51. Ukažte, že pro rheonomní vazby $f_k(\vec{x},t)=0$, $k\in\hat{r}$ platí: $\frac{\partial f_k}{\partial x_i}\frac{\partial \hat{x}_i}{\partial t}+\frac{\partial f_k}{\partial t}=0$ $\qquad 0=\frac{d f}{dt}$ $\Downarrow$

$x_i=\hat{x}_i(\vec{q},t)$

takaly splnily vazby $\rightarrow$ $\hat{f}_k(\vec{q},t)=f_k(\vec{x}(\vec{q},t),t)\equiv 0\ \forall\vec{q}\,\forall t$

$$0=\frac{\partial \hat{f}_k}{\partial t}=\frac{\partial f_k}{\partial x_i}\frac{\partial \hat{x}_i}{\partial t}+\frac{\partial f_k}{\partial t}\ \checkmark \rightarrow -\lambda\frac{\partial f_k}{\partial t}=\underbrace{\frac{\partial f_k}{\partial x_i}\dot{x}_i\lambda}_{\text{výkon vazebné síly}}$$

$$0=\frac{\partial \hat{f}_k}{\partial q_j}=\underbrace{\frac{\partial f_k}{\partial x_i}}_{(\nabla f)_i}\cdot\underbrace{\frac{\partial \hat{x}_i}{\partial q_j}}_{(j\text{-tý tečný vektor})_i}$$

49. Najděte výraz pro obecnou hybnost a obecnou energii nabité částice a v elektromagnetickém poli.

$$L=\frac{1}{2}m\vec{v}^2-e(\varphi(\vec{x},t)-\vec{v}\cdot\vec{A}(\vec{x},t))=\frac{1}{2}m\dot{x}_i^2-e(\varphi-\dot{x}_iA_i)$$

obecná hybnost $\quad p_j=\frac{\partial L}{\partial \dot{q}_j}=\frac{\partial L}{\partial \dot{x}_j}=\frac{1}{2}m2\dot{x}_i\delta_{ij}+e\delta_{ij}A_j$

$(q_j\equiv x_j\ \forall j=1,2,3)$

$$\boxed{p_j=m\dot{x}_j+eA_j}\neq \text{hybnost v MECH}$$

obecná energie

$$E=\left(\sum_{j=1}^{s}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right)-L=(m\dot{x}_j+eA_j)\dot{x}_j-\left[\frac{1}{2}m\dot{x}_i^2-e(\varphi-\dot{x}_iA_i)\right]=$$

$$=\underbrace{m\dot{x}_j^2}+\cancel{eA_j\dot{x}_j}-\underbrace{\frac{1}{2}m\dot{x}_i^2}+e\varphi-\cancel{e\dot{x}_iA_i}=\underline{\frac{1}{2}m\dot{x}_i^2+e\varphi}$$

54. Jak se změní zobecněná hybnost a zobecněná energie při změně Lagrangeovy funkce o $\frac{\hat{d}}{dt}h(\vec{q},t)$

$$L'=L+\frac{\hat{d}}{dt}h(\vec{q},t)=L+\frac{\partial h}{\partial q_i}\dot{q}_i+\frac{\partial h}{\partial t}$$

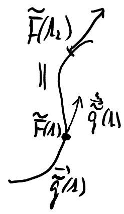
$L\rightarrow$ stejné pohyb. rce. (LR2D)

obecná hybnost $\quad p_j'=\frac{\partial L'}{\partial \dot{q}_j}=\underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{=p_j}+\frac{\partial h}{\partial q_i}\underbrace{\frac{\partial \dot{q}_i}{\partial \dot{q}_j}}_{=\delta_{ij}}+0=\underline{p_j+\frac{\partial h}{\partial q_j}}$

obecná energie $\quad E'=\frac{\partial L'}{\partial \dot{q}_j}\dot{q}_j-L'=\underbrace{\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j}+\cancel{\frac{\partial h}{\partial q_j}\dot{q}_j}-\underbrace{L}-\cancel{\frac{\partial h}{\partial q_i}\dot{q}_i}-\frac{\partial h}{\partial t}=\underline{E-\frac{\partial h}{\partial t}}$

Integrály Pohybu (I.P.)

Def Fce $F=F(\vec{q},\dot{\vec{q}},t)$ je I.P. $\iff\forall\vec{q}=\vec{q}(t)$ (Trajektorii) $\exists C\in\mathbb{R}$ takové, že $\overbrace{F(\vec{q}(t),\dot{\vec{q}}(t),t)}^{\tilde{F}(t)}=C\ \forall t$



$$\iff 0\overset{?}{=}\left.\frac{\hat{d}F}{dt}\right|_{\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)=0}=\frac{\partial F}{\partial q_j}\dot{q}_j+\frac{\partial F}{\partial \dot{q}_j}\ddot{q}_j+\frac{\partial F}{\partial t}=0$$

(Podél Trajektorie) $\Downarrow$ $\ddot{q}_j=G_j(\vec{q},\dot{\vec{q}},t)$ DOSADIT

Řešení lin. rovnic.

57. Dokažte, že funkce $F_1(x,\dot{x},t)=\dot{x}+gt$ $\ F_2(x,\dot{x},t)=\dot{x}^2+2gx$ jsou první integrály (I. P.) rovnice $\underline{\ddot{x}+g=0}$, $g=$konst.
a najděte pomocí nich trajektorii.

$\boxed{\ddot{x}=-g}$

$$\frac{dF_1}{dt}=\ddot{x}+g\ \overset{}{=}\ -g+g=0\ \checkmark \qquad \frac{dF_2}{dt}=2\dot{x}\ddot{x}+2g\dot{x}=2\dot{x}(\ddot{x}+g)=2\dot{x}\cdot 0=0\ \checkmark$$

$$\frac{dF_1}{dt}=\frac{\partial F_1}{\partial x}\dot{x}+\frac{\partial F_1}{\partial \dot{x}}\ddot{x}+\frac{\partial F}{\partial t}=0\cdot\dot{x}+1\cdot\ddot{x}+g=\ddot{x}+g \qquad \frac{dF_2}{dt}=\frac{\partial F_2}{\partial x}\dot{x}+\frac{\partial F_2}{\partial \dot{x}}\ddot{x}+\frac{\partial F_2}{\partial t}=2g\cdot\dot{x}+2\dot{x}\cdot\ddot{x}+0$$

Trajektorie $\quad 2s$ I.P. $\Rightarrow$ lze najít trajektorii $\vec{q}=\vec{q}(t)$ $\qquad 2s-1$ časově nezávislých I.P. $\Rightarrow$ Tvar Trajektorie

konst $F_1=\dot{x}+gt \quad\rightarrow\quad \dot{x}=F_1-gt$

$F_2=\dot{x}^2+2gx \qquad F_2=(F_1-gt)^2+2gx\ \Rightarrow\ x(t)=\frac{F_2-(F_1-gt)^2}{2g}=\underline{-\frac{1}{2}gt^2+F_1t+\frac{1}{2g}(F_2-F_1^2)}$

$F_2=\frac{2}{m}E$ (energie) $\qquad$ počáteční rychlost

58. DCV, $\ddot{x}+kx=0$

Jak Hledat I.P.

– podle toho co chybí v L za proměnné (Platí tehdy $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=\underline{\underline{0}}$ je $\vec{F}^{(nep)}=0$)

(1) $\frac{\partial L}{\partial t}=0$ $\ (L=L(\vec{q},\dot{\vec{q}}))$ $\Rightarrow$ obecná energie $E=\sum\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j-L$ je I.P.

(2) $\frac{\partial L}{\partial q_k}=0$ $\ k\in\hat{s}$ $\Rightarrow$ obecná hybnost $p_k=\frac{\partial L}{\partial \dot{q}_k}=$ konst je I.P. $\qquad$ Dk. $\frac{d}{dt}(p_k)=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)=\frac{\partial L}{\partial q_k}=0\ \checkmark$

q_k se nazývá cyklická souřadnice

~~(3) $\frac{\partial L}{\partial \dot{q}_k}=0$~~ ? (kvůli rychlosti $\dot{q}_j$ v L je vždy s)

55. Přepište Lagrangeovu funkci $L = \frac{1}{2}m(v_1^2+v_2^2+v_3^2) - mgx_3$ do cylindrických souřadnic a nejděte I. P.

a) Kartézské $s=3$ $\quad L = \underbrace{\frac{1}{2}m\dot{x}_i^2}_{T} - \underbrace{mgx_3}_{U}$

$\frac{\partial L}{\partial t} = 0 \Rightarrow E = \underbrace{\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i}_{2T} - L \underset{\text{zde}}{=} T + U = \frac{1}{2}m\dot{x}_i^2 + mgx_3$ je I.P.

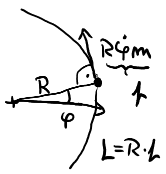
$\frac{\partial L}{\partial x_1} = 0 \Rightarrow p_1 = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$ je I.P. $\qquad \frac{\partial L}{\partial x_2} = 0 \Rightarrow p_2 = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 = \text{konst.}$ je I.P.

$\left.\begin{array}{l} E, p_1, p_2 \\ E, p_4 \end{array}\right\} \rightarrow \boxed{E, p_1, p_2, L_3}$

b) cylindrické $\quad L = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{z}^2) - mgz \qquad \frac{\partial L}{\partial t} = 0 \Rightarrow E = \ldots = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{z}^2) + mgz$

$x_1 = R\cos\varphi$
$x_2 = R\sin\varphi$
$x_3 = z$

φ je cyklická $\quad \frac{\partial L}{\partial \varphi} = 0 \Rightarrow p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \underbrace{mR^2\dot{\varphi}}_{\text{DCV.,} \; "L_3} = \text{konst.}$ je I.P. moment hybnosti,

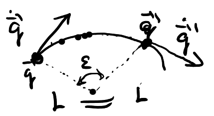


Teorém Noetherové

Transformace $\quad q_j' = q_j'(\vec{q}, \varepsilon) \in C^{(2)}$, invertibilní $\left|\frac{\partial q_j'}{\partial q_i}\right| \neq 0$, $\quad q_j'(\vec{q}, 0) = q_j$

$\varepsilon \in \mathbb{R}$ (funkce třídy; pro $\varepsilon = 0$ je to Identita)

$\dot{q}_j' = \frac{\hat{d}}{dt} q_j' = \frac{\partial q_j'}{\partial q_i}\dot{q}_i$



Invariance Lagrangeovy fce

$L(\vec{q}', \dot{\vec{q}}', t) = L(\vec{q}, \dot{\vec{q}}, t)$

$\Downarrow$

Veličina $\quad I = \sum_{k=1}^{s} \left(\frac{\partial L}{\partial \dot{q}_k}\right)\left(\frac{\partial q_k'}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I.P.

61. Nejděte integrály pohybu pro nabitou částici s nábojem e a hmotností m v homogenním magnetickém poli o indukci $\vec{B} = (0,0,B)$ s vektorovým potenciálem (a) $\vec{A} = \frac{1}{2}\underbrace{\vec{B} \times \vec{r}}_{(-Bx_2, Bx_1, 0)}$ $\quad$ (b) $\vec{A} = (0, Bx, 0)$ $\quad (\varphi = 0)$

konst.

$L = \frac{1}{2}m\vec{v}^2 - e(\varphi - \vec{v} \cdot \vec{A})$

(a) $\quad L = \underbrace{\frac{1}{2}m\dot{x}_i^2}_{T = \frac{1}{2}m\vec{v}^2} + \frac{1}{2}eB\underbrace{(-x_2\dot{x}_1 + x_1\dot{x}_2)}$

I.P.: $\quad \frac{\partial L}{\partial t} = 0 \Rightarrow E = \frac{\partial L}{\partial \dot{x}_j}\dot{x}_j - L = \ldots = \underline{\underline{\frac{1}{2}m\dot{x}_i^2}} \;//$ $\qquad \underset{\text{cyklická}}{\frac{\partial L}{\partial x_3}} = 0 \Rightarrow p_3 = \frac{\partial L}{\partial \dot{x}_3} = \underline{\underline{m\dot{x}_3}} \;//$

Teorém Noetherové ① Translace

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ konst. $\quad \varepsilon \in \mathbb{R}$

$x_i' = x_i + \varepsilon a_i \quad i = 1,2,3$
$\dot{x}_i' = \dot{x}_i + 0$

Invariance $L(\vec{x}', \dot{\vec{x}}', t) = L(\vec{x}, \dot{\vec{x}}, t)$

$L' = \frac{1}{2}m\dot{x}_i'^2 + \frac{1}{2}eB(-x_2'\dot{x}_1' + x_1'\dot{x}_2') = \frac{1}{2}m\dot{x}_i^2 + \frac{1}{2}eB(-(x_2 + \varepsilon a_2)\dot{x}_1 + (x_1 + \varepsilon a_1)\dot{x}_2)$

$= \underbrace{\frac{1}{2}m\dot{x}_i^2 + \frac{1}{2}eB(-x_2\dot{x}_1 + x_1\dot{x}_2)}_{L} + \underbrace{\frac{1}{2}eB(-a_2\dot{x}_1 + a_1\dot{x}_2)\varepsilon}_{\overset{!}{=}\, 0 \text{ chceme } \forall \varepsilon \forall \dot{x}_1 \forall \dot{x}_2}$

$\Rightarrow a_2\dot{x}_1 = a_1\dot{x}_2 \Rightarrow \boxed{a_1 = 0 = a_2}$

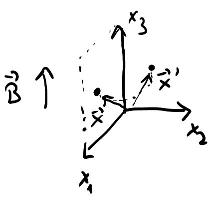
Pouze translace podél x_3 $\quad \vec{a} = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix}$ je symetrií L

I.P. $\quad I = \sum \underbrace{\left(\frac{\partial L}{\partial \dot{x}_k}\right)}_{p_k}\left(\frac{\partial x_k'}{\partial \varepsilon}\right)_{\varepsilon=0} = p_k a_k = 0 + 0 + p_3 a_3 = \underline{\underline{a_3 m\dot{x}_3}} \;// \quad a_3 \in \mathbb{R}$ konst.

$\left(\frac{\partial x_k'}{\partial \varepsilon}\right)_{\varepsilon=0} = (a_k)_{\varepsilon=0} = a_k$

② Rotace kolem x_3 $\rightarrow \underbrace{\left(\begin{array}{cc|c} \cos\varepsilon & -\sin\varepsilon & 0 \\ \sin\varepsilon & \cos\varepsilon & 0 \\ \hline 0 & 0 & 1 \end{array}\right)}$

$\vec{x}' = \mathbb{A}(\varepsilon)\vec{x}$



$x_1' = x_1\cos\varepsilon - x_2\sin\varepsilon$
$x_2' = x_1\sin\varepsilon + x_2\cos\varepsilon$
$x_3' = x_3$

($\varepsilon = 0$ je Identita)

$\dot{x}_1' = \dot{x}_1\cos\varepsilon - \dot{x}_2\sin\varepsilon$
$\dot{x}_2' = \dot{x}_1\sin\varepsilon + \dot{x}_2\cos\varepsilon$
$\dot{x}_3' = \dot{x}_3$

$Y_1 = \left(\frac{\partial x_1'}{\partial \varepsilon}\right)_{\varepsilon=0} = (-x_1\sin\varepsilon - x_2\cos\varepsilon)_{\varepsilon=0} = -x_2$

$Y_2 = \left(\frac{\partial x_2'}{\partial \varepsilon}\right)_{\varepsilon=0} = (x_1\cos\varepsilon - x_2\sin\varepsilon)_{\varepsilon=0} = x_1$

$Y_3 = \left(\frac{\partial x_3'}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$

Invariance $\quad L = \underbrace{\frac{1}{2}m\dot{x}_i^2}_{\text{je velikost vektoru}} + \frac{1}{2}eB(\underline{x_1\dot{x}_2 - x_2\dot{x}_1})$

(je Invariantní vůči rotaci)

$\longleftrightarrow \underline{x_1'\dot{x}_2' - x_2'\dot{x}_1'} = (x_1c\varepsilon - x_2s\varepsilon)(\dot{x}_1s\varepsilon + \dot{x}_2c\varepsilon) - (x_1s\varepsilon + x_2c\varepsilon)(\dot{x}_1c\varepsilon - \dot{x}_2s\varepsilon)$

$= \cancel{x_1\dot{x}_1 c\varepsilon s\varepsilon} - x_2\dot{x}_1 s^2\varepsilon + x_1\dot{x}_2 c^2\varepsilon - \cancel{x_2\dot{x}_2 s\varepsilon c\varepsilon}$

$- (\cancel{x_1\dot{x}_1 s\varepsilon c\varepsilon} - x_1\dot{x}_2 s^2\varepsilon + x_2\dot{x}_1 c^2\varepsilon - \cancel{x_2\dot{x}_2 c\varepsilon s\varepsilon}) = \underline{-x_2\dot{x}_1 + x_1\dot{x}_2}$ ✓

Integrál pohybu

$I = \sum \underbrace{\left(\frac{\partial L}{\partial \dot{x}_i}\right)}_{p_i}\underbrace{\left(\frac{\partial x_i'}{\partial \varepsilon}\right)_{\varepsilon=0}}_{Y_i} = p_i Y_i = p_1(-x_2) + p_2(x_1) + 0 = \underbrace{x_1p_2 - x_2p_1}_{L_3 \text{ (zobecněný moment hybnosti)}} = \underline{\underline{x_1(m\dot{x}_2 + \frac{1}{2}eBx_1) - x_2(m\dot{x}_1 - \frac{1}{2}eBx_2)}} \;//$