Introduction to Quantum Chromodynamics

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Chapter 1:
Elements of group theory and its applications to physics

Literature

Our discussion is based on

Quarks, partons and Quantum Chromodynamics

by Jiří Chýla

Available at http://www-hep.fzu.cz/ chyla/lectures/text.pdf

and

LIE ALGEBRAS in PARTICLE PHYSICS

by Howard Georgi

Adison-Wesley, 1982

Group and its representation

Group: a set $\mathbf{G}$ with binary operation "$\bullet$" satisfying:

1. $\forall x, y \in \mathbf{G} : x \bullet y \in \mathbf{G}$
2. $\exists e \in \mathbf{G} : \forall x \in \mathbf{G}, e \bullet x = x \bullet e = x$ ($\exists$ unit element e)
3. $\forall x \in \mathbf{G}\ \exists x^{-1} \in \mathbf{G} :\ x \bullet x^{-1} = x^{-1} \bullet x = e$ ($\exists$ inverse element)
4. $\forall x, y, z \in \mathbf{G} : x \bullet (y \bullet z) = (x \bullet y) \bullet z$ (associativity of $\bullet$)

Groups enter physics basically because they correspond to various symmetries. The concept crucial for the description of transformations of physical quantities under these symmetry transformations is that of the

Group representation: the mapping $\mathbf{D}$: $\mathbf{G} \mapsto \mathcal{L}_{\mathcal{H}}$ of the group $\mathbf{G}$ onto the space $\mathcal{L}_{\mathcal{H}}$ of linear operators on Hilbert space $\mathcal{H}$, which preserves the property of group multiplication "$\bullet$":

$$\forall x, y, z \in \mathbf{G} : x \bullet y = z \mapsto D(x) \cdot D(y) = D(z) \qquad (1)$$

Multiplication $\cdot$ is defined in the space $\mathcal{L}_{\mathcal{H}}$.
N.B. The representations will in general be denoted by boldface capital $\mathbf{D}$ with possible subscripts or superscripts, or as is common for SU(3) group, by a pair of nonnegative integers (i,j), specifying the so called *highest weight* of the representation (see Section 2.6). The image of a group element $g \in \mathbf{G}$ in such a representation will be denoted simply as $D(g)$.

Group and its representation: Examples

Examples:

- **G**=R, the set of real numbers with conventional addition as the group binary operation $\bullet$, $D(x) = \exp(i\alpha x)$ where α is a fixed real number. This representation plays a crucial role in the construction of abelian gauge theories, like QED.
- P_3, permutation group of three objects. Define (1,2) as permutation of 1,2 etc., (1,2,3) as simultaneous transpositions: 1→2, 2→3, 3→1 and e as a unit element. One of its representations is given by the following six matrices, with normal matrix multiplication defining the product of linear operators:

$$D(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(321) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Realization of operators by matrices:

any linear operator $\hat{O}$ acting on vectors $\mid j\rangle$ from a given normalized basis of $\mathcal{H}$ can be represented by means of the matrix O_{ij}:

$$O_{ji} \equiv \langle j \mid \hat{O} \mid i\rangle \Rightarrow \hat{O} \mid i\rangle = O_{ji} \mid j\rangle, \tag{2}$$

(the summation over the repeating indices is understood as usual).
N.B. In all the following applications, when talking about the group representation, we will always have in mind the above matrix representation.

Equivalence of representations:

the matrices O_{ij} depend on the chosen basis of $\mathcal{H}$.
$\Rightarrow$ representations $\mathbf{D_1}$ and $\mathbf{D_2}$ are equivalent if $\exists$ unitary operator $S \in \mathcal{L}_{\mathcal{H}}$ such that

$$\forall x \in \mathbf{G},\ D_1(x) = SD_2(x)S^{-1} \qquad = SD_2(x)S^{-1}SS^{+} = SD_2(x)S^{+}. \tag{3}$$

The transformation of $D(x)$ can be interpreted as resulting from the change of the basis of $\mathcal{H}$ induced by S.

Direct sum and product of representations

Direct sum of representations: $\mathbf{D} = \mathbf{D_1} \oplus \mathbf{D_2}$

- $\mathbf{D}_i$ act on corresponding Hilbert space $\mathcal{H}_i$ of dimension n_i, $i = 1, 2$,
- $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $dim(\mathcal{H}) = n_1 + n_2$ with basis

$$\underbrace{|\, e_1^1\rangle, |\, e_2^1\rangle, \cdots, |\, e_{n_1}^1\rangle}_{\text{basis of } \mathcal{H}_1}, \underbrace{|\, e_1^2\rangle, |\, e_2^2\rangle, \cdots, |\, e_{n_2}^2\rangle}_{\text{basis of } \mathcal{H}_2}. \quad (4)$$

- $\mathbf{D} = \mathbf{D_1} \oplus \mathbf{D_2}$ is made up from $(n_1 + n_2) \times (n_1 + n_2)$ block diagonal matrices

$$\begin{pmatrix} \mathrm{D}_1 & 0 \\ 0 & \mathrm{D}_2 \end{pmatrix}. \quad (5)$$

Direct product of representations: $\mathbf{D} = \mathbf{D_1} \otimes \mathbf{D_2}$

- the basis of the direct product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is formed by $n_1 \times n_2$ pairs of vectors of the form $|\, e_i^1\rangle \,|\, e_j^2\rangle$ where $|\, e_i^j\rangle \in \mathcal{H}_j, j = 1, 2$.
- $\mathbf{D} = \mathbf{D_1} \otimes \mathbf{D_2}$ is formed by the matrices

$$\underbrace{D_1 \otimes D_2(g)}_{\in \mathbf{D_1} \otimes \mathbf{D_2}} |\, e_i^1\rangle \,|\, e_j^2\rangle = (\underbrace{D_1(g)}_{\in \mathbf{D_1}} |\, e_i^1\rangle)(\underbrace{D_2(g)}_{\in \mathbf{D_2}} |\, e_j^2\rangle). \quad (6)$$

- The $(n_1 \times n_2) \times (n_1 \times n_2)$ dimensional matrices can be written as

$$[(D_1 \otimes D_2)(g)]_{ij,i'j'} = [D_1(g)]_{ii'} [D_2(g)]_{jj'}. \quad (7)$$

Reducibility of a representation

- Possibility to transform all elements of a given representation **D**, acting on the Hilbert space $\mathcal{H}$, by means of a unitary operator S to block–diagonal form:

$$\exists S \in \mathcal{L}_{\mathcal{H}} : \forall x \in \mathbf{G}, SD(x)S^{-1} = \begin{pmatrix} D_1(x) & 0 \\ 0 & D_2(x) \end{pmatrix}, \tag{8}$$

  where the matrices $D_1(x), D_2(x)$ act on Hilbert spaces $\mathcal{H}_1$ and H_2 respectively and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

- If such a unitary operator S does exist the representation **D** is called **reducible** and can be written as a direct sum $\mathbf{D} = \mathbf{D_1} \oplus \mathbf{D_2}$. If not **D** is said to be **irreducible**.

Reducibility example from QM2: Orbital momentum

- $[J_i, J_j] = i\epsilon_{ijk} J_k, \qquad \left[J^2, J_i\right] = 0, \qquad J^2 \equiv J_i J_i$
- $J^2 \mid \lambda m\rangle = \lambda \mid \lambda m\rangle, \qquad J_3 \mid \lambda m\rangle = m \mid \lambda m\rangle$
- $J_\pm \equiv J_1 \pm iJ_2, \qquad J_1 = \frac{1}{2}(J_+ + J_-), \qquad J_2 = \frac{1}{2i}(J_+ - J_-)$

- $J_\pm \mid jm\rangle = C_\pm(j,m) \mid jm \pm 1\rangle$
- $C_+(j,m) = \sqrt{(j-m)(j+m+1)} = \sqrt{j(j+1) - m(m+1)}$
- $C_-(j,m) = \sqrt{(j+m)(j-m+1)} = \sqrt{j(j+1) - m(m-1)}$

- $\langle j'm' \mid J^2 \mid jm\rangle = \delta_{j'j}\delta_{m'm} j(j+1), \qquad \langle j'm' \mid J_3 \mid jm\rangle = \delta_{j'j}\delta_{m'm} m$
- $\langle j'm' \mid J_1 \mid jm\rangle = \frac{\delta_{j'j}}{2}\{\delta_{m',m+1} C_+(j,m) + \delta_{m',m-1} C_-(j,m)\}$
- $\langle j'm' \mid J_2 \mid jm\rangle = \frac{-i\delta_{j'j}}{2}\{\delta_{m',m+1} C_+(j,m) - \delta_{m',m-1} C_-(j,m)\}$
- N.B. J_i and J^2 are **block diagonal** but only J_3 and J^2 are **really diagonal**.

- So explicitly for $j = 0, \frac{1}{2}, 1, \ldots$:

Reducibility example: J_3

$\vec{J}^2 \rightarrow$

j'm'\jm	(0,0)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	(0,0)	(1,0)	(1,-1)
(0,0)	0					
$(\frac{1}{2},\frac{1}{2})$		$\frac{3}{4}$				
$(\frac{1}{2},-\frac{1}{2})$			$\frac{3}{4}$			
(0,0)				2		
(1,0)					2	
(1,-1)						2

$$J_3 \rightarrow \left[\begin{array}{c|cc|ccc|c} 0 & & & & & & \\ \hline & \frac{1}{2} & & & & & \\ & & -\frac{1}{2} & & & & \emptyset \\ \hline & & & 1 & & & \\ & & & & 0 & & \\ & & & & & -1 & \\ \hline & \emptyset & & & & & \ddots \end{array}\right]$$

Reducibility example: J_1

$\vec{J}^2 \rightarrow$

j'm'\jm	(0,0)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	(0,0)	(1,0)	(1,-1)
(0,0)	0					
$(\frac{1}{2},\frac{1}{2})$		$\frac{3}{4}$				
$(\frac{1}{2},-\frac{1}{2})$			$\frac{3}{4}$			
(0,0)				2		
(1,0)					2	
(1,-1)						2

$$J_1 \rightarrow \begin{bmatrix} 0 & & & & & & \\ & 0 & \frac{1}{2} & & & & \\ & \frac{1}{2} & 0 & & & & \emptyset \\ & & & 0 & \frac{1}{\sqrt{2}} & 0 & \\ & & & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \\ & & & 0 & \frac{1}{\sqrt{2}} & 0 & \\ & \emptyset & & & & & \ddots \end{bmatrix}$$

Reducibility example: J_2

$\vec{J}^2 \rightarrow$

j'm'\jm	(0,0)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	(0,0)	(1,0)	(1,-1)
(0,0)	0					
$(\frac{1}{2},\frac{1}{2})$		$\frac{3}{4}$				
$(\frac{1}{2},-\frac{1}{2})$			$\frac{3}{4}$			
(0,0)				2		
(1,0)					2	
(1,-1)						2

$$J_2 \rightarrow \begin{bmatrix} 0 & & & & & & & \\ & 0 & \frac{-i}{2} & & & & \emptyset & \\ & \frac{i}{2} & 0 & & & & & \\ & & & 0 & \frac{-i}{\sqrt{2}} & 0 & & \\ & & & \frac{-i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} & & \\ & & & 0 & \frac{-i}{\sqrt{2}} & 0 & & \\ & & \emptyset & & & & \ddots & \\ & & & & & & & \ddots \end{bmatrix}$$

Groups: Further important concepts

- **Abelian group:** the group multiplication • is **commutative**.
- **Finite and infinite groups:** according to the number of independent elements.
- **Compact and noncompact groups:** Compact $\equiv$ closed and bounded. Compact groups have finite "volume" as measured by some measure on the group space, while the noncompact have infinite volume. Example: the group of phase factors introduced above is compact, as the unit circle in complex plane is the finite.

Lie groups and algebras

Lie groups:

- Group elements are labeled by a set of continuous parameters – coordinates in a subset of $\mathbf{R}^n$.
- "labeled by a set of continuous parameters" $\approx \exists$ a *local* one-to-one mapping between the group elements and points in some subset of $\mathbf{R}^n$ which is **continuous** in both ways and which allows us to translate all the operations and questions from the group space to analogous operations and questions in $\mathbf{R}^n$ where we know what to do.

Continuous group = Lie group $\Leftrightarrow$ the group multiplication $x \bullet y$ must be a continuous function of both x and y and the operation of taking the inverse x^{-1} must be continuous function of x.

In these lectures the group elements will always be matrices, parameterized by a few real numbers and for the vector space of matrices the topology is easily defined via the norm of a matrix.

Lie groups and algebras: propositions

For Lie groups, and in particular their matrix representations there is also intuitive understanding of the group "volume" and thus of the difference between the compact and noncompact groups. I shall now present, without proofs, several propositions that will be useful in further considerations.

Proposition

Any element of a **compact** *Lie group can be written in the form*

$$\forall g \in \mathbf{G}, g = g(\alpha_a) = \exp(i\alpha_a X_a),$$

where the operators X_a, called **generators** *of the Lie group* **G***, form the basis of a vector space* **X** *(over the field of complex mumbers) of dimension m with the operation of "adding", denoted as "+".*

The classification and properties of representations are simplest for the class of compact Lie groups, where all those needed in formulating the Standard Model do belong.

Lie groups and algebras: propositions

Proposition

All irreducible representations of compact Lie groups are finite dimensional.

Proposition

Finite dimensional representations of a compact Lie group **G** *are equivalent to representations by* unitary *operators, i.e. the generators X_a are* hermitian *operators.*

In the case of matrix groups, or representations, both the elements g of the group **G** and the generators X_a are again *matrices*. As a result, also the generators of any irreducible representation of a compact Lie group can be represented by finite dimensional hermitian matrices. Note that the generators of a given Lie group are not determined by this group *uniquely*, as any change of the basis vectors of the associated Hilbert space implies the change of these generators.

Algebra

Algebra: we shall restrict our discussion to algebras over the field $\mathcal{C}$ of complex numbers.

The $\mathcal{C}$-algebra is a vector space $\mathcal{A}$ over the field of complex numbers equipped with a bilinear binary operation (here denoted as simple multiplication) $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which means that $\forall x, y, z \in \mathcal{A}, a, b \in \mathcal{C}$

- (x+y)z=xz+yz
- x(y+z)=xy+xz
- (ax)(by)=(ab)(xy)

In the special case of **Lie algebra**, the binary operation, called "commutator", is *antisymmetric* and satisfies the Jacobi identity (12).

Algebra: Further important concepts

- **Abelian algebra:** Algebra with the property that all its elements *commute* with each other.
- **Invariant subalgebra $\mathcal{S}$ of algebra $\mathcal{A}$:** $\forall a \in \mathcal{S}, \forall x \in \mathcal{A} : [a, x] \in \mathcal{S}$.
- **Simple algebra:** algebra which contains no nontrivial invariant subalgebra.
- **Semi-simple algebra:** algebra which contains no *abelian* invariant subalgebra. This type of groups has great physical relevance in theories unifying various kinds of interactions, like the electroweak theory within the SM.
- **Rank of the algebra:** the maximal number of mutually commuting generators. Crucial characteristics of nonabelian algebras. Rank determines the number of independent quantum numbers, which uniquely characterize each state within a given irreducible representation, or, as is common to say, multiplet.

Structure constants

Structure constants: Due to the fact that the product of elements of a Lie group **G** like

$$\exp(i\lambda X_a)\exp(i\lambda X_b)\exp(-i\lambda X_a)\exp(-i\lambda X_b)$$

is also an element of this group, it must be expressible as $\exp(i\beta_c X_c)$. Using the Taylor expansion on both sides we find

$$[X_a, X_b] \equiv X_a X_b - X_b X_a = if_{abc} X_c; \ \ \beta_c = -\lambda^2 f_{abc}, \tag{9}$$

where f_{abc} are real numbers, called the **structure constants** of the Lie group **G**. They are by definition *antisymmetric* in first two indices. Similarly to generators they are, however, *not unique* and do depend on the choice of the basis in **X**. They can be used to express the product of two elements of the group as follows:

$$\exp(i\alpha_a X_a)\exp(i\beta_b X_b) = \exp(i\gamma_c X_c) \Rightarrow \gamma_c = \alpha_c + \beta_c - \frac{1}{2} f_{abc}\alpha_a\beta_b + \cdots \tag{10}$$

Structure constants

Proposition

The structure constants f_{abc} satisfy the following Jacobi identity:

$$f_{ade} f_{bcd} + f_{cde} f_{abd} + f_{bde} f_{cad} = 0. \tag{11}$$

Proof: (11) is a simple consequence of the following relation between the commutators of generators:

$$[X_a, [X_b, X_c]] + [X_c, [X_a, X_b]] + [X_b, [X_c, X_a]] = 0, \tag{12}$$

which is straightforward to verify. In this case the vector space is that of the generators X_a and the "commutator" is defined by the conventional commutator of matrices, representing the generators. Note that the elements of an algebra spanned on the generators of a compact Lie group contain also operators that are *not* hermitian!

Structure constants

Analogously as in the case of groups we can define representation of an algebra as a mapping of its elements to the operators on some Hilbert space, which conserves the commutator (on a Hilbert space of operators commutator of its elements A, B is simply $AB - BA$) and introduce the concepts of reducibility, irreducibility etc.. Generators of the direct product $\mathbf{D_1} \otimes \mathbf{D_2}$ of n_1 dimensional representation $\mathbf{D_1}$ and n_2 dimensional representation $\mathbf{D_2}$ are $(n_1 \times n_2) \times (n_1 \times n_2)$ dimensional matrices of the form

$$\left(X^{\mathbf{D}_1 \otimes \mathbf{D}_2}\right)_{ij,i'j'} = \underbrace{\left(X^{\mathbf{D_1}}\right)_{ii'}}_{\text{acts on } \mathcal{H}_1 \text{ only}} \delta_{jj'} + \delta_{ii'} \underbrace{\left(X^{\mathbf{D_2}}\right)_{jj'}}_{\text{acts on } \mathcal{H}_2 \text{ only}} . \tag{13}$$

Adjoint representation

- The $n \times n$ matrices, where $n = dim(\mathbf{X})$: $(T_a)_{bc} \equiv -if_{abc}$ (14)

  satisfy the same commutation relations as X_a themselves:
  $$[T_a, T_b] = if_{abc} T_c \tag{15}$$
  and therefore also form a representation of **G**.

- This is called **adjoint** representation and will be important in the description of particle multiplets in quark model and of gluons in QCD.

- Although we can choose any basis in **X**, particularly suitable is defined via normalization conditions:
  $$\mathrm{Tr}\left(X_a X_b\right) = \lambda \delta_{ab}, \quad \lambda = const. \tag{16}$$
  For compact Lie groups $\mathrm{Tr}(X_a X_b)$ is real symmetric tensor, it can always be diagonalized $\Rightarrow \lambda > 0$.

- In this normalization we have

$$f_{abc} = -\frac{i}{\lambda}\mathrm{Tr}\left(\left[X_a, X_b\right] X_c\right) = -\frac{i}{\lambda}\mathrm{Tr}\left(\left[X_c, X_a\right] X_b\right) = -\frac{i}{\lambda}\mathrm{Tr}\left(\left[X_b, X_c\right] X_a\right), \tag{17}$$

$\Rightarrow$ in this normalization f_{abc} is *fully antisymmetric* tensor.

Hilbert space of adjoint representation

- The Hilbert space associated to the adjoint representation can be constructed from the *vector* space $\mathbf{X}$ spanned on the generators of $\mathbf{G}$ by defining the binary operation of a "scalar" product of any two vectors $\mid X_a\rangle$, $\mid X_b\rangle$, associated to the generators X_a, X_b, in the following way:

$$\langle X_b \mid X_a\rangle \equiv \frac{1}{\lambda}\mathrm{Tr}\left(X_b^+ X_a\right), \tag{18}$$

  where "+" denotes the hermitian conjugation of the operator and $\langle a \mid, \mid a\rangle$ are the usual Dirac "bra" and "ket" vectors (see QM2).

- For the adjoint representation we have the following chain of equalities

$$T_a \mid T_b\rangle = (T_a)_{cb} \mid T_c\rangle = -if_{acb} \mid T_c\rangle = \mid if_{abc} T_c\rangle = \mid [T_a, T_b]\rangle, \tag{19}$$

  where the first equality is a consequence of the definition of action of the matrix T_a on the ket vector $\mid T_b\rangle$, the second uses just the definition (14), the third is trivial and the last one returns to (15). In other words the action of the operator (for us matrix) T_a on the ket vector $\mid T_b\rangle$ produces the vector, associated to the commutator of the matrices T_a, T_b!

SU(2) group and algebra

The simplest of nonabelian Lie groups, with plenty of applications in particle physics is the SU(2) group. Moreover, most of the techniques useful for the more complicated case of SU(3) and other groups can be generalized from the technically simpler case of SU(2).

Definition

SU(2) is formed by unitary 2×2 matrices with unit determinant. The associated Lie algebra is made out of traceless hermitian matrices 2×2.

There are 3 independent generators J_1, J_2, J_3 of SU(2), which form the basis of SU(2) albegra and which satisfy the well-known commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk} J_k. \tag{20}$$

Remark

In fact this algebra is equivalent to that of SO(3) group, indicating that the relation between the group and the associated algebra is not unique. This has to do with the fact that algebras express only the local properties of the groups, but don't describe the global ones.

Adjoint representations of SU(2)

- Generators of SU(2) can be written by means of Pauli matrices as $J_i = \frac{1}{2}\sigma_i$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (21)$$

(N.B. only σ_3 is diagonal!)

- Adjoint representation is formed by 3×3 matrices $(J_i^A)_{jk} = -i\varepsilon_{ijk}$

$$J_1^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_2^A = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, J_3^A = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (22)$$

- This is the spin 1 representation of the Lie algebra of the rotation group.

$$x_i^{'} = O_{ij}x_j = (e^{-i\boldsymbol{\alpha}\cdot\mathbf{J^A}})_{ij}x_j, \quad i,j = 1,2,3; \quad O^T O = 1, \quad O \epsilon SO(3) \quad (23)$$

Adjoint representations of SU(2)

- Let $x\epsilon$ **R** and define 2×2 matrix

$$X \equiv x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \mathbf{x}\cdot\boldsymbol{\sigma} = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} \tag{24}$$

i.e. $x_i = \frac{1}{2}Tr(\sigma_i X)$

- Since $DetX = -x^2 = -(x_1^2 + x_2^2 + x_3^2)$ the transformation

$$X^{'} = AXA^{-1}, \;\; A\epsilon SU(2) \Leftrightarrow A^{+} = A^{-1}, DetA = 1 \tag{25}$$

leaves $DetX = DetX^{'}$ invariant

- But $x_i^{'} = \frac{1}{2}Tr(\sigma_i X^{'}) = \frac{1}{2}Tr(\sigma_i AXA^{-1}) = \frac{1}{2}Tr(\sigma_i A\sigma_j A^{-1})x_j = O_{ij}x_j$

$$\Rightarrow \qquad O_{ij}(A) = \frac{1}{2}Tr(\sigma_i A\sigma_j A^{-1}) \tag{26}$$

$\Rightarrow \qquad O(A) = O(-A)$ i.e. one element of SO(3) corresponds to two elements of SU(2).

Aside: SL(2,C)

- Define 2×2 matrix

$$X \equiv x^{\mu}\sigma_{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad \mu = 0, 1, 2, 3, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (27)$$

- Since $x^{\mu}x_{\mu} = DetX$ the transformation

$$X^{'} = AXA^{+} \qquad (28)$$

where the matrices $A\epsilon SL(2, C)$: $|DetA| = 1$ leaves $x^{\mu}x_{\mu}$ invariant

- But $x^{'\mu} = \Lambda^{\mu}_{\nu}x^{\nu} = \frac{1}{2}Tr(\sigma_i X^{'}) = \frac{1}{2}Tr(\sigma_i AXA^{-1}) = -\frac{1}{2}Tr(\sigma_i A\sigma_j A^{-1})x^{\nu}$

$$\Rightarrow \qquad \Lambda(A) = \frac{1}{2}Tr(\sigma_i A\sigma_j A^{-1}) \qquad (29)$$

$\Rightarrow \qquad \Lambda(A) = \Lambda(-A))$ i.e. one element of SO(3,1) corresponds to two elements of SL(2,C).

- SL(2,C) is covering group of the Lorentz group SO(3,1) similarly as SU(2) is covering group of SO(3) group.

Construction:(using *the highest weight* method)

1. SU(2) is of rank 1, i.e. there is only one operator fully characterizing the state within a given multiplet. Let us choose J_3 and denote the state with the highest weight j as $\mid j\rangle$:

$$J_3 \mid j\rangle = j \mid j\rangle; \quad \langle i \mid j\rangle = \delta_{ij}. \tag{30}$$

2. Define the lowering and rising operators

$$J^{\pm} \equiv \frac{J_1 \pm \mathrm{i}J_2}{\sqrt{2}} \quad \Rightarrow [J_3, J^{\pm}] = \pm J^{\pm}, \quad [J^+, J^-] = J_3. \tag{31}$$

3. So if $J_3 \mid m\rangle = m \mid m\rangle$ from eq.(31) $\Rightarrow$

$$J_3(J^{\pm} \mid m\rangle) = (m \pm 1)J^{\pm} \mid m\rangle, \tag{32}$$

$\Rightarrow$ in any multiplet $\exists\, j$ such that $J^+ \mid j\rangle = 0$, (called the highest weight).

Representations of SU(2): Construction

4. Starting from the state of the highest weight (or from the state with lowest weight) we can construct all states of any multiplet.

5. Acting by J^- on the state with highest weight yields $J^- \mid j\rangle = N_j \mid j-1\rangle$ where the normalization factor N_j is:

$$N_j^* N_j \underbrace{\langle j-1 \mid j-1\rangle}_{1} = \langle j \mid \underbrace{J^+ J^-}_{J^- J^+ + J_3} \mid j\rangle = j \underbrace{\langle j \mid j\rangle}_{1}, \tag{33}$$

We have used $J^+ \mid j\rangle = 0$. As a result we find, using particular sign convention, $N_j = \sqrt{j}$.

6. In fact we can start with any state $\mid j-k\rangle$ for which we define the normalization factor as follows :

$$J^- \mid j-k\rangle \equiv N_{j-k} \mid j-k-1\rangle \Rightarrow J^+ \mid j-k-1\rangle = N_{j-k} \mid j-k\rangle. \tag{34}$$

Representations of SU(2): Construction

Repeating the above procedure of application of J^+J^- to $|\, j-k\rangle$ yields

$$N_{j-k}^2 = \langle j-k \,|\, \underbrace{J^+J^-}_{[J^+,J^-]+J^-J^+} \,|\, j-k\rangle = \underbrace{\langle j-k \,|\, J_3 \,|\, j-k\rangle}_{j-k} + \underbrace{\langle j-k \,|\, J^- \underbrace{J^+ \,|\, j-k\rangle}_{N_{j-k+1}\,|\, j-k+1\rangle}}_{N_{j-k+1}^2}, \tag{35}$$

$\Rightarrow$ the recurrence relation between the normalization coefficients:

$$N_{j-k}^2 - N_{j-k+1}^2 = j-k. \tag{36}$$

Representations of SU(2): Construction

- Writing the full sequence of such recurrence relations

$$\begin{array}{rcrcl} N_j^2 & - & N_{j+1}^2 & = & j \\ N_{j-1}^2 & - & N_j^2 & = & j-1 \\ & \vdots & & \vdots & \\ N_{j-k}^2 & - & N_{j-k+1}^2 & = & j-k \\ & \vdots & & \vdots & \\ N_{-j}^2 & - & N_{-j+1}^2 & = & -j \end{array} \tag{37}$$

- Summing the first $k+1$ lines on both sides of these relations (taking into account that $N_{j+1} = 0$):

$$N_{j-k}^2 = \sum_{i=0}^{k} (j-i) = (k+1)\left[j - \frac{k}{2}\right] = (k+1)\left[j - k + \frac{k}{2}\right]. \tag{38}$$

- N.B. Summing all the lines in (37) we get trivial identity $0 = 0$.

Representations of SU(2): Construction

$$N^2_{j-k} = \sum_{i=0}^{k} (j-i) = (k+1)\left[j - \frac{k}{2}\right] = (k+1)\left[j - k + \frac{k}{2}\right]$$

- For $k = 2j$ we get $\Leftrightarrow m = -j \Rightarrow N_{-j} = 0$

$\Rightarrow$
  - the multiplet with highest weight j has $2j + 1$ states
  - j has to be of the form $2j = l$ with l integer.

Remark

A general state within the multiplet with highest weight j, characterized by the eigenvalue m of J_3, will be denoted as $|\, j, m\rangle$. In terms of the numbers j, m the normalization factor (38) can be rewritten as

$$N^2_{j,m} = \frac{1}{2}(j - m + 1)(j + m). \tag{39}$$

Fundamental representation of SU(2)

- Exploiting the concepts of the **direct sum and product** of multiplets this representation can be used to construct all other multiplets of SU(2).
- Any multiplet of SU(2) can be constructed from the direct product of basic SU(2) doublets
$$\left[\underbrace{\mathbf{D}^{(1/2)} \otimes \mathbf{D}^{(1/2)} \ldots \otimes \mathbf{D}^{(1/2)}}_{n \text{ times}}\right]_{i_1,i_2,\cdots,i_n;j_1,j_2\cdots,j_n}, \qquad (40)$$
the matrices of which are given as
$$D^{(1/2)}_{i_1,j_1} D^{(1/2)}_{i_2,j_2} \cdots D^{(1/2)}_{i_n,j_n}, \qquad (41)$$
- The above direct product of multiplets is *reducible* and obviously symmetric under the permutations of the indices $1, 2, \cdots, n$. Individual irreducible components of this reducible representation are then characterized by particular type of symmetry of states on which they act.
- Starting from the symmetric state given as the direct product of n one particle states $|\,1/2, 1/2\rangle$ and applying the lowering operator J^- we get all states of the multiplet corresponding to the highest weight $j = n/2$.

Fundamental representation of SU(2)

- Thus subspace of fully symmetric states of n basic doublets corresponds to the multiplet with $j = n/2$.
- Physically, all these states describe the system of n **identical** noninteracting particles, each of them corresponding to the fundamental representation $\mathbf{D}^{(1/2)}$.
- For instance the direct products of two and three SU(2) doublets decomposes as:

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3}_s \oplus \mathbf{1}, \quad \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_s \oplus \mathbf{2}_{m,s} \oplus \mathbf{2}_{m,a} \tag{42}$$

  where the subscript "s" denotes representation symmetric under the permutations of the product representations whereas "ms" (mixed symmetric) and "ma" (mixed antisymmetric) denote two equivalent doublet representations, which differ in their symmetry properties under permutation of the first two doublets. The adjective "mixed" reflects the fact that as far as other permutations are concerned these representation possess no definite symmetry.

Application of SU(2) group: the isospin

- Consider a system of finite number of protons and neutrons, neglecting all effects of electromagnetic and weak interactions (like $m_p - m_n$ etc.).
- Starting from late forties, experiments had shown quite convincingly that under these assumptions strong interactions between protons and neutrons are **charge invariant**.
- There is a very close group theory analogy between the isospin symmetry of strong interactions and rotational symmetry of a two-particle spin-spin interaction Hamiltonian $H_{int} \propto \vec{s_1}\vec{s_2}$.
- A particularly suitable way how to describe such a system is based on the use of **creation** and **annihilation** operators (P^+_α, P_α for protons and N^+_α, N_α for neutrons in state α), which allow us to build all multiparticle states via their action on the vacuum state of the Hamiltonian, denoted as $\mid 0\rangle$ and excited states, denoted generically $\mid s\rangle$:

$$\begin{array}{rclcrcl} P^+_\alpha \mid s\rangle & = & \mid s + \text{proton in state } \alpha\rangle & , & P_\alpha \mid s\rangle & = & \mid s - \text{proton in state } \alpha\rangle \\ P_\alpha \mid 0\rangle & = & 0 & , & \langle 0 \mid P^+_\alpha & = & 0. \end{array} \tag{43}$$

Application of SU(2) group: the isospin

- The commutation relations of creation and annihilation operators are assumed to be the following:

$$\{P^+_\alpha, P_\beta\} \equiv P^+_\alpha P_\beta + P_\beta P^+_\alpha = \delta_{\alpha,\beta}, \tag{44}$$

$$\{P^+_\alpha, P^+_\beta\} = \{P_\alpha, P_\beta\} = 0, \tag{45}$$

$$\left[P^+_\alpha, N^+_\beta\right] = [P_\alpha, N_\beta] = \left[P^+_\alpha, N_\beta\right] = \left[P_\alpha, N^+_\beta\right] = 0. \tag{46}$$

- The first two of the above three equations enforce, when restricted to the case $\alpha = \beta$, the Pauli exclusion principle, which stipulates that there cannot be two identical fermions in any given state. For $\alpha \neq \beta$ they guarantee the antisymmetry of wave functions of systems of protons or neutrons.
- The third set of equalities tells us that protons are different from neutrons and so we can interchange the order of application of respective operators, be it creation or annihilation, with impunity.

Application of SU(2) group: the isospin

- These operators may be used to describe, for instance, the repulsion between two protons, with interaction Hamiltonian given as

$$H_{int} = \sum_{\alpha,\beta} P^+_\alpha P_\alpha V_{\alpha\beta}(r) P^+_\beta P_\beta, \tag{47}$$

- Out of $P^+_\alpha, P_\alpha, N^+_\alpha, N_\alpha$ we can construct the operators $J^\pm, J_3$ introduced earlier (denote them as $T^\pm$, T_3):

$$T^+ \equiv \frac{1}{\sqrt{2}} \sum_\alpha P^+_\alpha N_\alpha; \quad T^- \equiv \frac{1}{\sqrt{2}} \sum_\alpha N^+_\alpha P_\alpha, \tag{48}$$

$$T_3 \equiv \frac{1}{2} \sum_\alpha \left(P^+_\alpha P_\alpha - N^+_\alpha N_\alpha \right). \tag{49}$$

- Starting from the anticommutation relations for P^+_α etc.,

$$\left[T_3, T^\pm \right] = \pm T^\pm; \quad \left[T^+, T^- \right] = T_3 \tag{50}$$

as required for the generators of SU(2) group.

Application of SU(2) group: the isospin

- The group we have just constructed is called the **isospin** group.
- The invariance of strong interactions with respect to it implies that the appropriate Hamiltonian H_s commutes with all these generators:

$$\left[H_s, T^{\pm}\right] = [H_s, T_3] = 0. \tag{51}$$

- Writing the state vector of the nucleon in a two-component column form

$$\mid \psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1 \mid p\rangle + \psi_2 \mid n\rangle \tag{52}$$

  as a superposition of pure proton and neutron states $\mid p\rangle \mid n\rangle$, the interpretation of the transition operators $T^{\pm}$ becomes clear: they transform neutron to proton and vice versa.
- The relation between the third component T_3 of the isospin and the electric charge Q reads

$$Q = T_3 + \frac{B}{2}, \tag{53}$$

  where B is the baryon number of the system ($B = 1, -1, 0$ for the nucleon, antinucleon and pion respectively).

Application of SU(2) group: the isospin

- The power of this formalism as a way to describe charge invariance of strong interactions becomes highly nontrivial when it is extended to cover the interactions of nucleons with pions and other hadrons like, for instance,

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{n} + \pi^+$$

- Extend the above construction to pions: introduce $a_{\pi^+}, a_{\pi^-}, a_{\pi^0}$ for the annihilation operators and similarly for the creation ones.
- The commutation relations are the same as for the nucleons, with one very important change: the anticommutator in (45) etc. is replaced with the *commutator*!
- It is nontrivial, but easy to verify that in both cases one gets the same commutation relations for the generators of the SU(2) group.

Application of SU(2) group: the isospin

The expressions for the generators describing a system of both nucleons and pions then read

$$T^{+} \equiv \frac{1}{\sqrt{2}}\sum_{\alpha} P^{+}_{\alpha} N_{\alpha} + 1\sum_{\alpha}\left(a^{+}_{\pi^{+}}(\alpha)a_{\pi^{0}}(\alpha) + a^{+}_{\pi^{0}}(\alpha)a_{\pi^{-}}(\alpha)\right), \quad (54)$$

$$T^{-} \equiv \frac{1}{\sqrt{2}}\sum_{\alpha} N^{+}_{\alpha} P_{\alpha} + 1\sum_{\alpha}\left(a^{+}_{\pi^{-}}(\alpha)a_{\pi^{0}}(\alpha) + a^{+}_{\pi^{0}}(\alpha)a_{\pi^{+}}(\alpha)\right), \quad (55)$$

$$T_{3} \equiv \frac{1}{2}\sum_{\alpha}\left(P^{+}_{\alpha} P_{\alpha} - N^{+}_{\alpha} N_{\alpha}\right) + 1\sum_{\alpha}\left(a^{+}_{\pi^{+}}(\alpha)a_{\pi^{+}}(\alpha) - a^{+}_{\pi^{-}}(\alpha)a_{\pi^{-}}(\alpha)\right) \quad (56)$$

“1” in front of the second contributions expresses the fact that pions carry one unit of isospin.

Isospin conservation example

Example: the scattering of pions on nucleons. From the point of view of the isospin symmetry the system of one pion and one nucleon is described by means of the direct product of a doublet and a triplet of SU(2). The states of definite charge combinations can be written as follows

$$
\begin{aligned}
\mid \pi^+ p\rangle =\mid 1,1\rangle \mid \tfrac{1}{2},\tfrac{1}{2}\rangle &= \mid \tfrac{3}{2},\tfrac{3}{2}\rangle, & \langle \pi^+ p \mid S \mid \pi^+ p\rangle &= a_{3/2},\\
\mid \pi^+ n\rangle =\mid 1,1\rangle \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}} \mid \tfrac{3}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}} \mid \tfrac{1}{2},\tfrac{1}{2}\rangle, & \langle \pi^+ n \mid S \mid \pi^+ n\rangle &= \tfrac{1}{3}a_{3/2} + \tfrac{2}{3}a_{1/2},\\
\mid \pi^- p\rangle =\mid 1,-1\rangle \mid \tfrac{1}{2},\tfrac{1}{2}\rangle &= \sqrt{\tfrac{1}{3}} \mid \tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}} \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle, & \langle \pi^- p \mid S \mid \pi^- p\rangle &= \tfrac{1}{3}a_{3/2} + \tfrac{2}{3}a_{1/2},\\
\mid \pi^- n\rangle =\mid 1,-1\rangle \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle &= \mid \tfrac{3}{2},-\tfrac{3}{2}\rangle, & \langle \pi^- n \mid S \mid \pi^- n\rangle &= a_{3/2},\\
\mid \pi^0 p\rangle =\mid 1,0\rangle \mid \tfrac{1}{2},\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}} \mid \tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}} \mid \tfrac{1}{2},\tfrac{1}{2}\rangle, & \langle \pi^0 p \mid S \mid \pi^0 p\rangle &= \tfrac{2}{3}a_{3/2} + \tfrac{1}{3}a_{1/2},\\
\mid \pi^0 n\rangle =\mid 1,0\rangle \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}} \mid \tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}} \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle, & \langle \pi^0 n \mid S \mid \pi^0 n\rangle &= \tfrac{2}{3}a_{3/2} + \tfrac{1}{3}a_{1/2}.
\end{aligned}
\tag{57}
$$

Remark

The coefficients in front of the states with definite total isospin, called **Clebsh-Gordan** *coefficients, can be obtained directly by application of the above described technique of highest weight.*

Isospin conservation example

- The S-matrix elements of various channels are thus related as there are only two independent amplitudes, corresponding to full isospin equal to 1/2 and 3/2.
- These can also be used to write down the amplitudes of other processes like

$$\langle \pi^0 \mathrm{p} \mid \mathrm{S} \mid \pi^+ \mathrm{n} \rangle = \frac{\sqrt{2}}{3} \left(\mathrm{a}_{3/2} - \mathrm{a}_{1/2} \right) = \langle \pi^- \mathrm{p} \mid \mathrm{S} \mid \pi^0 \mathrm{n} \rangle. \tag{58}$$

- A simple manifestation of these relations is the ratio of the cross-sections to produce the Δ resonance in various pion-nucleon channels like, for instance,

$$r \equiv \frac{\sigma(\pi^+ \mathrm{p} \to \Delta^{++})}{\sigma(\pi^- \mathrm{p} \to \Delta^0)} = \frac{\sigma(\pi^+ \mathrm{p} \to \Delta^{++} \to \pi^+ \mathrm{p})}{\sigma(\pi^- \mathrm{p} \to \Delta^0 \to \pi^- \mathrm{p}) + \sigma(\pi^- \mathrm{p} \to \Delta^0 \to \pi^0 \mathrm{n})} = \frac{1}{\frac{1}{9} + \frac{2}{9}} = 3. \tag{59}$$

Reality of $SU(2)$ representations

(a) Let's show that SU(2) representations are equivalent to their complex conjugate representations.

1. First we'll show that $\forall U$ (U is 2×2 matrix, $UU^+ = 1,\ DetU = 1$) $\exists S$ connecting U with U^* via similarity transformation:
$$S^{-1}US = U^* \tag{60}$$
2. $U = e^{iH}$ where $H = H^+$, $Tr(H) = 0$:
$$\Rightarrow S^{-1}US = S^{-1}e^{iH}S = U^* = e^{-iH^*} \tag{61}$$
3. Due to hermiticity of H we can expand it in terms of Pauli matrices with **real** coefficients: $H = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$.
4. Since $Im(\sigma_1) = Im(\sigma_3) = 0$ but $Im(\sigma_2) \neq 0$ the complex conjugate matrix is:
$$H^* = a_1\sigma_1 - a_2\sigma_2 + a_3\sigma_3 \tag{62}$$
5. (61) $\Rightarrow S^{-1}HS = -H^*$ so:
$$S^{-1}\sigma_1 S = -\sigma_1, \quad S^{-1}\sigma_2 S = \sigma_2, \quad S^{-1}\sigma_3 S = -\sigma_3 \tag{63}$$

   and so (60) can be satisfied provided $S = const.\sigma_2$. Q.E.D.

Reality of $SU(2)$ representations

(b) Next we'll show that if ψ_1 and ψ_2 are the basis vectors of $\mathbf{D}^{(1/2)}$ representation of SU(2) such that: $J_3\psi_1 = \frac{1}{2}\psi_1 \quad J_3\psi_2 = -\frac{1}{2}\psi_2$ (64)

then $J_3\psi_1^* = -\frac{1}{2}\psi_1^* \quad J_3\psi_2^* = \frac{1}{2}\psi_2^*$ (65)

1. Denoting $\psi \equiv | \psi\rangle$ we have $\psi^{'} = U\psi$ and so for its complex conjugate:
$$\psi^{'*} = U^*\psi^* = (S^-1US)\psi^* \mathrm{or}(S\psi^{'*}) = U(S\psi^*) \quad (66)$$
i.e. $S\psi^*$ has the same transformation properties as ψ.
2. Writing it explicitly with $S = i\sigma_2$:
$$S\psi^* = \begin{pmatrix} 0 & 1 \\ \text{-1} & 0 \end{pmatrix} \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} \quad (67)$$
3. To say that ψ^* has the same transformation properties as ψ means for example that
$$J_3 \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & \text{-}1/2 \end{pmatrix} \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} \quad (68)$$
i.e. (65) is true. Q.E.D.

Reality of $SU(2)$ representations

- This shows that the $\mathbf{D^{(1/2)}}$ representation is equivalent to its complex conjugate representation. We say that it is a real representation.
- This property can be extended to all other representations of the SU(2) group, because all other representations can be obtained from the $\mathbf{D^{(1/2)}}$ representation by tensor product.
- Part (b) shows that the matrix S transforms any real diagonal matrix, e.g. σ_3, into the negative of itself. In other words, S will transform any eigenvalue to its negative.
- Thus the existence of such a matrix S requires that the eigenvalues of the hermitean-generating matrix occur in pairs of the form $\pm\alpha_1, \pm\alpha_2 \ldots$ (or are zero).
- It is then clear that for SU(N) groups with $N > 3$, such a matrix S cannot exist as eigenvalues of higher–rank special unitary groups do not have such a special pairwise structure.

Combining two fundamental isospin representations

- Let us write isospin doublet and its hermitean conjugate as

$$\mid \psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} \quad \text{and} \quad \langle \psi \mid = (\psi_1^* \psi_2^*) = (u^+ d^+) \tag{69}$$

- Let's find isospin of the product $\psi_i^* \psi_j$ where $i, j = 1, 2$.
- Defining $\psi^i \equiv \psi_i^*$ and $U_i^j \equiv U_{ij}$ (where $UU^+ = 1$) we can write:

$$\psi_i^{'} = U_{ij}\psi_j = U_i^j \psi_j \quad \text{and} \quad \psi_i^{'*} = U_{ij}^* \psi_j^* = \psi_j^* (U_j^i)^* = \psi^j U_i^j \tag{70}$$

- The combination $\psi^i \psi_i$ is SU(2) invariant (i.e. isoscalar $I = 0$):

$$\psi^{i'} \psi_i^{'} = \psi^j U_j^i U_i^k \psi_k = \psi_j U_{ij}^* U_{ik} \psi_k = \psi^j \delta_j^i \psi_k = \psi^j \psi_j \tag{71}$$

and can be removed. Remaining isovector:

$$T_j^i = \psi^i \psi_j - \tfrac{1}{2} \delta_j^i (\psi^k \psi_k) \tag{72}$$

transforms as $I = 1$ triplet and is traceless $T_i^i = 0$.

Combining two fundamental isospin representations

- The T^i_j components can be now written explicitly:

$$T^1_2 = \psi^1\psi_2 = u^+ d \sim \pi^- \quad \text{and} \quad T^2_1 = \psi^2\psi_1 \sim d^+ u \sim \pi^+ \qquad (73)$$

$$T^1_1 = \psi^1\psi_1 - \tfrac{1}{2}(\psi^1\psi_1 + \psi^2\psi_2) = \tfrac{1}{2}(\psi^1\psi_1 - \psi^2\psi_2) = \tfrac{1}{2}(u^+ u - d^+ d) \sim \tfrac{1}{\sqrt{2}}\pi^0 \quad (74)$$

- The full matrix reads: $$\hat{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \qquad (75)$$

- Summarizing: $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. The triplet is called the adjoint representation of SU(2).

Highest weight technique for general compact Lie groups

Divide the basis vectors of the associated vector space of a given compact Lie algebra $\mathcal{A}$ into:

1. $H_i, i = 1, \cdots m$ such that
$$[H_i, H_j] = 0, \forall i, j \tag{76}$$
where m, called the **rank** of the group, is the *maximal* number of mutually commuting elements of $\mathcal{A}$, forming the **Cartan subalgebra**. They can be chosen hermitian and thus interpreted as generators.
2. $n - m$ remaining elements E_α (n being the number of generators) such that
$$[H_i, E_\alpha] = \alpha_i E_\alpha, \tag{77}$$
where the m dimensional, *nonzero* vectors α are called the **roots** of the Lie algebra.

Adopt the following normalization of H_i, E_α (λ is defined in (16))

$$\begin{array}{lcl} \langle E_\alpha \mid E_\beta \rangle & = & \lambda^{-1}\mathrm{Tr}(E_\alpha^+ E_\beta) & = & \delta_{\alpha\beta} \\ \langle H_i \mid H_j \rangle & = & \lambda^{-1}\mathrm{Tr}(H_i H_j) & = & \delta_{ij}. \end{array} \tag{78}$$

Weights and roots of compact Lie algebras

- The Cartan subalgebra contains generators which are analogies of J_3 of SU(2), where there was just one of them.
- E_α are generalizations of the lowering and rising operators $J^\pm$.
- We can diagonalize all H_i simultaneously ($\Leftarrow [H_i, H_j] = 0, \forall i, j$), defining **real** vectors $\mu = (\mu_1, \cdots, \mu_m)$ of **weights** of a given multiplet **D**

$$H_i \mid \mu, \mathbf{D}\rangle = \mu_i \mid \mu, \mathbf{D}\rangle. \tag{79}$$

Their number is limited by the dimension of **D**.

There are several simple consequences of the above definitions:

1. Roots α are actually weights of the adjoint representation as in this representation:
$$H_i \mid E_\alpha\rangle = \mid [H_i, E_\alpha]\rangle = \alpha_i \mid E_\alpha\rangle. \tag{80}$$
Consequently α_i are vectors of *real numbers*.
2. Taking the hermitian conjugate of (77) we find
$$[H_i, E_\alpha]^+ = -[H_i, E_\alpha^+] = \alpha_i E_\alpha^+ \quad \Rightarrow \quad E_\alpha^+ = E_{-\alpha}, \tag{81}$$
i.e. E_α are **not** hermitian.

Weights and roots of compact Lie algebras

3. Using the commutation relations we easily get

$$H_i(E_{\pm\alpha} \mid \mu, \mathbf{D}\rangle) = (\mu \pm \alpha)_i E_{\pm\alpha} \mid \mu, \mathbf{D}\rangle \Rightarrow$$
$$E_{\pm\alpha} \mid \mu, \mathbf{D}\rangle = N_{\pm\alpha,\mu} \mid \mu \pm \alpha, \mathbf{D}\rangle, \quad N_{-\alpha,\mu} = N^*_{\alpha,\mu-\alpha} \tag{82}$$

   and analogously to the case of the SU(2) algebra, the normalization factors $N_{\alpha,\mu}$ can be chosen *real*.

4. As

$$H_i(E_\alpha \mid E_{-\alpha}\rangle) = E_\alpha \underbrace{H_i \mid E_{-\alpha}\rangle}_{-\alpha_i|E_{-\alpha}\rangle} + \underbrace{[H_i, E_\alpha]}_{\alpha_i E_\alpha} \mid E_{-\alpha}\rangle = 0 \tag{83}$$

   the state $E_\alpha \mid E_{-\alpha}\rangle$ has zero weight and can therefore be expressed as linear combination of the vectors $\mid H_i\rangle$. Using the normalization (16) one moreover finds (exercise 2.2)

$$E_\alpha \mid E_{-\alpha}\rangle = \mid [E_\alpha, E_{-\alpha}]\rangle = \sum_j \alpha_j \mid H_j\rangle \Leftrightarrow [E_\alpha, E_{-\alpha}] = \alpha_j H_j. \tag{84}$$

Construction of multiplets

- For general compact Lie group it follows closely that of the SU(2) group.
- Start with an arbitrary state $\mid \mu, \mathbf{D}\rangle$ of $\mathbf{D}$ and apply to it powers of the operators $E_{\pm\alpha}$. $dim(\mathbf{D}) < \infty \Rightarrow \exists\ p \geq 0, q \geq 0$ such that

$$E_{\alpha}^{p+1} \mid \mu, \mathbf{D}\rangle = E_{-\alpha}^{q+1} \mid \mu, \mathbf{D}\rangle = 0. \tag{85}$$

- Proceeding similarly as in the case of SU(2)

$$\underbrace{\langle \mu, \mathbf{D} \mid \underbrace{\underbrace{[E_\alpha, E_{-\alpha}]}_{\alpha_j H_j} \mid \mu, \mathbf{D}\rangle}_{\alpha\mu \mid \mu, \mathbf{D}\rangle}}_{\alpha\mu} = \underbrace{\langle \mu, \mathbf{D} \mid E_\alpha \underbrace{E_{-\alpha} \mid \mu, \mathbf{D}\rangle}_{N_{-\alpha,\mu} \mid \mu - \alpha, \mathbf{D}\rangle}}_{\mid N_{-\alpha,\mu} \mid^2} - \underbrace{\langle \mu, \mathbf{D} \mid E_{-\alpha} \underbrace{E_\alpha \mid \mu, \mathbf{D}\rangle}_{N_{\alpha,\mu} \mid \mu + \alpha, \mathbf{D}\rangle}}_{\mid N_{\alpha,\mu} \mid^2} \tag{86}$$

Construction of multiplets

we get the following set of equations

$$\begin{array}{lclcl} \mid N_{\alpha,\mu+\alpha(p-1)} \mid^2 & - & \overbrace{\mid N_{\alpha,\mu+\alpha p} \mid^2}^{0} & = & \alpha\cdot(\mu+p\alpha) \\ & \vdots & & \vdots & \\ \mid N_{\alpha,\mu} \mid^2 & - & \mid N_{\alpha,\mu+\alpha} \mid^2 & = & \alpha\cdot(\mu+\alpha) \\ \mid N_{\alpha,\mu-\alpha} \mid^2 & - & \mid N_{\alpha,\mu} \mid^2 & = & \alpha\cdot\mu \\ \mid N_{\alpha,\mu-2\alpha} \mid^2 & - & \mid N_{\alpha,\mu-\alpha} \mid^2 & = & \alpha\cdot(\mu-\alpha) \\ & \vdots & & \vdots & \\ \underbrace{\mid N_{\alpha,\mu-(q+1)\alpha} \mid^2}_{0} & - & \mid N_{\alpha,\mu-q\alpha} \mid^2 & = & \alpha\cdot(\mu-q\alpha). \end{array} \tag{87}$$

Summing the first $p+1$ of the above equations we get the explicit expression

$$N^2_{\alpha,\mu+\alpha(p-1)} - N^2_{\alpha,\mu} = (p+1)\left[\alpha\cdot\mu - \frac{p}{2}\alpha^2\right], \tag{88}$$

which reduces to (38) for the SU(2) group when we set $k=p$, $\mu=j-k$ and $\alpha=\pm1$.

Construction of multiplets

Summing all equations in (87) we get relation between the *roots* of the group and *weights* of its representations

$$0 = (p+q+1)(\alpha \cdot \mu) + \alpha^2 \left(\frac{p(p+1)}{2} - \frac{q(q+1)}{2} \right) \Rightarrow \frac{2\alpha \cdot \mu}{\alpha^2} = q - p. \quad (89)$$

Consider the adjoint representation (here weights equal to the roots themselves). We get the following condition for any pair of roots:

$$\frac{2\alpha \cdot \beta}{\alpha^2} = q - p = m; \quad \frac{2\beta \cdot \alpha}{\beta^2} = q' - p' = m' \quad \Rightarrow \frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} \equiv \cos^2 \vartheta = \frac{mm'}{4}, \quad (90)$$

where m, m' are integers, determining the angle between the roots considered as vectors in m-dimensional space. The integers p, q describe the shifting of the weight $\mu = \beta$ by means of the operator E_α, while in the case of p', q' the roles of α and β are reversed. The above formula implies that only those values of m, m' are allowed for which $mm' = 0, 1, 2, 3, 4$. For SU(2) we have $m = m' = 0, \pm 2$ as the roots corresponding to $J^{\pm}$ are one-dimensional and equal ± 1.

Composition of roots

Using the Jacobi identity:

$$[H_i, [E_\alpha, E_\beta]] = -[E_\beta, \underbrace{[H_i, E_\alpha]}_{\alpha_i E_\alpha}] - [E_\alpha, \underbrace{[E_\beta, H_i]}_{-\beta_i E_\beta}] = (\alpha+\beta)_i [E_\alpha, E_\beta], \tag{91}$$

we conclude that

- $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$ **if** the root $\alpha+\beta$ does exist,
- $[E_\alpha, E_\beta] = 0$ if $\alpha+\beta$ is not a root, but $\alpha \neq -\beta$,
- $[E_\alpha, E_{-\alpha}] = \alpha_j H_j$ according to (84).

Example: SU(3)

- Group of unitary matrices 3×3 with unit determinant.
- Its algebra is formed by 8 traceless hermitian matrices. 8 generators T_a:

$$T_a \equiv \frac{1}{2}\lambda_a \quad \Rightarrow \quad \mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{92}$$

can be chosen in many different ways, the simplest is Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \boldsymbol{\lambda_3} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{-1} & 0 \\ 0 & 0 & \mathbf{0} \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \boldsymbol{\lambda_8} = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{-2} \end{pmatrix} \tag{93}$$

- Two commuting generators, $H_1 \equiv T_3, H_2 \equiv T_8$ with common eigenvectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{94}$$

The eigenvalues (h_1, h_2) of H_1, H_2, corresponding to these states equal to $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, $(-\frac{1}{2}, \frac{1}{2\sqrt{3}})$,$(0, -\frac{1}{\sqrt{3}})$.

Example: SU(3)

$$\left[H_1, \frac{1}{\sqrt{2}}(T_1 \pm \mathrm{i} T_2)\right] = \pm 1 \left[\frac{1}{\sqrt{2}}(T_1 \pm \mathrm{i} T_2)\right], \quad \left[H_2, \frac{1}{\sqrt{2}}(T_1 \pm \mathrm{i} T_2)\right] = 0 \quad (95)$$

$$\left[H_1, \frac{1}{\sqrt{2}}(T_4 \pm \mathrm{i} T_5)\right] = \pm\frac{1}{2} \left[\frac{1}{\sqrt{2}}(T_4 \pm \mathrm{i} T_5)\right], \quad \left[H_2, \frac{1}{\sqrt{2}}(T_4 \pm \mathrm{i} T_5)\right] = \pm\frac{\sqrt{3}}{2} \left[\frac{1}{\sqrt{2}}(T_4 \pm \mathrm{i} T_5)\right] \quad (96)$$

$$\left[H_1, \frac{1}{\sqrt{2}}(T_6 \pm \mathrm{i} T_7)\right] = \mp\frac{1}{2} \left[\frac{1}{\sqrt{2}}(T_6 \pm \mathrm{i} T_7)\right], \quad \left[H_2, \frac{1}{\sqrt{2}}(T_6 \pm \mathrm{i} T_7)\right] = \pm\frac{\sqrt{3}}{2} \left[\frac{1}{\sqrt{2}}(T_6 \pm \mathrm{i} T_7)\right] \quad (97)$$

$\Rightarrow$

$$E_{\pm 1,0} = \frac{1}{\sqrt{2}}(T_1 \pm \mathrm{i} T_2); \quad E_{\pm 1/2, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(T_4 \pm \mathrm{i} T_5); \quad E_{\mp 1/2, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(T_6 \pm \mathrm{i} T_7). \quad (98)$$

- $SU_I(2) \subset SU(3)$ $\Rightarrow$ operators $E_{\pm 1,0}$ can be identified with operators $J^{\pm}$.
- $E_{\pm 1/2, \pm\sqrt{3}/2}$ and $E_{\mp 1/2, \pm\sqrt{3}/2}$ correspond to the other two SU(2) subgroups of SU(3), usually called U-spin and V-spin.
- $SU_I(2)$ subgroup is singled out by the particular choice (93) of the SU(3) generators by selecting as one of the diagonal matrices λ_3, corresponding to the projection of the isospin.

Example: SU(3)

- The basic triplet, represented by their weights, together with the roots is displayed in Fig. 1. According to (90) we find $mm' = 1$, i.e. the angles between roots of SU(3) algebra are multiples of 60° degrees.

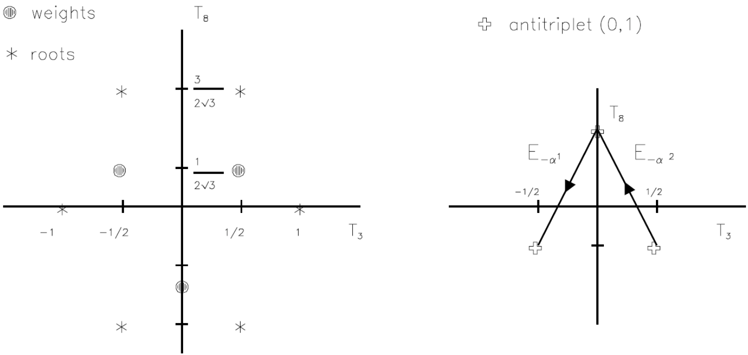

Figure: Roots and weights of the fundamental representations of SU(3).

Simple roots of simple Lie algebras

- One doesn't need all the transition operators E_α in order to sweep through a whole given multiplet. For SU(2) we had J^+ and J^-, but actually only one of them was really necessary (the other one working in opposite direction).
- The same occurs for general compact Lie groups.
- Define some **ordering** of operators, which will allow us to tell what "lowering" and "rising" means and thus distinguish between "positive" and "negative" roots. As the eigenvalues of operators from Cartan subalgebra form m–dimensional vectors $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$ we define

Definition

The m-dimensional vector $\mu = (\mu_1, \mu_2, \cdots, \mu_m)$ is called **positive** *if its first nonzero element is positive. Similarly for negative vectors.*

Corollary

The ordering of weights and roots is then defined via the relation

$$\mu^1 > \mu^2 \Leftrightarrow \mu^1 - \mu^2 > 0,$$

where the superscript labels different vectors.

Simple roots of simple Lie algebras

- It is obvious that for any **finite dimensional** representation **D** (N.B. for compact Lie groups all **irreducible** representations *are* finite dimensional) there *exists* a weight which may be called **highest** in the above defined sense. Similarly, all the roots can be divided into positive and negative ones the former called "rising" and the latter "lowering".

Definition

A positive root which cannot be expressed as a sum of two other positive roots is called a **simple** *root.*

Corollary

$\forall \alpha, \beta$ *of simple roots the difference* $\alpha - \beta$ **is not** *a root.* $\Rightarrow$

$$E_{-\alpha} \mid E_\beta\rangle = \mid \underbrace{[E_{-\alpha}, E_\beta]}_{0}\rangle = 0 \Rightarrow q = 0 \quad \Rightarrow \quad \cos\vartheta = -\frac{\sqrt{pp'}}{2} < 0 \qquad (99)$$

$\Rightarrow \vartheta \in (\pi/2, \pi)$.

For SU(3), which has 3 positive (and 3 negative) roots, two, $\alpha^1 = (1/2, \sqrt{3}/2)$ *and* $\alpha^2 = (1/2, -\sqrt{3}/2)$*, are simple, spanning the angle* 120°.

Properties of simple roots

Proposition

1. Simple roots are linearly independent.

Proposition

2. Each positive root Φ can be written as a sum

$$\Phi = \sum_{\alpha} k_\alpha \alpha$$

of simple roots α with **non-negative** *coefficients k_α.*

Proposition

3. The number k of simple roots of a simple Lie algebra is equal to its rank m.

- The first two are obvious and their proofs therefore left as an exercise.

Proof of 3^{rd} proposition

- Roots are m-dimensional vectors $\Rightarrow k \leq m$.
- Assume that $k < m$. In a suitable basis all simple roots will then have the first component equal to zero. This, however, means that the first component of every root Φ vanishes and we have

$$[H_1, E_\Phi] = \Phi_1 E_\Phi = 0.$$

- Consequently the generator H_1 commutes with *all* elements of the algebra and thus forms an invariant subalgebra by itself. This, however, is impossible in a simple algebra.

N.B. The general classification of compact Lie groups is based on systematic exploitation of the formula (99) which is conveniently expressed in the form of **Dynkin** diagrams. These diagrams describe the number and mutual orientation of all simple roots and are discussed in detail in **LIE ALGEBRAS in PARTICLE PHYSICS by Howard Georgi**.

Fundamental weights and fundamental representations

- Concepts which generalize the basic doublet representation $\mathbf{D}^{(1/2)}$ of SU(2) and its highest weight 1/2.
- Consider the highest weight μ of a representation **D** and form (90) for all **simple** roots:
$$\frac{2\alpha^i\mu}{(\alpha^i)^2} = q^i - \underbrace{p^i}_{0} = q^i \quad i = 1, \cdots, m, \tag{100}$$
where the set $q^i, i = 1, \cdots, m$ of nonnegative integers *fully* characterizes the highest weight of a representation **D** and thereby also the whole representation.
- Define a special class of highest weights μ^j by the condition
$$\frac{2\alpha^i\mu^j}{(\alpha^i)^2} = \delta_{ij}; \quad i, j = 1, \cdots, m \tag{101}$$
- A general highest weight μ can be expressed in terms of the weights μ^j as follows:
$$\mu = \sum_{i=1}^{m} q^i\mu^i; \quad \mu^i = (\mu^i_1, \mu^i_2, \cdots, \mu^i_m). \tag{102}$$

Fundamental weights and fundamental representations

Definition

The highest weights μ^j are called the fundamental weights *and the corresponding multiplets* $\mathbf{D^{(i)}}$ fundamental representations.

Denote the multiplets in either of the three equivalent ways:

- By means of its highest weight μ as $\mathbf{D}^{(\mu)}$,
- using the vector $q \equiv (q_1, q_2, \cdots, q_m)$,
- or by its dimensionality, like **8**.

The sum $\mu = \sum_{i=1}^{m} q^i \mu^i; \;\; \mu^i = (\mu_1^i, \mu_2^i, \cdots, \mu_m^i)$ (eq.(102)) corresponds to the fact that any irreducible representation $\mathbf{D}$, with highest weight μ, can be obtained as a multiple direct product of fundamental representations $\mathbf{D^{(i)}}$ of the form

$$\mathbf{D} = \underbrace{\mathbf{D^{(1)}} \otimes \cdots \otimes \mathbf{D^{(1)}}}_{q^1 \;\; times} \otimes \underbrace{\mathbf{D^{(2)}} \otimes \cdots \otimes \mathbf{D^{(2)}}}_{q^2 \;\; times} \cdots \underbrace{\mathbf{D^{(m)}} \otimes \cdots \otimes \mathbf{D^{(m)}}}_{q^m \;\; times}. \qquad (103)$$

Example: SU(3)

- For SU(3) there are *two* fundamental weights, $\Leftarrow$ there are two simple roots α^1, α^2:

$$\begin{aligned} \alpha^1 &= (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) &\Rightarrow\quad \mu^1 &= (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}), \\ \alpha^2 &= (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}) &\Rightarrow\quad \mu^2 &= (\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}). \end{aligned} \tag{104}$$

  1. The first corresponds to the basic representation $T_a = \frac{1}{2}\lambda_a$ given in (92), which acts on $\mathcal{H}$ spanned by the triplet of states e_1, e_2, e_3 (94).
  2. The second one corresponds to another fundamental representation, which from the point of view of group theory is equally "fundamental" as the first one.

- In view of its application to quark model let's call the first multiplet **quark** triplet and denote it as $\mathbf{3} \equiv (1,0)$.
- It will turn out that the second triplet $(0,1)$ can be identified with **antiquarks** and so let's call it **antitriplet** and denote $\overline{\mathbf{3}} \equiv (0,1)$.
- Both of these triplets are displayed in Fig. 2b.

Example: SU(3) - Explicit construction of $\overline{\mathbf{3}} \equiv (0,1)$

- Starting from the state of highest weight, i.e. the point with coordinates $(h_1, h_2) = \mu^2 = (1/2, -1/2\sqrt{3})$ and taking into account that for this weight $q^1 = 0$, only the application of $E_{-\alpha^2}$ leads to non-vanishing result, i.e. the point $(0, 1/\sqrt{3})$.
- Further application of $E_{-\alpha^2}$ leads to zero as $q^2 = 1$ and we are thus forced to apply $E_{-\alpha^1}$ which brings us to the leftmost point $(-1/2, -1/2\sqrt{3})$, where the procedure finally stops. One could attempt to apply E_{α^1} in the second stage instead of $E_{-\alpha^1}$, but this leads to zero due the fact that

$$E_{\alpha^1} E_{-\alpha^2} \mid \mu^2\rangle = E_{-\alpha^2} \underbrace{E_{\alpha^1} \mid \mu^2\rangle}_{0} + \underbrace{[E_{\alpha^1}, E_{-\alpha^2}]}_{0} \mid \mu^2\rangle = 0 \qquad (105)$$

where the first zero is obvious and the second one uses the basic property of simple roots as given in **Corollary 2.2**.

General strategy for constructing any multiplet from its highest weight μ

1. Apply to $\mid \mu, \mathbf{D}\rangle$ all possible combinations of lowering operators corresponding to **simple** roots α^i:
$$\prod_{i=1}^{k} E_{-\alpha^i} \mid \mu, \mathbf{D}\rangle.$$
2. At each step check if application of $E_{-\alpha^i}$ on the reached state $\mid \nu\rangle$ is "legal", i.e. whether α^i and ν satisfy the condition
$$\frac{2\alpha^i \nu}{(\alpha^i)^2} = q^i - p^i.$$
3. For weights, which can be reached via different paths determine whether they correspond to different states or not.

To solve this last problem the following lemma can be useful

Proposition

Let $\mid A\rangle, \mid B\rangle \in \mathcal{H}$ be two states from a Hilbert space $\mathcal{H}$. They are linearly dependent **if**
$$\frac{\langle A \mid B\rangle\langle B \mid A\rangle}{\langle A \mid A\rangle\langle B \mid B\rangle} = 1. \tag{106}$$

Weyl group of symmetries of multiplets

- Another useful tool for the construction of multiplets.
- This group exploits the existence in a compact Lie algebra of the SU(2) subalgebras spanned by the generators

$$\frac{\alpha_i H_i}{\alpha^2}, \quad \frac{E_{\pm\alpha}}{\sqrt{\alpha^2}}, \qquad (107)$$

  associated with any root α (not necessarily simple).
- Taking one of these subgroups and applying powers of $E_{\pm\alpha}$ successively to any weight μ leads in final effect to the "reflected" state, characterized by the weight

$$\mu^r = \mu - \frac{2(\alpha \cdot \mu)}{\sqrt{\alpha^2}} \frac{\alpha}{\sqrt{\alpha^2}}, \qquad (108)$$

  which reduces to trivial symmetry operation $-k = k - 2k$ for SU(2) group.

Weyl group of symmetries of multiplets

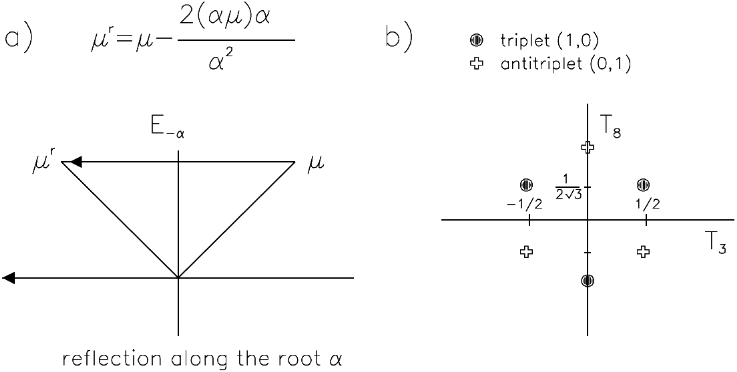

Figure: a) Weyl symmetry of weights, b) fundamental representations of SU(3)

The existence of the second fundamental representation of SU(3) $\overline{\mathbf{3}}$ and the relation of its weights to those of the defining triplet **3**

$$\begin{aligned} \mu^2 &= -(\mu^1 - \alpha^1 - \alpha^2) \\ (\mu^2 - \alpha^2) &= -(\mu^1 - \alpha^1) \\ \mu^2 - \alpha^2 - \alpha^1 &= -\mu^1 \end{aligned} \tag{109}$$

is an example of **complex representation**.

Weyl group of symmetries of multiplets

Complex representations: if the matrices T_a of a representation $\mathbf{D}$ satisfy $[T_a, T_b] = if_{abc}T_c$ (see eq.(9)) so do also the matrices

$$-(T_a)^*,$$

which form the so called *complex conjugate* representation $\overline{\mathbf{D}}$. This relation is *reflexive* as we have:

$$-\left(-(T_a)^*\right)^* = T_a.$$

Definition

The representation of a Lie algebra given by matrices T_a is said to be **real** *if it is equivalent to the complex conjugate representation of matrices $-T_a^*$.*

Corollary

As the elements of Cartan subalgebra are hermitian, their eigenvalues are the same for $\mathbf{D}$ *and* $\overline{\mathbf{D}}$ *and the weights related simply as* $\mu^{\mathbf{D}} = -\mu^{\overline{\mathbf{D}}}$.

- SU(2): As we saw already in (60) there is just one fundamental representation, which is *real*, as the Pauli matrices σ_i are equivalent to the negative of their complex conjugates.
- SU(3): The two fundamental representations are **not** equivalent, the fact that has important consequences in the quark model.

- There are several other multiplets of SU(3) which had played important role in the formulation and development of unitary symmetry and the quark model. We will discuss three of them in some detail and briefly mention several other.

Further SU(3) multiplets

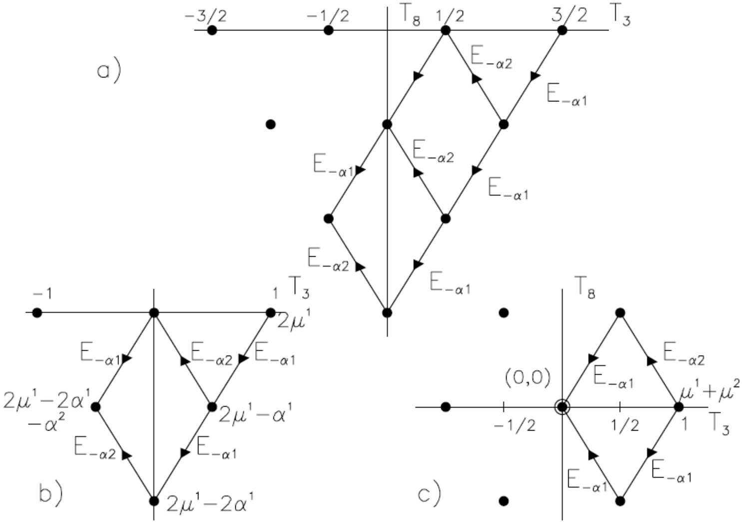

Figure: Further SU(3) multiplets; a) decuplet; b) sextet; c) octet.

Sextet

- The next simplest multiplet $\mathbf{6} \equiv (2,0)$ has the highest weight $2\mu^1 = (1, 1/\sqrt{3})$. All of its 6 states depicted in Fig. 3b are *unique*.
- This fact is not trivial, as there are two states, namely those with weights $(-1/2, -1/2\sqrt{3})$ and $(-1, 1/\sqrt{3})$, which can be reached via two different paths, as indicated in the figure. It is, however, easy to show, using the commutation relations between the lowering operators $E_{-\alpha^1}$ and $E_{-\alpha^2}$ that

$$E_{-\alpha^2} E_{-\alpha^1} E_{-\alpha^1} \mid 2\mu^1\rangle = 2E_{-\alpha^1} E_{-\alpha^2} E_{-\alpha^1} \mid 2\mu^1\rangle, \qquad (110)$$

  i.e. that the resulting states *are the same*.
- The same conclusion can be reached by means of (106). As the highest weight of this sextet is *symmetric* under the permutation of the two u-quarks which make it up, the whole multiplet consists of symmetrical combinations of the u, d, s quarks. There are 6 such symmetric combinations, the remaining three being *antisymmetric*.

Sextet

- Follow the action of the lowering operators on such quark combinations:

$$
\begin{array}{lll}
\mid uu\rangle & & \\
\downarrow E_{-\alpha^1} & & \\
N_{-\alpha^1,\mu^1} \mid us+su\rangle & \underrightarrow{E_{-\alpha^2}} & N_{-\alpha^1,\mu^1} N_{-\alpha^2,\mu^1-\alpha^1} \mid ud+du\rangle \\
E_{-\alpha^1} & & \downarrow E_{-\alpha^1} \\
\downarrow & & (N_{-\alpha^1,\mu^1})^2 N_{-\alpha^2,\mu^1-\alpha^1} \mid sd+ds\rangle \\
2(N_{-\alpha^1,\mu^1})^2 \mid ss\rangle & \underrightarrow{E_{-\alpha^2}} & 2(N_{-\alpha^1,\mu^1})^2 N_{-\alpha^2,\mu^1-\alpha^1} \mid sd+ds\rangle.
\end{array} \tag{111}
$$

- The same can be done for 3 antisymmetric combinations. Starting from $\mid us - su\rangle$ we find that:

$$
E_{-\alpha^1} E_{-\alpha^2} \mid us-su\rangle = E_{-\alpha^1} N_{-\alpha^2,\mu^1-\alpha^1} \mid ud-du\rangle = N_{-\alpha^2,\mu^1-\alpha^1} N_{-\alpha^1,\mu^1} \mid sd-ds\rangle \tag{112}
$$

with further application of either of E_α yielding zero.

- The above explicit construction shows that:

$$
\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}; \quad \overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \overline{\mathbf{6}} \oplus \mathbf{3}. \tag{113}
$$

- N.B. the weights corresponding to these combinations coincide with those of the antitriplet of antiquarks. $\Rightarrow$ The *antisymmetric* diquark combinations behave, from the point of view of SU(3)transformations (but not as far as the flavour quantum numbers are concern), as *antiquarks*.

Octet

- Denoted as $\mathbf{8} \equiv (1,1)$ is displayed in Fig. 3c.
- It had played the very central role in the discovery of SU(3) symmetry of hadrons and the subsequent formulation of the quark model.
- Also here are the states which can be reached from this μ in more then one way. Out of them only the one in the center of the octet is *not unique*. One can proceed exactly as in the previous case to show that the states with the weight $(0,0)$ arrived at by two different paths from that with highest weight $\mu = \mu^1 + \mu^2$ are different. The same can be proven also by means of the lemma (106) by showing that for the states:

$$\mid A\rangle \equiv E_{-\alpha^1} E_{-\alpha^2} \mid \mu\rangle; \;\; \mid B\rangle \equiv E_{-\alpha^2} E_{-\alpha^1} \mid \mu\rangle$$

one finds that:

$$\frac{\langle A \mid A\rangle\langle B \mid B\rangle}{\langle A \mid B\rangle\langle B \mid A\rangle} = 4 \quad \Rightarrow \mid A\rangle \neq \mid B\rangle. \tag{114}$$

Octet

- In the following we shall frequently need the fact that octet $\mathbf{8}$=$(1,1)$ appears in the decomposition of the direct product of the fundamental triplet and antitriplet as well as of three triplets

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{115}$$
$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_{\rm s} \oplus \mathbf{8}_{\rm ms} \oplus \mathbf{8}_{\rm ma} \oplus \mathbf{1}_{\rm a}, \tag{116}$$

where the subscripts “ms” (mixed symmetric) and “ma” (mixed antisymmetric) denote similarly as in the case of SU(2) in (42), two equivalent octet representations, which differ in their symmetry properties under permutation of the three triplets. Analogously for the product of antitriplets $\overline{\mathbf{3}}$.

Decuplet

The last of multiplets which played important role in the formulation of the quark model is the **decuplet** $\mathbf{10} \equiv (3,0)$ with the highest weight $\mu = 3\mu^1$, shown in Fig. 3a, which corresponds to fully symmetric combinations of three quarks in the decomposition (116). One can again ask about the uniqueness of its states and the answer is the same as for the sextet: *all are unique*.

Other products and multiplets

- Other multiplets that had played role in the discovery of unitary symmetry of hadrons and formulation of the quark model are those obtained in the reduction of the following direct products of two multiplets

$$\mathbf{3} \otimes \mathbf{6} = \mathbf{10} \oplus \mathbf{8}, \quad \mathbf{3} \otimes \mathbf{8} = \mathbf{15} \oplus \overline{\mathbf{6}} \oplus \mathbf{3}, \quad \overline{\mathbf{3}} \otimes \mathbf{6} = \mathbf{15} \oplus \mathbf{3}, \tag{117}$$

  and three triplets and/or antitriplets

$$\mathbf{3} \otimes \mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{3} \otimes (\mathbf{8} \oplus \mathbf{1}) = (\mathbf{3} \otimes \mathbf{8}) \oplus \mathbf{3} = \mathbf{15} \oplus \overline{\mathbf{6}} \oplus \mathbf{3} \oplus \mathbf{3} \tag{118}$$

- The above, as well as any other, direct products of the SU(3) multiplets can be reduced using the method of the highest weight.

Other products and multiplets

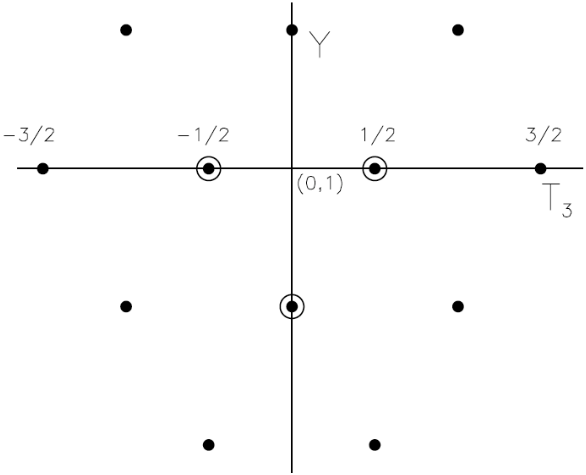

Figure: The 15-plet. Three inner three weights correspond to two different states.

Other products and multiplets

- The resulting decompositions may be quite complicated, but there is a method based on the so called **Young tableaux**, which allows fast determination of the results. It is a diagram associated with a given multiplet $\mathbf{D} = (p, q)$ describing its symmetry properties under the permutations of the fundamental SU(3) triplets and anti-triplets.
- The 15-plet, shown in Figure 4 and resulting from the decomposition (118) of the direct product of two triplets and one antitriplet, was used in the Sakata model, a predecessor of the so called *Eightfold way* (see the next Section for discussion of both schemes). Part of this multiplet has weights that coincide with those of the decuplet. The inner three weights correspond to two states.
- One of the most important characteristics of any SU(3) multiplet $\mathbf{D} = (p, q)$ is its dimension $D(p, q)$. This can also be read off the corresponding Young tableau with the result

$$D(p, q) = \frac{1}{2}(p+1)(q+1)(p+q+2). \tag{119}$$

Combining three fundamental isospin representations

- Let

$$| \psi \rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{120}$$

be SU(3) triplet.

- Let $\psi_i^* \equiv \psi^i$ and write SU(3) invariant trace:

$$\psi^i \psi_i = u^+ u + d^+ d + s^+ s \tag{121}$$

is an SU(3) invariant trace. The remaining 8 components transform as the octet representatiojn of SU(3):

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \tag{122}$$

- Following the same procedure as for SU(2) we can write the adjoint representation of SU(3) as:

$$A^i_j = \psi^i \psi_j - \frac{1}{3} \delta^i_j (\psi^k \psi_k) \tag{123}$$

Combining two fundamental isospin representations

- The A^i_j components can be now written explicitly:

$$A^1_2 = u^+ d \sim \pi^-, \qquad A^2_1 = d^+ u \sim \pi^+ \tag{124}$$

$$A^1_3 = u^+ s \sim K^-, \qquad A^3_1 = s^+ u \sim K^+ \tag{125}$$

$$A^3_2 = s^+ d \sim K^0, \qquad A^2_3 = d^+ s \sim \bar{K}^+ \tag{126}$$

and for the diagonal elements:

$$A^1_1 = u^+ u - \frac{1}{3}(u^+ u + d^+ d + s^+ s) \sim \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} \tag{127}$$

where

$$\pi^0 = \frac{1}{\sqrt{2}}(u^+ u - d^+ d), \quad \eta^0 = \frac{1}{\sqrt{6}}(u^+ u - d^+ d - 2s^+ s) \tag{128}$$

Similarly

$$A^2_3 \sim -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} \quad \text{and} \quad A^3_3 \sim -\frac{2\eta^0}{\sqrt{6}} \tag{129}$$

Combining two fundamental isospin representations

- These components can be put into traceless hermitian matrix:

$$\hat{A} =$$

$$\left(\begin{array}{ccc} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{array} \right) = \left(\begin{array}{ccc} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^+ & -\frac{2\eta^0}{\sqrt{6}} \end{array} \right) \quad (130)$$

- Since fundamental (defining) representation transforms as:

$$\psi_i \to \psi_i^{'} = U_i^j \psi_j, \quad \psi^i \to \psi i' = \psi_j U_j^i \quad (131)$$

- the adjoint representation transform as:

$$A_j^i \to A_j^{'i} U_k^i U_j^i A_j^k = (U_{ik})^* A_i^k (U) ij), \quad (132)$$

  or, in terms of matrix multiplication:

$$\hat{A} \to \hat{A}' = \hat{U}^{+} \hat{A} \hat{U} \quad (133)$$

- N.B. The matrix U is the defining representation of the SU(3) group.

Exercises

1. Evaluate Clebsh-Gordan coefficients of the direct product of SU(2) multiplets $\mathbf{1} \otimes \mathbf{1/2}$.
2. Prove (84).
3. Find the matrices λ_3, which correspond to the third projection of the U and V-spins. How do the matrices α_8 look like in these two cases?
4. Argue why only the simple roots are needed to reach from the state with the highest weight all the states of a given multiplet.
5. Prove the lemma (106).
6. Show that the fundamental representation of SU(2) is real.
7. Prove (114).

Exercises

8. Show that if we associate the basic doublet of SU(2) with $\begin{pmatrix} u \\ d \end{pmatrix}$ the pair of antiquarks which transforms according to identical representation is
$$\begin{pmatrix} \overline{d} \\ -\overline{u} \end{pmatrix}.$$
9. Prove that the elements of the Cartan subalgebra can be chosen as hermitian operators and the roots as real vectors.
10. Show that the three operators defined in (107) do, indeed, satisfy the SU(2) commutation relations (31).
11. Prove the uniqueness of all states in the sextet **6**=(2,0) representation of SU(3).
12. Show that the two states in the center of the octet of SU(3) are different.
13. Derive (116).