Introduction to Quantum Chromodynamics

Michal Šumbera

Nuclear Physics Institute ASCR, Prague

November 11, 2009

Literature

Our discussion is based on

Quarks, partons and Quantum Chromodynamics
by Jiří Chýla
Available at http://www-hep.fzu.cz/ chyla/lectures/text.pdf

Additional material comes from

A modern introduction to particle physics
by Fayyazuddin & Riazuddin
World Scientific Publishing 2000 (Second edition)

Additional material comes from

Quantum Field Theory in a Nutshell
by A. Zee
Princenton University Press 2003

Introduction

- Not a selfcontained introduction to Quantum Chromodynamics (QCD). Rather collection of remarks on points I consider important for understanding how perturbative QCD (pQCD) works and in particular how it accommodates QPM.
- First we concentrate on general principles of gauge theories, to which QCD, as well as the more familiar **Quantum Electrodynamics** (QED) belong.
- In most considerations we will stay within classical physics, avoiding the formal aspects of quantization of gauge theories.
- Basic features of the resulting Feynman rules will be shown to follow from the classical Lagrangian. The Feynman rules will then be used to evaluate matrix elements of some important parton level processes.
- Though most of presented material will cover pQCD we may try later on to discuss also few non–perturbative aspects of this theory: color confinement, hadron masses and wave functions and structure of the QCD vacuum.

Maxwell equations in the covariant form

- In classical physics the free electromagnetic field in the vacuum can be described by the electric and magnetic field strengths $\vec{E}(t,\vec{x})$ and $\vec{H}(t,\vec{x})$, which satisfy *Maxwell equations*:

$$\mathrm{rot}\vec{H} = \frac{\partial \vec{E}}{\partial t}, \quad \mathrm{div}\vec{E} = 0, \tag{1}$$

$$\mathrm{rot}\vec{E} = -\frac{\partial \vec{H}}{\partial t}, \quad \mathrm{div}\vec{H} = 0 \tag{2}$$

and appear in the expression for the force with which this field acts on a test particle with electric charge e and velocity v

$$\vec{F} = e\left[\vec{E} + \vec{v} \times \vec{H}\right]. \tag{3}$$

- The second pair of Maxwell equations (2) allows us to express $\vec{E}(x,t)$ and $\vec{H}(x,t)$ in terms of the scalar and vector potentials $\phi(x,t)$ and $\vec{A}(x,t)$:

$$\vec{E} = -\mathrm{grad}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{H} = \mathrm{rot}\vec{A}, \quad x^{\mu} = (t,\vec{x}), \quad x_{\mu} = (t,-\vec{x}) \tag{4}$$

because (4) automatically implies (2).

Maxwell equations in the covariant form

- Arranging $A_0 \equiv \phi$ and $\vec{A}$ as components of the four-potential
$$A^\mu \equiv (A^0, \vec{A}), \quad A^0 = A_0; \qquad A_\mu \equiv (A_0, -\vec{A}) \tag{5}$$
and introducing the antisymmetric tensor
$$F^{\mu\nu}(x) \equiv \frac{\partial A^\nu(x)}{\partial x_\mu} - \frac{\partial A^\mu(x)}{\partial x_\nu} \Rightarrow F^{0i}(x) = -E_i(x), \;\; F^{ij}(x) = -\epsilon_{ijk} H_k(x), \tag{6}$$
allows us to rewrite (1) in a manifestly covariant form
$$\partial_\mu F^{\mu\nu}(x) = 0. \tag{7}$$
- Within the framework of Lagrangian field theory (still at the classical level!) the above equations are just the *Euler equations* resulting from the Lagrangian
$$\mathcal{L}_{\mathrm{free}}^{\mathrm{Maxwell}} = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x). \tag{8}$$
- In classical physics introduction of $A_\mu(x)$ is nothing but convenient way how to reduce 8 equations (1-2) for 6 components E_i and H_i to only 4 in (7).
- On the other hand, we may in principle dispense with the A_μ as the expression for the Lorentz force (3) contains only $\vec{E}$ and $\vec{H}$.

Electromagnetic four-potential in quantum physics

- As we shall see below, in quantum field theory the four-potential $A_\mu(x)$ is indispensable for proper formulation of interacting electromagnetic field.
- Moreover as shown by Aharonov and Bohm in 1959 contrary to classical physics in quantum physics the gauge potential matters!
- Schrödinger equation for particle with electric charge e in potential $(\phi, \vec{A})$ reads:
$$-\frac{1}{2m}(\nabla - ie\vec{A})^2\Psi = (\frac{\partial}{\partial t} + ie\phi)\Psi \qquad (9)$$
- Taking $\vec{A} = const.(t)$ and putting $V = e\phi$ we try solution of (9) in the following form:
$$\Psi(\vec{r},t) = \Psi^0(\vec{r},t)e^{i\gamma(\vec{r})} \quad \text{where} \quad \gamma(\vec{r}) = e\int_P \vec{A}\cdot d\vec{r} \qquad (10)$$
Here Ψ can be regarded as a w.f. of a particle that goes from one place to another along a certain route P where a field A is present while Ψ^0 is the wave function for the same particle along the same route but with $\vec{A} = 0$.
- Thus (10) is a solution of (9) when $\vec{A}(\vec{r}) \neq 0$ if Ψ^0 satisfies
$$-\frac{1}{2m}(\nabla^2 + V)\Psi^0 = i\frac{\partial \Psi^0}{\partial t} \qquad (11)$$

Aharonov-Bohm experiment

- Consider a magnetic field B confined to a region Ω (see Fig. 1). Charged particle traveling along some path P in a region with zero magnetic field $\vec{B}$, but non zero $\vec{A}$ (by $\vec{B} = 0 = \nabla \times \vec{A}$), thus according to (10) acquires a phase shift γ

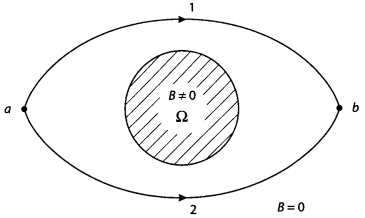


Figure 1: Aharonov-Bohm experiment.

- Two charged particles with the same start and end points, but traveling along two different routes will acquire a phase difference $\Delta\gamma$ determined by the magnetic flux Φ through the area between the paths 1 and 2. Stokes' theorem and $\nabla \times \vec{A} = \vec{B}$ gives:

$$\Delta\gamma = e \oint_C \vec{A}(\vec{r}) \cdot d\ell = e \int_S \vec{B} \cdot d\vec{\sigma} = e\Phi. \qquad (12)$$

Aharonov-Bohm experiment

- $\Delta\gamma$ can be observed by placing a solenoid between the slits of a double-slit experiment (Fig. 2).

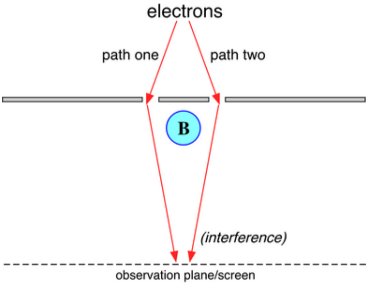


Figure 2: Schematic of double–slit experiment in which Aharonov–Bohm effect can be observed: electrons pass through two slits, interfering at an observation screen, with the interference pattern shifted when a magnetic field B is turned on in the cylindrical solenoid.

- In the interference region, the wave function is $\Psi = \Psi_1 + \Psi_2$ so that

$$|\Psi|^2 = |\Psi_1^0|^2 + |\Psi_2^0|^2 + 2|\Psi_1^0||\Psi_2^0| \cos(\gamma_1(\vec{r}) - \gamma_2(\vec{r})) \qquad (13)$$

Concepts of gauge theories

- Basic object of QED is the *local fermion field* $\Psi(x)$ describing one type of charged particle with spin 1/2 (lepton or quark). On the classical level it is just the solution of the corresponding Dirac equation. In QFT $\Psi(x)$ becomes a **local operator** satisfying certain **anticommutation** relations.
- Skipping the formalism of quantized fields (see KTP 81-150), we outline how the requirement of local gauge invariance, combined with Lorentz invariance determines the basic properties of electromagnetic interactions of charged fermions and photons in QED.
- Heuristic "derivation" of QED Lagrangian goes as follows:
  1. Start with the classical Lagrangian of a noninteracting fermion field $\Psi(x)$ of mass m

  $$\mathcal{L}_{\mathrm{free}}^{\mathrm{fermion}} = \overline{\Psi}(x)(\mathrm{i}\,\partial\!\!\!/ - m)\Psi(x), \tag{14}$$

  which generates, via the **Euler–Lagrange** equations of motion, the Dirac equation for a free fermion with spin 1/2

  $$(\mathrm{i}\,\partial\!\!\!/ - m)\,\Psi(x) = 0. \tag{15}$$

2. Define the **global gauge transformations** of this field as phase rotations:

$$\Psi' \equiv \exp(\mathrm{i}\alpha)\Psi(x), \tag{16}$$

where α is a *real number*. The set of such transformations forms a U(1) group. The free fermion Lagrangian (14) *is invariant* under this simple transformation.

3. Impose the requirement of **local gauge invariance**, i.e. invariance with respect to transformations (16) but for α *depending on x*. In that case the Lagrangian (14) is no longer invariant due to the term:

$$-\overline{\Psi}(x)\gamma^{\mu}\frac{\partial\alpha(x)}{\partial x^{\mu}}\Psi(x), \tag{17}$$

which comes from the partial derivative in (14) and vanishes only for constant α.

Derivation of QED Lagrangian

4. To recover local gauge invariance of one has to introduce another field, which *compensates* the noninvariance of (14). This is achieved by **gauge field** $A_\mu(x)$ – a **vector** field describing a particle with spin 1 and mass M. Its Lagrangian, including the possible mass term, is a simple generalization of (8)

$$\mathcal{L}_{\text{free}}^{\text{gauge}} = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) + \frac{1}{2}M^2 A_\mu(x)A^\mu(x), \tag{18}$$

where $F^{\mu\nu}(x)$ is given in (6). The corresponding equations of motion (so far on classical level) read

$$\partial_\mu F^{\mu\nu}(x) = -M^2 A^\nu(x). \tag{19}$$

To cancel the noninvariant term (17) this gauge field is *assumed* to interact with $\Psi(x)$ via the interaction term

$$\mathcal{L}_{\text{int}} = e\overline{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x) \equiv J^\mu A_\mu \tag{20}$$

and transform *simultaneously* with (16) as

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\frac{\partial\alpha(x)}{\partial x^\mu}, \tag{21}$$

where e, an arbitrary *real number*, determining the strength of the interaction (20), will be interpreted as the **electric charge** of the fermion field $\Psi(x)$.

4. (cont'd) Casualty of this compensation mechanism is the mass M of the gauge field. As the mass term $M^2 A_\mu A^\mu$ violates the invariance under the combined fermion and gauge field transformation only *massless* gauge field A_μ is compatible with local gauge invariance.
This completes construction of the full QED Lagrangian of interacting Dirac and Maxwell fields:

$$\mathcal{L}^{\mathrm{QED}} = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) + \overline{\Psi}(x)(\mathrm{i}\,\not{\partial} - m)\Psi(x) + e\overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x), \quad (22)$$

The coupled equations of motion satisfied by these fields read:

$$\left[\mathrm{i}\gamma_\mu \left(\partial^\mu - \mathrm{i}eA^\mu(x)\right) - m\right]\Psi(x) = 0, \quad \partial_\mu F^{\mu\nu}(x) = e\overline{\Psi}(x)\gamma^\nu \Psi(x). \quad (23)$$

Note that the Lagrangian (22) can be obtained from the sum of free lagrangians (14) and (18) by the replacement, called *minimal substitution*, of the partial derivative ∂_μ by the **covariant derivative** $D_\mu \equiv \partial_\mu - \mathrm{i}eA_\mu(x)$. The name comes from the fact that $D_\mu \psi(x)$ transforms in the same way (16) as fermion field itself

$$D'_\mu \psi'(x) = \exp(\mathrm{i}\alpha(x)) D_\mu \psi(x) \;\Rightarrow\; D'_\mu = \exp(\mathrm{i}\alpha(x)) D_\mu \exp(-\mathrm{i}\alpha(x)) \quad (24)$$

QED Lagrangian: Alternatives

- Quantization brings in some important conceptual changes, but doesn't modify the essence of the above "derivation". Moreover the form of the interaction term (20) suggests that the factor at the basic QED vertex e–γ–e will be proportional to $e\gamma^{\mu}$.
- N.B. The local gauge invariance alone is not sufficient to determine the QED Lagrangian uniquely.
- For instance we could add to (22) any power of the photon kinetic term $F^{\mu\nu}F_{\mu\nu}$, which by itself is locally gauge invariant and thus obtain *selfinteraction of electromagnetic field*!
- Similarly we could couple the field tensor $F^{\mu\nu}$ directly to the fermion tensor $\overline{\psi}\sigma_{\mu\nu}\psi$, which also represents locally gauge invariant interaction. In this case we would dispense with the gauge potential $A_{\mu}(x)$ and fermions would interact with photons via their magnetic moments, but this interaction would be of *finite range*.
- In both alternatives the kind of interaction we get at the classical level is significantly different from the *coulomb* forces observed in the nature.

QED Lagrangian: Alternatives

- The fact that the photon is massless corresponds to our experience that electromagnetic interactions are of *long* (actually *infinite*) range, but in classical theory none of the mentioned alternatives can be rejected merely on theoretical grounds.
- In QFT there is a powerful theoretical principle, which actually rules out both of the mentioned alternatives and which, combined with the requirement of local gauge invariance, determines the QED Lagrangian essentially *uniquely*: **the renormalizability of the theory**.
- On the other hand there is no a priori reason, why only locally gauge invariant field theories should exist in nature. For instance, it is possible to formulate a renormalizable theory of *massive* photons, interacting locally with fermions but such an interaction is again of *finite* range and thus doesn't describe the real world.
- The principle of local gauge invariance is probably the most fundamental principle of the current QFT. On the other hand it leads to complications in the description of the very simplest quantity concerning the photons: propagator of the free photon.

QED Lagrangian: Photon complications

- Consider the definition of a free photon propagator $D_{\mu\nu}(x,y)$, which follows from the Maxwell equations for the gauge field $A_\mu(x)$

$$\partial_\mu F^{\mu\nu}(x) = \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\mu(x) = 0 \tag{25}$$

and reads

$$\left(\partial^\mu \partial^\rho - \partial^2 g^{\mu\rho}\right) D_{\rho\nu}(x-y) = \mathrm{i}\delta_{\mu\nu}\delta(x-y). \tag{26}$$

- The Fourier transform of $D_{\mu\nu}(x-y)$, denoted $D_{\mu\nu}(k)$, satisfies, as a consequence of (26), the following matrix equation

$$\left(k^\mu k^\rho - k^2 g^{\mu\rho}\right) D_{\rho\nu}(k) = -\mathrm{i}g^\mu_\nu = -\mathrm{i}\delta_{\mu\nu}. \tag{27}$$

- However, as the matrix $(k_\mu k_\nu - k^2 g_{\mu\nu})$ is *singular* the equation (27) has no solution![1]

(1) For *massive* photons k^2 should be replaced in (27) by $k^2 - m^2$, which renders it regular and we find without problems $d_{\mu\nu} = -g_{\mu\nu} + k_\mu k_\nu / m^2$.

QED Lagrangian: Photon complications

- The reason: with each $A_\mu^{(0)}(x)$ satisfying the equation of motion (25), all the fields $A_\mu(x)$, gauge equivalent[1] to $A_\mu^{(0)}(x)$, do so as well.
- One way how to avoid this problem is to select from each class of gauge equivalent fields $A_\mu(x)$ merely those satisfying some additional **gauge fixing condition**,
$$f(A_\mu(x)) = 0. \tag{28}$$
and solve the equation (27) merely on the subset of functions satisfying (28).
- Alternatively we may give (26) a good meaning by adding to (22) the so called **gauge fixing term** of the form
$$-\frac{1}{2\alpha_G}\left(f(A_\mu(x))\right)^2, \tag{29}$$
where α_G is an arbitrary real number called **gauge parameter**.

(1) Gauge fields related by the gauge transformation (21) are called **gauge equivalent**.

QED Lagrangian: Gauge fixing

There are various classes of such gauge fixing terms, let us mention two of them, which are most often used in perturbation theory:

- The class of **covariant** gauges, where $f(A_\mu) \equiv \partial_\mu A^\mu(x)$. (30)

  In this case the addition of (29) modifies the equations of motion (25) to

  $$\partial_\mu F^{\mu\nu}(x) + \frac{1}{\alpha_G}\partial^\nu\left(\partial_\mu A^\mu\right) = \left(\partial^2 g^{\mu\nu} - (1 - 1/\alpha_G)\partial^\mu\partial^\nu\right) A_\mu(x) = 0 \quad (31)$$

  and the definition equation for the propagator in this gauge

  $$\left((1 - 1/\alpha_G)k^\mu k^\rho - k^2 g^{\mu\rho}\right) D_{\rho\nu}(k) = -\mathrm{i}g^\mu_\nu \quad (32)$$

  has a well–defined solution. Writing $D_{\mu\nu}(k)$ as

  $$D_{\mu\nu}(k) \equiv i\frac{d_{\mu\nu}}{k^2} \quad \Rightarrow \quad d_{\mu\nu} = -g_{\mu\nu} + (1 - \alpha_G)\frac{k_\mu k_\nu}{k^2}. \quad (33)$$

- Clearly *any* nonzero α_G is good for this purpose, but $\alpha_G = 1$, defining the so called **Feynman gauge**, makes the propagator particularly simple for calculations. Absence of the gauge fixing term, corresponding formally to the case $\alpha_G = \infty$, makes (33) obviously ill-defined.

QED Lagrangian: Gauge fixing

- Class of **axial** gauges with the gauge fixing term given as

$$f(A_\mu(x)) \equiv c_\mu A^\mu(x), \tag{34}$$

where c_μ is an arbitrary, but *fixed* four–vector with the dimension of mass.

- In this case the modified equations of motion read

$$\partial_\mu F^{\mu\nu}(x) - \frac{1}{\alpha_G} c^\nu \left(c_\mu A^\mu\right) = \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu - \frac{1}{\alpha_G} c^\mu c^\nu\right) A_\mu(x) = 0 \tag{35}$$

and the corresponding equation for the propagator

$$\left(k^\mu k^\rho + \frac{1}{\alpha_G} c^\mu c^\rho - k^2 g^{\mu\rho}\right) D_{\rho\nu}(k) = \mathrm{i} g^\mu_\nu \tag{36}$$

has again a well–defined solution with

$$d_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu \left(\alpha_G k^2 - c^2\right)}{(kc)^2} + \frac{k_\mu c_\nu + k_\nu c_\mu}{kc}. \tag{37}$$

QED Lagrangian: Gauge fixing

- For $c^2 = 0$ we talk about *lightlike* gauges. Setting in addition $\alpha_G = 0$ we get
$$d_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu c_\nu + k_\nu c_\mu}{kc}. \tag{38}$$
which is particularly suitable for perturbative calculations.
- In other words, in classical physics the addition of the gauge fixing term (29) to the Lagrangian (22) is *not equivalent* to selecting only those fields $A_\mu(x)$ satisfying the condition (28). It is inconsistent with gauge invariance and thus illegal.
- Situation is quite different in QFT. In QED the addition to the Lagrangian (22) of the gauge fixing term (29) *is consistent* with the gauge invariance of the quantized theory, but the proof that the results for **physical quantities** are independent of
  - the choice of the function $f(A_\mu)$ and
  - the value of gauge parameter α_G

  is by no means trivial and is an integral part of the whole quantization procedure!

QED perturbation theory

- Physical quantities are expressed as expansions in powers of the electric charge e^2, appearing in the Lagrangian (22). Individual terms of these power expansions are graphically represented by the **Feynman diagrams**, containing three basic elements:
  - External legs, describing the in and outgoing *real* particles.
  - Propagators, corresponding to *virtual* intermediate states.
  - Interaction vertices, describing the local interaction of quantized fields.
- Although "virtual" particles are just intermediate states in quantum evolution, they behave *nearly as real particles* if observed for sufficiently short time interval. This happens in the hard scattering processes (like those discussed in QPM) via the interaction with some "testing" particle, (e.g. lepton). The measure of the virtuality is basically particle off–mass shellness $m^2_{\mathrm{virt}} \equiv k^2 - m^2$, but what really determines whether this virtual particle behaves nearly like real is actually the ratio m_{virt}/Q, where "Q" denotes the "hardness" of the collision.[1] So even states far off mass-shell may behave nearly like real particles when observed for short enough time.

(1) In DIS "Q^2" is just the four-momentum transfer squared Q^2.

The rise and fall of quantum field theory

- Soon after the formulation of Quantum Electrodynamics by Dirac in 1928, theorists like Bethe, Heitler, Oppenheimer, Weisskopf and others have attempted to calculate quantum corrections (loop corrections) to lowest order perturbative results for several physical quantities.
- It took about two decades to develop a systematic procedure how to handle and remove infinities coming from contributions of very small distances – so called ultraviolet (UV) divergencies.
- For their decisive contributions towards the formulation of this procedure, called renormalization, Feynman, Schwinger and Tomonaga were awarded the 1965 Nobel Prize for physics.
- Although this procedure provided the framework for systematic removal of UV infinities to all orders of perturbation theory, it lacked clear physical interpretation and appeared mathematically rather arbitrary.
- As will be discussed later in more detail some of the nonabelian gauge theories have the property called "asymptotic freedom", which allows us to construct quantum field theories that are free of genuine UV infinities.

The rise and fall of QFT: Renormalization

- For Dirac and his contemporaries the electric charge appearing in the QED lagrangian was considered to have a given value, which might have been screened by quantum effects, but had nevertheless some definite value, and once this value was given the theory was fully defined.
- Modern view due to Kenneth Wilson and others says that we should express physical quantities not in terms of "fictitious" theoretical quantities but in terms of physically measurable quantities.
- In Wilson's approach construction of QFT starts with *discretized* space-time and involves, as a nontrivial step, the limiting procedure in which the lattice spacing goes to zero. A crucial feature of this limiting procedure concerns the fact that in order to get finite results for physical quantities, the parameters appearing in the original lagrangian must be nontrivial functions of the lattice spacing!
- N.B. Λ which parameterizes our ignorance of the next energy domain is roughly the inverse of the lattice spacing.

The rise and fall of QFT: Renormalization

- Replacing the lattice spacing with the radius of electron, the construction of lattice field theory follows closely the strategy employed by Landau, Pomeranchuk and their collaborators in the early fifties in their attempts to give a physical content and mathematical sense to the procedure of *renormalization* in QED.
- L& P put a finite electric charge e_0, usually called *"bare" charge*, on the sphere of radius r_0 placed in the QED vacuum, calculated how it appears to a test particle at a finite distance $r > r_0$ and investigated what happens when we sent $r_0 \to 0$, as we indeed, must if we want to have Lorentz invariant theory.
- If we do the same in classical physics, the answer is trivial. The force at a distance r induced by the bare charge can be characterized by the effective electric charge $e_{\mathrm{eff}}(r)$ which equals $e(r_0) = e_0$ for $r \geq r_0$ and vanishes for $r < r_0$. Consequently, if we shrink the electron radius to zero keeping $e_0 = e(r_0)$ fixed, the effective electron charge $e_{\mathrm{eff}}(r) = e_0$ for <u>all</u> distances r!
- In QFT vacuum is not "empty" but bustling with activity usually described as "virtual pair creation and annihilation".

The rise and fall of QFT: Renormalization

- Placing the bare electric charge of a finite radius into QED vacuum is similar to response of a dielectric medium to analogous action. Inserted bare charge polarizes elementary electric dipoles so that is charge is screened, i.e. its magnitude decreases with increasing distance r.
- Problem with this analogy is that the evaluation of the relation between $e_0 = e(r_0)$ (or, more conveniently, the combination $\alpha_0 \equiv e_0^2/4\pi$, called "couplant") and the effective couplant $\alpha(r)$ at the distance $r > r_0$, using standard QED perturbation theory, gives

$$\alpha(r) = \frac{\alpha(r_0)}{1 - \beta_0 \alpha(r_0) \ln(r_0/r)}, \quad \Rightarrow \quad \alpha(r_0) = \frac{\alpha(r)}{1 - \alpha(r)\beta_0 \ln(r/r_0)}. \tag{39}$$

  where $\beta_0 = 2/3\pi$.
- The above expression holds for $r \geq \lambda_e \doteq 400$ fm, whereas below this value $\alpha(r)$ becomes essentially constant, approaching the classical value $1/137$.
- For $r_0 \to 0$ and $\alpha_0 = \alpha(r_0)$ fixed, the first equation in (39) implies $\alpha(r) \to 0$ $\forall r > 0$! Shrinking the electron radius while keeping its charge fixed results in **free, non-interacting theory**!

The rise and fall of QFT: Renormalization

- For $\alpha(r)$ to be finite at all distances r, the electric charge placed on the sphere must grow with its decreasing radius r_0. From second equation in (39) we see that it rises and diverges at a finite distance $r_L \simeq \lambda_e \exp(-137/\beta_0)$.
- Shrinking r below r_L makes $\alpha(r)$ given by (39) negative, making the theory meaningless. Appearance of this so called **Landau singularity** had marred the attempts of Landau, Pomeranchuk and their school to formulate QED in a physically motivated and mathematically well-defined way. This, in turn had led to a temporary loss of interest in the lagrangian formalism of the local QFT.
- Since r_L is unimaginably small we can for all practical purposes forget about it. The failure to formulate the renormalization procedure in a fully consistent manner had not prevented the application of QED to evaluation of quantum corrections to various physical processes.
- Though the QFT fell into disrepute as the basic theoretical tool for the treatment of strong interaction, QED was widely used to describe, and with great success, the electromagnetic interactions.

The rise and fall of QFT: Renormalization

- Disregarding Landau singularity there were other reasons for disregard of QFT.
    1. Perturbation theory seemed inapplicable for large values of the coupling between hadrons within the framework of Yukawa theory or the Eightfold Way.
    2. QFT appeared entirely unsuitable to describe interactions between quarks within the quark model.
- Rejection of QFT as the basic theoretical tool for the description of strong interaction was accompanied by the formulation and development of the analytic S-matrix theory, which dispenses with the concept of local fields and works primarily with matrix elements describing transitions between various initial and final states of observable hadrons.
- This approach had dominated strong interaction theory since middle fifties until the arrival of QCD in 1973.

And its remarkable resurrection

- By the early 1973 the data had moreover provided compelling evidence for identification of charged partons with quarks and indirect one for the presence of neutral partons in the nucleon as well.
- By 1973 all the important ingredients of what we now call Quantum chromodynamics (QCD) were thus available. What remained to be done was to show that this quark-parton model with *approximate scaling* on one side *and* the *color confinement* on the other follows from some field theory.
- These seemingly contradictory requirements can be accommodated in a field theory that differs from QED in just one "minor" point: the sign of the coefficient β_0 appearing in (39).
- For $\beta_0 < 0$ situation changes dramatically: in order to keep corresponding effective couplant (denoted as $\alpha_s(r)$) finite as $r_0 \to 0$, the bare couplant $\alpha_s(r_0)$ does not have to diverge at some finite distance as in QED, but must vanish instead!
- This property of (non-abelian gauge) QFT is called *asymptotic freedom*.

Asymptotic freedom of non-abelian gauge theories

- This has three important consequences:
    1. Renormalization ala Landau can be pursued to arbitrarily small distances allowing construction of Lorentz invariant field theory.
    2. There are in fact not genuine ultraviolet infinities.
    3. $\alpha_s(r)$ depends on r in opposite manner that $\alpha(r)$ in QED: it decreases at small and grows at large distances, the latter behavior suggesting the quark confinement.
- In the other words, in theories with $\beta_0 < 0$ the effects of quantum fluctuations in the vacuum lead to *antiscreening* of the appropriate charge!
- It should be emphasized that the rise in such theories of the effective couplant at large distances must be taken as mere indication, not a real proof of the quark confinement.
- As in the cases of strangeness, Eightfold Way, the Quark model, also the discovery of asymptotic freedom was achieved simultaneously by two groups. Beside the duo Gross-Wilczek, the same result was obtained simultaneously and independently by Politzer. Both calculations were published in the same issue of PRL, heralding the dawn of a new age in physics.

- The chain of steps leading to its Lagrangian is similar but there is a *fundamental* difference due to the **nonabelian** character of the corresponding group of gauge transformations. This, in turn, is due to the fact that basic mathematical quantity describing a quark of a particular flavor (like u, d, s, etc.) is the matrix in the color space

$$\Psi(x) \equiv \begin{pmatrix} \Psi^1(x) \\ \Psi^2(x) \\ \Psi^3(x) \end{pmatrix} \tag{40}$$

and its *local* gauge transformations are multiplications by the 3×3 unitary, unimodulary matrices of the color SU(3) group:

$$\Psi'(x) = \underbrace{\exp(\mathrm{i}\alpha_a(x) T_a)}_{\in SU(3)} \Psi(x) = S\Psi(x), \tag{41}$$

where the sum in the exponent runs over the 8 generators T_a of $SU(3)$ and $\alpha_a(x), a = 1, \cdots 8$ are functions of x.

QCD Lagrangian

- Postulating the above form of gauge transformations of quark fields, the existence of 3 quark colors requires then introduction of 8 gauge fields A^a_μ, describing 8 **colored gluons** and conveniently represented by a column matrix

$$\vec{A}_\mu(x) \equiv \begin{pmatrix} A^1_\mu(x) \\ \vdots \\ A^8_\mu(x) \end{pmatrix}. \tag{42}$$

- There is an alternative representation of the gluon octet by the 3×3 matrix defined as

$$A_\mu(x) \equiv A^a_\mu(x) T^a, \tag{43}$$

- For each color index a we can define the tensor of field strength

$$F^{\mu\nu}_a(x) \equiv \frac{\partial A^\nu_a(x)}{\partial x_\mu} - \frac{\partial A^\mu_a(x)}{\partial x_\nu}. \tag{44}$$

QCD Lagrangian

- Taking into account these technical complications and following step by step the chain of considerations sketched above for QED, we write down the QCD Lagrangian in the form[1]:

$$\mathcal{L}^{QCD} = -\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu} + \overline{\Psi}(\mathrm{i}\,\partial\!\!\!/ - m_q)\Psi + g\overline{\Psi}\gamma_\mu\vec{T}\Psi\vec{A}^\mu, \qquad (45)$$

  where the scalar product of vectors is a shorthand for the sum over the color index a of the gluon fields A_a and the generators T_a.

- N.B. Scalar product in the third term of (45) *is* invariant under the simultaneous *global* (i.e. with α_a independent of x) gauge transformations (41) of the quark field and corresponding *global* rotations of gluon fields under the **adjoint** representation of SU(3):

$$\vec{A}'_\mu = \exp\left(\mathrm{i}\alpha_a F_a\right)\vec{A}_\mu, \quad \left(F_a\right)_{bc} \equiv \frac{1}{\mathrm{i}} f_{abc}. \qquad (46)$$

---

(1) As the gluons are "flavor blind", including arbitrary number of quark flavors is simple: the full Lagrangian is a sum of the second and third terms of (45), each corresponding to one particular flavor, to which *one* gluon kinetic term is added.

QCD Lagrangian

- Contrary to QED, the Lagrangian (45) is, however, not the end of the story. For the infinitesimal *local* gauge transformations of *nonabelian* gluon fields, defined in analogy with QED as

$$\left(A^a_\mu\right)' \equiv A^a_\mu + \alpha_b f_{bac} A^c_\mu + \frac{1}{g}\frac{\partial\alpha^a(x)}{\partial x^\mu} \Rightarrow A'_\mu = S A_\mu S^{-1} + \frac{\mathrm{i}}{g} S\partial_\mu S^{-1} \quad (47)$$

the last term cancels in the definition of the tensors $F^a_{\mu\nu}$ as in QED, but *noncomutativity* of color rotations make the gluonic kinetic energy term $F^a_{\mu\nu}F^{\mu\nu}_a$ $SU_c(3)$ noninvariant!

- The remedy simple: add to the r.h.s. of (44) the term which *compensates* this noninvariance, i.e. introduce

$$G^{\mu\nu}_a \equiv \frac{\partial A^\nu_a(x)}{\partial x_\mu} - \frac{\partial A^\mu_a(x)}{\partial x_\nu} + g f_{abc} A^\mu_b A^\nu_c \quad (48)$$

and use $G^a_{\mu\nu}$ instead of $F^a_{\mu\nu}$ in (45).

QCD Lagrangian

- Writing $G^a_{\mu\nu}G^{a\mu\nu}$ in terms of A^a_μ besides the expected kinetic term $\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}$, there are also two additional terms, which represent the *selfinteraction of three and four gluons*

$$gf_{abc}\left[\left(\frac{\partial A^a_\nu}{\partial x^\mu}-\frac{\partial A^a_\mu}{\partial x^\nu}\right)A^\mu_b A^\nu_c+\left(\frac{\partial A^{a\mu}}{\partial x_\nu}-\frac{\partial A^{a\nu}}{\partial x_\mu}\right)A_{b\mu}A_{c\nu}\right]+g^2 f_{abc}f_{ade}A^\mu_b A^\nu_c A_{d\mu}A_{e\nu}. \tag{49}$$

- This feature has profound consequences, to be discussed later.
- Compared to QED, quantization of full QCD Lagrangian brings another complication – appearance of so called **Faddeev–Popov ghosts** which *must* be introduced in certain gauges to guarantee the unitarity of the theory. From the point of view of Feynman diagram calculations, ghosts behave as *scalar* particles, coupling to gauge bosons, but appearing only in *propagators*.
- Ghosts are absent in axial gauges, but *do appear* in covariant gauges ⇒ axial gauges are particularly suitable for the construction and probabilistic interpretation of the ladder diagrams, used in the derivation of evolution equations for parton distribution functions, discussed later.

QCD Lagrangian: Conserved current

- Recall general method – Nöether's theorem. Suppose we have a set of fields $\phi_A(x)$ and corresponding Lagrangian $\mathcal{L}(\phi_A, \partial_\mu \phi_A)$.
- Consider infinitesinal gauge transformation:

$$\phi_A(x) \to \phi_A(x) + i\alpha_a(x)\,(T_a)_A^B\,\phi_B \tag{50}$$

where T_a are matrices corresponding to the non-Abelian gauge group and the representation to which the fields $\phi_A(x)$ belong. Variation of L gives:

$$\delta\mathcal{L} = \sum_\phi \frac{\partial \mathcal{L}}{\partial \phi_A}\delta\phi_A + \sum_\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)}\delta(\partial_\mu \phi_A) \tag{51}$$

- Using the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi_A} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)}\right) = 0 \tag{52}$$

and the fact that $\delta(\partial_\mu \phi_A) = \partial_\mu(\delta\phi_A)$ we have:

$$\delta\mathcal{L} = \sum_\phi \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)}\right)\delta\phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)}\delta(\partial_\mu \phi_A)\right] = \sum_\phi \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)}\right)\delta\phi_A\right] \tag{53}$$

QCD Lagrangian: Conserved current

- Using (50) so that $\delta\phi_A = i\alpha_a\,(T_a)_A^B\,\phi_B$ we get:

$$\delta\mathcal{L} = \sum_{\phi} \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)} i\alpha_a\,(T_a)_A^B\,\phi_B \right] \tag{54}$$

- If we take α independent of x then we can rewrite it as:

$$\delta\mathcal{L} = \partial_\mu \sum_{\phi} i \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\,(T_a)_A^B\,\phi_B \right] \alpha_a \equiv \partial_\mu J_a^\mu \alpha_a, \tag{55}$$

where

$$J_a^\mu = i \sum_{\phi} \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)} \right) (T_a)_A^B\,\phi_B \tag{56}$$

- Hence we have the Nöether's theorem. If the Lagrangian is invariant under the gauge transformation (50) with constant α, i.e. $\delta\mathcal{L} = 0$, then the current given in (56) is conserved: $\partial_\mu \vec{J}^\mu = 0$. There is one conserved current for every generator $\partial_\mu J_a^\mu = 0, a = 1, \ldots 8$.

QCD Lagrangian: Conserved current

- Let us apply this to the QCD Lagrangian (45). Here ϕ correspond to $\vec{A^{\mu}}$ and Ψ.
- For the gauge vector bosons which belong to the adjoint representation of $SU_c(3)$, we have $i(T_a)^c_b = -f_{bac}$ and for the quarks which belong to the triplet representation of $SU_c(3)$, $T_a = \frac{1}{2}\lambda_a$
- Using (48) in (45) gives:

$$J^{\mu}_a = \tfrac{1}{2}\overline{\Psi}\gamma^{\mu}\lambda_a\Psi + f_{abc}G^{\mu\nu}_b A^{\nu}_c \tag{57}$$

- In full analogy with (20) this current $\vec{J^{\mu}}$ is universally coupled to the gauge fields $\vec{A^{\mu}}$ with universal coupling g:

$$\mathcal{L}_{int} = \vec{J^{\mu}}\vec{A_{\mu}} = \overline{\Psi}\gamma^{\mu}\frac{\lambda_a}{2}\Psi A_{a\mu} + f_{abc}G^{\mu\nu}_b A^{\nu}_c A_{a\mu} \tag{58}$$

  where the second term is nothing but the self-gluon interaction part of the Lagragian given by (49)

Feynman diagram rules for QCD in the class of covariant gauges are listed. Ghost vertices and propagators appear in loops only. The letters i, j, k, l denote quark (triplet) color, a, b, c, d gluon (octet) color and Greek letters space–time coordinates. Small letters p, q, r denote momenta of corresponding lines.

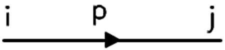
Rules for external particles:

- $u(p, s)$ for each incoming fermion with momentum p and spin s,
- $\overline{v}(p, s)$ for each incoming antifermion with momentum and spin s,
- $\overline{u}(p, s)$ for each outgoing fermion with momentum p and spin s,
- $v(p, s)$ for each outgoing antifermion with momentum p and spin s.

Statistical factors in loops:

- -1 for each closed fermion loop.
- $1/2!$ for each gluon loop with two internal gluons.
- $1/3!$ for each gluon loop with three internal gluons.

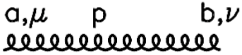
Feynman diagram rules in QCD: Propagators



$$\delta^{ij} \frac{i}{(\not{p} - m + i\varepsilon)}$$

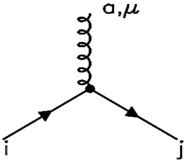
$$\delta^{ab} \frac{i}{(p^2 + i\varepsilon)}$$

ghost propagator in loops only

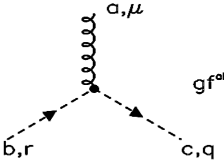


$$\delta^{ab} \frac{i}{(p^2 + i\varepsilon)} \left[-g_{\mu\nu} + (1 - \alpha_G) \frac{p_\mu p_\nu}{(p^2 + i\varepsilon)} \right]$$

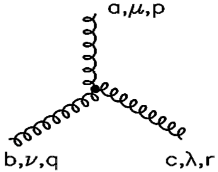
Feynman diagram rules in QCD: Vertices



$-ig(T^a)_{ji}\gamma_\mu$



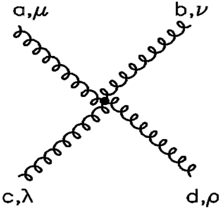
$gf^{abc}r^\mu$



all momenta outgoing

$-gf^{abc}C^{\mu\nu\lambda}(p,q,r)$

$C^{\mu\nu\lambda}(p,q,r)=g^{\mu\nu}(p-q)^\lambda+g^{\nu\lambda}(q-r)^\mu+g^{\lambda\mu}(r-p)^\nu$



$-ig^2f^{eac}f^{ebd}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\rho}g^{\nu\lambda})$

$-ig^2f^{ead}f^{ebc}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\lambda}g^{\nu\rho})$

$-ig^2f^{eab}f^{ecd}(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\rho}g^{\nu\lambda})$

Elementary calculations: Quark-quark scattering

- Consider the elastic scattering of two quarks with momenta, flavors and colors as indicated in Fig. 3 (with Greek and Latin letters labeling flavors and colors respectively), averaged over the spins and colors of quarks.

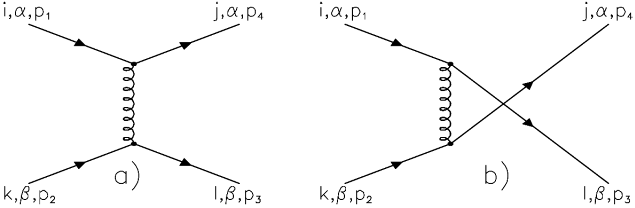


Figure 3: Lowest order Feynman diagrams for quark-quark elastic scattering.

- Working in the Feynman gauge we find for the spin (λ) and color (c) averaged square of the corresponding matrix element

$$\langle |M_{if}|^2 \rangle = \frac{1}{9}\frac{1}{4}\sum_{\lambda,c}\left(|M_t|^2 + |M_u|^2 + 2M_t^* M_u\right), \tag{59}$$

where the sum runs over spins λ and colors c of all the four quarks and the Mandelstam variables s, t, u are defined as

$$s \equiv (p_1+p_2)^2, \quad t \equiv (p_1-p_4)^2, \quad u \equiv (p_1-p_3)^2. \tag{60}$$

Elementary calculations: Quark-quark scattering

- Amplitudes M_t and M_u, corresponding to Fig. 3a and Fig. 3b, are:

$$M_t = \frac{\mathrm{i}g^2}{t}\left(T^a_{ji}T^a_{lk}\right)\left[\overline{u}_\alpha(p_4)\gamma^\mu u_\alpha(p_1)\right]\left[\overline{u}_\beta(p_3)\gamma_\mu u_\beta(p_2)\right], \tag{61}$$

$$M_u = \frac{\mathrm{i}g^2}{u}\left(T^a_{li}T^a_{jk}\right)\left[\overline{u}_\beta(p_3)\gamma^\mu u_\alpha(p_1)\right]\left[\overline{u}_\alpha(p_4)\gamma_\mu u_\beta(p_2)\right]. \tag{62}$$

- Considering first the square of M_t we find (*no* sum over t below)

$$\frac{1}{9}\frac{1}{4}\frac{g^4}{t^2}A_t B_t, \tag{63}$$

where the fractions $1/9$ and $1/4$ come from the color and spin averaging in the initial state. Final form of A_t, containing all the color factors, reads

$$A_t \equiv T^a_{ji}T^a_{lk}\underbrace{T^{b*}_{ji}}_{T^b_{ij}}\underbrace{T^{b*}_{lk}}_{T^b_{kl}} = \underbrace{T^a_{ji}T^b_{ij}}_{Tr(T_aT_b)}\underbrace{T^a_{lk}T^b_{kl}}_{Tr(T_aT_b)} = \frac{1}{4}(\delta_{ab})^2 = 2, \tag{64}$$

while working out bispinor traces is standard:

$$B_t \equiv \mathrm{Tr}\left(\not{p}_4\gamma_\mu\ \not{p}_1\gamma_\nu\right)\mathrm{Tr}\left(\not{p}_3\gamma^\mu\ \not{p}_2\gamma^\nu\right) = 32\left((p_1p_2)(p_3p_4)+(p_1p_3)(p_2p_4)\right) = 8(u^2+s^2). \tag{65}$$

Elementary calculations: Quark-quark scattering

- Putting all the relevant factors together yields

$$\langle |M_t|^2 \rangle = \frac{2}{9} g^4 \frac{2(u^2 + s^2)}{t^2}, \tag{66}$$

where the factor 2/9 comes from color factors.

- Proceeding similarly for the other two terms in (59) we get

$$\langle |M|^2 \rangle = \left(\frac{2}{9}\right)_c g^4 \left[\frac{2(u^2 + s^2)}{t^2} + \delta_{\alpha\beta} \frac{2(t^2 + s^2)}{u^2} - \delta_{\alpha\beta} \left(-\frac{1}{3}\right)_c \frac{4s^2}{tu}\right], \tag{67}$$

where the subscript "c" denotes factors coming from color traces.

- Effect of color is twofold: it changes the overall *magnitude* of the quark-quark interaction, but also *suppresses* the relative contribution of the interference term.

Elementary calculations: Quark-gluon scattering

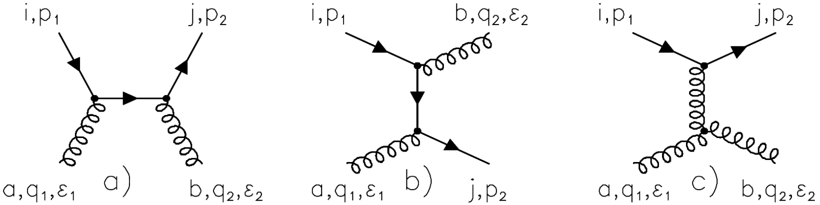


Figure 4: Lowest order Feynman diagrams for quark-gluon elastic scattering.

- This is the simplest process where the gluon selfinteraction vertex does appear. The appropriate Feynman diagrams are shown in Fig. 4 and the corresponding invariant amplitudes, given as

$$M_s = \frac{-ig^2}{s}\left(T^a_{li}T^b_{jl}\right)\left[\overline{u}^j(p_2)\,\not\epsilon_2(\not p_1+\not q_1)\,\not\epsilon_1 u^i(p_1)\right], \tag{68}$$

$$M_u = \frac{-ig^2}{u}\left(T^a_{jk}T^b_{ki}\right)\left[\overline{u}^j(p_2)\,\not\epsilon_1(\not p_1-\not q_2)\,\not\epsilon_2 u^i(p_1)\right], \tag{69}$$

where the polarization vectors ϵ_1, ϵ_2 describe the initial and final state gluons and all the colors and momenta upon which the amplitudes M_s, M_u depend were suppressed.

Quark-gluon scattering

- Compared to QED new in this channel is the contribution of the diagram in Fig. 4c, which contains the 3-gluon vertex

$$M_t = \frac{g^2}{t}\left[f_{abc}\left(T^c\right)_{ij}\right] C^{\lambda\mu\nu}(q_1 - q_2, -q_1, q_2)\epsilon_{1\mu}\epsilon_{2\nu}\left[\overline{u}^j(p_2)\gamma_\lambda u^i(p_1)\right]. \quad (70)$$

- Inserting into (70) the explicit form of $C^{\lambda\mu\nu}$ i.e. $C^{\lambda\mu\nu} = g^{\mu\nu}(p-q)^\lambda + g^{\nu\lambda}(q-r)^\nu + g^{\lambda\mu}(r-p)^\nu$, we get

$$M_t = \frac{g^2}{t}\left[f_{abc}\left(T^c\right)_{ji}\right]\times \quad (71)$$

$$\times\left\{\left[\overline{u}(p_2)\not\epsilon_1 u(p_1)\right](2q_1 - q_2)\epsilon_2 - \left[\overline{u}(p_2)(\not q_1 + \not q_2)u(p_1)\right](\epsilon_1\epsilon_2) + \left[\overline{u}(p_2)\not\epsilon_2 u(p_1)\right](2q_2 - q_1)\epsilon_1\right\}.$$

- Evaluation of traces of γ_μ is the same as in QED $\Rightarrow$ omitted. New with respect to QED are the factors due to color matrices. As an example consider the color trace in the spin and color average of $|M_s|^2$

$$\frac{1}{3}\frac{1}{8}T^a_{li}T^b_{jl}T^{b*}_{jk}T^{a*}_{ki} = \frac{1}{24}\mathrm{Tr}(T^a\underbrace{T^bT^b}_{(4/3)}T^a) = \frac{2}{9} \quad (72)$$

and similarly for M_u.

Quark-gluon scattering

- Factors $\frac{1}{3}$ and $\frac{1}{8}$ result from averaging over three colors of initial quark and eight of initial gluon. Analogous color factor in the case of M_t involves another kind of color traces:

$$\frac{1}{3}\frac{1}{8} f_{abc}\left(T^c\right)_{ji} f_{abd}\left(T^d\right)^*_{ji} = \frac{1}{24}\overbrace{f_{abc}f_{abd}}^{3\delta_{cd}}\underbrace{\mathrm{Tr}\left[T^cT^d\right]}_{(1/2)\delta_{cd}} = \frac{1}{2}. \qquad (73)$$

- Full result for spin and color averaged $|M_t|^2$ is listed in the Table 2 of subsection 4. Two aspects of this calculation merit a comment:
- Summing over polarizations of initial and final gluons we needed expression:

$$\sum_\lambda \epsilon_\mu(\lambda, q)\epsilon^*_\nu(\lambda, q), \qquad (74)$$

where the sum runs over all polarizations of the gluon with momentum q. This sum *depends* on the gauge we work in and so do the results of the individual contributions of squares and interference terms of M_s, M_u, M_t. The full results is, however, *independent* of this choice. Within the Feynman gauge, i.e. summing over all four gluon polarizations, the result for (74) is just $-g^{\mu\nu}$.

Quark-gluon scattering

- Recall that when summing over the *physical, transverse* polarizations of the gluon, we find:
$$\sum_\lambda \epsilon_\mu(\lambda, q)\epsilon^*_\nu(\lambda, q) = -g_{\mu\nu} + \frac{q_\mu p_\nu + q_\nu p_\mu}{qp}, \qquad (75)$$
where the fourvector p is defined as $p \equiv (q_0, -\vec{q})$.
- The division of the full invariant amplitude M_{if} into three M_s, M_t, M_u, corresponding to three diagrams in Fig. 4, *is not* gauge invariant and therefore has *no physical* meaning!
- Consider sum $M_s + M_u$ of the contributions corresponding to the diagrams in Fig. 4a and 4b, i.e. those which don't contain the 3–gluon vertex and which have their close analogies in QED. The decoupling of unphysical, longitudinal gluons implies that the full result for the $q + g$ scattering amplitude must vanish if either the incoming or outgoing gluon is longitudinal.

Quark-gluon scattering

- Assuming the second possibility, setting $\epsilon_2 = \lambda q_2$ and denoting the appropriate amplitudes $M_i(\lambda)$, $i = s, t, u$ we find that the sum

$$M_s(\lambda) + M_u(\lambda) = ig^2\lambda \underbrace{\left[T^a, T^b\right]_{ji}}_{\neq 0} [\overline{u}(p_2)\ \not{q}_1 u(p_1)] = -g^2\lambda f_{abc}\,(T_c)_{ji}\,[\overline{u}(p_2)\ \not{q}_1 u(p_1)] \tag{76}$$

does not vanish, due to nonabelian character of QCD!

- It is easy to check that for the same longitudinal polarization $\epsilon_2 = \lambda q_2$ the t–channel amplitude

$$M_t(\lambda) = -\lambda g^2 f_{abc} T^c_{ji} \left[\overline{u}^j(p_2)\ \not{q}_1 u(p_1)\right] \tag{77}$$

compensates (76) provided

$$\left[T^a, T^b\right] = if_{abc} T^c, \tag{78}$$

which, indeed, the matrices T^a satisfy!

Gluon-gluon scattering

- In QED the scattering of light on light is possible only in higher orders of perturbation theory via the box diagram in Fig. 5d and thus only thanks to the coupling of photons to charged particles.

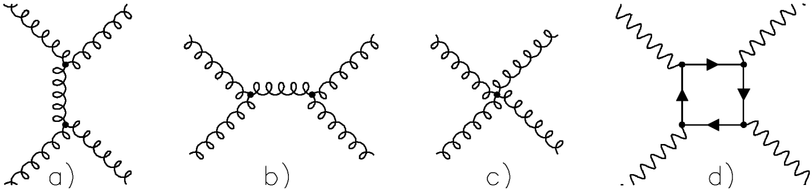


Figure 5: Lowest order Feynman diagrams for gluon–gluon elastic scattering in QCD (a-c), and for $\gamma - \gamma$ scattering in QED, (d). Neither momenta nor direction of the particles are shown. In the box diagram d), any charged particle can circulate around the loop.

- In QCD *gluons can interact* **directly**, even in absence of quarks, via the three and four gluon couplings (which means already at order g^2 in amplitude compared to order e^4 of the box diagram in Fig. 5d). Thus contrary to QED, there is a nontrivial theory called *gluodynamics*, which describes *only* the gluons and their selfinteractions.

Comparison of different parton subprocesses

Process	$\dfrac{\langle \mid M \mid^2 \rangle}{g^4}$
$q_\alpha q_\beta \to q_\alpha q_\beta$	$\frac{2}{9}\left[\frac{2(s^2+u^2)}{t^2} + \left(\frac{2(t^2+s^2)}{u^2} - \frac{1}{3}\frac{4s^2}{ut}\right)\delta_{\alpha\beta}\right]$
$q_\alpha \overline{q}_\beta \to q_\alpha \overline{q}_\beta$	$\frac{2}{9}\left[\frac{2(s^2+u^2)}{t^2} + \left(\frac{2(t^2+u^2)}{s^2} - \frac{1}{3}\frac{4u^2}{st}\right)\delta_{\alpha\beta}\right]$
$qg \to qg$	$\left[\left(1 - \frac{us}{t^2}\right) - \frac{4}{9}\left(\frac{s}{u} + \frac{u}{s}\right) - 1\right]$
$gg \to q\overline{q}$	$\frac{1}{6}\left[\frac{u}{t} + \frac{t}{u}\right] - \frac{3}{4}\left[1 - \frac{ut}{s^2}\right] + \frac{3}{8}$
$q\overline{q} \to gg$	$\frac{64}{9} M(gg \to q\overline{q})$
$gg \to gg$	$\frac{8}{9}\left[-\frac{33}{4} - 4\left(\frac{us}{t^2} + \frac{ut}{s^2} + \frac{st}{u^2}\right)\right] - \frac{9}{16}\left[45 - \left(\frac{s^2}{ut} + \frac{t^2}{us} + \frac{u^2}{ts}\right)\right]$

Results are for the spin and color averaged invariant amplitudes normalized as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi s^2}\langle |M|^2 \rangle. \qquad (79)$$

Comparison of different parton subprocesses

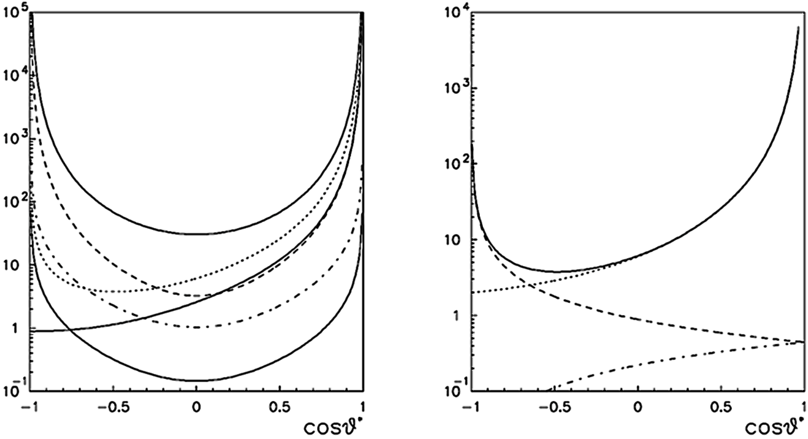


Figure 6: a): $\langle | M |^2 \rangle / g^4$ plotted for different 2 body processes as given in the Table below. The curves correspond, in decreasing order of their values for $\cos\vartheta = 0.5$, to $gg \to gg$, $qg \to qg$,$q\bar{q} \to q\bar{q}$, $qq \to qq$, $q\bar{q} \to gg$,$gg \to q\bar{q}$;b): Individual contributions of M_s (dash-dotted line),M_u (dashed line) and M_t (dotted line) to the full $qg \to qg$ cross-section (solid line)

Comparison of different parton subprocesses

- N.B. In the CMS of incoming particles (which are assumed massless) the Mandelstam variables s, t, u are related to the scattering angle ϑ^* as follows:
$$t = -2E^{*2}(1 - \cos\vartheta^*) \Rightarrow \mathrm{d}t = 2E^{*2}\mathrm{d}\cos\vartheta^*. \qquad (80)$$
- Numerical comparison of these cross-sections for fixed s as functions of $\cos\vartheta^*$ shown in Fig. 6 gives:
  - In all processes with t-channel gluon exchange contribution of corresponding Feynman diagram dominates in the forward direction.
  - $g + g$ channel gives by far the biggest cross-section in the whole angular range. This, however, *doesn't* imply that it gives also the biggest contribution to the cross–sections of hard collisions of hadrons!
  - Processes with t-channel gluon exchange are very steep at small angles (i.e. for $\cos\vartheta^* \to 1$) where the diagram corresponding to this exchange dominates the full cross-section. An example is given in Fig. 6b, where the three contributions to the quark-gluon cross-section are displayed separately, together with their sum. Note the marked difference between the behavior at small and large angles ϑ^*, where the exchange of quarks in the u-channel also leads to divergence, which, however, is weaker than that at small angles.

1. Show that longitudinal photon carries zero energy and is thus equivalent to nothing.
2. Construct interaction term between the Dirac fermion field Ψ and electromagnetic field which doesn't use the potential $A_\mu(x)$ but couples it directly the gauge invariant, observable tensor $F^{\mu\nu}$. Discuss physical interpretation of such an interaction.
3. Carry out explicitly the Fourier transformation of the equation (26).
4. Show that the expressions (33) and (37) are solutions of (32) and (36) respectively.
5. Show that the term $G^a_{\mu\nu}G^{\mu\nu}_a$ with $G^a_{\mu\nu}$ given in (48) is locally gauge invariant.
6. Use (49) to derive in a heuristic way the Feynman rules for 3 and 4 gluon vertex.
7. Carry out in detail the evaluation of the quark-gluon elastic cross-section.
8. Show in detail (76) and (77).