1) provedeme aproximaci $e_\varepsilon^{is\Delta} = \mathcal{F}^{-1} \circ M_{V(k,\varepsilon)} \circ \mathcal{F}$; $\varepsilon > 0$; $V(k,\varepsilon) = \exp(-i|k|^2 s - \varepsilon|y|^2)$

$\psi \in L^1 \cap L^2 \Rightarrow \hat{\psi} \in L^2$ a $\forall x \ \hat{\psi}(x) < \infty$

$e^{-\varepsilon|y|^2} \in L^2$, $\hat{\psi} \in L^2$; $\|\varphi\psi\|_{L^1} \le \|\varphi\|_{L^p} \|\psi\|_{L^q}$ $\frac{1}{p} + \frac{1}{q} = 1$ $p,q > 1$ $\Rightarrow e^{-\varepsilon|y|^2}\hat{\psi} \in L^1$

$\|e^{-\varepsilon|y|^2}\hat{\psi}\|_{L^2} \le \|\hat{\psi}\|_{L^2} \Rightarrow e^{-\varepsilon|y|^2}\hat{\psi} \in L^2 \cap L^1 \Rightarrow (e_\varepsilon^{is\Delta}\psi)(x) < \infty$ pro všechna x

$$(e_\varepsilon^{is\Delta}\psi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iyx} e^{-is|y|^2 - \varepsilon|y|^2} \int_{\mathbb{R}^n} e^{-iyu} \psi(u)\, du\, dy$$

$$= (2\pi)^{-n} \iint_{\mathbb{R}^n \mathbb{R}^n} e^{iy(x-u) - is|y|^2 - \varepsilon|y|^2} \psi(u)\, dy\, du$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \psi(u) \left[\int_{\mathbb{R}^n} e^{iy(x-u) - is|y|^2 - \varepsilon|y|^2} dy \right] du$$

$$= (2\pi)^{-n} \left(\frac{\pi}{\varepsilon + is}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \psi(u) \exp\left(-\frac{|x-u|^2}{4(\varepsilon+is)}\right) du \xrightarrow[\varepsilon \to 0]{} \left(\frac{1}{4\pi i s}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \psi(u) \exp\left(-\frac{|x-u|^2}{4is}\right) du$$

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kde jsme využili: $-y_j(iy_j s - i(x_j - u_j) + \varepsilon y_j) = -(\varepsilon + is)\left(y_j - \frac{i(x_j - u_j)}{2(\varepsilon + is)}\right)^2 - \frac{(x_j - u_j)^2}{4(\varepsilon + is)}$

$$\int_{\mathbb{R}} \exp\left[-(\varepsilon+is)\left(y_j - \frac{i(x_j-u_j)}{2(\varepsilon+is)}\right)^2\right] dy_j = \sqrt{\frac{\pi}{\varepsilon + is}}$$

$$(i)^{-\frac{n}{2}} = \left(\exp\left(i\frac{\pi}{2}\right)\right)^{-\frac{n}{2}} = \exp\left(-i\frac{\pi n}{4}\right)$$

2) ukážeme, že limita platí pro hustou podmnožinu $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ v $L^2(\mathbb{R}^n)$ a pomineme první část
toto je dostačující, protože $e^{i\Delta s}$ je omezený operátor

Lemma: $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ hustá v $L^2(\mathbb{R}^n)$

dk.: $\psi_\varepsilon \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$$\psi_\varepsilon := \underbrace{e^{-\varepsilon x^2}}_{\in L^2} \underbrace{\psi(x)}_{\in L^2} \in L^1$$

$\|\psi - \psi_\varepsilon\|_{L^2}^2 = \int (1 - e^{-\varepsilon|x|^2})^2 |\psi(x)|^2 dx \xrightarrow[\varepsilon \to 0]{} 0$ neboť lze zaměnit lim a $\int$ protože integrál je uniformně omezený $\|\cdot$

$(T_s^\varepsilon\psi)(y) = \exp(i\Delta s)\left[\exp(-\varepsilon|x|^2)\psi(x)\right](y)$

$$\left\|e^{i\Delta s}\psi - e^{-i\frac{\pi n}{4}} \int_{\mathbb{R}^n} \frac{\exp\left(i\frac{|x-y|^2}{4s} - \varepsilon|y|^2\right)}{(4\pi s)^{\frac{n}{2}}} \psi(y)\, dy\right\|_{L^2} = \|(T_s^0 - T_s^\varepsilon)\psi\|_{L^2} \le \|\exp(i\Delta s)\| \, \|(1 - e^{-\varepsilon|x|^2})\psi\|_{L^2} \le$$

$$\le \|e^{i\Delta s}\| \, \|1 - \exp(-\varepsilon|x|^2)\|_\infty \|\psi\|_{L^2} \xrightarrow[\varepsilon \to 0]{} 0$$

3) $i\partial_s \phi(s,x) \stackrel{?}{=} -\Delta_x \phi(s,x) \Leftrightarrow i\partial_s(e^{i\Delta s}\psi(x)) \stackrel{?}{=} -\Delta_x(e^{i\Delta s}\psi(x))$

$$i\partial_s(\exp(i\Delta s)\psi(x)) = i\exp\left(-i\frac{\pi}{4}n\right) \int_{\mathbb{R}^n} \left[\frac{-\frac{n}{2}\exp\left(i\frac{|x-y|^2}{4s}\right)}{(4\pi)^{\frac{n}{2}} s^{\frac{n}{2}+1}} - \frac{i\frac{|x-y|^2}{4s^2}\exp\left(i\frac{|x-y|^2}{4s}\right)}{(4\pi)^{\frac{n}{2}} s^{\frac{n}{2}}}\right] \psi(y)\, dy$$

$$\frac{\partial^2}{\partial x_j \partial x_j}\left(f\left((x_i-y_i)(x_i-y_i)\right)\right) = \frac{\partial}{\partial x_j}\left[f'\left((x_i-y_i)(x_i-y_i)\right) 2(x_i-y_i)\right]$$

$$= 2n f'\left((x_i-y_i)(x_i-y_i)\right) + 4 f''\left((x_i-y_i)(x_i-y_i)\right)(x_j-y_j)(x_j-y_j)$$

$\Downarrow$

$$\Delta_x\left(\exp\left(i\frac{|x-y|^2}{4t}\right)\right) = 2n\frac{i}{4t}\exp\left(i\frac{|x-y|^2}{4t}\right) + 4\left(\frac{i}{4t}\right)^2 \exp\left(i\frac{|x-y|^2}{4t}\right)|x-y|^2$$

$$-\Delta_x\left(e^{i\Delta t}\psi(x)\right) = -\exp\left(-i\frac{\pi}{4}n\right)\int_{\mathbb{R}^n}\frac{\psi(y)}{(4\pi t)^{\frac{n}{2}}}\left(2n\frac{i}{4t}\exp\left(i\frac{|x-y|^2}{4t}\right) + 4\left(\frac{i}{4t}\right)^2 \exp\left(i\frac{|x-y|^2}{4t}\right)|x-y|^2\right) dy$$

porovnáním členů dostaneme $i\,\partial_t \phi(t,x) = -\Delta_x \phi(t,x)$