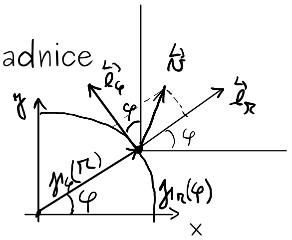
38. + 39. Křivočaré souřadnice



Polární

$x = r\cos\varphi \qquad r \in (0, +\infty)$

$y = r\sin\varphi \qquad \varphi \in \langle 0, 2\pi)$

Báze $B = (\vec{e}_x, \vec{e}_y)$

$\tilde{B} = (\vec{e}_r, \vec{e}_\varphi)$

$\tilde{B}^{ON} = (\vec{e}_r^{\,ON}, \vec{e}_\varphi^{\,ON})$

$\vec{e}_i^{\,ON} = \vec{e}_j S_{ji}^{ON}$

$\mathbb{S}^{ON} = {}^{\tilde{B}}(Id)^{B} = \left((\vec{e}_r^{\,ON})_B, (\vec{e}_\varphi^{\,ON})_B \right)$

$\mathbb{S}^{ON} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in SO(3)$

$v_x = \dot{x} = \dot{r}\cos\varphi + r(-\dot{\varphi}\sin\varphi)$

$v_y = \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

Souřadnicové křivky

$\gamma_\varphi : r \to \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

$\gamma_r : \varphi \to \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

$\vec{e}_r = \gamma_\varphi' = \dfrac{d\gamma_\varphi(r)}{dr} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = (\vec{e}_r)_B \qquad \vec{e}_r^{\,ON} = \dfrac{\gamma_\varphi'}{\|\gamma_\varphi'\|} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

$\| - \| - \| = 1$

$\vec{e}_\varphi = \gamma_r' = \dfrac{d\gamma_r(\varphi)}{d\varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix} = (\vec{e}_\varphi)_B \qquad \vec{e}_\varphi^{\,ON} = \dfrac{\gamma_r'}{\|\gamma_r'\|} = \dfrac{1}{r}\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$

$\| - \| - \| = \boxed{r \neq 0}$

$\mathbb{S} = {}^{\tilde{B}}(Id)^{B} = \left((\vec{e}_r)_B, (\vec{e}_\varphi)_B \right) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$

$\vec{\tilde{v}} = \mathbb{S}^{-1}\vec{v} \qquad (\mathbb{S}^{ON})^{-1} = (\mathbb{S}^{ON})^T$

$\vec{\tilde{v}}^{ON} = \begin{pmatrix} v_r^{ON} \\ v_\varphi^{ON} \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \end{pmatrix} \qquad \boxed{\begin{matrix} v_r^{ON} = \dot{r} \\ v_\varphi^{ON} = r\dot{\varphi} \end{matrix}}$

$\begin{pmatrix} a_r^{ON} \\ a_\varphi^{ON} \end{pmatrix} = (\mathbb{S}^{ON})^T \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} \Rightarrow \dots$

$\boxed{38.}$

Jacobiány

$B \xrightarrow{\mathbb{S}} \tilde{B} \qquad J = \det(\mathbb{S}) = \det\left(\dfrac{\partial x_i}{\partial \tilde{x}_j}\right) = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \neq 0$

$B \to \tilde{B}^{ON} \qquad J^{ON} = \det(\mathbb{S}^{ON}) = \det\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = 1$ ✓

"obecně" $\underset{B}{x_i} = x_i(\underset{\tilde{B}}{\tilde{x}_j}) \qquad (\vec{\tilde{e}}_j)_B = \dfrac{\partial \vec{x}}{\partial \tilde{x}_j} \quad \to \quad \boxed{\vec{\tilde{e}}_j = \dfrac{\partial}{\partial \tilde{x}_j}}$

$\vec{\tilde{e}}_j = \dfrac{\partial \square}{\partial \tilde{x}_j} = \underbrace{\dfrac{\partial x_i}{\partial \tilde{x}_j}}_{S_{ij}} \underbrace{\dfrac{\partial \square}{\partial x_i}}_{\vec{e}_i} = \vec{e}_i S_{ij}$

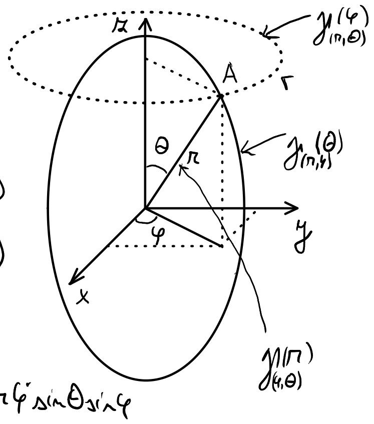
Jinak

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

tečné vektory k souřadnicovým křivkám

$(\vec{v})_{\tilde{B}} = \begin{pmatrix} v_r \\ v_\varphi \end{pmatrix} = \dfrac{d\vec{x}}{dt} = \dfrac{d}{dt}\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} = \underbrace{\dfrac{\partial}{\partial r}\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}}_{\vec{e}_r = \vec{\tilde{e}}_1} \cdot \dot{r} + \underbrace{\dfrac{\partial}{\partial \varphi}\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}}_{\vec{e}_\varphi = \vec{\tilde{e}}_2} \cdot \dot{\varphi} = \dot{r}\underbrace{\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}}_{\|\cdot\| = 1} + \dot{\varphi}\underbrace{\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix}}_{\|\cdot\| = r} = \overset{v_r^{ON}}{\dot{r}}\underbrace{\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}}_{\vec{e}_r^{\,ON}} + \overset{v_\varphi^{ON}}{\dot{\varphi} r}\underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}}_{\vec{e}_\varphi^{\,ON}}$

Sférické souřadnice



$\vec{x} \begin{cases} x = r\sin\theta\cos\varphi & \theta \in \langle 0, \pi \rangle \ \ (\, , \,) \\ y = r\sin\theta\sin\varphi & \varphi \in \langle 0, 2\pi) \\ z = r\cos\theta & r \in (0, +\infty) \end{cases}$

$\vec{e}_\theta = \gamma_{(r,\varphi)}'(\theta) = \dfrac{\partial \vec{x}}{\partial \theta} = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix} \qquad \vec{e}_\theta^{\,ON} = \dfrac{\vec{e}_\theta}{r} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}$

$\|\vec{e}_\theta\| = r$

$\vec{e}_\varphi = \gamma_{(r,\theta)}'(\varphi) = \dfrac{\partial \vec{x}}{\partial \varphi} = \begin{pmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix} \qquad \vec{e}_\varphi^{\,ON} = \dfrac{\vec{e}_\varphi}{r\sin\theta} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$

$\|\vec{e}_\varphi\| = r\sin\theta$

$\vec{e}_r = \gamma_{(\varphi,\theta)}'(r) = \dfrac{\partial \vec{x}}{\partial r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \vec{e}_r^{\,ON}$

$\|\vec{e}_r\| = 1$

$v_x = \dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$

$v_y = \dot{y} = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi$

$v_z = \dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$\underset{(x,y,z)}{B} \xrightarrow{\mathbb{S}} \underset{(\theta,\varphi,r)}{\tilde{B}^{ON}}$

$\mathbb{S}^{ON} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \in SO(3)$

$\vec{\tilde{v}} = \underset{= \mathbb{S}^T}{\mathbb{S}^{-1}} \vec{v}$

$(\vec{v})_{\tilde{B}^{ON}} = \begin{pmatrix} v_\theta \\ v_\varphi \\ v_r \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix}}^{(\mathbb{S}^{ON})^T} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} r\dot{\theta} \\ r\dot{\varphi}\sin\theta \\ \dot{r} \end{pmatrix}$

Jacobiány $\quad J^{ON} = \det(S^{ON}) = 1 \qquad J = \det\left(\frac{\partial x_i}{\partial \tilde{x}_j}\right) = (\vec{e}_\theta, \vec{e}_\varphi, \vec{e}_r) = r^2 \sin\theta \det S^{ON} = r^2 \sin\theta \neq 0$

$\vec{\tilde{x}} = \begin{pmatrix}\theta\\ \varphi\\ r\end{pmatrix}$ ... matice, není vektor!! $\qquad \Rightarrow r \neq 0 \wedge \theta \neq 0, \pi$

Lagrangeův Formalismus pro N hm. bodů v $\mathbb{R}^3 \to \mathbb{R}^{3N} \ni \vec{x}$

$\tilde{m}_1 \to m_1, m_2, m_3 \qquad T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha^2 = \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha i}^2$

① Volba kartézských souřadnic v inerciální soustavě

② Lagrangeova funkce v kartézských souřadnicích $\qquad L = T - U$ (kinetická energie; zobecněný potenciál)

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{\hat{d}}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right)$$

$$L(\vec{x}, \dot{\vec{x}}, t) = \underbrace{\frac{1}{2}\sum_{i=1}^{3N} m_i \dot{x}_i^2}_{T(\dot{\vec{x}})} - U(\vec{x}, \dot{\vec{x}}, t)$$

není určena jednoznačně $\quad L' = L + \frac{\hat{d}}{dt} h(\vec{x}, t)$

$L \to$ stejné rovnice

$$\boxed{\frac{\text{LR1.D}}{\forall j \in \widehat{3N}} \quad \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = F_j^{(\text{nep.})} + \underbrace{F_j^{(\text{vaz.})}}_{\sum_{k=1}^{r} \lambda_k \frac{\partial f_k}{\partial x_j}} \quad \wedge \quad f_k(\vec{x}, t) = 0 \quad \forall k \in \hat{r}}$$

nepotenciální $\quad$ vazební síly $\quad$ vazby

40. Napište Lagrangeovu funkci volného bezsilového hm. bodu v (a) kartéských (b) sférických (c) cylindrických souřadnicích.

$\quad$ = žádné vazby $\quad$ = žádné síly

(a) $L = T - \overset{=0}{U} = \underbrace{\frac{1}{2} m \dot{x}_i^2}_{} = \underbrace{\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}_{} = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m \dot{\vec{x}}^2$ $\qquad x = x_1,\ y = x_2,\ z = x_3$

(b) $L = T - U = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m (v_r^2 + v_\varphi^2 + v_\theta^2) = \underline{\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2 \sin^2\theta + r^2 \dot{\theta}^2)}$

$\quad = \frac{1}{2} m \dot{x}_i^2 = \qquad\qquad = //$

dosadil $\quad \dot{x}_1 = \dot{x} = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$

(c) $L = T - U = \frac{1}{2} m \dot{x}_i^2 - 0 = \frac{1}{2} m \left(\dot{R}^2(\cos^2\varphi + \sin^2\varphi) + R^2\dot{\varphi}^2(\sin^2\varphi + \cos^2\varphi) + \dot{z}^2\right) = \underline{\underline{\frac{1}{2} m (\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{z}^2)}}$

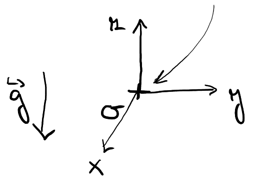
cylindrické $\quad x_1 = R\cos\varphi \quad \dot{x}_1 = \dot{R}\cos\varphi - R\dot{\varphi}\sin\varphi$

$\qquad x_2 = R\sin\varphi \quad \dot{x}_2 = \dot{R}\sin\varphi + R\dot{\varphi}\cos\varphi$

$\qquad x_3 = z \quad \dot{x}_3 = \dot{z}$

41. Napište Lagrangeovu funkci volného hm. bodu, na který působí homogenní gravitační pole a elastická centrální síla.

$\quad$ = bez vazeb



$\vec{x} = \vec{r} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$

① souřadnice $\quad \vec{F}_g = m\vec{g} \quad \vec{F}_e = -k\vec{x} = -k\vec{r} \quad \vec{g} = \begin{pmatrix}0\\ 0\\ -g\end{pmatrix}$

② $L = T - U \qquad U = U_g + U_e \qquad U_g = -m\vec{g}\cdot\vec{x} = -m(-g)z = mgz \qquad \vec{F}_g = -\nabla U_g$

$$\underline{L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz - \frac{1}{2} k (x^2 + y^2 + z^2)}$$

$U_e = \frac{1}{2} k \vec{x}^2 = \frac{1}{2} k (x^2 + y^2 + z^2) \qquad \vec{F}_e = -\nabla U_e$

LR1D. $\quad x: \ 0 + 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{1}{2} m 2\dot{x}\right) - \left(-\frac{1}{2} k 2x\right) = m\ddot{x} + kx$

$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{d}{dt}(m\dot{y}) - (-ky) = m\ddot{y} + ky$

$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \frac{d}{dt}(m\dot{z}) - (-mg - kz) = m\ddot{z} + mg + kz$

$\Leftrightarrow$ Newton $\quad m\ddot{\vec{x}} = \vec{F}_g + \vec{F}_e$

$$\boxed{\begin{aligned} m\ddot{x} &= 0 - kx \\ m\ddot{y} &= 0 - ky \\ m\ddot{z} &= -mg - kz \end{aligned}}$$