Pohybové rovnice (LR1D) pro neholonomní soustavu s vazbami lineárně závislými na rychlostech

vazby: r–holonomních (nezávislých) p–neholonomních (nezávislých)

$f_k(\vec x,t)=0\ \ \forall k\in\hat r \qquad h(\underbrace{\nabla f_1,\dots,\nabla f_r}_{\text{L.N.}})=r\ \ \forall \vec x\in M(t)$

$0=\delta f_k=\frac{\partial f_k}{\partial x_i}\delta x_i=\nabla f_k\cdot\delta\vec x$

$\sum_{i=1}^{3N} a_{li}(\vec x,t)\dot x_i+b_l(\vec x,t)=0\ \ \forall l\in\hat p \qquad \mathbb A=(a_{li})$ matice typu $p\times 3N$

$\mathbb A(\vec x,t)\dot{\vec x}+\vec b(\vec x,t)=0\ \ /\cdot dt \qquad h(\mathbb A(\vec x,t))=p\ \ \forall\vec x\in M(t)\ \forall t$

$\mathbb A(\vec x,t)\,d\vec x+\vec b(\vec x,t)\,dt=0 \leftarrow \dot{\vec x}\,dt=d\vec x\ \ /\ dt=0\Rightarrow d\vec x\to\delta\vec x$

$\mathbb A(\vec x,t)\,\delta\vec x=0 \qquad \sum_{i=1}^{3N} a_{li}(\vec x,t)\,\delta x_i=0\ \ \forall l\in\hat p$

d'Alembertův princip

$$\delta A_{ef}=\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\cdot\delta x_i=(\vec F-\mathbb M\ddot{\vec x})\cdot\delta\vec x=0 \quad (1)$$

$$\left.\begin{array}{l} f_k(\vec x,t)=0\ \ \forall k\in\hat r \\ \mathbb A(\vec x,t)\dot{\vec x}+\vec b(\vec x,t)=0\end{array}\right\}(3) \qquad \forall\delta\vec x\in\mathbb R^{3N}\ \left.\begin{array}{l}\nabla f_k\cdot\delta\vec x=0\\ \mathbb A\,\delta\vec x=0\end{array}\right\}(2)$$

$\forall\delta\vec x\in T_{\vec x}M(t)\cap\operatorname{Ker}\mathbb A(\vec x,t)=\operatorname{Ker}\mathbb B(\vec x,t)$

$\mathbb B(\vec x,t)$ matice $(r+p)\times 3N$ hodnosti $r+p$

BÚNO: regulární matice typu $(r+p)\times(r+p)$

$s=3N-r$

Metoda Lagrangeových multiplikátorů:
Místo vyjadření závislých posunutí složek δx_j pomocí $s-p$ nezávislých z podmínek (2) a jejich dosazení do rovnice (1) přenásobíme rovnice (2) Lagrangeovými multiplikátory λ_j,μ_j a přičteme k rovnici (1) a nastavíme λ_j,μ_j tak abychom vynulovali koeficienty u závislých δx_j

$\lambda_j,\mu_j \leftarrow$ funkce$(\vec x,\dot{\vec x},\ddot{\vec x},t)$

$$\underbrace{(\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_p)}_{\text{nastavíme tak aby}}\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{s-p+1}} & \cdots & \frac{\partial f_1}{\partial x_{3N}}\\ \vdots & & \vdots & & \vdots\\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_{s-p+1}} & \cdots & \frac{\partial f_r}{\partial x_{3N}}\\ a_{1,1} & \cdots & a_{1,s-p+1} & \cdots & a_{1,3N}\\ \vdots & & \vdots & & \vdots\\ a_{p,1} & \cdots & a_{p,s-p+1} & \cdots & a_{p,3N}\end{pmatrix}\begin{pmatrix}\delta x_1\\ \vdots\\ \delta x_{s-p}\\ \delta x_{s-p+1}\\ \vdots\\ \delta x_{3N}\end{pmatrix}\begin{array}{l}\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}\text{nezávislé}\\ \left.\begin{array}{l}\\ \\ \\ \end{array}\right\}\text{závislé}\end{array}=0$$

$$\overset{*}{=}\ (-F_1^{ef},\dots,-F_{s-p+1}^{ef},\dots,-F_{3N}^{ef})$$

$$0=\sum_{i=1}^{3N}\Big(F_i-m_i\ddot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\Big)\delta x_i=\sum_{i=1}^{s-p}\underbrace{\Big(F_i-m_i\ddot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\Big)}_{0''}\delta x_i+\sum_{i=s-p+1}^{3N}\underbrace{\Big(F_i-m_i\ddot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\Big)}_{0''}\delta x_i$$

$\delta x_1,\dots,\delta x_{3N}$ nejsou nezávislé $\quad 0'' \Leftarrow \delta x_1,\dots,\delta x_{s-p}$ jsou nezávislé $\quad 0''$ vynulujeme volbou λ_k a μ_l

Pohybové rovnice: $m_i\ddot x_i=F_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\ \ \forall i\in\widehat{3N} \qquad f_k(\vec x,t)=0\ \forall k\in\hat r \qquad \sum_{j=1}^{3N}a_{lj}(\vec x,t)\dot x_j+b_l(\vec x,t)=0\ \ \forall l\in\hat p$

Věta o energii: $\ /\dot x_i\ \ /\sum_{i=1}^{3N}$ $\qquad \frac{\hat d f_k}{dt}=\frac{\partial f_k}{\partial x_i}\dot x_i+\frac{\partial f_k}{\partial t}=0$ $\qquad$ pro konzervativní síly $F_i\dot x_i=-\frac{\partial U(\vec x)}{\partial x_i}\dot x_i=-\frac{\hat d U(\vec x)}{dt}$

$$m_i\ddot x_i\dot x_i=F_i\dot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}\dot x_i+\sum_{l=1}^{p}\mu_l a_{li}\dot x_i\ \Rightarrow\ \frac{\hat d T}{dt}=\vec F\cdot\dot{\vec x}-\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial t}-\sum_{l=1}^{p}\mu_l b_l\ \Rightarrow\ \frac{\hat d}{dt}(T+U)=-\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial t}-\sum_{l=1}^{p}\mu_l b_l$$

Odvození LR2D pro holonomní soustavu: $\ x_i=\hat x_i(\vec q,t) \qquad \delta x_i=\delta\hat x_i=\frac{\partial\hat x_i}{\partial q_j}\delta q_j$

$$0=\delta A_{ef}=\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\delta x_i=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[F_i-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\Big)\Big]\frac{\partial\hat x_i}{\partial q_j}\delta q_j=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[F_i\frac{\partial\hat x_i}{\partial q_j}-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\Big)\frac{\partial\hat x_i}{\partial q_j}\Big]\delta q_j=$$

$$=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[F_i\frac{\partial\hat x_i}{\partial q_j}-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\frac{\partial\hat x_i}{\partial q_j}\Big)+\frac{\partial T}{\partial\dot x_i}\frac{\hat d}{dt}\Big(\frac{\partial\hat x_i}{\partial q_j}\Big)\Big]\delta q_j=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[Q_j-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\frac{\partial\dot{\hat x}_i}{\partial\dot q_j}\Big)+\frac{\partial T}{\partial\dot x_i}\Big(\frac{\partial\dot{\hat x}_i}{\partial q_j}\Big)\Big]\delta q_j=\sum_{j=1}^{s}\Big[Q_j-\frac{\hat d}{dt}\Big(\frac{\partial\hat T}{\partial\dot q_j}\Big)+\frac{\partial\hat T}{\partial q_j}\Big]\underbrace{\delta q_j}_{\text{jsou nezávislé}}$$

$\frac{\hat d}{dt}\Big(\frac{\partial\hat T}{\partial\dot q_j}\Big)-\frac{\partial\hat T}{\partial q_j}=Q_j\ \ \forall j\in\hat s \qquad s=3N-r$

$$Q_j=Q_j^{(nep)}+Q_j^{(pot)}=Q_j^{(nep)}+\Big[-\frac{\partial U}{\partial x_i}+\frac{\hat d}{dt}\Big(\frac{\partial U}{\partial\dot x_i}\Big)\Big]\frac{\partial\hat x_i}{\partial q_j}=Q_j^{(nep)}-\frac{\partial U}{\partial x_i}\frac{\partial\hat x_i}{\partial q_j}+\frac{\hat d}{dt}\Big(\frac{\partial U}{\partial\dot x_i}\frac{\partial\hat x_i}{\partial q_j}\Big)-\frac{\partial U}{\partial\dot x_i}\frac{\hat d}{dt}\Big(\frac{\partial\hat x_i}{\partial q_j}\Big)=$$

$$=Q_j^{(nep)}-\frac{\partial U}{\partial x_i}\frac{\partial\hat x_i}{\partial q_j}-\frac{\partial U}{\partial\dot x_i}\frac{\hat d}{dt}\Big(\frac{\partial\hat x_i}{\partial q_j}\Big)+\frac{\hat d}{dt}\Big(\frac{\partial U}{\partial\dot x_i}\frac{\partial\dot{\hat x}_i}{\partial\dot q_j}\Big)=Q_j^{(nep)}-\frac{\partial\hat U}{\partial q_j}+\frac{\hat d}{dt}\Big(\frac{\partial\hat U}{\partial\dot q_j}\Big)$$

$\frac{\hat d}{dt}\Big(\frac{\partial\hat L}{\partial\dot q_j}\Big)-\frac{\partial\hat L}{\partial q_j}=Q_j^{(nep)} \leftarrow \hat L=\hat T-\hat U$

Pozn: Záměnnost variace a derivace $\frac{\hat d}{dt}(\delta\vec x)=\frac{\hat d}{dt}(\vec x'-\vec x)=\dot{\vec x}'-\dot{\vec x}=\delta\dot{\vec x}=\delta\Big(\frac{\hat d\vec x}{dt}\Big)$ platí i pro funkce $\frac{\hat d}{dt}\delta f=\delta\frac{\hat d f}{dt}$

Jourdainův princip $\qquad 0=\frac{\hat d}{dt}\delta A_{ef}=\sum_{i=1}^{3N}\frac{\hat d}{dt}(F_i-m_i\ddot x_i)\underbrace{\delta x_i}_{\text{zvolíme }=0}+\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\delta\dot x_i \qquad \boxed{\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\cdot\delta_1\dot x_i=0}$

Jourdainovy variace $\delta_1 t=0\ \ \delta_1 x_i=0$

Gaussův princip $\qquad 0=\sum_{i=1}^{3N}\frac{\hat d}{dt}(F_i-m_i\ddot x_i)\underbrace{\delta_2\dot x_i}_{\text{zvolíme }=0}+\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\delta_2\ddot x_i \qquad \boxed{\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\cdot\delta_2\ddot x_i=0}$

Gaussovy variace $\delta_2 t=0\ \ \delta_2 x_i=0\ \ \delta_2\dot x_i=0$

Ústřední rovnice Lagrangeova — jiný tvar d'Alembertova principu

$$0 = \delta A_{ef} = \sum_{i=1}^{3N}(F_i - m_i\ddot x_i)\delta x_i = \underbrace{\sum_{i=1}^{3N} F_i\,\delta x_i}_{\delta A \text{ virtuální práce akčních sil}} - \sum_{i=1}^{3N} m_i \ddot x_i\, \delta x_i = \delta A - \frac{\hat d}{dt}\Big(\sum_{i=1}^{3N} \underbrace{m_i \dot x_i}_{\frac{\partial T}{\partial \dot x_i}}\delta x_i\Big) + \delta\Big(\underbrace{\sum_{i=1}^{3N}\tfrac12 m_i\dot x_i^2}_{T}\Big) = \delta A - \frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\delta x_i\Big) + \delta T$$

$$\ddot x_i\,\delta x_i = \frac{\hat d}{dt}(\dot x_i\,\delta x_i) - \dot x_i \frac{\hat d}{dt}(\delta x_i) = \frac{\hat d}{dt}(\dot x_i\, \delta x_i) - \dot x_i\,\delta\dot x_i = \frac{\hat d}{dt}(\dot x_i\,\delta x_i) - \frac12\delta\dot x_i^2 \qquad \boxed{\delta T + \delta A = \frac{\hat d}{dt}\left(\frac{\partial T}{\partial \dot x_i}\delta x_i\right)}$$

v obecných souřadnicích: $x_i = \hat x_i(\vec q, t)$ $\quad \delta x_i = \delta\hat x_i = \frac{\partial \hat x_i}{\partial q_j}\delta q_j$ $\quad \hat T(\vec q,\dot{\vec q},t) = T(\dot{\hat{\vec x}}(\vec q,\dot{\vec q},t))$

$$\frac{\partial T}{\partial \dot x_i}\delta x_i = \frac{\partial T}{\partial\dot x_i}\frac{\partial \hat x_i}{\partial q_j}\delta q_j = \frac{\partial T}{\partial \dot x_i}\frac{\partial\dot{\hat x}_i}{\partial\dot q_j}\delta q_j = \frac{\partial\hat T}{\partial \dot q_j}\delta q_j \qquad \delta A = F_i\,\delta x_i = F_i\frac{\partial\hat x_i}{\partial q_j}\delta q_j = Q_j\,\delta q_j = \delta\hat A \qquad \boxed{\delta\hat T + \delta\hat A = \frac{\hat d}{dt}\left(\frac{\partial\hat T}{\partial\dot q_j}\delta q_j\right)}$$

$$\delta\hat A = Q_j^{(nep)}\delta q_j + Q_j^{(pot)}\delta q_j = Q_j^{(nep)}\delta q_j + \left(\frac{\hat d}{dt}\left(\frac{\partial\hat U}{\partial\dot q_j}\right) - \frac{\partial\hat U}{\partial q_j}\right)\delta q_j = Q_j^{(nep)}\delta q_j + \frac{\hat d}{dt}\left(\frac{\partial\hat U}{\partial\dot q_j}\delta q_j\right) - \frac{\partial \hat U}{\partial\dot q_j}\delta\dot q_j - \frac{\partial\hat U}{\partial q_j}\delta q_j = Q_j^{(nep)}\delta q_j + \frac{\hat d}{dt}\left(\frac{\partial\hat U}{\partial\dot q_j}\delta q_j\right) - \delta\hat U$$

$$\delta\hat T - \delta\hat U + Q_j^{(nep)}\delta q_j = \frac{\hat d}{dt}\left(\frac{\partial(\hat T-\hat U)}{\partial\dot q_j}\delta q_j\right) \qquad \boxed{\delta\hat L + Q_j^{(nep)}\delta q_j = \frac{\hat d}{dt}\left(\frac{\partial \hat L}{\partial\dot q_j}\delta q_j\right)}$$

Pozn: diferenciální počet

Funkce $\quad$ přírůstek funkce $\quad$ lineární část přírůstku = diferenciál funkce $\quad$ derivace $\quad$ stacionární hodnota

$f:\mathbb R\to\mathbb R \qquad \Delta f = f(x+\Delta x) - f(x) = \overbrace{A(x)}\Delta x + \underbrace{\omega(x,\Delta x)}_{\downarrow \text{ pro } \Delta x\to 0 \atop 0}\cdot\Delta x \qquad df = A(x)\Delta x = f'(x)dx \qquad f'(x) = \frac{df}{dx} = A(x) \qquad f'(x_0) = 0$

$f(x)=x \quad dx = \Delta x$

$f:\mathbb R^m\to\mathbb R \qquad \Delta f = f(\vec x+\Delta\vec x) - f(\vec x) = \sum_{j=1}^m A_j(\vec x)\Delta x_j + \underbrace{\omega(\vec x,\Delta\vec x)}_{\downarrow \text{ pro } |\Delta\vec x|\to 0 \atop 0}\cdot|\Delta\vec x| \qquad df = \underbrace{A(\vec x)\Delta\vec x}_{\nabla f(\vec x)\cdot d\vec x} = \sum_{j=1}^m\frac{\partial f(\vec x)}{\partial x_j}dx_j \qquad f'(\vec x) = A(\vec x)\in\mathcal L(\mathbb R^m,\mathbb R) \qquad A_j(\vec x) = \frac{\partial f}{\partial x_j} = \partial_j f = f_j \qquad \frac{\partial f(\vec x_0)}{\partial x_j} = 0 \quad \forall j\in\hat m$

pokud existuje, pak $df = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} f(\vec x + \varepsilon\, d\vec x)$

Variační počet $\quad$ body → křivky $\quad$ funkce → funkcionály

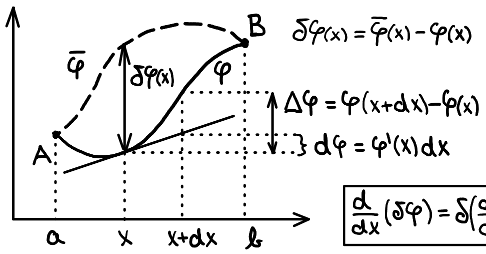
Křivka (třídy $n\in\mathbb N_0$) je spojité zobrazení $\varphi:\langle a,b\rangle\subset\mathbb R\to\mathbb R$ (třídy $C^{(n)}$ tj. má spojité derivace do řádu n).

$C^{(n)}\langle a,b\rangle$ mn. všech křivek třídy n tvoří vektorový prostor dim $+\infty$

s normou $\|\varphi\| = \max_{x\in\langle a,b\rangle}\{|\varphi(x)|,|\varphi'(x)|,\dots,|\varphi^{(n)}(x)|\}$ který označíme $\vec C^{(n)}\langle a,b\rangle$

$C^{(n)}_{(A,B)}\langle a,b\rangle = \{\varphi\in C^{(n)}\langle a,b\rangle \mid \varphi(a)=A \wedge \varphi(b)=B\}$ mn. křivek z A do B

$(C^{(n)}_{(A,B)}\langle a,b\rangle, -, \vec C^{(n)}_{(0,0)}\langle a,b\rangle)$ normovaný afinní prostor



$\boxed{\frac{d}{dx}(\delta\varphi) = \delta\left(\frac{d\varphi}{dx}\right)}$

Buď $\varphi\in C^{(n)}_{(A,B)}\langle a,b\rangle$ pak pro lib. $\bar\varphi\in C^{(n)}_{(A,B)}\langle a,b\rangle$ nazýváme $\delta\varphi = \bar\varphi-\varphi\in\vec C^{(n)}_{(0,0)}\langle a,b\rangle$ variací křivky φ s pevnými konci.

$\qquad\qquad\qquad\qquad\qquad\qquad C^{(n)}\langle a,b\rangle \qquad\qquad\qquad\qquad \vec C^{(n)}\langle a,b\rangle$ $\qquad$ s volnými konci.

Funkcionál je zobrazení z prostoru (např. vektorového) do reálných čísel.

Funkcionál $I: C^{(n)}\langle a,b\rangle\to\mathbb R$ je spojitý na křivce φ, pokud $\forall\varepsilon>0\ \exists\delta>0\ \forall\bar\varphi\in C^{(n)}\langle a,b\rangle \quad \|\bar\varphi-\varphi\|<\delta \Rightarrow |I(\bar\varphi) - I(\varphi)|<\varepsilon$

Funkcionál $I: C^{(n)}\langle a,b\rangle\to\mathbb R$ je diferencovatelný na křivce φ pokud existuje spojitý lineární funkcionál $\Phi: \vec C^{(n)}\langle a,b\rangle\to\mathbb R$ (nazývaný <u>variace I na křivce</u> φ značený $\delta I(\varphi)$) tak, že platí $\lim_{\|\delta\varphi\|\to 0}\frac{I(\varphi+\delta\varphi) - I(\varphi) - \Phi[\delta\varphi]}{\|\delta\varphi\|} = 0$.

tj. $\quad \Delta I = I(\varphi+\delta\varphi) - I(\varphi) = \underbrace{\Phi[\delta\varphi]}_{\text{lineární část přírustku } \delta I(\varphi)[\delta\varphi]} + \underbrace{\omega(\varphi,\delta\varphi)}_{\downarrow \text{ pro } \|\delta\varphi\|\to 0 \atop 0}\cdot\|\delta\varphi\|$

Pozn: existuje-li variace, lze ji najít takto $\eta\in\vec C^{(n)}\langle a,b\rangle \quad \hat I(\varepsilon) = I(\varphi+\overbrace{\varepsilon\eta}^{\delta\varphi}) \quad \forall\varepsilon\in\mathbb R \qquad \delta I(\varphi)[\delta\varphi] = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} I(\varphi+\varepsilon\,\delta\varphi)$

$$\frac{d\hat I}{d\varepsilon}\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac1\varepsilon\big(I(\varphi+\varepsilon\eta) - I(\varphi)\big) = \lim_{\varepsilon\to 0}\frac1\varepsilon\big(\Phi[\varepsilon\eta] + \omega(\varphi,\varepsilon\eta)\cdot\|\varepsilon\eta\|\big) = \lim_{\varepsilon\to0}\big(\Phi[\eta] + \mathrm{sgn}(\varepsilon)\,\omega(\varphi,\varepsilon\eta)\cdot\|\eta\|\big) = \Phi(\eta) = \delta I(\varphi)[\eta]$$

Funkcionál $I: C^{(n)}\langle a,b\rangle\to\mathbb R$ má na křivce φ – maximum (minimum) pokud $\forall\bar\varphi\in C^{(n)}\langle a,b\rangle \quad I(\varphi)\geq I(\bar\varphi) \quad (I(\varphi)\leq I(\bar\varphi))$

– stacionární hodnotu (φ je extremálou I) pokud $\delta I(\varphi) = 0$