24) ve cvičení 9 $\overline{\mathrm{Ran}(V)} \subseteq \sigma(M_m)$

$\overline{\mathrm{Ran}\,V} \supseteq \sigma(M_m)$: $\lambda \in \mathbb{C} \setminus \overline{\mathrm{Ran}(V)}$ pak $x \to \frac{1}{\lambda - m(x)}$ omezená a spojitá $\Rightarrow$

$\Rightarrow f \in X \qquad R_\lambda f := \frac{f(x)}{\lambda - m(x)} = M_{\frac{1}{\lambda - m}} f \in X$

R_λ lineární a $R_\lambda(\lambda I_X - M_m) = (\lambda I_X - M_m) R_\lambda = I_X$

$\Rightarrow \lambda \in \rho(M_m)$ a resolventa je $R_\lambda = \frac{1}{\lambda - m}$

tvr.: $\sigma(M_m) = \mathrm{Ran}(V) = \overline{\mathrm{Ran}(V)}$

σ_r, σ_c: $\lambda_0 \in \mathrm{Ran}\,V \Rightarrow \exists t_0 \in [0,1] \quad m(t_0) = \lambda_0$

nechť $g \in \mathrm{Ran}(\lambda_0 - M_m)$ pak existuje $f \in X$

$g(t_0) = [(\lambda_0 - M_m) f](t_0) = (\lambda_0 - m(t_0)) f(t_0) = 0$

$\Rightarrow \overline{\mathrm{Ran}(\lambda_0 - M_m)} \subseteq \overline{\{g \in X : g(t_0) = 0\}} \neq X \Rightarrow \sigma_r(M_m) = \mathrm{Ran}(m)$ a $\sigma_c(M_m) = \emptyset$

σ_p: $\lambda_0 \in \sigma_p(M_m) \Rightarrow f_0 \in X \setminus \{0\} \quad m f_0 = M_m f_0 = \lambda_0 f_0$

ale $f_0 \in C([0,1])$ a $f_0 \neq 0 \ \forall x \in [a,b]$, $0 \leq a < b \leq 1 \Rightarrow m(x) \overset{!}{=} \lambda_0 \ \forall x \in [a,b]$

Označme $P(m) := \{ \lambda \in \mathbb{C} : 0 \leq \underset{\exists}{\alpha < \beta} \leq 1$, taková, že $m(x) = \lambda \ \forall x \in [\alpha, \beta] \}$

$f_0 \in X$, $\lambda_0 \in P(m)$

$$f_0(x) := \begin{cases} 1 & x \in [\alpha, \beta] \\ 0 & x \notin [a,b] \\ [0,1] & \text{jinak} \end{cases} \qquad \begin{array}{l} 0 \leq a < \alpha < \beta < b \leq 1 \\ m(x) = \lambda_0 \quad \forall x \in [a,b] \end{array}$$

$(M_m f_0)(x) = m(x) f_0(x) = \lambda_0 f_0(x) \quad \forall x \in [0,1]$

$\Rightarrow \sigma_p(M_m) = P(m)$

15) $\mathcal{D}(H)=H^2(\mathbb{R})$, $\mathcal{D}(-\Delta)=H^2(\mathbb{R})$

$\|M_{\chi(-a,a)}\psi\| = 0\cdot\|-\Delta\psi\| + \|\psi\| \Rightarrow H$ je s.a. na $H^2(\mathbb{R})\subset L^2(\mathbb{R})$

$M_{\chi(-a,a)}$ rel. kompaktní operátor vůči $-\Delta$; $\sigma_{ess}(A+K)=\sigma_{ess}(A)$ A s.a., K rel. kompaktní vůči A ($K(A-i)^{-1}$ kompakt.)

$\sigma(-\Delta)=\sigma_{ess}(-\Delta)=\mathbb{R}^+$

$\sigma(H)\subseteq\overline{\Theta(H)}$ obor hodnot $\Theta(H)=\{(\psi,H\psi)\mid\psi\in\mathcal{D}(H), \|\psi\|=1\}$

$\sigma_{ess}(H)=\mathbb{R}^+$

zbývá zjistit $\sigma_d(H)=\emptyset$

$c\geq0 \Rightarrow H\geq0 \Rightarrow \sigma(H)\subseteq\mathbb{R}^+ \Rightarrow \sigma_d(H)=\emptyset$

$c<0$: musíme vyřešit spektrální úlohu

$-\psi''+cM_{\chi(-a,a)}\psi=-E\psi \qquad E>0$

$x\notin(-a,a)$: $-\psi''=-E\psi \Rightarrow \psi=e^{\pm\sqrt{E}x}$

$x\in(-a,a)$: $-\psi''=(E-c)\psi \Rightarrow \psi=c_1e^{\sqrt{E+c}\,x}+c_2e^{-\sqrt{E+c}\,x}$ pro $E+c>0$ i $E+c<$

nyní musím slepit ψ v $x=-a$ a $x=a$ tak, že $\psi\in H^2(\mathbb{R})$ tzn. ψ,ψ' absolutně spojité a $\psi\in L^2$:

$E+c<0$ jinak nelze diferencovatelně navázat $e^{\pm\sqrt{E}x}$ a $e^{\pm\sqrt{E+c}\,x}$

$\psi_1=c_1e^{\sqrt{E}x} \qquad x<-a$

$\psi_2=c_{2,s}\sin(\sqrt{|E+c|}\,x)+c_{2,c}\cos(\sqrt{|E+c|}\,x) \qquad x\in(-a,a)$

$\psi_3=c_3e^{-\sqrt{E}x} \qquad x>a$

$$c_1e^{-\sqrt{E}a}\overset{!}{=}c_{2,s}\sin(-\sqrt{|E+c|}\,a)+c_{2,c}\cos(-\sqrt{|E+c|}\,a)$$
$$c_3e^{-\sqrt{E}a}\overset{!}{=}c_{2,s}\sin(\sqrt{|E+c|}\,a)+c_{2,c}\cos(\sqrt{|E+c|}\,a)$$
$$\sqrt{E}c_1e^{-\sqrt{E}a}\overset{!}{=}\sqrt{|E+c|}\,c_{2,s}\cos(-\sqrt{|E+c|}\,a)-\sqrt{|E+c|}\,c_{2,c}\sin(-\sqrt{|E+c|}\,a)$$
$$-\sqrt{E}c_3e^{-\sqrt{E}a}\overset{!}{=}\sqrt{|E+c|}\,c_{2,s}\cos(\sqrt{|E+c|}\,a)-\sqrt{|E+c|}\,c_{2,c}\sin(\sqrt{|E+c|}\,a)$$
$$\Downarrow$$
$$c_{2,s}\sin(-\sqrt{|E+c|}\,a)+c_{2,c}\cos(-\sqrt{|E+c|}\,a)=\sqrt{\frac{|E+c|}{E}}\,c_{2,s}\cos(-\sqrt{|E+c|}\,a)-\sqrt{\frac{|E+c|}{E}}\,c_{2,c}\sin(-\sqrt{|E+c|}\,a)$$
$$c_{2,s}\sin(\sqrt{|E+c|}\,a)+c_{2,c}\cos(\sqrt{|E+c|}\,a)=-\sqrt{\frac{|E+c|}{E}}\,c_{2,s}\cos(\sqrt{|E+c|}\,a)+\sqrt{\frac{|E+c|}{E}}\,c_{2,c}\sin(\sqrt{|E+c|}\,a)$$
$$2c_{2,c}\cos(\sqrt{|E+c|}\,a)=2\sqrt{\frac{|E+c|}{E}}\,c_{2,c}\sin(\sqrt{|E+c|}\,a) \Rightarrow 1=\sqrt{\frac{|E+c|}{E}}\operatorname{tg}(\sqrt{|E+c|}\,a) \vee c_{2,c}=0$$
$$2c_{2,s}\sin(\sqrt{|E+c|}\,a)=-2\sqrt{\frac{|E+c|}{E}}\,c_{2,s}\cos(\sqrt{|E+c|}\,a) \Rightarrow -1=\sqrt{\frac{|E+c|}{E}}\operatorname{cotg}(\sqrt{|E+c|}\,a) \vee c_{2,s}=0$$

26) 1⇒2: ať $f_n \to f$ máme $f_n \xrightarrow{w} f$

Zavedeme pro $B_R(0) = \{x \in \mathbb{R}^d : |x| \le R\}$

$\chi_R(A) := 1_{B_R(0)}(A)$; $\overline{\chi}_R(A) := 1 - \chi_R(A)$ pro A s.a. operátor

pro $g \in L^2(\mathbb{R}^d)$

$$\int_{|x|>R} |g(x)|^2\,dx = \|\overline{\chi}_R(X) g\|_2^2$$

$\overline{\chi}_R(X)$ ortogonální projektor, X operátor polohy

$$\|\overline{\chi}_R(X) f_n\| \le \|\overline{\chi}_R(X) f\| + \|\overline{\chi}_R(X)(f_n - f)\| \le \|\overline{\chi}_R(X) f\| + \underbrace{\|f_n - f\|}_{\to 0}$$

$$\Rightarrow \limsup_{n\to\infty} \|\overline{\chi}_R(X) f_n\| \le \|\overline{\chi}_R(X) f\| \quad \forall R > 0$$

víme, že $\overline{\chi}_R(X) \to 0$ silně pro $R \to \infty$ $\Rightarrow$ $\lim_{R\to\infty} \limsup_{n\to\infty} \|\overline{\chi}_R(X) f_n\| = 0$ protože $\|\overline{\chi}_R(X) f\| \xrightarrow[R\to\infty]{} 0$

stejným způsobem pro $\int_{|k|>L} \hat{g}(k)\,dk$ s $\overline{\chi}_L(P)$ a $P = -i\nabla$

2⇒1: $f_n \xrightarrow{w} f \in L^2(\mathbb{R}^d)$

$$\begin{aligned}
\|f_n - f\| &\le \|\chi_R(X)(f_n - f)\| + \|\overline{\chi}_R(X)(f_n - f)\| \\
&\le \|\chi_R(X)\chi_L(P)(f_n - f)\| + \|\chi_R(X)\overline{\chi}_L(P)(f_n - f)\| + \|\overline{\chi}_R(X)(f_n - f)\| \\
&\le \|\chi_R(X)\chi_L(P)(f_n - f)\| + \|\overline{\chi}_L(P)(f_n - f)\| + \|\overline{\chi}_R(X)(f_n - f)\| \\
&\le \|\underbrace{\chi_R(X)\chi_L(P)}_{\text{Hilbert-Schmidtův operátor (ukážeme níže)}}(f - f_n)\| + \|\overline{\chi}_L(P) f_n\| + \|\overline{\chi}_L(P) f\| + \|\overline{\chi}_R(X) f\| + \|\overline{\chi}_R(X) f_n\|
\end{aligned}$$

$$f - f_n \xrightarrow{w} 0 \Rightarrow \|\chi_R(X)\chi_L(P)(f - f_n)\| \to 0$$

$$\limsup_{n\to\infty} \|f - f_n\| \le \|\overline{\chi}_L(P) f\| + \|\overline{\chi}_R(X) f\| + \limsup_{n\to\infty} \|\overline{\chi}_L(P) f_n\| + \limsup_{n\to\infty} \|\overline{\chi}_R(X) f_n\|$$

vezmeme $L, R \to \infty$ $f \in L^2(\mathbb{R}^d) \Rightarrow \limsup_{n\to\infty} \|f - f_n\| = 0 \Rightarrow f_n \to f$

$\chi_R(X)\chi_L(P) \in \mathcal{J}_2$: ukážeme $\|\chi_R(X)\chi_L(P)\|_{HS}^2 < \infty$

$$(\chi_R(X)\chi_L(P))(x,y) = \frac{1}{(2\pi)^{d/2}} \chi_R(x)\hat{\chi}_L(x-y)$$

$$\|\chi_R(X)\chi_L(P)\|_{HS}^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\chi_R(X)\chi_L(P)(x,y)|^2\,dx\,dy = \frac{1}{(2\pi)^d}\|\chi_R(X)\|_2^2\,\|\hat{\chi}_L(P)\|_2^2 = \frac{1}{(2\pi)^d}\|\chi_R(X)\|^2\,\|\chi_L(P)\|_2^2$$

$$\Rightarrow \|\chi_R(X)\chi_L(P)\|_{HS}^2 < \infty \quad \forall R, L < \infty$$