Introduction to Quantum Chromodynamics

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November 26, 2009

Ultraviolet renormalization – basic ideas and techniques

Literature

Our discussion is based on

Quarks, partons and Quantum Chromodynamics
by Jiří Chýla
Available at http://www-hep.fzu.cz/ chyla/lectures/text.pdf

Additional material comes from

Quantum Field Theory – A Modern Introduction
by Michio Kaku
Oxford University Press 1993

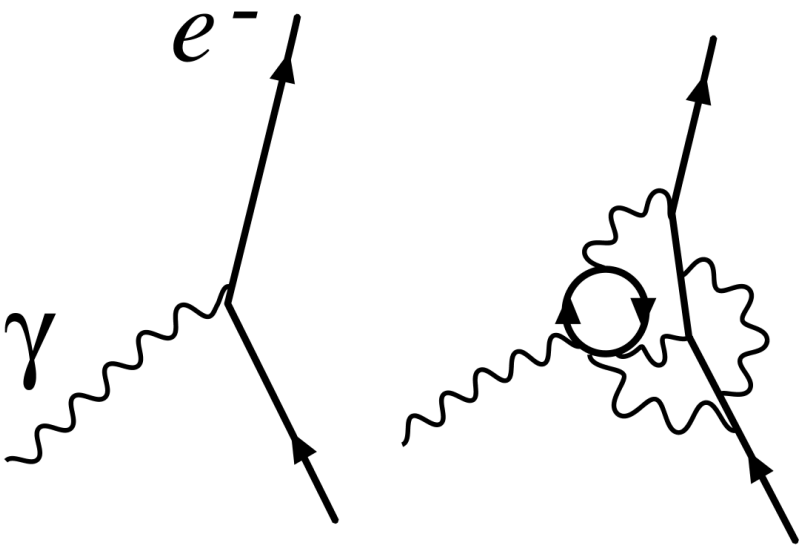
Additional material comes from

Quantum Field Theory in a Nutshell
by A. Zee
Princenton University Press 2003

Vacuum as a dielectric

- In *QED*: any charged particle is surrounded by a dense cloud of $e^- e^+$ virtual pairs that tend to screen the charge of a particle.

  At large distances – *effective coupling constant is reduced* by presence of this screening charge.
  At smaller distances – a probe can penetrate through this virtual cloud, and hence the QED *coupling constant gets larger* as we increase the energy of the probe.



Vacuum as a dielectric

- *Classically*, we can think of this in terms of the *dielectric constant of the vacuum*. Place a charge into a dielectric. $\Rightarrow$ Electric field of the dielectric will cause the dipoles within the dielectric medium to line up around the charge and decrease its value. $\Rightarrow$ the medium will acquire dielectric constant greater than one.
- *QCD*: we have color charges and color coupling constants. Each gluon carries both a color charge and an anti-color magnetic moment. The net effect of polarization of virtual gluons in the vacuum is not to screen the field, but to increase it and affect its color. This is sometimes called antiscreening.
- In *QCD*:

  At large distances (low energy) – presence of the cloud of virtual particles creates an antiscreening effect and the net *coupling constant gets larger*.
  At smaller distances – a probe that comes near a colored particle feels the *coupling constant decrease* at high energies. Thus, the dielectric constant of the vacuum is less than one for an asymptotically free theory.

Infinities in perturbative calculations

- Transition from a finite number of degrees of freedom in QM to an infinite number in QFT $\Rightarrow$ we must continually sum over an *infinite* number of internal modes in loop integrations, leading to divergences.
  Divergent nature of QFT reflects the fact that the UV region is sensitive to the infinite number of degrees of freedom of the theory.
- We'll discus UV renormalization of perturbative ϕ^4, QED and QCD.
- Renormalization procedure is not merely a technique of removing unpleasant UV infinities, but first of all an *effective description of quantum phenomena*.
- Essence of UV renormalization is common to QED and QCD $\Rightarrow$ we start with QED. Here the basic quantity governing the strength of interactions of charged particles are their **electric charges**. It is common to use *fine structure constant* $\alpha \equiv e^2/4\pi = 1/137$ (classically).
- $\mathcal{L}_{QED}$ leads to the pQFT, in which physical quantities are expressed as power expansions in α. It turns out that at higher orders the coefficients of these expansions, calculated according to the standard Feynman rules, come out formally infinite.

Divergences in integration over the loop momenta

- A: **Ultraviolet divergences** (UV), coming from integration over the *large* values of the *loop momenta*.
- B: **Mass divergences**, coming from integration over the region of *small virtualities*. These small virtualities appear in two different situations:
  - For vanishingly small energy and momentum of the virtual particles. These so called *infrared* (IR) singularities occur for the massless photon, independently of the electron mass.
  - When two of the three particles in the QED vertex e-γ-e are parallel to each other. This can happen only if $m_e = m_\gamma = 0$. $\Rightarrow$ *parallel* singularities are thus absent in QED. Trace of them shows up potentially large logarithms like $\alpha^k \ln^k(Q/m_e)$ (where Q is some external momentum) at k–th order of perturbation theory.

  Both types of mass singularities can be regularized by introduction of a fictitious mass of the photon.
- Here we shall analyze UV divergences which lead to the important concept of the **renormalized electric charge**. The mass singularities are dealt with in a completely different way and are discussed in the text of J. Chýla.

UV example: ϕ^4 theory

- Consider second order diagrams in ϕ^4. In addition to standard tree diagram (Fig.1, left) there is also so called loop diagram (Fig.1, right).
- Compare their propagators associated with internal line(s):

$$(-i\lambda)^2 \frac{i}{q^2 - m^2 + i\varepsilon} = (-i\lambda)^2 \frac{i}{(k_4 + k_5 + k_6)^2 - m^2 + i\varepsilon} \tag{1}$$

$$\frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k_1 + k_2 - k)^2 - m^2 + i\varepsilon} \sim \int \frac{d^4k}{k^4} \to \infty \tag{2}$$

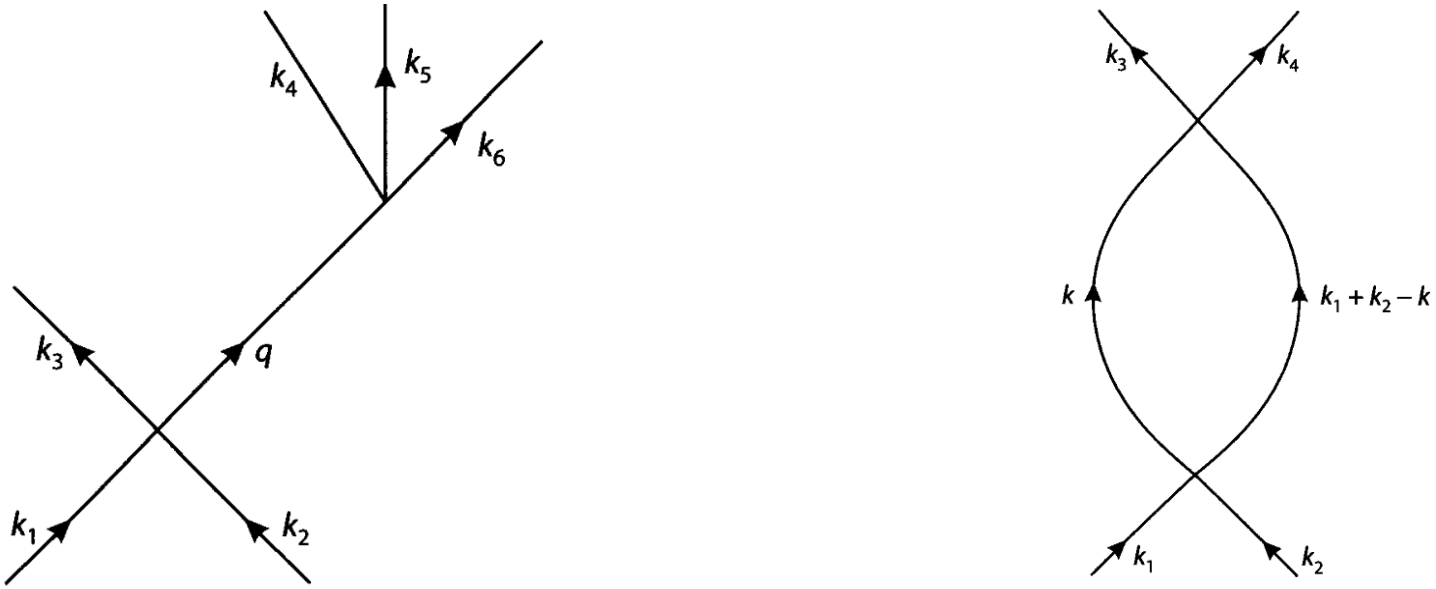


Figure 1: Feynman diagrams occuring in order g^2 in perturbation theory.

Regularization: ϕ^4 theory

- *Tree diagram* (Fig.1, left): internal line is associated with a virtual particle with $q^2 = (k_4 + k_5 + k_6)^2 \geq m^2$. The farther the momentum of the virtual particle is from the mass shell the smaller is the amplitude.
  $\Rightarrow$ *Virtual particle is penalized for not being real!*

- *Loop diagram* integrand is large only if one or the other or both of the virtual particles associated with the two internal lines are close to being real. Once again, there is a penalty for not being real. However for very large virtualities ($k \to \infty$) the integral in (2) *diverges logarithmically!*

- Thus, in evaluating $\int d^4k/(2\pi)^4$ in (2) we should integrate only up to Λ, known as a *cutoff*. The integral is said to have been *"regularized"*.
  Result is $2iC \ln \frac{\Lambda^2}{K^2}$ where $K \equiv k_1 + k_2$ and C is some numerical constant.

- *Modern view of divergences problem*: given model should be regarded as an *effective low energy theory, valid up to some energy scale* Λ. Going to higher and higher momentum scales may eventually bring us to completely new physics region where our simple ϕ^4 model is not anymore true!

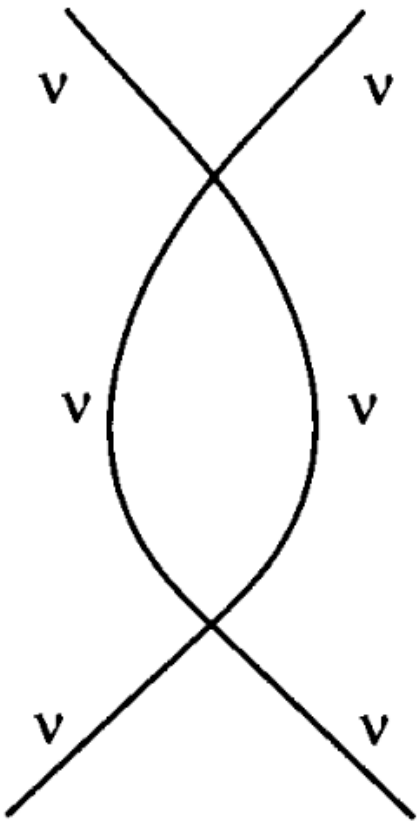
Renormalization and dimensional analysis

- Since $\hbar = c = 1 \;\Rightarrow\; S = \int d^4x\mathcal{L}$ is dimensionless: $[S] = 0 \;\Rightarrow\; [\mathcal{L}] = 4^*$. In this notation $[x] = -1$ and $[\partial] = 1$.
- In scalar ϕ^4 theory $\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] - \lambda\phi^4$. For term $(\partial\phi)^2$ to have the same dimension as $\mathcal{L}$ i.e. $\left[(\partial\phi)^2\right] = 4$ we must have $[\phi] = 1$ implying that $[\lambda] = 0$, i.e. *coupling is dimensionless*.
- For fermion field ψ we find its dimension from the free field lagrangian $\mathcal{L} = \overline{\psi} i\gamma^\mu\partial_\mu\psi + \ldots$ to be $[\psi] = \frac{3}{2}$.
- Looking at coupling $f\phi\overline{\psi}\psi \;\Rightarrow$ *Yukawa coupling f is also dimensionless*.
- From the Maxwell Lagrangian $\mathcal{L} = F^{\mu\nu}F_{\mu\nu}$ we see that $[F^{\mu\nu}] = [F_{\mu\nu}] = 2$ $\Rightarrow\; [A_\mu] = 1$ i.e. dimensions of $[A_\mu]$ and $[\partial_\mu]$ are the same.
- Consider QED interaction term $eA_\mu\overline{\psi}\gamma^\mu\psi$: $[A_\mu\overline{\psi}\gamma^\mu\psi] = 4 \;\Rightarrow\; [e] = 0$. N.B. The same conclusion can be also deduce from Coulomb's law written in natural units $V(r) = \alpha/r$, with the fine structure constant $\alpha = e^2/4\pi$.

(*) From now on $[A]$ denotes dimension of A in mass units.

Fermi theory of the weak interaction: M_{if} blows up

- In contrast 4-fermion interaction Lagrangian $\mathcal{L} = G\overline{\psi}\psi\overline{\psi}\psi$ has $[G] = -2$.
- Recall $d\sigma = \dfrac{(2\pi)^4}{\mid \vec{v}_1 - \vec{v}_2 \mid} \dfrac{1}{2E_1} \dfrac{1}{2E_2} |\mathcal{M}_{if}|^2 \prod_{i=3}^{n} \dfrac{\mathrm{d}\vec{p}_i}{(2\pi)^3 2E_i} \delta^4(p_1 + p_2 - \sum_{i=3}^{n} p_i)\ SF$
  $[\sigma] = -2, [v] = 0, [E] = 1 \ \Rightarrow\ [\mathcal{M}_{if}] = 0.$



- Calculate $\nu + \nu \to \nu + \nu$ scattering at $\sqrt{s} \ll \Lambda$. Since all masses and energies are by definition small compared to the cutoff Λ, we can simply set them equal to zero.
- In the lowest order: $\mathcal{M}_{if} \propto G$ and is *finite*.
- In the next order: $\mathcal{M}_{if} \propto G + G^2\Lambda^2 \Leftarrow [\mathcal{M}_{if}] = 0$ or by looking directly at the Feynman diagram (left) which goes as $G^2 \int^{\Lambda} d^4p(1/p)(1/p) \propto G^2\Lambda^2$.
- For $\Lambda \to \infty$ the theory is sick: $|\mathcal{M}_{if}|^2 \to \infty$.
  Infinity is predicted value for a physical quantity!
- Fermi theory and all other *non-renormalizable theories have the ability to announce its own eventual failure and hence their domains of validity*.

Renormalization

UV renormalization procedure consists of *two* distinct steps:

1) **Regularization**: Formally divergent integrals are *regularized*, i.e. basically *cut off* at high values of the loop momenta.
   There are numerous regularization techniques, for instance:
   - Simple cut-off.
   - Analytical regularization.
   - *Pauli–Villars* and its modifications: gives a clear physical interpretation of the renormalized electric charge while maintaining gauge invariance. In multiloop calculations this technique is, however, rather cumbersome.
   - *Dimensional regularization*: is much simpler and in nonabelian gauge theories, like QCD, it preserves gauge invariance. The renormalization scale, on which the renormalized electric charge does depend, enters in a rather artificial way and its physical interpretation is therefore rather obscure.

2) **Renormalization**: Divergent terms are absorbed in the newly defined **renormalized** quantities (couplant, mass and wave functions).

Example: Potential of infinite charged line

- Consider infinitely long wire carrying constant charge density ρ. Potential at at point P at distance R from the wire:

$$V(R) = \int \frac{\rho(r)}{r} dx = \rho \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{R^2 + x^2}} \tag{3}$$

  is logarithmically divergent $\Rightarrow$ *potential is ill defined*.

- Contrary to this electric field:

$$\vec{E} = -\vec{\nabla} V = V'(R) \propto \rho \int_{-\infty}^{+\infty} \frac{dx}{(R^2 + x^2)^{3/2}} < \infty. \tag{4}$$

- Regularizing (3) we obtain:

$$V_\Lambda(R) = \rho \int_{-\Lambda}^{+\Lambda} \frac{dx}{\sqrt{R^2 + x^2}} = \rho \ln \left[\frac{\sqrt{\Lambda^2 + R^2} + \Lambda}{\sqrt{\Lambda^2 + R^2} - \Lambda} \right] \tag{5}$$

  leading to slight redefinition of electric field as:

$$\vec{E} = \lim_{\Lambda \to \infty} \left[-\vec{\nabla} V_\Lambda(R) \right] = \lim_{\Lambda \to \infty} \hat{R} \frac{2\rho}{R} \frac{\Lambda}{\sqrt{\Lambda^2 + R^2}} \to \frac{2\rho}{R} \hat{R} \tag{6}$$

Example: Potential of infinite charged line

- We had to introduce new variable Λ with dimension of length i.e. $[\Lambda] = -1$.
- Notice that the difference: $\Delta = \lim_{\Lambda\to\infty} [V_\Lambda(r_2) - V_\Lambda(r_1)] = \rho \ln \frac{r_1^2}{r_2^2}$ (7)

  is well defined. So we can use it to *renormalize potential* by subtracting $V(R)$ at some fixed value of $R = R_0$ and taking the limit $\Lambda \to \infty$:

$$V(R) \to V(R) - V(R_0) = \rho \ln \frac{R_0^2}{R^2} \tag{8}$$

- Introduction of dimensionful parameter R_0 has caused non-physical infinities present in $V(R)$ and $V(R_0)$ to cancel each other, leaving a finite result with a non-trivial R-dependence. The cutoff Λ has disappeared!
- This example suggest a strategy for dealing with divergencies:

  1. Identify an appropriate way to *regularize* infinite integrals.
  2. Absorb the divergent terms into a redefinition of fields or parameters e.g. via *substractions*. This step is usually called *renormalization*.
  3. Make sure the procedure is *consistent*, by checking that the physical results do not depend on the regularization prescription.

Renormalization of ϕ^4 theory

- Divergent structure of any graph can be analyzed in terms of:

$$E = \text{number of external legs}$$
$$I = \text{number of internal lines}$$
$$V = \text{number of vertices}$$
$$L = \text{number of loops}$$

- Degree of its divergence D comes from power counting: each internal propagator contributes $1/p^2$, each loop contributes $d^4p \;\Rightarrow\; D = 4L - 2I$.
- Vertex has 4 lines connecting to it. Each of these lines, in turn, either ends on an external leg, or on one end of an internal leg, which has 2 ends. $\Rightarrow 4E = 2I + E$.
- Expression for the loop number $L = I - V + 1$ can be obtained as follows:
  - # of independent momenta = # of internal lines I - the constraints coming from momentum conservation.
  - There are V such momentum constraints, minus the overall momentum conservation from the entire graph.
  - # of independent momenta in a Feynman graph is also equal to the # of loop momenta.

Renormalization of ϕ^4 theory

- Inserting these graphical rules into our expression for D we have: $D = 4 - E$.
  $\Rightarrow$ *Degree of divergence of any graph in 4 dimensions depends only on the number of external lines, which is a necessary condition for renormalizability*.
  D is thus independent on number of internal loops in the graph[1].

- N.B. Situation in QED where $D = 4 - \frac{3}{2}E_\psi - E_A$ and $E_{\psi,A}$ is the number of external electron and photon legs, respectively, is very similar.

- In ϕ^4 theory only two-point and four-point graphs are divergent.
  $\Rightarrow$ We need to renormalize only two physical quantities: the mass and the coupling constant.

- Thus, by using only power counting arguments, in principle we can renormalize the entire theory with only two redefinitions corresponding to two physical parameters.

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(1) In d dimensions $D = d + (1 - d/2)E + (d - 4)V$. Because D increases with # of internal vertices, there are problems with renormalizing the theory in higher dimensions.

Renormalization in a nuttshell: What is actually measured

- Introduce cutt-off Λ into integral (2) and assume $m^2 \ll (k_1 + k_2)^2$ (so that we can neglect m^2 in the integrand). The scattering amplitude up to the second order in λ reads:

$$\mathcal{M}_{if} = -i\lambda + iC\lambda^2 L(s,t,u) + \mathcal{O}(\lambda^3) \tag{9}$$

where
$$L(s,t,u) \equiv \left[\ln\left(\frac{\Lambda^2}{s}\right) + \ln\left(\frac{\Lambda^2}{t}\right) + \ln\left(\frac{\Lambda^2}{u}\right)\right] \tag{10}$$

and $s \equiv K^2 = (k_1 + k_2)^2$, $t \equiv (k_1 - k_3)^2$ and $u \equiv (k_1 - k_4)^2$.

- $\mathcal{M}_{if}$ is supposed to be an actual (physical) scattering amplitude. $\Rightarrow \mathcal{M}_{if}$ should not depend on Λ. $\Rightarrow$ Any change of Λ must be always compensated by the shift of λ in such a way so that $\mathcal{M}_{if}$ does not change.
- Define *physical coupling* as: $$-i\lambda_P = -i\lambda + iC\lambda^2 L(s_0,t_0,u_0) + \mathcal{O}(\lambda^3) \tag{11}$$

and solve it for λ:

$$-i\lambda = -i\lambda_P - iC\lambda^2 L(s_0,t_0,u_0) + \mathcal{O}(\lambda^3) = -i\lambda_P - iC\lambda_P^2 L(s_0,t_0,u_0) + \mathcal{O}(\lambda_P^3) \tag{12}$$

where the second equality is allowed to the order of approximation indicated.

Renormalization in a nuttshell: What is actually measured

- Now plug this into (9):
$$\mathcal{M}_{if} = -i\lambda + iC\lambda^2 L(s,t,u) + \mathcal{O}(\lambda^3) = \\ = -i\lambda_P - iC\lambda_P^2 L(s_0,t_0,u_0) + iC\lambda_P^2 L(s,t,u) + \mathcal{O}(\lambda_P^3) \tag{13}$$
where, once again, all manipulations are legitimate up to the order of approximation indicated.
- Now in the scattering amplitude $\mathcal{M}_{if}$ we have the combination $L(s,t,u) - L(s_0,t_0,u_0) = [\ln(s_0/s) + \ln(t_0/t) + \ln(u_0/u)]$ and so the scattering amplitude comes out as:
$$\mathcal{M}_{if} = -i\lambda_P + iC\lambda_P^2 \left[\ln\left(\frac{s_0}{s}\right) + \ln\left(\frac{t_0}{t}\right) + \ln\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_P^3) \tag{14}$$
i.e. likewise (8) $\mathcal{M}_{if}$ is now expressed in terms of measurable quantity – in case of (14) it is the physical coupling constant λ_P.
The unmeasurable cutoff Λ has completely disappeared!
- <u>The lesson</u>: express physical quantities not in terms of fictitious theoretical quantities such as λ, but in terms of measurable quantities such as λ_P. Later is in the literature often denoted by λ_R and for historical reasons called the *renormalized coupling constant*.

Elements of dimensional regularization

- Consider again ϕ^4 theory in n=4 dimensional Minkowski space–time.

$$\mathcal{L}_\phi \equiv \frac{1}{2}\left(\partial_\mu \phi(x)\right)^2 - m^2\phi^2 - \frac{\lambda}{4!}\phi^4(x), \quad \lambda > 0. \tag{15}$$

- Simplest one–loop diagram – Fig. 2b – describes order λ^2 correction to the leading order diagram, depicted in Fig. 2a. Since basic four–particle vertex is associated with $-\mathrm{i}\lambda$ the amplitude corresponding to digram in Fig. 2b is:

$$I(p, n=4) \equiv \frac{\lambda^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{[k^2 - m^2 + \mathrm{i}\epsilon]}\frac{1}{[(k-p)^2 - m^2 + \mathrm{i}\epsilon]}, \tag{16}$$

where $p \equiv p_1 + p_2$. (16) is the same as (2) but in the other variables.

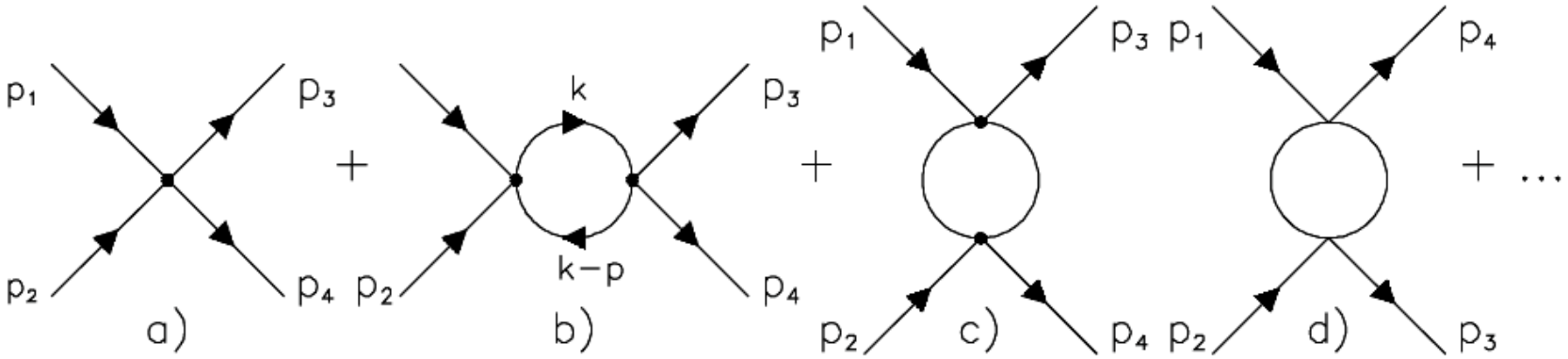

Figure 2: Lowest order Feynman diagrams for the elastic scattering of two scalar particles with mass m.

Dimensional regularization of ϕ^4 theory

- Basic idea of DR: in n dimensions the integral in (16) $\mathrm{d}^n k \propto k^{n-1}$. While in 4d it is logarithmically divergent, it converges in $n = 1, 2, 3$ dimensions.

- In $n = 1, 2, 3$ dimensions and in the rest frame of $p = (M, 0, 0, 0)$ (16) equals

$$I(p,n) = \frac{\lambda^2}{2(2\pi)^4} \int \mathrm{d}k_0 \int \mathrm{d}^{n-1}k \frac{1}{\left[k_0^2 - \vec{k}^2 - m^2 + \mathrm{i}\epsilon\right]} \frac{1}{\left[(k_0 - M)^2 - \vec{k}^2 - m^2 + \mathrm{i}\epsilon\right]}. \tag{17}$$

- The integrand depends now merely on $|\vec{k}|$ of $(n-1)$–dimensional vector $\vec{k}$.
  $\Rightarrow$ Integral over $\mathrm{d}^{n-1}k$ can therefore be performed in polar coordinates:

$$\mathrm{d}^n k = r^{n-1} \mathrm{d}r \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \cdots \sin\theta_2 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_1, \tag{18}$$

  where $r \equiv \sqrt{k^2}$ and $0 \leq \theta_1 \leq 2\pi$ is the "azimuthal" angle, all other angles $\theta_i \in (0, \pi)$ are generalizations of well known polar angles in 3 dimensions.

Dimensional regularization of ϕ^4 theory: $n < 4$

- Applying (18) to (17) and denoting $\omega \equiv \sqrt{\vec{k}^2}$ we get: $I(p,n) =$

$$\frac{\lambda^2}{2(2\pi)^4}\int \mathrm{d}k_0 \int_0^\infty \mathrm{d}\omega\omega^{n-2}\frac{1}{[k_0^2-\omega^2-m^2+\mathrm{i}\epsilon]}\frac{1}{[(k_0-M)^2-\omega^2-m^2+\mathrm{i}\epsilon]}W(n), \tag{19}$$

where $W(n)$ contains the integral over the angular coordinates

$$W(n) \equiv \int_0^{2\pi}\mathrm{d}\theta_1\int_0^\pi \mathrm{d}\theta_2 \sin\theta_2 \int_0^\pi \mathrm{d}\theta_3 \sin^2\theta_3 \cdots \int_0^\pi \mathrm{d}\theta_{n-2}\sin^{n-3}\theta_{n-2}. \tag{20}$$

- Angular part of $\int f(x)$ depending only on $r = \sqrt{x^2}$ of the n–dimensional vector x, can be carried out using the standard formulae

$$\int_0^\pi \mathrm{d}\theta \sin^m\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \quad \Rightarrow \int \mathrm{d}^n x f(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int \mathrm{d}r r^{n-1} f(r) \tag{21}$$

with the result: $I(p,n) =$

$$\frac{\lambda^2}{2(2\pi)^4}\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int \mathrm{d}k_0 \int_0^\infty \mathrm{d}\omega\omega^{n-2}\frac{1}{[k_0^2-\omega^2-m^2+\mathrm{i}\epsilon]}\frac{1}{[(k_0-M)^2-\omega^2-m^2+\mathrm{i}\epsilon]}. \tag{22}$$

Dimensional regularization of ϕ^4 theory: $n < 4$

- For $n = 4$ (22) reduces to (16), but it has also a well–defined meaning for <u>all real</u> $n \in (1, 4)$!
- With $I(p, n)$ defined on an open interval we can continue it analytically to the whole complex n plane. The singularity at $n = 4$ is a simple pole!
- To continue $I(p, n)$ below $n = 1$ we rewrite (22) using ℓ times a simple per partes integration and taking into account that $\partial/\partial\omega = 2\omega\partial/\partial\omega^2$

$$I(p, n) = \frac{\lambda^2}{2(2\pi)^4} \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2} + \ell)} \times$$

$$\int \mathrm{d}k_0 \int_0^\infty \mathrm{d}\omega \omega^{n-2+2\ell} \left(-\frac{\partial}{\partial\omega^2}\right)^\ell \frac{1}{[k_0^2 - \omega^2 - m^2 + \mathrm{i}\epsilon]} \frac{1}{[(k_0 - M)^2 - \omega^2 - m^2 + \mathrm{i}\epsilon]}, \tag{23}$$

which converges for all integer $1 - 2\ell < n < 4$.

- Rewrite $I(p, n)$ so that it coincides with original form (22) in the region $n \in (1, 4)$ but has also a well–defined meaning in a larger interval. Per partes integration does not change the behaviour of the integrand at large ω, but improves the convergence property at the origin $\omega = 0$, thereby extending the convergence of (23) to lower values of n.

Dimensional regularization of ϕ^4 theory: $n > 4$

- To continue analytically above $n = 4$ we insert into (22) the identity

$$1 = \frac{1}{2}\left(\frac{\mathrm{d}k_0}{\mathrm{d}k_0} + \frac{\mathrm{d}\omega}{\mathrm{d}\omega}\right). \tag{24}$$

- Performing the integrals over k_0 and ω again using per partes we arrive at:

$$I(p,n) = \frac{1}{2}(-n+6)I(p,n) + I'(p,n) \quad \Rightarrow I(p,n) = \frac{2}{n-4}I'(p,n), \tag{25}$$

where

$$I'(p,n) = \frac{\lambda^2}{2(2\pi)^4}\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int \mathrm{d}k_0 \int \mathrm{d}\omega\omega^{n-2}u(k_0,\omega,M), \tag{26}$$

$$u(k_0,\omega,M) \equiv \frac{2m^2}{\left(k_0^2 - m^2 - \omega^2 + \mathrm{i}\epsilon\right)^2\left[(k_0-M)^2 - \omega^2 - m^2 + \mathrm{i}\epsilon\right]} + \frac{2(k_0-M)M + 2m^2}{\left(k_0^2 - m^2 - \omega^2 + \mathrm{i}\epsilon\right)\left[(k_0-M)^2 - \omega^2 - m^2 + \mathrm{i}\epsilon\right]^2}. \tag{27}$$

- N.B. $I'(p,n)$ exists for all $1 < n < 5 \Rightarrow I(p,n)$ in (25), as a function of complex n, has a <u>simple pole</u> at $n = 4$!

Dimensional regularization of ϕ^4 theory

- Consider the diagram in Fig. 2c, where the bubble appears in the t–channel. First we combine the two propagators in (16) into one, using the identity

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x \frac{1}{[ax + b(1-x)]^2}, \tag{28}$$

which is a special case of **Feynman parametrization**. For (16) it gives

$$\begin{aligned} I(p,n) &= \frac{\lambda^2}{2} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \int_0^1 \frac{\mathrm{d}x}{[(k^2-m^2)x + ((k-p)^2-m^2)(1-x)]^2} \\ &= \frac{\lambda^2}{2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n l}{(2\pi)^n} \frac{1}{[l^2 - a^2(x) - \mathrm{i}\epsilon]^2}, \end{aligned} \tag{29}$$

where we have changed the order of integrations and subsituted

$$l \equiv k - p(1-x), \quad a^2(x) \equiv m^2 - p^2 x(1-x), \quad p^2 \equiv (p_1 - p_3)^2 . \tag{30}$$

- The inner integral in (29) yields:

$$I(p,n) = \frac{\lambda^2}{4} \frac{\mathrm{i}}{(4\pi)^2} (4\pi)^\varepsilon \Gamma(\varepsilon) \int_0^1 \mathrm{d}x \left[a^2(x)\right]^{-\varepsilon}, \quad \varepsilon \equiv 2 - \frac{n}{2}. \tag{31}$$

Dimensional regularization of ϕ^4 theory: mass parameter μ

- In n dimensions the coupling λ is no longer dimensionless, as in 4d, but has dimension $[\lambda] = 4 - n = 2\varepsilon$. The same dimension has also $I(p, n)$.
- For n close to the physical value $n = 4$ in (31) we can use:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \sum_{n=2}^{\infty} \frac{(-\varepsilon)^{n-1}}{n!}\zeta(n), \tag{32}$$

$$(4\pi)^{\varepsilon} = 1 + \varepsilon \ln 4\pi + \cdots, \tag{33}$$

where $\zeta(n)$ is the Riemann zeta function and $\gamma_E = 0.5772\cdots$

- We cannot, however, expand in a similar way directly $a^{-2\varepsilon}$ because $[a] = 2$. Physically well–defined expansion requires introducing some **mass parameter** μ that provides a scale for the expansion of the logarithm

$$\left(\frac{a^2(x)}{\mu^2}\right)^{-\varepsilon} = 1 - \varepsilon \ln \frac{a^2(x)}{\mu^2} + \cdots. \tag{34}$$

- Inserting identity $1 = \mu^{2\varepsilon}\mu^{-2\varepsilon}$ into (31) we get

$$I(p, n) = \mathrm{i}\frac{\lambda^2}{2}\frac{1}{(4\pi)^2}\mu^{-2\varepsilon}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \int_0^1 \mathrm{d}x \ln \frac{a^2(x)}{\mu^2} + \mathcal{O}(\varepsilon)\right]. \tag{35}$$

Coupling renormalization in ϕ^4 theory: λ_B and λ_R

- New parameter μ is an integral part of DR. $I(p,n)$ *does not depend* on it, provided we keep all the terms in expansion (34).
- *A truncated version* of (31), like the first two terms in the Laurent series in (35), which is the only part of (34) that survives in the limit $\varepsilon \to 0$, *still contains* μ despite the absence of the factor $\mu^{-2\varepsilon}$ to cancel this dependence.
- This apparent inconsistency disappears in the process of defining the *μ-dependent renormalized coupling*.
- Summing the contributions of diagrams in Fig. 2a,b we get*:

$$-\mathrm{i}\lambda_B\left[1-\frac{\lambda_B}{2}\frac{\mu^{-2\varepsilon}}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi-\int_0^1 \mathrm{d}x\,\ln\frac{a^2(x)}{\mu^2}\right)\right], \tag{36}$$

which to the order considered can be rewritten as

$$-\mathrm{i}\,\underbrace{\lambda_B\left[1-\frac{\lambda_B}{2}\frac{\mu^{-2\varepsilon}}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi\right)\right]}_{\equiv\,\mu^{2\varepsilon}\lambda_R(\mu)}\underbrace{\left[1+\frac{\lambda_B}{2}\frac{\mu^{-2\varepsilon}}{(4\pi)^2}\int_0^1 \mathrm{d}x\,\ln\frac{a^2(x)}{\mu^2}\right]}_{\text{finite terms}} \tag{37}$$

(*) From now on $\lambda_B \equiv \lambda$ to indicate its later interpretation as **bare** *coupling*.

Coupling renormalization in ϕ^4 theory: λ_B and λ_R

- **Renormalized coupling** $\lambda_R(\mu)$ is defined in terms of the bare coupling λ_B:

$$\lambda_R(\mu) \equiv \mu^{-2\varepsilon}\lambda_B\left[1-\lambda_B\frac{\mu^{-2\varepsilon}}{32\pi^2}\left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi\right)\right]. \tag{38}$$

- While $[\lambda_B]=2\varepsilon$ and is **held fixed** when μ is varied, $[\lambda_R(\mu)]=0$ and absorbs the singular term $1/\varepsilon$, eventually plus some finite terms, like $-\gamma_E+\ln 4\pi$ in (38).
- The convention defined in (38) is called $\overline{\mathrm{MS}}$, that corresponding to vanishing finite terms is called MS (minimal subtraction).
- Evaluating derivative of $\lambda_R(\mu)$ we find a *finite* result as $\varepsilon\to 0$:

$$\begin{aligned}\frac{\mathrm{d}\lambda_R(\mu)}{\mathrm{d}\ln\mu}=\mu\frac{\mathrm{d}\lambda_R(\mu)}{\mathrm{d}\mu} &= -2\varepsilon\lambda_B\mu^{-2\varepsilon}+\frac{1}{8\pi^2}\lambda_B^2\mu^{-4\varepsilon}+\mathcal{O}(\lambda_B^3) \\ &= -2\varepsilon\lambda_R+\frac{1}{16\pi^2}\lambda_R^2+\mathcal{O}(\lambda_R^3)\equiv\beta_\phi(\lambda_R,\varepsilon)\end{aligned} \tag{39}$$

- N.B. In the limit $\varepsilon\to 0$ term proportional to λ_B vanishes but *must be kept* if the renormalization procedure is carried out in terms of renormalized quantities and so called **counterterms**.

Coupling renormalization in ϕ^4 theory

- The second, finite, term comes from the product

$$-\lambda_B^2 \frac{1}{32\pi^2}\frac{1}{\varepsilon}\frac{\mathrm{d}\mu^{-4\varepsilon}}{\mathrm{d}\ln\mu} = -\lambda_B^2 \frac{1}{32\pi^2}\frac{1}{\varepsilon}(-4\varepsilon)\mu^{-4\varepsilon} = \frac{1}{8\pi^2}\lambda_B^2\mu^{-4\varepsilon} = \frac{\lambda_R^2}{8\pi^2}. \quad (40)$$

  where replacement $\lambda_B^2 \to \lambda_R^2$ in (40) is legal as these two expressions start to differ first at order λ_B^3.

- (40) determines implicit dependence of $\lambda_R(\mu)$ on μ, which cancels explicit dependence on μ of the finite terms in (37). This cancellation holds, however, only for physical quantities, like the cross–sections and only if perturbation theory is summed to all orders.

- Leading order β–function coefficient $1/16\pi^2$, obtained within DR in (40), is actually independent of the regularization method used in its derivation! For instance, in the conventional Pauli–Villars type of regularization we have instead of (38)

$$\lambda_R(\mu) \equiv \lambda_B\left[1 - \lambda_B \frac{1}{32\pi^2}\left(\ln\frac{M^2}{\mu^2} + \text{finite terms}\right)\right] \quad (41)$$

  where M is large cut–off to be sent to infinity. In this case $1/16\pi^2$ comes from derivative with respect to $\ln\mu$ of logarithmic term $\ln(M/\mu)$.

Electric charge renormalization in QED

- *$e + \mu$ scattering*: in the lowest order process is described by diagram in Fig. 3a, while the order $\mathcal{O}(\alpha^2)$ corrections correspond to diagrams in Figs. 3b-e.
- While diagrams on Fig. 3a,e give *finite* results the other are UV divergent.
- We'll work in the Feynman covariant gauge, i.e. set $\alpha_G = 1$ in $d_{\mu\nu} = -g_{\mu\nu} + (1 - \alpha_G)\frac{k_\mu k_\nu}{k^2}$.
- Quantity standing in each vertex on Fig. 3: $e_B \equiv$ *bare electric charge* .

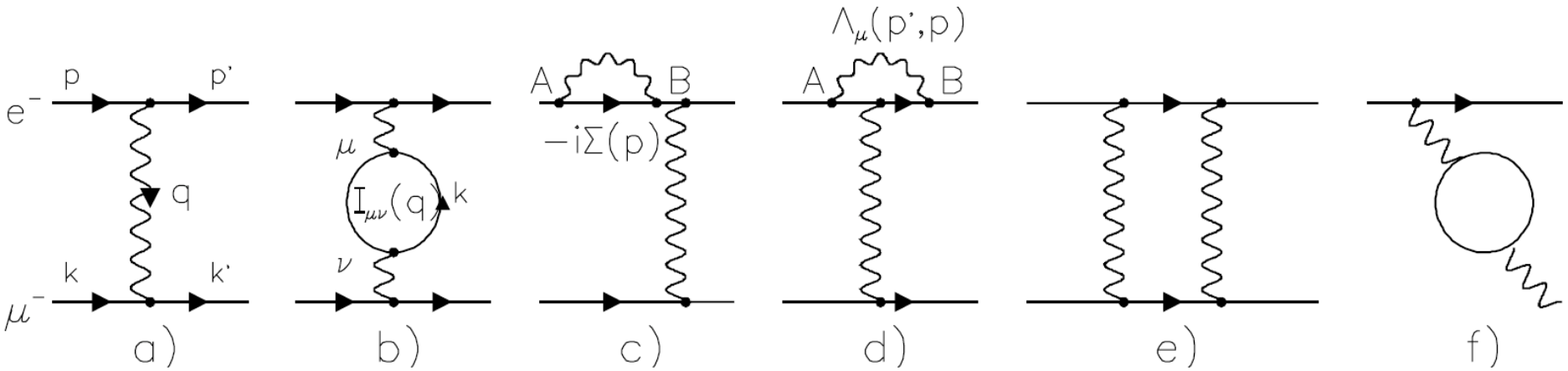

Figure 3: Feynman diagrams for electron–muon scattering in a leading (a) and next–to–leading order (b-d). The loop in c) can equally well be on any other of the four fermion legs while that in d) can circumvent the lower vertex as well.

Renormalization in QED – Electric charge renormalization

- Leading order (LO) diagram Fig. 3a gives a contribution to the invariant amplitude M_{if} (see Fig. 3 for notation)

$$\underbrace{[\overline{u}(p')(-\mathrm{i}\gamma_\mu)u(p)]\,[\overline{u}(k')(-\mathrm{i}\gamma_\nu)u(k)]}_{W_{\mu\nu}(k,p,q)}\underbrace{\left(-\frac{\mathrm{i}g_{\mu\nu}}{q^2}\right)}_{D_{\mu\nu}(q)}e_B^2, \tag{42}$$

where we have separated out the square of the bare electric charge and $D_{\mu\nu}$.

- Contribution of Fig. 3b has a similar structure:

$$W_{\mu\nu}(k,p,q)\left(\frac{-\mathrm{i}}{q^2}\right)I_{\mu\nu}(q,m_B)\left(\frac{-\mathrm{i}}{q^2}\right)e_B^2, \tag{43}$$

where the tensor:

$$I_{\mu\nu}(q,m_B)\equiv(-1)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[(-\mathrm{i}e_B\gamma_\mu)\frac{\mathrm{i}}{\not{k}-m_B+\mathrm{i}\epsilon}(-\mathrm{i}e_B\gamma_\nu)\frac{\mathrm{i}}{\not{k}-\not{q}-m_B+\mathrm{i}\epsilon}\right] \tag{44}$$

and factor (-1) comes from Pauli principle in the closed electron loop.

Renormalization in QED – Electric charge renormalization

- $m_B \equiv$ *bare electron mass*, will be throughout this subsection treated as a fixed number, although it also must be renormalized.

- Structure of $I_{\mu\nu}$ is dictated by the gauge invariance:

$$I_{\mu\nu}(q, m_B) = (-g_{\mu\nu} q^2 + q_\mu q_\nu) I(q^2, m_B) \tag{45}$$

  with $I(q^2, m_B)$ containing the divergent integral.

- Naive counting of the powers of k in the integrand of (44) suggests *quadratic* divergence of the integral, but because of (45) the integrand does not behave like $1/k^2$, where k is the loop momentum, but rather like q^2/k^4, thereby causing only the *logarithmic* divergence of $I(q^2, m_B)$.

- N.B. In the following we shall drop the term in (45) proportional to $q_\mu q_\nu$ as it vanishes after contraction with the tensor $W_{\mu\nu}$ in (43).

Renormalization in QED – Electric charge renormalization

- Regularization of diagram in Fig. 3b uses 4 technical ingredients:

  1. **Schwinger parameterization**

$$\frac{\mathrm{i}}{\not{k} - m + \mathrm{i}\epsilon} = \frac{\mathrm{i}(\not{k} + m)}{k^2 - m^2 + \mathrm{i}\epsilon} = (\not{k} + m) \int_0^\infty \mathrm{d}z e^{\mathrm{i}z(k^2 - m^2 + i\epsilon)} \tag{46}$$

  for both fermion propagators forming the closed loop adds two more integration variables z_1, z_2.

  2. Changing the original loop variable k to

$$\ell \equiv k - \frac{q z_2}{z_1 + z_2} = k - q + \frac{q z_1}{z_1 + z_2} \tag{47}$$

  and using the following identities to perform integrals over $\mathrm{d}^4\ell$

$$\int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \left[1, \ell_\mu, \ell_\mu \ell_\nu\right] \mathrm{e}^{\mathrm{i}\ell^2(z_1 + z_2 + \mathrm{i}\epsilon)} = \frac{1}{16\pi^2 \mathrm{i}(z_1 + z_2)^2} \left[1, 0, \frac{\mathrm{i}g_{\mu\nu}}{2(z_1 + z_2)}\right]. \tag{48}$$

Renormalization in QED – Electric charge renormalization

3. Using the identity $$1 = \int_0^\infty \frac{\mathrm{d}t}{t}\delta\left(1 - \frac{z_1 + z_2}{t}\right) \tag{49}$$ to trade one of the integrations over z_1 or z_2 for that over t.

4. The identity $$\int_0^\infty \frac{\mathrm{d}x}{x}\left(\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{\mathrm{i}bx}\right) = \ln\frac{b}{a}. \tag{50}$$

- Performing the first two steps we get:

$$I_{\mu\nu}(q) = \frac{-\mathrm{i}e_b^2}{4\pi^2}\int_0^\infty \mathrm{d}z_1 \int_0^\infty \frac{\mathrm{d}z_2}{(z_1+z_2)^2}\exp\left[\mathrm{i}\left(q^2\frac{z_1 z_2}{z_1+z_2} - (m_b^2 - \mathrm{i}\epsilon)(z_1+z_2)\right)\right] J_{\mu\nu}(q), \tag{51}$$

where the tensor $J_{\mu\nu}$ has two terms:

$$J_{\mu\nu}(q) \equiv 2(g_{\mu\nu}q^2 - q_\mu q_\nu)\frac{z_1 z_2}{(z_1+z_2)^2} \quad + \quad g_{\mu\nu}\left[\frac{-\mathrm{i}}{z_1+z_2} - \frac{q^2 z_1 z_2}{(z_1+z_2)^2} + m_B^2\right] \tag{52}$$

- First term satisfies the transversality condition $q_\mu I^{\mu\nu}(q, m_B) = 0$ that follows from gauge invariance, while the second term does not. Closer examination of the integral standing by $g_{\mu\nu}$ shows that it actually *vanishes*!

QED Electric charge renormalization: Pauli–Villars

- Using in the transverse term of (52), the identity (49) and carrying out the integral over z_1 or z_2, we get the following (still UV divergent!) expression:

$$I(q^2, m_B) = \frac{\mathrm{i}e_B^2}{2\pi^2} \int_0^1 \mathrm{d}z\, z(1-z) \int_0^\infty \frac{\mathrm{d}t}{t} \exp\left[\mathrm{i}t\left(q^2 z(1-z) - m_B^2 + \mathrm{i}\epsilon\right)\right], \quad (53)$$

  where the original logarithmic singularity of the integral over d^4k has been transformed into the same type of logarithmic singularity over the variable t.

- *Pauli–Villars* technique for regularization of integrals (53) consists in replacement

$$I(q^2, m_B) \to \overline{I}(q^2, m_B, M) \equiv I(q^2, m_B) - I(q^2, M), \quad (54)$$

  where the subtraction is understood to be carried using the identity (50). i.e. first subtracting the integrands and then performing the integration.

- Mass M acts as the UV regulator and at the end of the renormalization procedure should be sent to infinity.

- Usage of *fermion* mass M in the subtracted term $I(q^2, M)$ guarantees that this regularization technique *preserves gauge invariance*, which is of crucial importance for the proof of the full renormalizability of the theory.

Electric charge renormalization: Pauli–Villars

- Using (50) the regularized expression (54) equals

$$\bar{I}(q^2, m_B, M) = \frac{\mathrm{i}e_B^2}{2\pi^2} \int_0^1 \mathrm{d}z z(1-z) \ln\left(\frac{M^2}{m_B^2 - q^2 z(1-z)}\right). \tag{55}$$

- Isolating the logarithmically divergent term this can be rewritten as

$$-\mathrm{i}\bar{I}(q^2, m_B, M) = \frac{e_B^2}{12\pi^2} \ln\frac{M^2}{\mu^2} - \frac{e_B^2}{2\pi^2} \int_0^1 dz z(1-z) \ln\left(\frac{m_B^2 - q^2 z(1-z)}{\mu^2}\right). \tag{56}$$

- N.B. Integral over dimensionless parameter z is the trace of the original integration over the loop momentum k. Note that for any nonzero q^2 expression (55) is *regular* for $m_B \to 0$ and can thus be used even for massless fermions.
- In (56) new dimensional scale μ was introduced, which scales the cut–off parameter M in the singular term $\ln(M^2/\mu^2)$, but obviously $\bar{I}(q^2, m_B, M)$ is in fact *independent* of μ!

Renormalization in QED – Electric charge renormalization

- Once the regularization method has been chosen we can add contributions of diagrams in Fig. 3a,b obtaining (dropping the tensor $W_{\mu\nu}$)

$$-\frac{\mathrm{i}g_{\mu\nu}}{q^2}e_B^2\left[1-\frac{e_B^2}{12\pi^2}\ln\frac{M^2}{\mu^2}+\frac{e_B^2}{2\pi^2}\int_0^1 dz\, z(1-z)\ln\left(\frac{m_B^2-q^2z(1-z)}{\mu^2}\right)\right] \tag{57}$$

- Rewrite now (57) as a product of two terms

$$-\frac{\mathrm{i}g_{\mu\nu}}{q^2}\underbrace{e_B^2\overbrace{\left[1-\frac{e_B^2}{12\pi^2}\ln\frac{M^2}{\mu^2}\right]}^{Z_3^{-1}(A,e_B^2,M,\mu)}}_{\equiv e_R^2(A,\mu)}\times\left[\,1+\frac{e_B^2}{2\pi^2}\underbrace{\int_0^1 dz z(1-z)\ln\left(\frac{m_B^2-q^2z(1-z)}{\mu^2}\right)}_{C(A,q^2,\mu,m_B)}\,\right], \tag{58}$$

where UV divergence was absorbed into into newly defined *renormalized electric charge* $e_R^2(A,\mu)^*$ with the rest of the original $\mathcal{O}(e_B^2)$ term left in the *finite* contribution $C(A,q^2,\mu,m_B)$.

(*) Label "A" *etc.* is used to distinguish between different definitions of e_R.

Renormalization in QED – Electric charge renormalization

- In terms of this new parameter (58) can be written as

$$-\frac{\mathrm{i}g_{\mu\nu}}{q^2}e_R^2(A,\mu)\left[1+\frac{e_B^2}{2\pi^2}C(A,q^2,\mu,m_B)\right]\longrightarrow$$

$$\longrightarrow\quad -\frac{\mathrm{i}g_{\mu\nu}}{q^2}e_R^2(A,\mu)\left[1+\frac{e_R(A,\mu)}{2\pi^2}C(A,q^2,\mu,m_B)\right], \tag{59}$$

where the second expression, differing from the the first one by replacement $e_B^2 \longrightarrow e_R^2$ in the brackets, results if the renormalization procedure is carried out to order $\mathcal{O}(e_B^6)$.

- In two regions the integral over z in (58) can be performed analytically.
  - For large $Q^2 \equiv -q^2$, i.e. for $-q^2/m_B^2 \gg 1$ we find:

$$-\frac{\mathrm{i}g_{\mu\nu}}{q^2}e_R^2(A,\mu)\left[1+\frac{e_R^2(A,\mu)}{12\pi^2}\left(\ln\frac{-q^2}{\mu^2}+\frac{5}{3}\right)\right], \tag{60}$$

  - while for $-q^2/m_B^2 \ll 1$ we have:

$$-\frac{\mathrm{i}g_{\mu\nu}}{q^2}e_R^2(A,\mu)\left[1+\frac{e_R^2(A,\mu)}{12\pi^2}\ln\frac{m_B^2}{\mu^2}+\frac{e_R^2(A,\mu)}{60\pi^2}\frac{-q^2}{m_B^2}\right]. \tag{61}$$

Renormalization in QED – Electric charge renormalization

- In (58) singularity of the loop integral in the term proportional to $\ln(M^2/\mu^2)$ was isolated and included in definition of renormalized electric charge $e_R^2(\mu)$.
- This procedure is not unique – we can include in $e_R^2(\mu)$ arbitrary finite terms as well – e.g. we can separate integral over z in (57) into two parts

$$\underbrace{\frac{e_B^2}{2\pi^2}\int_0^1 \mathrm{d}z z(1-z)\ln\frac{M^2}{m_B^2+\mu^2 z(1-z)}}_{1-Z_3^{-1}(B,e_B^2,M,m_B,\mu)} + \frac{e_B^2}{2\pi^2}\underbrace{\int_0^1 \mathrm{d}z z(1-z)\ln\frac{m_B^2+\mu^2 z(1-z)}{m_B^2+(-q^2)z(1-z)}}_{C(B,q^2,\mu,m_B)}$$

and include in $e_R^2(\mu)\equiv e_B^2 Z_3^{-1}$ (denoted $e_R^2(B,\mu)$) the whole first term. (62)

- This term can be written as:

$$1-Z_3^{-1}(B,e_B^2,M,m_B,\mu)=\frac{e_B^2}{12\pi^2}\ln\frac{M^2}{\mu^2}-\frac{e_B^2}{2\pi^2}\int_0^1 \mathrm{d}z z(1-z)\ln\left(\frac{m_B^2}{\mu^2}+z(1-z)\right). \tag{63}$$

- Definition $e_R^2(\mu)\equiv e_B^2 Z_3^{-1}$ is motivated by the requirement that at $\mu^2=-q^2$ finite correction $C(B,q^2=-\mu^2,\mu)=0$, i.e. for $\mu^2=-q^2$ all the effects of the electron loop correction are included in $e_R^2(B,\mu)$.

Renormalization in QED – Electric charge renormalization

- For large μ, i.e. when $\mu^2 \gg m_B^2$, we find

$$1 - Z_3^{-1}(B) \to \frac{e_B^2}{12\pi^2}\left(\ln\frac{M^2}{\mu^2} + \frac{5}{3}\right), \tag{64}$$

which differs from previous case only by presence of additional constant $5/3$ in definition of $e_R^2(B,\mu)$, while in region $\mu \ll m_B$ we find a very different behavior:

$$1 - Z_3^{-1}(B) \to \frac{e_B^2}{12\pi^2}\ln\frac{M^2}{m_B^2} - \frac{e_B^2}{60\pi^2}\cdot\frac{\mu^2}{m_B^2}. \tag{65}$$

- For $\mu \to 0$, $e_R^2(B,\mu)$ *approaches a finite value* $e_R^2(B,0)$

$$e_R^2(B,\mu,m_B) = e_B^2\left(1 - \frac{e_B^2}{12\pi^2}\ln\frac{M^2}{m_B^2} + \frac{e_B^2}{60\pi^2}\frac{\mu^2}{m_B^2}\right) \longrightarrow$$

$$\longrightarrow e_B^2\left(1 - \frac{e_B^2}{12\pi^2}\ln\frac{M^2}{m_B^2}\right) \equiv e_R^2(B,0). \tag{66}$$

Electric charge renormalization: $e_R(A,\mu)$

- Basic difference between definitions A and B of $e_R(\mu)$ is how the effects of electron mass are taken into account. For $m_e = 0$ they obviously coincide.
- For this case *all the dependence* on m_B resides in (finite) correction $C(A, q^2, \mu, m_B)$. Note in particular the presence in (61) of the term proportional to $\ln(m_B^2/\mu^2)$, which would be arbitrarily large and therefore dangerous for $\mu \to 0$.
- $e_R^2(A,\mu)$ contains no information on m_B and therefore behaves exactly in the same way as for the massless electron. We shall therefore refer to $e_R^2(A,\mu)$ as **mass independent** *definition of renormalized charge*.
- Setting $\mu^2 = -q^2$ and approaching the limit $q^2 \to 0$ the expansion parameter $e_R^2(A, -q^2) \to 0^*$, but due to the presence of the term $\ln(-q^2/m_B)$ the coefficient function $C(A, q^2, \mu^2 = -q^2, m_B) \to \infty$ and so do in fact all higher order coefficients as well.

(*) For small μ $e_R^2(A,\mu)$ as defined in (58) turns negative and eventually blows up to $-\infty$ as $\mu \to 0$. However, taking into account higher order terms makes the couplant $e_R^2(A,\mu)$ indeed vanish at $\mu = 0$.

Electric charge renormalization: $e_R(B,\mu)$

- Situation is different as $e_R^2(B,\mu,m_B)$ absorbs part of the dependence on m_B and is therefore called **mass dependent** *renormalized charge*.
- For $\mu \gg m_B$ (apart from 5/3 in (64)) $e_R^2(B,\mu,m_B)$ behaves as in the case A, but below m_B μ–dependence changes dramatically, leading to the constant limit $e_R^2(B,0)$.
- Usual practice: all the mass effects are included in $e_R^2(B,\mu^2=-q^2)$, which has a finite limit for $q^2 \to 0$ and so has the coefficient function $C(B,q^2,\mu^2=-q^2,m_B)$!
- Nevertheless *in principle* both of these definitions of $e_R(\mu)$ are *equally legal*.
- μ can thus be interpreted as the *lower bound* of the logarithmically divergent integral of the type

$$\int_\mu^M \frac{\mathrm{d}k}{k} = \ln\frac{M}{\mu}. \tag{67}$$

- Heisenberg uncertainty relations $\Rightarrow$ $e_R(\mu) \approx$ charge (original plus the induced one) inside a sphere of the radius $r = 1/\mu$ around the center of an electron.

Scale independence and Renormalization group

- Arbitrary scale parameter μ appearing in both $e_R^2(\mu)$ and $C(q^2, \mu, m_B)$ is an *unphysical* parameter $\Rightarrow$ results for physical quantities *cannot* depend on μ.
- If we change μ, other parameters, like masses and coupling constants, must also change in order to compensate for this effect. To keep physics invariant, changing the subtraction point μ must be offset by changes in the renormalized physical parameters as a function of the energy.
- This natural requirement is the essence of the **renormalization group** (RG), introduced in the early fifties by Gell-Mann, Low, Bogoljubov and others. *RG expresses nothing else but* **internal consistency** *of renormalized pQFT*.
- Let R_1 represent some (unspecified) renormalization scheme. If Γ_B is unrenormalized quantity and Γ_{R_1} is same quantity renormalized by the scheme R_1, then:
$$\Gamma_{R_1} = Z(R_1)\Gamma_B \tag{68}$$
where $Z(R_1)$ is renormalization constant under renormalization scheme R.
- Let us now choose a different renormalization scheme R_2. Since the unrenormalized quantity Γ_B is independent of the renormalization scheme, then:
$$\Gamma_{R_2} = Z(R_2)\Gamma_B \tag{69}$$

The renormalization group

- Relationship between these two renormalized quantities is given by:

$$\Gamma_{R_2} = Z(R_2, R_1)\Gamma_B \quad \text{where} \quad Z(R_2, R_1) = Z(R_1)/Z(R_2) \tag{70}$$

- Trivially, this satisfies a group multiplication law:

$$Z(R_3, R_2)Z(R_2, R_1) = Z(R_3, R_1) \tag{71}$$

  where identity element is $Z(R, R) = 1$

- For example, in ϕ^4 theory, we have the following relationship between unrenormalized and renormalized quantities:

$$\Gamma_B^{(n)}(p_i, g_B, m_B) = Z_\phi^{-n/2}\Gamma_R^{(n)}(p_i, g, m, \mu) \tag{72}$$

- Since unrenormalized bare quantity is independent of μ, the derivative acting on the unrenormalized quantity must, by construction, be zero:

$$0 = \mu\frac{\partial}{\partial\mu}\Gamma_B^{(n)} = \left(\mu\frac{\partial}{\partial\mu}Z_\phi^{-n/2}\right)\Gamma_R^{(n)} + Z_\phi^{-n/2}\left(\mu\frac{\partial}{\partial\mu}\Gamma_R^{(n)}\right) \tag{73}$$

The renormalization group

- Using the chain rule for independent variables μ, g and m:

$$\frac{d}{d\mu} = \frac{\partial}{\partial\mu} + \frac{\partial g}{\partial\mu}\frac{\partial}{\partial g} + \frac{\partial m}{\partial\mu}\frac{\partial}{\partial m} \tag{74}$$

  we can rewrite (74) as:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + m\gamma_m(g)\frac{\partial}{\partial m}\right)\Gamma^{(n)}(p_i, g, m, \mu) = 0 \tag{75}$$

  where

$$\beta(g) \equiv \mu\frac{\partial g}{\partial\mu} \qquad \gamma(g) \equiv \mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi} \qquad m\gamma_m(g) \equiv \mu\frac{\partial m}{\partial\mu} \tag{76}$$

- These are the RG equations which express how renormalized vertex functions change when we make a change in scale μ.
- We can formally solve the expression (76) for the β function.

$$\frac{d}{d\mu} = \frac{dg}{\beta(g)} \Rightarrow \ln\frac{\mu}{\mu_0} = \int_{g(\mu_0)}^{g(\mu)}\frac{dg}{\beta(g)} \tag{77}$$

  where μ_0 is some arbitrary reference point.

Scale independence and Renormalization group

- Cancelation mechanism of μ–dependence in expansion parameter $e_R^2(\mu)$ and finite coefficient function $C(q^2, \mu, m_B)$ works in such a way that although the *full* sum of perturbation expansion *is μ-independent*, its approximation to order $e_R^4(\mu)$ (in fact, to any finite one) *is not*!
- This observation is the first signal of the *inevitable ambiguities* which appear when renormalized QED and QCD are considered to finite orders.

Electric charge renormalization – Additional comments

- In any practical calculation we therefore *have to choose* some value of μ!
  If the finite correction term, containing in both definitions the logarithm $\ln(-q^2/\mu^2)$ is to be reasonably "small" we should choose $\mu^2 \propto -q^2$.

- Proof that renormalization procedure can be carried out systematically to *all orders* of perturbation theory, so that infinities at all orders can be absorbed in the redefined renormalized couplant, masses and wave functions, is rather nontrivial ...

- Although the intermediate steps in the renormalization procedure depend on regularization technique used, the structure of the final results *does not!* There will always be some free scale parameter denoted μ, on which the redefined couplant as well as the new expansion coefficients will depend

- Once the renormalized perturbation expansions are expressed in terms of these free parameters and the mentioned invariants, they carry no trace of the regularization technique used.

Electric charge renormalization – Additional comments

- In this way the divergent term, containing the logarithm $\ln(M^2/\mu^2)$, is absorbed into definition of renormalized couplants $e_R^2(A,\mu)$ or $e_R^2(B,\mu)$

$$e_R^2(A,\mu) \equiv e_B^2 \left[1 - \frac{e_B^2}{12\pi^2} \ln \frac{M^2}{\mu^2}\right] \tag{78}$$

  and analogously for $e_R^2(B,\mu)$.

- In the final step $M \to \infty$ should be taken in (78). For *fixed* bare charge e_B^2 this is, however, impossible. However, since e_B^2 appears *always* in combination with the divergent logarithm $\ln(M/\mu)$ we can forget about their separate existence and consider the renormalized electric charge e_R as the basic QED parameter.

- This well–defined algorithm for evaluation of finite coefficients of perturbative expansions is clearly unsatisfactory on physical grounds. Many theorists, prominent among them Landau and his school, had addressed this problem asking following question:
  Can one define such a dependence $e_B^2(M)$ of the bare charge on the cut-off M, which would compensate in (78) the term proportional to $\ln M$, coming from the divergent integral, and allow the construction of the finite limit for the renormalized charge $e_R^2(\mu)$ when $M \to \infty$?

Electric charge renormalization – Higher orders

- Let us now improve a little bit the approximation we have so far worked in. Instead of a single loop in Fig. 3b consider the sum of terms corresponding to the subset of all Feynman diagrams shown in Fig. 4.

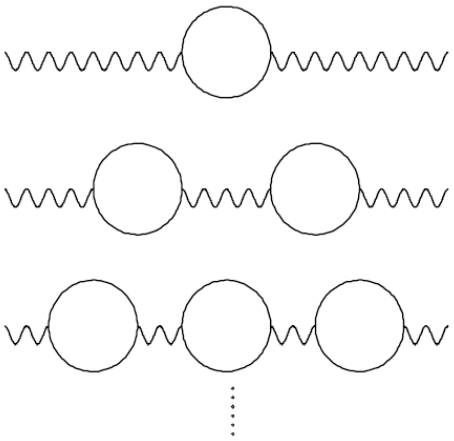

Figure 4: Subset of Feynman diagrams giving rise to a geometric series

- Bubbles are connected via single photon lines and each separately yields exactly the same result as the bubble in Fig. 3b.
  $\Rightarrow$ Summing this subset of Feynman diagrams leads to a *geometric series*, the leading term of which is just (78).
- This series is easily summed with the result:

$$\alpha_R(\mu) = \alpha_B(M)\left[1 + \alpha_B(M)\left(\beta_0 \ln \frac{\mu}{M} + \delta(x)\right) + \cdots\right] = \frac{\alpha_B(M)}{1 - \alpha_B(M)\left(\beta_0 \ln(\mu/M) + \delta(x)\right)}, \tag{79}$$

where $\beta_0 = 2/3\pi$ and $\delta(x)$ is:

$$\delta(A,x) = 0, \quad \delta(B,x) = \frac{2}{\pi}\int_0^1 \mathrm{d}z z(1-z) \ln\left(\frac{1}{x^2} + z(1-z)\right), \quad x \equiv \frac{\mu}{m_B}. \tag{80}$$

Electric charge renormalization – Higher orders

- For finite μ and $m_B \to 0$, i.e. for $x \to \infty$, $\delta(B,x) \to -5/3$ and thus

$$e_R^2(B,\mu) \equiv e_B^2 Z_3^{-1}(B) = e_B^2 \left[1 + \beta_0 \alpha_B \left(\ln \frac{\mu^2}{M^2} - \frac{5}{3}\right)\right] \tag{81}$$

  reflecting fact that massless fermions contribute to the renormalized $e_R^2(\mu)$ at all scales μ with the same strength.

- For fixed μ and $m_B \to \infty$ (80) diverges like $\ln m_B/\mu$.

  This looks strange and unphysical as one would expect that infinitely heavy electrons should not have any influence on the theory at finite energy scales μ.

  However, resuming all the bubbles in Fig. 4a the resulting $\alpha_R(B,\mu,m_B)$ has the property we wish it to have:
  In the limit $m_B \to \infty$ a heavy electron *decouples* from it.

Electric charge renormalization – The infrared limit

A closely related feature of (79) i.e. of

$$\alpha_R(\mu) = \frac{\alpha_B(M)}{1 - \alpha_B(M)\left[\beta_0 \ln(\mu/M) + \delta(x)\right]}$$

concerns its behavior as $\mu \to 0$.

A For the mass–independent definition of the renormalized couplant (79) implies $\alpha_R(A, \mu) \to 0$.

B For the mass–dependent definition the situation is differrent. Keeping the UV cut–off parameter M fixed we find

$$\beta_0 \ln \frac{\mu}{M} + \delta(B, x) \to \beta_0 \ln \frac{m_B}{M}, \tag{82}$$

$\Rightarrow$ for $m_B \neq 0$ the infrared limit of $\alpha_R(B, \mu)$ is determined by the *mass* m_B and coincides with that of massless electron at the scale $\mu = m_B$!

This is a manifestation of the fact that in QED (and in general in theories without confinement of elementary fermions) the IR limit of the gauge coupling is determined essentially by the fermion masses.

Dimensional transmutation

- Inverting (79) we have:

$$\alpha_B(M) = \frac{\alpha_R(\mu)}{1 - \alpha_R(\mu)\left[\beta_0 \ln(M/\mu) + \delta(x)\right]}. \tag{83}$$

- Thus for any *finite* α_R the bare couplant $\alpha_B(M)$ blows up to infinity at some *finite* value of M, corresponding to the pole of this expression.
- Note that even if we forget about $\alpha_B(M)$ and work directly with $\alpha_R(\mu)$ (79) implies that for *any* dependence of $\alpha_B(M)$ on M which has a *finite* limit as $M \to \infty$ the renormalized couplant $\alpha_R(\mu)$ *vanishes*!
- *Inconsistency:* a dimensionless quantity $\alpha(\mu)$ is expressed as a function of a dimensional one (μ or M).
- *Solution:* μ enters in the ratio with some other dimensional parameter, which will be denoted Λ.
- We should thus write more correctly

$$\alpha_R = \alpha_R(\mu/\Lambda), \quad \alpha_B = \alpha_B(M/\Lambda). \tag{84}$$

Dimensional transmutation

- Λ has nothing to do with the cut-off and represents a *fundamental scale parameter*, which appears in the theory entirely due to the renormalization procedure.
- This phenomenon is called **dimensional transmutation** and is typical *quantum phenomenon* valid for theories with dimensionless couplants.
- It is caused by the logarithmic divergences in one-loop diagrams which imply that this *constant* actually depends on the typical energy scale of the processes under considerations. The *running* is determined by the β-function and renormalization group.
- Appearance of Λ is an inevitable consequence of the renormalization procedure but its *numerical* value is not fixed by these considerations and must be determined from experimental data.
- Consequently, the strength of the interaction may be described by a *dimensionful parameter* – the energy scale where the interaction strength reaches the value 1. In the case of QCD, this energy scale is called the QCD scale and its value 150 MeV replaces the original dimensionless coupling constant.

Dimensional transmutation

- The standard way to define Λ *unambiguously* is to write down and solve the differential equations which follow from (79) and (83).

A we have simply

$$\frac{\mathrm{d}\alpha_R(A,\mu/\Lambda)}{\mathrm{d}\ln\mu} = \beta_0\alpha_R^2(A,\mu/\Lambda) \Rightarrow \alpha_R(A,\mu/\Lambda) = \frac{1}{\beta_0\ln(\Lambda/\mu)}, \tag{85}$$

$$\frac{\mathrm{d}\alpha_B(M/\Lambda)}{\mathrm{d}\ln M} = \beta_0\alpha_B^2(M/\Lambda) \Rightarrow \alpha_B(M/\Lambda) = \frac{1}{\beta_0\ln(\Lambda/M)}, \tag{86}$$

B the equation for $\alpha_R(B,\mu)$ is more complicated

$$\frac{\mathrm{d}\alpha_R(B,\mu/\Lambda)}{\mathrm{d}\ln\mu} = \left(\beta_0 + \frac{\mathrm{d}\delta(B,x)}{\mathrm{d}\ln\mu}\right)\alpha_R^2 = \beta_0\alpha_R^2(B,\mu/\Lambda)\underbrace{\int_0^1 \mathrm{d}z\frac{6x^2z^2(1-z)^2}{1+x^2z(1-z)}}_{h(x)}. \tag{87}$$

Dimensional transmutation

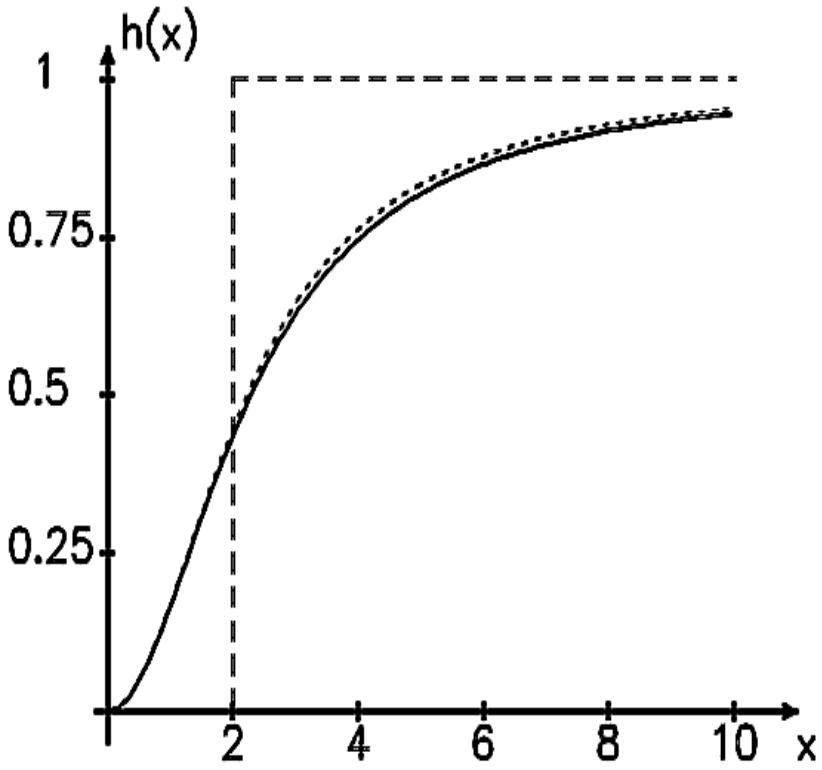

Figure 5: Shape of the function $h(x)$ together with its "step" approximation (dashed line) and the analytical approximation $x^2/(5+x^2)$ (dotted curve)

- Function $h(x)$ in 87 is:

$$h(x) = 1 - \frac{6}{x^2} + \frac{12}{x^3\sqrt{4+x^2}} \ln \frac{\sqrt{4+x^2}+x}{\sqrt{4+x^2}-x}$$
$$\doteq \frac{x^2}{5+x^2}$$

- This is illustrated in Fig. 5. We see that $h(x)$ approaches unity for $x \gg 1$ but vanishes
like x^2 as $x \to 0$.

- Consequently the solutions of (87) are essentially same in A and B in first region, but differ substantially in the second, where $\alpha_R(A, \mu/\Lambda) \to 0$ while $\alpha_R(B, \mu/\Lambda)$ flattens to a constant value, (see Fig. 6).

Dimensional transmutation

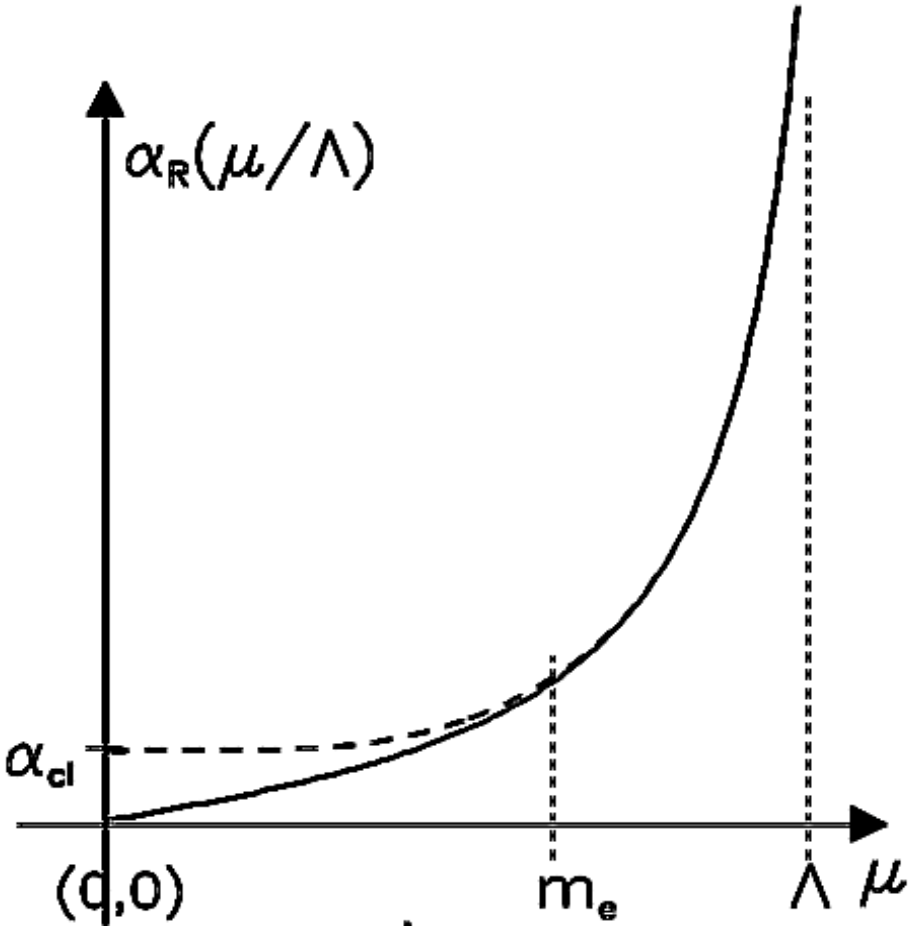

Figure 6: Sketch of dependence of $\alpha_R(\mu)$ on μ for both A and B definitions.

- In both cases Λ labels the elements of an infinite set of solutions of these equations.
- Note that after taking the derivative of (79) with respect to $\ln \mu$ the cut–off M has completely disappeared from the resulting expression (85) and (87)!
- Taking (85) or (87) as **definitions** of the renormalized couplant $\alpha_R(\mu/\Lambda)$ is the *crucial* step of the renormalization procedure.

Effective charge

- Since $\alpha_R \geq 0$ it makes sense only for $\mu < \Lambda$ and should be trusted only for $\mu \ll \Lambda$.
- So even if we start from (85) i.e. $\alpha_R(A, \mu/\Lambda) = [\beta_0 \ln(\Lambda/\mu)]^{-1}$ we encounter severe problems when we approach short distances .
- The renormalized couplant diverges at $\mu = \Lambda$ and we can, knowing α_R at some μ, evaluate Λ from the any of two expressions

$$\Lambda = \mu \exp\left[\frac{1}{\beta_0 \alpha_R(A, \mu/\Lambda)}\right] = M \exp\left[\frac{1}{\beta_0 \alpha_B(M/\Lambda)}\right]. \tag{88}$$

  For case B same relations hold for $x \gg 1$, while for $x \ll 1$ (88) are replaced by more complicated formulae.
- Thus we can alternatively determine Λ from the knowledge of the bare couplant α at some cut–off M.
- Dependence of $\alpha_R(\mu/\Lambda)$ on μ has also a very simple and intuitive physical interpretation: $e_R^2(\mu/\Lambda)$ is the charge contained inside the sphere of the radius $r = 1/\mu$ around the "core" of an electron. It can be viewed as the **effective charge** at a distance defined by the scale μ.

Large distance behavior of QED: $q^2 \to 0$

- Use of $\alpha_R(B,\mu)$ as an expansion parameter is appropriate even in this region as it gives a finite value for the contribution of the sum of diagrams in Fig. 3a,b, equal simply to $e_R^2(B,0)$.
- Including the (finite) contribution of the sum of other three diagrams in this figure leads to the final result proportional to $e_R^2(B,0)(1+r_1 e_R^2(B,0))$, where r_1 is some number. Recalling the way $e_R(B,\mu)$ was introduced, it is obvious that we could add to the quantity B and simultaneously subtract from $C(B,q^2,\mu,m_B)$ in (62) such finite term that the resulting r_1 actually *vanishes*!
- Denoting the corresponding renormalized charge $e_R(C,\mu)$, the $q^2 \to 0$ limit of the electron–muon scattering amplitude, evaluated using this definition of renormalized electric charge, would be proportional to $e_R^2(C,0)$, with *no higher order corrections*.
- Repeating this procedure for the case of Compton scattering on an electron leads to another renormalized charge: $e_R^2(D,\mu=0)$. According to the conventional definition of electric charge, this quantity coincides with the **fine structure constant** $\alpha_{cl} \equiv \left[e_R^2(D,0)\right]^2/4\pi = \frac{1}{137}$.

Technique of counterterms

- In QED at one loop all UV divergencies can be absorbed in the definition of the **renormalized**
  - electron charge $e_R^2(\mu)$,
  - electron mass $m_R(\mu)$,
  - electron and photon fields $A_R^\mu(x,\mu)$ and $\psi_R(x,\mu)$,

  which replace their "bare" analogues appearing in the original QED lagrangian. In terms of these quantities all physical observables become finite.

- At one loop this claim follows if we add the contributions from diagrams in Fig. 3a-d, with the lower vertex amputated. In terms of the renormalization factors Z_1, Z_{2F}, Z_3, Z_m, and writing out only the singular parts of the respective contributions we had from

$$\begin{aligned}
&\text{Fig. 3a: } -\mathrm{i}e_B\gamma_\mu,\\
&\text{Fig. 3b: } -\mathrm{i}e_B\gamma_\mu(Z_3-1),\\
&\text{Fig. 3c: } -\mathrm{i}e_B\gamma_\mu\left(1-Z_{2F}^{-1}\right),\\
&\text{Fig. 3d: } -\mathrm{i}e_B\gamma_\mu\left(Z_1^{-1}-1\right).
\end{aligned}$$

Technique of counterterms

- In summing the above expressions we have to keep in mind that in perturbation theory each renormalization factor Z_i has the form $Z_i = 1 + \mathcal{O}(e_B^2)$ so that to the leading order in e_B^2 the sum of divergent factors should be written as

$$\left[1 + (Z_3 - 1) + 2\left(1 - Z_{2F}^{-1}\right) + \left(Z_1^{-1} - 1\right)\right] = \frac{\left[1 + \left(Z_1^{-1} - 1\right)\right]\left[1 + (Z_3 - 1)\right]}{\left[1 + \left(Z_{2F}^{-1} - 1\right)\right]^2} = \tag{89}$$

  where the factor 2 in front of $(1 - Z_{2F}^{-1} - 1)$ reflects the fact that the loop in Fig. 3c can be on the outgoing electron leg as well.

- Multiplying this sum with $1/\sqrt{Z_3}$ for each external (from the point of view of the upper vertices in Fig. 3) photon and with $1/\sqrt{Z_{2F}}$ for each external electron line we end up with

$$-\mathrm{i}e_B\gamma_\mu\sqrt{Z_3}Z_{2F}Z_1^{-1} = -\mathrm{i}e_B\gamma_\mu\sqrt{Z_3} = -\mathrm{i}e_R\gamma_\mu, \tag{90}$$

  where we have now used the identity $Z_1 = Z_{2F}$. The diagram in Fig. 3c contributes also to the renormalization (redefition) of the electron mass, but this divergence can be ignored if we write electron propagators everywhere in Fig. 3 with the renormalized mass instead of the bare one.

Technique of counterterms

Systematic way to deal with UV divergencies at all orders of pQFT proceeds as:

- Bare quantities are expressed in terms of renormalized ones, by introducing for all of them appropriate renormalization factors

$$\psi_B \equiv \sqrt{Z_{2F}}\psi_R, \qquad A_B^\mu \equiv \sqrt{Z_3}A_R^\mu, \qquad m_B \equiv Z_m m_R, \qquad \alpha_B \equiv Z_\alpha \alpha_R. \tag{91}$$

  Each Z_i can be expanded in terms of renormalized coupling α_R as

$$Z_i(M/\mu) = 1 + \alpha_R\left((\text{singular terms}) + \text{finite terms}\right) + \mathcal{O}(\alpha_R^2). \tag{92}$$

- In DR structure of the counterterms reads:

$$Z_i(M/\mu) = 1 + \alpha_R\left(\frac{\gamma_1^{(1)}}{\varepsilon} + \gamma_0^1\right) + \alpha_R^2\left(\frac{\gamma_2^{(2)}}{\varepsilon^2} + \frac{\gamma_1^{(2)}}{\varepsilon} + \gamma_0^2\right) + \cdots, \tag{93}$$

  where all $\gamma_j^{(i)}$ are *finite* (calculable) numbers. Except for $\gamma_0^{(i)}$ which can be chosen completely *arbitrarily*, all those standing by the inverse powers of ε, are *uniquely defined* by requiring that they *cancel* the UV singularities in loops. In this approach we thus work from the very beginning exclusively with *renormalized* quantities and introduce the counterterms to cancel all the UV infinities coming from loops.

Technique of counterterms

- These counterterms formally appear when the original bare lagrangian is rewritten in terms of the renormalized quantities

$$\mathcal{L}^{\mathrm{QED}} = -\frac{1}{4} Z_3 F_R^{\mu\nu} F_{\mu\nu}^R + Z_{2F} \overline{\Psi}_R (\mathrm{i}\, \partial\!\!\!/ - Z_m m_R) \Psi_R + \sqrt{Z_\alpha} \sqrt{Z_3} Z_{2F} e_R \overline{\Psi}_R \gamma_\mu \Psi_R A_R^\mu \tag{94}$$

and then reorganized in the following equivalent way

$$\mathcal{L}^{\mathrm{QED}} = -\frac{1}{4} F_R^{\mu\nu} F_{\mu\nu}^R + \overline{\Psi}_R (\mathrm{i}\, \partial\!\!\!/ - m_R) \Psi_R + e_R \overline{\Psi}_R \gamma_\mu \Psi_R A_R^\mu + \mathcal{L}^{\mathrm{count}}, \tag{95}$$

where

$$\begin{aligned} \mathcal{L}^{\mathrm{count}} \equiv & -\frac{1}{4}(Z_3 - 1) F_R^{\mu\nu} F_{\mu\nu}^R + (Z_{2F} - 1) \overline{\Psi}_R \mathrm{i}\, \partial\!\!\!/ \Psi_R - (Z_{2F} Z_m - 1) m_R \overline{\Psi}_R \Psi_R + \\ & \left(\sqrt{Z_\alpha} \sqrt{Z_3} Z_{2F} - 1 \right) e_R \overline{\Psi}_R \gamma_\mu \Psi_R A_R^\mu. \end{aligned} \tag{96}$$

- Each of the so called **counterterms** in (96) is then treated as additional interaction term.

Feynman parametrization

- Formula (28) is simplest case of a general expression for the combination of arbitrary number of scalar propagators of the same type

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_n \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{1}{[x_1 a_1 + \cdots + x_n a_n]^n}. \tag{97}$$

- For combination of 2,3 and 4 propagators with general powers:

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xa+(1-x)b]^{\alpha+\beta}}, \tag{98}$$

$$\frac{1}{a^\alpha b^\beta c^\gamma} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 \mathrm{d}x x \int_0^1 \mathrm{d}y \frac{u_1^{\alpha-1} u_2^{\beta-1} u_3^{\gamma-1}}{[u_1 a + u_2 b + u_3 c]^{\alpha+\beta+\gamma}}, \tag{99}$$

where

$$u_1 = xy, \quad u_2 = x(1-y), \quad u_3 = 1-x,$$

$$\frac{1}{a^\alpha b^\beta c^\gamma d^\delta} = \frac{\Gamma(\alpha+\beta+\gamma+\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)} \int_0^1 \mathrm{d}x x^2 \int_0^1 \mathrm{d}y y \int_0^1 \mathrm{d}z \frac{u_1^{\alpha-1} u_2^{\beta-1} u_3^{\gamma-1} u_4^{\delta-1}}{[u_1 a + u_2 b + u_3 c + u_4 d]^{\alpha+\beta+\gamma+\delta}}, \tag{100}$$

and $$u_1 = 1-x, \quad u_2 = xyz, \quad u_3 = x(1-y), \quad u_4 = xy(1-z).$$

Feynman parametrization

- Another function encountered in evaluation of loop diagrams is

$$\int_0^1 \mathrm{d}x x^\alpha (1-x)^\beta = B(\alpha,\beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{101}$$

In all the above formulae we have used the standard definition of the Γ-function

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t, \tag{102}$$

which is related to another widely used ψ-function

$$\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi(1) = -\gamma_E. \tag{103}$$

Momentum integrals and related formulae

- Formulae needed for evaluation of one loop diagrams are (a is real number):

$$I(m,r) \equiv \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{(k^2)^r}{[k^2 - a^2 + \mathrm{i}\varepsilon]^m} = \frac{\mathrm{i}}{(4\pi)^{n/2}} (-1)^{r-m} \left(a^2\right)^{r-m+n/2} \frac{\Gamma(r+\frac{n}{2})\Gamma(m-r-\frac{n}{2})}{\Gamma(\frac{n}{2})\Gamma(m)} \tag{104}$$

$$\int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{[k^2 - a^2 + \mathrm{i}\varepsilon]^m} = \frac{1}{n} g_{\mu\nu} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k^2}{[k^2 - a^2 + \mathrm{i}\varepsilon]^m}, \quad \int \mathrm{d}^n k \frac{1}{k^2 + i\varepsilon} = 0. \tag{105}$$

- These formulae can be derived
  - using the so called **Wick rotation** write $I(m,r)$ as an integral over the momentum in <u>Euclidean</u> space (where $k_E^2 = \sum_{i=1}^n k_i^2$)

$$I(m,r) = \mathrm{i}(-1)^{r-m} \int \mathrm{d}^n k_E \frac{\left(k_E^2\right)^r}{[k_E^2 + a^2 - \mathrm{i}\varepsilon]^m}, \tag{106}$$

  - working out the angular integral according to (21) and
  - evaluate the remaining one–dimensional integral by means of per partes for integer n:

$$\int_0^\infty \mathrm{d}x \frac{x^{n-1+2r}}{[x^2 + a^2]^m} = \frac{1}{2} \left(a^2\right)^{r-m+n/2} \frac{\Gamma(r+\frac{n}{2})\Gamma(m-r-\frac{n}{2})}{\Gamma(m)}. \tag{107}$$

Dimensional regularization in QED and QCD

- Similarly to $\lambda\phi^4$ theory the way basic fields enter the QED lagrangian determines their dimensions, as well as that of the gauge couplant, in n–dimensional space. They are listed in the Table 1 and are the same in QED and QCD.

quantity	dimension
fermions	$(n-1)/2 = 3/2 - \varepsilon$
gauge bosons	$(n-2)/2 = 1 - \varepsilon$
gauge coupling	$(4-n)/2 = \varepsilon$

Table 1: Dimensions of the fields and the couplant in QED and QCD.

Clifford algebra in n-dimesions

- In QED and QCD there is additional complication related to the extension of γ matrices into n–dimensional space–time. To generalize the Clifford algebra, defined by the anticommutation relations,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad g_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu, \quad g_{00} = 1, \quad g_{ii} = -1 \quad \text{for} \quad i = 1, 2, \cdots n-1 \tag{108}$$

  to n dimension it is useful to start with integer n, where (108) makes sense mathematically and ask whether there is a **realization** of this Clifford algebra by some mathematical objects.

- The answer is not trivial even for integer n and has two parts:
  - **even n** $= 2k$: there are n matrices of the dimension 2^k satisfying (108) and the analogue of γ_5 in 4 dimensions is defined as

$$\gamma^* \equiv \mathrm{i}\gamma^0\gamma^1 \cdots \gamma^{n-1}, \tag{109}$$

  - **odd n** $= 2k+1$: one starts with even number of dimensions $n = 2k+2$ and takes $n-1 = 2k+1$ of the associated γ matrices, which obviously satisfy (108).

Clifford algebra in n-dimesions

- For integer n Clifford algebra can thus be realized by matrices of the dimension $2^{[(n+1)/2]}$ ($[x]$ is whole part of a real number x) , with following properties:

$$
\begin{aligned}
\mathrm{Tr}\ \mathbf{1} &= 2^{[(n+1)/2]}, \\
\mathrm{Tr}\ \gamma_\mu\gamma_\nu &= 2^{[(n+1)/2]} g_{\mu\nu}, \\
\mathrm{Tr}\ \gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta &= 2^{[(n+1)/2]} \left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}\right), \\
\gamma_\mu\gamma^\mu &= n, \\
\gamma_\mu\gamma^\alpha\gamma^\mu &= (2-n)\gamma^\alpha, \\
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= 4g^{\alpha\beta} + (n-4)\gamma^\alpha\gamma^\beta, \\
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\mu &= -2\gamma^\delta\gamma^\beta\gamma^\alpha + (4-n)\gamma^\alpha\gamma^\beta\gamma^\delta.
\end{aligned}
\tag{110}
$$

- Trace of an odd number of γ matrices vanishes in n dimensions as well. In the above relations n must obviously be an integer, but after the trace reductions and momentum integrations are done the resulting expressions will make sense for **arbitrary real n**. This is the essence of dimensional regularization.

Dimensional regularization in QED

- In QED the contribution of the basic vacuum polarization loop in Fig. 3b has in n dimensions the form similar to that of (16):

$$I_{\mu\nu}(q,n) = -e_B^2 \int \frac{\mathrm{d}k^n}{(2\pi)^n} \frac{L_{\mu\nu}(q,k,n)}{[k^2-m^2]\,[(k-q)^2-m^2]}. \tag{111}$$

  Note that contrary to the analogous quantity $I(p,n)$ in $\lambda\phi^4$ theory (111) is <u>dimensionless</u>.

- The numerators in (111) are easily recombined using the Feynman parametrization exactly as in (29)-(30) while the numerator equals

$$L_{\mu\nu}(k,q,n) = 4\left[2k_\mu k_\nu - (k_\mu q_\nu + k_\nu q_\mu) - (k^2 - kq - m^2)g_{\mu\nu}\right]$$

$$= 8x(1-x)\left[q^2 g_{\mu\nu} - q_\mu q_\nu\right] + 4g_{\mu\nu}\left[\left(\frac{2}{n}-1\right)l^2 + (m^2 - q^2 x(1-x))\right]. \tag{112}$$

Dimensional regularization in QED

- Using (104) it is straightforward to show that the second term in (112) vanishes, similarly as in the Pauli–Villars regularization technique, after integration over the momentum l, yielding

$$I_{\mu\nu}(q,n) = -e_B^2\left[q^2 g_{\mu\nu} - q_\mu q_\nu\right]\int_0^1 \mathrm{d}x 8x(1-x)\int \frac{\mathrm{d}l^n}{(2\pi)^n}\frac{1}{\left[l^2-a^2+\mathrm{i}\varepsilon\right]^2}, \quad (113)$$

where $a^2(x,q^2) = m^2 - q^2 x(1-x)$.

- Performing momentum integrals: $I_{\mu\nu}(q,n) = \left[q^2 g_{\mu\nu} - q_\mu q_\nu\right] I(q^2)$ (114)

where the scalar function $I(q^2)$ is given as

$$I(q^2,\varepsilon) \equiv -\mathrm{i}\mu^{-2\varepsilon}\frac{e_B^2}{2\pi^2}(2\pi)^\varepsilon\Gamma(\varepsilon)\int_0^1 \mathrm{d}x x(1-x)\left(a(x,q)\right)^{-\varepsilon}. \quad (115)$$

- Introducing the scale μ etc. as in $\lambda\phi^4$ theory by we get, keeping only terms that do not vanish in the limit $\varepsilon \to 0$

$$I(q,\varepsilon) = -\mathrm{i}\mu^{-2\varepsilon}\frac{e_B^2}{12\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - 6\int_0^1 \mathrm{d}x x(1-x)\ln\frac{\left(m^2 - q^2 x(1-x)\right)}{\mu^2}\right]. \quad (116)$$

Dimensional regularization in QED

- The sum of the diagrams in Fig. 3a,b thus equals (without the fermion bispinors and omitting the term $q_\mu q_\nu$ which vanishes after sandwiching between them)

$$\left(-\frac{\mathrm{i}}{q^2}\right) e_B^2 \left[1 + q^2 g_{\mu\nu} I(k,q,\varepsilon)\left(-\frac{\mathrm{i}}{q^2}\right)\right] =$$

$$\left(-\frac{\mathrm{i}}{q^2}\right) e_B^2 \left(1 - \frac{e_B^2 \mu^{-2\varepsilon}}{12\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - 6\int_0^1 \mathrm{d}x x(1-x) \ln \frac{m^2 - q^2 x(1-x)}{\mu^2}\right]\right). \tag{117}$$

- (117) has similar structure as (36) and leads to introduction of *renormalized electric charge*

$$e_R^2(A,\mu) \equiv e_B^2 \mu^{-2\varepsilon}\left(1 - \frac{e_B^2 \mu^{-2\varepsilon}}{12\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right]\right). \tag{118}$$

Dimensional regularization in QED

- Taking derivative of (118) w.r.t. $\ln\mu$ we get for $\varepsilon \to 0$ analogously to (40):

$$\frac{\mathrm{d}\alpha_R(A,\mu)}{\mathrm{d}\ln\mu} \equiv \beta_{\mathrm{QED}}(\alpha_R,\varepsilon) = -2\varepsilon\alpha_R(A,\mu) + \beta_0\alpha_R^2(A,\mu), \quad \alpha \equiv \frac{e^2}{4\pi}, \quad \beta_0 = \frac{2}{3\pi}, \tag{119}$$

  where, as in the case of ϕ^4 theory, the term, proportional to ε, has to be retained.

- In $\lambda\phi^4$ theory, QED as well as QCD (actually quite in general) we can therefore write β–functions in $n = 4 - 2\varepsilon$ dimensions as

$$\beta_\varepsilon(x,\varepsilon) = -2\varepsilon x + \beta(x), \tag{120}$$

  where $\beta(x)$ are β–functions in 4 dimensions.

- Moreover, the equation for the renormalized coupling is exactly the same as that obtained using the Pauli–Villars regularization. So long as the regularization technique is consistent with the basic requirements of relativistic and gauge invariance, the properties of *physical* renormalized quantities should not depend on its choice.

Renormalization in QCD

- Apart of color d.o.f. complications, basic novel features are related to gluon *selfinteraction*. The crucial one concerns the sign of the derivative of the renormalized color couplant $\alpha_s \equiv g^2/4\pi$ with respect to the scale μ.
- To illustrate this difference let us return to the physical quantity:

$$R(Q) \equiv \frac{\sigma(Q, \mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons})}{\sigma(Q, \mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)} = \overbrace{\left(3\sum_i e_i^2\right)}^{\mathrm{QPM}} [1 + r(Q)]\,, \qquad (121)$$

In QCD the simple QPM formula is just the leading order term, while QCD provides corrections, given by diagrams like those in Fig. 7b,c,d and contained in the quantity $r(Q)$.

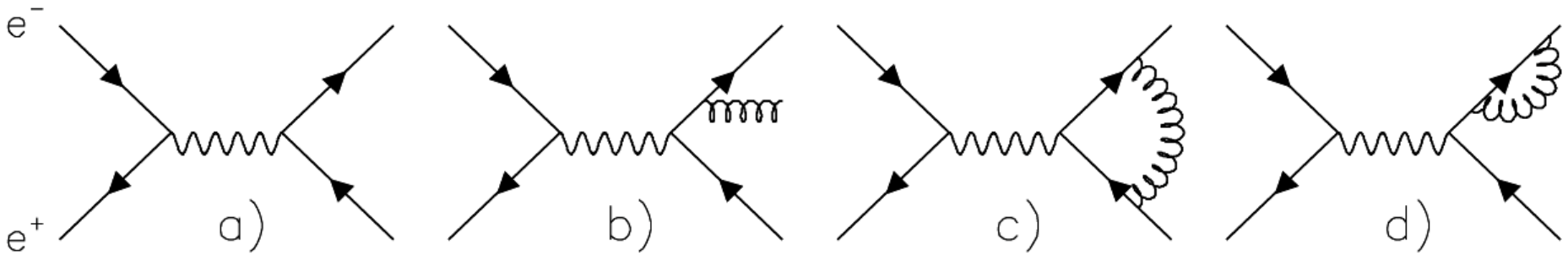

Figure 7: Examples of Feynman diagrams describing the $\mathrm{e}^+\mathrm{e}^-$ annihilation to hadrons in the QPM (a) and the lowest order of perturbative QCD (b-d).

Renormalization in QCD: Beyond the QPM

- $r(Q)$ can be expressed as a power expansion in $a(\mu) \equiv \alpha_s/\pi$:

$$r(Q) = a(\mu/\Lambda)\left[r_0 + r_1(\mu/Q)a(\mu/\Lambda) + r_2(\mu/Q)a^2(\mu/\Lambda) + \cdots\right]; \quad r_0 = 1 \qquad (122)$$

- Equation determining the μ–dependence of $a(\mu/\Lambda_{QCD})$ reads:

$$\frac{\mathrm{d}a(\mu, c_i)}{\mathrm{d}\ln\mu} = -ba^2(\mu, c_i)\left[1 + ca(\mu, c_i) + c_2 a^2(\mu, c_i) + \cdots\right], \qquad (123)$$

  where for brevity of this (and the following) formulae the QCD parameter Λ is always understood to scale μ and the subscript "R" standing for "renormalized" is dropped.

- The r.h.s. of (123), considered as a function of a, is called the **β–function** *of QCD*. For $m_q = 0$ and fixed α_G (appearing in the gauge fixing term $-\frac{1}{2\alpha_G}\left(f(A_\mu(x))\right)^2$) all the coefficients $b, c, c_i; i \geq 2$ are pure numbers.
- Coefficients b and c are *uniquely* determined by numbers of quark flavors (n_f) and colors (N_c):

$$b = \frac{11N_c - 2n_f}{6}; \quad c = \frac{51N_c - 19n_f}{22N_c - 4n_f}. \qquad (124)$$

Asymptotic freedom of QCD

- In realistic QCD, $N_c = 3$ and $n_f \leq 6$ and consequently the coefficient b on the r.h.s. of (123) is *positive*! Herein lies the basic difference from QED, where, on the contrary, $b < 0$.
- The fact that $b > 0$ and thus the leading term on the r.h.s. of (123) is *negative* implies that $a(\mu/\Lambda) \to 0$ as $\mu \to \infty$.
- This phenomenon, called **asymptotic freedom** is a fundamental feature of QCD and can be traced back to the selfinteraction of gluons. As a result the behavior of $a(\mu)$ changes dramatically compared to the situation in QED (Fig. 8b).

QED Problems in defining the theory at *short* distances but it smoothly joins the classical electrodynamics.

QCD No difficulties at *short* distances, where perturbation theory can be safely applied, but perturbation theory becomes inadequate at *large* ones.

- This becomes clear if we take LO solution of (123):

$$a(\mu/\Lambda) = \frac{1}{b \ln(\mu/\Lambda)}. \tag{125}$$

Asymptotic freedom of QCD

- We conclude that in QCD Λ provides the *lower* and not, as in QED, the *upper* bound on the meaningful values of μ! Graphically the situation is sketched in Fig. 8b, which could be obtained from Fig. 4b by transformation $\mu \to 1/\mu$.

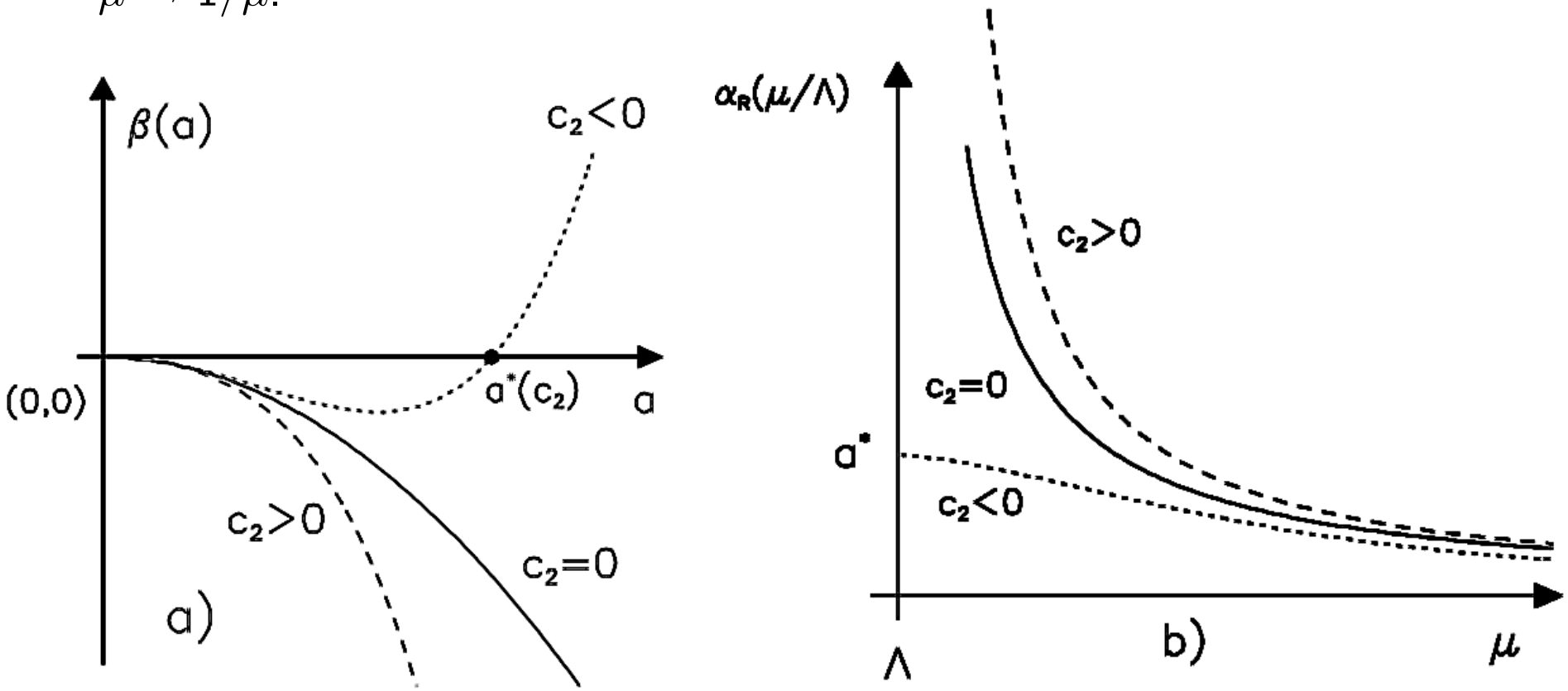

Figure 8: The shape of QCD β–function for three different cases of c_2 (a), and the behavior of the corresponding couplants as functions of the scale μ.

Asymptotic freedom of QCD

Several further aspects of the relations (122) and (123) merit a comment:

- Except for the first two coefficients b, c in (123) *all* the higher order ones $c_i; i \geq 2$ are *arbitrary* real numbers. The relation (123) should thus be considered as a **definition** of the couplant $a(\mu)$.
- The arbitrariness in $c_i; i \geq 2$ doesn't, however, mean that the full QCD results depend on their choice. Besides $a(\mu, c_i)$ also the expansion coefficients r_k *depend* on $c_i, i \leq k$ and these dependences cancel each other, provided we calculate (122) to *all* orders.
- Taking into account the next-to-leading order (NLO) term in (123) the equation for $a(\mu)$ can no longer be solved analytically, but the implicit equation

$$b \ln \frac{\mu}{\Lambda} = \frac{1}{a} + c \ln \frac{ca}{1 + ca} \tag{126}$$

can easily be solved numerically. It is quite common to use first two terms

$$a(\mu/\Lambda) = \frac{1}{b \ln(\mu/\Lambda)} - \frac{c}{b^2} \frac{\ln((b/c) \ln(\mu/\Lambda))}{\ln^2(\mu/\Lambda)} + \cdots \tag{127}$$

in the expansion of the exact solution of (126) in powers of the inverse logarithm $1/\ln(\mu/\Lambda)$.

Choice of the scale parameter in QCD

- Basically the dependence on the scale μ is a measure of our ignorance of uncalculated higher order terms. The more sensitive are the results to the variation of μ, the more important these higher order terms probably are.
- The conventional wisdom, based on years of experience with perturbative calculations: choose $\mu =$ **typical physical scale of the analyzed process**.
- In the case of (122) we could take for such a typical scale the square $\sqrt{s}$, in DIS either $\sqrt{Q^2}$ or $\sqrt{W^2}$ and for the Drell-Yan dilepton pair production $M_{\ell\ell}$.
- Question how far down with μ (i.e. to how large distances) we can go before running into troubles with perturbation expansions is difficult to answer quantitatively. However, we expect the behavior of the couplant in the infrared region to be related to the phenomenon of **color confinement**.
- The same effect which leads to the asymptotic freedom also causes the couplant to *increase* at large distances! Although we cannot believe perturbation theory in this region it is certainly an indication that QCD has at least a chance to describe *both* the success of QPM in hard scattering processes and the color confinement!

Further comments

1. Show that the second term in (52) does, indeed, vanish.
2. Taking into account that for μ of the order of electron mass $\alpha_R(\mu/\Lambda) = 1/137$, estimate the value of Λ in QED as given in (88).
3. Discuss the behavior of the exact solution to the NLO equation (126) in the limit $\mu \to \Lambda$. Where does it blow up? How does it differ from the behavior of the LO solution?
4. Show that the simultaneous changes of $a(\mu)$ and r_1 with μ cancel to the order a^2 in (122).
5. Extract the value of Λ corresponding to the RS where $r_1 = 1.411$, using the NLO approximation to (122) and the fact that for $Q = 35$ GeV its experimental value equals 0.05.
6. Express the IR fixed point $a^*(c_2)$ as a function of c_2 taking into account the first three terms in the expansion of $\beta(a)$.