1)$, find those whose vertical coordinate of the center of gravity $y_{CM} = \frac{1}{M}\int y dm = \frac{1}{l}\int_{x_0}^{x_1} y\sqrt{1 + y'^2}dx$ is the minimum.*

Cvičení 8.3 ** Brachistochrone problem. Find a plane curve connecting two points A, B in the vertical plane such that a mass point launched with zero initial velocity from point A and moving along this curve due to gravity reaches point B in the shortest time.*

9 Hamiltonian and Hamilton's equations

Cvičení 9.1 *Deduce Hamilton's equations directly by calculating the derivatives $\frac{\partial H}{\partial q_j}, \frac{\partial H}{\partial p_j}$.*

Cvičení 9.2 *Construct Hamilton's equations for the motion of a free mass point in a field of conservative forces (in Cartesian coordinates) and show that the equations obtained are equivalent to Newton's equations.*

Cvičení 9.3 *Write the Hamiltonian function of a linear harmonic oscillator.*

Cvičení 9.4 *Write the Hamiltonian function and construct the Hamilton's equations of a particle with charge e and mass m in a given external electromagnetic field with potentials $\varphi(\vec{r}, t), \vec{A}(\vec{r}, t)$.*

Cvičení 9.5 *Write the Hamiltonian function of a free mass point in Cartesian, spherical, and cylindrical coordinates in a field $U(\vec{x})$.*

Cvičení 9.6 *Construct Hamilton's equations for the motion of a free mass point under the influence of a central force with potential $U(r)$ in spherical coordinates. Determine the integrals of the motion.*

Cvičení 9.7 *The motion of mass point m is constrained to a cylindrical surface $x^2 + y^2 = R^2$ and moves along it under the influence of the central elastic force $\vec{F} = -k\vec{r}$. Find the Hamiltonian function, write the Hamilton's equations and solve them (in cylindrical coordinates).*

10 Poisson brackets and integrals of motion

Cvičení 10.1 *Compute* $\{e^{\alpha q}, e^{\beta p}\}$.

Cvičení 10.2 *Compute* $\{q_i, q_j\}$, $\{p_i, p_j\}$, $\{q_i, p_j\}$.

Cvičení 10.3 *Calculate the Poisson brackets for the components of the momentum* p_j *and angular momentum* $L_i = \varepsilon_{ijk} x_j p_k$ *of the particle i.e.* $\{L_i, p_j\}$ *and* $\{L_i, L_j\}$. *Will the same relations hold for the total momentum and total angular momentum of the particle system ?*

Cvičení 10.4 *Prove that if* L_1, L_2 *are integrals of motion, then* L_3 *is an integral of motion.*

Cvičení 10.5 ** Prove Poisson's theorem: the Poisson bracket of two integrals of motion is again an integral of motion.*

Cvičení 10.6 **Using Poisson's theorem, derive the next first integral of Hamilton's equations of motion (i.e., the integral of motion) in the case of a mass point under the influence of a central force in a rotating system. Suppose that the following integral of motion are given:* $u = \sum_{i=1}^{3} \frac{p_i^2}{2m} - \Omega \cdot (x_1 p_2 - x_2 p_1) + U(r)$, $v = \sum_{i=1}^{3} \frac{p_i^2}{2m} + U(r)$.

Cvičení 10.7 **Show that* $\{L_3, F\} = 0$, *where* $F = F(\vec{q} \cdot \vec{p})$ *is an arbitrary (sufficiently smooth) scalar function of the coordinates and momentum of the particle*

Cvičení 10.8 **Confirm that the components of the angular momentum* $\vec{L} = \vec{r} \times \vec{p}$ *and the Runge–Lenz vector* $\vec{A} = \frac{1}{m}\vec{p} \times \vec{L} + \alpha\frac{\vec{r}}{r}$ *satisfy the relations:* $\{L_i, A_j\} = \varepsilon_{ijk} A_k$, $\{A_i, A_j\} = -2H\varepsilon_{ijk} L_k$, *where* $H = \frac{\vec{p}^2}{2m} - \frac{\alpha}{r}$, $\alpha > 0$.

Cvičení 10.9 **Show that for a free particle with charge e in a magnetic field* $\vec{B}(\vec{x}, t)$, $\{m\dot{x}_i, m\dot{x}_j\} = e\varepsilon_{ijk} B_k$ *holds. Hint: In the Hamiltonian formalism, velocities* $\dot{x}_i$ *are functions on phase space, i.e., they depend on the variables* $\vec{x}, \vec{p}, t$

11 Canonical transformations

Cvičení 11.1 *Find the canonical transformations determined by the generating functions a)* $F_2 = \sum_k q_k P_k$, *b)* $F_2 = \sum_k f_k(\vec{q}, t) P_k$, *c)* $F_1 = \sum_k q_k Q_k$.

Cvičení 11.2 *Show that the transformation* $Q_j = p_j$, $P_j = -q_j$ *is canonical.*

Cvičení 11.3 *Show that the canonical transformation* $(x_1, x_2, x_3, p_1, p_2, p_3) \longrightarrow (R, \varphi, z, P_R, P_\varphi, P_z)$ *defined by the generating function* $F_2 = \sqrt{x_1^2 + x_2^2} P_R + \mathrm{arctg}(\frac{x_2}{x_1}) P_\varphi + x_3 P_z$ *converts Cartesian coordinates to cylindrical coordinates.*

Cvičení 11.4 *Show that the transformation* $Q = \mathrm{arctg}(\sqrt{km}\frac{q}{p})$, $P = \frac{1}{2}(\sqrt{km} q^2 + \frac{p^2}{\sqrt{km}})$ *is canonical. Find a generating function of the 1st kind for it. Use this transformation to solve the equations of motion of the harmonic oscillator* $H = \frac{p^2}{2m} + \frac{1}{2} k q^2$. *What is the physical meaning of the new variables* Q *and* P*?*

Cvičení 11.5 *Consider the transformation* $(q, p) \longrightarrow (Q, P)$, $Q = q^\alpha \cos(\beta p)$, $P = q^\alpha \sin(\beta p)$. *For which* $\alpha, \beta \in \mathbb{R}$ *is this transformation canonical? Find the appropriate generating function.*

12 Hamilton–Jacobi equation and Hamilton's principal function

Cvičení 12.1 *Solve the Hamilton–Jacobi equation for a free mass point described by the Hamiltonian function* $H = \sum_{i=1}^{3} \frac{p_i^2}{2m}$. *Show that the Hamilton–Jacobi equations for generating functions of the form* F_1 *and* F_2 *have solutions (complete integrals) of the forms* $S_1(q_i, t, Q_i) = \frac{m}{2t} \sum_{i=1}^{3} (q_i - Q_i)^2$ *and* $S_2(q_i, t, P_i) = -\sum_{i=1}^{3} \frac{P_i^2}{2m} t + \sum_{i=1}^{3} P_i q_i$, *respectively.*

Cvičení 12.2 *Construct the Hamilton's principal function* $S_1(q_i, t, Q_i) = \frac{m}{2t} \sum_{i=1}^{3} (q_i - Q_i)^2$ *by integrating the Lagrange function* $L = \frac{1}{2} \sum_{i=1}^{3} m \dot{q}_i^2$ *from* 0 *to* t *along the actual trajectory* $q_i(t) = Q_i + v_i t$ *of the force-free mass point.*

Cvičení 12.3 *Find the canonical transformations determined by the generating functions* $S_1(q_i, t, Q_i) = \frac{m}{2t} \sum_{i=1}^{3} (q_i - Q_i)^2$ *and* $S_2(q_i, t, P_i) = -\sum_{i=1}^{3} \frac{P_i^2}{2m} t + \sum_{i=1}^{3} P_i q_i$. *What is the geometric shape of the corresponding wavefronts* $S = konst$ *in configuration space?*

Cvičení 12.4 *Write and solve the Hamilton–Jacobi equation for the linear harmonic oscillator* $H = \frac{p^2}{2m} + \frac{kq^2}{2}$. *Construct the appropriate canonical transformation and find the phase trajectories.*

Cvičení 12.5 **Show that the function* $S(q, t, Q) = m\omega \frac{(q^2 + Q^2)\cos(\omega t) - 2qQ}{2\sin(\omega t)}$ *is a complete integral of the Hamilton–Jacobi equation(i.e. Hamilton's principal function) for the harmonic oscillator* ($\omega = \sqrt{\frac{k}{m}}$) *when* $0 < t < \pi$. *Use it to find the phase trajectories.*

13 Integrable systems and Noether's theorem

Cvičení 13.1 *Consider a system of* s *degrees of freedom whose phase space is* $\mathbb{R}^{2s}$ *(or an open subset thereof) and whose Hamiltonian function does not depend explicitly on time. Show that such a system can have at most* $2s - 1$ *mutually independent integrals of motion that do not depend explicitly on time.*

Cvičení 13.2 *Define an integrable system. Liouville's theorem*

Cvičení 13.3 *Tod's Molecule. Show that a linear three-atom molecule with Hamiltonian* $H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_1}$ *is an integrable system. Instructions: use the integrals of motion* H, $P = p_1 + p_2 + p_3$, $K = \frac{1}{9}(p_1 + p_2 - 2p_3)(p_2 + p_3 - 2p_1)(p_3 + p_1 - 2p_2) + (p_1 + p_2 - 2p_3)e^{q_1 - q_2} + (p_2 + p_3 - 2p_1)e^{q_2 - q_3} + (p_3 + p_1 - 2p_2)e^{q_3 - q_1}$.

Cvičení 13.4 *Show that the quantity* $L_3 = q_1 p_2 - q_2 p_1$ *is a rotation generator.*

Cvičení 13.5 *On the phase space of the system of* N *particles* $\alpha = 1, 2, \ldots, N$ *with coordinates* $q_{\alpha i}$ *and momenta* $p_{\alpha i}$, *where* $i = 1, 2, 3$ *is given by a function of the form* $G(q_{\alpha i}, p_{\alpha i}, t) = \sum_\alpha m_\alpha q_{\alpha 1} - \sum_\alpha p_{\alpha 1} t$ *as an infinitesimal transformation generator. Show that this function generates a special Galilei transform (along the 1st axis). Instructions: use* $\varepsilon = V$.