37) 1) použijeme následující

V (BEH - lin. op. v kor. f. - 7.4.4): T redukovaný ortogonální op. $E \Leftrightarrow ET \subset TE \Leftrightarrow ED_T \subset D_T$, $ETx = TEx$ pro každé $x \in D_T$

tvr.: chceme dokázat následující: $\left.\begin{matrix} ET \subset TE \\ T \text{ uzavíratelný} \end{matrix}\right\} \Rightarrow E D_{\overline{T}} \subset D_{\overline{T}}$, $E\overline{T}x = \overline{T}Ex$, $\forall x \in D_{\overline{T}}$

z předpokladů $ED_T \subset D_T$

T uzavíratelný $\to$ uzávěr v normě $\|\cdot\| + \|T\cdot\|$ $\Rightarrow ED_{\overline{T}} \subset D_{\overline{T}}$ (E omezený operátor)

mají ukázat $E\overline{T}x = \overline{T}Ex \quad \forall x \in D_{\overline{T}}$

z předpokladů máme $ETx = TEx \quad \forall x \in D_T$

zbývá ukázat „limitní body" $D_{\overline{T}} \setminus D_T$

$x_n \in D_T$

$ETx_n = TEx_n$

$\left.\begin{matrix} Tx_n \to y \\ x_n \to x \end{matrix}\right\} \exists x \in D_{\overline{T}} \ \overline{T}x = y$

limitní lim. dává $E\overline{T}x = \overline{T}Ex \Rightarrow \overline{T}$ redukuje E

2) A e.s.a. $\Leftrightarrow \overline{\operatorname{Ran}(A \pm i)} = \mathcal{H}$ (1)

E redukuje $A \Leftrightarrow ED_A \subset D_A$, $EAx = AEx \quad \forall x \in D_A$

chceme ukázat $\overline{\operatorname{Ran}(A \pm i)|_{E\mathcal{H}}} = E\mathcal{H}$

z (1) máme $\forall x \in \mathcal{H} \ \exists y_n : \lim_{n \to \infty} (A \pm i) y_n = x$

pro $x \in E\mathcal{H}$ máme $Ex = x$

$E(A \pm i)y_n = (A \pm i)\underbrace{Ey_n}_{\in E\mathcal{H}} \to Ex = x$

$\Rightarrow \overline{\operatorname{Ran}(A \pm i)|_{E\mathcal{H}}} = E\mathcal{H}$ tzn. $A|_{E\mathcal{H}}$ je e.s.a. na $E\mathcal{H}$

3) U unitární $\Rightarrow$ omezený; $EUx = UEx \quad \forall x \in \mathcal{H} \quad UU^* = U^*U = 1$

$EUx = UEx \qquad UU^{-1}Ex = EUU^{-1}x \Rightarrow UU^{-1}Ex = UEU^{-1}x \Rightarrow U^{-1}Ex = EU^{-1}x \Leftrightarrow U^*Ex = EU^*x$

$UE = EU \quad U : E\mathcal{H} \to E\mathcal{H}$

$U^*E = EU^* \quad U^* : E\mathcal{H} \to E\mathcal{H}$

$UU^*x = U^*Ux = x \quad \forall x \in E\mathcal{H} \Rightarrow U|$ unitární

38) 1) $\gamma = \mathrm{Tr}_{\mathcal{H}_1} P_2 \ \text{"}=\text{"} \sum_{k=1}^{\infty} \langle \phi_{k_1} \cdot | P_2 | \phi_{k_1} \cdot \rangle$

P_2 čistý stav $|\psi\rangle\langle\psi|$

smíšený stav $\sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$, $\lambda_k \geq 0$, $\sum_k \lambda_k = 1$

ONB: $\phi_1 = \phi$, ϕ_k ; výpočet provedem pro čistý stav, smíšený stav analogicky

$$\langle \phi, \gamma \phi \rangle = \mathrm{Tr}_{\mathcal{H}} (|\phi\rangle\langle\phi| \gamma)$$

$$= \sum_{k=1}^{\infty} \langle \phi \otimes \phi_k | P_2 | \phi \otimes \phi_k \rangle$$

2) $$\langle \phi, \gamma \phi \rangle = \sum_{k=1}^{\infty} \mathrm{Tr}_{\mathcal{H}^2} \left(|\phi_1 \otimes \phi_k\rangle\langle \phi_1 \otimes \phi_k | \psi\rangle\langle\psi| \right) \overset{P_2 \text{ pro fermiony}}{=} \sum_{k=2}^{\infty} \langle \phi_1 \otimes \phi_k | \psi \rangle\langle \psi | \phi_1 \otimes \phi_k \rangle$$

$$= \frac{1}{2} \left(\sum_{k=2}^{\infty} \langle \phi_1 \otimes \phi_k | \psi\rangle\langle\psi | \phi_1 \otimes \phi_k \rangle + \sum_{k=2}^{\infty} \langle \phi_1 \otimes \phi_k | \psi\rangle\langle\psi | \phi_1 \otimes \phi_k \rangle \right)$$

$$= \frac{1}{2} \left(\sum_{l=2}^{\infty} |\langle \phi_1 \otimes \phi_k | \psi \rangle|^2 + \sum_{k=2}^{\infty} |\langle \phi_m \otimes \phi_1 | \psi \rangle|^2 \right)$$

$$\leq \frac{1}{2} \left(\sum_{l,m=1}^{\infty} |\langle \phi_l \otimes \phi_m | \psi \rangle|^2 \right) = \frac{1}{2}$$

kde jsme využili $\langle \psi , \sum_{l,m} |\phi_l \otimes \phi_m\rangle\langle \phi_l \otimes \phi_m | \psi \rangle = \langle \psi | \mathbb{1} \psi \rangle = \langle \psi, \psi \rangle = 1$

$$\mathrm{Tr}_{\mathcal{H}^1} (|\phi\rangle\langle\phi| \gamma) = \mathrm{Tr}_{\mathcal{H}^2} \left((|\phi\rangle\langle\phi| \otimes \mathbb{1}) |\psi\rangle\langle\psi| \right) \qquad \mathbb{1} = \sum_m |\phi_m\rangle\langle\phi_m|$$

3) stejný postup jako v 3)

$$\langle \phi, \gamma \phi \rangle = \sum_{j=1}^{N-1} \sum_{k=1}^{\infty} \langle \phi \otimes \phi_{k_1} \otimes \cdots \otimes \phi_{k_{N-1}} , P_N \, \phi \otimes \phi_{k_1} \otimes \cdots \otimes \phi_{k_{N-1}} \rangle$$

$$= \sum_{1 \neq k_j \neq k_i \neq 1} \langle \phi \otimes \phi_{k_1} \otimes \cdots \otimes \phi_{k_{N-1}} , P_N \, \phi \otimes \phi_{k_1} \otimes \cdots \otimes \phi_{k_{N-1}} \rangle$$

$$\leq \frac{1}{N} \sum_{k_1 \dots k_N = 1}^{\infty} |\langle \phi_{k_N} \otimes \phi_{k_1} \otimes \cdots \otimes \phi_{k_{N-1}} | \psi \rangle|^2 = \frac{1}{N}$$

4) $$\gamma_\psi^1 = N \, \mathrm{Tr}_{\mathcal{H}^{N-1}} P_N \Rightarrow \underset{\gamma_\psi^1 \text{ adjungovaný operátor}}{0 \leq} \gamma_\psi^1 \overset{\text{z } 3)}{\leq} 1$$

39) 1) zavedeme matici

$$F=\begin{pmatrix} f_1(x_1) & \cdots & f_N(x_1) \\ \vdots & & \vdots \\ f_1(x_N) & \cdots & f_N(x_N) \end{pmatrix}$$

$$\det F=\sum_{\pi\in S_N}(-1)^{\operatorname{sgn}\pi}\prod_{i=1}^{N}F_{i\,\pi(i)}=\det F^T=\sum_{\pi\in S_N}(-1)^{\operatorname{sgn}\pi}\prod_{i=1}^{N}F_{\pi(i)\,i}$$

$$(f_1\wedge f_2\wedge\ldots\wedge f_N)(x_1\ldots x_N)=\frac{1}{\sqrt{N!}}\det F$$

2) $\langle f_i,f_j\rangle=\delta_{ij}$

$$\|f_1\wedge\ldots\wedge f_N\|_2^2=\frac{1}{N!}\int_{\mathbb{R}^{dN}}\Big[\sum_{\pi\in S^N}(-1)^{\operatorname{sgn}\pi}\prod_{j=1}^{N}f_{\pi(j)}(x_j)\Big]\Big[\sum_{\rho\in S^N}(-1)^{\operatorname{sgn}\rho}\prod_{k=1}^{N}f_{\rho(k)}(x_k)\Big]\prod_{l=1}^{N}dx_l^d$$

$$=\frac{1}{N!}\sum_{\pi,\rho\in S^N}(-1)^{\operatorname{sgn}\pi+\operatorname{sgn}\rho}\prod_{j=1}^{N}\int_{\mathbb{R}^d}f_{\pi(j)}(x_j)f_{\rho(j)}(x_j)\,dx_j^d$$

$$\prod_{j=1}^{N}a_j\prod_{k=1}^{N}b_k=\prod_{j=1}^{N}a_jb_j$$

$$=\frac{1}{N!}\sum_{\pi,\rho\in S^N}(-1)^{\operatorname{sgn}\pi+\operatorname{sgn}\rho}\delta_{\pi,\rho}=\frac{1}{N!}\sum_{\pi\in S^N}(-1)^{2\operatorname{sgn}\pi}=\frac{N!}{N!}=1$$

3)
$$\langle f_1\wedge\ldots\wedge f_N,g_1\wedge\ldots\wedge g_N\rangle=\frac{1}{N!}\int_{\mathbb{R}^{dN}}\Big[\sum_{\pi\in S^N}(-1)^{\operatorname{sgn}\pi}\prod_{j=1}^{N}f_{\pi(j)}(x_j)\Big]\Big[\sum_{\rho\in S^N}(-1)^{\operatorname{sgn}\rho}\prod_{k=1}^{N}f_{\rho(k)}(x_k)\Big]\prod_{l=1}^{N}dx_l^d$$

$$=\frac{1}{N!}\sum_{\pi,\rho\in S^N}(-1)^{\pi+\rho}\prod_{j=1}^{N}\langle f_{\pi(j)},g_{\rho(j)}\rangle\underset{\pi=\rho\tilde\pi}{=}\frac{1}{N!}\sum_{\tilde\pi,\rho\in S^N}(-1)^{2\rho+\tilde\pi}\prod_{j=1}^{N}\langle f_{\rho\tilde\pi(j)},g_{\rho(j)}\rangle$$

$$=\frac{1}{N!}\sum_{\tilde\pi,\rho\in S^N}(-1)^{\tilde\pi}\prod_{j=1}^{N}\langle f_{\tilde\pi(j)},g_j\rangle=\sum_{\tilde\pi\in S^N}(-1)^{\tilde\pi}\prod_{j=1}^{N}\langle f_{\tilde\pi(j)},g_j\rangle=\det\big[\langle f_i,g_j\rangle\big]$$

4)
$$F=\begin{pmatrix} f_1(x_1) & \cdots & f_N(x_1) \\ \vdots & & \vdots \\ f_1(x_N) & \cdots & f_N(x_N) \end{pmatrix}\qquad H=\begin{pmatrix} h_1(x_1) & \cdots & h_N(x_1) \\ \vdots & & \vdots \\ h_1(x_N) & \cdots & h_N(x_N) \end{pmatrix}\qquad H=AF$$

$$(h_1\wedge\ldots\wedge h_N)(x_1\ldots x_N)=\frac{1}{N!}\det H=\frac{1}{N!}\det AF=\det A\,\frac{1}{N!}\det F=\det A\,(f_1\wedge\ldots\wedge f_N)(x_1,\ldots,x_N)$$

40) 2 ⇒ 1 : BÚNO $f_N = \sum_{j=1}^{N-1} a_j f_j$

$$f_1 \wedge \dots \wedge f_{N-1} \wedge f_N = \sum_{j=1}^{N-1} a_i \left(f_1 \wedge \dots \wedge f_{N-1} \wedge f_j \right) = *$$

$$f_1 \wedge \dots \wedge f_{i-1} \wedge f_i \wedge f_{i+1} \wedge \dots \wedge f_{N-1} \wedge f_i \underset{\uparrow}{=} - f_1 \wedge \dots \wedge f_{i-1} \wedge f_i \wedge f_{i+1} \wedge \dots \wedge f_{N-1} \wedge f_i = 0$$

prohození 2 sloupců v determinantu (i-tý a N-tý)

$$* = \sum_{i=1}^{N} a_i \cdot 0 = 0$$

1 ⇒ 2 : ekvivalentní k $(\neg 2) \Rightarrow (\neg 1)$ ⟺ f_i jsou lineárně nezávislé ⇒ $f_1 \wedge \dots \wedge f_N \neq 0$

f_i lin nezávislé ⇒ $\|f_i\| \neq 0$

BÚNO $\|f_i\| = 1$

užitím Gram-Schmidtova ortogonalizačního procesu na f_i dostaneme g_i, $\langle g_i, g_j \rangle = \delta_{ij}$

užitím vz. 39-2) $\|g_1 \wedge \dots \wedge g_N\| = 1$

užitím vz 39-2) $\|f_1 \wedge \dots \wedge f_N\| = |\det A| \, \|g_1 \wedge \dots \wedge g_N\|$, kde A je matice přechodu od g_j na f_j

⇒ $\|f_1 \wedge \dots \wedge f_N\| \neq 0$ ⇒ $f_1 \wedge \dots \wedge f_N \neq 0$